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ORIGINAL PAPER

Graph theory and combinatorial calculus: an early approach to enhance robust understanding

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Abstract

The objective of this work is to show an educational path for combinatorics and graph theory that has the aim, on one hand, of helping students understand some discrete mathematics properties, and on the other, of developing modelling skills through a robust understanding. In particular, for the path proposed to middle-school students, we used a connection between *k*-permutations and colourings of graphs: we indicated a way to solve problems related to counting all the possible arrangements of given objects in a *k*-tuple under given constraints. We solve this kind of problem by associating a graph with the constraints related to the *k*-tuple and by using graphs' colourings, in which every colour is associated with one of the objects. The number of arrangements is given by finding the number of colourings through an algorithm called the Connection-Contraction Algorithm. The educational path is set within the Teaching for Robust Understanding framework and the goal, from the mathematical skills perspective, is to enhance modelling, passing from real situations (the fish problem in our experiment) to mathematical problems (the graph's colouring in our experiment) and vice versa through the use of technology (the Connection-Contraction Algorithm with yEd editor, in our experiment), by using an extended modelling cycle. The meetings with students were videotaped and some results of the experimentation are given.

Keywords Graph colourings · *k*-permutations · Graph theory algorithms · TRU framework · Modelling

1 Introduction

Graph theory is a relatively new branch of mathematics that has emerged increasingly often on the international research scene for its countless applications (Derrible & Kennedy, 2011; Hart, 2008). However, it is also true that it can be used for a better understanding of mathematical concepts in the field of education: the understanding of combinatorics concepts often presents itself as quite challenging for secondary school students (Hart & Martin, 2018), and some concepts from graph theory can help their understanding.

In this paper, after a review of relevant literature and introducing the research question, we present the theoretical framework to which we adhered and the mathematical content that led us to the innovative educational path that we brought to the classroom. We show some of the results obtained in the experimentation with students through a qualitative analysis, in terms of teaching for robust understanding (Schoenfeld, 2014, 2016). We also illustrate how these activities can foster mathematical skills such as modelling (Greefrath, 2011).

2 Graph theory and combinatorics in mathematics education

Research in mathematics education includes studies on the teaching and learning of discrete mathematics. Let us consider this subject in papers of the 13th International Congress on Mathematical Education (Hart & Sandefur, 2018). Topics of discrete mathematics can be useful in the study of disciplines like computer science or operations research (see, e.g., Beineke & Wilson, 1997), and even if "discrete mathematics is a robust field with many modern applications" (Hart & Martin, 2018), in the U.S., for example, as well as in many other countries, "the

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Common Core State Standards for Mathematics essentially excludes discrete mathematics" (Rosenstein, 2018).

In particular, graph theory, which is one of the topics of discrete mathematics (Hart, Sandefur, & Ouvrier-Buffet, 2017), also plays a significant role in engineering and economics, and, of course, all of these topics are relevant for mathematics students (González, Muñoz-Escolano, & Oller-Marcén, 2019; Kolman, Zach, & Holoubek, 2013; Milková, 2009; Vidermanová & Melušová, 2011).

Despite its 'recent' birth (in the second half of the 1700s) and the origin of its development (more closely related to games than to mathematical matters), graph theory is nowadays studied for both theoretical and practical reasons (Voloshin, 2009). In fact, graphs are useful tools for modelling real-life problems related to transportation networks, telecommunications, social networks, or big data (Derrible & Kennedy, 2011; Hart, 2008). Moreover, since some graph theory topics do not require prior knowledge to be mastered, several experiments involving these topics have been carried out in primary and secondary schools (Cartier, 2008; Ferrarello & Mammana, 2018; Niman, 1975; Oller-Marcén & Muñoz-Escolano, 2006; Santoso, 2018; Wasserman, 2017). Gonzàles, Muñoz-Escolano, and Oller-Marcén (2019) provided a theoretical analysis of the reasoning processes students used when solving graph-theory problems, in which they classified four levels of reasoning (recognition, use and formulation of definitions, classification, and proof), most of which are applicable also in primary and middle schools.

"Combinatorics might be considered the mathematical art of counting. Combinatorial reasoning is the skill of reasoning about the size of sets, the process of counting, or the combinatorial setting to answer the question 'How many?'" (Hart & Sandefur, 2018, p. vi). Combinatorics does not depend on calculus, offers challenging problems that can be discussed with pupils, and can be used to train students in enumeration and generalisation and to present many applications (Kapur, 1970). At the same time, combinatorics is a field that most students find very difficult; most combinatorial problems do not have readily available solution methods (Batanero et al., 1997). Students often have difficulties working with combinatorial problems (Eisenberg & Zaslavsky, 2003; Fischbein & Gazit, 1988). Several studies over the years have promoted approaches to enhance students' capabilities in solving combinatorial problems, from primary children (English, 1991; Hoeveler, 2018; Zak, 2020) to middle- and high-school students (Ďuriš et al., 2021). In our study, we aimed to address some difficulties with combinatorial problems by creating an educational path that takes graph theory into account as a support in solving the problems.

3 Theoretical background

We chose to design the activity and record the results using the Teaching for Robust Understanding (TRU) framework proposed by Schoenfeld (2013, 2014, 2016) and the modelling cycle introduced by Blum and Leiß (2007) and extended by Greefrath (2011).

The TRU framework consists of five dimensions for powerful classrooms, described in Table 1.

The framework identifies these five dimensions, which raise a truly effective teaching/learning context and foster deep student understanding, thereby achieving ambitious, robust teaching. Briefly, "if the content is rich, the students get to engage, they get powerful ideas, they build on each other's ideas, they can build positive identities with the teacher adjusting the level of instruction so that it is right for the students to engage productively" (Schoenfeld, video in https://truframework.org/).

The *content* we choose can provide opportunities to learn; in particular, it can support the important disciplinary idea of mathematical modelling.

As mentioned, we refer to the modelling cycle introduced by Blum and Leiß (2007) and extended by Greefrath (2011) (Fig. 1). Other modelling cycles were presented by Vorhölter et al. (2019). The modelling process is divided into various phases: there is the pole of reality (on the left), the one of mathematics (in the middle), and the one of technology (on the right). The real situation, given in the original problem, is translated into a real model and transferred into the *mathematics realm* in a mathematical model. The mathematical model is then technologically modelled and solved in the *technology realm*. Once the technological results are obtained, they are translated into mathematical solutions, re-interpreted in terms of real results, and given back to the *rest of the world*. The use of the technological realm could be useful for all students, and could aid students with difficulties, thereby aiming for an *Equitable Access to Content*. In the passage from the problem to the mathematical solution, the argumentation (Toulmin, 1958) plays an important role. It consists of one or more linked steps of reasoning that lead from an initial input to a conclusion by means of guaranteed rules. During this process, students realise not only that "the property is true", but also "*why* it is true"; it contributes not only to knowledge construction (Mariotti, 2008, p. 189), but also to explaining already-acquired knowledge to others. Personal knowledge construction makes students able to acquire *Ownership* of the content, even if C*ognitive Demand* is challenging. Argumentation is useful for a *Formative Assessment*, because the teacher can give students opportunities to deepen understanding by listening to and analysing their reasoning, rather than judging only

Table 1 The five dimensions of the TRU model

The five dimensions of powerful classrooms

The Content	Cognitive Demand	Equitable Access to Content	Agency, Ownership, and Identity	Formative Assessment
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful disciplinary thinkers Discussions are focused and coherent, providing opportunities to learn disciplinary ideas, techniques, and perspectives, make connections, and develop productive disciplinary habits of mind	The extent to which students have opportunities to grapple with and make sense of important disciplinary ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conductive to what has been called a "productive struggle"	The extent to which classroom activity structures invites and support the active engagement of all of the students in the classroom with the core disciplinary content being addressed by the class. Classrooms in which small number of students get most of the "air time" are not equitable, no matter how rich the content, all students need to involved in meaningful ways	The extent to which students are provided opportunities to "walk the walk and talk the talk"—to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs- in ways that contribute to their- in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners	The Extent to which classroom activities elicit student thinking and the subsequent interactions in response to those idea, building on productive beginnings and addressing emerging misunderstanding. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understanding

the final results. At this point, students are ready to solve problems and use their acquired abilities and knowledge in different contexts, eventually making abstractions and generalisations, as well as connections to different topics.

4 Research question

The motivations for this work are based on the idea that difficulties often encountered in understanding and dealing with certain topics, may originate not only in the lack of prerequisites, but also in the teaching method of the teacher. In fact, we want to show that in order to teach certain topics, it is insufficient merely to know the subject matter, but, taking into account the knowledge and peculiarities of the students, it is also necessary to know what cognitive mechanisms may or may not lead to understanding of a particular topic, along with some appropriate teaching methods to deal with it. In practice, we want to show that what you teach, how you teach, and to whom you teach are all equally important. In particular, we wanted to test whether the *robust understanding* framework (TRU, see Sect. 3) could be useful in *how* to reach those topics perceived as difficult, such as combinatorial ones (*what*), with middle school pupils (*to whom*). To accomplish this goal, we wanted to use graph theory and technological tools, aiming to translate combinatorial problems into graph problems that can be solved algorithmically, following the *Extended modelling cycle* (see Sect. 3). Thus we posed the following research question (RQ): "Is it possible for 8th grade students to reach a *Robust Understanding* of challenging combinatorial topics, using graph theory and technological tools, to enhance modelling skills?" We divided this question into the following sub-questions, each one regarding one the five dimensions of the *Robust Understanding* framework. RQ1:"Is it possible for 8th grade students to understand rich *Content*, in our case graph theory, combinatorics, and the connection between them?" RQ2: "Would 8th grade students be able to positively answer such challenging *Cognitive Demand*?" RQ3: "Could a path aimed at challenging demand be for all the students, thanks to the use of technology, guaranteeing an *Equitable Access to Content*?" RQ4: "Could an approach using 'real objects' to represent 'mathematical objects' have an impact on *Agency, Ownership, and Identity* in students?" RQ5: "Is an approach based on *Formative Assessment* useful to help students in understanding?". In order to answer the research question RQ we report on our analysis of data in Sect. 8, according to the five dimensions of the TRU model, and taking into account, in several of the investigated dimensions, also the development of the Extended Modelling Cycle.

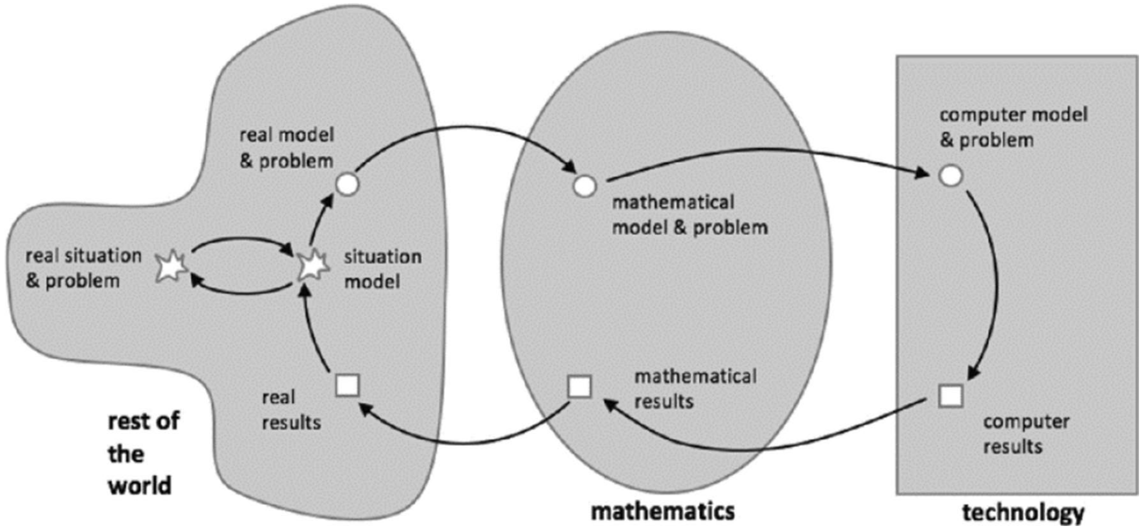


Fig. 1 Extended modelling cycle

5 Mathematical content

Combinatorics is one of the arch enemies of students in the high-school mathematics curriculum, and a new approach to teaching and learning it could be useful. There are several interesting connections between graph theory and combinatorics; the one expressed by Gionfriddo (2011) inspired the educational path that is the focus of this work.

We briefly mention here some definitions and properties of graph theory that we used in our work with students. (For more details, see Voloshin, 2009).

A graph is a pair $G = (V, E)$, where V is a nonempty set of n elements called vertices and E is a set of pairs of distinct elements of V called edges. If x, y are two vertices such that {x, y} is an edge of G, then x, y are said to be adjacent. A graph with n vertices K_n is *complete* if E is the set of all pairs of distinct elements of V. A graph with n vertices Ω_n is *empty* if E is the empty set. A vertex colouring (or simply a colouring) of a graph G is a mapping $f : V \to C$, where C is a set of colours, such that $f(x) \neq f(y)$ for every pair of adjacent vertices x, y. Two *colourings* $f : V \to C$ and $g : V \to C$ of G are said to be *distinct* if there exists at least one vertex $x \in V$ such that $f(x) \neq g(x)$. The *chromatic polynomial* of G is defined to be a function $P(G, \lambda)$ that expresses the number of distinct colourings of G by at most λ colours for each positive integer λ. The *chromatic number* of G is the smallest number of colours necessary to colour a graph.

It is possible to represent a graph graphically by associating each vertex with a point on the plane and each edge with a line joining adjacent vertices (Table 2). Graph G in the table has been vertex-coloured.

Note that, when colouring a complete graph K_n, all vertices must have different colours and that n colours are needed. Moreover, $P\left(K_n, k\right) = P_{k,n} = k(k-1)\dots(k-n+1)$, where $P_{k,n}$ is the number of simple permutations of k objects in n places, with $n \leq k$.

Two vertices x and y are *connected* if there exists an ordered $(2L + 1)$-tuple

$$C(x, y) = \left(x = x_1, s_1, x_2, s_2, \dots, x_L, s_L, x_{L+1} = y\right)$$

Table 2 A null graph, a complete graph, and a coloured graph

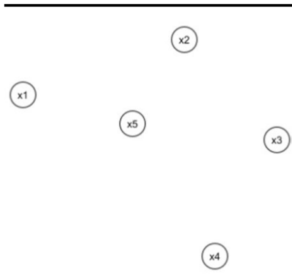
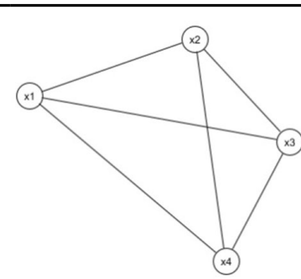
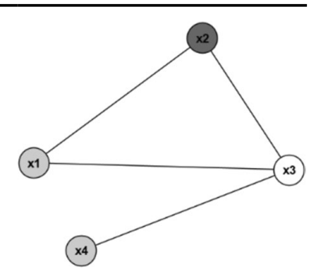
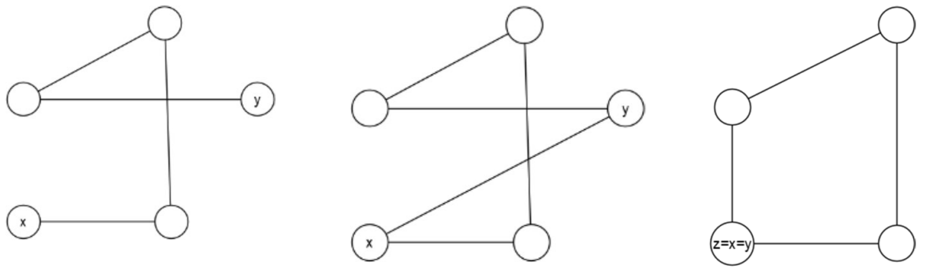
Ω_5	K_4	Graph G
		

Table 3 Graphs G, G + xy, and G\xy



G	$G+xy$	$G\backslash xy$

such that: $x_i \in V, \forall i = 1, 2, \ldots, L+1$, and $s_i \in E, s_i = \{x_i, x_{i+1}\}, \forall i = 1, 2, \ldots, L$.

A graph is said to be *connected* if any pair of vertices is connected.

For our purposes, we used only connected graphs.

In the following, we introduce two graphs, *connection graph* and *contraction graph*, that will be used in the *Connection-Contraction Algorithm*: this algorithm provides all possible colourings of a graph with a given number of colours.

Let $G = (V, E)$ be a graph that is not a complete graph, i.e., $G \neq K_n$. Let $x, y \in V$ be such that x and y are not adjacent vertices. We define the following two graphs (Table 3):

- *connection graph*: $G + xy = (V, E \cup \{x, y\})$;

- *contraction graph:* $G\backslash xy$, obtained from the graph G by substituting the vertices x and y with one vertex $z = x = y$ that is adjacent to all the vertices adjacent to x and all the vertices adjacent to y in the graph G.

We are now ready for the *Connection-Contraction Algorithm (C–C A)*:

1. Suppose G is not a complete graph and let x and y be two non-adjacent vertices.
2. Generate the graphs
   (a) $G + xy$, connected, and
   (b) $G\backslash xy$, contracted.
3. If $G + xy$ (or $G\backslash xy$) is complete, we stop.

If $G + xy$ (or $G\backslash xy$) is not complete, we generate two new graphs from $G + xy$ (or $G\backslash xy$) as in 1. and 2.

We stop only when we obtain complete graphs.

This algorithm produces the chromatic polynomial of a graph (giving all possible colourings of a graph with a given number of colours) and the chromatic number of the graph (the fewest number of colours needed to colour the graph). To understand why this is so, see the example below and consider the following reasoning.

Let f be a colouring of a graph G, and x and y be two non-adjacent vertices of G. If $f(x) \neq f(y)$. Then f is a colouring of $G + xy$, while if $f(x) = f(y)$ then f is a colouring of $G\backslash xy$. Therefore, *if G is not a complete graph, and x and y are non-adjacent vertices, then* $P(G, \lambda) = P(G + xy, \lambda) + P(G\backslash xy, \lambda)$. Now, since the algorithm ends when both $G + xy$ and $G\backslash xy$ have been transformed into complete graphs, we get the chromatic polynomial of G to be:

$P(G, \lambda) = P(K_1, \lambda) + P(K_2, \lambda) + \ldots + P(K_t, \lambda)$, where $K_1, K_2, \ldots, K_t$ are the complete graphs obtained from the previous algorithm.

The chromatic number of the graph (the fewest number of colours needed to colour the graph), is then given by the smallest n such that K_n is one of the complete graphs obtained at the end of the algorithm.

As an example, we can apply the *Connection Contraction Algorithm* in order to know the number of different colourings of the graph in Fig. 2, with, for example, 4 colours (Fig. 3).

The chromatic polynomial of the graph G then turns out to be

$$P(G, \lambda) = P(K_5, \lambda) + 4P(K_4, \lambda) + 2P(K_3, \lambda),$$

from which

$$\begin{aligned} P(G, 4) &= P(K_5, 4) + 4P(K_4, 4) + 2P(K_3, 4) \\ &= 0 + 4P(K_4, 4) + 2P(K_3, 4) \\ &= 4 \cdot (4 \cdot 3 \cdot 2 \cdot 1) + 2 \cdot (4 \cdot 3 \cdot 2) = 144. \end{aligned}$$

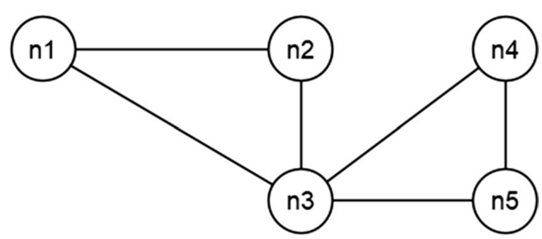


Fig. 2 A graph

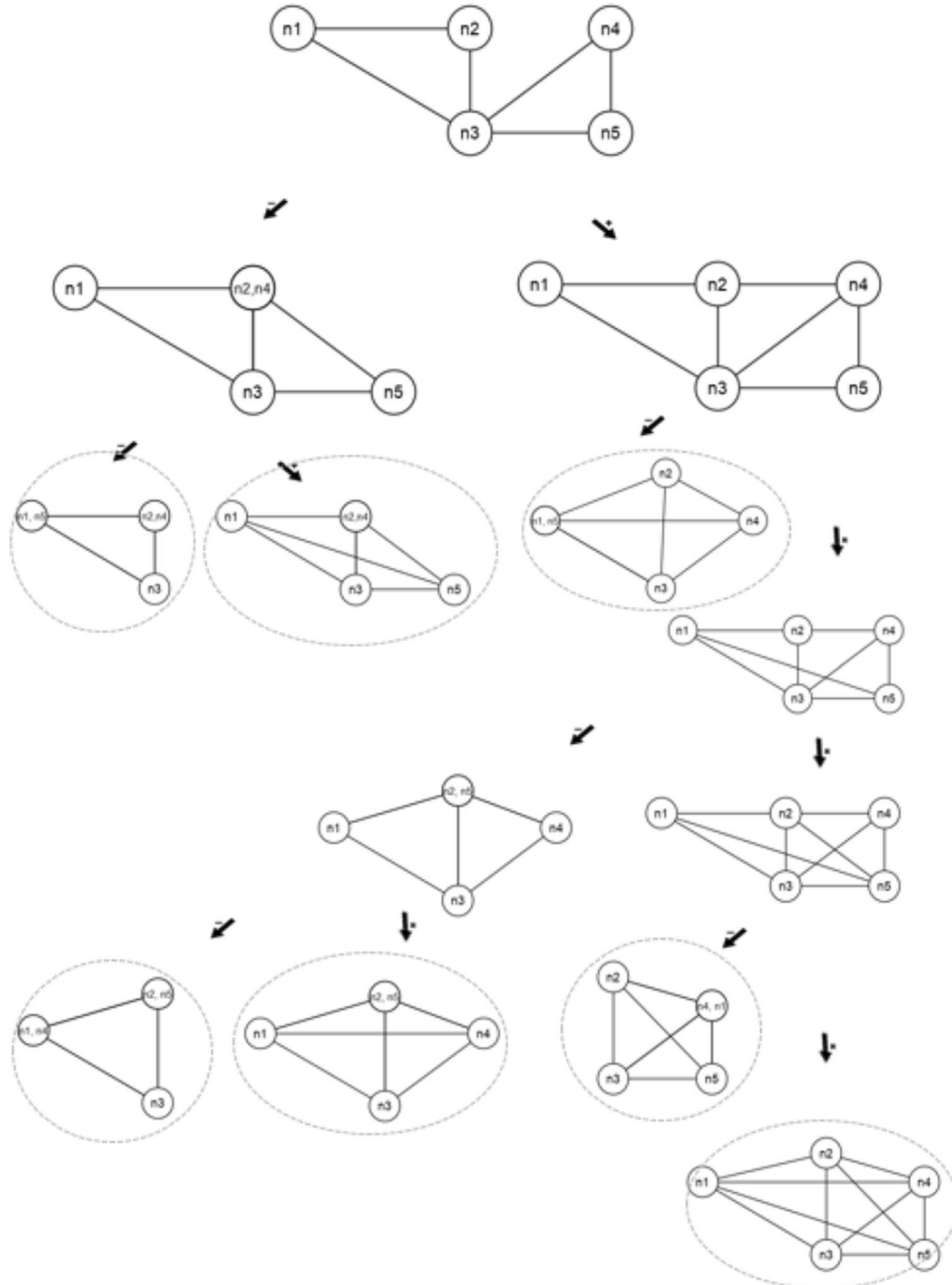


Fig. 3 The Connection-Contraction Algorithm

6 Method

6.1 Study design

The authors proposed the activity in an 8th grade class (in the 'Padre Pio da Pietralcina' school in Misterbianco, Italy); our challenge was to experiment with such topics in middle school, even if it is not in the regular curriculum, in order to start with some important ideas and habits of mind of mathematics, such as modelling. We were not acquainted with the students. We proposed the path to the teacher, who helped us set the activity up in a was that was suitable for her students. We assert that it is important to maintain a very close collaboration between researchers and teachers because "the results of the research will be directly applicable (instead of merely potentially relevant) to practice", as argued by Stylianides and Stylianides (2013, p. 334). Bishop (1998) posed the problem that the research community had not sufficiently answered real problems in real classrooms and claimed that "researchers need to engage more with practitioners' knowledge, perspectives, and work and activity situations, with actual materials and actual constraints and within actual social and institutional contexts" (p. 36). Other researchers have since made similar observations. For instance, Wiliam and Lester (2008) claimed that research needed a radical shift towards interventions taken by researchers and teachers directly, which have been taken into account increasingly in recent years. For example, Ferrara and Ferrari (2020) designed and experimented an intervention, studying the impact of such an intervention when learners are engaged with new situations by thinking mathematically, while furthering and planning the activities with the regular classroom teacher.

6.2 Sample

The class we dealt with was composed of 26 students, many of whom were attentive and well-disposed towards learning mathematics, often actively participating during the lectures. However, a few students had some cognitive difficulties, and we wanted to propose to *all* the students a non-trivial topic, indeed a very rich, potentially difficult one.

6.3 Study method

To achieve our goal, we designed several activities based on the TRU framework, involving the use of bodies, paper, pencils, technological devices (interactive whiteboard and tablets), and specific software to draw graphs (yEd, a graph editor). The classroom was arranged in 'islands' (Fig. 4) where students sat in circles so that they could collaborate and help each other.

The intervention was short. We agree with Stylianides and Stylianides (2013), who argued it is possible to design interventions of short duration in mathematics education that can alleviate typical problems of classroom practice: teachers can benefit from the observed methodology without messing up their curriculum structures. The whole path consisted of three meetings: the first two were held by the researchers and lasted 2 h each; the third one, to consolidate the concepts, was held by the teacher of the class. The researchers were present during the activities to introduce problems and lead collective discussion, but also to observe and interact with students. They video-taped and took pictures of most of the activities with a camera. Students' parents agreed to have their children video-taped by signing a consent form. In particular, the researchers videotaped all their interventions (there were two researchers in the classroom: if one of us was talking, the other one or the regular class teacher videotaped) with special attention to the mathematical discussion and to any speech arising from students. Moreover, they went around the classroom when students were working and videotaped students, paying particular attention to those having some difficulties and/or some good intuitions. With three of us (two researchers and the teacher), we could have a look at each 'island' (Fig. 4), to see how each group was working. The researchers then collected the video data to analysed them later on. Data for our analyses come from the transcriptions of videos and the pictures of students' productions in their exercise books and/or on devices: the authors conducted a content analysis with a directed approach, as a qualitative research technique, to support the theory chosen as a theoretical framework (TRU framework with Extended modelling cycle). Since we used a directed approach, the analysis starts with the chosen theory as guidance for coding. In the qualitative content analysis, in fact, the interpretation of the content of text data (transcriptions from videos, in our case) was done through the classification process of coding. We proceeded as follows: the authors viewed all the videos and selected those most interesting for the research, with open coding, and as a result of this process categories were formulated and revised. This ongoing method aims at a true description of the investigated phenomenon, without

Fig. 4 Classroom disposition

preconceptions of the researcher, to really understand the data, as explained by Mayring (2014, p. 79). One of the authors tagged each video with an initial code containing information with the following six tags: Content_modelling; Content_generalization; Cognitive demand; Equity; Ownership; Formative assessment. In this way, 12 of the 28 initial videos were selected. Then a second author viewed the 12 videos, tagging again each one with one of the six codes. Once the authors agreed on the tags and the videos were definitively selected and tagged, two authors viewed again them, commented on them, and caused transcripts of them to be made. The comments that arose from the authors are reported in Sect. 8, in order to acertain whether the five dimensions of the TRU framework, together with the Modelling cycle, were satisfied.

7 The educational path

As mentioned, the educational path was supported using the yEd software. Concerning the choice of this software, yEd is free software designed to create and manipulate diagrams, and therefore also graphs, supported by most operating systems. It is easy to use; one can decide how to draw each vertex (shape node, Fig. 5), the type of line for an edge, or rearrangement by dragging.

The software seemed especially useful to the researchers in explaining the C–C Algorithm on the interactive whiteboard, and to the students in practicing the algorithm on the tablets, as connected and contracted graphs are easily created using *Copy* and *Paste* commands, and 'dragging' one vertex to another, or connecting two vertices, is done easily and quickly. The software dynamically manages the space on the virtual sheet (that is, the user has a potentially infinite sheet), which is especially useful considering that the complete evolution of the steps of the algorithm is not known a priori; moreover, it is easy to modify the colour of the vertices, helpful for explaining the colouring on the interactive whiteboard.

7.1 First meeting

In the first meeting, we dealt with three activities: *Who is on the podium?*, *The fish problem*, and *Draw the relationship*.

The first one, *Who is on the podium*?, was designed to increase understanding of the mathematical concept of a simple k-permutation (useful, as shown in the following, for calculating the chromatic polynomial). The activity consisted of counting the arrangement of n students in k places. This was done using the classroom's chairs and the students. To start with, the number of chairs was 2, representing a podium with gold and silver medals; we counted how many possible podiums can be obtained with 3 classmates. We then added another classmate, and afterward, another chair: mathematically speaking, we required simple 2-permutations of 3 objects, simple 2-permutations of 4 objects, and simple 3-permutations of 4 objects, respectively.

In the end, we asked the students to generalise the results to obtain the number of simple k-permutations of n objects.

In the second activity, *The fish problem*, we introduced the *leitmotiv* of the whole path:

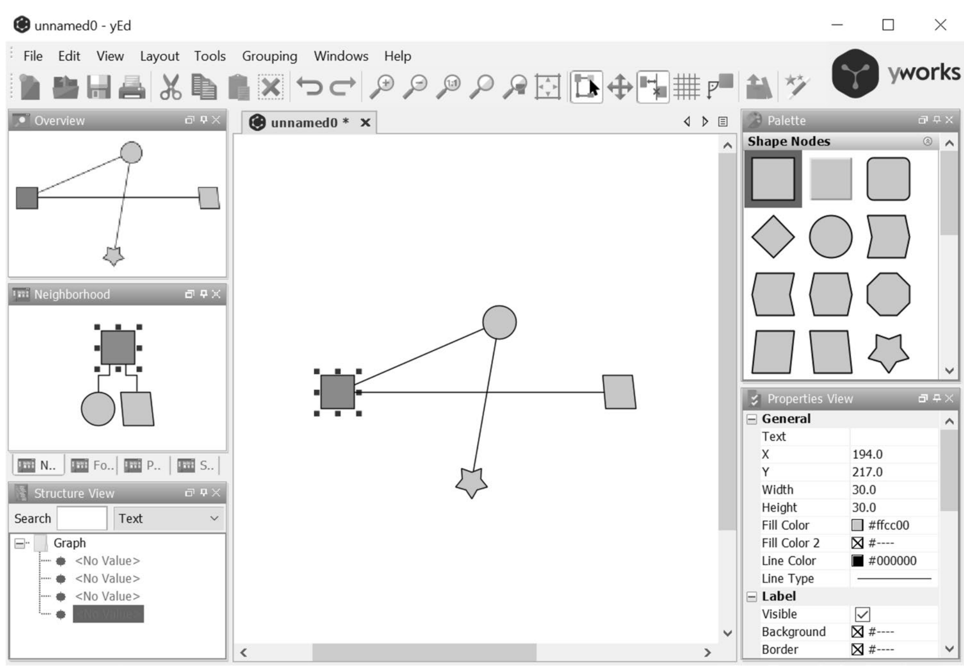


Fig. 5 The yEd overview

The fish problem

The owner of an aquarium store received the new fish he had ordered and now must arrange them in empty tanks. At the moment, there are only 4 empty tanks in the store, all with a coloured lid. The newly arrived fish are of 4 different varieties: regal tang, magnificent fire fish, clownfish, octopus. The shopkeeper knows very well that some of these fish cannot stay together in the same tank, as it would create a prey-predator relationship. In fact:

1. Regal tang fish cannot stay together with magnificent fire fish and octopuses
2. Magnificent fire fish cannot stay together with regal tang fish and octopuses
3. Clownfish cannot stay together with octopuses
4. Octopuses cannot stay together with regal tang fish, magnificent fire fish and clown fish

Keeping in mind that the owner does not have to use all the tanks, in how many ways can he put the fish inside them? Remember: the shopkeeper wants the fish of the same kind to stay together in the same tank!

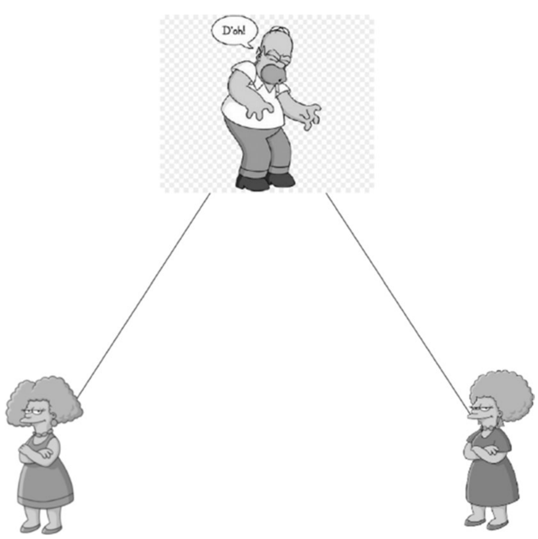


Fig. 6 Simpsons' graph

Students tried to solve the problem by using the newly discovered rules on k-permutations, but they immediately realised that the rules were not suitable for the problem, and that while simple permutations are useful, they cannot be used to solve every kind of problem involving arrangements.

Then we started with the apparently separate topic of a *graph* with the third activity, *Draw the relationship*. We presented graphs of several situations representing relationships among people (brothers, friends practicing the same sport, etc.), asking students, "How can you draw this situation graphically?" Students started drawing the situation on paper. Afterwards, the researchers started using the yEd editor, giving an opportunity to use several images of people as vertices, as well as an opportunity to use any picture as a vertex by simply dragging it into the sheet. In Fig. 6, we show the graph representing the relationships between Homer Simpson and his sisters-in-law, defined by 'being in conflict': Homer is connected by an edge to each one of the sisters-in-law because they do not get along together, while

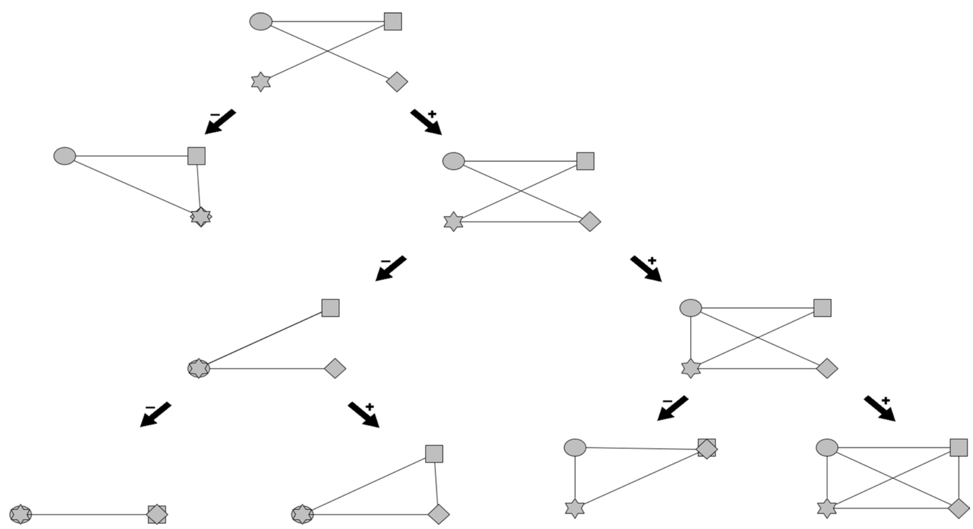


Fig. 7 Example of the Connection-Contraction Algorithm

there is no edge between the two women because they are not in conflict. We decided to use this example because we found it useful to imagine that two people in conflict want to stay in different places (or different colours, in term of the graphs' colourings), as in our fish problem (keeping fish in different tanks). If we have to colour the Simpsons' graph we should use at least two colours, one for Homer and a different one for the women. In general, graphs are used to model relationships and graph colouring is primarily used to model the conflict relationship, to help solve problems where you want to manage conflicts, as in this simple example.

7.2 Second meeting

In the second meeting, after recalling the graph's topic, we dealt with vertex colourings of graphs. We pointed out how easy it is to find the number of all possible n-colourings of a complete graph: it suffices to count the number k of vertices of the graph, then the number of k-permutations of n objects. Then, by using yEd at the interactive whiteboard, we explained the C–C A starting from a specific graph, together with the representations of all the steps of the algorithm, (Fig. 7).

The class was invited to practice the algorithm, applying it to some graphs we provided. Each student chose to practice in the way he/she personally thought best: some used the yEd software installed on their tablets, others used pen and paper, another used clipped paper. All of them managed to master the use of the passages sufficiently.

Afterwards, we discussed the usefulness of the algorithm, emphasising that, thanks to the resulting complete graphs, we can determine the chromatic number of each graph exactly, but above all, we are able to determine the chromatic polynomial of each graph. We also discussed the advantages of using the algorithm to obtain absolute certainty of having exhausted all possible colourings, as opposed to solving the problems by repeated attempts. Finally, students were guided to model *The fish problem* in terms of graphs and to see the connection between the two topics (combinatorics and graphs). They easily determined the particular chromatic polynomial associated with the graph of *The fish problem* (Fig. 8), obtaining as a result the value 72 as the total number of colourings of the graph with 4 colours from the chromatic polynomial $n^4 - 4n^3 + 5n^2 - 2$.

Now the class was able to solve the problem easily, taking very little time, and succeeding, without too much effort, in carrying out the generalisation of the result to find the chromatic polynomial.

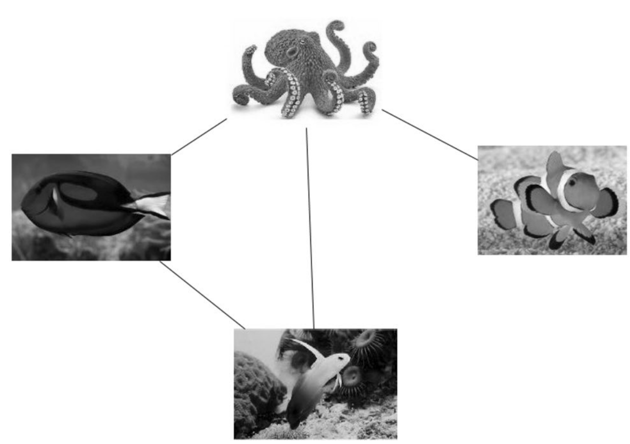


Fig. 8 Graph associated with *The fish problem*

In the third meeting, the general concept of the chromatic polynomial was consolidated and the path ended with a connection to algebra topics (polynomials) that students had been working on before starting this path.

8 Analysis of the educational path

Our activity is in the TRU framework because we proposed a rich topic (The *Content*) with several connections and realistic examples that foster modelling and conduct students to a productive struggle (*Cognitive Demand*). All students had personal devices, could work in groups, and had access to different ways of working, according to what was most suitable to them (*Equitable Access*). In this situation, there was active participation by students who were free to model the problem as they preferred and to argue for their choices (*Agency, Ownership, Identity*). The teachers (here, the researchers) played a central role: they met students 'where they were', collected their ideas, built on their beginnings, and addressed their misunderstandings (*Formative Assessment*).

Before going into the details of the analysis, we want to describe the class environment fostered by the teacher, Maria, to help explain how the class was prepared for the experiment. As mentioned, the students were attentive and active. This is also due to Maria, who got her students used to thinking about, arguing, and practicing mathematical concepts, rather than explaining prepared mathematical topics. During her lectures, Maria often asks students what they thought, to give their ideas, and to share possible solutions of tasks with their classmates. The content of the experiment was very rich and potentially difficult, and we had doubts about obtaining helpful results. During the experimentation in class, we videotaped the meetings and, in the end, used the videos to categorise the results that were obtained in terms of the efficacy of the TRU framework, considering also the extended modelling cycle (Greefrath, 2011). Here we deal with the five dimensions of TRU, making our considerations with respect to each one, answering each sub-research question, which is reported at the beginning of each subsection. In the following, quotes of students or researchers are written in italics in the text.

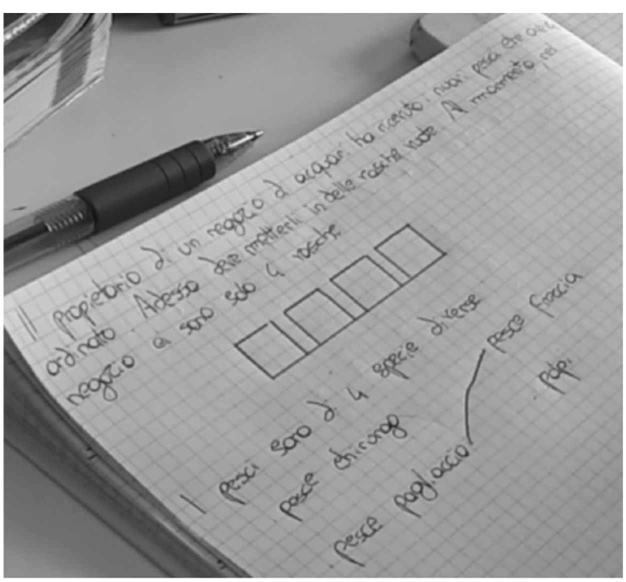


Fig. 9 Dario's 'scheme' of the fish problem

8.1 Dimension 1: the content

RQ1: 'Is it possible for 8th grade students to understand rich *Content*, in our case graph theory, combinatorics, and the connection between them?'.

Too often, at least in Italy, mathematics is presented as a set of separate chapters, and rarely do teachers work on connections among mathematical topics. The content we dealt with, instead, is rich in connections among topics and with the theme the class was working on before our experiment, namely, early algebra. We arrived at algebraic concepts only at the end of the path, passing over two apparently separated topics (combinatorics and graphs). Rather than being superficial, our content is rich indeed (it can be taught at the university level). Moreover, it can provide one of the important mathematical skills, namely, modelling. We helped students with modelling using a mathematical concept that is easy to grasp, namely, a graph. So easy, in fact, that when we posed *The fish problem* (activity 2), a student had already drawn a graph before knowing the mathematical concept (Fig. 9).

The student Dario drew the scheme shown in Fig. 9. Here is a dialog between Dario (D) and the researcher (R), which anticipates the vertex-colouring topic:

D: "*If the clownfish can stay together with magnificent fire fish, they are no more 4, but 3. It is like we have 3 varieties of fish in 4 tanks*."

R: "*And if, instead, I leave them separated, they are 4*."

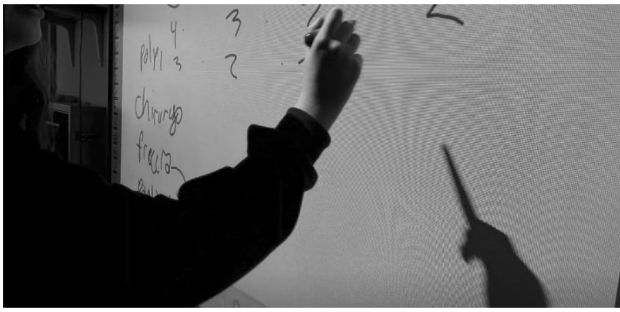
D: "*Yes*."

R: "*So, I can do this: I can consider them equals, right? … or I can consider them separated. Very good. This will be very useful, especially next time.*"

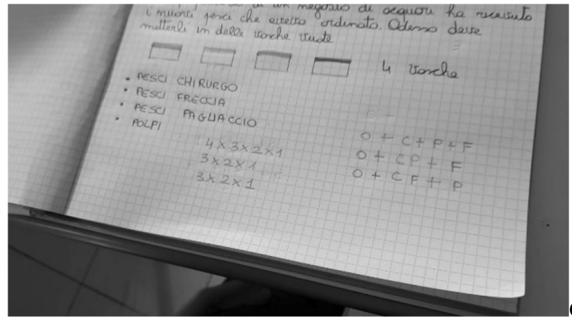
Dario's scheme is the complement of the graph we then used to solve *The fish problem* (he joined the fish that can stay together instead of the fish that cannot), but the researcher here was focused only on how to draw the situation given by relationships using a graph. The researchers, at this point, were happy to see the choice made to use the correct way (graphs) to grasp potentially difficult content.

As for modelling, here Dario is still in the *rest of the world* realm (Fig. 1), handling the *real situation & problem*.

Table 4 Alice's and Giulia's attempts



Alice tries to solve the fish problem



Giulia tries to solve the fish problem

In fact, he talks about fish and tanks, but he is starting to face the problem in a mathematical way by schematising the problem on paper.

8.2 Dimension 2: cognitive demand

RQ2: 'Could 8th grade students be able to positively answer such a challenging *Cognitive Demand*?'.

Even if we discussed only the usefulness of an intuitive object, like a graph, we emphasise that the posed task was not trivial to solve. It requires a struggle. It was a challenge. At the beginning, when we asked how many possibilities the owner of an aquarium had wherewith to arrange the fish, students were not able to answer correctly. They tried to use what we had just explained: simple k-permutations (Table 4).

In particular, two students, Alice and Giulia, counted 24 ways to arrange the 4 fish varieties in 4 tanks (simple 4-permutations of 4 objects) and 12 ways (6 + 6) to arrange 3 fish varieties (identifying two varieties that can stay in the same tanks) in 3 tanks (simple 3-permutations of 3 objects). In this last calculation, they should have counted two simple 3-permutations of 4 objects instead.

By the end of the path, the students understood how to count all possible arrangements, winning the challenge. We relate a discussion after dealing with vertex colourings and the algorithm, as recorded by the researcher in class:

R: "*How did we solve the fish problem? Do you remember?*".

Student: "*Yes, we have 4 empty tanks and 4 varieties of fish.*"

R: "*Yes. So, how is the graph? We have 4 tanks, which means … What does it mean to have 4 tanks?*".

Student: "*4 … 4 colours*". (Several students said, together, "*4 colours*".)

R: "*4 colours: red tank, green tank, …. And the fish are: octopus, magnificent fire fish, clownfish and regal tang fish* (drawing on the blackboard 4 points with the names of fish) *and what about the links? We joined … whom?*".

Student: "*Who cannot stay in the same tank.*"

Fig. 10 Students guide the researchers to draw the 'fish graph'

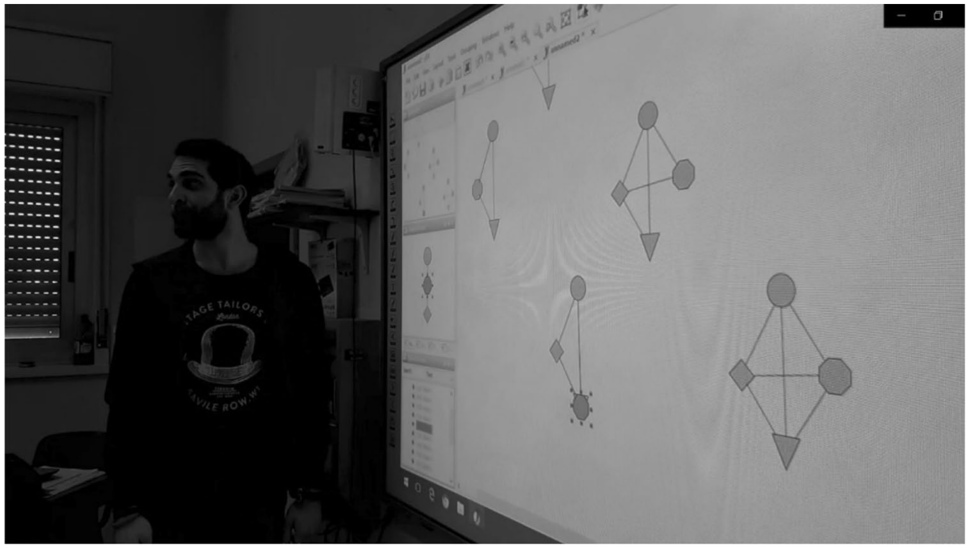


Fig. 11 Complete graphs at the end of the Connection-Contraction Algorithm

Fig. 12 Final counting for the fish problem

R: "*Who cannot stay in the same tank, in such a way that they have a different colour.*"

Students were able to pass from a *real model & problem* to a *mathematical model & problem* (as in the Blum and Leiß cycle, shown in Fig. 1). In fact, they easily translated real objects (fish and tanks) into mathematical objects (vertices of a graph and colours of vertices) and a real relationship (a food chain) into a mathematical relationship (edges of the graph). While one of the researchers drew the graph suggested by students with the fish names (Fig. 10), another one drew the graph with the yEd editor, anticipating the application of the C-CAlgorithm, arriving at the *computer model & problem* in the *technology realm* (Fig. 1), making sure that students understood that, when colouring a graph, two unconnected vertices can have different colours (connection) or the same colour (contraction). After using the algorithm, students were invited to observe the complete graphs obtained (Fig. 11) and count the arrangements. The researcher invited students to move from the *computer*

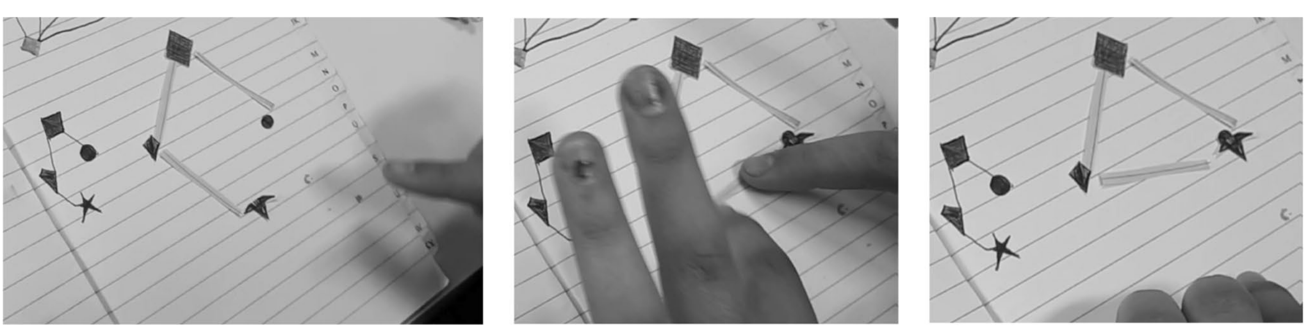


Fig. 13 Asia contracts a graph using physical movements

results of the technology realm to the *mathematical results* of the mathematics one (Fig. 1):

R: “*What have we obtained?*”.

Student: “*2, 2 C_3 and 1 C_4*” (students called complete graphs C_n, instead of K_n, as we did in Sect. 5, because ‘C’ is the first letter of the Italian word for ‘complete’).

R.: “*And then how many ways do we have to colour, i.e., to put the fish in the tanks?*”.

Students correctly counted, together, “*for every C_3 with 4 colours,* 4 • 3 • 2*, twice, plus* 4 • 3 • 2 • 1.” The researcher wrote what they suggested on the blackboard (Fig. 12). By the end, students had arrived at the *rest of the world* realm for the final *real results* (Fig. 1), after struggling in the beginning with a non trivial task.

8.3 Dimension 3: equitable access to mathematics

RQ3: ‘Could a path aimed at challenging demand be for all the students, thanks to the use of technology, guarantee *Equitable Access to Content*?’.

The class was composed of students with different aptitudes, but all very accustomed to technology. So, we decided to set up the educational path with an emphasis on technology, but we also wanted to leave students free to use the approach they felt would be best. We supported students in their method of choice, both in posing the problem and in practicing the algorithm. Some of them preferred technological devices; others, paper and pencil; still others cut the paper to create vertices and edges. Moreover, we let students work and went around the desks to support them in the mathematical activities. Here we show Asia’s behaviour: she decided to cut the paper, emphasising how technology could be useful in such cases; the student decided to contract and connect graphs by using and rearranging physical objects she created herself with coloured paper (Fig. 13).

The researcher aided Asia with the task, but noticed that she had some difficulties, beginning with the first steps of the algorithm. The girl correctly understood the algorithm, but in moving the vertices she could represent only one situation (the one in which she was presently working), losing track of any previous steps. In fact, from the second step forward, she always connected vertices not yet joined, but forgot to contract them. Passing from the *mathematical model* to the *computer model* (Fig. 1) can help in solving the task, because technology, in this case, could have been used as an extension of memory: the software does not think for the person, but can leave the person’s mind free from remembering previous steps and free to focus on the important mathematical concepts. Indeed, we asked Giulia, another student, how she preferred to work and why. She answered: “*Using the software, because it is more usable and quicker* (than using paper and pen). *Because you can do more in less time*.”

We noticed, moreover, that in each step, one can pass from a specific graph to a more general set of generated graphs, using the *Zoom* command in the software, keeping the whole algorithm under control. This is what Dario and Giulia did, for instance. See Fig. 14 for the *Zoom* command used by Dario.

Another technological feature that can help students access mathematical activities is the virtual approach: the software gives the opportunity to manipulate the vertices as if they were real objects, taking advantage of the ease in obtaining and dragging copies. This is evident in what Dario, who used yEd, said: “*We start from this one, the main* (graph)*, where the square is joined to the octagon and the circle is alone. I copied it and overlapped the circle and the square, obtaining a complete graph. Instead, this side I joined the circle and the square.”* Dario talked about movements—“I overlapped” and “I joined”—referring to virtual, not physical, objects.

8.4 Dimension 4: agency, ownership, and identity

RQ4: ‘Could an approach using ‘real objects’ to represent ‘mathematical objects’ facilitate *Agency, Ownership, and Identity* in students?’.

We wanted the students to be the main actors in the educational path; we wanted them to be active with their minds, hands, and entire bodies. This is why we engaged them from the beginning with activities involving their bodies, real objects that they can touch, instead of mathematical abstract objects, as in the first task, used for simple *k*-permutations

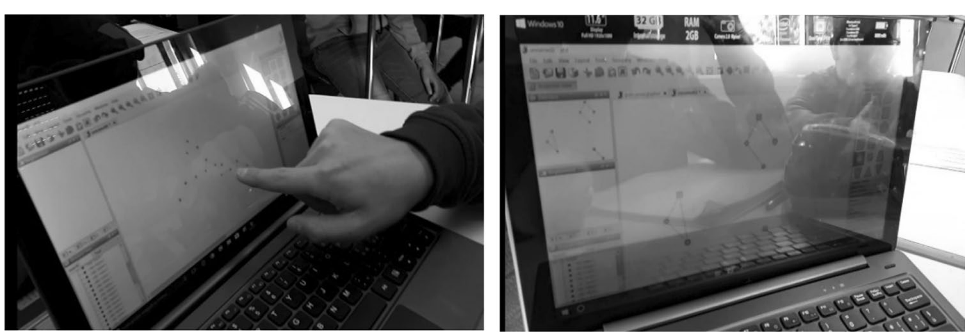


Fig. 14 Dario uses the Zoom command to focus on the general and the specific

(*Who is on the podium?*). Being active in both mind and body was useful in generalising simple *k*-permutations, as shown in the following dialogue between Graziano (G) and the researcher:

G: "*If we have 5 people in 3 places, we have* 5 • 4 • 3*.*"

R: "*Good! That's right. Could you tell me 'mathematically?'*".

G: "*The number of factors depends on the number of places*."

R: "*And why do you start from 5?*".

G: "*Because if we add a person, we have 5 possibilities from which to choose for the first place, and in the second place we will put 4, and in the third, 3.*"

Graziano talked in terms of people and also used gestures to simulate the movements of a person added to the group of classmates on the;podium'. Dario and other students referred to the activity using physical bodies in the second meeting, during a discussion on graph colouring:

R: "*If I have a complete graph with 2 vertices, and I have 5 colours, how many colourings can I have?*".

D: "*20, because we have to compute* 5 • 4."

R: "*How did you come up with it so quickly?*".

G: "*Because we remember the example from our last meeting, concerning first and second place*."

In the second and third activities (*The fish problem* and *Draw the relationship*), students dealt with fish, friends, and cartoons, rather than with 'abstract' points (this choice was helpful to promote students' ownership). They were invited to avoid considering already-proven mathematics and to think and learn by their own 'creations'. They showed ownership of the content, easily passing from one representation of the topic to another. We relate a discussion that emphasises this ability to pass between real and mathematical topics and move easily within the realm of the extended modelling cycle (Fig. 1):

R: "*We have a complete graph with 2 vertices and 5 colours. How many colours do we have?*".

Student: "*20, because we have 2 tanks and 5 varieties of fish.*"

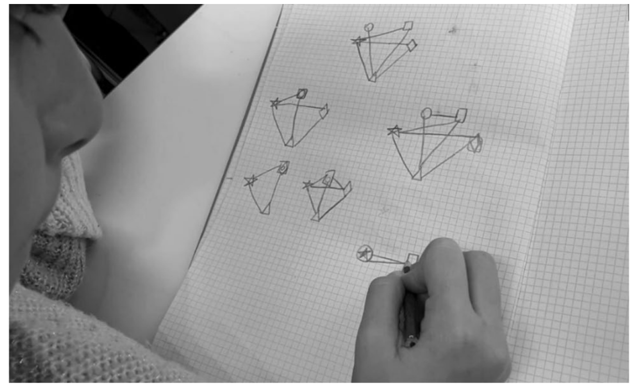


Fig. 16 Roberta reports 2 copies of an edge in the contracted graph

Even if the student is confused about tanks (which should be the number of colours, 5) and varieties of fish (which should be the number of vertices, 2), he realises that the boundary between math and real topics is porous. The researcher asked about graphs, and they answered in terms of tanks and fish. Students had no problem assigning equal/different colourings to equal/different tanks in the fish problem. They showed ownership of the topic, seeing it from different points of view.

8.5 Dimension 5: formative assessment

RQ5: 'Is an approach based on a *Formative Assessment*, useful to help students in understanding?'.

We met the students 'where they were', leaving them time and space to think and produce according to their own ideas. For such a teaching method, it is neither easy nor helpful to provide a summative assessment. We did not want a quick, accurate answer from students, but rather questions, ideas, and strategies. To encourage students, it can be useful to accompany them in their current path, suggesting ways to set up and/or fix the steps in such a way as to guide them in the right direction. We have some videos showing students who applied the C–C Algorithm correctly and quickly, making

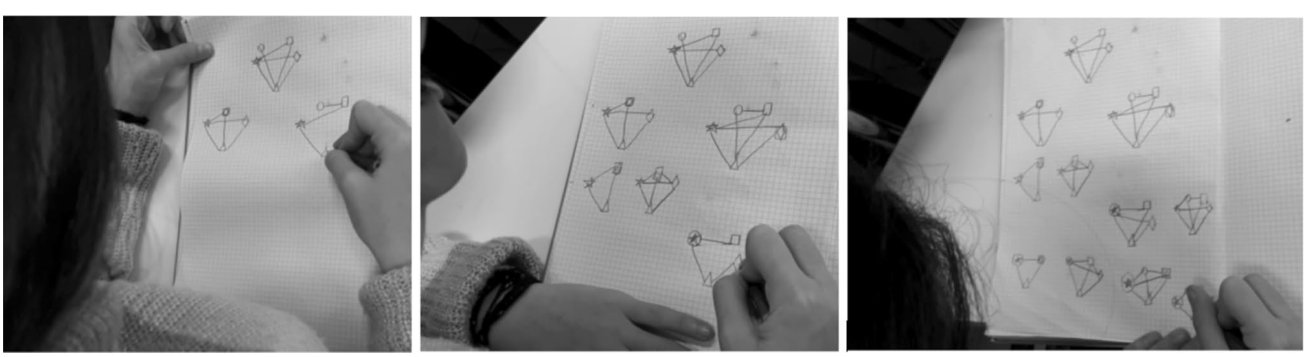


Fig. 15 Roberta applies the algorithm with paper and pencil

use of the yEd software. Here, however, we want to relate what happened with Roberta, who decided to work on paper and needed some suggestions. First of all, we noticed that, as shown in the third picture of Fig. 15, Roberta filled the whole sheet before completing the algorithm. This problem could have been avoided by using the potentially infinite sheet in the yEd editor. However, the researcher did not want to force the student to use the software, because of our preference to respect the method students chose to use. This led the researcher to accompany the girl, helping her fix the problem, by essentially discarding graphs.

In particular, she often forgot the connected graph. Moreover, in the contracted graph, whenever she overlapped two vertices, she tended to draw 2 copies of the edges (Fig. 16).

Again, it would have been helpful to suggest the use of the software, because it would have made all the steps ‘automatic’, but the researcher respected Roberta’s choice. She needed the constant presence of the researcher to guide her in various steps, even if she understood all the steps of the algorithm. All the corrections were made to help her keep the steps in mind, rather than to assign her a grade. Moreover, the help and interaction in class was not only between students and researchers, but also among classmates arranged in ‘islands’ (Fig. 4).

We want to emphasise that although no grades were given to the students, they remembered in the second meeting what they had done and learned the previous week. Their aim was not related to a performance goal, but rather to a mastery goal. Students were genuinely interested in solving the fish problem and engaged in the whole activity, while also having fun.

9 Conclusions

The educational path presented in this paper was designed with inspiration taken from Gionfriddo (2011). The mathematical content was aimed at linking combinatorics and graph theory. While neither topic is part of the school curriculum, arguments can be made for including both topics (Sandefur, Lockwood, Hart, & Greefrath, 2022), and we were convinced to embrace this challenging task. Moreover, we also connected the topics to algorithms, computer use and modelling.

The educational path is divided into phases marked by three activities (*Who is on the podium?*, *The fish problem*, and *Draw the relationship*) and a meeting on vertex colouring and the C–C Algorithm. The activities were configured as a tool to introduce the topic to students and initiate knowledge processes that would unfold through discussions, comparisons, and reasoning, with the help of digital and non-digital technologies. The modelling activity was prevalent throughout the entire process. In the first activity, *Who is on the podium?*, the modelling activity occured through consultation with peers. Subsequently, in *The fish problem*, students immediately modelled the problem using graph theory and solved the question with the help of technology (yEd), then returned to the solution of the problem by implementing the extended modelling cycle of Greefrath (2011).

Upon finally solving *The fish problem*, the researcher asked, "What are the tanks?" Students immediately answered “*the colours*:” not only was the modelling implemented, but there was also an awareness of the analogy between the problem posed and the mathematical tool used to solve it. The educational path in the classroom was carried out in the spirit of *learning by doing*: by doing, I discover, I think, I verify, I try, I argue my position. The evolution of the students' argumentative competence (Toulmin, 1958) is evident from their productions. Dario supported his thesis with conviction by exposing the different possibilities of colouring the graph, first with three, then with two colours, indicating the vertices he can colour in red and those he can colour in green.

Students were able to apply the algorithm and justify why the algorithm works, observing contracting two vertices means to make them of the same colour, and connecting two vertices by an edge means they are coloured with different colours.

The dimensions of the TRU found application in this pathway, in which, as we saw in a previous section, *content*, *involvement*, *challenges*, and *new problems* combined to create that *mathematical identity* appropriate for *each* student, also enhancing modelling skills, since students were able to pass among *Rest of the world*, *Mathematics* and *Technology* realms of the Extended Modelling cycle.

With activities of this type, one has the opportunity to give meaning to mathematical concepts that often remain abstract and generally considered difficult; all students can be creative and small mathematicians. In the end of the experimentation, when asking to students what graphs might be useful for, Giulia answered “*We need them to solve problems. When we have a schema represented by a graph, we understand what we have to do. We were asked to work on the fish problem and we could not solve it at first. Today, with the graphs, we completed it*.” In Giulia’s response we see the evolution of the whole activity: modelling (by graphs), solving problems (with the algorithm), and the possibility to re-use what they learned (to solve other problems).

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