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Oscillatory Behavior of Second-Order Neutral Differential Equations

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Abstract
In this paper, we study oscillatory properties of neutral differential equations. Moreover, we discuss some examples that show the effectiveness and the feasibility of the main results.

Keywords Oscillation · Non-oscillation · Neutral differential equations · Second order

Mathematics Subject Classification 34C10 · 34K11

1 Introduction

Delay differential equations are widely used in mathematical modeling to describe physical and biological systems, by inducing oscillatory behavior.

In the literature, several mathematical models with different levels of complexity have been proposed for delay differential equations in order to represent for example the cardiovascular system (CVS).

The pioneering and remarkable paper of Ottesen (1997) shows how to use delay differential equations to solve a cardiovascular model that has a discontinuous derivative. Ottesen (1997) also illustrated that complex dynamic interactions between nonlinear

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behaviors and delays associated with the autonomic-cardiac regulation may cause instability (Ataeea et al. 2015).

Moreover, a model-based approach to stability analysis of autonomic-cardiac regulation was studied in Ataeea et al. (2015); specifically, it is important to underline that the autonomic-cardiac regulation operates by the interaction between autonomic nervous system (ANS) and cardiovascular system (CVS) (Ataeea et al. 2015).

It is clear that mathematical analysis related to physics-based models can be a versatile tool in examining delay differential equations from the point of view of medical and biological systems.

In this paper we consider the following equation of neutral type

$$\left(a(y)\left(w'(y)\right)^{\gamma}\right)' + \sum_{j=1}^{m_2} q_j(y)x^{\beta_j}\left(\vartheta_j(y)\right) = 0, \quad y \geq y_0, \tag{1.1}$$

belonging to those families used to model problems that arise in the biological sciences. Our aim is to study the oscillatory behavior of (1.1) where $w(y) = x(y)+\sum_{i=1}^{m_1} p_i(y)x^{\alpha_i}(\varsigma_i(y)), \alpha_i, \gamma$ and β_j, for all $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$, are quotients of odd positive integers.

Moreover, many researchers study qualitative properties of delay mathematical models examining oscillation and nonoscillation properties of different delay logistic models and their modifications (Agarwal et al. 2014c). These studies are concerned also with the investigation of local and global stability. Mainly the oscillation properties are investigated for models with delayed feedback, hyperlogistic models and models with varying capacity. For further details regarding the techniques and other applications to Biology we refer the reader to Agarwal et al. (2014a, b, c, 2015, 2016), Baculíková et al. (2011), Džurina et al. (2020), Fisnarova and Marik (2017), Grace et al. (2018), Li and Rogovchenko (2014, 2015, 2017), Li et al. (2015), Pinelas and Santra (2018), Qian and Xu (2011), Santra (2016, 2017, 2019a, b, 2020a, b); Santra and Dix (2020) Tripathy and Santra (2020), Zhang et al. (2015); Bazighifan (2020a, b); Chatzarakis et al. (2019b), Moaaz et al. (2017), Bazifghifan and Ramos (2020) and Bazighifan et al. (2020a).

For a recent review on the asymptotic properties for functional differential equations (FDEs), we suggest to the reader the interesting book Berezansky et al. (2020).

2 Mathematical Background and Hypotheses

Throughout this work, we assume that the following assumptions are fulfilled for Eq. (1.1):

(A1) $\vartheta_j, \varsigma_i \in C([y_0,\infty),\mathbb{R}_+)$, $\varsigma_i \in C^2([y_0,\infty),\mathbb{R}_+)$, $\vartheta_j(y) < y$, $\varsigma_i(y) < y$, $\lim_{y\to\infty}\vartheta_j(y) = \infty$, $\lim_{y\to\infty}\varsigma_i(y) = \infty$ for all $i = 1, 2, \ldots, m_1$ and $j = 1, 2, \ldots, m_2$;

(A2) $a \in C^1([y_0,\infty),\mathbb{R}_+)$, $q_j \in C([y_0,\infty),\mathbb{R}_+)$; $0 \leq q_j(y)$, for all $y \geq 0$ and $j = 1, 2, \ldots, m_2$; $\sum_{j=1}^{m_2} q_j(y)$ is not identically zero in any interval $[b,\infty)$;

(A3) $\lim_{y\to\infty} A(y) = \infty$, where $A(y) = \int_{y_0}^{y} a^{-1/\gamma}(\eta)\,d\eta$;

(A4) $p_i : [y_0, \infty) \to \mathbb{R}^+$ are continuous functions for $i = 1, 2, \ldots, m$;
(A5) there exists a differentiable function $\vartheta_0(y)$ satisfying the properties $0 < \vartheta_0(y) = \min_{j=1,\ldots,m_2}\{\vartheta_j(y) : y \geq y^* > y_0\}$ and $\vartheta_0'(y) \geq \vartheta_0$ for $y \geq y^* > y_0$, $\vartheta_0 > 0$.

Now we recall some basic definitions.

Definition 2.1 A function $x(y) : [y_x, \infty) \to \mathbb{R}$, $y_x \geq y_0$ is said to be a *solution* of (1.1) if $x(y)$ and $a(y)\left(w'(y)\right)^\gamma$ are continuously differentiable for all $y \in [y_x, \infty)$ and it satisfies the equation (1.1) for all $y \in [y_x, \infty)$.

We assume that (1.1) admits a solution in the sense of Definition 2.1.

Definition 2.2 A solution $x(y)$ of (1.1) is said to be *non-oscillatory* if it is eventually positive or eventually negative; otherwise, it is said to be *oscillatory*.

Definition 2.3 Equation (1.1) is said to be *oscillatory* if all of its solutions are oscillatory.

In this paper, we restrict our attention to study oscillation and non-oscillation of (1.1). First of all, it is interesting to make a review in the context of functional differential equation.

Brands (1978) proved that for each bounded delay $\vartheta(y)$, the equation

$$x''(y) + q(y)x(y - \vartheta(y)) = 0$$

is oscillatory if and only if the equation

$$x''(y) + q(y)x(y) = 0$$

is oscillatory. Chatzarakis et al. (2019a) and Chatzarakis and Jadlovská (2019) considered the following more general equation

$$\left(a(x')^\beta\right)'(y) + q(y)x^\beta(\vartheta(y)) = 0 \tag{2.1}$$

and established new oscillation criteria for (2.1) when $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$.

Wong (2000) has obtained oscillation conditions of

$$\left(x(y) + px(y - \varsigma)\right)'' + q(y)f\left(x(y - \vartheta)\right) = 0, \quad -1 < p < 0$$

in which the neutral coefficient and delays are constants. In Baculíková and Džurina (2011a) and Džurina (2011), the authors studied the equation

$$\left(a(y)\left(w'(y)\right)^\gamma\right)' + q(y)x^\beta(\vartheta(y)) = 0, \quad w(y) = x(y) + p(y)x(\varsigma(y)), \quad y \geq y_0, \tag{2.2}$$

and established the oscillation of solutions of (2.2) using comparison techniques when $\gamma = \beta = 1$, $0 \leq p(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$. Using the same technique, Baculíková and Džurina (2011b) considered (2.2) and obtained oscillation conditions of (2.2) considering the assumptions $0 \le p(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$. Tripathy et al. (2016), studied (2.2) and established several conditions of the solutions of (2.2) considering the assumptions $\lim_{y\to\infty} A(y) = \infty$ and $\lim_{y\to\infty} A(y) < \infty$ for different values of the neutral coefficient p. Bohner et al. (2017) obtained sufficient conditions for the oscillation of the solutions of (2.2) when $\gamma = \beta$, $\lim_{y\to\infty} A(y) < \infty$ and $0 \le p(y) < 1$. Grace et al. (2018) studied the oscillation of (2.2) when $\gamma = \beta_j$, assuming that $\lim_{y\to\infty} A(y) < \infty$, $\lim_{y\to\infty} A(y) = \infty$ and $0 \le p(y) < 1$. Li et al. (2019) established sufficient conditions for the oscillation of the solutions of (2.2), under the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $p(y) \ge 0$. Karpuz and Santra (2019) studied the equation

$$\left(a(y)(x(y) + p(y)x(\varsigma(y)))'\right)' + q(y) f\left(x(\vartheta(y))\right) = 0,$$

considering the assumptions $\lim_{y\to\infty} A(y) < \infty$ and $\lim_{y\to\infty} A(y) = \infty$, for different values of p.

For any positive, continuous and decreasing to zero function $\rho : [y_0, \infty) \to \mathbb{R}^+$, we set

$$P(y) = \left(1 - \sum_{i=1}^{m} \alpha_i p_i(y) - \frac{1}{\rho(y)} \sum_{i=1}^{m} (1 - \alpha_i) p_i(y)\right);$$

$$Q_1(y) = \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right);$$

$$Q_2(y) = \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) \rho^{\beta_j - 1}\left(\vartheta_j(y)\right);$$

$$Q_3(y) = \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j - 1}\left(\vartheta_j(y)\right);$$

$$Q_4(y) = \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j}(\vartheta_j(y));$$

$$U(y) = \int_y^{\infty} \sum_{j=1}^{m_2} q_j(\zeta) x^{\beta_j}(\vartheta_j(\zeta))\, d\zeta\,.$$

Let us assume that $P(y)$ and $U(y)$ are non-negative in $[y_0, \infty)$.

We now recall the technical lemmas and the main results contained in Bazighifan et al. (2020b).

Lemma 2.1 *Let* (A1)–(A4) *hold for* $y \ge y_0$. *If a solution* x *of* (1.1) *is eventually positive, then* w *satisfies*

$$w(y) > 0, \quad w'(y) > 0, \quad \text{and} \quad \left(a(w')^{\gamma}\right)'(y) \le 0 \quad \text{for} \quad y \ge y_1. \tag{2.3}$$

Lemma 2.2 *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1.1) *is eventually positive, then* w *satisfies*

$$w(y) \geq (a(y))^{1/\gamma}\, w'(y)A(y) \quad \text{for } y \geq y_1.$$

and

$$\frac{w(y)}{A(y)} \text{ is decreasing for } y \geq y_1.$$

Lemma 2.3 *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1.1) *is eventually positive, then* w *satisfies*

$$x(y) \geq P(y)w(y) \quad \text{for} \quad y \geq y_1. \tag{2.4}$$

Lemma 2.4 *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If a solution* x *of* (1.1) *is eventually positive, then there exist* $y_1 > y_0$ *and* $\delta > 0$ *such that*

$$0 < w(y) \leq \delta A(y) \text{ and} \tag{2.5}$$

$$A(y)U^{1/\gamma}(y) \leq w(y) \tag{2.6}$$

hold for all $y \geq y_1$.

Theorem 2.4 *Assume that there exists a constant* δ_1, *quotient of odd positive integers, such that* $0 < \beta_j < \delta_1 < \gamma$, *and* (A1)–(A4) *hold for* $y \geq y_0$. *If*

(A6) $\int_0^{\infty} Q_4(\eta)\, d\eta = \infty$.

holds, then every solution of (1.1) *is oscillatory.*

Theorem 2.5 *Assume that there exists a constant* δ_2, *quotient of odd positive integers, such that* $\gamma < \delta_2 < \beta_j$. *Furthermore, assume that* (A1)–(A5) *hold for* $y \geq y_0$ *and* $a(y)$ *is non-decreasing. If*

(A7) $\int_0^{\infty} \left[\frac{1}{a(\eta)} \int_\eta^{\infty} Q_1(\zeta)\, d\zeta\right]^{1/\gamma} d\eta = \infty$

holds, then every solution of (1.1) *is oscillatory.*

3 Oscillation Criteria for (1.1)

In this section we discuss our main results. The oscillation criteria in this paper complete the study started in Bazighifan et al. (2020b) but it is important to underline that the criteria discussed in Bazighifan et al. (2020b) differ from those examined in this work in terms of assumptions. Precisely, both the main results of Bazighifan et al. (2020b) (Theorem 1 and 2), require the existence of two constants δ_1 and δ_2 that are quotients of odd positive integers and the bounds for b_j involve such constants. The results presented in this paper do not involve the existence of auxiliary constants and under fewer hypotheses guarantee the oscillatory behavior of the equations under consideration.

Theorem 3.1 *Let* (A1)–(A4) *hold for* $y \geq y_0$. *If*

(A6) $\int_0^\infty Q_1(\eta)d\eta = \infty$

holds, then every solution of (1.1) *is oscillatory.*

Proof Let the solution x be eventually positive. Then there exists $y_0 > 0$ such that $x(y) > 0$, $x(\varsigma_i(y)) > 0$ and $x\big(\vartheta_j(y)\big) > 0$ for all $y \geq y_0$ and for all $i = 1, 2, \ldots, m_1$ and $i = 1, 2, \ldots, m_2$. Applying Lemmas 2.1 and 2.3 for $y \geq y_1 > y_0$ we conclude that w satisfies (2.3), w is increasing and $x(y) \geq P(y)w(y)$ for all $y \geq y_1$. From (1.1), we have

$$\Big(a(y)\big(w'(y)\big)^\gamma\Big)' + \sum_{j=1}^{m_2} q_j(y)P^{\beta_j}\big(\vartheta_j(y)\big)w^{\beta_j}\big(\vartheta_j(y)\big) \leq 0 \tag{3.1}$$

for $y \geq y_1$. Applying (2.3) we conclude that $\lim_{y\to\infty}\Big(a(y)\big(w'(y)\big)^\gamma\Big)$ exists, and there exist $y_2 > y_1$ and a number $c > 0$ such that $w(y) \geq c$ for $y \geq y_2$. Integrating (3.1) from y_2 to y, for a suitable constant $\tilde{c}$, we have

$$\tilde{c}\int_{y_2}^{y}\sum_{j=1}^{m_2} q_j(\eta)P^{\beta_j}\big(\vartheta_j(\eta)\big)d\eta \leq -\Big[a(\eta)\big(w'(\eta)\big)^\gamma\Big]_{y_2}^{y} < \infty \quad \text{as} \quad y \to \infty,$$

which is a contradiction to (A6).

The case where x is an eventually negative solution is similar and we omit it here. Thus, the proof is complete. □

Remark Theorem 3.1 holds for any β_j and γ.

Theorem 3.2 *Let* (A1)–(A4) *hold for* $y \geq y_0$ *and* $\beta_j > 1$. *If*

(A7) $\int_0^\infty Q_2(\eta)d\eta = \infty$

holds, then every solution of (1.1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.1 we obtain (3.1). Since $w(y)$ is positive and increasing, $\rho(y)$ is positive and decreasing to zero, there exists $y_0 \geq y_1$ such that

$$w(y) \geq \rho(y) \quad \text{for} \quad y \geq y_1. \tag{3.2}$$

Applying (3.2) in (3.1) we have

$$\Big(a(y)\big(w'(y)\big)^\gamma\Big)' + \sum_{j=1}^{m_2} q_j(y)P^{\beta_j}\big(\vartheta_j(y)\big)\rho^{\beta_j-1}\big(\vartheta_j(y)\big)w\big(\vartheta_j(y)\big) \leq 0. \tag{3.3}$$

The rest of the proof is similar to that of Theorem 3.1 and hence it is omitted. □

Theorem 3.3 *Let* (A1)–(A4) *hold for* $y \geq y_0$ *and* $0 < \beta_j < 1$. *If*

(A8) $\int_0^\infty Q_3(\eta)d\eta = \infty$

holds, then every solution of (1.1) *is oscillatory.*

Proof Proceeding as in the proof of Theorem 3.1 we obtain (3.1). Now (3.1) can be written as

$$\left(a(y)\left(w'(y)\right)^{\gamma}\right)' + \sum_{j=1}^{m_2} q_j(y) P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j-1}\left(\vartheta_j(y)\right) \frac{w^{\beta_j-1}\left(\vartheta_j(y)\right)}{A^{\beta_j-1}\left(\vartheta_j(y)\right)} w\left(\vartheta_j(y)\right) \le 0 \tag{3.4}$$

for $y \ge y_2 > y_1$. Since $\frac{w(y)}{A(y)}$ is decreasing, there exists a constant k such that

$$\frac{w(y)}{A(y)} \le k \quad \text{for} \quad y \ge y_2. \tag{3.5}$$

Using (3.5) and $\beta_j < 1$ in (3.4), we have

$$\left(a(y)\left(w'(y)\right)^{\gamma}\right)' + \sum_{j=1}^{m_2} q_j(y) \frac{P^{\beta_j}\left(\vartheta_j(y)\right) A^{\beta_j-1}\left(\vartheta_j(y)\right)}{k^{1-\beta_j}} w\left(\vartheta_j(y)\right) \le 0.$$

The rest of the proof is similar to that of Theorem 3.2 and hence it is omitted. □

4 Examples

We conclude the paper presenting some examples that show the effectiveness and the feasibility of the main results.

Example 4.1 Let us consider the differential equation

$$\left(y\left(\left(x(y) + \frac{1}{y}x^{\frac{1}{3}}\left(\frac{y}{2}\right) + \frac{1}{y^2}x^{\frac{1}{5}}\left(\frac{y}{3}\right)\right)'\right)^3\right)' + y^6 x^3\left(\frac{y}{3}\right) + y^7 x^3\left(\frac{y}{4}\right) = 0 \quad \text{for} \quad y \ge 4, \tag{4.1}$$

where $a(y) :\equiv y$, $p_i(y) :\equiv \frac{1}{y^i}$, $\alpha_i :\equiv \frac{1}{2i+1}$, $\varsigma_i(y) :\equiv \frac{y}{i+1}$, $\beta_j = \gamma = 3$, $q_j(y) :\equiv y^{j+5}$ and $\vartheta_j(y) :\equiv \frac{y}{j+2}$ for $i = 1, 2$, $j = 1, 2$ and $y \ge 4$. All the assumptions of Theorem 3.1 are fulfilled with $i = 1, 2$, $j = 1, 2$. Hence, due to Theorem 3.1, equation (4.1) is oscillatory in the sense of Definition of 2.3.

Example 4.2 Let us consider the differential equation

$$\left(y\left(\left(x(y) + \frac{1}{y}x^{\frac{1}{3}}\left(\frac{y}{3}\right) + \frac{1}{y^2}x^{\frac{1}{5}}\left(\frac{y}{4}\right)\right)'\right)^5\right)' + y^{\frac{6}{5}} x\left(\frac{y}{2}\right) + y^{\frac{7}{6}} x\left(\frac{y}{3}\right) = 0 \quad \text{for} \quad y \ge 2, \tag{4.2}$$

where $a(y) :\equiv y$, $p_i(y) :\equiv \frac{1}{y^i}$, $\alpha_i :\equiv \frac{1}{2i+1}$, $\varsigma_i(y) :\equiv \frac{y}{i+2}$, $\beta_j = 1 < \gamma = 5$, $q_j(y) :\equiv y^{\frac{j+5}{j+4}}$ and $\vartheta_j(y) :\equiv \frac{y}{j+1}$ for $i = 1, 2$, $j = 1, 2$ and $y \geq 2$. All the assumptions of Theorem 3.1 are fulfilled with $i = 1, 2$, $j = 1, 2$. Hence, due to Theorem 3.1, equation (4.2) is oscillatory in the sense of Definition of 2.3.

Example 4.3 Let us consider the differential equation

$$\left(y^2\left(\left(x(y)+\frac{1}{y^2}x^{\frac{1}{5}}\left(\frac{y}{3}\right)+\frac{1}{y^4}x^{\frac{1}{9}}\left(\frac{y}{5}\right)\right)'\right)^3\right)'+y^7x^3\left(\frac{y}{4}\right)+y^9x^3\left(\frac{y}{6}\right)=0 \quad \text{for} \quad y \geq 6, \tag{4.3}$$

where $a(y) :\equiv y^2$, $p_i(y) :\equiv \frac{1}{y^{2i}}$, $\alpha_i :\equiv \frac{1}{4i+1}$, $\varsigma_i(y) :\equiv \frac{y}{2i+1}$, $\beta_j = 3 > 1$, $\gamma = 3$, $q_j(y) :\equiv y^{2j+5}$ and $\vartheta_j(y) :\equiv \frac{y}{2j+2}$ for $i = 1, 2$, $j = 1, 2$ and $y \geq 6$. All the assumptions of Theorem 3.2 are fulfilled with $i = 1, 2$, $j = 1, 2$ and $\rho(y) = \frac{1}{y}$. Hence, due to Theorem 3.2, equation (4.3) is oscillatory in the sense of Definition of 2.3.

Example 4.4 Let us consider the differential equation

$$\left(y\left(\left(x(y)+\frac{1}{y^{1/2}}x^{\frac{1}{5}}\left(\frac{y}{3}\right)+\frac{1}{y}x^{\frac{1}{9}}\left(\frac{y}{5}\right)\right)'\right)^3\right)'+y^5x^{1/5}\left(\frac{y}{4}\right)+y^6x^{1/5}\left(\frac{y}{5}\right)=0 \quad \text{for} \quad y \geq 5, \tag{4.4}$$

where $a(y) :\equiv y$, $p_i(y) :\equiv \frac{1}{y^{i/2}}$, $\alpha_i :\equiv \frac{1}{4i+1}$, $\varsigma_i(y) :\equiv \frac{y}{2i+1}$, $\beta_j = \frac{1}{5} < 1$, $\gamma = 3$, $q_j(y) :\equiv y^{j+4}$ and $\vartheta_j(y) :\equiv \frac{y}{j+3}$ for $i = 1, 2$, $j = 1, 2$ and $y \geq 5$. All the assumptions of Theorem 3.3 are fulfilled with $i = 1, 2$, $j = 1, 2$ and $A(y) = \frac{5}{2}(y^{2/5} - y_0^{2/5})$. Hence, due to Theorem 3.3, equation (4.4) is oscillatory in the sense of Definition of 2.3.

Example 4.1 and 4.2 show that Theorem 3.1 can be applied for any γ and β_j. Example 4.3 is valid for $\gamma > 1$ and $\rho(y) = \frac{1}{y}$, and Example 4.4 is valid for $\gamma < 1$.

5 Conclusions

In this work we established several oscillation criteria for second-order nonlinear neutral differential equations. Our results complete the research started in Bazighifan et al. (2020b). For the sake of completeness, we presented some examples related to the main results of the paper.

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Declarations

Conflict of interests The authors declare no conflict of interest.

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