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Properties and applications of the Apéry set of good semigroups in $\mathbb{N}^d$

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Abstract
In this article, we discuss some applications of the construction of the Apéry set of a good semigroup in $\mathbb{N}^d$ given in (Commun Algebra 49(10):4136–4158, 2021). In particular, we study the duality of a symmetric and almost symmetric good semigroup, the Apéry set of non-local good semigroups and the Apéry set of value semigroups of plane curves.

Keywords Good semigroups · Apéry set · Symmetric and almost symmetric semigroups · Plane curves

Mathematics Subject Classification 20M10 · 20M14 · 20M25

1 Introduction

In this article, we discuss several applications of the construction of the Apéry set of a good semigroup given in [18]. Good semigroups form a family of submonoids of $\mathbb{N}^d$ defined axiomatically in [3], in order to study Noetherian analytically unramified one-dimensional semilocal reduced rings, e.g., the local rings arising from curve singularities and their blowups. Indeed, the family of good semigroups contains all value semigroups of such algebroid curves. The concept of value semigroup has been already known long time before the definition of good semigroup was given, and, often in the literature, properties of algebroid curves and of the corresponding rings have

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been translated and studied at semigroup level [6, 8–11, 15, 16, 19–21]. For instance, it is well known that a one-dimensional analytically unramified local ring is Gorenstein if and only if its value semigroup is symmetric. The integer d represents the number of branches of the curve. For $d = 1$, good semigroups coincide with numerical semigroups and have been extensively studied in connection with many subjects including algebraic geometry, commutative algebra, factorization theory, coding theory [2, 25]. More recently, good semigroups in $\mathbb{N}^d$ with $d \geq 2$ have been studied, still in connection with the geometric and algebraic theory of curve singularities, but also with the purpose of extending pure combinatoric properties of numerical semigroups to this more general setting.

The concept of Apéry set, a classical notion in the theory of numerical semigroups, has been extended to the "good" case, first in [5] for value semigroups of plane curves with two branches then for arbitrary good semigroups in $\mathbb{N}^2$ in [12], and for any good semigroup and any d in [18].

This notion has been a fundamental tool to generalize various features of the numerical setting, obtaining new characterization of classes of good semigroups, such as symmetric and almost symmetric, and studying important invariants, such as type, embedding dimension, genus [12, 13, 22, 23].

Unfortunately, for non-numerical good semigroups, the Apéry set is an infinite set, but it can be partitioned canonically in a finite number of subsets, called levels. Properties of such levels reflect the behavior of the semigroup and have particularly nice applications.

In this paper, we consider some of these various applications, with the idea of both extending results from the case $d \leq 2$ to arbitrary d and also to cover complementary results which have not considered in the previous papers. Usually, instead of considering only the Apéry sets, when possible we prove results for complements of good ideals (see definitions in Sect. 2), since this approach is more general and it is needed in some of the cases we consider.

After a preliminary section (Sect. 2), in which we recall all the main definitions and results we are going to use, we consider in Sect. 3 the duality property of Apéry sets of symmetric and almost symmetric good semigroups. Such semigroups are interesting since when they are value semigroups of algebroid curves, they correspond to those algebroid curves having, respectively, Gorenstein and almost Gorenstein ring (for references on the almost Gorenstein case see [3, 7]). We show how duality properties on the levels of the Apéry set characterize these classes extending to any $d \geq 2$ the results obtained in [12, 13] for $d = 2$ (and more classical results for $d = 1$).

In Sect. 4, we consider the complement of good ideals (and Apéry sets) of non-local good semigroups (when they are value semigroups they correspond to non-local rings of algebroid curves). In this case, a more precise description of the partition in levels can be obtained by splitting the semigroup as direct product of smaller good semigroups. This description will be also useful in Sect. 6.

In Sect. 5, we study a class of complements of good ideals that we call *well-behaved* and that includes the Apéry sets of plane curves. Also in this case, we can give a better description of the structure and prove some more interesting results related to the content of the last section.

Finally, in Sect. 6 we prove a result about the Apéry set of a plane curve and its blowup, which generalizes and reinterprets a result in [5]. To motivate this to the reader, we present a more detailed overview on the content of this result and on its historical background.

Let $\mathcal{O} = \mathbb{K}[[X, Y]]/(F_1 \dots F_d)$, with F_i irreducible polynomials, be the ring of a plane algebroid curve. Let $S = \mathrm{v}(\mathcal{O})$ be its value semigroup. The minimal nonzero element $\boldsymbol{e}$ of S is called *multiplicity* and it is an invariant related to the multiplicity of the ring $\mathcal{O}$. A fundamental invariant involved in the study of the equivalence classes of algebroid curves is the sequence of multiplicities of the successive blowups of the ring $\mathcal{O}$. Two algebroid plane curves are formally equivalent if they have the same value semigroup [26], and it is well known that two plane algebroid branches (i.e., plane curves in the case $d = 1$) have the same value semigroup if and only if they have the same multiplicity sequence [27]. Hence, the problem of classification of plane curves can be considered equivalently in a semigroup setting.

In [1], Apéry showed that the numerical semigroups of a plane branch and of its blowup can be determined by studying their respective Apéry sets (see also [4]). Denote by $\mathcal{O}'$ the blowup of $\mathcal{O}$. Apéry precisely proved that if $A = \{\omega_1, \dots, \omega_e\}$ is the Apéry set of $S = \mathrm{v}(\mathcal{O})$, then $A' = \{\omega_1, \omega_2 - e, \omega_3 - 2e, , \dots, \omega_e - (e-1)e\}$ is the Apéry set of $S' := \mathrm{v}(\mathcal{O}')$ with respect to the same element e. This theorem does not hold in the case of arbitrary numerical semigroups not associated with plane branches.

For $d > 1$, in [3], it is shown how to associate a multiplicity tree to a good semigroup. The multiplicity tree is a tree where the vertices are the multiplicities of the value semigroups of iterated blowups of $\mathcal{O}$ and edges represent consecutive blowups. After a finite number of blowups of $\mathcal{O}$, one gets a semilocal non-local ring that is expressed as direct product of local rings. Until all the blowup rings (and equivalently their value semigroups) are local, the multiplicity tree is a path containing their multiplicity vector. When the blow up gives a non-local ring, the associated value semigroup S' is also non-local and can be written as direct product of local good semigroups. In this point, the multiplicity tree branches out and each branch contains all the multiplicity vectors of the direct components of S' and of their blowups.

In the same paper [3], using the concept of Arf closure and Arf semigroup, the authors observed that any two algebroid curves are formally equivalent if they have the same multiplicity tree.

In [5], generalizing Apéry's theorem, the authors proved that, if $d = 2$, $\mathcal{O}$ and its blowup $\mathcal{O}'$ are both local rings, and A_i and A'_i denote, respectively, the i-th level of the Apéry set of $\mathrm{v}(\mathcal{O})$ and of $\mathrm{v}(\mathcal{O}')$, then $A_i = A'_i + (i-1)\boldsymbol{e}$.

As a consequence, they showed how to determine the multiplicity tree of a good semigroup of a plane algebroid curve with two branches using the levels of the Apéry sets of iterated blowups. Moreover, they showed also how to determine the value semigroup starting from a multiplicity tree. In order to do this, they use a result by García [17], which allows to determine the local value semigroup of a plane curve in $\mathbb{N}^2$ knowing its numerical projections (see [5, Proposition 4.2]).

Our purpose here is to give a proof of Apéry's theorem for value semigroups of plane curves with two branches also in the case S is local and S' is not. The key ingredient of this proof is the description of the Apéry sets of non-local good semigroups, which we provide in Sect. 4. Observe, indeed, that in [5] the Apéry set was defined only in the local case. Our method allows to get the same result on the multiplicity tree without using García's result. Hopefully, the method we use here can be extended also to the case of an arbitrary number of branches d, for which a property analogous to that proved by García is unknown.

2 Preliminaries

In this section, we fix some notations, recall some known results and demonstrate some preliminary results that will be used in the following sections.

We use the symbol $\leq$ to denote the partial ordering in $\mathbb{N}^d$: setting $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d)$, then $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, d\}$.

Through this paper, if not differently specified, when referring to minimal or maximal elements of a subset of $\mathbb{N}^d$, we refer to minimal or maximal elements with respect to $\leq$.

The element $\boldsymbol{\delta}$ such that $\delta_i = \min(\alpha_i, \beta_i)$ for every $i = 1, \ldots, d$ is called the the infimum of the set $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and will be denoted by $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$.

Let S be a submonoid of $(\mathbb{N}^d, +)$. We say that S is a *good semigroup* if

(G1) For every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in S$;
(G2) Given two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S$ such that $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ and $\alpha_i = \beta_i$ for some $i \in \{1, \ldots, d\}$, then there exists $\boldsymbol{\epsilon} \in S$ such that $\epsilon_i > \alpha_i = \beta_i$ and $\epsilon_j \geq \min\{\alpha_j, \beta_j\}$ for each $j \neq i$ (and if $\alpha_j \neq \beta_j$ the equality holds).
(G3) There exists an element $\boldsymbol{c} \in S$ such that $\boldsymbol{c} + \mathbb{N}^d \subseteq S$.

A good semigroup is said to be *local* if $\mathbf{0} = (0, \ldots, 0)$ is its only element with a zero component.

By property (G1), it is always possible to define the element $\boldsymbol{c} := \min\{\boldsymbol{\alpha} \in \mathbb{Z}^d \mid \boldsymbol{\alpha} + \mathbb{N}^d \subseteq S\}$; this element is called *conductor* of S. We set $\boldsymbol{\gamma} := \boldsymbol{c} - \mathbf{1}$.

A subset $E \subseteq \mathbb{N}^d$ is a *relative ideal* of S if $E + S \subseteq E$ and there exists $\boldsymbol{\alpha} \in S$ such that $\boldsymbol{\alpha} + E \subseteq S$. A relative ideal E contained in S is simply called an ideal. An ideal E satisfying properties (G1), (G2) is called a *good ideal* (notice that all ideals satisfy (G3) by definition). The minimal element $\boldsymbol{c}_E$ such that $\boldsymbol{c}_E + \mathbb{N}^d \subseteq E$ is called the *conductor* of E. As for S, we set $\boldsymbol{\gamma}_E := \boldsymbol{c}_E - \mathbf{1}$.

We denote by $\boldsymbol{e} = (e_1, e_2, \ldots, e_d)$ the minimal element of S such that $e_i > 0$ for all $i \in I$. The set $\boldsymbol{e} + S$ is a good ideal of S and its conductor is $\boldsymbol{c} + \boldsymbol{e}$. Similarly for every $\boldsymbol{\omega} \in S$, the principal good ideal $E = \boldsymbol{\omega} + S$ has conductor $\boldsymbol{c}_E = \boldsymbol{c} + \boldsymbol{\omega}$.

We will use through all paper the following notation holding for any arbitrary subset $S \subseteq \mathbb{N}^d$. We denote by I the set of indexes $\{1, \ldots, d\}$. Given $F \subseteq I$, $\boldsymbol{\alpha} \in \mathbb{N}^d$, we set:

$$\Delta_F^S(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ for } i \in F \text{ and } \beta_j > \alpha_j \text{ for } j \notin F\}.$$
$$\widetilde{\Delta}_F^S(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ for } i \in F \text{ and } \beta_j \geq \alpha_j \text{ for } j \notin F\} \setminus \{\boldsymbol{\alpha}\}.$$
$$\Delta_i^S(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in S \mid \beta_i = \alpha_i \text{ and } \beta_j > \alpha_j \text{ for } j \neq i\}.$$
$$\Delta^S(\boldsymbol{\alpha}) = \bigcup_{i=1}^{d} \Delta_i^S(\boldsymbol{\alpha}).$$

In particular, for $S = \mathbb{N}^d$, we set $\Delta_F(\boldsymbol{\alpha}) := \Delta_F^{\mathbb{N}^d}(\boldsymbol{\alpha})$.

In general, given $F \subseteq I$, we denote by $\widehat{F}$ the set $I \setminus F$.

We recall here some general properties concerning good semigroups and complementary sets of a good ideals proved in Section 1 and Section 2 of the paper [18]. We refer to that paper for all the necessary proofs. These properties will be widely used throughout the article; for this reason, we suggest to read these sections of [18], to find there further details and see graphical representations related to these properties.

For any subset $A \subseteq S$, we say that two elements $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$ are *consecutive* in A if whenever $\boldsymbol{\alpha} \leq \boldsymbol{\delta} \leq \boldsymbol{\beta}$ for some $\boldsymbol{\delta} \in A$, then $\boldsymbol{\delta} = \boldsymbol{\alpha}$ or $\boldsymbol{\delta} = \boldsymbol{\beta}$.

Lemma 2.1 *Let $E \subseteq S$ be a proper good ideal. Then,*

1. *Let $\boldsymbol{\alpha} \in S \setminus E$. Assume $\Delta_F^E(\boldsymbol{\alpha}) \neq \emptyset$. As a consequence of property (G1) of the good ideal E, $\widetilde{\Delta}_{\widehat{F}}^E(\boldsymbol{\alpha}) = \emptyset$.*
2. *Let $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\beta} \in \Delta_F^E(\boldsymbol{\alpha})$ and $\boldsymbol{\theta} \in \Delta_G^E(\boldsymbol{\alpha})$. If $F \cup G \subsetneq I$, then $\boldsymbol{\beta} \wedge \boldsymbol{\theta} \in \Delta_{F \cup G}^E(\boldsymbol{\alpha})$, while if $F \cup G = I$, then $\boldsymbol{\beta} \wedge \boldsymbol{\theta} = \boldsymbol{\alpha}$.*
3. *Let $\boldsymbol{\alpha} \in E$ and $\boldsymbol{\beta} \in \Delta_F^E(\boldsymbol{\alpha})$ be consecutive to $\boldsymbol{\alpha}$ in E. Then $\Delta_H^E(\boldsymbol{\alpha}) = \emptyset$ for every $H \supsetneq F$.*
4. *Let $\boldsymbol{\alpha} \in \mathbb{N}^d$. Assume there exists $\boldsymbol{\beta} \in \Delta_F^E(\boldsymbol{\alpha})$ and that $\Delta_H^E(\boldsymbol{\alpha})$ is non-empty for some $H \subsetneq F$. Then, there exists $T \subsetneq F$ such that $T \supseteq (F \setminus H)$ and $\Delta_T^E(\boldsymbol{\alpha}) \neq \emptyset$.*

Let us now recall the definition of complete infimum; these elements are crucially involved in the definition of the Apéry set of a good semigroup $S \subseteq \mathbb{N}^d$ with $d > 2$.

Definition 2.2 Let $S \subseteq \mathbb{N}^d$ be a good semigroup and let $A \subseteq S$ be any subset. We say that $\boldsymbol{\alpha} \in A$ is a *complete infimum* in A if there exist $\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(r)} \in A$, with $r \geq 2$, satisfying the following properties:

1. $\boldsymbol{\beta}^{(j)} \in \Delta_{F_j}^S(\boldsymbol{\alpha})$ for some non-empty set $F_j \subsetneq I$.
2. For every $j \neq k \in \{1, \ldots, r\}$, $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(j)} \wedge \boldsymbol{\beta}^{(k)}$.
3. $\bigcap_{k=1}^r F_k = \emptyset$.

In this case we write $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \mathbin{\widetilde{\curlywedge}} \boldsymbol{\beta}^{(2)} \cdots \mathbin{\widetilde{\curlywedge}} \boldsymbol{\beta}^{(r)}$.

In some proofs, we will need to write an element as complete infimum of elements in specific directions. Let us therefore recall the following proposition, which descends directly by property (G2).

Proposition 2.3 *Let $S \subseteq \mathbb{N}^d$ be a good semigroup, $E \subseteq S$ a good ideal and $\boldsymbol{\alpha} \in E$. Suppose that there exists $\boldsymbol{\beta} \in \Delta_F^E(\boldsymbol{\alpha})$ for some $F \subsetneq I$. Then, there exist $\boldsymbol{\beta}^{(1)}, \ldots, \boldsymbol{\beta}^{(r)}$ with $1 \leq r \leq |F|$, such that*

$$\boldsymbol{\alpha} = \boldsymbol{\beta} \mathbin{\widetilde{\curlywedge}} \boldsymbol{\beta}^{(1)} \mathbin{\widetilde{\curlywedge}} \boldsymbol{\beta}^{(2)} \mathbin{\widetilde{\curlywedge}} \cdots \mathbin{\widetilde{\curlywedge}} \boldsymbol{\beta}^{(r)}.$$

In particular $\boldsymbol{\beta}^{(i)} \in \Delta_{G_i}^E(\boldsymbol{\alpha})$, with $G_i \supseteq \widehat{F}$ and $G_1 \cap G_2 \cap \cdots \cap G_r = \widehat{F}$.

From the proof of [18, Proposition 1.7], it follows also that we can choose each G_i such that $\Delta_H^E(\boldsymbol{\alpha}) = \emptyset$ for every $H \supsetneq G_i$.

Using the definition of complete infimum, we can define a canonical partition of the complementary set A of a given good ideal E.

Let E a good ideal of a good semigroup S and $A = S \backslash E$. Given $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_d)$ in $\mathbb{N}^d$, we say that $\boldsymbol{\alpha} \leq\leq \boldsymbol{\beta}$ if and only if either $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or $\alpha_i < \beta_i$ for every $i \in \{1, \dots, d\}$. In the second case, we say that $\boldsymbol{\beta}$ *dominates* $\boldsymbol{\alpha}$ and use the notation $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$.

Definition 2.4 Define A as above. Set:

$$\begin{aligned}
B^{(1)} &:= \{\boldsymbol{\alpha} \in A : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq\},\\
C^{(1)} &:= \{\boldsymbol{\alpha} \in B^{(1)} : \boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \widetilde{\wedge} \boldsymbol{\beta}^{(2)} \cdots \widetilde{\wedge} \boldsymbol{\beta}^{(r)} \text{ for } 1 < r \leq d \text{ and } \boldsymbol{\beta}^{(k)} \in B^{(1)}\},\\
D^{(1)} &:= B^{(1)} \backslash C^{(1)}.
\end{aligned}$$

For $i > 1$ assume that $D^{(1)}, \dots, D^{(i-1)}$ have been defined and set inductively:

$$\begin{aligned}
B^{(i)} &:= \{\boldsymbol{\alpha} \in A \backslash \left(\bigcup_{j<i} D^{(j)} \right) : \boldsymbol{\alpha} \text{ is maximal with respect to } \leq\leq\},\\
C^{(i)} &:= \{\boldsymbol{\alpha} \in B^{(i)} : \boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \widetilde{\wedge} \boldsymbol{\beta}^{(2)} \cdots \widetilde{\wedge} \boldsymbol{\beta}^{(r)} \text{ for } 1 < r \leq d \text{ and } \boldsymbol{\beta}^{(k)} \in B^{(i)}\},\\
D^{(i)} &:= B^{(i)} \backslash C^{(i)}.
\end{aligned}$$

By construction $D^{(i)} \cap D^{(j)} = \emptyset$, for any $i \neq j$ and, since the set $S \backslash A$ has a conductor, there exists $N \in \mathbb{N}_+$ such that $A = \bigcup_{i=1}^{N} D^{(i)}$. As in [18] we enumerate the sets in this partition in increasing order setting $A_i := D^{(N+1-i)}$. Hence, $A = \bigcup_{i=1}^{N} A_i$. We call the sets A_i the *levels* of A.

Given $\boldsymbol{\omega} \in S$, we can consider the good ideal $E = \boldsymbol{\omega} + S$. In this case its complement $A = S \backslash E = \mathrm{Ap}(S, \boldsymbol{\omega})$ is the Apéry set of S with respect to $\boldsymbol{\omega}$. The main theorem of [18] describes the number of levels of sets of the form $\mathrm{Ap}(S, \boldsymbol{\omega})$. We have:

Theorem 2.5 [18, Theorem 4.4] *Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_d)$ be a nonzero element of a good semigroup S. Then, the number of levels of the Apéry set* $\mathrm{Ap}(S, \boldsymbol{\omega})$ *is equal to* $\sum_{i=1}^{d} \omega_i$.

We recall that, if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$, $\boldsymbol{\alpha} \ll \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \in A_i$, then $\boldsymbol{\beta} \in A_j$ for some $j > i$. Moreover, the last set of the partition is $A_N = \Delta(\boldsymbol{\gamma}_E) = \Delta^S(\boldsymbol{\gamma}_E)$. If S is local, then $A_1 = \{\mathbf{0}\}$.

Several basic properties of the Apéry set and of this partition in levels are listed in the [18, Lemma 2.3].

Now we restate two key theorems, which will be used in many of the subsequent proofs. These are very helpful to control the levels of different elements.

Theorem 2.6 *Let S be a good semigroup, $E \subseteq S$ a good ideal and $A = S \backslash E$. Let $\boldsymbol{\delta} \in S$, $\boldsymbol{\theta} \in \Delta_G^S(\boldsymbol{\delta}) \cap A_h$ and assume $\Delta_{\widehat{G}}^S(\boldsymbol{\delta}) \subseteq A$. Let $\boldsymbol{\beta} \in \Delta_F^S(\boldsymbol{\delta})$ with $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ consecutive and $F \supseteq \widehat{G}$.*

1. *If $\widetilde{\Delta}_F^S(\boldsymbol{\delta}) \subseteq A$, then $\boldsymbol{\beta} \in A_i$ with $i \leq h$;*

2. *If* $\boldsymbol{\delta} \in A$ *and* $\widetilde{\Delta}^S_{\widehat{G}}(\boldsymbol{\delta}) \subseteq A$ *then* $\boldsymbol{\delta} \in A_i$ *with* $i < h$.

Theorem 2.7 *Let* S *be a good semigroup,* $E \subseteq S$ *a good ideal and* $A = S \setminus E$. *Let* $\boldsymbol{\alpha} \in A_i$ *and let* $\boldsymbol{\theta} \in \Delta^S_G(\boldsymbol{\alpha})$ *be consecutive to* $\boldsymbol{\alpha}$. *Assume that* $\widetilde{\Delta}^E_{\widehat{G}}(\boldsymbol{\alpha}) \neq \emptyset$. *Then* $\boldsymbol{\theta} \in A_i$.

Next results have not been proved previously; hence, we include also a proof of them.

Lemma 2.8 *Let* S *be a good semigroup and let* $A = \bigcup_{i=1}^N A_i$ *be the complement of a good ideal* $E \subseteq S$. *Let* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_i$, $\boldsymbol{\delta} \in S$, *and assume* $\boldsymbol{\alpha} < \boldsymbol{\delta} < \boldsymbol{\beta}$. *Then* $\boldsymbol{\delta} \in A_i$.

Proof It suffices to show $\boldsymbol{\delta} \in A$. By way of contradiction suppose $\boldsymbol{\delta} \in E$. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are in the same level and they are comparable, they must share at least a coordinate. Hence say that $\boldsymbol{\beta} \in \Delta^S_H(\boldsymbol{\alpha})$ and $\boldsymbol{\delta} \in \Delta^S_F(\boldsymbol{\alpha})$ with $F \supseteq H$. Since $\boldsymbol{\delta} \in E$, by Lemma 2.1, $\widetilde{\Delta}^S_{\widehat{F}}(\boldsymbol{\alpha}) \subseteq A$. By property (G2), applied to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ following Proposition 2.3, we can write $\boldsymbol{\alpha} = \boldsymbol{\beta} \,\widetilde{\curlywedge}\, \boldsymbol{\beta}^1 \,\widetilde{\curlywedge}\, \ldots \,\widetilde{\curlywedge}\, \boldsymbol{\beta}^r$ with $\boldsymbol{\beta}^j \in \Delta^S_{G_j}(\boldsymbol{\alpha})$ and $\bigcap_{j=1}^r G_j = \widehat{H} \supseteq \widehat{F}$. Moreover, we can assume each $\boldsymbol{\beta}_j$ to be consecutive to $\boldsymbol{\alpha}$ and therefore by Theorem 2.7 applied to $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$, we get $\boldsymbol{\beta}_j \in A_i$. This is a contradiction since also $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_i$. □

Lemma 2.9 *Let* $S \subseteq \mathbb{N}^d$ *be a good semigroup, and let* $A = \bigcup_{i=1}^N A_i$ *be the complement of a good ideal* $E \subseteq S$. *Let* $\boldsymbol{\alpha} \in \mathbb{N}^d$ *and suppose* $\Delta^S(\boldsymbol{\alpha}) \subseteq A$ *and it is non-empty. Then, the minimal elements of* $\Delta^S(\boldsymbol{\alpha})$ *are all in the same level.*

Proof By property (G1), every set of the form $\Delta^S_k(\boldsymbol{\alpha})$ has only one minimal element. Thus, we assume that at least two sets of the form $\Delta^S_k(\boldsymbol{\alpha})$ are non-empty otherwise there is nothing to prove. By possible permuting the indexes, suppose $\Delta^S_1(\boldsymbol{\alpha}), \Delta^S_2(\boldsymbol{\alpha}) \neq \emptyset$ and call $\boldsymbol{\beta}, \boldsymbol{\theta}$ their minimal elements. Thus, $\boldsymbol{\delta} := \boldsymbol{\beta} \wedge \boldsymbol{\theta} \in \Delta^S_{1,2}(\boldsymbol{\alpha})$. Say that $\boldsymbol{\beta} \in \Delta^S_F(\boldsymbol{\delta})$ and $\boldsymbol{\theta} \in \Delta^S_G(\boldsymbol{\delta})$ with $1 \in \widehat{G} \subseteq F$ and $2 \in \widehat{F} \subseteq G$. Suppose that $\boldsymbol{\beta} \in A_i$ and $\boldsymbol{\theta} \in A_h$ with $h \leq i$. To prove that $h = i$, we apply Theorem 2.6 to the pair $\boldsymbol{\theta}, \boldsymbol{\delta}$ to show that $h \geq i$ (observe that by definition $\boldsymbol{\beta}$ is a minimal element in $\Delta^S_F(\boldsymbol{\delta})$). Hence, we need to verify that the assumption of Theorem 2.6 is satisfied and show that $\Delta^S_{\widehat{G}}(\boldsymbol{\delta}) \cup \widetilde{\Delta}^S_F(\boldsymbol{\delta}) \subseteq A$. Pick $\boldsymbol{\eta} \in \Delta^S_{\widehat{G}}(\boldsymbol{\delta}) \cup \widetilde{\Delta}^S_F(\boldsymbol{\delta})$. Then, $\eta_1 = \delta_1 = \alpha_1$ and $\eta_j \geq \delta_j > \alpha_j$ for $j \neq 1, 2$. If $\eta_2 > \delta_2$, then $\boldsymbol{\eta} \in \Delta^S_1(\boldsymbol{\alpha}) \subseteq A$. In particular, since $2 \notin F$, it follows that $\Delta^S_{\widehat{G}}(\boldsymbol{\delta}) \cup \Delta^S_U(\boldsymbol{\delta}) \subseteq \Delta^S_1(\boldsymbol{\alpha}) \subseteq A$ for every $U \supseteq F$ such that $2 \notin U$.

Suppose there exists $\boldsymbol{\eta} \in \Delta^S_U(\boldsymbol{\delta})$ with $F \cup \{2\} \subseteq U$. We show that this will contradict the minimality of $\boldsymbol{\beta}$ in $\Delta^S_1(\boldsymbol{\alpha})$. Applying property (G2) to $\boldsymbol{\eta}$ and $\boldsymbol{\delta}$ as in Proposition 2.3, we find elements in some sets $\Delta^S_{H_1}(\boldsymbol{\delta}), \ldots, \Delta^S_{H_r}(\boldsymbol{\delta})$ such that $H_1 \cap \ldots \cap H_r = \widehat{U}$. In particular, since $2 \notin \widehat{U}$, we can find some element in a set $\Delta^S_H(\boldsymbol{\delta})$ such that $H \cup U = I$ and $2 \notin H$. Hence, $H \nsubseteq F$. It follows that $F \subsetneq (H \cup F) \subseteq I \setminus \{2\}$ and by Lemma 2.1, $\Delta^S_{H \cup F}(\boldsymbol{\delta}) \neq \emptyset$. A minimal element $\boldsymbol{\omega}$ in $\Delta^S_{H \cup F}(\boldsymbol{\delta})$ is now an element of $\Delta^S_1(\boldsymbol{\alpha})$. Pick $j \in H \setminus F$. Observing that $\beta_j > \omega_j = \delta_j$, we contradict the minimality of $\boldsymbol{\beta}$. □

While it follows easily by definition, that if $\boldsymbol{\alpha} \in A_i$, then there exists $\boldsymbol{\theta} \in A_{i+1}$ such that $\boldsymbol{\theta} \geq \boldsymbol{\alpha}$, it is not straightforward to see that there always exists also $\boldsymbol{\beta} \in A_{i-1}$ such that $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$ (if $d = 2$ this is proved in [12, Proposition 4]).

Proposition 2.10 *Let S be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. For $i > 1$, given $\boldsymbol{\alpha} \in A_i$ there exists always some $\boldsymbol{\beta} \in A_{i-1}$, $\boldsymbol{\beta} < \boldsymbol{\alpha}$.*

Proof We can restrict to assume $\boldsymbol{\alpha}$ to be a minimal element in A_i with respect to $\leq$. Looking for a contradiction we suppose that for every $\boldsymbol{\beta} \in A_{i-1}$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \neq \boldsymbol{\beta}$. It is always possible to find a $\boldsymbol{\beta} \in A_{i-1}$ such that $\boldsymbol{\delta} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is maximal. To conclude, we have to show the existence of $\boldsymbol{\theta} \in A_{i-1}$ such that $\boldsymbol{\theta} \wedge \boldsymbol{\alpha} > \boldsymbol{\delta}$. Say that $\boldsymbol{\alpha} \in \Delta_F^S(\boldsymbol{\delta})$ and $\boldsymbol{\beta} \in \Delta_G^S(\boldsymbol{\delta})$ with $G \supseteq \widehat{F}$. We consider two different possible cases:

Case 1 $\Delta_H^E(\boldsymbol{\beta}) \neq \emptyset$ for some $H \nsubseteq F$.

Let $\boldsymbol{\omega} \in \Delta_H^E(\boldsymbol{\beta})$ and apply property (G2) as in Proposition 2.3 to $\boldsymbol{\beta}$ and $\boldsymbol{\omega}$ to find non-empty sets $\Delta_{V_1}^S(\boldsymbol{\beta}), \dots, \Delta_{V_r}^S(\boldsymbol{\beta})$ such that $V_1 \cap \cdots \cap V_r = \widehat{H}$. Since $\widehat{F} \nsubseteq \widehat{H}$, we can find $\boldsymbol{\theta} \in \Delta_V^S(\boldsymbol{\beta})$ with $\widehat{H} \subseteq V$, $\widehat{F} \nsubseteq V$. By Lemma 2.1, $\widetilde{\Delta}_{\widehat{H}}^S(\boldsymbol{\beta}) \subseteq A$. Hence, $\boldsymbol{\theta} \in A$ and we may assume without restrictions $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ to be consecutive. Theorem 2.7 implies $\boldsymbol{\theta} \in A_{i-1}$. By construction $\boldsymbol{\theta} \geq \boldsymbol{\beta}$ and there exists $k \notin F$ such that $\theta_k > \beta_k$. Therefore, $\boldsymbol{\theta} \wedge \boldsymbol{\alpha} > \boldsymbol{\delta}$.

Case 2 $\Delta_H^E(\boldsymbol{\beta}) = \emptyset$ for every $H \nsubseteq F$.

Notice that this hypothesis implies $\widetilde{\Delta}_{\widehat{F}}^S(\boldsymbol{\beta}) \subseteq A$. Pick $\boldsymbol{\eta}$ such that $\boldsymbol{\delta} \leq \boldsymbol{\eta} < \boldsymbol{\beta}$ and $\boldsymbol{\eta}, \boldsymbol{\beta}$ and are consecutive. Say that $\boldsymbol{\beta} \in \Delta_U^S(\boldsymbol{\eta})$ with $U \supseteq \widehat{F}$. Apply again property (G2) using Proposition 2.3 to $\boldsymbol{\eta}$ and $\boldsymbol{\beta}$ to find an element $\boldsymbol{\theta} \in \Delta_V^S(\boldsymbol{\eta})$ with $V \supseteq \widehat{U}$ and $\widehat{F} \nsubseteq V$. Following Proposition 2.3, we can also assume $\Delta_H^S(\boldsymbol{\eta}) = \emptyset$ for every $H \supsetneq V$ and therefore that $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ are consecutive. Since $\widehat{V} \nsubseteq F$, we get $\widetilde{\Delta}_{\widehat{V}}^S(\boldsymbol{\beta}) \subseteq A$. We claim that this implies also $\widetilde{\Delta}_{\widehat{V}}^S(\boldsymbol{\eta}) \subseteq A$. Indeed if $\boldsymbol{\tau} \in \widetilde{\Delta}_{\widehat{V}}^E(\boldsymbol{\eta})$, we would have $\boldsymbol{\tau} \wedge \boldsymbol{\beta} \in \widetilde{\Delta}_{\widehat{V} \cup U}^E(\boldsymbol{\eta}) = \widetilde{\Delta}_U^E(\boldsymbol{\eta})$. Since $\boldsymbol{\beta}$ and $\boldsymbol{\eta}$ are consecutive, $\boldsymbol{\tau} \wedge \boldsymbol{\beta} = \boldsymbol{\beta}$ and hence $\boldsymbol{\tau} \in \widetilde{\Delta}_{\widehat{V}}^E(\boldsymbol{\beta})$, a contradiction.

Now, if $\boldsymbol{\eta} \in A_{i-1}$ we can just replace $\boldsymbol{\beta}$ by $\boldsymbol{\eta}$ and iterate, possibly using again Case 1 (notice that also $\boldsymbol{\alpha} \wedge \boldsymbol{\eta} = \boldsymbol{\delta}$). Hence, suppose $\boldsymbol{\eta} \in A_{i-2} \cup E$. If $\boldsymbol{\eta} \in A_{i-2}$, since $\boldsymbol{\eta}, \boldsymbol{\beta}$ are consecutive, by Theorem 2.7 we must have $\widetilde{\Delta}_{\widehat{U}}^S(\boldsymbol{\eta}) \subseteq A$. If instead $\boldsymbol{\eta} \in E$, using that $\widetilde{\Delta}_{\widehat{V}}^S(\boldsymbol{\eta}) \cup \widetilde{\Delta}_U^S(\boldsymbol{\eta}) \subseteq A$, by [18, Proposition 1.4], we get $\Delta_{\widehat{U}}^S(\boldsymbol{\eta}) \cup \Delta_V^S(\boldsymbol{\eta}) \subseteq A$. Since $\Delta_H^S(\boldsymbol{\eta}) = \emptyset$ for every $H \supsetneq V$, it follows $\widetilde{\Delta}_V^S(\boldsymbol{\eta}) \subseteq A$. In both cases $\boldsymbol{\theta} \in A$ and the assumptions of Theorem 2.6 are satisfied both if we apply it to $\boldsymbol{\theta}, \boldsymbol{\eta}$ or to $\boldsymbol{\beta}, \boldsymbol{\eta}$. This implies that $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ are in the same level. Finally, observe that $\theta_j \geq \eta_j \geq \delta_j$ and, by the choice of V, there exists $k \notin F$ such that $\theta_k > \beta_k = \delta_k$. Thus, we can conclude as in Case 1. □

3 Duality of the Apéry sets of symmetric and almost symmetric good semigroups

In this section, we extend to good semigroups in $\mathbb{N}^d$ the results on the duality of levels of Apéry set of symmetric and almost symmetric good semigroups, proved in the case $d = 2$ in [12, Section 5] and [13, Section 5].

Let S be a good semigroup. Recall that an element $\boldsymbol{\alpha} \in S$ is *absolute* (sometimes also called maximal) if $\Delta^S(\boldsymbol{\alpha}) = \emptyset$. Then, S is symmetric if and only if for every

$\boldsymbol{\alpha} \in \mathbb{Z}^d$, $\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$. In a symmetric good semigroup, the absolute elements are dual with respect to $\boldsymbol{\gamma}$ in the sense that if $\boldsymbol{\alpha}$ is absolute then also $\boldsymbol{\gamma} - \boldsymbol{\alpha} \in S$ and it is absolute.

The set of pseudo-Frobenius element of S is the set $\mathrm{PF}(S)$ of elements $\boldsymbol{\alpha} \in \mathbb{N}^d \setminus S$ such that $\boldsymbol{\alpha} + \boldsymbol{\beta} \in S$ for every nonzero element $\boldsymbol{\beta} \in S$. A good semigroup S is almost symmetric if and only if

$$\mathrm{PF}(S) = \Delta(\boldsymbol{\gamma}) \cup \{\boldsymbol{\alpha} \in (\mathbb{N}^2 \setminus S) \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}.$$

For an overview on properties of symmetric and almost symmetric good semigroup in connection with the Apéry set, we refer to [12, Section 4] and [13, Section 4].

In the case $d = 1$, it is well known that symmetric and almost symmetric numerical semigroups are characterized by duality properties on the elements of Apéry set, or on the pseudo-Frobenius elements, with respect to the largest element in the set, see [25, Proposition 4.10] and [24, Theorem 2.4].

In the case of symmetric and almost symmetric good semigroups some correspondent, but less intuitive, duality relations do exist for the levels of the partition of $\mathrm{Ap}(S)$. In general if $A = \bigcup_{i=1}^{N} A_i$ is the complement of a good ideal E, for $\boldsymbol{\omega} \in \mathbb{N}^2$, define $\boldsymbol{\omega}' := \boldsymbol{\gamma}_E - \boldsymbol{\omega}$. For each level A_i, set

$$A_i' := \left(\bigcup_{\boldsymbol{\omega} \in A_i} \Delta^S(\boldsymbol{\omega}') \right) \setminus \left(\bigcup_{\boldsymbol{\omega} \in A_j, j<i} \Delta^S(\boldsymbol{\omega}') \right).$$

We want to prove the following theorem for arbitrary $d \geq 2$. For a good semigroup S having Apéry set $\mathrm{Ap}(S) = \bigcup_{i=1}^{e} A_i$, define

$$Z := \mathrm{PF}(S) \cup \{\mathbf{0}\}, \quad W := \{\mathbf{0}\} \cup \Delta(\boldsymbol{\gamma} + \boldsymbol{e}) \cup \{\boldsymbol{\alpha} \in \bigcup_{i=2}^{e-1} A_i \mid \boldsymbol{\alpha} - \boldsymbol{e} \notin \mathrm{PF}(S)\}.$$

It can be shown that the sets Z and W are complement of good ideals of some opportune semigroup, exactly as in [13, Proposition 5.3]. Hence, we can write their partitions $Z = \bigcup_{h=1}^{n} Z_h$ and $W = \bigcup_{i=1}^{m} W_i$. Then,

Theorem 3.1 *Let $S \subseteq \mathbb{N}^d$ be a good semigroup and let $A = \mathrm{Ap}(S)$. Then:*

- *S is symmetric if and only if $A_i' = A_{e-i+1}$ for every $i = 1, \ldots, e$.*
- *S is almost symmetric if and only if $Z_h' = Z_{n-h+1}$ for every $h = 1, \ldots, n$ and $W_i' = W_{m-i+1}$ for every $i = 1, \ldots, m$.*

Proof This result can be proved exactly with the same method used in [12, Theorem 9] and [13, Theorem 5.6] after proving the next general result that we state as Theorem 3.4. □

Example 3.2 Let us consider the good semigroup $S \subseteq \mathbb{N}^3$, having elements $\ll \boldsymbol{c} + \mathbf{1}$ equal to

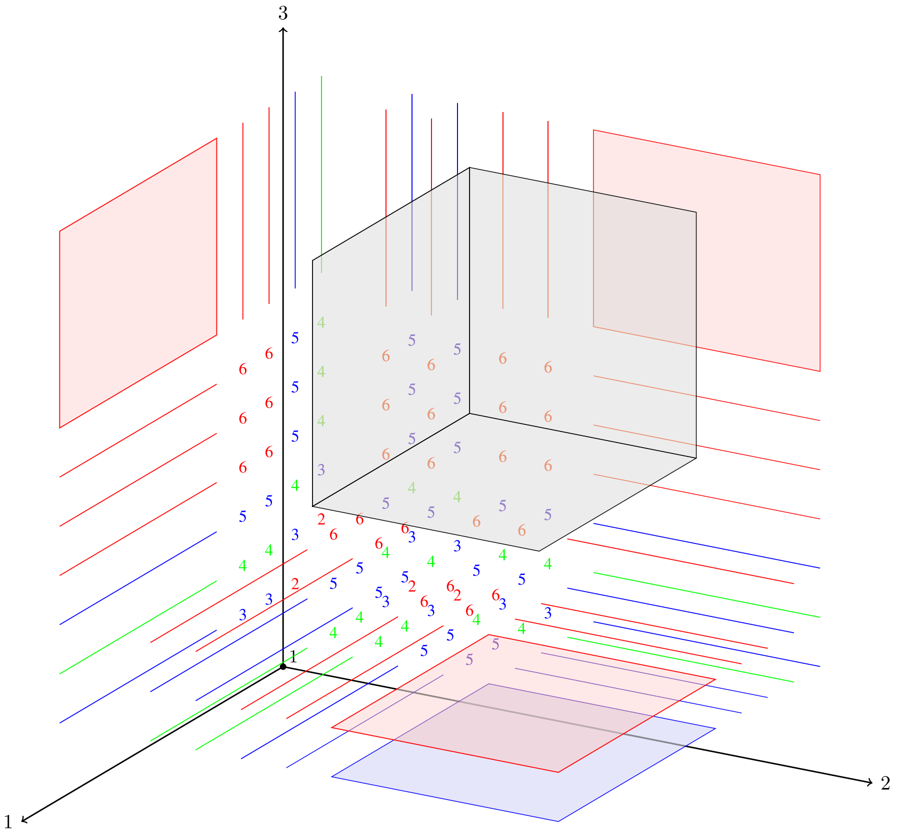


Fig. 1 The representation of the Apéry set of the good semigroup S. Levels 1–7 are black; levels 2–6 are red; levels 3–5 are blue; level 4 is green (Color figure online)

$$\begin{aligned}\text{Small}(S) =\{&(2,2,3),(2,2,4),(2,2,5),(2,2,6),(2,4,3),(2,4,4),(2,4,5),\\&(2,4,6),(2,5,5),(2,5,6),(2,6,3),(2,6,4),(2,6,5),(2,6,6),\\&(3,2,3),(3,2,4),(3,4,3),(3,4,4),(3,4,5),(3,4,6),(3,5,3),\\&(3,5,4),(3,5,5),(3,5,6),(3,6,5),(4,2,3),(4,2,4),(4,2,5),\\&(4,2,6),(4,4,3),(4,4,4),(4,4,5),(4,5,4),(4,5,5),(4,6,3),\\&(4,6,4),(4,6,6)\}\end{aligned}$$

Using the procedure described in [23, Proposition 1.6], it is possible to see that the *length* and *genus* of the good semigroup are both equal to the half of the sum of the components of the conductor. This implies that S is a symmetric good semigroup (see [16, Theorem 2.3]). In Fig. 1 is represented the Apéry set of S. The levels that correspond to each other following the duality relation of Theorem 3.1 are represented with the same color.

We give a general definition.

Definition 3.3 Let S be a good semigroup (not necessarily local) and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. We say that A is a *symmetric complement* of E if:

- $A_1 = \{\mathbf{0}\}$.
- $\boldsymbol{\alpha} \in E$ if and only if $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha}) = \emptyset$.
- $\boldsymbol{\alpha} \in A$ if and only if $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha}) \subseteq A$ and it is non-empty.

As a consequence of [12, Section 4] and [13, Section 4], it follows that the Apéry set of a symmetric good semigroups and the sets Z and W in the case of an almost symmetric good semigroups are symmetric complements. This is proved for $d = 2$, but the proof does not depend on the number d; hence, it follows by the same argument for any d. Hence to prove Theorem 3.1 it is sufficient to prove next theorem.

Theorem 3.4 *Let S be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. Suppose A is a symmetric complement. Then, $A'_i = A_{N-i+1}$ for every $i = 1, \dots, N$.*

Proof This proof can be done with the same exact method used for $d = 2$ in [12, Theorem 9], after proving the results in Lemma 3.7 and Proposition 3.8. Such results generalize [12, Lemma 4 and 5] to the case of arbitrary $d \geq 2$. □

Hence, we dedicate the remaining part of this section to prove Lemma 3.7 and Proposition 3.8. First, we need some preliminary technical result.

Lemma 3.5 *Let $S \subseteq \mathbb{N}^d$ be a good semigroup, and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subsetneq S$. Let $\boldsymbol{\alpha} \in A_i$ for $i < N$. For every $k = 1, \dots, d$ there exists $\boldsymbol{\beta}^{(k)} \in A_{i+1}$ such that $\boldsymbol{\beta}^{(k)} \geq \boldsymbol{\alpha}$ and $\beta_k^{(k)} > \alpha_k$.*

Proof If there exists $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$ we are done. Hence, suppose this is not the case.

It follows that $\boldsymbol{\alpha}$ must be a complete infimum of elements in A that are either in A_i or in A_{i+1}, but not all in A_i. By definition of complete infimum, we can find some element $\boldsymbol{\theta} \in A_i \cup A_{i+1}$ such that $\boldsymbol{\theta} \geq \boldsymbol{\alpha}$ and $\theta_k > \alpha_k$. If $\boldsymbol{\theta} \in A_{i+1}$ we are done. Assume then $\boldsymbol{\theta} \in A_i$. Now, if $\boldsymbol{\theta}$ was dominated by some element of A_{i+1}, the same element would dominate also $\boldsymbol{\alpha}$. Thus also $\boldsymbol{\theta}$ is a complete infimum of elements in $A_i \cup A_{i+1}$. Since at least one of such elements is in A_{i+1}, we conclude by choosing that element. □

Lemma 3.6 *Let S be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. Suppose that A is a symmetric complement, $\boldsymbol{\beta} \in A$ and $\widetilde{\Delta}_G^E(\boldsymbol{\beta}) = \emptyset$. Then, there exists $k \in G$ such that $\Delta_k^S(\boldsymbol{\gamma}_E - \boldsymbol{\beta}) \neq \emptyset$.*

Proof Set $\boldsymbol{\beta}' := \boldsymbol{\gamma}_E - \boldsymbol{\beta}$. For every $\boldsymbol{\delta} \in \mathbb{Z}^d$ such that $\boldsymbol{\beta}' \in \widetilde{\Delta}_G(\boldsymbol{\delta})$, then $\boldsymbol{\delta}' = \boldsymbol{\gamma}_E - \boldsymbol{\delta} \in \widetilde{\Delta}_G(\boldsymbol{\beta})$. Hence, by definition of symmetric complement, $\Delta^S(\boldsymbol{\delta}) \neq \emptyset$ for any of such $\boldsymbol{\delta}$. Moreover, since $\boldsymbol{\beta} \in A$, then $\Delta^S(\boldsymbol{\beta}') \neq \emptyset$. Hence, the set $U = \{j \in I : \Delta_j^S(\boldsymbol{\beta}') \neq \emptyset\}$ is non-empty. To get the thesis, we have to prove that $U \cap G \neq \emptyset$. By way of contradiction, suppose $U \subseteq \widehat{G}$.

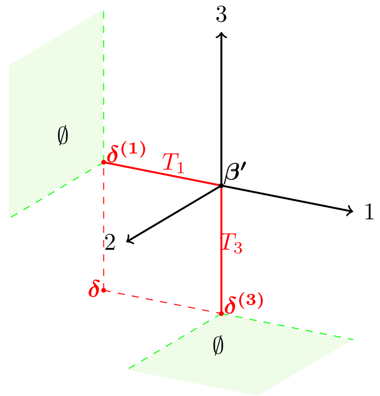


Fig. 2 A graphical representation of costruction explained above. In this picture: $d = 3$, $G = \{2\}$, $U = \{1, 3\}$

For each j, consider the half-line $T_j := \{\boldsymbol{\alpha} \in \mathbb{Z}^d : \boldsymbol{\beta}' \in \Delta^S_{I\setminus\{j\}}(\boldsymbol{\alpha})\}$. For $j \in U$ we notice that, since $G \subseteq I\setminus\{j\}$, each element $\boldsymbol{\delta} \in T_j$ is such that $\Delta^S(\boldsymbol{\delta}) \neq \emptyset$. By construction, the j-th coordinate of the elements of T_j can be arbitrarily small (possibly negative); therefore, there exists a maximal element $\boldsymbol{\delta}^{(j)} \in T_j$ such that $\Delta^S_j(\boldsymbol{\delta}^{(j)}) = \emptyset$. Set $\boldsymbol{\delta} := \bigwedge_{j\in U} \boldsymbol{\delta}^{(j)}$. It can be easily seen that $\boldsymbol{\beta}' \in \Delta_{\widehat{U}}(\boldsymbol{\delta})$. Thus, $\Delta^S(\boldsymbol{\delta}) \neq \emptyset$, since $\widehat{U} \supseteq G$. (see Fig. 2 for a graphical representation in case $d = 3$)

We divide now the proof in three parts, proving the following claims:

1. $\Delta^S_j(\boldsymbol{\delta}) = \emptyset$ for every $j \in U$.
2. Given $\boldsymbol{\theta} \in \Delta_{\widehat{U}}(\boldsymbol{\delta})$ such that $\boldsymbol{\theta} \leq \boldsymbol{\beta}'$, then $\Delta^S_{\{k\}\cup H}(\boldsymbol{\theta}) = \emptyset$ for every $k \notin U$, $H \subseteq U$ (including $H = \emptyset$, but excluding the case $\{k\} \cup H = I$).
3. Conditions 1. and 2. together yield a contradiction.

Let us prove claim 1. If $|U| = 1$, then $\boldsymbol{\delta} = \boldsymbol{\delta}^{(j)}$ and the claim follows by definition of $\boldsymbol{\delta}^{(j)}$. Otherwise, assume by way of contradiction there exists $\boldsymbol{\omega} \in \Delta^S_j(\boldsymbol{\delta})$. Then, $\omega_j = \delta_j = \delta^{(j)}_j < \beta'_j$ and $\omega_l > \delta_l = \delta^{(j)}_l = \beta'_l$ for $l \notin U$. Also we can suppose $\boldsymbol{\omega}$ to be minimal in $\Delta^S_j(\boldsymbol{\delta})$ with respect to the coordinates in $\widehat{U}$. Since $\Delta^S_j(\boldsymbol{\delta}^{(j)}) = \emptyset$, we must have $\omega_i \leq \delta^{(j)}_i = \beta'_i$ for some $i \in U$, $i \neq j$. We want to find an element $\boldsymbol{\theta} \in S$, such that $\boldsymbol{\theta} > \boldsymbol{\omega}$, $\theta_i > \omega_i$, and $\theta_j = \omega_j$. Iterating this process we find eventually an element in $\Delta^S_j(\boldsymbol{\delta}^{(j)})$ and this is impossible. The element $\boldsymbol{\theta}$ is constructed using property (G2) as follows.

Notice that for each $k \in U\setminus\{j\}$ we can find $\boldsymbol{\tau}^{(k)}$ in the line $T_k \cup \{\boldsymbol{\beta}'\} \cup \Delta_{I\setminus\{k\}}(\boldsymbol{\beta}')$ such that $\tau^{(k)}_k = \omega_k$ and $\boldsymbol{\tau}^{(k)} > \boldsymbol{\delta}^{(k)}$. Define $\boldsymbol{\tau} := \bigwedge_{k\in U\setminus\{j\}} \boldsymbol{\tau}^{(k)}$. Observe that $\boldsymbol{\tau} \leq \boldsymbol{\tau}^{(i)} \leq \boldsymbol{\beta}'$ and $\tau_k = \min\{\beta'_k, \omega_k\}$. Hence, $\delta^{(k)}_k < \tau_k \leq \beta'_k$ for every $k \in U\setminus\{j\}$. By definition of $\boldsymbol{\delta}^{(k)}$ and by the assumption that $\Delta^S_k(\boldsymbol{\beta}') \neq \emptyset$ for every $k \in U$, we obtain that for each $k \in U\setminus\{j\}$, $\Delta^S_k(\boldsymbol{\tau}) \neq \emptyset$. Therefore by property (G1), $\Delta^S_{U\setminus\{j\}}(\boldsymbol{\tau}) \neq \emptyset$. Pick $\boldsymbol{\alpha} \in \Delta^S_{U\setminus\{j\}}(\boldsymbol{\tau})$. Since $\tau_k \geq \omega_k$ for each $k \in U$ and $\tau_l = \beta'_l = \delta_l$ for each $l \notin U$, we get that $\boldsymbol{\alpha} \wedge \boldsymbol{\omega} \in \Delta^S_j(\boldsymbol{\delta})$. The choice of the element $\boldsymbol{\omega}$ minimal with respect to the coordinates in $\widehat{U}$ implies that $\boldsymbol{\alpha} > \boldsymbol{\omega}$. Thus $\boldsymbol{\alpha} \in \Delta^S_V(\boldsymbol{\omega})$ with $V \supseteq U\setminus\{j\}$, and $j \notin V$

because $\tau_j = \beta'_j > \delta^{(j)}_j = \omega_j$. Define $\boldsymbol{\theta}$ by applying property (G2) to $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ in such a way to find $\boldsymbol{\theta} \in \Delta^S_H(\boldsymbol{\omega})$, with $j \in \widehat{V} \subseteq H$ and $i \notin H$.

To prove claim 2, observe that such condition is true for $\boldsymbol{\beta}'$ by an iterated application of Lemma 2.1 choosing S as good ideal of itself. Indeed, by the fact that $\Delta^S_j(\boldsymbol{\beta}') \neq \emptyset$ if and only if $j \in U$, we get that $\Delta^S_{\{k,j\}}(\boldsymbol{\beta}') = \emptyset$ for every $k \notin U$, $j \in U$. Iterating, we obtain also $\Delta^S_{\{k\}\cup H}(\boldsymbol{\beta}') = \emptyset$ for $k \notin U$ and $H \subseteq U$. Let now $\boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_d$ be the elements of the canonical basis of $\mathbb{Z}^d$ as $\mathbb{Z}$-module. Starting from $\boldsymbol{\beta}'$, we proceed inductively assuming our claim true for $\boldsymbol{\theta} + \boldsymbol{\epsilon}_l$ with $l \in U$ and proving it for $\boldsymbol{\theta}$.

We only need to show that $\Delta^S_k(\boldsymbol{\theta}) \neq \emptyset$ if and only if $k \in U$, and then apply again Lemma 2.1 in the same way as done for $\boldsymbol{\beta}'$. In the case when $k \notin U$, if there were some $\boldsymbol{\omega} \in \Delta^S_k(\boldsymbol{\theta})$, we would have $\boldsymbol{\omega} \in \Delta^S_{\{k,l\}}(\boldsymbol{\theta}+\boldsymbol{\epsilon}_l) \cup \Delta^S_{\{k\}}(\boldsymbol{\theta}+\boldsymbol{\epsilon}_l)$, which contradicts the inductive hypothesis. For $j \in U$, observe that by construction $\boldsymbol{\theta} = \bigwedge_{j\in U} \boldsymbol{\theta}^{(j)}$ for some $\boldsymbol{\theta}^{(j)} \in T_j \cup \{\boldsymbol{\beta}'\}$ such that $\boldsymbol{\theta}^{(j)} > \boldsymbol{\delta}^{(j)}$ and $\theta^{(j)}_j = \theta_j$. By definition of $\boldsymbol{\delta}^{(j)}$, $\Delta^S_j(\boldsymbol{\theta}^{(j)}) \neq \emptyset$, and therefore also $\Delta^S_j(\boldsymbol{\theta}) \neq \emptyset$.

Finally, we prove claim 3. By 1., since $\Delta^S(\boldsymbol{\delta}) \neq \emptyset$, necessarily there must exist $k \notin U$ such that $\Delta^S_k(\boldsymbol{\delta}) \neq \emptyset$. Pick $\boldsymbol{\alpha} \in \Delta^S_k(\boldsymbol{\delta})$. Let $\boldsymbol{\theta} \in \mathbb{Z}^d$ be defined such that $\theta_j = \beta'_j = \delta_j$ if $j \in \widehat{U}$ and $\theta_j = \alpha_j \wedge \beta'_j$ if $j \in U$. By construction, $\boldsymbol{\theta} \in \Delta_{\widehat{U}}(\boldsymbol{\delta})$ and $\boldsymbol{\theta} \leq \boldsymbol{\beta}'$. Moreover, $\boldsymbol{\alpha} \in \Delta^S_{\{k\}\cup H}(\boldsymbol{\theta})$ for some $H \subseteq U$. If $\{k\} \cup H \neq I$, we find a contradiction by 2. Otherwise, we must have $H = U = I \backslash \{k\}$ and $\boldsymbol{\theta} = \boldsymbol{\alpha} \in S$. But, as observed in the proof of 2., $\Delta^S_j(\boldsymbol{\theta}) \neq \emptyset$ for every $j \in U$, and by property (G1) (Lemma 2.1), $\Delta^S_U(\boldsymbol{\theta}) \neq \emptyset$. By property (G2), $\widetilde{\Delta}^S_{\widehat{U}}(\boldsymbol{\theta}) = \widetilde{\Delta}^S_k(\boldsymbol{\theta}) \neq \emptyset$. Again this contradicts 2. ☐

We are finally ready to prove Lemma 3.7 and Proposition 3.8.

Lemma 3.7 *Let S be a good semigroup and let $A = \bigcup_{i=1}^N A_i$ be the complement of a good ideal $E \subseteq S$. Suppose A to be a symmetric complement. Let $\boldsymbol{\alpha} \in A_i$, then*

$$\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha}) \cap A_j = \emptyset, \text{ for every } j < N - i + 1.$$

Proof We work by decreasing induction on i starting by $i = N$. In the basis case, there is nothing to prove. Assume the thesis to be true for the elements in the level A_{i+1}. Pick $\boldsymbol{\theta} \in \Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha})$ and without loss of generality say that $\boldsymbol{\theta} \in \Delta^S_1(\boldsymbol{\gamma}_E - \boldsymbol{\alpha})$. Since A is a symmetric complement, we can say that $\boldsymbol{\theta} \in A_h$ for some h. By Lemma 3.5, we can find $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\beta} > \boldsymbol{\alpha}$ and $\beta_1 > \alpha_1$. To conclude, it suffices to prove that there exists some element in $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\beta}) \cap A_t$ with $t < h$. Indeed, assuming by way of contradiction $h < N - i + 1$, one would get $t < N - i$ contradicting the inductive hypothesis. Clearly, $\boldsymbol{\gamma}_E - \boldsymbol{\beta} < \boldsymbol{\gamma}_E - \boldsymbol{\alpha}$. Let $\boldsymbol{\delta}$ be a minimal element in $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\beta})$ and suppose $\boldsymbol{\delta} \in \Delta^S_k(\boldsymbol{\gamma}_E - \boldsymbol{\beta})$. It is easy to see that $\boldsymbol{\theta} \gg \boldsymbol{\gamma}_E - \boldsymbol{\beta}$. Hence $\boldsymbol{\delta} \wedge \boldsymbol{\theta} \in \Delta^S_k(\boldsymbol{\gamma}_E - \boldsymbol{\beta})$ and, by minimality of $\boldsymbol{\delta}$, we get $\boldsymbol{\delta} \leq \boldsymbol{\theta}$. If $\boldsymbol{\delta} \ll \boldsymbol{\theta}$, then $\boldsymbol{\delta} \in A_t$ with $t < h$. Otherwise, $\boldsymbol{\theta} \in \Delta^S_H(\boldsymbol{\delta})$ with $k \notin H$. Notice now that

$$\widetilde{\Delta}^S_{\widehat{H}}(\boldsymbol{\delta}) \subseteq \Delta^S_k(\boldsymbol{\gamma}_E - \boldsymbol{\beta}) \subseteq A.$$

By application of Theorem 2.6 to $\boldsymbol{\theta}$ and $\boldsymbol{\delta}$, we get that $\boldsymbol{\delta}$ is in a level strictly smaller than the level of $\boldsymbol{\theta}$. □

Proposition 3.8 *Let S be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. Suppose A to be a symmetric complement. Let $\boldsymbol{\alpha} \in A_i$, then the minimal elements of $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha})$ are in A_{N-i+1}.*

Proof We work by increasing induction on i. If $i = 1$, since $A_1 = \{\mathbf{0}\}$, we have $\Delta^S(\boldsymbol{\gamma}_E - \mathbf{0}) = A_N$ and the thesis is true. Hence, we assume the thesis to be true for the elements in the level A_{i-1} and we prove it for A_i. By Lemma 3.7, $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha}) \cap A_j = \emptyset$ for every $j < N - i + 1$. By Proposition 2.10, there exists $\boldsymbol{\beta} \in A_{i-1}$ such that $\boldsymbol{\beta} < \boldsymbol{\alpha}$. We can also assume that there are no other elements of A_{i-1} strictly between $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$. Say that $\boldsymbol{\alpha} \in \Delta_F^S(\boldsymbol{\beta})$ with possibly $F = \emptyset$ (i.e., $\boldsymbol{\alpha} \gg \boldsymbol{\beta}$). We claim then that $\widetilde{\Delta}_{\widehat{F}}^E(\boldsymbol{\beta}) = \emptyset$. Indeed, if this were not the case, using property (G2) as in Proposition 2.3, we can express $\boldsymbol{\beta}$ as complete infimum of (consecutive) elements in some directions $\Delta_{H_1}^S(\boldsymbol{\beta}), \ldots, \Delta_{H_t}^S(\boldsymbol{\beta})$ with $\bigcap_{l=1}^{t} H_l = \widehat{V}$ for some $V \supseteq \widehat{F}$. Hence, there exists $H = H_l$ such that $H \cup F \neq I$. By Theorem 2.7, we could find $\boldsymbol{\theta} \in A_{i-1} \cap \Delta_H^S(\boldsymbol{\beta})$. Hence, $\boldsymbol{\beta} < \boldsymbol{\theta} \wedge \boldsymbol{\alpha} \leq \boldsymbol{\theta}$, and by Lemma 2.8 $\boldsymbol{\theta} \wedge \boldsymbol{\alpha} \in A_{i-1}$, a contradiction with the assumption on $\boldsymbol{\beta}$.

Now, recall that by Lemma 2.9 all the minimal elements of $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha})$ are in the same level, and thus, it is enough to determine the level of only one of them.

Let $\boldsymbol{\omega}$ by a minimal element of $\Delta^S(\boldsymbol{\gamma}_E - \boldsymbol{\alpha})$. By Lemma 3.6, since $\widetilde{\Delta}_{\widehat{F}}^E(\boldsymbol{\beta}) = \emptyset$, then there exists $k \in \widehat{F}$ such that $\Delta_k^S(\boldsymbol{\gamma}_E - \boldsymbol{\beta}) \neq \emptyset$. Therefore, we can find a minimal element $\boldsymbol{\delta} \in \Delta_k^S(\boldsymbol{\gamma}_E - \boldsymbol{\beta})$. By inductive hypothesis, $\boldsymbol{\delta} \in A_{N-i+2}$. We need to show that $\boldsymbol{\omega}$ is in a level strictly smaller than the level of $\boldsymbol{\delta}$. This will imply $\boldsymbol{\omega} \in A_{N-i+1}$ by Lemma 3.7. By assumption on $\boldsymbol{\delta}$, we have that $\boldsymbol{\delta} \gg \boldsymbol{\gamma}_E - \boldsymbol{\alpha}$. This implies the result by the same exact argument used at the end of the proof of Lemma 3.7. □

4 The Apéry set of a non-local good semigroup in $\mathbb{N}^d$

The partition in level of the complement of a good ideal described in [18] and recalled here in Sect. 2 is perfectly well-defined also for non-local good semigroup. From [3, Theorem 2.5], every good semigroup can be expressed as a direct product of local good semigroups. The nice structure of non-local good semigroups as direct products allow us to give a more precise description of the levels of the partition in terms of the levels of partitions in the direct factors. We do this in Theorem 4.5.

Our method consists of proving all the results for a direct product of two arbitrary good semigroups (not necessarily local). Everything proved in this setting can be extended by a finite number of iterations to any non-local good semigroup.

Our setting is the following: Let $d_1, d_2 \geq 1$ and let $S_1 \subseteq \mathbb{N}^{d_1}$, $S_2 \subseteq \mathbb{N}^{d_2}$ be two good semigroups, not necessarily local. Consider the non-local good semigroup $S := S_1 \times S_2 \subseteq \mathbb{N}^d$ where $d = d_1 + d_2$. Each element of S is expressed in the form $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(1)} \in S_1$, $\boldsymbol{\alpha}^{(2)} \in S_2$. Let us also define set of indexes $I_1 := \{1, \ldots, d_1\}$, $I_2 := \{d_1 + 1, \ldots, d_1 + d_2\}$. We characterize good ideals of S.

Lemma 4.1 *Let E be a good ideal of S. Then $E = E_1 \times E_2$ where $E_1 \subseteq S_1$ and $E_2 \subseteq S_2$ are good ideals. Conversely, if E_1, E_2 are good ideals, respectively, of S_1 and S_2 then $E_1 \times E_2$ is good ideal of S.*

Proof If E is a good ideal of S, define E_1 and E_2 to be the projections of E on S_1 and S_2. Then clearly $E \subseteq E_1 \times E_2$. Let now $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in E_1 \times E_2$. We can find an element $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\beta}) \in E$ with $\boldsymbol{\beta} \gg \boldsymbol{\alpha}^{(2)}$ (this follows since E is an ideal and S contains elements of the form $(\mathbf{0}_{S_1}, \boldsymbol{\beta})$ for any $\boldsymbol{\beta} \in S_2$). Similarly we can find an element $(\boldsymbol{\delta}, \boldsymbol{\alpha}^{(2)}) \in E$ with $\boldsymbol{\delta} \gg \boldsymbol{\alpha}^{(1)}$. By property (G1), $\boldsymbol{\alpha} \in E$. It is easy to check that the projections of a good ideal are good ideals and the direct product of two good ideals is a good ideal. □

Given a good ideal $E = E_1 \times E_2$ of S, let us fix the notation $A := S \backslash E$, $A^{(1)} := S_1 \backslash E_1$, and $A^{(2)} := S_2 \backslash E_2$. Then,

$$A = (A^{(1)} \times S_2) \cup (S_1 \times A^{(2)}).$$

We prove the following lemmas.

Lemma 4.2 *Let $S = S_1 \times S_2$, $E \subsetneq S$, and A be defined as above. Pick $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in S$. Then, $\boldsymbol{\alpha}^{(2)} \in E_2$ if and only if there exists $\boldsymbol{\eta}^{(1)} > \boldsymbol{\alpha}^{(1)}$ such that $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in E$.*

Proof Since E_1 is a good ideal, we can always find $\boldsymbol{\eta}^{(1)} \in E_1$ such that $\boldsymbol{\eta}^{(1)} > \boldsymbol{\alpha}^{(1)}$. Both implications follow now since $E = E_1 \times E_2$. □

Lemma 4.3 *Let $S = S_1 \times S_2$, $E \subsetneq S$, and A be defined as above. Fix an element $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$. Let $\boldsymbol{\theta}^{(2)} \le \boldsymbol{\alpha}^{(2)}$ be two consecutive elements of S_2. Then, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A$ and they are consecutive in S. Suppose $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \in A_i$. Then, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i$ if and only if there exists $\boldsymbol{\delta}^{(2)} \in E_2$ such that $\boldsymbol{\delta}^{(2)} \wedge \boldsymbol{\alpha}^{(2)} = \boldsymbol{\theta}^{(2)}$ (in particular, also if $\boldsymbol{\delta}^{(2)} = \boldsymbol{\theta}^{(2)} \in E_2$). Otherwise $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_{i+1}$.*

Proof Since $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$, then $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A$ and they are clearly consecutive since so are $\boldsymbol{\alpha}^{(2)}$ and $\boldsymbol{\theta}^{(2)}$ in S_2. Hence, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i \cup A_{i+1}$. Looking at the coordinates we can find a set of indexes $F \supseteq I_1$ such that $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in \Delta_F^S(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$.

Assume there exists $\boldsymbol{\delta}^{(2)} \in E_2$ such that $\boldsymbol{\delta}^{(2)} \wedge \boldsymbol{\alpha}^{(2)} = \boldsymbol{\theta}^{(2)}$. Thus, by Lemma 2.9 there exists $G \supseteq I_1$ such that $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\delta}^{(2)}) \in \Delta_G^S(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \cup \{(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})\}$ and $F \cup G = I$. By Lemma 4.2, there exists $\boldsymbol{\eta}^{(1)} > \boldsymbol{\alpha}^{(1)}$ such that $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\delta}^{(2)}) \in E$. This is saying that $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\delta}^{(2)}) \in \Delta_H^E(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$ with $H \not\supseteq I_1$ and $H \cap I_2 = G \cap I_2$. In particular, $F \supseteq \widehat{H}$. Since $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\delta}^{(2)}) \in E$, by Lemma 2.1, $\widetilde{\Delta}_{\widehat{H}}^S(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \subseteq A$. By Theorem 2.7 $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i$.

Conversely, suppose that no element $\boldsymbol{\delta}^{(2)} \in E_2$ satisfying such property exists. Let G be a set of indexes such that $F \cup G = I$. If there were some element $(\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) \in \Delta_G^E(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$, then $\boldsymbol{\eta}^{(2)} \in \Delta_{G \cap I_2}^{E_2}(\boldsymbol{\theta}^{(2)})$, and this would be a contradiction (notice that $(F \cap I_2) \cup (G \cap I_2) = I_2$). Therefore, $\widetilde{\Delta}_{\widehat{F}}^S(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \subseteq A$. By Theorem 2.6, the level of $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$ is strictly smaller than the level of $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$. This proves $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_{i+1}$. □

We define now a level function for all the elements of an arbitrary good semigroup T, which extend the notion of level for the elements not in the set A. This needs to be done since an element is in A if and only if at least one of its components is in the corresponding $A^{(j)}$ for $j = 1, 2$. Therefore, there are elements in A with a component not in a set of the form $A^{(j)}$.

Definition 4.4 Let T be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal. If T is numerical, $A = \{w_1, \dots, w_N\}$ is finite and we set $A_i = \{w_i\}$. We define a level function $\lambda : T \to \{1, \dots, N+1\}$ in the following way:

- If $\boldsymbol{\alpha} \in A_i$, $\lambda(\boldsymbol{\alpha}) = i$.
- If $\boldsymbol{\alpha} \notin A$, $\lambda(\boldsymbol{\alpha}) = 1 + \max\{i$ such that $\boldsymbol{\alpha} > \boldsymbol{\theta}$ for some $\boldsymbol{\theta} \in A_i\}$.

Theorem 4.5 *Let $S = S_1 \times S_2$, $E \subsetneq S$, and A be defined as above. Then, given $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A$, the level of $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$ in A is equal to*

$$\lambda(\boldsymbol{\alpha}^{(1)}) + \lambda(\boldsymbol{\alpha}^{(2)}) - 1.$$

Proof It is always true that $\mathbf{0} = (\mathbf{0}_{S_1}, \mathbf{0}_{S_2}) \in A_1$. Clearly, $1 = \lambda(\mathbf{0}_{S_1}) + \lambda(\mathbf{0}_{S_1}) - 1$ and the theorem is true for this element. We can thus work by iterating the following procedure: we assume the result true for an element of A and we prove it for another element consecutive to it in A. Observe that any two consecutive elements in A are either of the form $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}), (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)})$ with $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$ or $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}), (\boldsymbol{\eta}^{(1)}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(2)} \in A^{(2)}$. Such elements are also consecutive in S.

Fix $\boldsymbol{\alpha}^{(1)} \in A^{(1)}$ and consider $\boldsymbol{\alpha}^{(2)} \in S_2$, $\boldsymbol{\alpha}^{(2)} \neq \mathbf{0}_{S_2}$. Let $\boldsymbol{\theta}^{(2)} \in A^{(2)}$ be a maximal element such that $\boldsymbol{\theta}^{(2)} < \boldsymbol{\alpha}^{(2)}$ and $\lambda(\boldsymbol{\theta}^{(2)})$ is also maximal among the elements of $A^{(2)}$ smaller than $\boldsymbol{\alpha}^{(2)}$. Pick a chain of consecutive elements of S_2

$$\boldsymbol{\theta}^{(2)} \leq \boldsymbol{\theta}_1^{(2)} \leq \cdots \leq \boldsymbol{\theta}_c^{(2)} \leq \boldsymbol{\alpha}^{(2)}$$

and consider the corresponding chain of consecutive elements of S

$$(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \leq (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}_1^{(2)}) \leq \cdots \leq (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}_c^{(2)}) \leq (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}).$$

By our choice of $\boldsymbol{\theta}^{(2)}$, we have $\boldsymbol{\theta}_1^{(2)}, \dots, \boldsymbol{\theta}_c^{(2)} \in E_2$. An iterated application of Lemma 4.3 implies that $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$ is in the same level of $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}_1^{(2)})$, and we can therefore restrict to assume that $\boldsymbol{\theta}^{(2)}$ and $\boldsymbol{\alpha}^{(2)}$ are consecutive.

Say that $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\theta}^{(2)}) \in A_i$. Hence, $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i \cup A_{i+1}$. By inductive hypothesis, $i = \lambda(\boldsymbol{\alpha}^{(1)}) + \lambda(\boldsymbol{\theta}^{(2)}) - 1$. We know that $\lambda(\boldsymbol{\alpha}^{(2)}) \in \{\lambda(\boldsymbol{\theta}^{(2)}), \lambda(\boldsymbol{\theta}^{(2)}) + 1\}$. To conclude, it is sufficient to prove that $\lambda(\boldsymbol{\alpha}^{(2)}) = \lambda(\boldsymbol{\theta}^{(2)})$ if and only if $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i$. We apply Lemma 4.3. Consider first the case $\boldsymbol{\alpha}^{(2)} \in E_2$. By definition of λ and by construction of $\boldsymbol{\theta}^{(2)}$, in this case $\lambda(\boldsymbol{\alpha}^{(2)}) = 1 + \lambda(\boldsymbol{\theta}^{(2)})$. If there exists $\boldsymbol{\delta}^{(2)} \in E_2$ such that $\boldsymbol{\delta}^{(2)} \wedge \boldsymbol{\alpha}^{(2)} = \boldsymbol{\theta}^{(2)}$, we would get $\boldsymbol{\theta}^{(2)} \in E_2$ that is a contradiction. This implies $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_{i+1}$ by Lemma 4.3.

The other case to consider is when $\boldsymbol{\alpha}^{(2)} \in A^{(2)}$. If $\boldsymbol{\alpha}^{(2)} \gg \boldsymbol{\theta}^{(2)}$, then $\lambda(\boldsymbol{\alpha}^{(2)}) = 1 + \lambda(\boldsymbol{\theta}^{(2)})$ and clearly there cannot exist any element $\boldsymbol{\delta}^{(2)} \neq \boldsymbol{\theta}^{(2)}$ such that $\boldsymbol{\delta}^{(2)} \wedge$

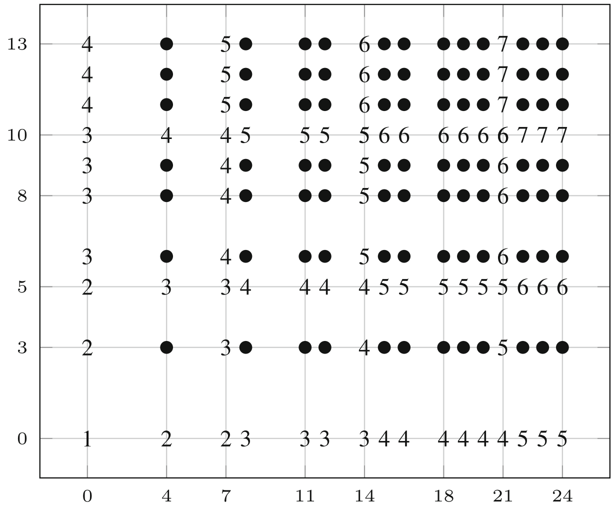


Fig. 3 The Apéry set of the non-local good semigroup $S = S_1 \times S_2$ with respect to the element $\boldsymbol{e} = (4, 3)$. Here, $S_1 = \langle 4, 7\rangle$, $S_2 = \langle 3, 5\rangle$, and $\mathrm{Ap}(S_1) = \{0, 7, 14, 21\}$, $\mathrm{Ap}(S_2) = \{0, 5, 10\}$. The levels are indicated by distinct numbers. Black marks indicate the elements of $\boldsymbol{e} + S$

$\boldsymbol{\alpha}^{(2)} = \boldsymbol{\theta}^{(2)}$. Since $\boldsymbol{\theta}^{(2)} \notin E_2$, this again shows $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_{i+1}$. Otherwise $\boldsymbol{\alpha}^{(2)} \in \Delta_F^{S_2}(\boldsymbol{\theta}^{(2)})$ for some non-empty set of indexes $F \supsetneq I_2$. Now the existence of $\boldsymbol{\delta}^{(2)} \in E_2$ such that $\boldsymbol{\delta}^{(2)} \wedge \boldsymbol{\alpha}^{(2)} = \boldsymbol{\theta}^{(2)}$ (which by Lemma 4.3 is equivalent to have $(\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}) \in A_i$) is equivalent to have $\widetilde{\Delta}_{\widehat{F}}^{E_2}(\boldsymbol{\theta}^{(2)}) \neq \emptyset$. The application of Theorems 2.6 and 2.7 shows that this is equivalent to have $\lambda(\boldsymbol{\alpha}^{(2)}) = \lambda(\boldsymbol{\theta}^{(2)})$ and concludes the proof. □

As application, we obtain a nice description of the levels of the Apéry set of a non-local good semigroup in $\mathbb{N}^2$, see also Fig. 3. A similar description can be obtained also in the case of good semigroup in $\mathbb{N}^d$, which splits completely as direct product of d numerical semigroups.

Corollary 4.6 *Let $S = S_1 \times S_2 \subseteq \mathbb{N}^2$ be a non-local good semigroup and let $\boldsymbol{\omega} = (w_1, w_2) \in S$. Write $\mathrm{Ap}(S_1, w_1) = \{u_1, u_2, \ldots, u_{w_1}\}$ and $\mathrm{Ap}(S_2, w_2) = \{v_1, v_2, \ldots, v_{w_2}\}$ with the elements listed in increasing order. Set formally $u_{w_1+1} = v_{w_2+1} = \infty$. Then:*

- *The level A_1 of $\mathrm{Ap}(S, \boldsymbol{\omega})$ only consists of the element $(0, 0)$.*
- *For $1 \le i \le w_1$, $1 \le j \le w_2$, the level A_{i+j} of $\mathrm{Ap}(S, \boldsymbol{\omega})$ is equal to the set*

$$\{(u_i, b) : v_j < b \le v_{j+1}\} \cup \{(a, v_j) : u_i < a \le u_{i+1}\}.$$

5 Well-behaved Apéry sets

In this section, we consider a particularly nice class of complement of good ideals which includes the Apéry sets of local value semigroups of plane curves. Value semi-

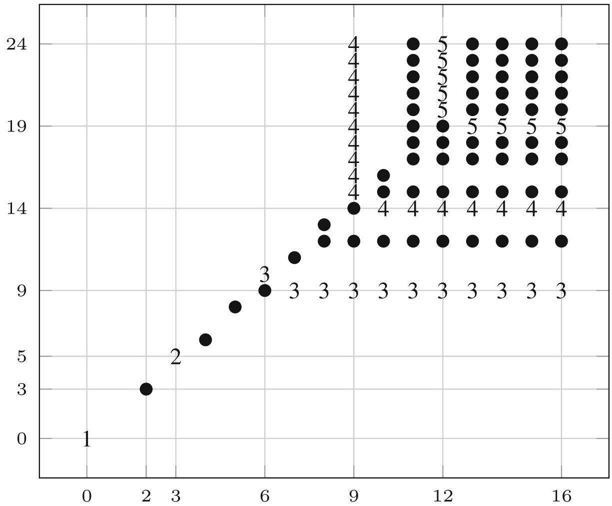


Fig. 4 The Apéry set of the good semigroup S associated with the plane curve $\mathbb{K}[[X,Y]]/((X^3 - Y^2)\cdot(X^5 - Y^3)) \cong \mathbb{K}[[(t^2, u^3), (t^3, u^5)]]$ with respect to the element $\boldsymbol{e} = (2, 3)$. Here, $S \subseteq S_1 \times S_2$ with $S_1 = \langle 2, 3\rangle$, $S_2 = \langle 3, 5\rangle$, and $\mathrm{Ap}(S_1) = \{0, 3\}$, $\mathrm{Ap}(S_2) = \{0, 5, 10\}$. The levels are indicated by distinct numbers. Black marks indicate the elements of $\boldsymbol{e} + S$

groups of plane curves with more then one branch have been considered in [8, 17, 26], and their Apéry set has been defined in the case of two branches in [5]. The definition of Apéry set given in [5] is slightly different from that one given for general good semigroups in [12] and [18], but they intuitively seem to agree on value semigroups of plane curves. We give an explicit proof that they coincide in Proposition 5.1.

Let S be the local value semigroup of a plane curve and let A be the Apéry set of S with respect to some nonzero element $\boldsymbol{\omega}$. Then, A can be partitioned as $\bigcup_{i=1}^{N} A_i$ as defined in [18] and recalled in Sect. 2 of this paper, or as $\bigcup_{i=1}^{M} B_i$ following the definition in [5, Section 3].

Such definition can be summarized as follows. The set B_M consists of all the maximal elements of A with respect to the order relation $\ll$. The set B_j for $j < M$ is defined inductively as the set of all maximal elements of $A\backslash(\bigcup_{i=j+1}^{M} B_i)$ with respect to the order relation $\ll$. In [5] is proved that M is equal to the sum of the components of $\boldsymbol{\omega}$. Hence, by [18, Theorem 4.4], $N = M$.

Observing the specific case where $d = 2$, we see that the main reason for which the two partitions of the Apéry set coincide is due to the fact that, if S is the value semigroup of a plane curve, then all the complete infimums of elements in the Apéry set of S are not in the Apéry set (see Fig. 4 for a graphical interpretation of this property).

This fact may be proved with valuation theoretic arguments following the notations and the method developed in [5]. In this paper, we rather prefer to give a proof using an approach based only on the combinatorics of good semigroups. To do this, we need to recall some key facts.

In [5, Lemma 3.3], it is proved that if $\boldsymbol{\alpha} \in B_i$ (and $i < M$), then there always exists $\boldsymbol{\beta} \in B_{i+1}$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$.

In [5, Proposition 3.10], it is proved that if $\boldsymbol{\alpha} \in B_i$, then $\Delta^S(\boldsymbol{\gamma}+\boldsymbol{\omega}-\boldsymbol{\alpha}) \subseteq B_{M-i+1}$.

All the results in [5] are proved in the case $d = 2$, but the authors mention explicitly in the introduction of the paper that each result until Theorem 4.1 (in particular all those in Sect. 3) can be proved with the same identical arguments for any number of branches d. Therefore, we can state and prove next proposition for arbitrary d. Recall that value semigroups of plane curves are symmetric, since rings of plane curves are always Gorenstein.

Proposition 5.1 *Let S be the local value semigroup associated with a plane curve with d branches. Write $\mathrm{Ap}(S) = \bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$, where A_i and B_i are the partitions defined, respectively, in [5, 18]. Then $A_i = B_i$ for every $i = 1, \dots, N$.*

Proof First we show by decreasing induction on $i = N, \dots, 1$ that $A_i \cap B_j = \emptyset$ for every $j < i$. By the definitions, it is clear that $A_N = B_N = \Delta(\boldsymbol{\gamma}+\boldsymbol{\omega})$ (notice that since S is local and symmetric, then $\boldsymbol{\gamma}+\boldsymbol{\omega} \in \boldsymbol{\omega}+S$). Hence for every $j < N$, we get $A_N \cap B_j = B_N \cap B_j = \emptyset$. Suppose now the result true for every $h > i$ and say that, by way of contradiction, there exists an element $\boldsymbol{\alpha} \in A_i \cap B_j$ with $j < i$. By [5, Lemma 3.3], we can find $\boldsymbol{\beta} \in B_i$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$. Hence $\boldsymbol{\beta} \in A_h$ for some $h > i$. But by inductive hypothesis $A_h \cap B_i = \emptyset$, and this is a contradiction.

After setting this fact, assume to have $A_i \neq B_i$ for some i and $A_j = B_j$ for every $j > i$. By considering the maximal elements with respect to $\ll$, after removing the levels $A_N, \dots, A_{i+1}$, we get that necessarily there must exist one element $\boldsymbol{\alpha} \in A_{i-1} \cap B_i$ which is a complete infimum of elements in $A_i \cap B_i$. Let now $\boldsymbol{\theta}$ be a minimal element in $\Delta^S(\boldsymbol{\gamma}+\boldsymbol{\omega}-\boldsymbol{\alpha})$. By [5, Proposition 3.10], $\boldsymbol{\theta} \in B_{N-i+1}$, while by Proposition 3.8, $\boldsymbol{\theta} \in A_{N-i+2}$. This contradicts the fact we proved in the first paragraph of this proof. □

The following definition of well-behaved set aims to describe in a more general setting, and for an arbitrary number of branches, this specific behavior of the Apéry sets of plane curves with respect to infimums. Through this section, we describe properties of these well-behaved sets. An application to plane curves is discussed in the next section.

Definition 5.2 Let $S \subseteq \mathbb{N}^d$ be a good semigroup and let A be the complement of a good ideal $E \subseteq S$. We say that A is *well-behaved* if whenever $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \widetilde{\wedge} \cdots \widetilde{\wedge} \boldsymbol{\beta}^{(r)}$ with $\boldsymbol{\beta}^{(j)} \in \widetilde{\Delta}^S_{G_j}(\boldsymbol{\alpha}) \subseteq A$ for every j, then $\boldsymbol{\alpha} \in E$.

If $d = 2$, this corresponds to say that whenever $\Delta^S(\boldsymbol{\alpha}) \subseteq A$ and it is non-empty, then $\boldsymbol{\alpha} \notin A$.

The Apéry sets of value semigroups of plane curves are well-behaved as a consequence of next proposition. Again we use the the fact proved in [5, Lemma 3.3] and recalled previously.

Proposition 5.3 *Let S be a good semigroup and let $A = \bigcup_{i=1}^{N} A_i$ be the complement of a good ideal $E \subseteq S$. Suppose that for every $i = 1, \dots, N-1$ and for every $\boldsymbol{\alpha} \in A_i$, there exists $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$. Then A is well-behaved.*

Proof Pick $\boldsymbol{\alpha} \in A_i$ and assume by way of contradiction $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} \,\widetilde{\curlywedge}\, \cdots \,\widetilde{\curlywedge}\, \boldsymbol{\beta}^{(r)}$ with $\boldsymbol{\beta}^{(j)} \in \widetilde{\Delta}^S_{G_j}(\boldsymbol{\alpha}) \subseteq A$ for every j. This assumption allows to further assume that all the $\boldsymbol{\beta}^{(j)}$ are consecutive to $\boldsymbol{\alpha}$ and at least one of them is in A_{i+1}. By hypothesis, we can find $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$. For every j, since $\boldsymbol{\alpha} < \boldsymbol{\beta} \wedge \boldsymbol{\beta}^{(j)} \leq \boldsymbol{\beta}^{(j)}$, then $\boldsymbol{\beta} \wedge \boldsymbol{\beta}^{(j)} = \boldsymbol{\beta}^{(j)}$ implying $\boldsymbol{\beta} > \boldsymbol{\beta}^{(j)}$. Choose now j such that $\boldsymbol{\beta}^{(j)} \in A_{i+1}$. Hence, $\boldsymbol{\beta} \not\gg \boldsymbol{\beta}^{(j)}$ and $\boldsymbol{\beta} \in \Delta^S_{F_j}(\boldsymbol{\beta}^{(j)})$ with $G_j \subseteq \widehat{F}_j$. It follows that $\widetilde{\Delta}^S_{\widehat{F}_j}(\boldsymbol{\beta}^{(j)}) \subseteq \widetilde{\Delta}^S_{G_j}(\boldsymbol{\alpha}) \subseteq A$. Theorem 2.6 applied to $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{(j)}$ forces $\boldsymbol{\beta}^{(j)}$ to be in a level strictly smaller than the level of $\boldsymbol{\beta}$. This is a contradiction. □

Next lemma shows that, if A is well-behaved, then any subspace of $\mathbb{N}^d$ whose intersection with S is non-empty and contained in A, has to be contained in a unique level.

Lemma 5.4 *Let $S \subseteq \mathbb{N}^d$ be a good semigroup and let A be the complement of a good ideal $E \subseteq S$. Suppose A to be well-behaved. Let $\boldsymbol{\omega} \in \mathbb{Z}^d$ be any element and assume $\Delta^S_F(\boldsymbol{\omega}) \subseteq A$ (and it is non-empty). Then $\Delta^S_F(\boldsymbol{\omega}) \subseteq A_i$ for some i.*

Proof It is sufficient to show that any two consecutive elements in $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Delta^S_F(\boldsymbol{\omega})$ are in the same level. Say that $\boldsymbol{\beta} \in \Delta^S_H(\boldsymbol{\alpha})$ with $H \supseteq F$ and suppose $\boldsymbol{\alpha} \in A_i$ and $\boldsymbol{\beta} \in A_{i+1}$. This implies that $\widetilde{\Delta}^E_{\widehat{H}}(\boldsymbol{\alpha}) = \emptyset$ otherwise we would get a contradiction with Theorem 2.7. Using property (G2), we can write $\boldsymbol{\alpha} = \boldsymbol{\beta} \,\widetilde{\curlywedge}\, \boldsymbol{\beta}^{(2)} \,\widetilde{\curlywedge}\, \cdots \,\widetilde{\curlywedge}\, \boldsymbol{\beta}^{(r)}$ with, for $j = 2, \ldots, r$, $\boldsymbol{\beta}^{(j)} \in \widetilde{\Delta}^S_{G_j}(\boldsymbol{\alpha})$ and $G_j \supseteq \widehat{H}$. Hence, $\widetilde{\Delta}^S_{G_j}(\boldsymbol{\alpha}) \subseteq \widetilde{\Delta}^S_{\widehat{H}}(\boldsymbol{\alpha}) \subseteq A$. By Definition 5.2, we get a contradiction since $\boldsymbol{\alpha} \in A$. □

As a consequence of Theorem 2.6, we also have that:

Corollary 5.5 *With the same notation of Lemma 5.4, if $d = 2$ and $\Delta^S(\boldsymbol{\omega})$ is non-empty and contained in A, then $\Delta^S(\boldsymbol{\omega}) \subseteq A_i$ for some i.*

As a main consequence of the previous lemma, for $A = S \backslash E$ a well-behaved set, we give a criterion describing the level of all the elements having some coordinate not in the projection of E. Our notation is similar to that used in Sect. 4 in the case of non-local good semigroups.

Let $S \subseteq \mathbb{N}^d$ be a good semigroup, and let $d_1, d_2 \geq 1$ be positive integers such that $d_1 + d_2 = d$. Write a partition of the set of indexes $I = \{1, \ldots, d\} = I_1 \cup I_2$, with $|I_j| = d_j$. Define S_1 to be the canonical projection of S on the set of indexes I_1 and S_2 to be the canonical projection of S on the set of indexes I_1. The set S_1 and S_2 are also good semigroups $S \subseteq S_1 \times S_2$. For each element $\boldsymbol{\alpha} \in S$, we write $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)})$ with $\boldsymbol{\alpha}^{(j)} \in S_j$ and we use the corresponding notation also for $\boldsymbol{\alpha} \in \mathbb{N}^d \subseteq \mathbb{N}^{d_1} \times \mathbb{N}^{d_2}$. Let E_1, E_2 be the projections of E and $A^{(h)} := S_h \backslash E_h$ for $h = 1, 2$. We denote by $A^{(h)}_i$ the i-th level of the partition of $A^{(h)}$ in the semigroup S_h.

Theorem 5.6 *Let $S \subseteq \mathbb{N}^d$ be a local good semigroup and let A be the complement of a good ideal $E \subsetneq S$. Suppose A to be well-behaved. Define S_1, S_2 as above. Pick $\boldsymbol{\alpha}^{(1)} \in A^{(1)}_i$. Then*

$$\Gamma(\boldsymbol{\alpha}^{(1)}) := \{\boldsymbol{\alpha} \in \mathbb{N}^d : \boldsymbol{\alpha}^{(2)} \in S_2\} \cap S \subseteq A_i.$$

The analogous result holds for elements in the projection S_2 by switching the coordinates.

Proof Since S is local, $A_1 = \{\mathbf{0}\}$ and therefore the result is true for $\boldsymbol{\alpha}^{(1)} = \mathbf{0}_{S_1}$. Thus, we can reduce to prove that, if $\boldsymbol{\beta}^{(1)} < \boldsymbol{\alpha}^{(1)}$ are consecutive in $A^{(1)}$ and the thesis is true for $\boldsymbol{\beta}^{(1)}$, then it is true for $\boldsymbol{\alpha}^{(1)}$. Clearly, for $\boldsymbol{\alpha}^{(1)} > \mathbf{0}_{S_1}$, $\Gamma(\boldsymbol{\alpha}^{(1)}) = \Delta^S_{I_1}(\boldsymbol{\alpha}^{(1)}, \mathbf{0}_{S_2}) \subseteq A$. By Lemma 5.4, $\Gamma(\boldsymbol{\alpha}^{(1)}) \subseteq A_h$ for some h. To show that $h = i$, it is sufficient to show that the minimal element of $\Gamma(\boldsymbol{\alpha}^{(1)})$, that we call $\boldsymbol{\alpha}$, is in A_i. We consider now different cases:

Case 1 $\boldsymbol{\beta}^{(1)}, \boldsymbol{\alpha}^{(1)}$ are consecutive in S_1.

Let $\boldsymbol{\beta}'$ be the minimal element of $\Gamma(\boldsymbol{\beta}^{(1)})$. If $\boldsymbol{\beta}' < \boldsymbol{\delta} < \boldsymbol{\alpha}$, then since $\boldsymbol{\beta}^{(1)}, \boldsymbol{\alpha}^{(1)}$ are consecutive in S_1 and by minimality of $\boldsymbol{\alpha}^{(1)}$, necessarily $\boldsymbol{\delta}^{(1)} = \boldsymbol{\beta}^{(1)}$. Hence, we can find an element $\boldsymbol{\beta} \in \Gamma(\boldsymbol{\beta}^{(1)})$ such that $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are consecutive in S. Say that $\boldsymbol{\beta}^{(1)} \in A_k^{(1)}$ with $k \in \{i-1, i\}$. Assuming by induction the thesis true for $\boldsymbol{\beta}^{(1)}$, we get $\boldsymbol{\beta}', \boldsymbol{\beta} \in A_k$. Say that $\boldsymbol{\alpha} \in \Delta^S_F(\boldsymbol{\beta})$ with $F \not\supseteq I_1$ and possibly $F = \emptyset$. Then $\boldsymbol{\alpha}^{(1)} \in \Delta^{S_1}_{F\cap I_1}(\boldsymbol{\beta}^{(1)})$. Notice that $h \in \{k, k+1\}$. We have to show that $k = i-1$ if and only if $h = k+1$. This will imply $h = i$.

First suppose $k = i-1$. Then either $F \cap I_1 = \emptyset$ (i.e. $\boldsymbol{\beta}^{(1)} \ll \boldsymbol{\alpha}^{(1)}$) or, by Theorem 2.7, $\widetilde{\Delta}^{S_1}_{\widehat{F}\cap I_1}(\boldsymbol{\beta}^{(1)}) \subseteq A^{(1)}$. If $F \cap I_1 \neq \emptyset$ this implies that $\widetilde{\Delta}^S_{\widehat{F}}(\boldsymbol{\beta}) \subseteq A$, since E_1 is the projection of E. If instead, $F \cap I_1 = \emptyset$, if also $F = \emptyset$ we are done since $\boldsymbol{\beta} \ll \boldsymbol{\alpha}$, otherwise $F \subseteq I_2$. In the case $F \subseteq I_2$, it follows that

$$\widetilde{\Delta}^S_{\widehat{F}}(\boldsymbol{\beta}) \subseteq \widetilde{\Delta}^S_{I_1}(\boldsymbol{\beta}) \subseteq \Gamma(\boldsymbol{\beta}^{(1)}) \subseteq A.$$

In all such cases, by Theorem 2.6, $\boldsymbol{\beta}$ is in a level strictly smaller than $\boldsymbol{\alpha}$, implying $h = k + 1 = i$.

Conversely, suppose $k = i$ and show $h = i$. In this case, clearly $F \cap I_1 \neq \emptyset$ and by Theorem 2.6 $\widetilde{\Delta}^{E_1}_{\widehat{F}\cap I_1}(\boldsymbol{\beta}^{(1)}) \neq \emptyset$. Pick $\boldsymbol{\delta}^{(1)} \in \widetilde{\Delta}^{E_1}_{\widehat{F}\cap I_1}(\boldsymbol{\beta}^{(1)})$ and let $\boldsymbol{\delta}$ be the minimal element of $\Gamma(\boldsymbol{\delta}^{(1)}) \cap E$. By minimality of $\boldsymbol{\beta}'$, since $\boldsymbol{\beta}^{(1)} < \boldsymbol{\delta}^{(1)}$, then $\boldsymbol{\beta}' < \boldsymbol{\delta}$. Thus, $\boldsymbol{\delta} \in \Delta^S_G(\boldsymbol{\beta}')$ with $G \cap I_1 \supseteq \widehat{F} \cap I_1$. Applying property (G2) to $\boldsymbol{\delta}$ and $\boldsymbol{\beta}'$, we write $\boldsymbol{\beta}' = \boldsymbol{\delta} \,\widetilde{\wedge}\, \boldsymbol{\omega}^1 \,\widetilde{\wedge}\, \cdots \,\widetilde{\wedge}\, \boldsymbol{\omega}^r$ with $\boldsymbol{\omega}^j \in \Delta^S_{H_j}(\boldsymbol{\beta})$ and $\bigcap_{j=1}^r H_j = \widehat{G}$. As usual we can assume that $\boldsymbol{\omega}^j$ are consecutive to $\boldsymbol{\beta}'$ and by Theorem 2.7, $\boldsymbol{\omega}^j \in A_i$. We prove that there exists j such that $H_j \cup F \not\supseteq I_1$. Indeed, if for every $j = 1, \ldots, r$, $I_1 \subseteq H_j \cup F$, we would have $I_1 \subseteq (\bigcap_{j=1}^r H_j \cup F) = \widehat{G} \cup F$. Hence,

$$I_1 = I_1 \cap (\widehat{G} \cup F) = (I_1 \cap \widehat{G}) \cup (I_1 \cap F) = (I_1 \cap F)$$

and this is a contradiction since $F \not\supseteq I_1$. Choose now this j such that $H_j \cup F \not\supseteq I_1$ and call $\boldsymbol{\omega} := \boldsymbol{\omega}^j$ and $H := H_j$. Since $\boldsymbol{\beta}' \leq \boldsymbol{\beta} < \boldsymbol{\alpha}$, $\boldsymbol{\alpha} \in \Delta^S_U(\boldsymbol{\beta}')$ with $U \subseteq F$. Furthermore, since $I_1 \nsubseteq H \cup F$, then $I_1 \nsubseteq H \cup U$, and $H \cup U \neq I$. Observing that $\boldsymbol{\omega} \wedge \boldsymbol{\alpha} \in \Delta^S_{H\cup U}(\boldsymbol{\beta}')$ we get that $\boldsymbol{\beta}' < \boldsymbol{\omega} \wedge \boldsymbol{\alpha} \leq \boldsymbol{\alpha}$. But the condition $I_1 \nsubseteq H \cup U$ implies that $\boldsymbol{\omega} \wedge \boldsymbol{\alpha} \notin \Gamma(\boldsymbol{\beta}^{(1)})$. Thus, since $\boldsymbol{\beta}^{(1)}, \boldsymbol{\alpha}^{(1)}$ are consecutive in S_1, $(\boldsymbol{\omega} \wedge \boldsymbol{\alpha})^{(1)} = \boldsymbol{\alpha}^{(1)}$, and by minimality of $\boldsymbol{\alpha}$ in $\Gamma(\boldsymbol{\alpha}^{(1)})$ we get $\boldsymbol{\omega} \wedge \boldsymbol{\alpha} = \boldsymbol{\alpha}$. This implies $\boldsymbol{\omega} \geq \boldsymbol{\alpha}$ and $\boldsymbol{\alpha} \in A_i$.

Case 2 There exists $\boldsymbol{\theta}^{(1)} \in E_1$ such that $\boldsymbol{\beta}^{(1)} < \boldsymbol{\theta}^{(1)} < \boldsymbol{\alpha}^{(1)}$.

Denote by $\boldsymbol{\beta}$ the minimal element of of $\Gamma(\boldsymbol{\beta}^{(1)})$ and assume the thesis true for $\boldsymbol{\beta}^{(1)}$. By Lemma 2.8, $\boldsymbol{\beta}^{(1)} \in A_{i-1}^{(1)}$ and hence $\boldsymbol{\beta} \in A_{i-1}$. Say again that $\boldsymbol{\alpha} \in \Delta_F^S(\boldsymbol{\beta})$ with $F \not\supseteq I_1$ and possibly $F = \emptyset$. Thus, $\boldsymbol{\alpha}^{(1)} \in \Delta_{F \cap I_1}^{S_1}(\boldsymbol{\beta}^{(1)})$ and $\boldsymbol{\theta}^{(1)} \in \Delta_{V \cap I_1}^{S_1}(\boldsymbol{\beta}^{(1)})$ with $V \supseteq F$. If $F \cap I_1 \neq \emptyset$, using that $\boldsymbol{\beta}^{(1)} \in A^{(1)}$, by property (G1) of the good ideal E_1, we get $\widetilde{\Delta}_{\widehat{V} \cap I_1}^{E_1}(\boldsymbol{\beta}^{(1)}) = \emptyset$. Therefore, $\widetilde{\Delta}_{\widehat{F} \cap I_1}^{S_1}(\boldsymbol{\beta}^{(1)}) \subseteq \widetilde{\Delta}_{\widehat{V} \cap I_1}^{S_1}(\boldsymbol{\beta}^{(1)}) \subseteq A^{(1)}$. Now both in this case and in the case when $F \cap I_1 = \emptyset$ we conclude proceeding as in Case 1, in the subcase $k = i - 1$. □

As a corollary, we describe the case $d = 2$ adding an observation on upper bounds for the maximal coordinate of elements in the first levels. Again the reader can compare this statement with the representation in Fig. 4.

Corollary 5.7 *Let $S \subseteq S_1 \times S_2 \subseteq \mathbb{N}^2$ be a local good semigroup and pick $\boldsymbol{\omega} = (\omega_1, \omega_2) \in S$. Suppose $A = \mathrm{Ap}(S, \boldsymbol{\omega})$ to be well-behaved. Let $\{u_1, \ldots, u_{\omega_1}\}$ be the Apéry set of S_1 with respect to ω_1. Then for every $i = 1, \ldots \omega_1$,*

$$\Delta_1^S(u_i, -1) \subseteq A_i.$$

Moreover if $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$, then $a_1 \leq u_i$. The analogous result holds for the projection S_2 by switching the coordinates.

Proof Clearly the set $\Delta_1^S(u_i, -1)$ is non-empty and by Lemma 5.4 is contained in the level A_i. Call $\boldsymbol{\beta} = (u_i, b_2)$ the minimal element of $\Delta_1^S(u_i, -1)$. Assume there exists $\boldsymbol{\alpha} \in A_i$ such that $a_1 > u_i$. Then $a_2 \leq b_2$ since $\boldsymbol{\alpha} \not\gg \boldsymbol{\beta}$. By property (G1), the minimality of $\boldsymbol{\beta}$ excludes the case $a_2 < b_2$. If $a_2 = b_2$, by property (G2) we find $\boldsymbol{\theta} \in \Delta_1^S(\boldsymbol{\beta}) \subseteq A_i$. Hence, $\boldsymbol{\beta} = \boldsymbol{\alpha} \wedge \boldsymbol{\theta}$ and this is a contradiction since they are all elements of A_i. □

We conclude this section with some further results for the case $d = 2$, which will be needed to prove the main theorem of next section. First we have a lemma describing an equivalent condition of being well-behaved.

Lemma 5.8 *Let $S \subseteq \mathbb{N}^2$ and let A be the complement of a proper good ideal of S. The following assertions are equivalent:*

1. *A is well-behaved.*
2. *For every level A_i and every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_i$, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in A$ if and only if $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$.*
3. *For every level A_i with $i < N$ and every $\boldsymbol{\alpha} \in A_i$, there exists $\boldsymbol{\beta} \in A_{i+1}$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$.*

Proof 1. ⇒ 2. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A_i$, $\boldsymbol{\theta} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ with $\boldsymbol{\alpha} \in \Delta_1^S(\boldsymbol{\theta})$ and $\boldsymbol{\beta} \in \Delta_2^S(\boldsymbol{\theta})$. Suppose $\boldsymbol{\theta} \neq \boldsymbol{\alpha}, \boldsymbol{\beta}$. If $\Delta^S(\boldsymbol{\theta}) \subseteq A$ we are done. Hence assume $\Delta_1^E(\boldsymbol{\theta}) \neq \emptyset$ (if $\Delta_2^E(\boldsymbol{\theta}) \neq \emptyset$ we use the analogous argument) and $\boldsymbol{\theta} \in A$. Therefore, $\boldsymbol{\theta} \in A_h$ with $h < i$. By property (G1) used on E, we get $\Delta_2^S(\boldsymbol{\theta}) \subseteq A$ and by Theorem 2.7 we can find $\boldsymbol{\omega} \in A_h \cap \Delta_2^S(\boldsymbol{\theta})$. We can suppose $\boldsymbol{\omega}$ to be the maximal element in A_h such that $\boldsymbol{\theta} < \boldsymbol{\omega} < \boldsymbol{\beta}$. It follows that $\Delta_1^S(\boldsymbol{\omega}) \subseteq A$ otherwise we would contradict such maximality by Theorem 2.7.

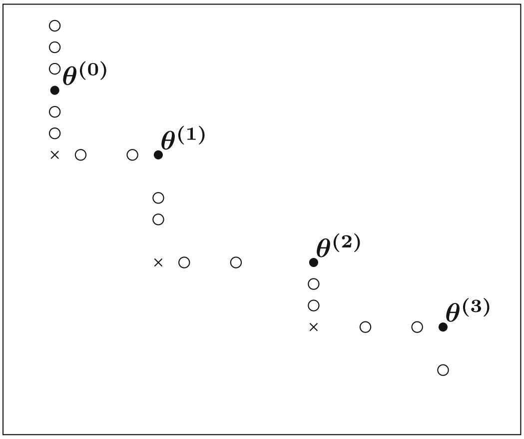


Fig. 5 In this figure is represented the level A_i of a well-behaved set A in a good semigroup $S \subseteq \mathbb{N}^2$, as described by Proposition 5.9. The elements $\{\theta^{(0)}, \theta^{(1)}, \theta^{(2)}, \theta^{(3)}\}$ are denoted by $\bullet$; all other elements of A_i are denoted by $\circ$. The elements marked by $\times$ are not in the level A_i. Here A_i is bounded with respect to coordinate 1 and infinte with respect to coordinate 2

Thus $\Delta^S(\boldsymbol{\omega}) \subseteq A$ and it is non-empty. This contradicts the hypothesis of having A well-behaved.

2. $\Rightarrow$ 3. Suppose there exists $\boldsymbol{\alpha} \in A_i$ with $i < N$ that is not dominated by any element of A_{i+1}. Necessarily, by definition of the partition in levels, $\boldsymbol{\alpha} = \boldsymbol{\theta} \,\widetilde{\curlywedge}\, \boldsymbol{\omega}$ with $\boldsymbol{\theta}, \boldsymbol{\omega} \in A_{i+1}$. This contradicts 2.

3. $\Rightarrow$ 1. This is Lemma 5.3. □

Next result gives strong restrictions on the areas of $\mathbb{N}^2$ where the elements of a fixed level of a well-behaved set can exist. This description is done in terms of the absolute elements of S which are also in that level. We recall that $\boldsymbol{\alpha} \in S$ is an absolute element if $\Delta^S(\boldsymbol{\alpha}) = \emptyset$.

In general, we say that a level A_i is *bounded with respect to the coordinate h* if there exists $n \in \mathbb{N}$ such that for every $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$, $a_h < n$. In the opposite case, we say that A_i is *infinite with respect to the coordinate h*.

Let $c_E = (q_1, q_2)$ be the conductor of $E := S \setminus A$. Fixed a level A_i, let $\{\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(r)}\}$ be all the absolute element of S in the level A_i. We assume them to be ordered increasingly with respect to the first component (thus decreasingly with respect to the second one). Moreover, if A_i is infinite with respect to the coordinate 1, we define $\boldsymbol{\theta}^{(0)} := (q_1, s_2)$ such that $\Delta_1(\boldsymbol{\theta}^{(0)}) \subseteq A_i$. Similarly, if A_i is infinite with respect to the coordinate 2, define $\boldsymbol{\theta}^{(r+1)} = (s_1, q_2)$ such that $\Delta_2(\boldsymbol{\theta}^{(r+1)})) \subseteq A_i$.

The following proposition describes the structure of the levels in a well-behaved set, in a good semigroup $S \subseteq \mathbb{N}^2$ (see Fig. 5 for a graphical representation).

Proposition 5.9 *Let $A \subseteq S \subseteq \mathbb{N}^2$ be a well-behaved set. Let $\boldsymbol{\theta}^{(k)}$ for $k = 0, \ldots, r+1$ be defined as above for a fixed level A_i. Let $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$. Then one of the following assertions holds:*

(i) $\boldsymbol{\alpha} \in \Delta_1(\boldsymbol{\theta}^{(0)}) \cup \Delta_2(\boldsymbol{\theta}^{(r+1)})$.
(ii) *There exists $k \in \{0, \ldots, r\}$ such that $\boldsymbol{\theta}^{(k)} \wedge \boldsymbol{\theta}^{(k+1)} < \boldsymbol{\alpha} \leq \boldsymbol{\theta}^{(k)}$.*
(iii) *There exists $k \in \{1, \ldots, r+1\}$ such that $\boldsymbol{\theta}^{(k-1)} \wedge \boldsymbol{\theta}^{(k)} < \boldsymbol{\alpha} \leq \boldsymbol{\theta}^{(k)}$.*
(iv) *A_i is bounded with respect to 1 and $\boldsymbol{\theta}^{(r)} \in \Delta_1^S(\boldsymbol{\alpha})$.*
(v) *A_i is bounded with respect to 2 and $\boldsymbol{\theta}^{(1)} \in \Delta_2^S(\boldsymbol{\alpha})$.*

In particular $\boldsymbol{\alpha}$ *shares at least one coordinate with some* $\boldsymbol{\theta}^{(k)}$.

Proof If $a_h > q_h$ for some $h = 1, 2$, we are necessarily in the situation described in the first item (see [12, Lemma 1, items 7–8]). Thus we can restrict to assume $a_h < q_h$ in both coordinates and $\boldsymbol{\alpha} \neq \boldsymbol{\theta}^{(k)}$ for every k. First consider the case $\theta_1^{(k)} \leq a_1 < \theta_1^{(k+1)}$. Since two distinct elements of the same level are incomparable with respect to the order relation $\ll$, we must have $\theta_2^{(k)} \geq a_2 \geq \theta_2^{(k+1)}$. Furthermore, since A is well-behaved, by Lemma 5.8, $\boldsymbol{\delta} := \boldsymbol{\theta}^{(k+1)} \wedge \boldsymbol{\theta}^{(k)} \in E$. We only have to exclude the case $\boldsymbol{\alpha} \gg \boldsymbol{\delta}$. For this, we can clearly suppose $\boldsymbol{\alpha}$ to be a maximal element in A_i such that $\boldsymbol{\alpha} \gg \boldsymbol{\delta}$. Indeed, since $\boldsymbol{\theta}^{(k+1)}, \boldsymbol{\theta}^{(k)}$ are absolute elements in A_i, there cannot exist infinitely many element $\boldsymbol{\alpha} \in A_i$ such that $\boldsymbol{\alpha} \gg \boldsymbol{\delta}$. But $\boldsymbol{\alpha}$ cannot be another absolute element in A_i. Thus $\Delta^S(\boldsymbol{\alpha}) \neq \emptyset$. If $\Delta^E(\boldsymbol{\alpha}) \neq \emptyset$ we contradict the maximality of $\boldsymbol{\alpha}$ using Theorem 2.7. If instead $\Delta^E(\boldsymbol{\alpha}) = \emptyset$, we contradict the hypothesis of having A well-behaved.

The only case we still need to discuss is when $a_1 < \theta_1^{(1)}$ and A_i is bounded with respect to the second coordinate (the other cases are obtained by analogy switching the coordinates). Also in this case, if $a_2 > \theta_2^{(1)}$, since A_i is bounded, we may say that $\boldsymbol{\alpha}$ is maximal with respect to satisfy such property. The same argument as above forces $\boldsymbol{\alpha}$ to be a new absolute element, which is a contradiction. □

6 An application to plane curves

In this section we prove a result for value semigroups of plane curves with two branches which extends [5, Theorem 4.1] in the non-local case and provides an alternative method of reconstructing the value semigroup from the multiplicity sequence of the curve.

The setting is the following. Let $S = v(\mathcal{O}) \subseteq S_1 \times S_2$ be the local value semigroup of a plane curve and S_1 and S_2 be its numerical projections. Clearly S_1 and S_2 are value semigroups of plane branches. Let $\boldsymbol{e} = (e_1, e_2)$ be the minimal nonzero element of S and let $A = \bigcup_{i=1}^{e} A_i$ be the Apéry set with respect to $\boldsymbol{e}$, where $e = e_1 + e_2$.

Let S' be the value semigroup of the blow up of $\mathcal{O}$ and suppose that S' is not local. In this case, $S' = S_1' \times S_2'$ and where S_1' and S_2' are the respective blowups of S_1 and S_2.

We recall that all the above semigroups are value semigroups of some plane curve; hence, they are symmetric and their Apéry sets are well-behaved as a consequence of Proposition 5.3.

By Apéry's theorem in the case of plane branches, after setting $\mathrm{Ap}(S_1) = \{u_1, u_2, \ldots, u_{e_1}\}$ and $\mathrm{Ap}(S_2) = \{v_1, v_2, \ldots, v_{e_2}\}$, we get

$$\mathrm{Ap}(S_1', e_1) = \{u_1, u_2 - e_1, u_3 - 2e_1, \ldots, u_{e_1} - (e_1 - 1)e_1\},$$
$$\mathrm{Ap}(S_2', e_2) = \{v_1, v_2 - e_2, v_3 - 2e_2, \ldots, v_{e_2} - (e_2 - 1)e_2\}.$$

Let $A' = \bigcup_{i=1}^{E} A_i'$ be the Apéry set of S' with respect to $\boldsymbol{e}$. The levels of the Apéry set of S' are described in Theorem 4.5 and Corollary 4.6. We aim to prove next theorem:

Theorem 6.1 *Let $S = v(\mathcal{O}) \subseteq S_1 \times S_2$ be a local value semigroup of a plane curve and suppose $S' = S'_1 \times S'_2$ to be not local. For every $i = 1, \dots, e$ we have $A_i = A'_i + (i-1)\boldsymbol{e}$.*

Before the proof, we need to discuss some other results. First, since we are dealing also with numerical value semigroups of plane branches, we recall their properties (see for instance [14, Definition 1.3]).

Remark 6.2 Let S be a numerical semigroup minimally generated by $g_1, \dots, g_n$. For $i = 2, \dots, n$ define τ_i to be the minimal positive integer h such that $(h+1)g_i \in \langle g_1, \dots, g_{i-1} \rangle$. Let $\mathrm{Ap}(S)$ be the Apéry set of S with respect to the minimal nonzero element $e = g_1$.

The semigroup S is the value semigroup of a plane branch if

$$\mathrm{Ap}(S) = \left\{ \sum_{i=2}^{n} \lambda_i g_i | 0 \leq \lambda_i \leq \tau_i \right\}$$

and $(\tau_i + 1)g_i < g_{i+1}$ for every $i = 2, \dots, n-1$. It can be observed that if S is the value semigroup of a plane branch and $\mathrm{Ap}(S) = \{\omega_1, \dots, \omega_e\}$, then $\omega_i > (i-1)e$ and $\omega_i - \omega_{i-1} > e$.

By [5, Proposition 2.2.c], the fact that S' is not local is equivalent to say that $\Delta^S(\boldsymbol{e}) \neq \emptyset$. This implies the following property.

Lemma 6.3 *Let $S \subseteq \mathbb{N}^2$ be a symmetric local good semigroup such that $\Delta^S(\boldsymbol{e}) \neq \emptyset$. Then, all the absolute elements of S are in $A := \mathrm{Ap}(S)$.*

Proof Since S is local and symmetric, $\boldsymbol{\gamma} \in S$. Symmetry together with the fact that $\Delta^S(\boldsymbol{e}) \neq \emptyset$ implies that $\boldsymbol{\gamma} \in A$. Now if $\boldsymbol{\alpha}$ is an absolute element of S, by symmetry $\boldsymbol{\gamma} - \boldsymbol{\alpha} \in S$ and it is another absolute element. Since $\boldsymbol{\alpha} + (\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \boldsymbol{\gamma} \in A$, necessarily $\boldsymbol{\alpha} \in A$. □

Next proposition describes how the absolute elements of S are related to elements in S'. Define the following elements of S'. For $j = 1, \dots, e_1$, $k = 1, \dots, e_2$, let

$$\boldsymbol{\omega}_{j,k} := (u_j - (j-1)e_1, v_k - (k-1)e_2).$$

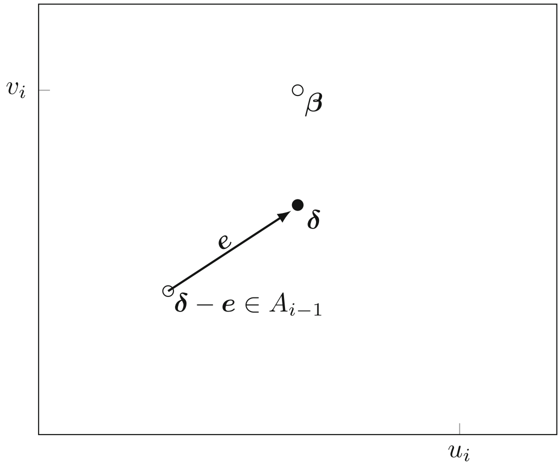
We show that these elements come from the absolute elements of S after blowup.

Proposition 6.4 *Let $S = v(\mathcal{O}) \subseteq S_1 \times S_2 \subseteq \mathbb{N}^2$ be the local value semigroup of a plane curve and suppose $S' = S'_1 \times S'_2$ to be not local. Let $i \in \{1, \dots, e\}$ Then:*

- *The absolute elements of the level A_i of $\mathrm{Ap}(S)$ are the elements $\boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$ such that $j + k - 1 = i$.*
- *If $i > 1$ and $\boldsymbol{\alpha}$ is an absolute element of S in the level A_{i-1}, then $\Delta^S(\boldsymbol{\alpha} + \boldsymbol{e})$ contains some absolute element of S in the level A_i.*

Proof Since $\boldsymbol{\omega}_{1,1} + (1-1)\boldsymbol{e} = \boldsymbol{0} \in A_1$, by induction we can assume the thesis true for all the levels indexed by numbers smaller than i. Take j, k such that $j + k - 1 = i \geq 2$

Fig. 6 In this figure $\boldsymbol{\delta} - \boldsymbol{e}$ is an absolute element in A_{i-1}, $\boldsymbol{\delta}$ is an absolute element in $S + \boldsymbol{e}$



and call $\boldsymbol{\beta} := \boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$. First consider the case $j = 1, k \geq 2$, see Fig. 6 (the case $k = 1$, $j \geq 2$ is analogous). By inductive hypothesis $\boldsymbol{\omega}_{1,k-1} + (i-2)\boldsymbol{e}$ is an absolute element of S in the level A_{i-1}. Hence $\boldsymbol{\delta} := \boldsymbol{\omega}_{1,k-1} + (i-1)\boldsymbol{e}$ is an absolute element of E and $\boldsymbol{\beta} \in \Delta_1(\boldsymbol{\delta})$.

By Lemma 6.3, an absolute element of E cannot be an absolute element of S, therefore $\Delta^S(\boldsymbol{\delta})$ is non-empty and contained in A. In particular, by Corollary 5.5 it is all contained in the same level, say A_h with $h \geq i$. Observe that in this case, since $i = k$,

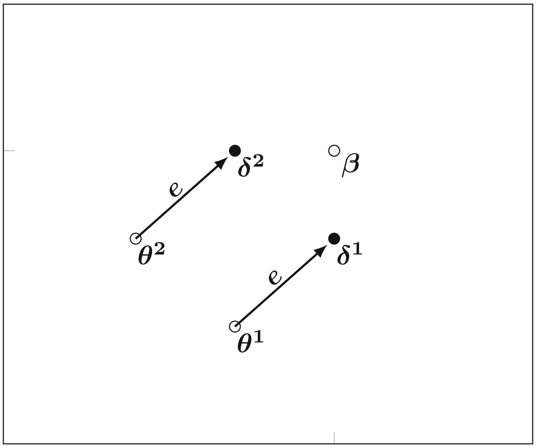
$$\boldsymbol{\beta} = (u_1 + (i-1)e_1, v_k - (k-1)e_2 + (i-1)e_2) = ((i-1)e_1, v_i) \in \Delta_2(0, v_i).$$

By Corollary 5.7, $\Delta_2^S(0, v_i) \subseteq A_i$ and it is non-empty. Inductively, we can assume also that $\Delta_2^S(0, v_p)$ contains only one element for every $p < i$ (this is obviously true for $p = 1$). We aim to show that the only element in $\Delta_2^S(0, v_i)$ is $\boldsymbol{\beta}$, proving that indeed it is an absolute element of S in A_i.

Suppose there exists a maximal (thus absolute) element $\boldsymbol{\eta}^1 \in \Delta_2^S(0, v_i)$, $\boldsymbol{\eta}^1 < \boldsymbol{\beta}$. Hence, since S_1 is a plane branch semigroup, $\eta_1^1 < (i-1)e_1 < u_i$. Now, again by Corollary 5.7, also the set $\Delta_1^S(u_i, 0) \subseteq A_i$ and it is non-empty. Therefore it must contain an absolute element $\boldsymbol{\eta}^2$, otherwise we would find elements in A_i dominating $\boldsymbol{\eta}^1$. Recalling that A is well-behaved and using Lemma 5.8, necessarily $\boldsymbol{\eta}^1 \wedge \boldsymbol{\eta}^2 \in E$. We can then find an element $\boldsymbol{\eta} \in E$, such that $\Delta^S(\boldsymbol{\eta}) \subseteq A$, and $\boldsymbol{\eta}^1 \in \Delta_1^S(\boldsymbol{\eta})$ (such element exists by property (G2) of E). It follows that $\boldsymbol{\eta} - \boldsymbol{e}$ is an absolute element of S. By Lemma 6.3, we can say that $\boldsymbol{\eta} - \boldsymbol{e} \in A_p$ with $p < i$. Since, by induction, $\boldsymbol{\delta} - \boldsymbol{e} = \boldsymbol{\omega}_{1,k-1} + (i-2)\boldsymbol{e}$ is the only element in $\Delta_2^S(0, v_{i-1})$, necessarily $\boldsymbol{\eta} - \boldsymbol{e} \ll \boldsymbol{\delta} - \boldsymbol{e}$. Indeed, the first coordinate is strictly smaller by assumption and the second one must be smaller, otherwise property (G1) would contradict the uniqueness of $\boldsymbol{\delta} - \boldsymbol{e}$ on its horizontal line.

Hence $p < i - 1$. If $i = 2$ this is impossible. Otherwise, by inductive hypothesis on the second statement of the theorem, this implies that $\Delta^S(\boldsymbol{\eta})$ contains some absolute

Fig. 7 In this figure $\boldsymbol{\theta}^1, \boldsymbol{\theta}^2$ are absolute elements in A_{i-1}, $\boldsymbol{\delta}^1, \boldsymbol{\delta}^2$ are absolute elements in $S + \boldsymbol{e}$



element in the level $p + 1$. This is also impossible since $\boldsymbol{\eta}^1 \in \Delta_1^S(\boldsymbol{\eta}) \cap A_i$. Hence we cannot have $\boldsymbol{\eta}^1 < \boldsymbol{\beta}$.

Suppose either $\boldsymbol{\eta}^1 > \boldsymbol{\beta}$ or there are infinitely many elements in $\Delta_2^S(0, v_i)$. If $\boldsymbol{\beta} \in S$, then automatically $\boldsymbol{\beta} \in A$ and we get a contradiction since $\Delta^S(\boldsymbol{\beta}) \subseteq \Delta_1^S(\boldsymbol{\delta}) \cup \Delta_2^S(0, v_i) \subseteq A$ and A is well-behaved. If $\boldsymbol{\beta} \notin S$, there would exist an element in $\Delta_2(0, v_i) \cap A_i$ dominating the maximal element in $\Delta_1(\boldsymbol{\delta})$, which is in the level A_h with $h \geq i$. Again a contradiction. This proves $\boldsymbol{\beta} \in A_i$, $\Delta^S(\boldsymbol{\beta}) = \emptyset$ and $\boldsymbol{\beta}$ is the only element in $\Delta_2^S(0, v_i)$.

We deal now with the case $j, k \geq 2$ (see Fig. 7). Similarly as before, we start by the inductive hypothesis that $\boldsymbol{\theta}^1 := \boldsymbol{\omega}_{j,k-1} + (i-2)\boldsymbol{e}$ and $\boldsymbol{\theta}^2 := \boldsymbol{\omega}_{j-1,k} + (i-2)\boldsymbol{e}$ are absolute elements of S in the level A_{i-1}. Hence $\boldsymbol{\delta}^1 := \boldsymbol{\omega}_{j,k-1} + (i-1)\boldsymbol{e}$ and $\boldsymbol{\delta}^2 := \boldsymbol{\omega}_{j-1,k} + (i-1)\boldsymbol{e}$ are absolute elements of E and $\boldsymbol{\beta} \in \Delta_1(\boldsymbol{\delta}^1) \cap \Delta_2(\boldsymbol{\delta}^2)$. Again $\Delta_1^S(\boldsymbol{\delta}^1)$, $\Delta_2^S(\boldsymbol{\delta}^2) \neq \emptyset$ and they are contained in A. This, together with A well-behaved, shows that, if $\boldsymbol{\beta} \in S$, then it is absolute.

Suppose now that $i = j + k - 1 \leq e_2$ (or analogously $i \leq e_1$). Starting by $\boldsymbol{\omega}_{1,i} + (i-1)\boldsymbol{e}$, which has been treated previously, we can suppose working by induction on j that we already proved that $\boldsymbol{\omega}_{j-1,k+1} + (i-1)\boldsymbol{e}$ is an absolute element and it is in A_i. Since $\boldsymbol{\omega}_{j-1,k+1} + (i-1)\boldsymbol{e} \in \Delta_1^S(\boldsymbol{\delta}^2)$, by Corollary 5.5, we get $\Delta_2^S(\boldsymbol{\delta}^2) \subseteq A_i$. Hence, if $\boldsymbol{\beta} \in S$, then it is in A_i.

As before, suppose $\boldsymbol{\beta} \notin S$ and there exists an absolute element $\boldsymbol{\eta}^1 \in \Delta_2^S(\boldsymbol{\delta}^2) \subseteq A_i$ and $\boldsymbol{\eta}^1 < \boldsymbol{\beta}$. If also $e_1 \geq i$, by the previous case $\boldsymbol{\eta}^2 := \boldsymbol{\omega}_{i,1} + (i-1)\boldsymbol{e} \in A_i$ is an absolute element and $\eta_1^2 > \beta_1$. If instead $i > e_1$, there exists an element $\boldsymbol{\eta}^2 \in A_i$ such that $\Delta_2(\boldsymbol{\eta}^2)$ is an infinite horizontal line in A_i (for this see [12, Lemma 1, items 7–8 and Theorem 5]). In both case $\boldsymbol{\eta}^1 \wedge \boldsymbol{\eta}^2 \in E$ and, as in the previous case, we can find an absolute element of E, called $\boldsymbol{\eta}$, such that $\boldsymbol{\eta}^1 \in \Delta_1^S(\boldsymbol{\eta})$. This yields to a contradiction. Indeed $\boldsymbol{\eta} - \boldsymbol{e} = (s_1, s_2)$ is an absolute element of S in a level A_p with $p < i$. But, exactly as in the previous case, we cannot have $p < i - 1$. Thus $p = i - 1$, but $\theta_1^2 < s_1 < \theta_1^1$, $s_2 < \theta_2^2$, and this contradicts the inductive hypothesis describing all the absolute elements of A_{i-1}. Concluding as in the case $j = 1$, we get $\boldsymbol{\beta} \in A_i$ is absolute.

The remaining case has the assumption $i > e_1, e_2$. Following the same process above, if we could say that $\Delta_2^S(\boldsymbol{\delta}^2) \subseteq A_i$, we would be able to conclude exactly in the same way. Call again $\boldsymbol{\eta}^1$ the maximal element in $\Delta_2^S(\boldsymbol{\delta}^2)$. Say that $\boldsymbol{\eta}^1 \in A_h$ with $h \geq i$. We use now the duality property of the levels of A (see [5, Proposition 3.10] or use Proposition 3.8 together with A well-behaved). Thus, $\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\delta}^2 \in \Delta^S(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\eta}^1) \subseteq A_{e-h+1}$ and it is an absolute element, since $\boldsymbol{\delta}^2$ is absolute in E and S is symmetric. Hence $(\boldsymbol{\gamma} + \boldsymbol{e} - \boldsymbol{\delta}^2) + \boldsymbol{e}$ is an absolute element of E. Since $e - h + 1 \leq e - i + 1 < \min\{e_1 + 1, e_2 + 1\}$, by [12, Theorem 5] A_{e-h+1} is a finite level. All the finite levels can be considered in the previous cases, therefore $\Delta^S(\boldsymbol{\gamma} + 2\boldsymbol{e} - \boldsymbol{\delta}^2) \subseteq A_{e-h+2}$. But at the same time $\boldsymbol{\gamma} + 2\boldsymbol{e} - \boldsymbol{\delta}^2 = \boldsymbol{\gamma} + \boldsymbol{e} - (\boldsymbol{\delta}^2 - \boldsymbol{e})$ and $\boldsymbol{\delta}^2 - \boldsymbol{e}$ is in A_{i-1}. Thus $\Delta^S(\boldsymbol{\gamma} + 2\boldsymbol{e} - \boldsymbol{\delta}^2) \subseteq A_{e-i+2}$ implying $h = i$.

The last thing to do is proving by induction that the elements $\boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$ such that $j + k - 1 = i$ are the only absolute elements in the level A_i. Say that $\boldsymbol{\alpha}$ is another absolute element in the level A_i. If there exist $\boldsymbol{\theta}^1 = \boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$ and $\boldsymbol{\theta}^2 = \boldsymbol{\omega}_{j-1,k+1} + (i-1)\boldsymbol{e}$ such that $\boldsymbol{\alpha} \gg (\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2)$ we easily get a contradiction using property (G1) since, following our construction, $\Delta^S(\boldsymbol{\theta}^1 \wedge \boldsymbol{\theta}^2) \subseteq A_i$ and A is well-behaved. The other possible situation arises if $\boldsymbol{\alpha}$ is minimal among the absolute elements of A_i with respect to one coordinate, say the first one. Let $\boldsymbol{\theta}$ be the absolute element of type $\boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$ minimal with respect to the first coordinate. By construction $\boldsymbol{\theta} \in \Delta_1^S(\boldsymbol{\delta})$ where $\boldsymbol{\delta} = \boldsymbol{\theta}' + \boldsymbol{e}$ and $\boldsymbol{\theta}'$ is the absolute element in the level A_{i-1} minimal with respect to the first coordinate. But $\boldsymbol{\theta} \wedge \boldsymbol{\alpha} \in E$, and thus there exists an element $\boldsymbol{\eta}$, absolute in E, such that $\boldsymbol{\alpha} \in \Delta_1^S(\boldsymbol{\eta})$. It follows that $\boldsymbol{\eta} - \boldsymbol{e}$ is an absolute element and it is in some level A_p with $p < i$. If $p < i - 1$, this contradicts the inductive hypothesis as before, while if $p = i - 1$, it contradicts the minimality of $\boldsymbol{\theta}'$. □

Remark 6.5 Assume the same setting and notation of Theorem 6.1. Let $\boldsymbol{\alpha} \in A_i$ such that $\boldsymbol{\alpha} \ll \boldsymbol{\gamma} + \boldsymbol{e}$ (obviously $i < e$). By Proposition 5.9, $\boldsymbol{\alpha}$ shares a coordinate with an absolute element in A_i. By what observed in the proof of Proposition 6.4, the cases (iv)-(v) of Proposition 5.9 can be excluded since for $i \leq e_2$, $j \leq e_1$, $\Delta_2^S(0, v_i)$ and $\Delta_1^S(u_j, 0)$ contains only one element. Hence, using the notation of Proposition 5.9, $\boldsymbol{\alpha} \in \Delta^S(\boldsymbol{\delta})$ where $\boldsymbol{\delta}$ is either the minimum of two "consecutive" absolute elements $\boldsymbol{\theta}^{(k)}, \boldsymbol{\theta}^{(k+1)} \in A_i$ or the minimum of an absolute element in A_i and of the elements of an infinite line in A_i. Moreover, $\boldsymbol{\delta}$ is an absolute element of E.

We are ready to prove Theorem 6.1.

Proof (of Theorem 6.1) For this proof, we will use the structure of Apéry set of a non-local good semigroup described in Theorem 4.5 and Corollary 4.6. Proposition 6.4 describes the absolute elements of S in the level A_i as exactly the opportune translation of the elements $\boldsymbol{\alpha}$ in $\mathrm{Ap}(S')$ such that $\boldsymbol{\alpha} \in A_i$ and $\Delta^S(\boldsymbol{\alpha}) \subseteq \mathrm{Ap}(S)$. Equivalently such $\boldsymbol{\alpha}$'s are the elements (a_1, a_2) such that $a_1 \in \mathrm{Ap}(S_1', e_1)$ and $a_2 \in \mathrm{Ap}(S_2', e_2)$.

Notice that by Lemma 6.3, $\boldsymbol{\gamma} \in \mathrm{Ap}(S)$. If $\boldsymbol{\alpha} \neq \boldsymbol{\gamma}$ is an absolute element, then $\boldsymbol{\gamma} \gg \boldsymbol{\alpha}$. Hence $\boldsymbol{\gamma}$ is in a strictly larger level than all the other absolute elements of S. Clearly $A_e = \Delta(\boldsymbol{\gamma} + \boldsymbol{e})$ does not contain any absolute element. By Proposition 6.4, the level A_{e-1} contains only one absolute element, namely $\boldsymbol{\omega}_{e_1,e_2} + (e-2)$. Thus $\boldsymbol{\gamma} = \boldsymbol{\omega}_{e_1,e_2} + (e-2)$.

Now we are able to show that the infinite lines in $\text{Ap}(S)$ come from the infinite lines in $\text{Ap}(S')$ according to the formula $A_i = A'_i + (i-1)\boldsymbol{e}$. We do this for the vertical lines, but the argument for the horizontal is analogous switching the components. Using [12, Lemma 1 items 7–8 and Theorem 5], it is sufficient to consider the elements $\boldsymbol{\alpha}^i = (s_i, \gamma_2+e_2+1) \in A_i$ for $i = e_2+1, \dots, e$. Set $j := i - e_2$. Since the coordinates of the infinite vertical lines in $\text{Ap}(S')$ correspond to the elements of $\text{Ap}(S'_1)$, we have to prove that

$$s_i = (u_j - (j-1)e_1) + (i-1)e_1 = u_j + e_1e_2.$$

By [12, Theorem 8], the (first) coordinates of infinite vertical lines in $\text{Ap}(S)$ are dual with respect to $\gamma_1 + e_1$ to the coordinates of $\text{Ap}(S_1)$. Thus, we know that $s_i = \gamma_1 + e_1 - u_{e_1-j+1}$. But since $\boldsymbol{\gamma} = \boldsymbol{\omega}_{e_1,e_2} + (e-2)\boldsymbol{e}$, we get

$$\gamma_1 = (u_{e_1} - (e_1-1)e_1) + (e_1+e_2-2)e_1 = u_{e_1} - e_1 + e_1e_2.$$

To conclude, we just need to recall that, since S_1 is symmetric, $u_{e_1} = u_j + u_{e_1-j+1}$.

Finally, it remains to prove that $A_i = A'_i + (i-1)\boldsymbol{e}$ only for elements $\ll \boldsymbol{\gamma} + \boldsymbol{e}$ that are not absolute (the behavior of the absolute elements follows by Proposition 6.4). Since the level A_e does not contain any element of this kind, we can assume inductively that for all the levels A_p with $p > i$ the thesis is true.

Let $\boldsymbol{\alpha} = (a_1, a_2) \in A_i$. By Proposition 5.9, $\boldsymbol{\alpha}$ shares a coordinate with an absolute element in A_i. By Remark 6.5, suppose that $\boldsymbol{\alpha} \in \Delta^S(\boldsymbol{\delta})$ where $\boldsymbol{\delta}$ is an absolute element of E.

Say that $\boldsymbol{\alpha} \in \Delta_2^S(\boldsymbol{\delta})$. Thus, again by Remark 6.5, $\Delta_2^S(\boldsymbol{\alpha})$ contains either an absolute element or an infinite line in A_i. In both cases, it follows by the previous part of the proof that $a_2 - (i-1)e_2 \in S'_2$. We want to show that also $a_1 - (i-1)e_1 \in S'_1$. If $\Delta_1^S(\boldsymbol{\alpha})$ contains some element of A, then it must contain some element in A_h with $h > i$. By inductive assumption, the thesis is true for the level A_h and we get $a_1 - (i-1)e_1 = a_1 - (h-1)e_1 + (h-i)e_1 \in S'_1$.

Otherwise, $\Delta_1^S(\boldsymbol{\alpha})$ contains only elements of E which eventually form an infinite line by Lemma 6.3. Hence, $a_1 = s_p + me_1$, where $m \geq 1$ and $s_p = u_{p-e_2} + e_1e_2$ is the coordinate in S_1 of the infinite vertical line of S contained in the level A_p for some p.

Suppose $p \leq i$. This implies that the level A_i contains an infinite vertical line of coordinate $s_i = u_{i-e_2} + e_1e_2$. By Remark 6.2, since S_1 is a plane branch, for every $h > 1$, $u_h - u_{h-1} > e_1$. Clearly $s_i < a_1$, thus

$$0 < a_1 - s_i = -(u_{i-e_2} - u_{p-e_2}) + me_1 < (-i+p+m)e_1.$$

Therefore $p > i - m$. Again this shows $a_1 - (i-1)e_1 = a_1 - (p+m-1)e_1 + (p+m-i)e_1 \in S'_1$.

Suppose now $p > i$. In this case, we get a contradiction finding an element in $A_p \cap \Delta_1^S(\boldsymbol{\alpha})$. By Lemma 5.8, there exists $\boldsymbol{\beta} \in A_p$ such that $\boldsymbol{\beta} \gg \boldsymbol{\alpha}$. Since $s_p < a_1$, using the notation of Proposition 5.9, there exist two elements $\boldsymbol{\theta}^{(t)}, \boldsymbol{\theta}^{(t+1)} \in A_p$ with $t \geq 0$, such that $\theta_1^{(t)} < \alpha_1 < \theta_1^{(t+1)}$. The element $\boldsymbol{\omega} := (a_1, \theta_2^{(t+1)}) \in \Delta_1^S(\boldsymbol{\alpha}) \cap \Delta^S(\boldsymbol{\theta}^{(t)} \wedge \boldsymbol{\theta}^{(t+1)})$. By Remark 6.5, $\boldsymbol{\omega} \in A_p$.

In all the above cases $\boldsymbol{\alpha} - (i-1)\boldsymbol{e} \in S_1' \times S_2' = S'$. Combining now Theorem 4.6, Proposition 6.4 and the definition of $\boldsymbol{\omega}_{j,k}$, it follows that $\boldsymbol{\alpha} - (i-1)\boldsymbol{e} \in A_i'$.

To prove the opposite inclusion, let $\boldsymbol{\beta} \in A_i'$ such that $\boldsymbol{\beta} \ll \boldsymbol{\gamma}(S') + \boldsymbol{e}$. If $\boldsymbol{\beta} = \boldsymbol{\omega}_{j,k}$ with $j + k - 1 = i$, clearly we are done by Proposition 6.4. Hence assume that $\boldsymbol{\beta} = (b_1, b_2) < \boldsymbol{\omega}_{j,k}$ and $\boldsymbol{\beta} \in \Delta_2^S(\boldsymbol{\omega}_{j-1,k})$ with $j + k - 1 = i$ (or similarly in $\Delta_1^S(\boldsymbol{\omega}_{j,k-1})$). By Proposition 6.4, if $\boldsymbol{\beta} + (i-1)\boldsymbol{e} \in S$, then it must belong to the level A_i. It suffices then to show $\boldsymbol{\beta} + (i-1)\boldsymbol{e} \in S$.

Suppose not. Since $b_1 \in S_1'$ and $u_{j-1} - (j-2)e_1 < b_1 < u_j - (j-1)e_1$, then $b_1 = (u_h - (h-1)e_1) + me_1$ with $h \leq j-1, m \geq 1$. Since $i \geq j > h, b_1 + (i-1)e_1 = b_1 + (h-1)e_1 + (i-h)e_1 \in S_1$. Thus, there exists some element in S having the first coordinate equal to $b_1 + (i-1)e_1$. Since $\boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e} \in \Delta_2^S(\boldsymbol{\beta} + (i-1)\boldsymbol{e})$, there must exist an absolute element $\boldsymbol{\theta} = (t_1, t_2) \in S$ with $t_1 = b_1 + (i-1)e_1$ and $t_2 < b_2 + (i-1)e_2$ (otherwise, property (G1) would contradict the assumption $\boldsymbol{\beta} + (i-1)\boldsymbol{e} \notin S$). Since $\boldsymbol{\theta} \ll \boldsymbol{\omega}_{j,k} + (i-1)\boldsymbol{e}$, then $\boldsymbol{\theta} \in A_p$ with $p < i$. Thus, $t_1 - (p-1)e_1 \in \mathrm{Ap}(S_1', e_1)$. But $t_1 - (p-1)e_1 = b_1 + (i-1)e_1 - (p-1)e_1 = b_1 + (i-p)e_1 \notin \mathrm{Ap}(S_1', e_1)$ since $p < i$. This is a contradiction and shows $\boldsymbol{\beta} + (i-1)\boldsymbol{e} \in S$. □

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