



# A sustainable dynamic closed-loop supply chain network equilibrium for collectibles markets

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## Abstract

In this paper, we present a sustainable closed-loop supply chain network equilibrium problem for collectible items in a time-dependent framework. In particular, we consider a network consisting of manufacturers, retailers, demand markets, and an online second-hand platform involved in forward and reverse logistics competition. Manufacturers as well as retailers try to control the amount of emissions generated in transaction processes. We analyze the optimal behaviors of all the decision-makers, and we give the governing closed-loop supply chain network equilibrium conditions. Then, we introduce the associated evolutionary variational inequality and the related infinite-dimensional projected dynamical system, which are convenient to solve such a problem. Finally, using a discrete-time approximation procedure, we solve a numerical example considering eBay as online platform.

**Keywords** Evolutionary variational inequality · Reverse logistics · Second-hand market · Closed-loop supply chain · Online platform · Pollution control

## 1 Introduction

Nowadays, the importance of environmental concern and sustainable consumption has drawn extensive attention. Re-manufacturing is one of the most used ways of product recovery, which allows us to reduce the exploitation of natural resources and recover value from used products. Recently, second-hand trading has raised wide interest in the opportunity of extending the life span of products and reducing

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environmental stresses for the purchase of new products. Moreover, online auctions and trading platforms, such as eBay or Vinted, are increasing the opportunities for sustainable consumption, and are changing the traditional interactions among the tiers of supply chain networks. In fact, exchange sites, auctions, or other Internet-based trading platforms have transformed consumers from buyers into active sellers.

In particular, collectibles markets are becoming more and more dynamic. Collectibles are valuable items because of their rarity and popularity, and their monetary value is higher than their original sale value. A collector may spend a lot of time collecting a particular item of his interest, and stores it in the appropriate atmosphere to preserve it. A pandemic-driven increase in online sales of collectibles sent the global collectibles markets to a record high in 2020. According to Market (2021), collectibles market size was estimated at \$ 412 billion in 2020 and is expected to reach \$ 628 billion by 2031. Today, apart from the classical items such as coins, stamps and antiquities, the product segments of the collectibles market include sports memorabilia, military items, music NFT, movie avatar NTF, art NTF, and digital NFT.

In this paper, we present a closed-loop supply chain (CLSC) network equilibrium model with online trading of high-value products, considering time-dependent demands and returns. In particular, we consider a CLSC network consisting of manufacturers, retailers, demand markets, and one online platform, in which some consumers purchase new products and collect them. Then, collectors sell the products to consumers through the online platform. This study also integrates the environmentally friendly attitude of manufacturers and retailers. We study the optimal behaviors of all the decision-makers, formulate the equilibrium conditions governing the CLSC network as an evolutionary variational inequality, and provide the associated projected dynamical systems.

In the literature, we find substantial contributions on CLSCs. These studies highlighted the analysis of CLSC networks, including manufacturers, retailers and consumer markets. For instance, authors in Nagurney and Toyasaki (2003) developed a supply chain network model for decision-making considering environmental criteria. In Nagurney and Toyasaki (2005), the authors used a variational inequality approach to study a reverse supply chain network. In Shen et al. (2020), the authors explored the sale of second-hand items using an online platform on a supply chain consisting of contributors, one online platform, and one supplier. They also considered different scenarios of CLSC structure and block-chain. In Wang et al. (2019), the authors provided a variational inequality model to study the CLSC network with consideration to the waste of electrical and electronic equipment. In Zhang et al. (2014), the authors examined a CLSC network equilibrium problem in multiperiod planning horizons, considering product lifetime and carbon emission constraints. The governing CLSC network equilibrium model established were given by variational inequalities and complementary theory. In Fu et al. (2021), the authors studied the reverse channel in dynamic CLSC system which consists of manufacturers and retailers, and searched the decisions and profits of CLSC members in different reverse channels, that consider the quantitative characteristic of products. There is also notable research on sustainable supply chain. In Nagurney and Nagurney (2010), the authors dealt with the design of sustainable supply chain networks. Specifically, they focused on a firm that minimizes the total costs associated with design/construction

and operation, but also minimizes the emissions generated. In Nagurney et al. (2007b), the authors developed a sustainable supply chain model, in which manufacturers, retailers, as well as the consumers at the demand markets follow some environmental criteria. In Colajanni and Daniele (2018), the authors introduced the distinction by brand of the products of manufacturers, and added the e-commerce to the traditional physical links for the shipments from manufacturers to demand markets. Moreover, to the forward chain they added a reverse chain model where manufacturers, using the unsold product given back from retailers, after reworking, produce a new commodity which will be sold to new retailers. For a systematic review of the sustainable supply chain management literature, we address the reader to Brandenburg et al. (2014), Seuring and Müller (2008). In contrast to the existing competition equilibrium models, we construct a dynamic equilibrium model of the CLSC network, and extend the model in Fargetta and Scrimali (2022). Our model captures the time associated with the activities of the various tiers of the CLSC and takes into account time-varying demands. As a consequence, the costs and revenues all depend on time. Moreover, we introduce transaction costs of manufacturers and retailers that reduce the environmental impact. This policy may include the acquisition of raw materials, internal logistics, packaging, recycling, reuse, efficiency in the use of resources and proper disposal of waste, see Zsidisin and Siferd (2001).

The time-dependent framework for network equilibrium problem has been investigated in several papers. For example, in Chan et al. (2018), the authors introduced a dynamic equilibrium model of oligopolistic CLSC network taking into account the seasonality of demand. The dynamic Cournot-Nash equilibrium of the oligopolistic network was constructed by evolutionary variational inequality and projected dynamical system. In Feng et al. (2014), the authors developed a CLSC supernetwork model with suppliers, manufacturers, retailers and demand markets, in which the demand for a product is seasonal. The equilibrium condition of the CLSC was based on the evolutionary variational inequalities and the projected dynamical systems. In Daniele (2010), the author presented a supply chain network model in the case when prices and shipments depend on time. Using the infinite dimensional duality theory, the equilibrium conditions for the representatives of each tier of the network were given, as well as the time-dependent variational inequality governing the complete supply chain. In Nagurney et al. (2007a), the authors proposed a dynamic electric power supply chain network model in which the demand varies over time using an evolutionary variational inequality formulation. In Fargetta and Scrimali (2023), the authors developed a dynamic network model of the competition of digital contents on social media platforms. The problem is formulated as a time-dependent generalized Nash equilibrium for which the associated evolutionary variational inequality, using the variational equilibrium concept, is provided (see Fargetta and Scrimali 2021 for the static decision network).

Thus, motivated by the above results, we propose a CLSC for second-hand collectible items using online trading platforms and in a time-dependent setting. The main contributions of this paper are:

- Modeling the second-hand market in the reverse logistics in a dynamic environment;

- Studying the competition among the members of the same tiers as well as one between adjacent tiers with time-varying data;
- Formulating the equilibrium conditions of the CLSC network as an evolutionary variational inequality and a projected dynamical system;
- Integrating environmental and social sustainability into the process of transactions among the layers of the CLSC network;
- Including consumers' risk-aversion to purchasing second-hand products, and platform's risk-aversion to transacting with collectors.

The paper is organized as follows. In Sect. 2, we introduce the CLSC model and present the manufacturers' and the retailers' competitive behavior, as well as the interactions with the demand markets and the online platform. In Sect. 3, we state the governing equilibrium conditions of the CLSC network. Then, we provide an evolutionary variational inequality formulation of the optimal behavior of decision-makers, and give the associated projected dynamical system. In Sect. 4, we discuss a numerical example. Finally, in Sect. 5, we summarize our results and draw our conclusions.

## 2 The CLSC network

In this section, we illustrate the equilibrium network model of a CLSC in a dynamic framework. Let us consider a CLSC network consisting of multiple manufacturers, retailers and demand markets in which some consumers purchase new items to collect them. The second-hand products are distributed by collectors through an online platform for the aim of gaining profit on the resale. Each decision-maker in the network tries to maximize his own total profit. We indicate with  $M$  the set of manufacturers (let  $m$  be the typical manufacturer), with  $R$  the set of retailers (let  $r$  be the typical retailer), with  $K$  the set of demand markets (let  $k$  be the typical demand market), and we consider a single online platform as eBay, Amazon, Marketplace by Facebook, Vinted, etc... We underline that, without risk of confusion, we use an abusing notation for the same symbols here to denote the sets  $M$ ,  $R$ ,  $K$  and their cardinalities. Furthermore, we define the set of collectors  $K_c$ , which represents the set of consumers who decide to resell their items, i.e. collectibles ( $K_c$  is less than the number of all the consumers at the demand markets). The planning horizon of our model is the interval  $[0, T]$ ,  $T > 0$ . The topology of the network, which does not change in  $[0, T]$ , is illustrated in Fig. 1.

The network is divided into two parts: the forward chain formed by manufacturers, retailers and demand markets, and the reverse chain formed by collectors, the online platform and demand markets. In Fig. 1, the links from the manufacturing nodes (left-hand nodes  $1, \dots, m, \dots, M$ ) are connected to the retailer nodes (middle nodes  $1, \dots, r, \dots, R$ ). The links from the retailer nodes are, in turn, connected to the demand market nodes (right-hand nodes  $1, \dots, k, \dots, K$ ). Moreover, the links from every demand market node  $k$ ,  $k = 1, \dots, K$ , of the reverse supply chain market are connected to the online second-hand platform. Finally, the links from the online

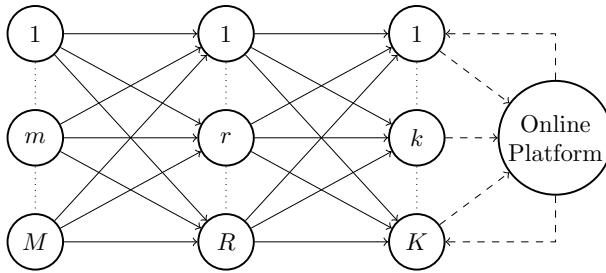


Fig. 1 The CLSC network

platform are connected to demand markets to complete a single loop. Thus, our CLSC network combines the forward supply chain activities, such as production, delivery and distribution, with the reverse supply chain ones, such as collecting and delivery, to form a closed-loop network. The forward and the reverse chains are connected by the collectors and the online platform, and create the CLSC network. In Fig. 1, the forward transactions are represented by the solid line and the reverse ones by the dashed lines. The items considered are divided into the new ones denoted by index  $n = 1, \dots, N$ , and the second-hand ones denoted by index  $u = 1 \dots U$ .

For any positive integer  $w$ , we denote by  $L^2([0, T], \mathbb{R}_+^w)$  the Hilbert space of square-integrable functions from  $[0, T]$  into  $\mathbb{R}_+^w$ . For any  $f, g \in L^2([0, T], \mathbb{R}_+^w)$ , the inner product is denoted by

$$\langle f, g \rangle_{L^2} = \int_0^T \langle f(t), g(t) \rangle dt,$$

where  $\langle \cdot, \cdot \rangle$  represents the inner product in the corresponding Euclidean space.

Finally, we recall the standard form of an evolutionary variational inequality which is the problem to determine  $X^* \in \mathcal{K}$ , such that

$$\langle F(X^*), X - X^* \rangle_{L^2} = \int_0^T \langle F(t, X^*(t)), X(t) - X^*(t) \rangle dt \geq 0, \quad \forall X \in \mathcal{K}, \quad (1)$$

where  $\mathcal{K} \subset L^2([0, T], \mathbb{R}_+^w)$  is a closed and convex set, and  $F : [0, T] \times \mathcal{K} \rightarrow L^2([0, T], \mathbb{R}_+^w)$ .

In the following subsections, we present the evolutionary CLSC problem and give the corresponding equilibrium conditions. Firstly, we focus on the behavior of the manufacturers, subsequently turn to the behavior of the retailers, then to the demand markets, and, finally, to the platform. The complete equilibrium model is constructed as an evolutionary variational inequality in Sect. 3.

### 2.1 The optimal behavior of the manufacturers

We now present the problem of determining the production quantities of each new product so that all the manufacturers' profit is maximized in the planning horizon  $[0, T]$ . Let  $x_{mr}^n \in L^2([0, T], \mathbb{R}_+)$  be such that  $x_{mr}^n(t)$  is the quantity of new item  $n$  sold by

manufacturer  $m$  to retailer  $r$  at time  $t \in [0, T]$ . We group all the  $n$  and  $r$  quantities into the function  $x_m \in L^2([0, T], \mathbb{R}_+^{NR})$ , and then we group  $x_m$  for all  $m$  into the function  $x^M \in L^2([0, T], \mathbb{R}_+^{NMR})$ . Let  $x_m^{max} \in L^2([0, T], \mathbb{R}_+)$  be such that  $x_m^{max}(t)$  is the production capacity of manufacturer  $m$  at time  $t$ . In order to maximize his own profit, each manufacturer  $m$  must decide the quantity  $x_{mr}^n(t)$  of new item  $n$  to be sold to retailer  $r$ .

We introduce:

- $c_m : L^2([0, T], \mathbb{R}_+^{NMR}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $c_m(x^M(t))$  is the production cost of manufacturer  $m$  at time  $t$ ;
- $t_{mr} : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $t_{mr}(x_{mr}^n(t))$  is the transaction cost from manufacturer  $m$  to retailer  $r$  at time  $t$ ;
- $g_{mr} : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $g_{mr}(x_{mr}^n(t))$  is the emission-control at time  $t$ ;
- $p_{mr}^{*n} \in L^2([0, T], \mathbb{R}_+)$ , where  $p_{mr}^{*n}(t)$  is the selling price of a new product at time  $t$ .

We emphasize that a manufacturer is concerned with the emissions generated in transaction of the product to the various retailers in the forward logistics, and this is represented by the function  $g_{mr}(x_{mr}^n(t))$ .

To guarantee the existence of the equilibrium, the following additional assumptions hold:

- (M1)  $c_m(x^M(t))$ ,  $t_{mr}(x_{mr}^n(t))$  and  $g_{mr}(x_{mr}^n(t))$  are continuously differentiable and convex functions a.e. in  $[0, T]$ ;
- (M2)  $p_{mr}^{*n}(t)$  is continuous a.e. in  $[0, T]$ .

Given the above notation, each manufacturer  $m$  wishes to maximize his profit as follows:

$$\max \int_0^T \left\{ \sum_{n \in N} \sum_{r \in R} p_{mr}^{*n}(t) x_{mr}^n(t) - c_m(x^M(t)) - t_{mr}(x_{mr}^n(t)) - g_{mr}(x_{mr}^n(t)) \right\} dt \quad (2)$$

subject to:

$$\sum_{n \in N} \sum_{r \in R} x_{mr}^n(t) \leq x_m^{max}(t), \quad x_{mr}^n(t) \geq 0, \text{ a.e. in } [0, T], \forall n, r. \quad (3)$$

Problem (2) represents the goal of each manufacturer that seeks to maximize his profit given other manufacturers' decisions. All the manufactures compete in a non-cooperative fashion and their profits are equal to sales revenue minus costs associated with transaction, production and pollution control. The first constraint in (3) represents the production capacity of manufacturer  $m$ .

We consider the feasible set

$$S^M = \left\{ x^M \in L^2([0, T], \mathbb{R}_+^{NMR}) : \sum_{n \in N} \sum_{r \in R} x_{mr}^n(t) \leq x_m^{max}(t), \text{ a.e. in } [0, T], \forall m \in M \right\}, \quad (4)$$

that is a nonempty, convex, closed and bounded set.

The following theorem holds (see also Daniele 2010):

**Theorem 1** Under the assumptions (M1) and (M2),  $x^{*M} \in S^M$  is a solution to problem (2)–(3) if and only if it is a solution to the evolutionary variational inequality:

$$\int_0^T \langle F^M(t, x^{*M}(t)), x^M(t) - x^{*M}(t) \rangle dt \geq 0, \quad \forall x^M \in S^M, \tag{5}$$

where

$$F^M(t, x^{*M}(t)) = \begin{bmatrix} F_{nmr}^M(t, x^{*M}(t)) \end{bmatrix} \begin{matrix} n \in N \\ m \in M \\ r \in R \end{matrix};$$

$$F_{nmr}^M(t, x^{*M}(t)) = \frac{\partial c_m(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial t_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} + \frac{\partial g_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} - p_{mr}^{*n}(t).$$

We note that the price  $p_{mr}^{*n}(t)$  is one of the data of our model.

### 2.2 The optimal behavior of the retailers

Both manufacturers and consumers at demand markets interact with retailers. In particular, retailers decide the amount of products to order from the manufacturers so as to transact with the consumers, while seeking to maximize their profit. Let  $x_{rk}^n \in L^2([0, T], \mathbb{R}_+)$  be such that  $x_{rk}^n(t)$  is the product shipment of a new item  $n$  between retailer  $r$  and consumers at demand market  $k$  at time  $t$ ; the product shipments for all  $n$  and  $k$  are then grouped into the function  $x_r \in L^2([0, T], \mathbb{R}_+^{NK})$  and, further, into the function  $x^R \in L^2([0, T], \mathbb{R}_+^{NRK})$ .

We introduce:

- $c_r : L^2([0, T], \mathbb{R}_+^{NMR}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $c_r(x^M(t))$  is the management cost related to the items in stock at time  $t$ ;
- $\hat{c}_{mr}^n : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{c}_{mr}^n(x_{mr}^n(t))$  is the transportation cost at time  $t$ ;
- $t_{rk}^n : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $t_{rk}^n(x_{rk}^n(t))$  is the transaction costs at time  $t$ , when selling new products to consumers at demand markets;
- $\hat{g}_{mr}^n : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{g}_{mr}^n(x_{mr}^n(t))$  is the emission-control at time  $t$  to reduce the emissions associated with transacting with consumers at demand markets;
- $p_{rk}^{*n} \in L^2([0, T], \mathbb{R}_+)$ , where  $p_{rk}^{*n}(t)$  is the price of new items that retailers fix for consumers at demand markets at time  $t$ .

Let  $s_r^n \in L^2([0, T], \mathbb{R}_+^{NRK}) \rightarrow L^2([0, T], \mathbb{R}_+)$  be such that  $s_r^n(x^R(t))$  is the revenue from advertising on the social network profiles of each retailer, for instance, Instagram

stories or TikTok videos based on the launch of a new product. This is a total revenue because they have no costs to make this videos.

Finally, we suppose that the following additional assumptions hold:

- (R1)  $c_r(x^M(t))$ ,  $\hat{c}_{mr}^n(x_{mr}^n(t))$ ,  $\hat{g}_{mr}^n(x_{mr}^n(t))$ ,  $t_{rk}^n(x_{rk}^n(t))$  and  $s_r^n(x^R(t))$  are continuously differentiable and convex functions a.e. in  $[0, T]$ ;
- (R2)  $p_{rk}^{*n}(t)$  and  $p_{mr}^{*n}(t)$  are continuous a.e. in  $[0, T]$ .

Each retailer  $r$  seeks to maximize his profit function as follows:

$$\max \int_0^T \left\{ \sum_{n \in N} \left( \sum_{k \in K} p_{rk}^{*n}(t)x_{rk}^n(t) - \sum_{m \in M} \hat{c}_{mr}^n(x_{mr}^n(t)) - \sum_{m \in M} \hat{g}_{mr}^n(x_{mr}^n(t)) - \sum_{k \in K} t_{rk}^n(x_{rk}^n(t)) - \sum_{m \in M} p_{mr}^{*n}(t)x_{mr}^n(t) \right) + s_r^n(x^R(t)) - c_r(x^M(t)) \right\} dt \tag{6}$$

subject to:

$$\sum_{n \in N} \sum_{k \in K} x_{rk}^n(t) \leq \sum_{n \in N} \sum_{m \in M} x_{mr}^n(t), \text{ a.e. in } [0, T], \forall n, m, k, \tag{7}$$

$$x_{mr}^n(t) \geq 0, x_{rk}^n(t) \geq 0, \text{ a.e. in } [0, T], \forall n, m, k. \tag{8}$$

Problem (6) expresses the profit maximization of each retailer, where the profit is equal to sales revenues minus costs associated with the transportation, the management, the transaction, the payout to the manufacturers and the pollution control. Constraints in (7) state that consumers cannot purchase more from a retailer than is held in stock. Conditions (8) express the non-negative constraints. The optimality conditions for all the retailers can be described simultaneously using an evolutionary variational inequality, see (Cojocaru et al. 2008; Nagurney et al. 2007a), as all the retailers compete in a non-cooperative fashion. Let the feasible set be:

$$S^R = \left\{ (x^R, x^M) \in L^2([0, T], \mathbb{R}_+^{NRK+NM}) : \sum_{n \in N} \sum_{k \in K} x_{rk}^n(t) \leq \sum_{n \in N} \sum_{m \in M} x_{mr}^n(t), \text{ a.e. in } [0, T], \forall r \in R \right\}. \tag{9}$$

We observe that  $S^R$  is a nonempty, convex, closed and bounded set.

The following theorem holds (see also Daniele 2010):

**Theorem 2** *Under the assumptions (R1) and (R2), the pair  $(x^{*R}, x^{*M}) \in S^R$  is a solution to problem (6)–(8) if and only if it is a solution to the evolutionary variational inequality:*

$$\int_0^T \langle F^R(t, X^{*R}(t)), X^R(t) - X^{*R}(t) \rangle dt \geq 0, \quad \forall X^R \in S^R, \tag{10}$$



where

$$\begin{aligned}
 X^R &= (x^M, x^R); \\
 F^R(t, X^{*R}(t)) &= \left( F_1^R(t, X^{*R}(t)), F_2^R(t, X^{*R}(t)) \right); \\
 F_1^R(t, X^{*R}(t)) &= \left[ F_{1mr}^R(t, X^{*R}(t)) \right]_{\substack{m \in M \\ r \in R}}; \quad F_2^R(t, X^{*R}(t)) = \left[ F_{2mr}^R(t, X^{*R}(t)) \right]_{\substack{r \in R \\ k \in K}}; \\
 F_{1mr}^R(t, X^{*R}(t)) &= p_{mr}^{*n}(t) + \frac{\partial c_r(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial \hat{c}_{mr}^n(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} + \frac{\partial \hat{g}_{mr}^n(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)}; \\
 F_{2mr}^R(t, X^{*R}(t)) &= \frac{\partial t_{rk}^n(x_{rk}^{*n}(t))}{\partial x_{rk}^n(t)} - p_{rk}^{*n}(t) - \frac{\partial s_r^n(x^{*R}(t))}{\partial x_{rk}^n(t)}.
 \end{aligned}$$

We remark that the prices  $p_{mr}^{*n}(t)$  and  $p_{rk}^{*n}(t)$  are data of our model.

### 2.3 The optimal behavior of the consumers

Although, generally, consumers show high levels of concern for the environment, when they have to buy or resell products, their actions could be not sustainable. On the one hand, a product may be preferred as it offers satisfactory performance or a convenient price, even if it has high environmental impact. On the other hand, the limited number of transactions of collectors prevent them from adopting expensive environmentally friendly measures. For these reasons, we assume that consumers do not have environmental attitudes.

Retailers interact with consumers at the demand markets and consumers, in turn, negotiate with the online platform. In particular, consumers buy new products in the forward supply chain; then, some consumers collect items and resell them on the online platform. Finally, consumers purchase collectibles in the reverse supply chain. We analyze the forward and the reverse logistics separately.

#### 2.3.1 The consumers in the forward logistics

We now describe the behavior of the consumers at demand markets involved in the forward supply chain processes. We introduce:

- $\hat{c}_{rk}^n : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{c}_{rk}^n(x_{rk}^n(t))$  is the transportation cost for new product  $n$  sold by retailer  $r$  to demand market  $k$  at time  $t$ ;
- $d_k^n : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $d_k^n(p_k^n(t))$  is the price-demand of new item  $n$  at demand market  $k$  at time  $t$ ;
- $p_k^n \in L^2([0, T])$ , where  $p_k^n(t)$  is the price that the consumers are willing to pay for new product  $n$  at demand market  $k$  at time  $t$ .

We group all prices  $p_k^n$  into  $p_k \in L^2([0, T], \mathbb{R}_+^N)$ , then into the function  $p^N \in L^2([0, T], \mathbb{R}_+^{NK})$ . It is worth noting that the price  $p_k^n(t)$ , as we will show, will be endogenously determined by the equilibrium conditions of the CLSC.

The consumers take into account the price  $p_{rk}^{*n}(t)$  charged by the retailers for the product and the transaction cost to obtain the product. The equilibrium conditions for consumers at demand market  $k$  are (see Nagurney et al. 2002, 2007a):

$$p_{rk}^{*n}(t) + \hat{c}_{rk}^n(x_{rk}^{*n}(t)) \begin{cases} = p_k^{*n}(t) & \text{if } x_{rk}^{*n}(t) > 0, \\ \geq p_k^{*n}(t) & \text{if } x_{rk}^{*n}(t) = 0, \end{cases} \quad \text{a.e. in } [0, T], \forall r, k, n. \quad (11)$$

$$d_k^n(p_k^{*n}(t)) \begin{cases} = \sum_{r \in R} x_{rk}^{*n}(t) & \text{if } p_k^{*n}(t) > 0, \\ \leq \sum_{r \in R} x_{rk}^{*n}(t) & \text{if } p_k^{*n}(t) = 0, \end{cases} \quad \text{a.e. in } [0, T], \forall k, n. \quad (12)$$

Conditions (11) and (12) correspond to the well-known spatial price equilibrium conditions, see Nagurney (1998). In particular, inequality (11) states that if the consumers at demand market  $k$  purchase the products from retailer  $r$ , namely,  $x_{rk}^{*n}(t) > 0$ , then the sum of the amount that they pay for the product plus the transportation cost will be equal to the price that the consumers are willing to pay. Equation (12) states that if the equilibrium price that the consumers are willing to pay for the new products at the demand market is positive, then the amount of new products bought from the retailers will be exactly equal to the demand.

### 2.3.2 The consumers in the reverse logistics

Now, we focus on the behaviour of those consumers who decide to resell collectibles to the demand markets through the online platform. Let  $x_{k_c, k}^u \in L^2([0, T], \mathbb{R}_+)$  be such that  $x_{k_c, k}^u(t)$  is the shipment of second-hand product  $u$  between collector  $k_c$  and consumers at demand market  $k$  at time  $t$ , using the online platform. We group the product shipments  $x_{k_c, k}^u$ , for all  $u$  and  $k$  into the function  $x_{k_c} \in L^2([0, T], \mathbb{R}_+^{UK})$  and, further, into the function  $x^{K_c} \in L^2([0, T], \mathbb{R}_+^{UK_c K})$ .

Let  $Q_{k_c} \in L^2([0, T])$  be such that  $Q_{k_c}(t)$  is the amount of items in the collection of collector  $k_c$  at time  $t$ . Collectors choose to sell on the online platform, because it can give higher visibility to the collectible items and, as a consequence, it can be more profitable, despite the platform retains a portion of the sale price. We denote by  $\gamma = 0.1$  the transaction price coefficient. For instance, on eBay it amounts to the 10% of the selling price. We consider:

- $c_{k_c} : L^2([0, T], \mathbb{R}_+^{UK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $c_{k_c}(x_{k_c}(t))$  is the maintenance and restoring cost of the collector  $k_c$  at time  $t$ , depending on the amount of items that he resells on the online platform;
- $\hat{c}_{rk_c}^n : L^2([0, T], \mathbb{R}_+^{NRK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{c}_{rk_c}^n(x^R(t))$  is the transportation cost from retailer  $r$  for new item  $n$  to collector  $k_c$  at time  $t$ ;

- $p_{k_c}^{*u} \in L^2([0, T], \mathbb{R}_+)$ , where  $p_{k_c}^{*u}(t)$  is the price charged by the collector  $k_c$  for second-hand items at time  $t$ ;
- $x_{rk_c}^n \in L^2([0, T], \mathbb{R}_+)$ , where  $x_{rk_c}^n(t)$  is the product shipment of a new item  $n$  between retailer  $r$  and collector  $k_c$  at time  $t$ ;
- $p_{rk_c}^{*n} \in L^2([0, T], \mathbb{R}_+)$ , where  $p_{rk_c}^{*n}(t)$  is the price of new items that retailers fix for collector  $k_c$  at time  $t$ .

The product shipments  $x_{rk_c}^n(t)$  for all  $n$  and  $k_c$  are then grouped into the function  $x_{rc} \in L^2([0, T], \mathbb{R}_+^{NK_c})$  and, further, into the function  $x^{RK_c} \in L^2([0, T], \mathbb{R}_+^{NRK_c})$ .

We denote by  $\mu_{k_c} \in (0, 1]$  the portion of second-hand products that collector  $k_c \in K_c$  decides to sell on the platform. Each collector  $k_c \in K_c$  seeks to maximize his profit function as follows:

$$\max \int_0^T \left\{ \sum_{u \in U} \sum_{k \in K} (1 - \gamma) p_{k_c}^{*u}(t) x_{k_c,k}^u(t) - c_{k_c}(x_{k_c}(t)) - \sum_{n \in N} \sum_{r \in R} \left( p_{rk_c}^n(t) x_{rk_c}^n(t) - \hat{c}_{rk_c}^n(x^R(t)) \right) \right\} dt \tag{13}$$

subject to

$$\sum_{u \in U} \sum_{k \in K} x_{k_c,k}^u(t) \leq \mu_{k_c} Q_{k_c}(t), \quad x_{k_c,k}^u(t), x_{rk_c}^n(t) \geq 0, \text{ a.e. in } [0, T], \forall n, u, r, k. \tag{14}$$

Problem (13) is the profit maximization of each collector, where the profit is equal to sales revenues minus costs associated with restoring, purchasing and transportation. The first constraint in (14) expresses that the amount of products that collector  $k_c$  decides to sell should be less than or equal to the amount of collectibles in  $k_c$ 's collection.

The transactions between the platform and the demand market  $k$  is analyzed as follow. We consider:

- $\hat{c}_{k_c,k}^u : L^2([0, T], \mathbb{R}_+^{UK_cK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{c}_{k_c,k}^u(x^U(t))$  is the transportation cost from collector  $k_c$  to consumers at demand market  $k$  for used product  $u$  purchased on the platform at time  $t$ ;
- $\rho_k^u \in L^2([0, T], \mathbb{R}_+)$ , where  $\rho_k^u(t)$  is the willingness to pay second-hand products for all consumers at demand market  $k$  at time  $t$ ;
- $d_k^u \in L^2([0, T], \mathbb{R}_+)$ , where  $d_k^u(\rho^U(t))$  is the price-demand of second-hand items for all consumers at demand market  $k$  at time  $t$ .

Then, we group all  $\rho_k^u$  into  $\rho_k \in L^2([0, T], \mathbb{R}_+^U)$ , then into the function  $\rho^U(t) \in L^2([0, T], \mathbb{R}_+^{UK})$ .

It is interesting to investigate the risk associated with purchasing second-hand items from the trading platform. Indeed, each consumer exhibits risk aversion which may depend on the flows controlled by the other consumers. Hence, the

risk aversion function (see Nagurney et al. 2005) can be expressed as the function  $\pi_k^u : L^2([0, T], \mathbb{R}_+^{UK_cK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\pi_k^u(x^U(t))$  is the risk aversion of consumer  $k$  for the used products at time  $t$ . The equilibrium conditions for consumers at demand market  $k$  in the reverse supply chain are given by:

$$p_{k_c}^{*u}(t) + \hat{c}_{k_c,k}^u(x^{*U}(t)) + \pi_k^u(x^{*U}(t)) \begin{cases} = \rho_k^{*u}(t) & \text{if } x_{k_c,k}^{*u}(t) > 0, \\ \geq \rho_k^{*u}(t) & \text{if } x_{k_c,k}^{*u}(t) = 0, \end{cases} \quad \text{a.e. in } [0, T], \forall k_c, u; \tag{15}$$

$$d_k^u(\rho^{*U}(t)) \begin{cases} = x_{k_c,k}^{*u}(t) & \text{if } \rho_k^{*u}(t) > 0, \\ \leq x_{k_c,k}^{*u}(t) & \text{if } \rho_k^{*u}(t) = 0, \end{cases} \quad \text{a.e. in } [0, T], \forall k, u; \tag{16}$$

$$p_{rk}^{*n}(t) - \rho_k^{*u}(t) \begin{cases} < 0 & \text{if } x_{rk}^{*n}(t) = 0, \\ \geq 0 & \text{if } x_{rk}^{*n}(t) > 0, \end{cases} \quad \text{a.e. in } [0, T], \forall r, k, u, n. \tag{17}$$

Equality (15) states that if the consumers at demand market  $k$  purchase the product on the online platform, then the sum of the price charged by the collector  $k_c$  for second-hand items, of the transportation cost, and of the risk incurred by the consumer is equal to the price that the consumer is willing to pay. Condition in (16) states that if the equilibrium price that the consumers at demand market  $k$  are willing to pay for the second-hand product is positive, then the amount purchased of second-hand product should exactly be equal to the demand of this second-hand item. The last condition in (17) means that the unitary price of a second-hand collectible is higher than the unitary price of a new collectible that is totally sold out.

We suppose that the following assumptions hold:

- (C1)  $c_{k_c}(x_{k_c}(t))$ ,  $\hat{c}_{rk_c}^n(x^R(t))$  and  $\pi_k^u(x^U(t))$  are continuously differentiable and convex functions a.e. in  $[0, T]$ ;
- (C2)  $p_{k_c}^{*u}(t)$ ,  $d_k^n(p_k^n(t))$ ,  $\hat{c}_{rk}^n(x_{rk}^n(t))$ ,  $\hat{c}_{k_c,k}^u(x_{k_c,k}^u(t))$ ,  $\pi_k^u(x^U(t))$  and  $d_k^u(\rho^U(t))$  are continuous a.e. in  $[0, T]$ .

### 2.3.3 The consumers' equilibrium conditions

We combine consumers' behaviors in both forward and reverse supply chain, and we express the equilibrium conditions for all the demand markets using an evolutionary variational inequality, see (Nagurney et al. 2007a). Let the feasible set be given by:

$$S^K = \left\{ (p^N, x^{Kc}, \rho^U, x^R, x^{RK_c}) \in L^2\left([0, T], \mathbb{R}_+^{KN+2UK+UK_cK+NRK}\right) : \right. \\ \left. \sum_{u \in U} \sum_{k \in K} x_{k_c,k}^u(t) \leq \mu_{k_c} Q_{kc}(t), \text{ a.e. in } [0, T] \right\},$$

where  $S^K$  is a nonempty, convex, closed and bounded set.

The following theorem holds (see also Daniele 2010):

**Theorem 3** Under assumptions (C1) and (C2), the vector  $(p^{*N}, x^{*Kc}, \rho^{*U}, x^{*R}, x^{*RKc}) \in S^K$  is a solution to problem (11)–(17) if and only if it is a solution to the evolutionary variational inequality:

$$\int_0^T \langle F^K(t, X^{*K}(t)), X^K(t) - X^{*K}(t) \rangle dt \geq 0, \quad \forall X^K \in S^K, \tag{18}$$

where

$$X^K = (p^N, x^{Kc}, \rho^U, x^R, x^{RKc});$$

$$F^K(t, X^{*K}(t)) = \left( F_1^K(t, X^{*K}(t)), F_2^K(t, X^{*K}(t)), \dots, F_5^K(t, X^{*K}(t)) \right);$$

$$F_1^K(t, X^{*K}(t)) = \left[ F_{1nk}^K(t, X^{*K}(t)) \right]_{\substack{n \in N \\ k \in K}}; \quad F_2^K(t, X^{*K}(t)) = \left[ F_{2uk_c}^K(t, X^{*K}(t)) \right]_{\substack{u \in U \\ k_c \in K_c}};$$

$$F_3^K(t, X^{*K}(t)) = \left[ F_{3uk}^K(t, X^{*K}(t)) \right]_{\substack{u \in U \\ k \in K}}; \quad F_4^K(t, X^{*K}(t)) = \left[ F_{4nrk}^K(t, X^{*K}(t)) \right]_{\substack{n \in N \\ r \in R \\ k \in K}};$$

$$F_5^K(t, X^{*K}(t)) = \left[ F_{5nr}^K(t, X^{*K}(t)) \right]_{\substack{n \in N \\ r \in R}};$$

$$F_{1nk}^K(t, X^{*K}(t)) = \sum_{r \in R} x_{rk}^{*n}(t) - d_k^n(p_k^{*n}(t));$$

$$F_{2uk_c}^K(t, X^{*K}(t)) = \frac{\partial c_{k_c}(x_{k_c}^{*u}(t))}{\partial x_{k_c,k}^{*u}(t)} + \gamma p_{k_c}^{*u}(t) + \hat{c}_{k_c,k}^u(x_{k_c,k}^{*u}(t)) + \pi_k^u(x^{*U}(t)) - \rho_k^{*u}(t);$$

$$F_{3uk}^K(t, X^{*K}(t)) = \sum_{k_c \in K_c} x_{k_c,k}^{*u}(t) - d_k^u(\rho^{*U}(t));$$

$$F_{4nrk}^K(t, X^{*K}(t)) = 2p_{rk}^{*n}(t) + \hat{c}_{rk}^n(x_{rk}^{*n}(t)) - p_k^{*n}(t) - \sum_{u \in U} \rho_k^{*u}(t) - \frac{\partial \hat{c}_{rk_c}^n(x^{*R}(t))}{\partial x_{rk}^n(t)};$$

$$F_{5nr}^K(t, X^{*K}(t)) = \sum_{k_c \in K_c} p_{rk_c}^{*n}(t).$$

### 2.4 The behavior of the online platform

Now, we present the behavior of the online platform as an intermediary that matches consumers and collectors. As an intermediary, the platform is involved in transactions with both the collectors and the consumers at the demand markets. We introduce:

- $C^u : L^2([0, T], \mathbb{R}_+^{UK_cK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $C^u(x^U(t))$  is the management costs of second-hand product  $u$  at time  $t$ , including processing and advertisement;
- $\hat{\gamma}_k^u : L^2([0, T], \mathbb{R}_+) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\hat{\gamma}_k^u(x_k^u(t))$  is the transaction cost function between the platform and demand market  $k$  at time  $t$ ;
- $p_k^u \in L^2([0, T], \mathbb{R}_+)$ , where  $p_k^u(t)$  is the unitary price of second-hand products at time  $t$ ;
- $\pi_k^u : L^2([0, T], \mathbb{R}_+^{UK_cK}) \rightarrow L^2([0, T], \mathbb{R}_+)$ , where  $\pi_k^u(x^U(t))$  is the risk function associated with online platform at time  $t$ .

We set  $x_k^u(t) = \sum_{k_c} x_{k_c,k}^u(t)$ . We note that the online platform does not affect the collectors' sales choices. However, the platform, like the user who buys, risks owning / buying fake objects or with descriptions that do not correspond to the real conditions of the object. Thus, the platform incurs the risks associated with transacting with the various collectors and with the demand markets. We assume that

- (P1)  $C^u(x^U(t))$ ,  $\hat{\gamma}_k^u(x_k^u(t))$  and  $\pi_k^u(x^U(t))$  are continuously differentiable and convex, for all  $k \in K$  and  $k_c \in K_c$ ;
- (P2)  $p_k^u(t)$  is continuous a.e. in  $[0, T]$ .

We define  $Q^u(t) \in L^2([0, T], \mathbb{R}_+)$  as the total amount of item  $u$  on the online platform. Each online platform makes his optimal decisions based on maximizing the following profit function:

$$\max \int_0^T \left\{ \sum_{u \in U} \left( \sum_{k_c \in K_c} \sum_{k \in K} \gamma p_{k_c}^u(t) x_{k_c,k}^u(t) - C^u(x^U(t)) - \sum_{k \in K} \hat{\gamma}_k^u(x_k^u(t)) - \pi_k^u(x^U(t)) \right) \right\} dt \tag{19}$$

subject to

$$\sum_{k \in K} x_k^u(t) \leq Q^u(t), \quad x_k^u(t), x_{k_c,k}^u(t) \geq 0, \quad \forall u, \forall k, \text{ a.e. in } [0, T]. \tag{20}$$

Problem (19) is the profit maximization of the online platform, where the profit is equal to a percentage of the profit of sale of the product minus the management, transaction costs and the risk. The constraint in (20) states that the total amount of each second-hand item bought by all consumers at demand market  $k$  on the platform should be less than or equal to the availability of item  $u$ . The optimality conditions for the online platform can be expressed as an evolutionary variational inequality, see (Nagurny et al. 2007a), with feasible set

$$S^P = \left\{ (x^U, x^{K_c}) \in L^2\left([0, T], \mathbb{R}_+^{U K_c K + UK}\right) : \sum_{k \in K} x_k^u(t) \leq Q^u(t), \forall u \in U \text{ a.e. in } [0, T] \right\}, \tag{21}$$

where  $S^P$  is a nonempty, convex, closed and bounded set. The following theorem holds (see also Daniele 2010):

**Theorem 4** *If the assumptions (P1) and (P2) hold, then  $(x^{*U}, x^{*K_c}) \in S^P$  is a solution to problem (19)–(20) if and only if it is a solution to the evolutionary variational inequality:*

$$\int_0^T \langle F^P(t, X^{*P}(t)), X^P(t) - X^{*P}(t) \rangle dt \geq 0, \quad \forall X^P \in S^P, \tag{22}$$

where

$$\begin{aligned} X^P &= (x^U, x^{K_c}); \\ F^P(t, X^{*P}(t)) &= \left( F_1^P(t, X^{*P}(t)), F_2^P(t, X^{*P}(t)) \right); \\ F_1^P(t, X^{*P}(t)) &= \left[ F_{1u k_c k}^P(t, X^{*P}(t)) \right]_{\substack{u \in U \\ k_c \in K_c \\ k \in K}}; \quad F_2^P(t, X^{*P}(t)) = \left[ F_{2uk}^P(t, X^{*P}(t)) \right]_{k \in K}; \\ F_{1u k_c k}^P(t, X^{*P}(t)) &= \frac{\partial C^u(x^{*U}(t))}{\partial x_{k_c k}^u(t)} + \frac{\partial \pi_k^u(x^{*U}(t))}{\partial x_{k_c k}^u(t)} - \gamma p_{k_c}^{*u}(t); \\ F_{2uk}^P(t, X^{*P}(t)) &= \frac{\partial \hat{r}_k^u(x_k^{*u}(t))}{\partial x_k^u(t)}. \end{aligned}$$

We remark that the price  $p_{k_c}^{*u}(t)$  is given in our model.

### 3 The equilibrium conditions of the close-loop supply chain

For all manufacturers, all retailers, all consumers at the demand markets and the online platform, the optimality conditions at equilibrium must be satisfied simultaneously. Specifically, we provide the CLSC network equilibrium and give an equivalent evolutionary variational inequality formulation.

**Definition 1** *(A CLSC network equilibrium)* The CLSC network is at equilibrium if the forward and reverse flows between the tiers of the decision-makers coincide and the product flows and prices satisfy the sum of optimal conditions in (5), (10), (18) and (22).

Using arguments as in Nagurney et al. (2002), it can be proved that the equilibrium conditions governing the CLSC network model with competition are equivalent to solve a single variational inequality problem. We can establish the following theorem:

**Theorem 5** (Variational Inequality Formulation) *The equilibrium conditions governing the CLSC network model with competition are equivalent to the solution of the evolutionary variational inequality, given by:*

$$\int_0^T \langle F(t, X^*(t)), X(t) - X^*(t) \rangle \geq 0, \forall X \in \mathcal{K}, \tag{23}$$

where

$$\begin{aligned} X &= (x^M, x^R, p^N, \rho^U, x^{Kc}, x^{RKc}, x^U); \\ F(t, X^*(t)) &= (F_1(t, X^*(t)), F_2(t, X^*(t)), \dots, F_7(t, X^*(t))); \\ F_1(t, X^*(t)) &= F^M(t, X^{*M}(t)) + F_1^R(t, X^{*R}(t)); \\ F_2(t, X^*(t)) &= F_2^R(t, X^{*R}(t)) + F_4^K(t, X^{*K}(t)); \\ F_3(t, X^*(t)) &= F_1^K(t, X^{*K}(t)); \\ F_4(t, X^*(t)) &= F_3^K(t, X^{*K}(t)); \\ F_5(t, X^*(t)) &= F_2^K(t, X^{*K}(t)) + F_1^P(t, X^{*P}(t)); \\ F_6(t, X^*(t)) &= F_5^K(t, X^{*K}(t)); \\ F_7(t, X^*(t)) &= F_2^P(t, X^{*P}(t)); \\ \mathcal{K} &= S^M \times S^R \times S^K \times S^P. \end{aligned}$$

**Proof** Firstly, we prove that the equilibrium conditions (5), (10), (18), and (22) imply the evolutionary variational inequality (23). In fact, the summation of (5), (10), (18), and (22) yields, after algebraic simplification, to (23).

Now, we verify that a solution to (23) satisfies the sum of the inequalities (5), (10), (18), and (22) and hence it is an equilibrium according to Definition 1. For this aim, in (23), we add the vector  $(p_{mr}^{*n}(t) - p_{mr}^{*n}(t))_{n,m,r}$  to  $F_1(t, X^*(t))$  and the vector  $(p_{rk}^{*n}(t) - p_{rk}^{*n}(t))_{n,r,k}$  to  $F_2(t, X^*(t))$ . Finally, we add the vector  $(\gamma p_k^{*u}(t) - \gamma p_k^{*u}(t))_{u,k}$  to  $F_5(t, X^*(t))$ . Such terms do not change the value of the evolutionary variational inequality, since they are identically equal to zero. As a result, we obtain the following variational inequality in extended form:



$$\begin{aligned}
 & \int_0^T \left\{ \sum_{n \in N} \sum_{m \in M} \sum_{r \in R} \left( p_{mr}^{*n}(t) - p_{mr}^{*n}(t) + \frac{\partial c_m(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial t_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} + \frac{\partial g_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} \right. \right. \\
 & \quad + \frac{\partial c_r(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial \hat{c}_{mr}^n(x_{mr}^*(t))}{\partial x_{mr}^n(t)} + \left. \frac{\partial \hat{g}_{mr}^n(x^{*M}(t))}{\partial x_{mr}^n(t)} \right) (x_{mr}^n(t) - x_{mr}^{*n}(t)) \\
 & \quad + \sum_{n \in N} \sum_{r \in R} \sum_{k \in K} \left( p_{rk}^{*n}(t) - p_{rk}^{*n}(t) + \frac{\partial t_{rk}(x_{rk}^{*n}(t))}{\partial x_{rk}^n(t)} + \hat{c}_{rk}^n(x_{rk}^{*n}(t)) - p_k^{*n}(t) + p_{rk}^{*n}(t) \right. \\
 & \quad - \sum_{u \in U} \rho_k^{*u}(t) + \frac{\partial \hat{c}_{rk}^n(x^{*R}(t))}{\partial x_{rk}^n(t)} - \left. \frac{\partial s_r^n(x^{*R}(t))}{\partial x_{rk}^n(t)} \right) (x_{rk}^{*n}(t) - x_{rk}^{*n}(t)) + \sum_{n \in N} \sum_{k \in K} \left( \sum_{r \in R} x_{rk}^n(t) \right. \\
 & \quad - d_k^{*n}(p_k^{*n}(t)) \left. \right) (p_k^n(t) - p_k^{*n}(t)) + \sum_{u \in U} \sum_{k \in K} \left( x_{k_c,k}^{*u}(t) - d_k^u(\rho^{*U}(t)) \right) (p_k^u(t) - \rho_k^{*u}(t)) \\
 & \quad + \sum_{u \in U} \sum_{k \in K} \sum_{k_c \in K_c} \left( \gamma p_k^{*u}(t) - \gamma p_k^{*u}(t) + \frac{\partial c_{k_c}(x_{k_c}^*(t))}{\partial x_{k_c,k}^u(t)} + \hat{c}_{k_c,k}^u(x_{k_c,k}^{*u}(t)) + \pi_k^u(x^{*U}(t)) + \pi_k^u(x^{*U}(t)) \right. \\
 & \quad - \rho_k^{*u}(t) + \left. \frac{\partial C^u(x^{*U}(t))}{\partial x_{k_c,k}^u(t)} + \frac{\partial \pi_k^u(x^{*U}(t))}{\partial x_{k_c,k}^u(t)} \right) (x_{k_c,k}^u(t) - x_{k_c,k}^{*u}(t)) \\
 & \quad \left. + \sum_{n \in N} \sum_{r \in R} p_{rk_c}^{*n}(t) (x_{rk_c}^n(t) - x_{rk_c}^{*n}(t)) + \sum_{u \in U} \sum_{k \in K} \left( \frac{\partial \pi_k^u(x^{*U}(t))}{\partial x_k^u(t)} \right) (x_k^u(t) - x_k^{*u}(t)) \right\} dt \geq 0, \\
 & \forall (x^M, x^R, p^N, \rho^U, x^{K_c}, x^{RK_c}, x^U) \in \mathcal{K}.
 \end{aligned} \tag{24}$$

Rearranging the terms, we find

$$\begin{aligned}
 & \int_0^T \left\{ \sum_{n \in N} \sum_{m \in M} \sum_{r \in R} \left( \frac{\partial c_m(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial t_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} + \frac{\partial g_{mr}(x_{mr}^{*n}(t))}{\partial x_{mr}^n(t)} \right. \right. \\
 & \quad - p_{mr}^{*n}(t) \left. \right) (x_{mr}^n(t) - x_{mr}^{*n}(t)) \\
 & \quad + \sum_{n \in N} \sum_{m \in M} \sum_{r \in R} \left( p_{mr}^{*n}(t) + \frac{\partial c_r(x^{*M}(t))}{\partial x_{mr}^n(t)} + \frac{\partial \hat{c}_{mr}^n(x_{mr}^*(t))}{\partial x_{mr}^n(t)} + \frac{\partial \hat{g}_{mr}^n(x^{*M}(t))}{\partial x_{mr}^n(t)} \right) (x_{mr}^n(t) - x_{mr}^{*n}(t)) \\
 & \quad + \sum_{n \in N} \sum_{r \in R} \sum_{k \in K} \left( \frac{\partial t_{rk}(x_{rk}^{*n}(t))}{\partial x_{rk}^n(t)} - p_{rk}^{*n}(t) - \frac{\partial s_r^n(x^{*R}(t))}{\partial x_{rk}^n(t)} \right) (x_{rk}^n(t) - x_{rk}^{*n}(t)) \\
 & \quad + \sum_{n \in N} \sum_{r \in R} \sum_{k \in K} \left( p_{rk}^{*n}(t) + \hat{c}_{rk}^n(x_{rk}^{*n}(t)) - p_k^{*n}(t) + p_{rk}^{*n}(t) - \sum_{u \in U} \rho_k^{*u}(t) + \frac{\partial \hat{c}_{rk}^n(x^{*R}(t))}{\partial x_{rk}^n(t)} \right) \\
 & \quad \times (x_{rk}^n(t) - x_{rk}^{*n}(t)) + \sum_{n \in N} \sum_{k \in K} \left( \sum_{r \in R} x_{rk}^{*n}(t) - d_k^{*n}(p_k^{*n}(t)) \right) (p_k^n(t) - p_k^{*n}(t)) \\
 & \quad + \sum_{u \in U} \sum_{k \in K} \left( x_{k_c,k}^{*u}(t) - d_k^u(\rho^{*U}(t)) \right) (p_k^u(t) - \rho_k^{*u}(t)) \\
 & \quad + \sum_{u \in U} \sum_{k \in K} \sum_{k_c \in K_c} \left( \frac{\partial c_{k_c}(x_{k_c}^*(t))}{\partial x_{k_c,k}^u(t)} + \gamma p_{k_c}^{*u}(t) + \hat{c}_{k_c,k}^u(x_{k_c,k}^{*u}(t)) + \pi_k^u(x^{*U}(t)) - \rho_k^{*u}(t) \right) \\
 & \quad \times (x_{k_c,k}^u(t) - x_{k_c,k}^{*u}(t)) + \sum_{u \in U} \sum_{k \in K} \sum_{k_c \in K_c} \left( \frac{\partial C^u(x^{*U}(t))}{\partial x_{k_c,k}^u(t)} + \frac{\partial \pi_k^u(x^{*U}(t))}{\partial x_{k_c,k}^u(t)} - \gamma p_{k_c}^{*u}(t) \right) \\
 & \quad \times (x_{k_c,k}^u(t) - x_{k_c,k}^{*u}(t)) \\
 & \quad + \sum_{n \in N} \sum_{r \in R} p_{rk_c}^{*n}(t) (x_{rk_c}^n(t) - x_{rk_c}^{*n}(t)) + \sum_{u \in U} \sum_{k \in K} \left( \frac{\partial \pi_k^u(x^{*U}(t))}{\partial x_k^u(t)} \right) (x_k^u(t) - x_k^{*u}(t)) \left. \right\} dt \geq 0, \\
 & \forall (x^M, x^R, p^N, \rho^U, x^{K_c}, x^{RK_c}, x^U) \in \mathcal{K}.
 \end{aligned} \tag{25}$$

Thus, we conclude that (25) is equivalent to the sum of the conditions (5), (10), (18), and (22), which is exactly Definition 1 of a sustainable dynamic CLSC network for reselling collectibles. This completes the proof.  $\square$

### 3.1 Existence of solutions

To ensure the existence of solutions, we may apply the results in Maugeri and Raciti (2009). We first recall some definitions.

Let  $E$  be a reflexive Banach space with dual space  $E^*$  and  $S \subset E$  a closed convex set.

**Definition 2** A mapping  $A : S \rightarrow E^*$  is said to be

- pseudomonotone in the sense of Brezis (shortly denoted by B-pseudomonotone) if
  - (a) for each sequence  $u_n$  weakly converging to  $u$  in  $S$  and such that  $\limsup_n \langle Au_n, u_n - u \rangle \leq 0$  it results  $\liminf_n \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle$ ,  $\forall v \in S$ ;
  - (b) for each  $v \in S$  the function  $u \mapsto \langle Au, u - v \rangle$  is lower bounded on the bounded subsets of  $S$ ;
- pseudomonotone in the sense of Karamardian (shortly denoted by K-pseudomonotone) if for all  $u, v \in S$

$$\langle Av, u - v \rangle \geq 0 \Rightarrow \langle Au, v - u \rangle \geq 0.$$

**Definition 3** The map  $A : S \rightarrow E^*$  is said to be

- Hemicontinuous in the sense of Fan (shortly denoted by F-hemicontinuous) if for all  $v \in S$  the function  $u \mapsto \langle Au, v - u \rangle$  is weakly lower semicontinuous on  $S$ ;
- Lower hemicontinuous along line segments if the function:  $\xi \mapsto \langle A\xi, u - v \rangle$  is lower semicontinuous for all  $u, v \in S$  on the line segments  $[u, v]$ .

**Theorem 6** Let us assume that the map  $A : S \rightarrow E^*$  be B-pseudomonotone or F-hemicontinuous and there exist  $u_0 \in S$  and  $R > \|u_0\|$  such that

$$\langle Av, v - u_0 \rangle \geq 0, \quad \forall v \in S \cap \{v \in E : \|v\| = R\}. \quad (26)$$

Then, the variational inequality  $\langle Au, v - u \rangle \geq 0, \forall v \in S$ , admits solutions.

**Theorem 7** Let  $A : S \rightarrow E^*$  be a K-pseudomonotone map which is lower hemicontinuous along line segments. Let us assume that condition (26) holds true. Then, the variational inequality  $\langle Au, v - u \rangle \geq 0, \forall v \in S$ , admits solutions.

We recall that condition (26) is satisfied if the coercivity condition is verified:

$$\lim_{\substack{\|u\| \rightarrow \infty \\ u \in S}} \frac{\langle Au, u - u_0 \rangle}{\|u\|} = +\infty. \tag{27}$$

If we set:

$$\begin{aligned} u &= X^*(t), v = X(t); \\ Au &:= F(t, X^*(t)); \\ L^2([0, T], \mathbb{R}_+^{NMR+NRK+KN+2UK+UK_cK+UK}) &= \mathcal{L}; \\ F &: [0, T] \times \mathcal{L} \rightarrow \mathcal{L}; \\ S &= \mathcal{K}; \end{aligned}$$

then we can apply the above relevant theorems and ensure the existence of the solution to (23), see also (Cojocaru et al. 2008; Nagurney et al. 2007a).

**Theorem 8** *If  $F$  in (23) is  $B$ -pseudomonotone or  $F$ -hemicontinuous and (26) or (27) holds true. Then, the evolutionary variational inequality (23) admits solutions.*

Moreover, we note that the lower hemicontinuity along line segments can be ensured if  $F$  is a Carathéodory function such that

$$\exists \phi \in L^2([0, T]) : \|F(t, X(t))\| \leq \phi(t)\|X(t)\|, \quad \text{a.e. in } [0, T], \forall X \in \mathcal{L}, \tag{28}$$

where  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^{NMR+NRK+KN+2UK+UK_cK+UK}$ .

**Theorem 9** *Let us assume that operator  $F$  of (23) is  $K$ -pseudomonotone, condition (28) verified, and (26) or (27) holds true. Then, the evolutionary variational inequality (23) admits solutions.*

Following (Cojocaru et al. 2006), if the operator of (23) is strictly  $K$ -pseudomonotone on  $\mathcal{K}$ , namely for all  $X, Y \in \mathcal{K}$ ,  $\langle F(X), Y - X \rangle_{L^2} \geq 0 \rightarrow \langle F(Y), Y - X \rangle_{L^2} > 0$ , then (23) admits a unique solution.

### 3.2 Projected dynamical system

In this section, we provide the connection between the variational inequality (23) and a particular projected dynamical system. Specifically, in Cojocaru et al. (2008), the authors shows that the following projected dynamical system (PDS) with state variable  $X(t, \tau)$  can be associated with problem (23) in the Hilbert space  $\mathcal{L}$ :

$$\frac{X(t, \tau)}{d\tau} = \Pi_{\mathcal{K}}(X(t, \tau), -F(t, X(t, \tau))), \quad X(t, 0) \in \mathcal{K}, \tag{29}$$

where

$$\Pi_{\mathcal{K}}(y, -F(t, y)) = \lim_{\delta \rightarrow 0^+} \frac{P_{\mathcal{K}}(y - \delta F(t, y)) - y}{\delta}, \quad \forall y \in \mathcal{K}, \quad (30)$$

and  $P_{\mathcal{K}} : \mathcal{L} \rightarrow \mathcal{K}$  is the projection operator given by

$$\|P_{\mathcal{K}}(z) - z\| = \inf_{y \in \mathcal{K}} \|y - z\|.$$

As discussed in Cojocaru et al. (2008), we remark that the state variable depends on both  $t$  and  $\tau$ . In fact, at each instant  $t \in [0, T]$ , the solution of (23) represents a static state of the underlying system. As  $t$  varies over the interval  $[0, T]$ , the static states describe one or more curves of equilibria. Meanwhile,  $\tau \in [0, \infty]$  is the time that describes the dynamics of the system until it reaches one of the equilibrium solutions of the curves.

An important feature of (locally) projected dynamical systems is that the set of stationary points coincides with the set of solutions of the evolutionary variational inequality, as shown in Cojocaru et al. (2008), Nagurny et al. (2007a)). Thus, we find:

**Theorem 10** *We assume that  $\mathcal{K} \subseteq \mathcal{L}$  is non-empty, closed and convex, and  $F : [0, T] \times \mathcal{K} \rightarrow \mathcal{L}$  is a pseudo-monotone Lipschitz continuous vector function. Then, the solution of (23) is the same as the critical point of the projected differential equation (29), that is, there exists  $X \in \mathcal{K}$ , such that*

$$\Pi_{\mathcal{K}}(X(t), -F(t, X(t))) = 0, \quad (31)$$

and viceversa.

We note that in our model the feasible set  $\mathcal{K}$  is non-empty closed and convex. Moreover, it is reasonable to assume that  $F$  is continuous and monotone, so that the assumptions of Theorem 10 are verified.

The following result gives the uniqueness of solutions of PDS (see Cojocaru et al. 2008).

**Theorem 11** *We assume that  $\mathcal{K} \subseteq \mathcal{L}$  is a non-empty, closed and convex subset. Let  $F : [0, T] \times \mathcal{K} \rightarrow \mathcal{L}$  be a Lipschitz continuous vector field and  $X_0 \in \mathcal{K}$ . Then, for each  $t \in [0, \tau]$ , problem (29) has a unique solution for  $\tau \in [0, \infty]$ .*

## 4 Numerical example and discussion

In this section, we present a CLSC network consisting of two manufacturers ( $m = 1, 2$ ), two retailers ( $r = 3, 4$ ), two demand markets ( $k = 5, 6$ ), one collector ( $k_c = 5$ ), one online platform, see Fig. 2.

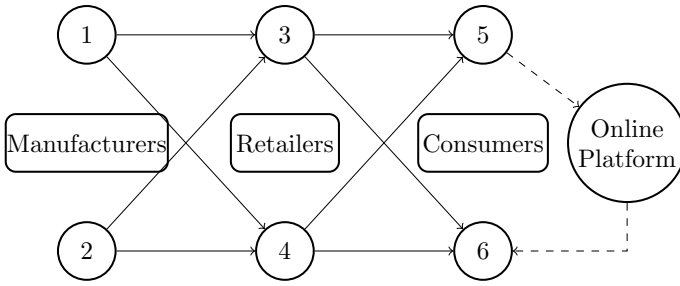


Fig. 2 The CLSC network

We focus only on one type of item, that is new and, in a second moment, becomes a second-hand item, i.e.  $n = 1$  and  $u = 1$  (for simplicity, we omit the apexes  $n$  and  $u$ ). Furthermore, we underline that we have limited ourselves to considering only one product for purely illustrative purposes. However, the example can be easily extended to the case of several new products, and, consequently, to more used products, which belong to the same kind of production or type of object.

**4.1 Procedure**

We now consider the evolutionary variational inequality (23). Let the operator  $F$  be strictly monotone, so that the solution  $X^*$  is unique and assume that, under suitable assumptions (see Barbagallo 2006, 2007, 2008), the solution is continuous, namely,  $X^* \in C([0, T], \mathbb{R}^{\bar{n}})$ , where  $\bar{n} = NMR + NRK + KN + 2UK + UK_cK + UK$ .

Therefore, we may solve

$$\langle F(t, X^*(t)), X(t) - X^*(t) \rangle \geq 0, \quad \forall t \in [0, T]. \tag{32}$$

In the case where the vector field  $F(t, X(t)) = A(t)X(t) + B(t)$  is a linear operator,  $A(t)$  is a continuous and positive definite matrix in  $[0, T]$ , and  $B(t)$  is a continuous vector, continuity results have been obtained in Barbagallo (2006), Barbagallo (2007).

For the aim of solving the evolutionary variational inequality (23), we refer to the procedure described in Barbagallo (2007), Cojocaru et al. (2008), Cojocaru et al. (2006), in which the time horizon is discretized, and at each fixed time the associated static projected dynamical system is solved. Therefore, the procedure is as follows:

- we partition the time interval  $[0, T]$  into  $\{t_0, t_1, \dots, t_v, \dots, t_N\}$  such that  $0 = t_0 < t_1 < \dots < t_v < \dots < t_N = T$ ;
- for each  $t_v, v = 0, \dots, N$ , we solve the following static variational inequality:

$$\langle F(t_v, X^*(t_v)), X(t_v) - X^*(t_v) \rangle \geq 0, \quad \forall X(t_v) \in \mathcal{K}(t_v), \tag{33}$$

where

$$\begin{aligned} \mathcal{K}(t_v) = & \left\{ (x^M(t_v), x^R(t_v), p^N(t_v), \rho^U(t_v), x^{K_c}(t_v), x^{RK_c}(t_v), x^U(t_v)) \in \mathbb{R}_+^{\bar{n}} : \right. \\ & \sum_{n \in N} \sum_{r \in R} x_{mr}^n(t_v) \leq x_m^{max}(t_v), \forall m \in M, \\ & \sum_{n \in N} \sum_{k \in K} x_{rk}^n(t_v) \leq \sum_{n \in N} \sum_{m \in M} x_{mr}^n(t_v), \forall r \in R, \\ & \left. \sum_{u \in U} \sum_{k \in K} x_{k_c k}^u(t_v) \leq \mu_{k_c} Q_{k_c}(t_v), \forall k_c \in K_c, \sum_{k \in K} x_k^u(t_v) \leq Q^u(t_v), \forall u \in U \right\}. \end{aligned}$$

We can compute the unique solution of the finite-dimensional variational inequality (33) by means of the critical point of the projected dynamical system:

$$\frac{X(t_v, \tau)}{d\tau} = \Pi_{\mathcal{K}}(X(t_v, \tau), -F(t_v, X(t_v, \tau))), \quad X(t_v, 0) = X(t_v) \in \mathcal{K}(t_v); \tag{34}$$

- we apply the extragradient method in Korpelevich (1977) to (33) to find the critical points of the system (34);
- we construct an approximated equilibrium solution by linear interpolation.

To apply the extragradient method to (33) (see also Barbagallo 2007), we start from any  $X^0(t_v) \in \mathcal{K}(t_v)$  and a fixed  $\alpha \in (0, 1/L)$ , where  $L$  is the Lipschitz constant, we set the iteration counter  $k = 1$ , then, to get the next iterate  $X^{k+1}$ , we compute

$$Y^k(t_v) = P_{\mathcal{K}}(X^k(t_v) - \alpha F(t_v, X^k(t_v))), \tag{35}$$

$$X^{k+1}(t_v) = P_{\mathcal{K}}(X^k(t_v) - \alpha F(t_v, Y^k(t_v))). \tag{36}$$

The solutions  $Y^k(t_v)$  and  $X^{k+1}(t_v)$  are obtained, respectively, by solving the variational inequalities:

$$\langle Y^k(t_v) + \alpha F(t_v, X^k(t_v)) - X^k(t_v), X(t_v) - Y^k(t_v) \rangle \geq 0, \quad \forall X(t_v) \in \mathcal{K}(t_v), \tag{37}$$

$$\langle X^{k+1}(t_v) + \alpha F(t_v, Y^k(t_v)) - X^k(t_v), X(t_v) - X^{k+1}(t_v) \rangle \geq 0, \quad \forall X(t_v) \in \mathcal{K}(t_v). \tag{38}$$

The conditions for the convergence of the modified projection method are that the operator  $F$  entering variational inequality (33) is Lipschitz continuous and monotone. We note that these are reasonable conditions for our problem. Thus,  $X^k(t_v)$  converges to the critical points of the system (34) as  $k \rightarrow \infty$ . After that we have obtained the equilibrium solutions at  $t = t_0, t_1, \dots, t_v, \dots, t_N$ , we can construct a function, by linear interpolation, to obtain the curve of the dynamic equilibrium of the CLSC network problem.

The data for this example were constructed so that the assumptions of Theorem 10 are satisfied, and the continuity of the solution is ensured. For easy interpretation purposes, we fixed the time points as  $t = 0, \dots, 7$  to point out seven days. The algorithm was implemented in Matlab and was tested on MacBook Air (2021), processor Apple

M1 8 Core, 3.2 GHz, RAM 8 GB. The convergence criterion used was in the order of  $10^{-4}$ .

### 4.2 Example

We consider the production cost functions for the manufacturers given by

$$\begin{aligned}
 c_1(x_{13}(t), x_{14}(t), x_{23}(t), x_{24}(t)) &= 0.025(x_{13}(t) + x_{14}(t))^2 + 0.01(x_{13}(t) + x_{14}(t)) \\
 &\quad \cdot (x_{23}(t) + x_{24}(t)) + 0.005(x_{13}(t) + x_{14}(t)), \\
 c_2(x_{13}(t), x_{14}(t), x_{23}(t), x_{24}(t)) &= 0.025(x_{23}(t) + x_{24}(t))^2 + 0.01(x_{13}(t) + x_{14}(t)) \\
 &\quad \cdot (x_{23}(t) + x_{24}(t)) \\
 &\quad + 0.005(x_{23}(t) + x_{24}(t)).
 \end{aligned}$$

We introduce the pollution control functions  $g_{mr}(x_{mr}^n(t))$ , for each manufacturer  $m = 1, 2, n = 1$  and each retailer  $r = 3, 4$ , as

$$\begin{aligned}
 g_{13}(x_{13}(t)) &= 0.01x_{13}(t), \quad g_{14}(x_{14}(t)) = 0.01x_{14}(t), \\
 g_{23}(x_{23}(t)) &= 0.01x_{23}(t), \quad g_{24}(x_{24}(t)) = 0.01x_{24}(t).
 \end{aligned}$$

The transaction cost functions faced by the manufacturers and associated with transacting with the retailers are given by

$$\begin{aligned}
 t_{13}(x_{13}(t)) &= 0.005x_{13}(t)^2 + g_{13}(x_{13}(t)), \quad t_{14}(x_{14}(t)) = 0.005x_{14}(t)^2 + g_{14}(x_{14}(t)), \\
 t_{23}(x_{23}(t)) &= 0.005x_{23}(t)^2 + g_{23}(x_{23}(t)), \quad t_{24}(x_{24}(t)) = 0.005x_{24}(t)^2 + g_{24}(x_{24}(t)).
 \end{aligned}$$

We define the unitary price of the quantity of item  $n$ , which does not depend on time  $t$ , that retailer  $r$  has to pay to manufacturer  $m$  for new item  $n = 1$  as

$$p_{13}^* = 2.8, \quad p_{23}^* = 2.6, \quad p_{14}^* = 3, \quad p_{24}^* = 2.9.$$

The retailers' management cost functions are

$$\begin{aligned}
 c_3(x_{13}(t), x_{23}(t)) &= 0.005(x_{13}(t) + x_{23}(t))^2, \\
 c_4(x_{14}(t), x_{24}(t)) &= 0.005(x_{14}(t) + x_{24}(t))^2.
 \end{aligned}$$

The transportation costs including the pollution control functions between the manufacturers and the retailers, that belong to retailers are given by

$$\begin{aligned}
 \hat{c}_{13}(x_{13}(t)) &= 0.05x_{13}(t)^2 + 0.2 + \hat{g}_{13}(x_{13}(t)), \\
 \hat{c}_{14}(x_{14}(t)) &= 0.05x_{14}(t)^2 + 0.3 + \hat{g}_{14}(x_{14}(t)), \\
 \hat{c}_{23}(x_{23}(t)) &= 0.05x_{23}(t)^2 + 0.25 + \hat{g}_{23}(x_{23}(t)), \\
 \hat{c}_{24}(x_{24}(t)) &= 0.05x_{24}(t)^2 + 0.4 + \hat{g}_{24}(x_{24}(t)),
 \end{aligned}$$

where

$$\hat{g}_{13}(x_{13}(t)) = 0.25x_{13}(t), \hat{g}_{14}(x_{14}(t)) = 0.25x_{14}(t),$$

$$\hat{g}_{23}(x_{23}(t)) = 0.25x_{23}(t), \hat{g}_{24}(x_{14}(t)) = 0.25x_{24}(t).$$

Considering the behavior of manufacturers, we set  $x_m^{max}(t) = 9.15t$ .

The transaction costs between the retailers and the consumers at the demand markets are given by

$$t_{35}(x_{35}(t)) = 0.75x_{35}(t)^2, t_{36}(x_{36}(t)) = 0.75x_{36}(t)^2,$$

$$t_{45}(x_{45}(t)) = 0.75x_{45}(t)^2, t_{46}(x_{46}(t)) = 0.75x_{46}(t)^2,$$

and the unitary price between the retailers and the consumers at the demand markets are given by

$$p_{35}^* = p_{36} = 9, \quad p_{36}^* = p_{46} = 8.$$

The linear transportation cost functions from retailers to consumers are given by

$$\hat{c}_{35}(x_{35}(t), x_{36}(t)) = 0.01(x_{35}(t) + x_{36}(t)),$$

$$\hat{c}_{36}(x_{35}(t), x_{36}(t)) = 0.01(x_{35}(t) + x_{36}(t)),$$

$$\hat{c}_{45}(x_{45}(t), x_{46}(t)) = 0.01(x_{45}(t) + x_{46}(t)),$$

$$\hat{c}_{46}(x_{45}(t), x_{46}(t)) = 0.01(x_{45}(t) + x_{46}(t)).$$

In addition, the social network advertisement functions, i.e. the revenue from advertising on the social network profiles of each retailer are given by

$$s_3(x_{35}(t), x_{36}(t)) = 18(x_{35}(t) + x_{36}(t)), \quad s_4(x_{45}(t), x_{46}(t)) = 18(x_{45}(t) + x_{46}(t)).$$

For completeness, we present the other function used:

$$d_5^n(p_5(t)) = 3p_5(t), \quad d_6^n(p_6(t)) = 4p_6(t), \quad d_6^u(\rho_6(t)) = 0.04\rho_6(t),$$

which are the demands for the new and the second-hand product, and the price for this item from retailers to collector, respectively. The risk-aversion function to the purchasing second-hand item from  $k = 6$  towards the online platform, the transportation cost from collector  $k_c = 5$  to consumer  $k = 6$ , and the maintenance and restoring cost of the collector  $k_c$  are given, respectively, by

$$\pi_k(x_{56}(t)) = x_{56}(t) + 0.5, \quad \hat{c}_{56}(x_{56}(t)) = 10x_{56}(t), \quad c_5(x_{56}(t)) = 4.5x_{56}^2(t).$$

The management costs of the second-hand product and the transaction cost for the online platform and the total amount of item  $u$  on the platform are given, respectively, by

$$C(x_{56}(t)) = 0.5x_{56}^2(t),$$

$$\hat{t}_6(x_{56}(t)) = 0.5x_{56}^2(t) + 1.5x_{56}(t), \quad Q^u(t) = \mu_5 \cdot 2t, \text{ where } \mu_5 = 0.1.$$

Therefore, we describe the curves of optimal solution in Fig. 3, 5, 7.



Figure 3 shows the curves of the quantity of product supplied by manufacturers to retailers. We note that the amount of the new item sent by manufacturers to retailers  $x_{mr}^n(t)$  grows with time. This happens because the demand for a collectible product increases with time, as well as the maximum production changes over time. In particular, the increase in the demand over time is often dictated by advertising from influencers, who sponsor the product through Instagram or other social platforms. We use the same color for the retailers and the same hatch for the manufacturers considered. We also note that manufacturer 1 mainly supplies retailer 3, while manufacturer 2 mainly supplies retailer 4. Manufacturer 2 reaches the highest market value at  $t = 5$ , when the sales of that particular product have stabilized. As a consequence of the higher sales of manufacturer 2 to both retailers, manufacturer 2 has a higher profit than manufacturer 1, as shown in Fig. 4.

Figure 5 illustrates the values of the profit function for retailers. We highlight the retailers with the same color as in Fig. 3 and the same consumer with the same hatch. We note that the quantities sold by retailer 4 to market 6 at equilibrium are the highest among those considered. We can also observe that the customer demand after a certain instant  $t$  stabilizes around the optimal value. This stability can be interpreted as a consumer's subscription to that product or to loyalty to that particular store. In Fig. 6, the initial growth in functions shows how an advertising reminder through an Instagram story or similar can accelerate the retailers' earnings in the days immediately following the arrival of the new product in the store. In addition, reseller 4's sales are, on average, slightly higher than reseller 3's ones, and reseller 3 sells the item at a higher price than retailer 4. Therefore, although seller 3 sells fewer items, his strategy turns out to be the most profitable.

Finally, Fig. 7 establishes the variation of price for new and second-hand items. The solution found, i.e.  $x_{56}^*(7) = 0.985$ , shows that when the end of the week approaches, the collector 5 will tend to sell an item to the consumer 6, using the online platform. We notice that the solution  $x_{56}^*(7)$  converges to 1. This behavior reflects the limited availability of the product that the collector 5 can sell (due to its difficult availability). This explains the increase in the purchase price of second-hand items via the online platform in Fig. 7. Indeed, we notice that the price of the new product  $p_6^*$  is lower than the price  $\rho_6^*$  that consumer 6 is willing to pay for the second-hand product for all time  $t$ , and this distance increases with time. Furthermore, the prices at the equilibrium  $p_5^*$  and  $p_6^*$  of the new product for consumers  $k = 5, 6$  are initially different and increasing over time, but after some instants of time, as  $t = 5$ , the prices tend to remain at a certain range of value, because the market price and how much a particular consumer is willing to pay are stabilized.

## 5 Conclusions

In this paper, we study the equilibrium model of a CLSC network consisting of manufacturers, retailers, demand markets, and one online platform with time-varying data. In our model, some consumers purchase new products and collect

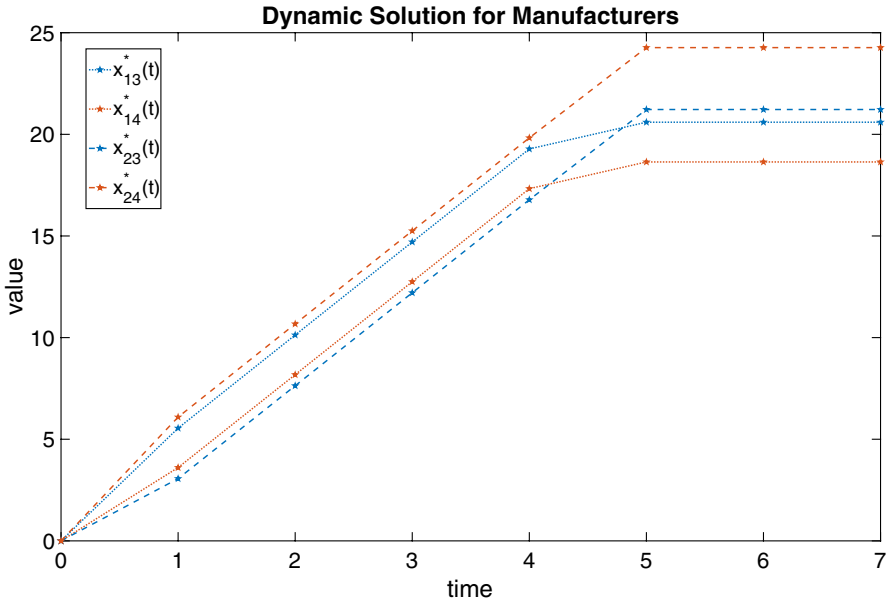


Fig. 3 Optimal Solutions for manufacturers  $m = 1, 2$  to retailers  $r = 3, 4$

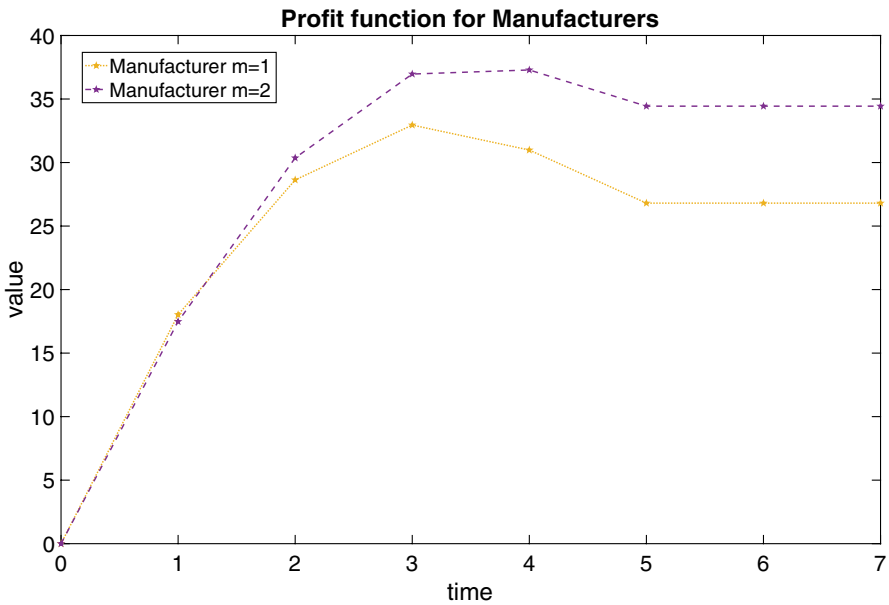


Fig. 4 Trend of the objective function of the manufactures  $m = 1, 2$  over time

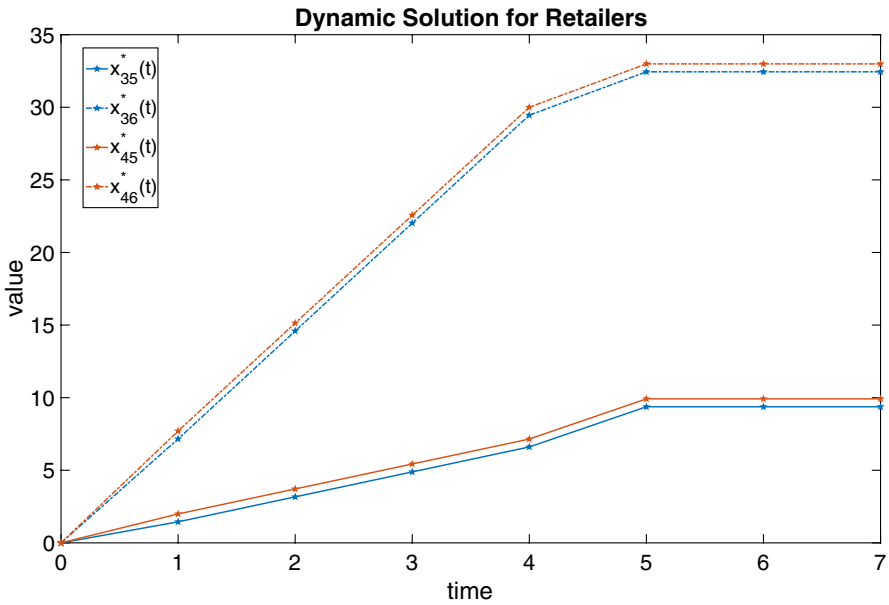


Fig. 5 Optimal Solutions for retailers  $r = 3, 4$  to consumers  $k = 5, 6$

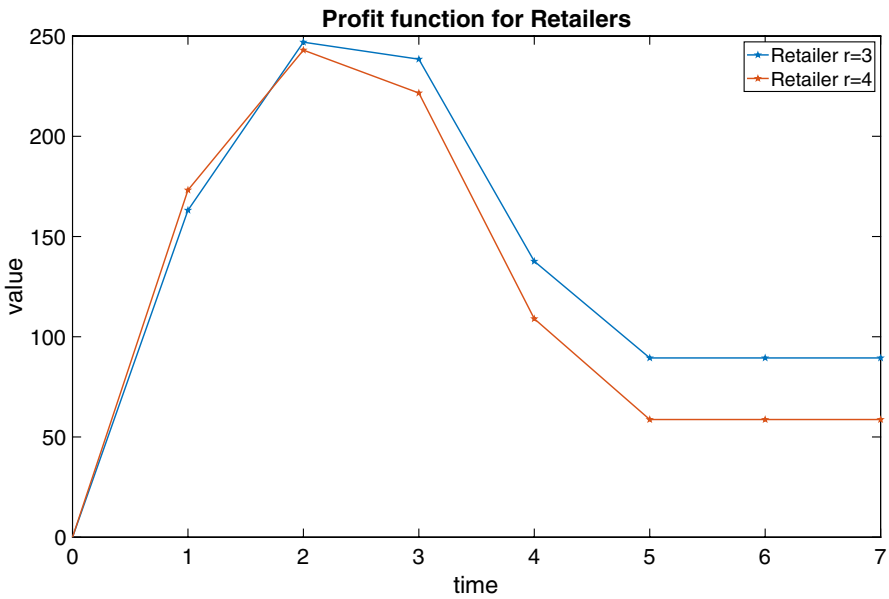


Fig. 6 Trend of the objective function of the retailers  $r = 3, 4$  over time

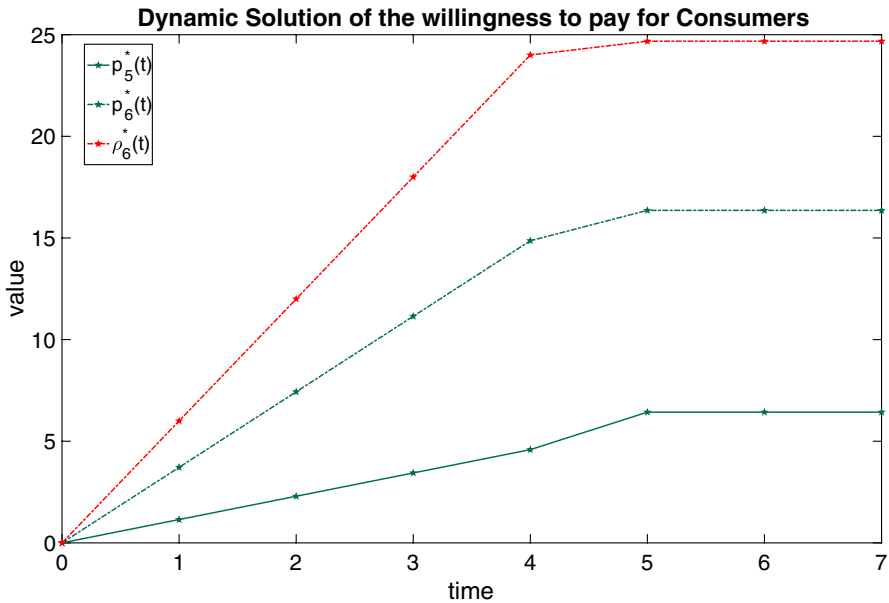


Fig. 7 Optimal Solutions for price of new object and second-hand item

them; then, they may decide to resell them on the online platform. Therefore, the network can be divided into two parts: the forward chain formed by the manufacturers, the retailers and the consumers, and the reverse chain formed by the collectors, the online platform and the consumers. The collectors and the online platform link the forward and the reverse chains and form the closed-loop network. We also take into account capacity constraints of manufacturers and retailers, as well as consumers' risk-aversion to purchasing second-hand products, and platform's risk-aversion to transacting with collectors. Moreover, manufacturers as well as retailers try to control the amount of emissions generated in transaction processes. The optimal behaviors of all the decision-makers are analyzed, and the governing CLSC network equilibrium conditions are provided. Then, we introduce the associated evolutionary variational inequality, and the related projected dynamical system that allows us to construct the dynamic equilibrium solution. Numerical results show that the optimal decisions of each tier the CLSC network at equilibrium changes according to the demands. Our results also reveal the role of advertising of influencers who sponsor the product through Instagram or other social platforms in determining the increase in sales and profits of retailers.

Our contributions to the literature lie in advancing the state-of-the-art of CLSC network in a time-dependent framework, as well as the applications of evolutionary variational inequalities. Our work can provide analytical tools for investigating the market equilibrium when collectors engage in the second-hand business in a time-varying environment. We emphasize that adopting closed-loop business models can be an effective way to encourage sustainable consumption.

Reducing the use of available resources will maximize the benefits for the entire community.

This model could be extended in future research. We could explore the case of random demands and the associated stochastic optimization model. The extension to a two-stage stochastic problem where we consider two stages of information is another future research opportunity.

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