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Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrödinger–Kirchhoff-type

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In this article, we obtain the existence and infinitely many nontrivial solutions for a class of nonlinear critical anisotropic elliptic equations involving variable exponents and two real parameters, via combining the variational method, and the concentration-compactness principle for anisotropic variable exponent under suitable assumptions on the nonlinearities.

KEYWORDS

anisotropic variable mean curvature operator, anisotropic variable exponent Sobolev spaces, concentration-compactness principle, Dirichlet boundary conditions, Schrödinger–Kirchhoff-type problems, $\vec{p}(x)$-Laplacian

MSC CLASSIFICATION

35B33, 35D30, 35J20, 35J60, 46E35

1 | INTRODUCTION

In recent years, anisotropic partial differential equations have gained attention from several researchers due to their applicability in various fields of science. For example, in the early 1990s, Perona and Malik [1] proposed the first anisotropic partial differential equation (PDE) model, which was used for image enhancement and denoising by preserving significant image features while utilizing anisotropic PDEs, as detailed in Tschumperlé and Deriche [2]. Anisotropic problems also appear in physics models that describe the dynamics of fluids with different conductivities in different directions. Additionally, anisotropic equations can be applied in models that describe the spread of epidemic diseases in heterogeneous environments, for more details on the mentioned applications, see, for example, Bear [3] and Antontsev et al. [4].

On the other hand, function spaces and differential equations that involve variable exponents and nonstandard growth conditions have attracted multiple researchers from various fields (see, e.g., Diening et al. [5] and Papageorgiou et al. [6]), as these structures have been increasingly beneficial in applications such as nonlinear elasticity problems (see, e.g., Zhikov [7]), electrorheological fluids (see, e.g., Acerbi and Mingione [8, 9] and Růžička [10]), thermorheological fluids (see, e.g., Antontsev et al. [11]), image restoration (see, e.g., Chen et al. [12]), and contact mechanics (see, e.g., Boureanu et al. [13]).

In this work, we are concerned with a class of critical anisotropic Schrödinger–Kirchhoff equations with variable exponents. This type of nonlinear partial differential equation describes the behavior of waves in an anisotropic (direction-dependent) system by combining two well-known models in physics and engineering: The Schrödinger equation, which describes the behavior of quantum mechanical waves (that it plays the role of Newton's laws and conservation of energy in classical mechanics), and Kirchhoff's laws of mechanics, which describe the behavior of mechanical waves in elastic bodies. The inclusion of variable exponent terms allows for a more flexible and accurate description of wave behavior in anisotropic systems, particularly in complex and dynamic media. The critical nonlinearities in these equations can lead to important wave behaviors such as singularities, wave collapse, and shock wave generation. Understanding these behaviors is essential for studying waves in complex media and predicting how they interact with light. Solving the anisotropic Schrödinger–Kirchhoff equation with variable exponents provides valuable information about wave behavior in anisotropic systems, including wave frequency, velocity, and intensity. This information is useful for studying physical phenomena such as the propagation of light or sound waves, wave behavior in heterogeneous materials, and wave response to external perturbations and can lead to new developments in the design and engineering of materials and devices that utilize wave propagation. For further details, we refer to Ablowitz et al. [14], Kirchhoff [15], Repovš [16], Schrödinger [17], Stanway et al. [18], Sulem [19], Sun et a.[20], Lv et al.[21], and Vetro [22], and the references therein.

More precisely, we show existence and multiplicity results of solutions for the critical problem given by

$$\begin{aligned} -M\left(\int_\Omega \sum_{i=1}^N A_i(x,\partial_{x_i}u) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\right)\left(\sum_{i=1}^N \partial_{x_i}a_i(x,\partial_{x_i}u) - b(x)|u|^{p_M(x)-2}u\right) + \lambda|u|^{r(x)-2}u \\ = |u|^{q(x)-2}u + \beta f(x,u) \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $b \in L^\infty(\Omega)$ satisfies $b_0 := \operatorname{ess\,inf}_{x\in\Omega} b(x) > 0$, while λ and β are real parameters such that β is positive, and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function with the potential $F(x,\xi) = \int_0^\xi f(x,t)dt$, which satisfies some conditions that will be specified later. r, q and p_i, $i = 1, 2, \ldots, N$ are continuous functions such that

$$1 < p_m^- \leq p_i(x) \leq p_M^+ < r^- \leq r(x) \leq r^+ < q^- \leq q(x) \leq q^+ \leq p_m^*(x) < \infty, \text{for all } \ x \in \overline{\Omega},$$

where p_m^-, p_M^+, q^-, q^+ and $p_m^*(x)$ are defined in Section 2. The differential operator $\sum_{i=1}^N \partial_{x_i}a_i(x,\partial_{x_i}u)$ was introduced by Boureanu and Rădulescu [23], and it is a more general type of Laplacian operator, while the functions $a_i(x,\xi)$ are the continuous derivative with respect to ξ of the mapping $A_i : \Omega\times\mathbb{R} \to \mathbb{R}$, $A_i = A_i(x,\xi)$, that is, $a_i(x,\xi) = \frac{\partial}{\partial\xi}A_i(x,\xi)$.

Throughout this paper, we shall assume that the following hypotheses are fulfilled for all $1 \leq i \leq N$:

($\boldsymbol{A}_1$) There exist $c_{a_i} > 0$ such that

$$|a_i(x,\xi)| \leq c_{a_i}\left(g_i(x) + |\xi|^{p_i(x)-1}\right), \text{for a.e. } \ x \in \Omega \ \text{ and all } \ \xi \in \mathbb{R},$$

where the nonnegative functions g_i, $i = 1, 2, \ldots, N$ belong to $L^\infty(\Omega)$ and there exists g_0 such that $g_i(x) \geq g_0$ for all $x \in \Omega$.

($\boldsymbol{A}_2$) There exist positive constants k_i such that

$$k_i|\xi|^{p_i(x)} \leq a_i(x,\xi)\xi \leq p_i(x)\,A_i(x,\xi), \ \ \text{for a.e. } \ x \in \Omega \ \text{ and all } \ \xi \in \mathbb{R}.$$

($\boldsymbol{A}_3$) The functions a_i satisfy

$$(a_i(x,\xi) - a_i(x,\eta))\,(\xi - \eta) > 0, \text{for a.e. } \ x \in \Omega \ \text{ and all } \ \xi, \eta \in \mathbb{R} \ \text{ with } \ \xi \neq \eta.$$

$M : \mathbb{R}_0^+ \to \mathbb{R}^+$ be a nondecreasing and continuous Kirchhoff function, satisfying

($\boldsymbol{M}_1$) There exists $\mathbf{M}_0 > 0$ such that $M(\xi) \geq \mathbf{M}_0 = M(0)$, for all $\xi \in \mathbb{R}_0^+$.

($\boldsymbol{M}_2$) There exists $\gamma \in (\frac{p_M^+}{q^-}, 1]$ such that

$$\widehat{M}(\xi) \geq \gamma M(\xi)\xi, \ \ \text{for all } \ \xi \in \mathbb{R}_0^+, \ \text{ where } \ \widehat{M}(\xi) := \int_0^\xi M(s)ds.$$

As it is well known, there are many examples of function M that satisfy the assumptions ($\boldsymbol{M}_1$)–($\boldsymbol{M}_2$), for example,

$$M(\xi) = \mathbf{M}_1^0 + \mathbf{B}_1 \xi^{\frac{1}{\gamma}}, \text{with } \mathbf{M}_1^0, \mathbf{B}_1 \geq 0, \mathbf{M}_1^0 + \mathbf{B}_1 > 0 \text{ and } \gamma \leq 1.$$

In particular, when $\mathbf{M}_1^0 = 0$ and $\mathbf{B}_1 > 0$, the Kirchhoff equation associated with M is said to be degenerate. On the other hand, when $\mathbf{M}_1^0 > 0$ and $\mathbf{B}_1 \geq 0$, the Kirchhoff equation associated with M is said to be nondegenerate; in this case, when $\mathbf{B}_1 = 0$, the Kirchhoff equation associated with M (is a constant) reduces to a local quasi-linear elliptic problem.

The study of critical anisotropic elliptic equations with variable exponents is a relatively recent development in the field of nonlinear partial differential equations. The origins of the field can be traced back to the work of Brezis and Nirenberg [24] in the 1983s, who introduced f critical exponent in the study of nonlinear elliptic equations involving the Laplacian operator. Since then, there have been extensions of [24] in many directions, for example, Alessio and Valdinoci [25], Alves et al. [26], and Servadei and Valdinoci [27, 28].

The main difficulty in elliptic problems involving critical growth is the lack of compactness of the embedding of Sobolev spaces into Lebesgue spaces arising in connection with the variational approach. To overcome this difficulty, Lions [29] introduced in 1985 the so-called concentration-compactness principle (CCP, for short) to prove that a minimizing sequence or a Palais–Smale (PS, for short) sequence is precompact. The isotropic variable exponent version of the Lions concentration-compactness principle for a bounded domain was independently obtained by Bonder and Silva [30] and Fu [31]. Following that, many authors have applied these results to critical elliptic problems involving variable exponents (see, e.g., Alves and Ferreira [26], Alves and Barreiro [32], Chems Eddine et al. [33–35], Fu and Zhang [36], Ho and Sim [37], and Hurtado et al. [38], and the references therein). Moreover, when p_i are constant functions for all $i \in \{1, 2, \dots, N\}$, El Hamidi and Rakotoson [39] have extended a concentration-compactness principle to the anisotropic Sobolev space with constant exponents. Using this new concentration-compactness principle, they showed that a certain critical best Sobolev constant is achieved, and many authors have successfully dealt with critical problems involving the $\vec{p}$-Laplacian operator (see, e.g., Alves and El Hamidi [40] and Figueiredo et al. [41, 42]).

On the other hand, anisotropic type problems in the subcritical case have received specific attention in recent decades. We refer the reader to the papers Boureanu et al. [23, 43], Fan [44], Ji [45], Mihăilescu et al. [46, 47], Ourraoui and Ragusa [48], and Rădulescu and Repovš [49], and the references therein.

When $M \equiv 1$ and $\lambda = 0$ in the subcritical case, in a recent paper [50], Afrouzi et al. have studied the following anisotropic Schrödinger elliptic problem

$$-\sum_{i=1}^{N} \partial_{x_i} a_i(x, \partial_{x_i} u) + b(x)|u|^{p_M^+ - 2} u = \beta f(x, u) \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega,$$

with nonstandard growth conditions and $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary. By using variational methods, they obtained existence results.

Our objective in this paper is to study the existence and multiplicity of solutions for a class of anisotropic Schrödinger–Kirchhoff-type equations with variable exponents problem (1.1) under the critical growth condition $\left\{ x \in \Omega : q(x) = \frac{N p_m(x)}{N - p_m(x)}, \text{where } p_m(x) = \min_{1 \leq i \leq N} \{p_i(x)\} \right\} \neq \emptyset$. Our approach to this is variational and uses minimax critical point theorems. More precisely, our main results of this work extend, complement, and complete several of the above works.

As we shall see in the next sections, there are main difficulties in our situation. The principal difficulties are that problem (1.1) involves the nonlocal term $M\left(\int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)} |u|^{p_M(x)} \mathrm{d}x \right)$, which prevents us from applying the methods as before, and due to the lack of compactness of the embedding $W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{p_m^*(x)}(\Omega)$ and the Palais–Smale condition for the corresponding energy functional could not be checked directly. To overcome these difficulties, we use some variational technical calculus and the new version of the Lions concentration-compactness principle for the anisotropic variable exponent Sobolev spaces extended by Chems Eddine et al. [51].

Throughout this paper, we shall assume that f satisfies the following conditions:

($\boldsymbol{f}_1$) There exist a positive function $\ell \in C(\overline{\Omega})$ and a positive constant C_f such that

$$|f(x, \xi)| \leq C_f (1 + |\xi|^{\ell(x) - 1}) \text{ for all } (x, \xi) \in \Omega \times \mathbb{R},$$

where $p_M^+ < \ell(x) < p_m^{*-}$ for all $x \in \overline{\Omega}$

($\boldsymbol{f}_2$) There are $R > 0$ and $\theta_\lambda \geq \max\{p_M^+/\gamma, r^+\}$ (resp. $p_M^+/\gamma \leq \theta_\lambda \leq r^-$) if $\lambda \geq 0$ (resp. $\lambda < 0$), such that for any ξ with $|\xi| \geq R$ and $x \in \Omega$, we have

$$0 < \theta_\lambda F(x, \xi) \leq \xi f(x, \xi).$$

($\boldsymbol{f}_3$) $f(x, \xi) = \circ(|\xi|^{p_M^+})$ as $\xi \to 0$ and uniformly for $x \in \Omega$.

($\boldsymbol{f}_4$) f is odd in ξ, that is, $f(x, -\xi) = -f(x, \xi)$, for any $(x, \xi) \in \Omega \times \mathbb{R}$.

The main results of the paper are as follows:

Theorem 1.1. *Assume that the assumptions ($\boldsymbol{A}_1$)–($\boldsymbol{A}_3$), ($\boldsymbol{M}_1$)–($\boldsymbol{M}_2$), and ($\boldsymbol{f}_1$)–($\boldsymbol{f}_3$) hold. Then, for any $\lambda \in \mathbb{R}$ and $\beta > 0$, problem (1.1) has at least a nontrivial weak solution.*

Theorem 1.2. *Assume that the assumptions ($\boldsymbol{A}_1$)–($\boldsymbol{A}_3$), ($\boldsymbol{M}_1$)–($\boldsymbol{M}_2$), and ($\boldsymbol{f}_1$)–($\boldsymbol{f}_4$) hold. Then, for any $\lambda \in \mathbb{R}$ and $\beta > 0$, problem (1.1) has infinitely many weak solutions.*

The paper is organized as follows: In Section 2, we give some preliminary results of the variable exponent spaces. In Section 3, we provide proof for the main results, after we have verified the Palais–Smale condition at some special energy levels, by using the concentration-compactness principle. Finally, in Section 4, we illustrate the degree of generality of the kind of problems we studied in this paper.

2 | FUNCTIONAL FRAMEWORK

In this section, we set up the notations and collect the necessary basic results on the theory of variable exponent function spaces that will be frequently used throughout the rest of the paper.

Everywhere in this paper, let Ω be a bounded Lipschitz (nonempty) domain in $\mathbb{R}^N$ ($N \geq 2$). Let $C_+(\overline{\Omega}) = \{p : p \in C(\overline{\Omega}), p(x) > 1 \text{ for a.e. } x \in \overline{\Omega}\}$. Denote the set of all functions $p \in C_+(\overline{\Omega})$ that are log-Holder continuous by $C_+^{\log}(\Omega)$, that is, $\sup\{x, y \in \overline{\Omega} : |p(x) - p(y)| \log \frac{1}{|x-y|} < \infty, 0 < |x - y| < \frac{1}{2}.\}$ For any $p \in C_+(\overline{\Omega})$, denote $p^+ = \sup_{x\in\Omega} p(x)$ and $p^- = \inf_{x\in\Omega} p(x)$. For any $p \in C_+(\overline{\Omega})$, define the variable exponent Lebesgue space as $L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function and } \rho_p(u) < \infty\}$, where the functional $\rho_p : L^{p(x)}(\Omega) \to \mathbb{R}$ is defined as $\rho_p(u) := \int_\Omega |u(x)|^{p(x)} dx$. The functional ρ_p is called the $p(x)$-modular of the $L^{p(x)}(\Omega)$ space, and it plays an important role in manipulating the generalized Lebesgue–Sobolev spaces. We endow the space $L^{p(x)}(\Omega)$ with the Luxemburg norm $\|u\|_{L^{p(x)}(\Omega)} := \inf\left\{\tau > 0 : \rho_p\left(\frac{|u(x)|}{\tau}\right) \leq 1\right\}$. Then $(L^{p(x)}(\Omega), \|u\|_{L^{p(x)}(\Omega)})$ is a separable and reflexive Banach space (see, e.g., Kováčik and Rákosník [52, Theorem 2.5, Corollary 2.7]). Let us now recall more basic properties concerning the Lebesgue spaces.

Proposition 2.1 (Kováčik and Rákosník [52, Theorem 2.8]). *Let p and h be variable exponents in $C_+(\overline{\Omega})$ such that $p \leq h$ in Ω. Then the embedding $L^{h(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous.*

Moreover, the following Hölder-type inequality

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)} \tag{2.1}$$

holds for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ (see, e.g., Kováčik and Rákosník [52, Theorem 2.1]), where $L^{p'(x)}(\Omega)$ is the conjugate space (or the topological dual space) of $L^{p(x)}(\Omega)$, obtained by conjugating the exponent pointwise, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ (see, e.g., Kováčik and Rákosník [52, Corollary 2.7]). Furthermore, if $p, q, h : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{h(x)} = 1$, then for each $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$, $w \in L^{h(x)}(\Omega)$, we have $\int_\Omega |u(x)v(x)w(x)|dx \leq \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{h^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{q(x)}(\Omega)} \|w\|_{L^{h(x)}(\Omega)}$. In addition, if $u \in L^{p(x)}(\Omega)$ and $p < \infty$, we have the following properties (see, e.g., Fan and Zhao [53, Theorem 1.3, Theorem 1.4]):

$$\|u\|_{L^{p(x)}(\Omega)} < 1\ (= 1; > 1) \text{ if and only if } \rho_p(u) < 1\ (= 1; > 1), \tag{2.2}$$

$$\text{if} \|u\|_{L^{p(x)}(\Omega)} > 1 \text{ then } \|u\|^{p^-}_{L^{p(x)}(\Omega)} \le \rho_p(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)}, \tag{2.3}$$

$$\text{if} \|u\|_{L^{p(x)}(\Omega)} < 1 \text{ then } \|u\|^{p^+}_{L^{p(x)}(\Omega)} \le \rho_p(u) \le \|u\|^{p^-}_{L^{p(x)}(\Omega)}. \tag{2.4}$$

As a result, we get

$$\|u\|^{p^-}_{L^{p(x)}(\Omega)} - 1 \le \rho_p(u) \le \|u\|^{p^+}_{L^{p(x)}(\Omega)} + 1, \text{ for all } u \in L^{p(x)}(\Omega). \tag{2.5}$$

As a consequence, we find also the equivalence between norm convergence and modular convergence

$$\|u\|_{L^{p(x)}(\Omega)} \to 0 \text{ (resp. } \to \infty) \text{ if and only if } \rho_p(u) \to 0 \text{ (resp. } \to \infty). \tag{2.6}$$

Now, let us pass to the (isotropic) Sobolev space with variable exponent, that is, $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \partial_{x_i} u \in L^{p(x)}(\Omega), i \in \{1, \dots, N\}\}$, where $\partial_{x_i} u$, $i \in \{1, \dots, N\}$, represent the partial derivatives of u with respect to x_i in the weak sense. This space is endowed with the norm $\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}$, for all $u \in W^{1,p(x)}(\Omega)$. The Sobolev space with zero boundary values $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and is equipped with the norm $\|u\|_{p(x)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$, for all $u \in W_0^{1,p(x)}(\Omega)$. It is well known that $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces (see, e.g., Kováčik and Rákosník [52, Theorem 3.1]).

Next, for all $x \in \Omega$, denote the critical Sobolev exponent of $p(x)$ by

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

We recall the following crucial embeddings of $W^{1,p(x)}(\Omega)$.

Proposition 2.2 (Lars Diening et al. [5], Edmunds and Rakosnik [54]). *Let $p \in C_+^{\log}(\overline{\Omega})$ be such that $p^+ < N$, and $h \in C(\overline{\Omega})$ such that $1 \le h(x) \le p^*(x)$ for all $x \in \overline{\Omega}$. Then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$. If we assume in addition that $1 \le h(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then this embedding is also compact.*

For detailed properties of the variable exponent Lebesgue–Sobolev spaces, we refer the reader to Diening et al. [5] and Kováčik and Rákosník [52].

Next, we introduce the anisotropic Sobolev space $W^{1,\vec{p}(x)}(\Omega)$, where $\vec{p} : \overline{\Omega} \to \mathbb{R}^N$ is the vector function $\vec{p}(x) = (p_1(x), \dots, p_N(x))$ and $p_i \in C_+(\overline{\Omega})$ with $1 < p_i^- \le p_i^+ < \infty$ for all $i \in \{1, \dots, N\}$, and we set $p_m(x) = \min\{p_1(x), \dots, p_N(x)\}$, $p_M(x) = \max\{p_1(x), \dots, p_N(x)\}$. The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(x)}(\Omega)$ is defined as $W^{1,\vec{p}(x)}(\Omega) = \{u \in L^{p_M(x)}(\Omega) : \partial_{x_i} u \in L^{p_i(x)}(\Omega) \text{for all } i \in \{1, \dots, N\}\} = \{u \in L^1_{loc}(\Omega) : u \in L^{p_i(x)}(\Omega) \text{and } \partial_{x_i} u \in L^{p_i(x)}(\Omega) \text{for all } i \in \{1, \dots, N\}\}$ and is endowed with the norm $\|u\|_{W^{1,\vec{p}(x)}(\Omega)} = \|u\|_{L^{p_M(x)}(\Omega)} + \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}$. The space $\left(W^{1,\vec{p}(x)}(\Omega), \|\cdot\|_{W^{1,\vec{p}(x)}(\Omega)}\right)$ is reflexive Banach (see, e.g., Fan [44, Theorems 2.1 and 2.2]).

The anisotropic variable exponent Sobolev space with zero boundary values $W_0^{1,\vec{p}(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$, under the norm $\|u\|_{\vec{p}(x)} = \sum_{i=1}^N \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}$. Moreover, the space $W_0^{1,\vec{p}(x)}(\Omega)$ allows the adequate treatment of the existence of the weak solutions for problem (1.1) and can be considered a natural generalization of the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. On the other hand, the space $W_0^{1,\vec{p}(x)}(\Omega)$ can be considered also a natural generalization of the classical anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$ where $\vec{p}$ is the constant vector $(p_1, \dots, p_N)$.

Furthermore, in what follows, we recall the following anisotropic embedding theorem.

Proposition 2.3 (Chems Eddine et al. [51]). *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$, with $p_m \in C_+^{\log}(\overline{\Omega})$ such as $p_m^+ \le N$. If $q \in C(\overline{\Omega})$ satisfies the condition*

$$1 \le q(x) \le p_m^*(x) \text{ for all } x \in \overline{\Omega},$$

then, is true the continuous imbedding

$$W^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

If we assume in addition that $1 \le q(x) < p_m^*(x)$ for all $x \in \overline{\Omega}$, then the above imbedding is compact.

The following proposition is a consequence of Proposition 2.3

Proposition 2.4. *Let $p_i \in C_+(\overline{\Omega})$ for all $i \in \{1, \dots, N\}$ such that $p_M(x) < p_m^*(x)$ for all $x \in \overline{\Omega}$.*

For every $u \in W_0^{1,\vec{p}(x)}(\Omega)$, the Poincaré-type inequality

$$\|u\|_{L^{p_M(x)}(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)} \text{ for all } u \in W_0^{1,\vec{p}(x)}(\Omega) \tag{2.7}$$

holds with a positive constant C independent of u.

Proof. Suppose, by contradiction, that inequality (2.7) does not hold. Then, there exists a sequence $\{u_n\} \subset W_0^{1,\vec{p}(x)}(\Omega)$ such that

$$\|u_n\|_{L^{p_M(x)}(\Omega)} \geq n \sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}.$$

Without loss of generality, we can assume that $\|u_n\|_{L^{p_M(x)}(\Omega)} = 1$. Then

$$\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} \leq \frac{1}{n} \text{ for all } n = 1, 2, \dots, \text{and } \{u_n\} \text{is bounded in } W_0^{1,\vec{p}(x)}(\Omega).$$

So, by using Proposition 2.3, there exists a subsequence of $\{u_n\}$, still labeled $\{u_n\}$ such that $\{u_n\}$ is convergent in $L^{p_M(x)}(\Omega)$. Thus, $\{u_n\}$ is a Cauchy sequence in $W_0^{1,\vec{p}(x)}(\Omega)$, and hence there exists $u_0 \in W_0^{1,\vec{p}(x)}(\Omega)$ as $n \to \infty$. Since $\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} \leq \frac{1}{n}$ and $\partial_{x_i} u_n \to \partial_{x_i} u_0$ in $L^{p_i(x)}(\Omega)$ for all $i = 1, 2, \dots, N$, we have

$$\sum_{i=1}^{N} \|\partial_{x_i} u_0\|_{L^{p_i(x)}(\Omega)} = \lim_{n\to\infty} \sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} = 0,$$

and consequently $\nabla u_0 = 0$. It follows from $u_0 \in W_0^{1,\vec{p}(x)}(\Omega) \subset W_0^{1,1}(\Omega)$ that $u_0 = 0$, which contradicts with that $\|u_0\|_{L^{p_i(x)}(\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^{p_i(x)}(\Omega)} = 1$. The proof is complete. □

As is well known, the use of critical points theory needs the well-known Palais–Smale condition ($(PS)_c$ for short), which plays a central role.

Definition 2.5. Consider a function $E : X \to \mathbb{R}$ of class C^1, where X is a real Banach space. We call a sequence $\{u_m\}$ a Palais–Smale sequence ((PS)-sequence, for short) on X if $E(u_m)$ is bounded and $E'(u_m) \to 0$in X'. If it happens that $E(u_m) \to c$ for some $c \in \mathbb{R}$, the (PS)-sequence will be called a $(PS)_c$-sequence. Moreover, if every $(PS)_c$-sequence for E has a strongly convergent subsequence in X, then we say that E satisfies the Palais–Smale condition at level c (or E is $(PS)_c$, for short).

Now, we recall the classical mountain pass theorem and Rabinowitz's $\mathbb{Z}_2$-symmetric version that we use to prove our most important results in Section 3, recalled respectively in the next theorems.

Theorem 2.6 (Rabinowitz [55]). *Let X be a real infinite dimensional Banach space and let $E : X \to \mathbb{R}$ of class C^1 and satisfy the $(PS)_c$ such that $E(0_X) = 0$. Suppose that*

($\mathscr{I}_1$) *there exist two positive constants $\mathscr{R}$ and ρ such that $E(u) \geq \mathscr{R}$ and for any $u \in X$ with $\|u\|_X = \rho$.*
($\mathscr{I}_2$) *there exists an element $z \in X$ such that $\|z\|_X > \rho$ and $E(z) < 0$.*

Then, E has a critical value $c \geq \mathscr{R}$, which can be characterized as

$$c := \inf_{\phi\in\Gamma} \max_{\delta\in[0,1]} E(\phi(\delta)),$$

where

$$\Gamma = \{\phi : [0,1] \to X, \phi \text{ is a continuous with } \phi(0) = 0_X, E(\phi(1)) < 0\}.$$

Theorem 2.7 (Rabinowitz [55]). *Let X be a real infinite dimensional Banach space and $E : X \to \mathbb{R}$ of class C^1 be even, satisfying $(PS)_c$ and $E(0_X) = 0$. Suppose that assumption ($\mathscr{I}_1$) holds in addition to the following:*

$(\mathcal{I}_2')$ *For each finite dimensional subspace* $X_1 \subset X$*, the set* $\mathcal{S}_1 := \{u \in X_1 : E(u) \geq 0\}$ *is bounded in* X*.*

Then, E *has an unbounded sequence of critical values.*

To prove our existence result, since we have lost the compactness in the inclusions $W_0^{1,\vec{p}(x)}(\Omega) \hookrightarrow L^{p_m^*(x)}(\Omega)$, we can no longer expect the Palais–Smale condition to hold. Nevertheless, we can prove a local Palais–Smale condition that will hold for the energy functional corresponding to problem (1.1) below a certain value of energy, by using the principle of concentration compactness for the variable exponent Sobolev space $W_0^{1,\vec{p}(x)}(\Omega)$. For reader's convenience, we state this result in order to prove Theorem 1.1; see Chems Eddine et al. [51] for the proof.

Theorem 2.8 (Chems Eddine et al. [51]). *Let* $q(x)$ *and* $p_i(x)$ *be continuous functions such that* $1 < \inf_{x\in\Omega} p_i(x) \leq \sup_{x\in\Omega} p_i(x) < N$ *for all* $i \in \{1, 2, \dots, N\}$ *and* $1 \leq q(x) \leq p_m^*(x)$ *in* Ω*, with* $p_m \in C_+^{\log}(\overline{\Omega})$*. Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a weakly convergent sequence in* $W_0^{1,\vec{p}(x)}(\Omega)$ *with weak limit* u *and such that* $|\partial_{x_i} u_n|^{p_i(x)} \overset{*}{\rightharpoonup} \mu_i$ *in* $\mathscr{M}(\overline{\Omega})$ *and* $|u_n|^{q(x)} \overset{*}{\rightharpoonup} \nu$ *in* $\mathscr{M}(\overline{\Omega})$*. Also, suppose that the set* $\mathscr{A} = \{x \in \Omega : q(x) = p_m^*(x)\}$ *is nonempty. Then there exist* $\{x_j\}_{j\in J} \subset \mathscr{A}$ *of distinct points and* $\{\mu_j\}_{j\in J}, \{\nu_j\}_{j\in J} \subset (0, \infty)$*, where* J *is countable index set, such that*

$$\nu = |u|^{q(x)} + \sum_{j\in J} \nu_j \delta_{x_j}, \tag{2.8}$$

$$\mu \geq \sum_{i=1}^{N} |\partial_{x_i} u|^{p_i(x)} + \sum_{j\in J} \mu_j \delta_{x_j}, \tag{2.9}$$

$$N^{1-p_M^+} S_h \nu_j^{\frac{1}{p_m^*(x_j)}} \leq \max\left\{ \left(\mu_j\right)^{1/p_M^+}, \left(\mu_j\right)^{1/p_m^-} \right\}. \forall j \in J. \tag{2.10}$$

where δ_{x_j} *is the Dirac mass at* x_j*,* $\mu = \sum_{i=1}^{n} \mu_i$ *and*

$$S_q := \inf_{u\in \in C_0^\infty(\Omega)} \frac{\|u\|_{\vec{p}(x)}}{\|u\|_{L^{q(x)}(\Omega)}} \tag{2.11}$$

is the best constant in the Gagliardo–Nirenberg–Sobolev inequality for anisotropic variable exponents.

The following result is an extension of the Brezis–Lieb lemma to variable exponent Lebesgue spaces.

Lemma 2.9 (Bonder and Silva [31, Lemma 2.1]). *Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^{q(x)}(\Omega)$ *and* $u_n(x) \to u(x) \in L^{q(x)}(\Omega)$ *for a.e* $x \in \Omega$*. Then,*

$$\lim_{n\to\infty} \left(\int_\Omega |u_n|^{q(x)} dx - \int_\Omega |u - u_n|^{q(x)} dx \right) = \int_\Omega |u|^{q(x)} dx.$$

Notations. Strong (resp. weak, weak-*) convergence is denoted by $\to$ (resp., $\rightharpoonup$, $\overset{*}{\rightharpoonup}$), C_i, c_i, and c_i' denote positive constants, which may vary from line to line and can be determined in concrete conditions. We denote by X the anisotropic variable exponent space $W_0^{1,\vec{p}(x)}(\Omega)$ and X^* denotes the dual space of X, and δ_{x_j} is the Dirac mass at x_j. For any $\rho > 0$ and for any $x \in \Omega$, $B(x, \rho)$ denotes the ball of radius ρ centered at x.

3 | PROOF OF THE MAIN THEOREMS

In this section, we shall establish the existence and multiplicity of nontrivial solutions for problem (1.1).

Definition 3.1. We say that $u \in X$ is a weak solution of (1.1) if

$$M\left(\int_\Omega \sum_{i=1}^{N} A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)} |u|^{p_M(x)} \mathrm{d}x \right) \left(\int_\Omega \sum_{i=1}^{N} a_i(x, \partial_{x_i} u) \partial_{x_i} v + b(x)|u|^{p_M(x)-2} uv\, \mathrm{d}x \right)$$
$$+ \lambda \int_\Omega |u|^{r(x)-2} uv\, \mathrm{d}x = \int_\Omega |u|^{q(x)-2} uv\, \mathrm{d}x + \beta \int_\Omega f(x, u) v\, \mathrm{d}x,$$

for all $v \in X$.

The energy functional associated with problem (1.1) is defined by $E_{\lambda,\beta} : X \to \mathbb{R}$, where

$$\begin{aligned} E_{\lambda,\beta}(u) = \widehat{M}\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\right) + \int_\Omega \frac{\lambda}{r(x)}|u|^{r(x)}\,\mathrm{d}x \\ - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x - \beta\int_\Omega F(x,u)\,\mathrm{d}x. \end{aligned} \tag{3.1}$$

By a standard calculus, one can see that $E_{\lambda,\beta} \in C^1(X,\mathbb{R})$ and the Frechet derivative is

$$\begin{aligned} \langle E'_{\lambda,\beta}(u), v\rangle = M\left(\int_\Omega \sum_{i=1}^N A_i(x, \partial_{x_i} u) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\right)\left(\int_\Omega \sum_{i=1}^N a_i(x, \partial_{x_i} u)\partial_{x_i} v + b(x)|u|^{p_M(x)-2}uv\,\mathrm{d}x\right) \\ + \lambda\int_\Omega |u|^{r(x)-2}uv\,\mathrm{d}x - \int_\Omega |u|^{q(x)-2}uv\,\mathrm{d}x - \beta\int_\Omega f(x,u)v\,\mathrm{d}x, \end{aligned} \tag{3.2}$$

for all $u, v \in X$. Thus, the weak solutions of (1.1) coincide with the critical points of $E_{\lambda,\beta}$.

For the proof of Theorem 1.1, we shall use the mountain pass theorem 2.6. We first start with the following lemmas.

Lemma 3.2. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a (PS)-sequence for the functional $E_{\lambda,\beta}$. If ($\boldsymbol{A}_1$)–($\boldsymbol{A}_2$), ($\boldsymbol{M}_1$)–($\boldsymbol{M}_2$), and ($\boldsymbol{f}_2$) hold, then for any $\lambda \in \mathbb{R}$, $\{u_n\}_{n\in\mathbb{N}}$ is bounded.*

Proof. be a (PS)-sequence. Then we have

$$E_{\lambda,\beta}(u_n) = C_{\lambda,\beta} + o_n(1) \quad \text{and} \quad \langle E'_{\lambda,\beta}(u_n), v\rangle = o_n(1) \text{for all } \ v \in X. \tag{3.3}$$

Then, by using assumptions ($\boldsymbol{M}_1$)–($\boldsymbol{M}_2$) and ($\boldsymbol{f}_2$), we have for large n

$$\begin{aligned} C_{\lambda,\beta} + o_n(1) &\geq E_{\lambda,\beta}(u_n) - \frac{1}{\theta_\lambda}\langle E'_{\lambda,\beta}(u_n), u_n\rangle \\ &\geq \mathbf{M}_0\left(\int_\Omega \left[\gamma\left(\sum_{i=1}^N A_i(x, \partial_{x_i} u_n) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right) - \frac{1}{\theta_\lambda}\left(\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n + b(x)|u_n|^{p_M(x)}\right)\right]\mathrm{d}x\right) \\ &\quad + \lambda\left(\frac{1}{\bar{r}} - \frac{1}{\theta_\lambda}\right)\int_\Omega |u_n|^{r(x)}\,\mathrm{d}x + \int_\Omega\left(\frac{1}{\theta_\lambda} - \frac{1}{q^-}\right)|u_n|^{q(x)}\,\mathrm{d}x + \beta\int_\Omega\left(F(x,u_n) - f(x,u_n)\frac{u_n}{\theta_\lambda}\right)\mathrm{d}x \\ &\geq \mathbf{M}_0\left(\int_\Omega \left[\gamma\left(\sum_{i=1}^N A_i(x, \partial_{x_i} u_n) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right) - \frac{1}{\theta_\lambda}\left(\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n + b(x)|u_n|^{p_M(x)}\right)\right]\mathrm{d}x\right), \end{aligned}$$

with $\bar{r} :\equiv r^+$ if $\lambda \geq 0$ and $\bar{r} :\equiv r^-$ if $\lambda < 0$. From assumption ($\boldsymbol{A}_2$), we have

$$C_{\lambda,\beta} + o_n(1) \geq \mathbf{M}_0\left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\int_\Omega\left(\sum_{i=1}^N a_i(x, \partial_{x_i} u_n)\partial_{x_i} u_n + b(x)|u_n|^{p_M(x)}\right)\mathrm{d}x.$$

By relations (2.3) and (2.4), we get

$$\int_\Omega b(x)|u_n|^{p_M(x)}\mathrm{d}x \geq b_0 \min\left\{\|u_n\|^{P_M^+}_{L^{p_M(x)}(\Omega)}, \|u_n\|^{P_M^-}_{L^{p_M(x)}(\Omega)}\right\} \geq 0 \text{ for all } \ X; \tag{3.4}$$

hence, by using again assumption ($\boldsymbol{A}_2$) and inequality (3.4), we have

$$C_{\lambda,\beta} + o_n(1) \geq \mathbf{M}_0 \min_{1\leq i\leq N} k_i\left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\sum_{i=1}^N \int_\Omega |\partial_{x_i} u_n|^{p_i(x)}\mathrm{d}x. \tag{3.5}$$

For any $n \in \mathbb{N}$, we denote by I_{n_1} and I_{n_2} the indices sets

$$I_{n_1} = \{i \in \{1,2,\dots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} \le 1\} \text{ and } \mathcal{I}_{n_2} = \{i \in \{1,2,\dots,N\} : \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)} > 1\}.$$

Applying relations (2.3) and (2.4) and inequality (3.5), we find

$$\begin{aligned} C_{\lambda,\beta} + o_n(1) &\ge \mathbf{M}_0 \min_{1\le i\le N} k_i \left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\left(\sum_{i\in\mathcal{I}_{n_1}} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_M^+} + \sum_{i\in\mathcal{I}_{n_2}} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-}\right) \\ &= \mathbf{M}_0 \min_{1\le i\le N} k_i \left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\left[\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - \sum_{i\in\mathcal{I}_{n_1}} \left(\|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_M^+}\right)\right] \\ &\ge \mathbf{M}_0 \min_{1\le i\le N} k_i \left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\left(\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-} - N\right). \end{aligned}$$

On the other hand, by using Jensen's inequality on the convex function $h : \mathbb{R}^+ \to \mathbb{R}^+, h(t) = t^{p_m^-}, p_m^- > 1$, we find that

$$\frac{\|\partial_{x_i} u_n\|_{\vec{p}(x)}^{p_m^-}}{N^{p_m^- - 1}} = N\left(\frac{\sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}}{N}\right)^{p_m^-} \le \sum_{i=1}^{N} \|\partial_{x_i} u_n\|_{L^{p_i(x)}(\Omega)}^{p_m^-}. \tag{3.6}$$

So, by relation (3.6), we get that

$$C_{\lambda,\beta} + o_n(1) \ge \mathbf{M}_0 \min_{1\le i\le N} k_i \left(\frac{\gamma}{p_M^+} - \frac{1}{\theta_\lambda}\right)\left(\frac{\|\partial_{x_i} u_n\|_{\vec{p}(x)}^{p_m^-}}{N^{p_m^- - 1}} - N\right).$$

Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X and the proof is completed. □

Lemma 3.3. *Let $\{u_n\}_{n\in\mathbb{N}} \subset X$ be a (PS)-sequence with energy level $C_{\lambda,\beta}$, if*

$$C_{\lambda,\beta} < L \min\left\{\inf_{j\in J}\left((\boldsymbol{M}_0 \min_{1\le i\le N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}}, \inf_{j\in J}\left((\boldsymbol{M}_0 \min_{1\le i\le N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\},$$

where $L = \left(\frac{1}{\theta_\lambda} - \frac{1}{q_{\mathcal{A}}^-}\right)$, $\mathcal{A}$ and S_q are defined in Theorem 2.8. Then, there exists a subsequence strongly convergent in X.

Proof. We divide the proof into two claims.

Claim 1. We claim that $u_n \to u$ strongly in $L^{q(x)}(\Omega)$ as $n \to \infty$. By Lemma 3.2, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Passing to a subsequence, still labeled $\{u_n\}_{n\in\mathbb{N}}$, it is weakly convergent in X. Therefore, there exist positive bounded measures $\mu_i, \nu \in \Omega$ such that

$$\sum_{i=1}^{N} |\partial_{x_i} u_n|^{p_i(x)} \overset{*}{\rightharpoonup} \mu = \sum_{i=1}^{N} \mu_i \text{ and } |u_n|^{q(x)} \overset{*}{\rightharpoonup} \nu. \tag{3.7}$$

Hence, by Theorem 2.8, if $J = \emptyset$, then $u_n \to u$ in $L^{q(x)}(\Omega)$. Let us show that if

$$C_{\lambda,\beta} < L \min\left\{\inf_{j\in J}\left((\mathbf{M}_0 \min_{1\le i\le N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}}, \inf_{j\in J}\left((\mathbf{M}_0 \min_{1\le i\le N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\},$$

and $\{u_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence with energy level $C_{\lambda,\beta}$, then $J = \emptyset$. In fact, we assume that $J \neq \emptyset$, for all $j \in J$, and let $x_j \in \mathcal{A}$ be a singular point of the measures μ_i and ν.

We consider $\psi \in C_0^\infty(\mathbb{R}^N)$, such that $0 \le \psi(x) \le 1$, $\psi(0) = 1$, $\mathrm{supp}\psi \subset B(0,1)$ and $\|\nabla\psi\|_\infty \le 2$. For any $j \in J$ and any $\varepsilon > 0$, we define the function $\psi_{j,\varepsilon} := \psi\left(\frac{x-x_j}{\varepsilon}\right)$, for all $x \in \mathbb{R}^N$. Notice that $\psi_{j,\varepsilon} \in C_0^\infty(\mathbb{R}^N, [0,1])$, $\|\nabla\psi_{j,\varepsilon}\|_\infty \le \frac{2}{\varepsilon}$ and

$$\psi_{j,\varepsilon}(x) = \begin{cases} 1, & x \in B(x_j,\varepsilon), \\ 0, & x \in \mathbb{R}^N \setminus B(x_j, 2\varepsilon). \end{cases}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X, the sequence $\{\psi_{j,\varepsilon} u_n\}_{n\in\mathbb{N}}$ is also bounded in X. Thus, $\langle E'_{\lambda,\beta}(u_n)(\psi_{j,\varepsilon}u_n)\rangle = o_n(1)$, that is,

$$\begin{aligned} &M\left(\int_\Omega\left(\sum_{i=1}^N A_i\left(x,\partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\int_\Omega \psi_{j,\varepsilon}\left(\sum_{i=1}^N a_i(x,\partial_{x_i}u_n)\partial_{x_i}u_n + b(x)|u_n|^{p_M(x)}\right)\mathrm{d}x \\ &+\lambda\int_\Omega |u|^{r(x)}\psi_{j,\varepsilon}\,\mathrm{d}x = -M\left(\int_\Omega\left(\sum_{i=1}^N A_i\left(x,\partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\sum_{i=1}^N\int_\Omega u_n a_i\left(x,\partial_{x_i}u_n\right)\partial_{x_i}\psi_{j,\varepsilon}\,\mathrm{d}x \\ &+\int_\Omega |u_n|^{q(x)}\psi_{j,\varepsilon}\,\mathrm{d}x + \beta\int_\Omega f(x,u_n)\psi_{j,\varepsilon}u_n\,\mathrm{d}x + o_n(1). \end{aligned} \tag{3.8}$$

Now, we shall prove that

$$\lim_{\varepsilon\to0}\left[\limsup_{n\to\infty}\left|\sum_{i=1}^N M\left(\int_\Omega\left(\sum_{i=1}^N A_i\left(x,\partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\sum_{i=1}^N\int_\Omega u_n a_i\left(x,\partial_{x_i}u_n\right)\partial_{x_i}\psi_{j,\varepsilon}\,\mathrm{d}x\right|\right] = 0. \tag{3.9}$$

We remark that, due to the hypotheses $(\boldsymbol{A}_1)$, it suffices to show that

$$\lim_{\varepsilon\to0}\left[\limsup_{n\to\infty} M\left(\int_\Omega\left(\sum_{i=1}^N A_i\left(x,\partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\left|\sum_{i=1}^N\int_\Omega \bar{c}_i g_i(x) u_n\partial_{x_i}\psi_{j,\varepsilon}\,\mathrm{d}x\right|\right] = 0, \tag{3.10}$$

and

$$\lim_{\varepsilon\to0}\left[\limsup_{n\to\infty} M\left(\int_\Omega\left(\sum_{i=1}^N A_i(x,\partial_{x_i}u_n) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\left|\sum_{i=1}^N\int_\Omega u_n\left|\partial_{x_i}u_n\right|^{p_i(x)-1}\partial x_i\psi_{j,\varepsilon}\,\mathrm{d}x\right|\right] = 0. \tag{3.11}$$

First, by using the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in X, we obtain

$$\begin{aligned} &\left|\int_\Omega u_n\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial x_i u_n\partial x_i\psi_{j,\varepsilon}\,\mathrm{d}x\right| \le \int_\Omega \left|\partial_{x_i}u_n\right|^{p_i(x)-1}\left|u_n\partial x_i\psi_{j,\varepsilon}\right|\,\mathrm{d}x \\ &\le 2\left\||\partial x_i u_n|^{p_i(x)-1}\right\|_{L^{\frac{p_i(x)}{p_i(x)-1}}(\Omega)}\left\|\partial x_i\psi_{j,\varepsilon}u_n\right\|_{L^{p_i(x)}(\Omega)} \\ &\le C\max\left\{\left(\int_\Omega |u_n|^{p_i(x)}|\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right)^{\frac{1}{p_i^-}}, \left(\int_\Omega |u_n|^{p_i(x)}|\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right)^{\frac{1}{p_i^+}}\right\}. \end{aligned}$$

So, by using Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} &\left|\int_\Omega u_n\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial x_i u_n\partial x_i\psi_{j,\varepsilon}\,\mathrm{d}x\right| \\ &\le C\max\left\{\left(\int_\Omega |u|^{p_i(x)}|\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right)^{\frac{1}{p_i^-}}, \left(\int_\Omega |u|^{p_i(x)}|\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right)^{\frac{1}{p_i^+}}\right\}. \end{aligned}$$

Furthermore, by the Hölder inequality,

$$\int_\Omega |u|^{p_i(x)}|\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\,\mathrm{d}x \le C\left\||u|^{p_i(x)}\right\|_{L^{\frac{N}{N-p_i(x)}}(B(x_j,2\varepsilon))}\left\||\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right\|_{L^{\frac{N}{p_i(x)}}(B(x_j,2\varepsilon))}.$$

From $\int_{B(x_j,2\varepsilon)}|\partial x_i\psi_{j,\varepsilon}|^N \,\mathrm{d}x = \int_{B(0,2)}|\partial x_i\psi_{j,\varepsilon}|^N \,\mathrm{d}x$, we derive

$$
\begin{aligned}
&\left\||\partial x_i\psi_{j,\varepsilon}|^{p_i(x)}\right\|_{L^{\frac{N}{p_i(x)}}(B(x_j,2\varepsilon))} \\
&\leq \max\left\{\left(\int_{B(x_j,2\varepsilon)}|\partial x_i\psi_{j,\varepsilon}|^N \mathrm{d}x\right)^{\frac{1}{\left(\frac{N}{p_i(x)}\right)^+}}, \left(\int_{B(x_j,2\varepsilon)}|\partial x_i\psi_{j,\varepsilon}|^N \mathrm{d}x\right)^{\frac{1}{\left(\frac{N}{p_i(x)}\right)^-}}\right\} \leq C,
\end{aligned}
$$

for some positive constant C, which is independent of ε. Therefore,

$$
\limsup_{n\to\infty}\left|\int_\Omega u_n\left|\partial_{x_i}u_n\right|^{p_i(x)-2}\partial x_i u_n \partial x_i\psi_\varepsilon \,\mathrm{d}x\right| \leq C\left\{\left\||u|^{p_i(x)}\right\|^{\frac{1}{p_i^-}}_{L^{\frac{N}{N-p_i(x)}}(B(x_j,2\varepsilon))}, \left\||u|^{p_i(x)}\right\|^{\frac{1}{p_i^+}}_{L^{\frac{N}{N-p_i(x)}}(B(x_j,2\varepsilon))}\right\}.
$$

However,

$$
\left\||u|^{p_i(x)}\right\|_{L^{\frac{N}{N-p_i(x)}}(B(x_j,2\varepsilon))} \leq \max\left\{\left(\int_{B(x_j,2\varepsilon)}|u|^{p_i^*(x)}\mathrm{d}x\right)^{\frac{1}{\left(\frac{N}{N-p_i(x)}\right)^+}}, \left(\int_{B(x_j,2\varepsilon)}|u|^{p_i^*(x)}\mathrm{d}x\right)^{\frac{1}{\left(\frac{N}{N-p_i(x)}\right)^-}}\right\},
$$

so it follows that

$$
\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_\Omega u_n\left|\partial_{x_i}u_n\right|^{p_i(x)-1}\partial x_i\psi_{j,\varepsilon}\,\mathrm{d}x\right| = 0, \text{ for all } i = 1, 2, \dots, N. \tag{3.12}
$$

Since $\{u_n\}$ is bounded in X, we may assume that $(\int_\Omega \left(\sum_{i=1}^N A_i\left(x, \partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right) dx \to t_0 \geq 0$ as $n \to \infty$. So, by the continuity of M, we have

$$
M\left(\int_\Omega\left(\sum_{i=1}^N A_i\left(x, \partial_{x_i}u_n\right) + \frac{v(x)}{p_M(x)}|u_n|^{p_M(x)}dx\right) \to M(t_0) \geq \mathbf{M}_0, \text{ as } n \to \infty.
$$

Hence, from this and (3.12), we deduce (3.11). A similarly, we can check (3.10). Hence, we have completed the proof of (3.9).

We obtain, by using assumption $(\boldsymbol{f}_1)$, Theorem 2.3, and Lebesgue's dominated convergence theorem, that

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega|u_n|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x = \int_\Omega|u|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x, \ \lim_{n\to\infty}\int_\Omega|u_n|^{r(x)}\psi_{j,\varepsilon}\,\mathrm{d}x = \int_\Omega|u|^{r(x)}\psi_{j,\varepsilon}\,\mathrm{d}x, \\
&\text{and } \lim_{n\to\infty}\int_\Omega f(x,u_n)u_n\psi_{j,\varepsilon}\,\mathrm{d}x = \int_\Omega f(x,u)u\psi_{j,\varepsilon}\,\mathrm{d}x.
\end{aligned} \tag{3.13}
$$

Thus, when $\varepsilon \to 0$, we get

$$
\lim_{\varepsilon\to 0}\int_\Omega|u|^{p_M(x)}\psi_{j,\varepsilon}\,\mathrm{d}x = 0, \ \lim_{\varepsilon\to 0}\int_\Omega|u|^{r(x)}\psi_{j,\varepsilon}\,\mathrm{d}x = 0, \text{ and } \lim_{\varepsilon\to 0}\int_\Omega f(x,u)u\psi_\varepsilon\,\mathrm{d}x = 0. \tag{3.14}
$$

On the other hand, we have

$$
\lim_{\varepsilon\to 0}\int_\Omega \psi_{j,\varepsilon}d\mu_i = \mu_i\psi(0) \text{ and } \lim_{\varepsilon\to 0}\int_{\partial\Omega}\psi_{j,\varepsilon}d\nu = \nu\psi(0).
$$

Since $\psi_{j,\varepsilon}$ has compact support, by letting the limit $n \to \infty$ and $\varepsilon \to 0$ in (3.8), from relations (3.9), (3.13), and (3.14), we get

$$
\lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(M\left(\int_\Omega\left(\sum_{i=1}^N\frac{1}{p_i(x)}|\partial_{x_i}u_n|^{p_i(x)} + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\right)dx\right)\sum_{i=1}^N\int_\Omega\psi_{j,\varepsilon}a_i\left(x,\partial_{x_i}u_n\right)\partial_{x_i}u_n\,\mathrm{d}x\right)\right] = \nu_j.
$$

By assumptions ($\boldsymbol{A}_2$) and ($\boldsymbol{M}_1$), we have

$$\lim_{\varepsilon\to 0}\left[\limsup_{n\to+\infty}\left(\sum_{i=1}^{N}\mathbf{M}_0\int_\Omega k_i\psi_{j,\varepsilon}|\partial_{x_i}u_n|^{p_i(x)}\,\mathrm{d}x\right)\right]\le \nu_j; \tag{3.15}$$

hence,

$$\mathbf{M}_0\min_{i\in\{1,\dots,N\}}k_i\mu_j\le\nu_j \text{ for any } j\in J. \tag{3.16}$$

Thus, invoking relation (2.10), we can conclude that

$$\nu_j\ge\min\left\{\left((\mathbf{M}_0\min_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},\left((\mathbf{M}_0\min_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\}.$$

Consequently,

$$\begin{aligned}\int_\Omega|u_n|^{q(x)}\mathrm{d}x\to\int_\Omega\mathrm{d}\nu\ge&\int_\Omega|u|^{q(x)}\mathrm{d}x\\&+\min\left\{\left((\mathbf{M}_0\inf_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},\left((\mathbf{M}_0\inf_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\}\sum_{j\in J}\delta_{x_j}\\\ge&\int_\Omega|u|^{q(x)}\mathrm{d}x\\&+\min\left\{\left((\mathbf{M}_0\inf_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},\left((\mathbf{M}_0\inf_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\}\text{card } J.\end{aligned}$$

If card $J=\infty$, we get contradiction. On the other hand, by assumptions ($\boldsymbol{A}_2$) and ($\boldsymbol{f}_2$), we have

$$\begin{aligned}E_{\lambda,\beta}(u_n)-\frac{1}{\theta_\lambda}\langle E'_{\lambda,\beta}(u_n),u_n\rangle\ge&\ \mathbf{M}_0\inf_{1\le i\le N}k_i\left(\frac{\gamma}{p_M^+}-\frac{1}{\theta_\lambda}\right)\sum_{i=1}^{N}\int_\Omega\left|\partial_{x_i}u_n\right|^{p_i(x)}\mathrm{d}x+\lambda\left(\frac{1}{\bar r}-\frac{1}{\theta_\lambda}\right)\int_\Omega|u_n|^{r(x)}\,\mathrm{d}x\\&+\int_\Omega\left(\frac{1}{\theta_\lambda}-\frac{1}{q^-}\right)|u_n|^{q(x)}\,\mathrm{d}x+\beta\int_\Omega\left(F(x,u_n)-f(x,u_n)\frac{u_n}{\theta_\lambda}\right)\mathrm{d}x\\\ge&\left(\frac{1}{\theta_\lambda}-\frac{1}{q^-}\right)\int_\Omega|u_n|^{q(x)}\,\mathrm{d}x.\end{aligned}$$

Now, setting $\mathscr{A}_\delta=\cup_{x\in\mathscr{A}}(\mathbf{B}(x,\delta)\cap\Omega)=\{x\in\Omega:\ \mathrm{dist}(x,\mathscr{A})<\delta\}$, when $n\to+\infty$, we find

$$\begin{aligned}C_{\lambda,\beta}\ge&\left(\frac{1}{\theta_\lambda}-\frac{1}{q^-_{\mathscr{A}_\delta}}\right)\left(\int_\Omega|u|^{q(x)}\,\mathrm{d}x+\sum_{j\in J}\nu_j\delta_{x_j}\right)\\\ge&\left(\frac{1}{\theta_\lambda}-\frac{1}{q^-_{\mathscr{A}_\delta}}\right)\min\left\{\inf_{j\in J}\left((\mathbf{M}_0\min_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}},\inf_{j\in J}\left((\mathbf{M}_0\min_{1\le i\le N}k_i)^{\frac{1}{p_M^+}}N^{1-p_M^+}S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\}.\end{aligned}$$

Since $\delta > 0$ and arbitrary, and q is continuous, we obtain

$$C_{\lambda,\beta} \geq \left(\frac{1}{\theta_\lambda} - \frac{1}{q^-_{\mathscr{A}}}\right) \min\left\{\inf_{j\in J}\left((\mathbf{M}_0 \min_{1\leq i\leq N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}}, \inf_{j\in J}\left((\mathbf{M}_0 \min_{1\leq i\leq N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\}.$$

Therefore, if

$$C_{\lambda,\beta} < \left(\frac{1}{\theta_\lambda} - \frac{1}{q^-_{\mathscr{A}}}\right) \min\left\{\inf_{j\in J}\left((\mathbf{M}_0 \min_{1\leq i\leq N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_M^+}{p_m^*(x_j)-p_M^+}}, \inf_{j\in J}\left((\mathbf{M}_0 \min_{1\leq i\leq N} k_i)^{\frac{1}{p_M^+}} N^{1-p_M^+} S_q\right)^{\frac{p_m^*(x_j)p_m^-}{p_m^*(x_j)-p_m^-}}\right\},$$

then the index set J is empty and, consequently, $\rho_q(u_n) \to \rho_q(u)$ as $n \to \infty$. So, by applying Lemma 2.9 and relation (2.6), we find that $u_n \to u$ strongly in $L^{q(x)}(\Omega)$ as $n \to \infty$. This concludes the proof of Claim 1.

Claim 2. We claim that $u_n \to u$ strongly in X as $n \to +\infty$.

$\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Also, X is a reflexive space, so there exists a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$ and $u \in X$ such that

$$u_n \rightharpoonup u \text{ weakly in } X. \tag{3.17}$$

By Theorem 2.3, we know that X is compactly embedded in $L^{h(x)}(\Omega)$, where $1 \leq h(x) \leq p_m^*(x)$. Therefore, since $u_n \rightharpoonup u$ in the Banach space X, we can infer that

$$u_n \to u \text{ in } L^{h(x)}(\Omega). \tag{3.18}$$

Using (3.3) and (3.17) and the fact that $|\langle E'_{\lambda,\beta}(u_n), u_n - u\rangle| \leq \|E'_{\lambda,\beta}(u_n)\|_{X^*}\,\|u_n - u\|_X$, we get that $\lim_{n\to\infty}|\langle E'_{\lambda,\beta}(u_n), u_n - u\rangle| = 0$, that is,

$$\lim_{n\to\infty}\Bigg[M\left(\int_\Omega \sum_{i=1}^N A_i\left(x, \partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}dx\right)\int_\Omega\left(\sum_{i=1}^N a_i(x,\partial_{x_i}u)\left(\partial_{x_i}u_n - \partial_{x_i}u\right) + b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\right)dx$$
$$+\lambda\int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)\,dx - \int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)\,dx - \beta\int_\Omega f(x,u_n)(u_n-u)\,dx\Bigg] = 0. \tag{3.19}$$

Therefore, by applying the Hölder inequality and the fact that $v \in L^\infty(\Omega)$, we get that

$$\left|\int_\Omega b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\,dx\right| \leq 2\|b\|_{L^\infty(\Omega)}\left\||u_n|^{p_M(x)-1}\right\|_{L^{p'_M(x)}(\Omega)}\|u_n - u\|_{L^{p_M(x)}(\Omega)}. \tag{3.20}$$

We assume by contradiction that $\left\||u_n|^{p_M(x)-1}\right\|_{L^{p'_M(x)}(\Omega)} \to \infty$. Then by relation (2.6), we have $\int_\Omega\left(|u_n|^{p_M(x)-1}\right)^{p'_M(x)}dx \to +\infty$ if and only if $\int_\Omega\left(|u_n|^{p_M(x)}\right)dx \to +\infty$ if and only if $\|u_n\|_{L^{p_M(x)}(\Omega)} \to +\infty$. However, $\|u_n\|_{L^{p_M(x)}(\Omega)} \to \|u\|_{L^{p_M(x)}(\Omega)}$, which is a contradiction. Consequently, by relations (3.20) and (3.18), we get

$$\lim_{n\to\infty}\int_\Omega b(x)|u_n|^{p_M(x)-2}u_n(u_n-u)\,dx = 0. \tag{3.21}$$

Similarly, we can show that

$$\lim_{n\to\infty}\int_\Omega |u_n|^{r(x)-2}u_n(u_n-u)\,dx = 0. \tag{3.22}$$

By Claim 1, $u_n \to u$ in $L^{q(x)}(\Omega)$, we have

$$\lim_{n\to\infty}\int_\Omega |u_n|^{q(x)-2}u_n(u_n-u)\,dx = 0. \tag{3.23}$$

In addition, by assumption ($\boldsymbol{f}_2$) and the Hölder inequality, we find that

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n-u)\,dx\right| &\leq \int_\Omega |f(x,u_n)||u_n-u|\,dx\\ &\leq C_f\int_\Omega |u_n-u|\,dx + C_f\int_\Omega |u_n|^{\ell(x)-1}|u_n-u|\,dx\\ &\leq C_f\|u_n-u\|_{L^1(\Omega)} + 2C_f\||u_n|^{\ell(x)-1}\|_{L^{\ell'(x)}(\Omega)}\|u_n-u\|_{L^{\ell(x)}(\Omega)}.\end{aligned}$$

So, by relations (3.18) and (2.6), we find as above

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)(u_n-u)\,dx = 0. \tag{3.24}$$

Therefore, by combining relations (3.19), (3.21), (3.22), (3.23), and (3.24), we find

$$\lim_{n\to\infty} M\left(\int_\Omega \sum_{i=1}^N A_i\left(x,\partial_{x_i}u_n\right) + \frac{b(x)}{p_M(x)}|u_n|^{p_M(x)}\,dx\right)\int_\Omega \sum_{i=1}^N a_i\left(x,\partial_{x_i}u_n\right)\left(\partial_{x_i}u_n-\partial_{x_i}u\right)dx = 0.$$

By assumptions ($\boldsymbol{A}_2$) and ($\boldsymbol{M}_1$), we obtain

$$\lim_{n\to\infty}\mathbf{M}_0\min_{1\leq i\leq N}k_i\int_\Omega\sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n\left(\partial_{x_i}u_n-\partial_{x_i}u\right)dx \leq 0. \tag{3.25}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ converges weakly to u in X, we have

$$\lim_{n\to\infty}\mathbf{M}_0\min_{1\leq i\leq N}k_i\int_\Omega\sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\left(\partial_{x_i}u_n-\partial_{x_i}u\right)dx \leq 0. \tag{3.26}$$

By combining relations (3.25) and (3.26), we can infer that

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega\left(|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n - |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right)\left(\partial_{x_i}u_n-\partial_{x_i}u\right)dx \leq 0. \tag{3.27}$$

Therefore, invoking some elementary inequalities (see, e.g., DiBenedetto [56, Chapter I]), for all $\sigma > 1$, there exists a positive constant C_σ such that

$$\langle|\xi|^{\sigma-2}\xi-|\varrho|^{\sigma-2}\varrho,\xi-\varrho\rangle \geq \begin{cases} C_\sigma|\xi-\varrho|^\sigma & \text{if } \sigma\geq 2\\ C_\sigma\frac{|\xi-\varrho|^2}{(|\xi|+|\varrho|)^{2-\sigma}},(\xi,\varrho)\neq(0,0) & \text{if } 1<\sigma<2\end{cases} \tag{3.28}$$

for any $\xi,\varrho\in\mathbb{R}$.

So, by relation (3.27) and inequalities (3.28), we see that

$$\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega\left|\partial_{x_i}u_n-\partial_{x_i}u\right|^{p_i(x)}dx = 0. \tag{3.29}$$

Therefore, we can conclude that $u_n\to u$ strongly in X. □

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is an immediate consequence of the mountain pass theorem 2.6, Lemma 3.2, and Lemma 3.3. Precisely, it suffices to verify that $E_{\lambda,\beta}$ has the mountain pass geometry and that $E_{\lambda,\beta}(\kappa u) < 0$ for some

$\kappa > 0$. From assumption $(\boldsymbol{f}_2)$, there exists $C = C(x) > 0$ such that

$$F(x,\xi) \geq C|\xi|^{\theta_\lambda}, \ \forall x \in \Omega, \text{and } \ \forall \xi \in \mathbb{R} \ \text{ with } \ |\xi| \geq R, \tag{3.30}$$

By assumption $(\boldsymbol{M}_2)$, we can obtain for $t > t_0$

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{\gamma}}} t^{\frac{1}{\gamma}} \leq ct^{\frac{1}{\gamma}}. \tag{3.31}$$

Then, due to assumption $(\boldsymbol{A}_1)$ and inequalities (3.30)–(3.31), for $u \in X$ and any $\kappa > 1$, we have

$$\begin{aligned} E_{\lambda,\beta}(\kappa u) \leq{}& c\left(\int_\Omega \sum_{i=1}^N c_{a_i}\left(|g_i(x)||\partial_{x_i}(\kappa u)| + \frac{|\partial_{x_i}(\kappa u)|^{p_i(x)}}{p_i(x)}\right) + \frac{b(x)}{p_M(x)}|\kappa u|^{p_M(x)}dx\right)^{\frac{1}{\gamma}} + \lambda\int_\Omega \frac{1}{r(x)}|\kappa u|^{r(x)}\,\mathrm{d}x \\ & - \int_\Omega \frac{1}{q(x)}|\kappa u|^{q(x)}\,\mathrm{d}x - \beta\int_{\{x\in\Omega:\,|u(x)|>R\}} C|\kappa u|^{\theta_\lambda}\,\mathrm{d}x - \beta\text{meas}(\Omega)\inf\{F(x,s)\,:\, x\in\Omega,\, |s|\leq R\} \\ \leq{}& c\kappa^{\frac{p_M^+}{\gamma}}\left(\int_\Omega \sum_{i=1}^N c_{a_i}\left(|g_i(x)||\partial_{x_i}(u)| + \frac{|\partial_{x_i}(u)|^{p_i(x)}}{p_i(x)}\right) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}dx\right)^{\frac{1}{\gamma}} + \lambda\kappa^{\bar r}\int_\Omega \frac{1}{r(x)}|u|^{r(x)}\,\mathrm{d}x \\ & - \frac{\kappa^{q^-}}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - C\kappa^{\theta_\lambda}\int_\Omega |u|^{\theta_\lambda}\,\mathrm{d}x - \beta\text{meas}(\Omega)\inf\{F(x,s)\,:\, x\in\Omega,\, |s|\leq R\}, \end{aligned}$$

with again $\bar r :\equiv r^+$ if $\lambda \geq 0$ and $\bar r :\equiv r^-$ if $\lambda < 0$. Since $\max\{\frac{p_M^+}{\gamma}, \bar r\} < q^-$, we infer that, for any $\lambda \in \mathbb{R}$, $E_{\lambda,\beta}(\kappa u) \to -\infty$ as $\kappa \to +\infty$.

On the other hand, from assumptions $(\boldsymbol{f}_1)$ and $(\boldsymbol{f}_3)$, for all $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ such that

$$|F(x,\xi)| \leq \varepsilon|\xi|^{p_M^+} + C(\varepsilon)|\xi|^{\ell(x)} \text{ for a.e. } x \in \Omega \ \text{ and all } \ \xi \in \mathbb{R}. \tag{3.32}$$

Therefore, by assumptions $(\boldsymbol{M}_1)$–$(\boldsymbol{M}_2)$ and $(\boldsymbol{A}_2)$ and inequalities (3.4) and (3.32), we have

$$\begin{aligned} E_{\lambda,\beta}(u) \geq{}& \frac{\mathbf{M}_0\gamma}{p_M^+}\left(\sum_{i=1}^N \int_\Omega k_i\left|\partial_{x_i}u\right|^{p_i(x)}\mathrm{d}x + \int_\Omega b(x)|u|^{p_M(x)}\,\mathrm{d}x\right) + \lambda\int_\Omega \frac{1}{r(x)}|u|^{r(x)}\,\mathrm{d}x \\ & - \frac{1}{q^-}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - \beta\int_\Omega (\epsilon|u|^{p_M^+} + C(\epsilon)|u|^{\ell(x)})\,\mathrm{d}x \\ \geq{}& \frac{\mathbf{M}_0\gamma \min\limits_{1\leq i\leq N} k_i}{p_M^+ N^{p_M^+-1}}\int_\Omega \sum_{i=1}^N \left|\partial_{x_i}u\right|^{p_i(x)}\mathrm{d}x - \frac{|\lambda|}{r^-}\int_\Omega |u|^{r(x)}\,\mathrm{d}x \\ & - \frac{1}{q^-}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - \beta\int_\Omega \epsilon|u|^{p_M^+}\,\mathrm{d}x - \beta\int_\Omega C(\epsilon)|u|^{\ell(x)}\,\mathrm{d}x. \end{aligned} \tag{3.33}$$

Consider $0 < \|u\|_X = \varrho < 1$. By using relations (2.3) and (2.4) and Theorem 2.3, we have

$$\begin{aligned} E_{\lambda,\beta}(u) \geq{}& \frac{\mathbf{M}_0\gamma \min\limits_{1\leq i\leq N} k_i}{p_M^+ N^{p_M^+-1}}\|u\|_X^{P_M^+} - \frac{|\lambda|C}{r^-}\max\left\{\|u\|_X^{r^+}, \|u\|_X^{r^-}\right\} - \frac{1}{h^-}C_1'\|u\|_X^{h^-} \\ & - \beta\varepsilon C_2'\|u\|_X^{P_M^+} - \beta C(\varepsilon)C_3'\|u\|_X^{\ell^-} \\ \geq{}& \|u\|_X\left[\left(\frac{\mathbf{M}_0\gamma \min\limits_{1\leq i\leq N} k_i}{p_M^+ N^{p_M^+-1}} - \beta\varepsilon C_2'\right)\|u\|_X^{P_M^+-1} - \frac{|\lambda|C}{r^-}\max\left\{\|u\|_X^{r^+-1}, \|u\|_X^{r^--1}\right\}\right. \\ & \left. - \frac{C_1'}{h^-}\|u\|_X^{h^--1} - \beta C(\varepsilon)C_3'\|u\|_X^{\ell^--1}\right]. \end{aligned} \tag{3.34}$$

Take $\varepsilon = \frac{\mathbf{M}_0\gamma \min_{1\le i\le N}\{k_i\}}{2\beta C_2' p_M^+ N^{p_M^+-1}}$ and set

$$\Phi(t) = \frac{\min_{1\le i\le N}\{k_i\}}{2p_M^+ N^{p_M^+-1}} t^{p_M^+-1} - \frac{|\lambda| C}{r^-}\max\left\{t^{r^+-1}, t^{r^--1}\right\} - \frac{C_1'}{h^-}t^{h^--1} - \beta C(\varepsilon)C_3' t^{\ell^--1}.$$

Since $p_M^+ < \min\{r^-, \ell^-\} < h^-$, we see that there exists $\rho > 0$ such that $\max_{t\ge 0}\Phi(t) = \Phi(\rho)$. Then, by (3.34), there exists $\rho > 0$ such that $E_{\lambda,\beta}(u) \ge \mathscr{R} > 0$ as $\|u\|_X = \varrho$.

This yields the existence of an element $\bar{u}$ of X such that $E_{\lambda,\beta}(\bar{u}) < 0$. Consequently, the critical value is $C_{\lambda,\beta} := \inf_{\phi\in\Gamma}\max_{t\in[0,1]} E_{\lambda,\beta}(\phi(t))$, where $\Gamma = \{\phi \in C([0,1], X) : \phi(0) = 0, \phi(1) = \bar{u}\}$. This concludes the proof. □

Next we will prove under some symmetry condition on the function f that (1.1) possesses infinitely many nontrivial solutions.

Proof. (Proof of Theorem (1.2) We will use a $\mathbb{Z}_2$-symmetric version of the mountain pass theorem 2.7, to accomplish the proof of Theorem 1.2. By assumption $(\boldsymbol{f}_4)$, the function f is even, the functional $E_{\lambda,\beta}$ is even too. Considering the proof of Theorem 1.2, we need only check the condition $(\mathscr{I}_2')$. In fact by using condition $(\boldsymbol{f}_2)$, we have

$$F(x,\xi) \ge C_1|\xi|^{\theta_\lambda} - C_2, \text{ for any } (x,\xi) \in \Omega \times \mathbb{R}.$$

Then, by (3.31), there exist constants $C_5', C_6' > 0$ such that

$$\begin{aligned} E_{\lambda,\beta}(u) \le\ & c\left(\int_\Omega \sum_{i=1}^N c_{a_i}\left(|g_i(x)||\partial_{x_i}u| + \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)}\right) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}dx\right)^{\frac{1}{\gamma}} + \frac{\lambda}{\hat{r}}\int_\Omega |u|^{r(x)}\,\mathrm{d}x \\ & - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - C_5'\|u\|^{\theta_\lambda}_{L^{\theta_\lambda}(\Omega)} - C_6', \end{aligned}$$

where $\hat{r} = r^-$ if $\lambda > 0$ and $\hat{r} = r^+$ if $\lambda \le 0$. On the other hand, the functions b and $g_i, i = 1, 2, \dots, N \in L^\infty(\Omega)$ and $\partial_{x_i}u \in L^{p_i}(\Omega) \hookrightarrow L^1(\Omega)$, we deduce that $\int_\Omega |\partial_{x_i}u|\,\mathrm{d}x \le \int_\Omega |\partial_{x_i}u|^{p_i(x)}\mathrm{d}x$ for all $i = 1, 2, \dots, N$, and by using the anisotropic Sobolev embdding, the Poincaré inequality (2.7), and the following inequality

$$\sum_{i=1}^N \|\partial_{x_i}u\|^{p_M^+}_{L^{p_i(x)}(\Omega)} \le C\left(\sum_{i=1}^N \|\partial_{x_i}u\|_{L^{p_i(x)}(\Omega)}\right)^{p_M^+},$$

where C is a positive constant, we obtain in the case $\lambda > 0$ that

$$E_{\lambda,\beta}(u) \le C'\|u\|_X^{p_M^+/\gamma} + C_4\|u\|_X^{r^+} - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - C_5'\|u\|_X^{\theta_\lambda} - C_6'.$$

Let $u \in X$ be arbitrary but fixed. We denote by

$$U = \{x \in \Omega : |u(x)| < 1\}, \text{and } V = \Omega \setminus U.$$

Then, we have

$$\begin{aligned} E_{\lambda,\beta}(u) &\le C'\|u\|_X^{p_M^+/\gamma} + C_4\|u\|_X^{r^+} - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x - C_5'\|u\|_X^{\theta_\lambda} - C_6 \\ &\le C'\|u\|_X^{p_M^+/\gamma} + C_4\|u\|_X^{r^+} - \frac{1}{q^+}\int_V |u|^{q^-}\,\mathrm{d}x - C_2'\|u\|_X^{\theta_\lambda} \\ &\le C'\|u\|_X^{p_M^+/\gamma} + C_4\|u\|_X^{r^+} - \frac{1}{q^+}\int_\Omega |u|^{q^-}\,\mathrm{d}x + \frac{1}{q^+}\int_U |u|^{q^-}\,\mathrm{d}x - C_5'\|u\|_X^{\theta_\lambda}. \end{aligned}$$

But there exists positive constants C_7 such that

$$\frac{1}{q^+}\int_U |u|^{q^-}\,\mathrm{d}x \le C_7.$$

Thus, we have

$$E_{\lambda,\beta}(u) \le C'\|u\|_X^{p_M^+/\gamma} + C_4\|u\|_X^{r^+} - \frac{1}{q^+}\int_\Omega |u|^{q^-}\,\mathrm{d}x - C_5'\|u\|_X^{\theta_\lambda} + C_9$$

for all $u \in X$. The functional $|.|_{q^-} : X \to \mathbb{R}$ defined by

$$|u|_{q^-} := \left(\int_\Omega |u|^{q^-}\,dx\right)^{1/q^-}$$

is a norm on X. Let X_1 be a fixed finite dimensional subspace of X. Then, $|.|_{q^-}$ and $\|.\|_X$ are equivalent norms, so there exists a positive constant $C_q = C(X_1)$ such that

$$\|u\|_X^{q^-} \le C_q\,|u|_{q^-}^{q^-}\text{, for all }\ u \in X_1.$$

Consequently, we have that there exist positive constants C_q' and C_t' such that

$$0 \le E_{\lambda,\beta}(u) \le C'\|u\|_X^{p_M^+} + C_4\|u\|_X^{r^+} - C_q'\,\|u\|_X^{q^-} - C_5'\|u\|_X^{\theta_\lambda} + C_9$$

since $\theta_\lambda > r^+$ and $q^- > p_M^+/\gamma$, we conclude that $\mathcal{S}_1$ is bounded in X.

Thus, by Theorem 2.7, $E_{\lambda,\beta}$ has an unbounded sequence of critical values and consequently problem (1.1) has infinitely many weak solutions in X. The proof of Theorem 1.2 is complete. □

4 | SOME EXAMPLES

In this section, we will focus on studying a particular class of equations that have both mathematical significance and practical applications in fields such as physics and other related areas. These equations belong to the general class of equations that we have studied in this paper, but with appropriate assumptions on the functions a_i.

Example 4.1. Considering $a_i(x,\xi) := |\xi|^{p_i(x)-2}\xi$ for all $i \in \{1,\dots,N\}$, we have that $A_i(x,\xi) = \frac{1}{p_i(x)}|\xi|^{p_i(x)}$ and a_i satisfies the $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$ and $(\boldsymbol{A}_3)$ for all $i \in \{1,\dots,N\}$. Hence, Equation (1.1) has become

$$\begin{gathered}-M\left(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}|\partial_{x_i}u|^{p_i(x)} + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\right)\left(\Delta_{\vec{p}(x)}(u) - b(x)|u|^{p_M(x)-2}u\right)\\ +\lambda|u|^{r(x)-2}u = |u|^{q(x)-2}u + \beta f(x,u)\ \text{in}\ \Omega,\\ u = 0\ \text{on}\ \ \partial\Omega.\end{gathered} \tag{4.1}$$

The operator $\Delta_{\vec{p}(x)}(u) := \sum_{i=1}^N \partial_{x_i}\left(\left|\partial_{x_i}u\right|^{p_i(x)-2}\partial_{x_i}u\right)$ is so-called $\vec{p}(x)$-Laplacian operator, when $p_i(x) = p(x)$ for any $i = 1,2,\dots,N$, the operator $\Delta_{\vec{p}(x)}u$ is the $p(x)$-Laplacian operator, that is, $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which coincides with the usual p-Laplacian when $p(x) = p$, and with the Laplacian when $p(x) = 2$.

Example 4.2. Considering $a_i(x,\xi) := (1+|\xi|^2)^{\frac{p_i(x)-2}{2}}\xi$ for all $i \in \{1,\dots,N\}$, we have that $A_i(x,\xi) = \frac{1}{p_i(x)}\left((1+|\xi|^2)^{\frac{p_i(x)}{2}} - 1\right)$ and a_i satisfies the $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$ and $(\boldsymbol{A}_3)$ for all $i \in \{1,\dots,N\}$. Hence, Equation (1.1) has become

$$\begin{gathered}-M\left(\int_\Omega \sum_{i=1}^N \frac{1}{p_i(x)}\left((1+|\partial_{x_i}u|^2)^{\frac{p_i(x)}{2}} - 1\right) + \frac{b(x)}{p_M(x)}|u|^{p_M(x)}\mathrm{d}x\right)\left(\sum_{i=1}^N \partial_{x_i}\left(\left(1+|\partial_{x_i}u|^2\right)^{(p_i(x)-2)/2}\partial_{x_i}u\right)\right.\\ \left. - b(x)|u|^{p_M(x)-2}u\right) + \lambda|u|^{r(x)-2}u = |u|^{q(x)-2}u + \beta f(x,u)\ \text{in}\,\Omega,\\ u = 0\ \text{on}\,\partial\Omega.\end{gathered} \tag{4.2}$$

The operator $\sum_{i=1}^{N} \partial_{x_i} \left(\left(1 + |\partial_{x_i} u|^2\right)^{(p_i(x)-2)/2} \partial_{x_i} u \right)$ is so-called the anisotropic variable mean curvature operator.

Let a and b be positive constants. Set $M(\xi) = a + b\xi$ in the previous two examples. Then, $M(\xi) \geq a$ and

$$\widehat{M}(\xi) = \int_0^1 M(s)ds = a\xi + \frac{1}{2}b\xi^2 \geq \frac{1}{2}(a + b\xi)\xi = \gamma M(\xi)\xi,$$

where $\gamma = 1/2$. Hence, the conditions $(\boldsymbol{M}_1)$ and $(\boldsymbol{M}_2)$ are satisfied. For this case, a typical example of a function satisfying the conditions $(\boldsymbol{f}_1)$–$(\boldsymbol{f}_3)$ is given by

$$f(x, \xi) = \sum_{i=1}^{k} d_i(x)|\xi|^{\ell_i(x)-2}\xi,$$

where $k \geq 1$, $p_M^+ < \ell_i(x) < p_m^{*-}$ and $d_i(x) \in C(\overline{\Omega})$.

Assumptions $(\boldsymbol{A}_1)$, $(\boldsymbol{A}_2)$, and $(\boldsymbol{A}_3)$ are satisfied by the anisotropic divergence operators presented in Examples 4.1 and 4.2. As a consequence of Theorems 1.1 and 1.2, we obtain the following corollary.

Corollary 4.3. *Assume that assumptions $(\boldsymbol{f}_1)$–$(\boldsymbol{f}_3)$ hold. Then, for any $\lambda \in \mathbb{R}$ and $\beta > 0$, problems (4.1) and (4.2) have at least a nontrivial weak solution.*

Moreover, if also assumption $(\boldsymbol{f}_4)$ hold, then, for any $\lambda \in \mathbb{R}$ and $\beta > 0$, problems (4.1) and (4.2) have infinitely many weak solutions.

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CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interest.

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