Commun. Optim. Theory 2025 (2025) 5 https://doi.org/10.23952/cot.2023.5

MATHRES
Communications in Optimization Theory
Available online at http://cot.mathres.org

QUALITATIVE PROPERTIES OF SOLUTIONS OF SOME QUASILINEAR EQUATIONS RELATED TO BINGHAM FLUIDS

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Dedicated to the memory of Professor Roland Glowinski

Abstract. We consider a quasilinear parabolic equation and its associate stationary problem which correspond to a simplified formulation of a Bingham flow and we mainly study two qualitative properties. The first one concerns with the *absence and, respectively, disappearance in finite time, of the movement.* We show that there is a suitable balance between the L^1-norm of the forcing datum f_∞ and the measure of the spatial domain Ω (essentially saying that the forcing daum must be small enough) such that the corresponding solution $u_\infty(x)$ of the stationary problem is such that $u_\infty \equiv 0$ a.e. in Ω (even if $f_\infty \neq 0$). Moreover, if f_∞ is also the forcing term of the parabolic problem, and if the above mentioned balance is strict, for any $u_0 \in L^\infty(\Omega)$ there exists a finite time $T_{u_0,f_\infty} > 0$ such that the unique solution $u(t,x)$ of the parabolic problem globally stops after T_{u_0,f_∞}, in the sense that $u(t,x) \equiv 0$ a.e. in Ω, for any $t \geq T_{u_0,f_\infty}$. The second property concerns with the *formation of a positively measure "solid region"*. We show that if the above balance condition fails (i.e., when the forcing datum is large enough) then the solution $u_\infty(x)$ of the stationary problem satisfies that $u_\infty \neq 0$ in Ω and its "*solid region"* (defined as the set $\mathscr{S}(u_\infty) = \{x \in \Omega : \nabla u_\infty(x) = 0\}$) has a positive measure. Similar results are obtained for the symmetric solutions $u(t)$ of the parabolic problem. In addition, the convergence $u(t) \to u_\infty$ in $H_0^1(\Omega)$, as $t \to +\infty$, does not take place in any finite time.

Keywords. Bingham flows; Finite stopping time; Rearrangements comparison; Solid region; Variational inequalities.

2020 Mathematics Subject Classification. 35K59, 35K55, 35K92.

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Received July 21, 2022; Accepted October 21, 2022.

1. Introduction

The main goal of this paper is to establish some qualitative properties of solutions of the following nonlinear parabolic equation

$$\begin{cases} \partial_t u - \kappa\Delta u - g\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right) = f(t,x) & \text{in } \Omega\times(0,T), \\ u=0 & \text{on } \partial\Omega\times(0,T), \\ u(0)=u_0, & \text{on } \Omega, \end{cases} \tag{1.1}$$

and its associate stationary problem

$$\begin{cases} -\kappa\Delta u_\infty - g\nabla\cdot\left(\frac{\nabla u_\infty}{|\nabla u_\infty|}\right) = f_\infty(x) & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problems (1.1) and (1.2) are scalar problems which arise, for instance, in the study of some Bingham fluids after suitable simplifications. It corresponds to materials which behave as rigid bodies at low shear stress but flow as viscous fluids at high shear stress (see, e.g., the formulation presented in [19]). We recall that the problem is associated to Eugene C. Bingham (1878-1945) who, for the first time, in 1916, proposed a mathematical description for this visco-plastic behavior [6]. Common examples of Bingham fluids are tooth paste and paint. The Bingham model has also been used to describe the blood flow in small vessels, such as arterioles and capillaries, where the size of the vessel diameter is comparable to the size of blood cells; see, e.g., [36]. The isothermal flow of an incompressible Bingham visco-plastic medium, is modeled by a system of equations (of the *Navier-Stokes system* type) and it can be shown that, if for instance, the spatial domain is a cylinder then the problem can be reduced to some scalar problems as (1.1) and (1.2). The constants κ, g are the viscosity and the plasticity constants, respectively. In some sense, the above continuos medium behaves like a viscous fluid (of viscosity κ) in the *"viscous region"* $\mathscr{V}(u(t,.)) = \{x\in\Omega : |\nabla u(t,x)|>0\}$ and like a rigid medium in the *"solid region"* $\mathscr{S}(u(t,.)) = \{x\in\Omega : |\nabla u(t,x)|=0\}$.

We also mention that problems of this nature also arise in other different frameworks, as, for instance, in image processing where mainly $\kappa=0$. This leads to very delicate regularity questions (see, e.g., [1], [10], and [26]). In addition, problem (1.2) is a very good example of the family of (p,q)-double phase problems (see, e.g., the survey [29]).

The existence and uniqueness of L^2-weak solutions to problems (1.1) and (1.2) are today well-known results under different regularity assumptions on the data (see, e.g., [7], [21], [22], [23], [24], and the references therein). We point out that weaker notion of solutions can be considered under more general conditions on the data, nevertheless, since this paper is mainly devoted to the study of some qualitative properties we will not deal with other weaker notion of solutions. We will assume that Ω is a bounded regular open subset of $\mathbb{R}^N$, κ and g are positive constants and at least $f\in L^2(0,T:L^2(\Omega))$, $u_0\in L^\infty(\Omega)$ and $f_\infty\in L^2(\Omega)$. Thus the weak formulation of problems (1.1) and (1.2) can be expressed in terms of the following variational inequalities:

$$\begin{cases} u \in C([0,T]:L^2(\Omega)) \cap L^2(0,T:H_0^1(\Omega)), \partial_t u \in L^2(0,T:H^{-1}(\Omega)), \\ u(0) = u_0 \quad \text{on } \Omega, \\ \langle \partial_t u(t), v - u(t) \rangle + \int_\Omega \nabla u(t) \cdot \nabla (v - u(t)) \mathrm{d}x + g(j(v) - j(u(t))) \geq \int_\Omega f(t)(v - u(t)) \mathrm{d}x \\ \forall v \in H_0^1(\Omega) \text{ and a.e. } t \in (0,T), \end{cases} \tag{1.3}$$

and, respectively,

$$\begin{cases} u_\infty \in H_0^1(\Omega) \\ \kappa \int_\Omega \nabla u_\infty \cdot \nabla (v - u_\infty) \mathrm{d}x + g(j(v) - j(u_\infty)) \geq \int_\Omega f_\infty (v - u_\infty) \mathrm{d}x, \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{1.4}$$

where $\langle , \rangle$ denotes the duality between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, $j(v) = \int_\Omega |\nabla v| \mathrm{d}x$ for $v \in H_0^1(\Omega)$.

Note that both variational inequalities can be formulated also (see [7]) in terms of the subdifferential of the convex function $\varphi(v)$ given by

$$\varphi(v) = \begin{cases} \frac{\kappa}{2} \int_\Omega |\nabla v|^2 \mathrm{d}x + g \int_\Omega |\nabla v| \mathrm{d}x & \text{if } v \in H_0^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

as

$$\partial_t u + \partial \varphi(u) \ni f(t),$$

and

$$\partial \varphi(u_\infty) \ni f_\infty,$$

respectively. It was shown in [7] (see Theorem 15) that the domain of $(\partial\varphi)$ is $D(\partial\varphi) = H^2(\Omega) \cap H_0^1(\Omega)$. Nevertheless, we know that, in general, $u_\infty \notin C^2(\Omega)$, $u_\infty \notin H^3(\Omega)$ (see [24]). In addition, an equivalent formulation can be given in terms of suitable Lagrange multipliers (see [21] and [23])). Namely, concerning the parabolic problem (1.1), there exists $\lambda(t,x)$ such that

$$\begin{cases} \lambda \in [L^\infty((0,T) \times \Omega)]^N, \\ |\lambda| \leq 1, \ \lambda \cdot \nabla u = |\nabla u| \quad \text{a.e. in } (0,T) \times \Omega, \end{cases} \tag{1.5}$$

and

$$\partial_t u - \kappa \Delta u + g \operatorname{div} \lambda = \mathrm{f(t,x)}.$$

Analogously, concerning the associate stationary problem (1.2), there exists $\lambda_\infty(x)$ such that

$$\begin{cases} \lambda_\infty \in L^\infty(\Omega)^N, \\ |\lambda_\infty| \leq 1, \ \lambda_\infty \cdot \nabla u_\infty = |\nabla u_\infty| \quad \text{a.e. in } \Omega, \end{cases} \tag{1.6}$$

and

$$-\kappa \Delta u_\infty + g \operatorname{div} \lambda_\infty = \mathrm{f}_\infty(\mathrm{x}).$$

We also recall that under some additional conditions on f and u_0 it can be proved (see, e.g. [7], [19], and [20]) that if $f(t) \to f_\infty$ in $L^2(\Omega)$ as $t \to +\infty$, then $u(t) \to u_\infty$ in $H_0^1(\Omega)$, as $t \to +\infty$, where $u_\infty(x)$ is the (unique) solution of the associated stationary problem.

In order to simplify the exposition, we will assume that

$$f(t,x) \equiv f_\infty(x), \text{ a.e. in } \Omega,$$

and

$$f_\infty \neq 0 \text{ a.e. in } \Omega. \tag{1.7}$$

Notice that f_∞ is a simplified representation of the addition of the gradient pressure and the external forcing in the vectorial Navier-Stokes type formulation. Thus the condition (1.7) is very natural since the case $f_\infty \equiv 0$ a.e. in Ω has a restrictive meaning in fluid mechanics.

In order to simplify the exposition we will also assume that

$$f_\infty \gneqq 0 \text{ a.e. in } \Omega. \tag{1.8}$$

In this paper we are interested, among other things, in proving the two following different qualitative properties.

Property 1: *Absence and, respectively, disappearance in finite time, of the movement.* We will show that if

$$\|f_\infty\|_{L^1(\Omega)} \leq g\left(\frac{|\Omega|}{\omega_N}\right)^{(N-1)/N}, \tag{1.9}$$

where $|\Omega|$ and ω_N denote the measure of the Ω and the unit ball of $\mathbb{R}^N$, respectively, then the unique solution $u_\infty(x)$ of problem (1.2) satisfies $u_\infty \equiv 0$ a.e. in Ω (even if $f_\infty \neq 0$). Moreover, if the balance is strict

$$\|f_\infty\|_{L^1(\Omega)} < g\left(\frac{|\Omega|}{\omega_N}\right)^{(N-1)/N}, \tag{1.10}$$

for any $u_0 \in L^\infty(\Omega)$ there exists a finite time $T_{u_0,f_\infty} > 0$ such that the unique solution $u(t,x)$ of problem (1.1) globally stops after T_{u_0,f_∞}, in the sense that $u(t,x) \equiv 0$ a.e. in Ω, for any $t \geq T_{u_0,f_\infty}$.

Property 2: *Formation of a positively measure "solid region".* We will show that if the above balance condition fails, i.e.

$$\|f_\infty\|_{L^1(\Omega)} > g\left(\frac{|\Omega|}{\omega_N}\right)^{(N-1)/N}, \tag{1.11}$$

then the unique solution $u_\infty(x)$ of problem (1.2) satisfies $u_\infty \neq 0$ a.e. in Ω but its associated "*solid region*" (defined as the set $\mathscr{S}(u_\infty) = \{x \in \Omega : \nabla u_\infty(x) = 0\}$) has a positive measure. Moreover, there is a large class of initial data, $u_0 \in L^\infty(\Omega)$, for which the convergence $u(t) \to u_\infty$ in $H_0^1(\Omega)$, as $t \to +\infty$, does not take place in any finite time.

Concerning the stationary problem, the previous two properties can be stated as follows:

Theorem 1.1. *Assume $f_\infty \in L^2(\Omega)$ such that (1.8) holds. Let $u_\infty(x)$ be the unique solution of problem (1.2). Then:*

i) *If the balance condition (1.9) holds then $u_\infty \equiv 0$ a.e. in Ω.*

ii) *If the opposite balance condition (1.11) holds then $u_\infty \neq 0$, $u_\infty \in L^\infty(\Omega)$, and its associated "solid region" $\mathscr{S}(u_\infty)$ has a positive measure. In fact*

$$\mathscr{S}(u_\infty) \subset \left\{x \in \Omega\colon u_\infty(x) = \|u_\infty\|_{L^\infty(\Omega)}\right\}.$$

Our strategy to prove this and others qualitative properties relies on the use of classical symmetrization techniques. Namely, we consider the symmetrized problem

$$\begin{cases} -\kappa\Delta U_\infty - g\nabla\cdot\left(\dfrac{\nabla U_\infty}{|\nabla U_\infty|}\right) = f_\infty^*(x) & \text{in } \Omega^*, \\ U_\infty = 0 & \text{on } \partial\Omega^*, \end{cases} \tag{1.12}$$

and the associate parabolic problem

$$\begin{cases} \partial_t U - \kappa\Delta U - g\nabla\cdot\left(\frac{\nabla U}{|\nabla U|}\right) = f^*_\infty(x) & \text{in } \Omega^*\times(0,T), \\ U=0 & \text{on } \partial\Omega^*\times(0,T), \\ U(0)=U_0=u_0^*, & \text{on } \Omega^*, \end{cases} \tag{1.13}$$

where Ω^* is the ball centered at the origin with the same measure as Ω, u_0^* and f^*_∞ are the spherically symmetric rearrangement of the data u_0 and f_∞, respectively (see definition and properties in Section 2). Then we will start by proving the qualitative properties firstly to the symmetric problems and then we will deduce them for the non necessarily symmetric problem.

Concerning the parabolic problem, the qualitative properties we will get are the following:

Theorem 1.2. *Assume $f_\infty \in L^2(\Omega)$ such that (1.8) holds and let $u_0\in L^\infty(\Omega)$. Let $u(t,x)$ be the unique solution of problem (1.1).Then:*

i) *if the balance condition (1.10) holds then there exists a finite time $T_{u_0,f_\infty}>0$ such that $u(t,x)\equiv 0$ a.e. in Ω, for any $t\geq T_{u_0,f_\infty}$.*

ii) *If the opposite balance condition (1.11) holds and $u_0^*\in H^2(B(0,R))\cap H^1_0(B(0,R))$ is such that*

$$\frac{\kappa}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{du_0^*}{dr}(r))+\frac{g}{r^{N-1}}\frac{d}{dr}(r^{N-1}\lambda_0(r))+f_\infty(r)\geq 0 \text{ in } H^{-1}(0,R), \tag{1.14}$$

for some $\lambda_0(r)\in \operatorname{sign}\left(\frac{du_0}{dr}(r)\right)$ a.e. in $(0,R)$, with $S(u_0^*)\subset B(0,R_0)$ then we have

$$S(U(t))\subset B(0,R_0) \quad \text{for any } t\geq 0,$$

with

$$U(t,r)=\|U(t)\|_{L^\infty(B(0,R))} \quad \text{for any } r \text{ in } S(U(t)).$$

Moreover,

$$U(t)\to U_\infty \quad \text{in } H^1_0(B(0,R)) \quad \text{as } t\to+\infty, \tag{1.15}$$

where U_∞ is the unique solution of problem (1.12), and there exits a $R^*\in(0,R)$ such that

$$\|U(t)-U_\infty\|_{C^0([R^*,R])}>0 \quad \text{for any } t>0. \tag{1.16}$$

In addition, for any $t\geq 0$ we have that

$$\left|\{x\in\Omega:\ u(t,x)=\|u(t)\|_{L^\infty(\Omega^*)}\}\right| \leq \left|\{x\in\Omega^*:\ U(t,x)=\|U(t)\|_{L^\infty(\Omega^*)}\}\right|. \tag{1.17}$$

The useful tool we will use to reduce the study to the symmetric case is the following set of comparison results:

Theorem 1.3. *We have:*

1) *Let $u_\infty(x)$ be the unique solution of (1.2). Then*

$$u^*_\infty(x)\leq U_\infty(x) \quad a.e\ \ x\in\Omega^*, \tag{1.18}$$

and

$$|\nabla u^*_\infty(x)|\leq|\nabla U_\infty(x)| \quad a.e\ \ x\in\Omega^*. \tag{1.19}$$

2) *Let $u(t,x)$ be the unique solution of (1.1). Then, for any $t \in [0,T]$ we have the mass comparison of u and U*

$$\int_{B(0,r)} u^*(t,x)dx \leq \int_{B(0,r)} U(t,x)dx, \forall t > 0, \forall r \in [0,R], \tag{1.20}$$

assumed that $\Omega^* = B(0,R)$.

As already mentioned, after proving this result, the proof of the above qualitative properties is reduced to the establishment of the properties for solutions of problems (1.12) and (1.13) in the framework of symmetric solutions. This will be done in Section 3 where we will present also some other remarks. In this way, we improve many previous results in the literature, most of them dealing with the case $f_\infty = c$, where c is a positive constant (see, e.g., [19], [23], and [24]). We also recall that most of the previous results on the existence of a finite stopping time T_{u_0,f_∞} (see part i) of Theorem 1.2) hold assuming $f_\infty = 0$ and under the important restriction on the dimension of the space $N = 2$ (other conditions on the dimension of the space N when dealing with other non-Newtonian flows can be found in [2] and [11]).

2. Symmetrization: rearrangement comparison results

We begin by recalling some basic definitions from the theory of rearrangements of functions. Given a real valued measurable function u defined in a measurable subset Ω of $\mathbb{R}^N$, the distribution function of u is the function defined by

$$\mu(\tau) = |\{x \in \Omega : |u(x)| > \tau\}|. \tag{2.1}$$

The decreasing rearrangement of u is the smallest decreasing function $\widetilde{u}$ from $[0,\infty]$ into $[0,\infty]$ such that $\widetilde{u}(\mu(\tau)) \geq \tau$ for every τ. Equivalently

$$\widetilde{u}(s) = \inf\{\tau \geq 0 : \mu(\tau) < s\}. \tag{2.2}$$

The spherically symmetric rearrangement of u is the function $u^* : \Omega^* \to \mathbb{R}$ defined by

$$u^*(x) = \widetilde{u}(\omega_N |x|^N) = u^*(r), \ r = |x|, \tag{2.3}$$

where ω_N is the measure of the N-dimensional unit ball and Ω^* is the ball centered at the origin having the same measure as Ω, that is, $\Omega^* = B(0,R)$ with $R = \left(\frac{|\Omega|}{\omega_N}\right)^{1/N}$.

In order to prove Theorem 1.3, a suitable approximation of problem (1.2) is considered: given $f_\infty \in L^2(\Omega)$ and $1 < p < 2$, let us consider the following non-linear problem

$$\begin{cases} -\kappa\Delta u_p - g\Delta_p u_p = f_\infty & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where Δ_p is the p-Laplacian operator defined as

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v).$$

The existence and uniqueness of a solution u_p of (2.4) in the space $H_0^1(\Omega)$ can be deduced from classical results in the literature (see, e.g., [21]). In addition, we know that u_p is the solution of the variational problem

$$\min_{v\in H_0^1(\Omega)}\left(\frac{\kappa}{2}\int_\Omega |\nabla v|^2+\frac{g}{p}\int_\Omega |\nabla v|^p-\int_\Omega f_\infty v\right).$$

Moreover, since $f_\infty \gneqq 0$ a.e. in Ω we know that $u_p\geq 0$ a.e. on Ω (see, e.g., [14]).

It will be useful to apply the following result:

Lemma 2.1. *We have*

$$\lim_{p\to 1}\left\|u_p-u_\infty\right\|_{H_0^1(\Omega)}=0. \tag{2.5}$$

Proof. The solution $u_p\in H_0^1(\Omega)$ of (2.4) satisfies

$$\kappa\int_\Omega \nabla u_p(\nabla v-\nabla u_p)+\frac{g}{p}\int_\Omega |\nabla v|^p-\frac{g}{p}\int_\Omega \left|\nabla u_p\right|^p\geq \int_\Omega f_\infty(v-u_p)\quad \forall v\in H_0^1(\Omega). \tag{2.6}$$

Thus, by taking $v=u_\infty$ in (2.6) we obtain the following estimate:

$$\kappa\int_\Omega \nabla u_p(\nabla u_\infty-\nabla u_p)+\frac{g}{p}\int_\Omega |\nabla u_\infty|^p-\frac{g}{p}\int_\Omega \left|\nabla u_p\right|^p\geq \int_\Omega f_\infty(u_\infty-u_p). \tag{2.7}$$

In the same way, choosing $v=u_p$ as test function in (1.4), it follows:

$$\kappa\int_\Omega \nabla u_\infty(\nabla u_p-\nabla u_\infty)+g\int_\Omega \left|\nabla u_p\right|-g\int_\Omega |\nabla u_\infty|\geq \int_\Omega f_\infty(u_p-u_\infty). \tag{2.8}$$

From (2.7) and (2.8) we deduce:

$$\kappa\int_\Omega \left|\nabla u_\infty-\nabla u_p\right|^2+g\int_\Omega\left(\frac{1}{p}\left|\nabla u_p\right|^p-\left|\nabla u_p\right|\right)\leq g\int_\Omega\left(\frac{1}{p}|\nabla u_\infty|^p-|\nabla u_\infty|\right). \tag{2.9}$$

Since the following estimate holds

$$\int_\Omega\left(\frac{1}{p}\left|\nabla u_p\right|^p-\left|\nabla u_p\right|\right)\geq\left(\frac{1}{p}-1\right)|\Omega|, \tag{2.10}$$

inequality (2.9) gives

$$\kappa\left\|u_\infty-u_p\right\|_{H_0^1}^2\leq g\left(1-\frac{1}{p}\right)|\Omega|+g\int_\Omega\left(\frac{1}{p}|\nabla u_\infty|^p-|\nabla u_\infty|\right).$$

Taking the limit as $p\to 1$ in the above inequality we obtain (2.5).■

Now we will state some other technical lemmas which will be used later. Let Q be the function defined in $(0,+\infty)$ by

$$Q(r)=\kappa+g\,|r|^{p-2}, \tag{2.11}$$

and let

$$B(r)=\kappa r+g\,|r|^{p-2}\,r.$$

Lemma 2.2. *Let $u_p\in H_0^1(\Omega)$ be the non-negative solution of problem (2.4). Then the decreasing function*

$$\tau\to\int_{\{u_p>\tau\}}Q(\left|\nabla u_p\right|)\left|\nabla u_p\right|^2dx$$

is Lipschitz continuous and the following inequality holds a.e. $\tau > 0$,

$$0 \leq -\frac{d}{d\tau}\int_{\{u_p>\tau\}} Q(|\nabla u_p|)\,|\nabla u_p|^2\,dx \leq \int_0^{\mu_p(\tau)} \widetilde{f_\infty}(s)ds, \tag{2.12}$$

where $\mu_p(\tau)$ *denotes the distribution function of* u_p *and* $\widetilde{f_\infty}$ *is the decreasing rearrangement of* f_∞.

Lemma 2.3. *The following estimate holds a.e.* $\tau > 0$

$$1 \leq -\mu_p'(\tau)[N\omega_N^{\frac{1}{N}}\mu_p(\tau)^{1-\frac{1}{N}}]^{-1}B^{-1}\left(\frac{-1}{N\omega^{\frac{1}{N}}\mu_p(\tau)^{1-\frac{1}{N}}}\right)\frac{d}{d\tau}\int_{\{u_p>\tau\}} Q\left(|\nabla u_p|\right)|\nabla u_p|^2\,dx. \tag{2.13}$$

Lemma 2.4. *The decreasing rearrangement* $\widetilde{u}_p$ *of* u_p *is absolutely continuous in* $(0,|\Omega|)$ *and satisfies*

$$-\frac{d\widetilde{u}_p}{ds}(s) \leq \frac{1}{\alpha(s)}B^{-1}\left(\frac{1}{\alpha(s)}\int_0^s \widetilde{f}_\infty(\theta)d\theta\right), \tag{2.14}$$

where

$$\alpha(s) = N\omega_N^{\frac{1}{N}}s^{\frac{N-1}{N}}. \tag{2.15}$$

Moreover, if $f_\infty = f_\infty^*$, *we have*

$$-\frac{d\widetilde{u}_p}{ds}(s) = \frac{1}{\alpha(s)}B^{-1}\left(\frac{1}{\alpha(s)}\int_0^s \widetilde{f}_\infty(\theta)d\theta\right). \tag{2.16}$$

For the proofs of Lemmas 2.2, 2.3 and 2.4 we refer to [34], [35] and [14]. We are now in conditions to prove part 1) of Theorem 1.3.

Proof of part 1) of Theorem 1.3. Let $U_p \in H_0^1(\Omega^*)$ be the solution of the symmetrized problem

$$\begin{cases} -\kappa\Delta U_p - g\Delta_p U_p = f_\infty^* & \text{in } \Omega^*, \\ U_p = 0 & \text{on } \partial\Omega^*. \end{cases} \tag{2.17}$$

Since f_∞^* is non-increasing, we deduce (see, e.g., [14]) that

$$U_p^* = U_p \quad \text{on } \Omega^*.$$

On the other hand, from Lemmas 2.2, 2.3 and 2.4 we deduce

$$-\alpha(s)\frac{d\widetilde{u}_p(s)}{ds} \leq -\alpha(s)\frac{d\widetilde{U}_p(s)}{ds} \quad \text{a.e } s \in (0,|\Omega|].$$

From this inequality, using the definition of symmetric rearrangement, we obtain the estimate

$$|\nabla u_p^*(x)| \leq |\nabla U_p(x)| \quad \text{a.e. } x \in \Omega^*. \tag{2.18}$$

Furthermore, from Theorem 1 of [35] we have

$$u_p^*(x) \leq U_p(x) \quad \text{a.e. } x \in \Omega^*. \tag{2.19}$$

Using Lemma 2.1 we get that

$$u_p^* \to u_\infty^* \text{ and } U_p \to U_\infty \text{ strongly in } H_0^1(\Omega^*). \tag{2.20}$$

From (2.20), passing to the limit, as $p \to 1$, in (2.18) and (2.19) we get estimates (1.18) and (1.19).■

Proof of part 2) of Theorem 1.3. It follows the same global idea of the proof of part 1) but with important additional arguments, as usual in the application of rearrangement techniques to nonlinear parabolic problems. Several alternatives are possible. Here we will follow the approach indicated by the second author in [17] (see also [15], [16], and [18]) since it leads to some quantitative inequalities which are not evident to be obtained through the implicit time discretization used in the theory of accretive operators. We start by approximating the diffusion operator by the degenerate quasilinear operator given in terms of the p-Laplacian operator for $p \in (1,2)$. Thus, we consider the parabolic problem

$$\begin{cases} \partial_t u - \kappa \Delta u - g\Delta_p u = f_\infty(x) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{on } \Omega, \end{cases} \tag{2.21}$$

and its symmetrized problem

$$\begin{cases} \partial_t U - \kappa \Delta U - g\Delta_p U = f^*_\infty(x) & \text{in } \Omega^* \times (0,T), \\ U = 0 & \text{on } \partial\Omega^* \times (0,T), \\ U(0) = U_0 = u_0^*, & \text{on } \Omega^*. \end{cases} \tag{2.22}$$

It can be easily shown (using the techniques of [7]: see also Chapter 4 of [14]) that the associated diffusion operator is the subdifferential of a convex function $\partial\varphi_p(u)$ on the Hilbert space $H = L^2(\Omega)$, with $\varphi_p(u)$ given by

$$\varphi_p(v) = \begin{cases} \dfrac{\kappa}{2}\displaystyle\int_\Omega |\nabla v|^2 dx + g\int_\Omega |\nabla v|^p dx & \text{if } v \in H_0^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we know (see Theoreme 3.6 of [9]) that, without loss of generality, we can assume that $u_0 \in D(\partial\varphi_p)$ and that we are dealing with strong solutions, in the sense that $u_t(t) \in L^2(\Omega)$ for a.e. $t > 0$ (and, similarly, $U_t(t) \in L^2(\Omega^*)$). Then we can apply the main result of [30] proving that the following identity holds

$$\int_{u>\theta} \frac{\partial u}{\partial t}(t,x)dx = \int_0^{\mu(\theta)} \frac{\partial \widetilde{u}(t,\sigma)}{\partial t} d\sigma = \frac{\partial k}{\partial t}(t,\mu(\theta))$$

where

$$k(t,s) = \int_0^s \widetilde{u}(t,\sigma)d\sigma,$$

and $\widetilde{u}(t,\cdot)$ is the scalar decreasing rearrangement of the solution u. Then applying the same techniques of the proof of part 1) we get that $k(t,s)$ satisfies

$$(FN^*)\begin{cases} \dfrac{\partial k}{\partial t} - \kappa a_2(s)\dfrac{\partial^2 k}{\partial s^2} + g a_p(s)\left|\dfrac{\partial^2 k}{\partial s^2}\right|^{p-2}\dfrac{\partial^2 k}{\partial s^2} \le \displaystyle\int_0^s \widetilde{f}_\infty(\sigma)d\sigma, & s \in (0,|\Omega|), t \in (0,T) \\ k(t,0) = 0, \quad k(t,|\Omega|) = 0, & t \in (0,T), \\ k(0,s) = \displaystyle\int_0^s \widetilde{u}_0(\sigma)d\sigma & s \in (0,|\Omega|), \end{cases}$$

where, for $p \in (1,2]$,

$$a_p(s) := \left[N\omega_N^{1/N} s^{(N-1)/N}\right]^p.$$

Analogously, by defining

$$K(t,s)=\int_0^s \widetilde{U}(t,\sigma)d\sigma,$$

we get that

$$(FN)\begin{cases} \dfrac{\partial K}{\partial t}-\kappa a_2(s)\dfrac{\partial^2 K}{\partial s^2}+ga_p(s)\left|\dfrac{\partial^2 K}{\partial s^2}\right|^{p-2}\dfrac{\partial^2 K}{\partial s^2}=\displaystyle\int_0^s \widetilde{f}_\infty(\sigma)d\sigma, & s\in(0,|\Omega|),t\in(0,T)\\ K(t,0)=0, \quad K(t,|\Omega|)=0, & t\in(0,T),\\ K(0,s)=\displaystyle\int_0^s \widetilde{u}_0(\sigma)d\sigma & s\in(0,|\Omega|). \end{cases}$$

As in [17], using that the corresponding fully nonlinear operator is T-accretive in the Banach space $L^\infty(0,|\Omega|)$, we get the comparison

$$k(t,s)\le K(t,s) \qquad \forall t\in[0,T], \quad \forall s\in(0,|\Omega|),$$

which implies the inequality (1.20).
The convergence of solutions when $p\searrow 1$ can be obtained through a very easy modification of the proof given in part 1) since we have the convergence of the subdifferentials $\partial\varphi_p(u)\to\partial\varphi(u)$ in the sense of resolvents $(I+\delta\partial\varphi_p(u))^{-1}z\to(I+\delta\partial\varphi(u))^{-1}z$, for any $z\in L^2(\Omega)$ and any $\delta>0$ (see Proposition 3.14 of [9]).■

Remark 2.5. *The above convergence argument holds under much more general conditions (see, e.g.,* [26], [33]*, and their many references).*

3. Qualitative property of the radially symmetric problems and proofs of the main results

3.1. **The stationary symmetric problem.** In this subsection we shall consider only the radially symmetric case in which $\Omega=B(0,R)$, the open ball of radius R centered at the origin, and the datum of the stationary problem (1.2) f_∞ is assumed to be a radially symmetric and nonnegative function.

The uniqueness of solutions implies that the problem can be formulated in the following terms: given

$$f_\infty\ge 0 \quad \text{with} \quad \int_0^R [f_\infty(r)]^2 r^{N-1}dr<+\infty, \tag{3.1}$$

find $u_\infty\in H_0^1(B(0,R))$ such that

$$\begin{cases} -\dfrac{\kappa}{r^{N-1}}\dfrac{d}{dr}(r^{N-1}\dfrac{du_\infty}{dr}(r))-\dfrac{g}{r^{N-1}}\dfrac{d}{dr}(r^{N-1}\lambda_\infty(r))=f_\infty(r), \text{ for } r\in(0,R),\\ u_\infty(R)=0 \quad \text{and} \quad \dfrac{du_\infty}{dr}(0)=0, \end{cases} \tag{3.2}$$

for some scalar-valued function $\lambda_\infty\in L^\infty(0,R)$ satisfying

$$|\lambda_\infty(r)|\le 1 \quad \text{and} \quad \lambda_\infty(r)\frac{du_\infty}{dr}(r)=\left|\frac{du_\infty}{dr}(r)\right| \quad \text{a.e. in } (0,R). \tag{3.3}$$

Note that, by the regularity proved in [7], we know that $u_\infty \in H^2(B(0,R))$. In fact, this implies that $u_\infty \in C^1([0,R))$ and that $r^{N-1}\lambda_\infty(r)$ is an element of $H^1(B(0,R))$ and, that, in particular, $\lambda_\infty \in C^0(0,R)$. We also mention that condition (3.3) can be equivalently written as

$$\lambda_\infty(r) \in \operatorname{sign}\left(\frac{du_\infty}{dr}(r)\right) \text{ a.e. in } (0,R),$$

where sign denotes the maximal monotone graph of $\mathbb{R}$ given by $\operatorname{sign}(s) = +1$ if $s > 0$, $\operatorname{sign}(s) = -1$ if $s < 0$ and $\operatorname{sign}(0) = [-1,+1]$.

We are interested in studying the *solid region* generated by the solution,

$$S(u_\infty) = \left\{ r \in [0,R) : \frac{du_\infty}{dr}(r) = 0 \right\},$$

and its dependence with respect to the data f_∞, R, κ and g.

As a matter of fact, due to applicability of symmetrization techniques, we want to know sufficient conditions in order to get a nontrivial (radially symmetric) solution $u_\infty(r) > 0$ and non-increasing for $r \in (0,R)$. Thus, we shall consider only the case in which the *solid region* generated by the solution, $S(u_\infty)$ contains a neighborhood of the origin. As we shall see, in our case it is related to the monotonicity of the function $\lambda_\infty(r)$. In this framework, the interesting case arises when function $f_\infty(r)$ satisfies an additional condition:

$$f_\infty(r) \text{ is a non-increasing function of } r. \tag{3.4}$$

We have

Proposition 3.1. *Assume f_∞ satisfying (3.1) and (3.4). Then:*

a) *if*

$$\frac{1}{gR^{N-1}} \int_0^R s^{N-1} f_\infty(s)\,ds \le 1 \tag{3.5}$$

the non-increasing solution $u_\infty(r)$ of (3.2) is the trivial solution $u_\infty(r) \equiv 0$ and $\lambda_\infty(r)$ is the decreasing function given by

$$\lambda_\infty(r) = -\frac{1}{gr^{N-1}} \int_0^r s^{N-1} f_\infty(s)\,ds, \quad \text{for any } r \in (0,R], \tag{3.6}$$

b) *if we assume that there exists a $R_0 \in (0,R)$ (R_0 depending on g) such that*

$$\frac{1}{gR_0^{N-1}} \int_0^{R_0} s^{N-1} f_\infty(s)\,ds = 1, \tag{3.7}$$

then $u_\infty \in L^\infty(B(0,R))$ and the non-increasing profile of the solution $u_\infty(r)$ is given by

$$u_\infty(r) = \begin{cases} M_\infty & \text{if } r \in (0,R_0), \\ \dfrac{1}{\kappa} \displaystyle\int_r^R \left(\frac{1}{\sigma^{N-1}} \int_0^\sigma s^{N-1} f_\infty(s)\,ds \right) d\sigma & \text{if } r \in (R_0,R), \end{cases}$$

with

$$M_\infty = \frac{1}{\kappa} \int_{R_0}^R \left(\frac{1}{\sigma^{N-1}} \int_0^\sigma s^{N-1} f_\infty(s)\,ds \right) d\sigma = \|u_\infty\|_{L^\infty(B(0,R))},$$

and $\lambda_\infty(r)$ is given by the non-increasing function

$$\lambda_\infty(r) = \max\left\{-\frac{1}{gr^{N-1}}\int_0^r s^{N-1} f_\infty(s)ds,\ -1\right\} \quad \text{for any } r \in (0,R]. \tag{3.8}$$

Proof. We start by assuming *a strict balance i.e. we assume the stronger condition*

$$\frac{1}{gR^{N-1}}\int_0^R s^{N-1} f_\infty(s)ds < 1. \tag{3.9}$$

Given $g > 0$ and $f_\infty(s)$ satisfying (3.1), let us introduce the function

$$\psi(r) = \frac{1}{gr^{N-1}}\int_0^r s^{N-1} f_\infty(s)ds.$$

Then, by differentiation with respect to r we can see that $\psi(r)$ is a strictly increasing function. Indeed, this is equivalent to have the following condition on $f_\infty(r)$

$$f_\infty(r) > \frac{(N-1)}{r^N}\int_0^r s^{N-1} f_\infty(s)ds \text{ a.e. } r \in (0,R). \tag{3.10}$$

If we define the functions

$$\alpha(r) = \frac{1}{gr^{N-1}} \text{ and } \beta(r) = \int_0^r s^{N-1} f_\infty(s)ds,$$

then (3.10) implies that $\alpha'(r)\beta(r) + \alpha(r)\beta'(r) > 0$ a.e. $r \in (0,R)$. Note that condition (3.10) is an easy consequence of the assumption (3.4) since given $r \in (0,R)$ we have $f_\infty(r) \geq f_\infty(s)$ a.e. $s \in (0,r)$ and integrating we get that

$$f_\infty(r) \geq \frac{N}{r^N}\int_0^r s^{N-1} f_\infty(s)ds \text{ a.e. } r \in (0,R),$$

which, in turn, implies (3.10).

Moreover, by l'Hôpital rule, $\psi(0) = 0$, and so, $\psi(r) > 0$ for any $r \in (0,R]$. We also recall that, since $|\lambda_\infty(r)| \leq 1$, then

$$-\lambda_\infty(r) = \min\{\psi(r), 1\} \text{ for any } r \in (0,R], \tag{3.11}$$

which will lead to the expression (3.8). On any positively measured subset of the solid region $S(u_\infty)$ the equation in (3.2) reduces to the condition

$$-\frac{g}{r^{N-1}}\frac{d}{dr}(r^{N-1}\lambda_\infty(r)) = f_\infty(r),\ r \in S(u_\infty).$$

Moreover, $\frac{du_\infty}{dr}(0) = 0$ and if we denote by R_0 (with $R_0 \in (0,R]$) to the boundary of $S(u_\infty)$, we get (since the profile $u_\infty(r)$ is non-decreasing) that necessarily

$$\lambda_\infty(r) = -\psi(r) = -\frac{1}{gr^{N-1}}\int_0^r s^{N-1} f_\infty(s)ds \text{ for any } r \in [0,R_0]. \tag{3.12}$$

Now, to prove a) it suffices to use the fact that condition (3.5) implies that $\lambda_\infty(R) = -\psi(R) \in (-1,0)$. Thus, $\lambda_\infty(r) \in \text{sign}(0)$ a.e. in $(0,R)$ and the choice $u_\infty(r) \equiv 0$ satisfies all the requirements as to be a solution of problem (3.2). Moreover, by the uniqueness of solutions, $u_\infty(r) \equiv 0$ is the unique choice satisfying all the conditions of weak solution of (3.2).

If we assume now condition (3.5) then it is clear that we can approximate f_∞ by a sequence of functions $f_{\infty,n}$ satisfying *(3.1), (3.4)* and the strict balance (3.9). Then, the respective solutions of the problems (3.2) must satisfy that $u_{\infty,n} \equiv 0$ on Ω and as the convergence $f_{\infty,n} \to f_\infty$ in

$L^2(B(0,R))$ implies the convergence $u_{\infty,n} \to u_\infty$ in $L^2(B(0,R))$ (recall that the operator is the subdifferential of a convex functional on $L^2(B(0,R))$) we finally deduce that $u_\infty \equiv 0$ on Ω.
To prove b) we assume conditions (3.10) and (3.7). Then, the expression (3.12) and the facts that $\psi(r)$ is a strictly increasing function and that we must have $|\lambda_\infty(r)| \leq 1$ for any $r \in [0,R]$ imply that, necessarily, $\lambda_\infty(r) > -1$ for any $r \in (0,R_0)$ and $\lambda_\infty(r) = -1$ for any $r \in [R_0,R]$ (see (3.11)).
Finally, once we have determined the function $\lambda_\infty(r)$ on $[0,R]$, the (unique) expression for $u_\infty(r)$ is found by integrating twice in the equation

$$-\frac{d}{dr}\left(r^{N-1}\frac{du_\infty}{dr}(r)\right) = \frac{r^{N-1}}{\kappa}f_\infty(r),$$

and using the fact that $u_\infty(R) = 0$ and $\frac{du_\infty}{dr}(r) = 0$ for any $r \in [0,R_0]$.■

Remark 3.2. *The above result gives a necessary and sufficient condition in order to have a trivial solution $u_\infty(r) \equiv 0$ of problem (3.2), once we assume the additional condition (3.4). Notice that condition (3.5) holds when $\|f_\infty\|_{L^1(B(0,R))}$ is small enough, for fixed values of g and R. In the special case of $f_\infty \equiv c$ and $N = 2$ condition (3.5) coincides with the condition*

$$c \leq \frac{2g}{R}, \tag{3.13}$$

assumed in [24]. Notice that now condition (3.5) is stated in the more general terms of the L^1 norm of function $f_\infty(r)$ and that the balance condition is independent of κ. In fact the above characterization remains true for the limit case $\kappa = 0$ but in this case, as in the paper [1], the solution u_∞ must be searched in the class of bounded variation functions.

Remark 3.3. *For the case $\kappa = 0$ the paper [32] proves that if $f_\infty \in L^N(B(0,R))$ then the rigid region $S(u_\infty)$ have a positive measure. In fact, by using the results of [1] it is easy to see that the regularity $f_\infty \in L^N(B(0,R))$ is necessary, since in [1] some explicit examples are given showing that the conclusion fails for some special symmetric functions f_∞ such that $f_\infty \notin L^N(B(0,R))$.*

Remark 3.4. *If $f_\infty \equiv c$ and $N = 2$ an explicit solution was given in the [24] (and later collected also in [23] and [22]) when*

$$c > \frac{2g}{R}, \tag{3.14}$$

(which is a special case of (3.7)). In that case

$$u_\infty(r) = \begin{cases} \left(\frac{R-R_0}{2\kappa}\right)\left(\frac{c}{2}(R+R_0)-2g\right) & \text{if } r \in (0,R_0), \\ \left(\frac{R-r}{2\kappa}\right)\left(\frac{c}{2}(R+r)-2g\right) & \text{if } r \in (R_0,R). \end{cases}$$

Other properties of the solid region $S(u_\infty)$ (and its complementary: $\Omega^+ = \Omega - S(u_\infty)$), for instance in the case in which Ω is a ring, can be found in [31].

Remark 3.5. *It is remarkable that the measure of the solid region (in other words, the value of R_0) is independent of R (once that R is large enough).This is entirely different to the case of the free boundary raised in the problem*

$$\begin{cases} -\Delta_p u + u = 1 & \text{in } \Omega = B(0,R), \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $p > 2$. In that case the "solid region" is defined as $\Omega_1 = \{x \in \Omega : u = 1\}$ and it is well-known that $|\Omega_1|$ increases when R increases (see, e.g. [14] *and* [28]*).*

Remark 3.6. *Some numerical experiences can be found in* [12], [19], [24], *and* [27],

3.2. **Proof of Theorem 1.1.** Now we consider the elliptic non-radially symmetric problem.
Proof of i). By definition we know that f_∞^* satisfies (3.4). Moreover, by virtue of the properties of the decreasing rearrangement we deduce that

$$\|f_\infty\|_{L^1(\Omega)} = \|f_\infty^*\|_{L^1(\Omega^*)} = \int_0^R s^{N-1} f_\infty^*(s)ds$$

and

$$R = \left(\frac{|\Omega|}{\omega_N}\right)^{1/N}.$$

Thus assumption (1.9) is equivalent to the condition (3.5). By Proposition 3.1, we get that the solution U_∞ of (1.12) satisfies $U_\infty \equiv 0$ in Ω^*. Then, from inequality (1.18) we deduce that $u_\infty^*(x) \equiv 0$ in Ω^*, which implies that $u_\infty(x) \equiv 0$ in Ω.
Proof of ii). We note that assumption (1.11) implies that

$$\frac{1}{gR^{N-1}} \int_0^R s^{N-1} f_\infty^*(s)ds > 1.$$

Thus, since $\psi(r)$ is strictly increasing and $\psi(0) = 0$, necessarily, there is a unique $R_0 \in (0, R)$ such that

$$\frac{1}{gR_0^{N-1}} \int_0^{R_0} s^{N-1} f_\infty^*(s)ds = 1, \tag{3.15}$$

and condition (3.7) holds. Then, by Proposition 3.1 we know that $\nabla U_\infty = 0$ on the ball $B(0, R_0)$. But, from (1.19) we get that $\nabla u_\infty^*(x) = 0$ on the ball $B(0, R_0)$. This implies that $u_\infty^*(x)$ is a positive constant on the ball $B(0, R_0)$, and then the "*solid region*" (the set $\mathscr{S}(u_\infty) = \{x \in \Omega : \nabla u_\infty(x) = 0\}$) has a positive measure: in fact

$$|\mathscr{S}(u_\infty)| \geq \omega_N R_0^N.$$

Moreover, from the proof of *ii)* we know that $\mathscr{S}(u_\infty^*) = \left\{x \in \Omega\text{: } u_\infty^*(x) = \|u_\infty^*\|_{L^\infty(\Omega^*)}\right\}$ and since $\|u_\infty^*\|_{L^\infty(\Omega^*)} = \|u_\infty\|_{L^\infty(\Omega)}$ we conclude that $\mathscr{S}(u_\infty) \subset \left\{x \in \Omega\text{: } u_\infty(x) = \|u_\infty\|_{L^\infty(\Omega)}\right\}$. ■

3.3. **The parabolic symmetrized problem.** We consider now the parabolic problem (1.1) for $\Omega = B(0, R)$ and radially symmetric, with non-increasing profiles data, u_0 and f_∞. As mentioned in the Introduction, given u_0, $f_\infty \in L^2(B(0, R))$ the existence and uniqueness of solutions $u \in C([0, +\infty) : L^2(B(0, R)))$ is a direct consequence of the results of [7].

Let us start by considering Property 1. Notice that the parabolic problem can be formulated as a nice special case of the abstract Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) + Bu(t) \ni f(t) & \text{in } X, \\ u(0) = u_0, \end{cases}$$

where X is a Banach space and $B : D(B) \subset X \to \mathscr{P}(X)$ is an accretive operator. The general question of the possible finite extinction time of the solution, when operator B is multivalued for $u=0$, was analyzed firstly in Brézis ([8]) for the case of $X=H$ a Hilbert space and then in [13] for the case of a general Banach space. It was shown there that the necessary condition

$$f(t)\in B(0) \text{ for } t \text{ large enough},$$

is almost sufficient. In our case we have a complete description of operator B

$$B:D(B)\subset L^2(\Omega)\to\mathscr{P}(L^2(\Omega)),$$
$$Bu=-\kappa\Delta u-g\text{div}\left(\frac{\nabla u}{|\nabla u|}\right),\ D(B)=H^2(\Omega)\cap H_0^1(\Omega),$$

nevertheless, the set $B(0)$ is defined in a very implicit way and thus the abstract results of [8], [17] for multivalued operators can not be directly applied. As in [1], it can be shown that

$$B(0)\supset\{c\in\mathbb{R}:\ |c|\le g\frac{N}{R}\},$$

A more direct formulation of the problem is the following:

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,r)-\dfrac{\kappa}{r^{N-1}}\dfrac{\partial}{\partial r}(r^{N-1}\dfrac{\partial u}{\partial r}(t,r))-\dfrac{g}{r^{N-1}}\dfrac{\partial}{\partial r}(r^{N-1}\lambda(t,r))=f_\infty(r), \text{ for } t>0, r\in(0,R),\\ u(t,R)=0 \text{ and } \dfrac{\partial u}{\partial r}(t,0)=0, \qquad \text{for } t>0,\\ u(t,R)=u_0(r) \qquad r\in(0,R),\end{cases}\tag{3.16}$$

for some scalar-valued function $\lambda\in L^2_{loc}(0,+\infty:L^\infty(0,R))$ satisfying

$$|\lambda(t,r)|\le 1 \text{ and } \lambda(t,r)\frac{\partial u}{\partial r}(t,r)=\left|\frac{\partial u}{\partial r}(t,r)\right| \quad \text{a.e. } r\in(0,R),\, t>0.\tag{3.17}$$

Proposition 3.7. *Let $u_0\in L^\infty(B(0,R))$, $f_\infty\in L^\infty(B(0,R)), u_0,f_\infty\ge 0$, satisfying (3.1) and (3.4) Let $u(t,r)$ be the unique solution of (3.2). Then:*

a) *If*

$$\psi(R):=\frac{1}{gR^{N-1}}\int_0^R s^{N-1}f_\infty(s)ds<1\tag{3.18}$$

then, for any $u_0\in L^\infty(B(0,R))$ there exists a finite time $T_{u_0,f_\infty}>0$ such that $u(t,r)$ globally stops after T_{u_0,f_∞}, in the sense that $u(t,r)\equiv 0$ a.e. in $B(0,R)$, for any $t\ge T_{u_0,f_\infty}$. Moreover

$$T_{u_0,f_\infty}\le\frac{R\|u_0\|_{L^\infty(B(0,R))}}{gN(1-\psi(R))}.$$

b) *Assume (3.4) and that there exists a $R_0\in(0,R)$ (R_0 depending on g) such that*

$$\frac{1}{gR_0^{N-1}}\int_0^{R_0}s^{N-1}f_\infty(s)ds=1.\tag{3.19}$$

Let $u_0\in H^2(B(0,R))\cap H_0^1(B(0,R))$, u_0 symmetric, non-negative and with non-increasing profile, such that

$$\frac{\kappa}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{du_0}{dr}(r))+\frac{g}{r^{N-1}}\frac{d}{dr}(r^{N-1}\lambda_0(r))+f_\infty(r)\ge 0 \text{ in } H^{-1}(0,R),\tag{3.20}$$

for some $\lambda_0(r) \in \operatorname{sign}\left(\frac{du_0}{dr}(r)\right)$ *a.e. in* $(0,R)$ *and with* $S(u_0) \subset B(0,R_0)$*. Then* $\frac{\partial u}{\partial t}(t,r) \geq 0$ *a.e.* $r \in (0,R)$ *and a.e.* $t > 0$*, the profile of the solution* $u(t,r)$ *is non-increasing in* r*, for any* $t \geq 0$*, and we have*

$$S(u(t)) \subset B(0,R_0) \text{ for any } t \geq 0,$$

with

$$u(t,r) = \|u(t)\|_{L^\infty(B(0,R))} \text{ on } S(u(t)), \text{ for any } t \geq 0.$$

c) *Under the same conditions than in b),* $u(t) \to u_\infty$ *in* $H_0^1(B(0,R))$ *as* $t \to +\infty$*, where* u_∞ *is the unique solution of problem (3.2), and there exits a* $R^* \in (0,R)$ *such that*

$$\|u(t) - u_\infty\|_{C^0([R^*,R])} > 0 \text{ for any } t > 0.$$

Proof. To prove part a) we will use the comparison principle. Let us show that we can construct a supersolution $\overline{u}(t,r) = \alpha(t)$ such that $\alpha(t) = 0$ for $t \geq T_{u_0,f_\infty}$ for some $T_{u_0,f_\infty} > 0$. Then from the comparison inequalities $0 \leq u(t,r) \leq \alpha(t)$ for any $t > 0$ and a.e. $r \in (0,R)$ we deduce our conclusion.

The non-trivial part of the proof is to characterize the condition on $\alpha(t)$ in order to know that $\overline{u}(t,r)$ is a supersolution of problem (3.16). Since by construction $S(\overline{u}(t)) = B(0,R)$, then, as in the proof of i) of Proposition 3.1, we must have

$$-\frac{g}{r^{N-1}}\frac{\partial}{\partial r}(r^{N-1}\overline{\lambda}(t,r)) \geq f_\infty(r) - \alpha'(t), \; r \in (0,R),$$

for some $\overline{\lambda}(t,r)$ such that $\left|\overline{\lambda}(t,r)\right| \leq 1$ a.e. $(t,r) \in (0,+\infty) \times (0,R)$. Then

$$\overline{\lambda}(t,r) \leq -\frac{1}{gr^{N-1}}\int_0^r s^{N-1} f_\infty(s)ds + \frac{\alpha'(t)r}{gN}.$$

Then, in order to know that $\left|\overline{\lambda}(t,r)\right| \leq 1$ it is enough to have that

$$-\psi(R) + \frac{\alpha'(t)R}{gN} \geq -1$$

(with $\psi(R)$ given in (3.18)) to conclude that $\overline{u}(t,r)$ is a supersolution (see, e.g., the proof of Theorem 4 of [1]). Thus we must have

$$\alpha'(t) \geq -\frac{gN}{R}(1 - \psi(R)).$$

But from the balance condition (3.18) we know that $\psi(R) - 1 < 0$. Thus we can take

$$\alpha(t) = \left[K - \frac{gN}{R}(1 - \psi(R))t\right]_+ \text{ for any } t \geq 0,$$

for some $K > 0$. To check that

$$u_0(r) \leq \overline{u}(0,r) \text{ a.e. on } B(0,R),$$

it is enough to take

$$K = \|u_0\|_{L^\infty(B(0,R))},$$

and the conclusion follows with

$$T_{u_0,f_\infty} \leq \frac{R\|u_0\|_{L^\infty(B(0,R))}}{gN(1-\psi(R))}.$$

Let us prove part b). As usual in the rearrangement theory, it is easy to see that the profile of $u(t,r)$ is non-decreasing in $r \in (0,R)$. Moreover, the assumptions on $u_0(r)$ implies that $\frac{\partial u}{\partial t}(t,r) \geq 0$ a.e. $t > 0$ and a.e. $r \in (0,R)$ (this is a classical result for m-accretive operators in a Banach lattice (see, e.g., Proposition 1 of [4]). Then, on any positively measured subset of the solid region $S(u(t))$ the equation in (3.16) reduces to the condition

$$-\frac{g}{r^{N-1}}\frac{\partial}{\partial r}(r^{N-1}\lambda(t,r)) = f_\infty(r) - \frac{\partial u}{\partial t}(t,r),\ r \in S(u(t)).$$

Moreover, $\frac{\partial u_\infty}{\partial r}(t,0) = 0$ and if we denote by $R_0(t)$ (with $R_0(t) \in (0,R]$) to the boundary of $S(u(t))$, we get (since the profile of $u(t,r)$ is non-decreasing) that necessarily

$$\lambda(t,r) = -\psi(t,r) = -\frac{1}{gr^{N-1}}\int_0^r s^{N-1}(f_\infty(s) - \frac{\partial u}{\partial t}(t,r))ds \ \text{ a.e. } r \in (0,R_0). \tag{3.21}$$

Moreover, if $\psi_\infty(r)$ is the function defined by (3.12) in the proof of Proposition 3.1, then $\psi_\infty(r)$ is strictly increasing (since we are assuming (3.4)), and so

$$\psi(t,r) \leq \psi_\infty(r) < 1 \text{ a.e. } r \in (0,R_0) \text{ a.e. } t > 0,$$

since $\frac{\partial u}{\partial t}(t,r) \geq 0$. In particular

$$\lambda(t,r) \geq \lambda_\infty(r) > -1 \text{ a.e. } r \in (0,R_0) \text{ a.e. } t > 0.$$

This implies that $S(u(t)) \subset B(0,R_0)$ for any $t \geq 0$.
The statement of part c) is an special case of Theorem 4 of [19].■

Remark 3.8. *Notice that in the parabolic case it is assumed a strict inequality in the balance. Among other difficulties, notices that the estimate* $T_{u_0,f_\infty} \leq \frac{R\|u_0\|_{L^\infty(B(0,R))}}{gN(1-\psi(R))}$ *is not useful when* $\psi(R) = 1$.

Remark 3.9. *Part a) of Proposition 3.7 holds if we replace* $f_\infty(r)$ *by a time dependent function* $f(t,r) \geq 0$, $f \in L^1_{loc}(0,+\infty : L^\infty(B(0,R))$ *such that*

$$f(t,r) = f_\infty(r) \text{ for a.e. } t \geq T_f > 0$$

with a function $f_\infty(r)$ *satisfying the conditions of Proposition 3.7. In that case we must take*

$$K = \|u_0\|_{L^\infty(B(0,R)} + \int_0^{T_f} \|f(t,r)\|_{L^\infty(B(0,R))}dt.$$

It suffices to use the fact that the realization of operator B on $L^\infty(B(0,R))$ *is T-accretive (see, [5]).*

Remark 3.10. *Notice that, from the proof of part b) we also know that* $\lambda_\infty(r) = -1$ *for any* $r \in [R_0,R]$ *(thanks to the assumption (3.4)) but now, in contrast with the stationary case, we do not know if* $\lambda(t,r) = -1$ *on* $[R_0,R]$ *since the monotonicity of the function* $\lambda(t,r)$ *is not evident (due to the presence of the time derivative of the unknown).*

3.4. **Proof of Theorem 1.2.** *Proof of Theorem 1.2* To prove part *i)* we apply part 2 of Theorem 1.3. Then it suffices to show that if $U(t,x)$ is the unique solution of problem (1.13) then there exists a finite time $T_U > 0$ such that $U(t) \equiv 0$ for any $t \geq T_U$. This is a consequence of part a) of Proposition 3.7. Indeed, if $R > 0$ is such that $\Omega^* = B(0,R)$, from elementary properties of the rearrangements, we know that u_0^*, $f_\infty^* \in L^\infty(B(0,R))$, $u_0^*, f_\infty^* \geq 0$ and satisfy (3.1) and (3.4) and the same for u_0^*. We recall that, as in the proof of Theorem 1.1, the assumption (3.18) is equivalent to the condition (1.10). Then we deduce that

$$T_U \leq T_{u_0,f_\infty} \leq \frac{R\|u_0\|_{L^\infty(B(0,R))}}{gN(1-\psi^*(R))},$$

where ψ^* is the function ψ corresponding to the datum f_∞^*.
To prove (1.17) we recall that by a well-known result due to Hardy, Littlewood and Polya [25], the inequality (1.20) implies that for any continuous, nondecreasing convex function $\Phi : [0,+\infty) \to [0,+\infty)$ we have

$$\int_{B(0,r)} \Phi(u^*(t,x))dx \leq \int_{B(0,r)} \Phi(U(t,x))dx, \ \forall t > 0, \ \forall r \in [0,R].$$

In particular, for any $p \in [1,+\infty]$

$$\|u(t)\|_{L^p(\Omega)} \leq \|U(t)\|_{L^p(\Omega^*)} \ \forall t \geq 0. \tag{3.22}$$

In addition, given $M > 0$. For any $\varepsilon \in (0,M)$, consider any continuous, nondecreasing convex function $\Phi\varepsilon : [0,+\infty) \to [0,+\infty)$ such that

$$\Phi\varepsilon(s) = 0 \text{ if } s \in [0, M-\varepsilon] \text{ and } \Phi\varepsilon(M) = 1.$$

Then, by taking $M = M_U(t) = \|U(t)\|_{L^\infty(\Omega^*)}$ we have that, for any $\varepsilon \in (0, M_U(t))$

$$\int_{B(0,r)} \Phi_\varepsilon(u^*(t,x))dx \leq \int_{B(0,r)} \Phi_\varepsilon(U(t,x))dx, \forall t > 0, \ \forall r \in [0,R].$$

Then, taking $r = R$

$$\int_{\{x:\, u^*(t,x) \geq M_U(t)-\varepsilon\}} \Phi_\varepsilon(u^*(t,x))dx \leq \int_{\{x:\, \Phi_\varepsilon(U(t,x)) \geq M_U(t)-\varepsilon\}} dx.$$

Letting $\varepsilon \downarrow 0$ we obtain in the limit that, given $t \geq 0$

$$\left|\{x \in \Omega^* :\ u^*(t,x) = M_U(t)\}\right| \leq \left|\{x \in \Omega^* :\ U(t,x) = M_U(t)\}\right|.$$

Then, from (3.22) we get that

$$\left|\{x \in \Omega^* :\ u^*(t,x) = \|u^*(t)\|_{L^\infty(\Omega^*)}\}\right| \leq \left|\{x \in \Omega^* :\ U(t,x) = M_U(t)\}\right..$$

Finally, since $\|u^*(t)\|_{L^\infty(\Omega^*)} = \|u(t)\|_{L^\infty(\Omega)}$ we arrive to the desired conclusion.
The proof of part *ii))* of Theorem 1.2 was given in the previous section (see Proposition 3.7).■

Remark 3.11. *We do not know if in case i) the profile of the solution $u(t,.)$ may have a solid region before the finite time stopping time T_{u_0,f_∞}. In the special case of the Total Variation problem ($\kappa = 0$) it was shown in* [1] *(see also* [3]*) that the answer is affirmative but this is possible since the solution $u(t,.)$ is merely in $BV(\Omega)$: the case $\kappa > 0$ leads to more regular solutions and the study made in* [1] *is not applicable.*

Acknowledgements
The research of the first author has been supported by the project EEEP&DLaD — Pia.Ce.Ri, Piano della Ricerca di Ateneo 2020-2022 University of Catania. She is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilitá e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The research of JID was partially supported by the project ref. PID2020-112517GB-I00 of the AEI (Spain) and the Research Group MOMAT (Ref. 910480) of the UCM.

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