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Differential identities and almost polynomial growth. Star algebras and cocharacters.

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To my mother
and my brother

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Contents

Introduction

This thesis is devoted to the study of some interesting and challenging aspects of PI-theory, i.e., the theory of algebras that satisfy a non-trivial polynomial identity (PI-algebras).

Let A be an associative algebra over an infinite field F. A polynomial identity of A is a polynomial in non-commuting variables vanishing under all evaluations in A. We denote by $\mathrm{Id}(A)$ the T-ideal of all polynomial identities of A.

The description of the identities of an algebra is in general a hard problem. In fact, even if every T-ideal is finitely generated (see [34]), the polynomial identities are far from being understood. Also it is quite impossible in general to deduce from the generators of $\mathrm{Id}(A)$ information on the polynomials of A of a given degree. To overcome some of these difficulties it is natural to introduce some numerical invariants allowing to give a quantitative description of the growth of the polynomial identities of A.

Alongside the ordinary polynomial identities, it is often convenient to study the polynomial identities of algebras with an additional structure such as group-graded algebras, algebras with an action of a group by automorphism and anti-automorphism, algebras with an action of a Lie algebra by derivations, or more in general algebras with a generalized Hopf algebra action (see, for instance, [4, 6, 7, 19, 21, 36]). In fact, such identities theoretically determine the ordinary ones and also they allow to construct finer invariants that can be related to the ordinary ones.

The purpose of this thesis is the study of some invariants of polynomial identities of algebras with derivation (algebras with an action of a Lie algebra by derivations) and of algebras with involution (algebras with an action of an anti-automorphism of order two).

A very useful numerical invariant measuring the growth of the polynomial identities of A is the codimensions sequence, $\{c_n(A)\}_{n\geq 1}$. In general $c_n(A)$ is bounded from above by $n!$, but in case A is a PI-algebra a celebrated theorem of Regev asserts that $\{c_n(A)\}_{n\geq 1}$ is exponentially bounded (see [47]). Later Kemer (see [35]) showed that, given any PI-algebra A, $\{c_n(A)\}_{n\geq 1}$ cannot have intermediate growth, i.e., either is polynomially bounded or grows exponentially. Moreover, Giambruno and Zaicev in [22] and [23] computed the exponential rate of growth of a PI-algebra and proved that it is a non-negative integer.

In this context it is often convenient to use the language of varieties of algebras. Given a variety of algebras $\mathcal{V}$, the growth of $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}(A)$. The algebra of 2×2 upper triangular matrices $UT_2(F)$ and the infinite dimensional Grassmann algebra G are crucial in the investigation of the growth of the polynomial identities of algebras. In fact, a well known theorem of Kemer (see [33]) states that $\mathrm{var}(UT_2(F))$ and $\mathrm{var}(G)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

In light of the above, it seems interesting to study the structure of the polynomial identities of the algebras $UT_2(F)$ and G with an additional structure. In this perspective, one of the aims of this thesis is to study the growth of the differential identities of these two algebras.

In case the base field is of characteristic zero, there is another useful invariant that can be attached to the identities of an algebra A, the so-called cocharacter sequence. Since the base field is of characteristic zero, every T-ideal is completely determined by its multilinear elements. Hence, one considers, for every $n \geq 1$, the space P_n of all multilinear polynomials in a given fixed set of n variables and acts on it with the symmetric group S_n. The space P_n modulo $\mathrm{Id}(A)$ becomes an S_n-module, its character, $\chi_n(A)$, is called the nth-cocharacter of A and $\{\chi_n(A)\}_{n\geq 1}$ is the cocharacter sequence of A. By complete reducibility we can write the nth-cocharacter of A as a sum of irreducible characters with corresponding multiplicities.

We already know that, in case A is a PI-algebra, the multiplicities of its cocharacter are polynomially bounded. Thus it seems interesting to characterize the cocharacter sequence when stronger conditions hold for the multiplicities. Consequently in this thesis we characterize the cocharacter sequence of algebras with involution when the corresponding multiplicities are bounded by a constant.

The first chapter of this thesis is preliminary and contains the basic definitions and results needed for the further exposition. We introduce the algebras with polynomial identity by giving their basic definitions and properties. Then we give a brief introduction to the classical representation theory of the symmetric group via the theory of Young diagrams. In the last part of the chapter we deal with the basic numerical invariants of the polynomial identities of a given algebras: the codimensions sequence and cocharacter sequence of the T-ideal an the algebra. Moreover we present some important results about the asymptotic behaviour of the sequence of codimensions.

In the second chapter we extend our approach to algebras with derivations and to algebras with involution. We first give a complete view of algebras with a Hopf algebra action and their identities. Then, as a particular case, we introduce the differential identities. Finally, in order to include the algebras with involution, we extend the Hopf algebra action to a more general action and then we present some relevant results concerning algebras with involution.

In the third chapter we study in detail the differential polynomial identities of the algebra of 2×2 upper triangular matrices over a field of characteristic zero when two distinct Lie algebras of derivations act on it. We explicitly determine a basis of the corresponding differential identities, the sequence of codimensions and the sequence of cocharacters in both cases. Furthermore, we study the growth of differential identities in both cases. In particular we prove that when the Lie algebra L of all derivations acts on $UT_2(F)$, then the variety of differential algebras with L action generated by $UT_2(F)$ has no almost polynomial growth (unlike the ordinary case and the graded case); nevertheless we exhibit a subvariety of almost polynomial growth.

The fourth chapter is devote to the study of the differential identities of the infinite dimensional Grassmann algebra over a field F of characteristic different from two with respect to the action of a finite dimensional Lie algebra L of inner derivations. We explicitly construct a set of generators for the ideal of differential identities of G and also we compute its differential codimensions. As a consequence it turns out that the growth of the differential identities of G is exponential, as in the ordinary case. However, we prove that unlike the ordinary case G with the action of a finite dimensional Lie algebra of inner derivations does not generate a variety of almost polynomial growth; in fact we exhibit a subvariety of exponential growth. Furthermore, when the base field is of characteristic zero, we determine the decomposition of the differential cocharacter of G in its irreducible components by computing all the corresponding multiplicities.

Finally, in the fifth chapter we introduce the Grassmann envelope of a superalgebra with superinvolution and we describe a useful connection between varieties of algebras with involution and vareities of superalgebras with superinvolution. Then we study and characterize the algebras with involution over a field F of characteristic zero satisfying a polynomial identity such that the multiplicities in the corresponding $*$-cocharacter are bounded by a constant.

Chapter 1

A general setting

In this first chapter we introduce the main object of study, i.e., PI-algebras and we give their basic properties.

In the first section we present the notions of T-ideal of the free algebra and of variety of algebras. We also discuss the so-called multilinear polynomials and some of their properties.

In the second section, we introduce ordinary representation theory of the symmetric group S_n through the theory of Young tableaux. We also define a natural action of the symmetric group S_n on the space of multilinear polynomials in n variables which has the basic property of leaving T-ideals invariant. Our main objective in this setting is to understand the decomposition of the corresponding module into irreducibles.

As a consequence in the last section we introduce two numerical invariants of a T-ideal: the sequence of codimensions and the sequence of cocharacters. We give two typical examples of PI-algebras: the Grassmann algebra and $UT_2(F)$, the upper triangular matrices of order 2 over a field F, that we shall use later.

Finally, we present some celebrated theorems about the growth of the codimension sequence of a PI-algebra.

1.1 Basic definitions

We start with the definition of free algebra. Let F be a field and $X = \{x_1, x_2, \dots\}$ a countable set. The free associative algebra on X over F is the algebra $F\langle X\rangle$ of polynomials in the non-commuting indeterminates $x \in X$. A basis of $F\langle X\rangle$ is given by all words in the alphabet X, adding the empty word 1. Such words are called monomials and the product of two monomials is defined by juxtaposition. The elements of $F\langle X\rangle$

are called polynomials and if $f \in F\langle X \rangle$, then we write $f = f(x_1, \ldots, x_n)$ to indicate that $x_1, \ldots, x_n \in X$ are the only indeterminates occurring in f.

We define $\deg u$, the degree of a monomial u, as the length of the word u. Also $\deg_{x_i} u$, the degree of u in the indeterminate x_i, is the number of the occurrences of x_i in u. Similarly, the degree $\deg f$ of a polynomial $f = f(x_1, \ldots, x_n)$ is the maximum degree of a monomial in f and $\deg_{x_i} f$, the degree of f in x_i, is the maximum degree of $\deg_{x_i} u$, for u monomial in f.

The algebra $F\langle X \rangle$ is defined, up to isomorphism, by the following universal property: given an associative F-algebra A, any map $X \to A$ can be uniquely extended to a homomorphism of algebras $F\langle X \rangle \to A$. The cardinality of X is called the rank of $F\langle X \rangle$.

Definition 1.1.1. *Let A be an associative F-algebra and $f = f(x_1, \ldots, x_n) \in F\langle X \rangle$. We say that f is a polynomial identity for A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$.*

We shall usually say also that A satisfies $f \equiv 0$ or, sometimes, that f itself is an identity of A. Since the trivial polynomial $f = 0$ is an identity for any algebra A, we say that A is a PI-algebra if it satisfies a non-trivial polynomial identity.

For $a, b \in A$, let $[a, b] = ab - ba$ denote the Lie commutator of a and b. Now we are able to give some examples of PI-algebras.

Example 1.1.1. *If A is a commutative algebra, then A is a PI-algebra since it satisfies the identity $[x_1, x_2] \equiv 0$.*

Example 1.1.2. *If A is a nilpotent algebra, with $A^n = 0$, then A is a PI-algebra since it satisfies the identity $x_1 \cdots x_n \equiv 0$.*

Example 1.1.3. *Let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices over F. Then $UT_n(F)$ satisfies the identity:*

$$[x_1, x_2] \ldots [x_{2n-1}, x_{2n}] \equiv 0.$$

Example 1.1.4. *Let G be the Grassmann algebra on a countable dimension vector space over a field F of characteristic different from 2. Then G satisfies the identity $[[x, y], z] \equiv 0$.*

A central role in the theory of PI-algebras is played by the T-ideal of polynomial identities of an algebra A over a field F.

Definition 1.1.2. *Given an algebra A, the two-sided ideal of polynomial identities of A is defined as*

$$Id(A) = \{f \in F\langle X\rangle | f \equiv 0 \text{ on } A\}.$$

Recalling that an ideal I of $F\langle X\rangle$ is a T-ideal if $\varphi(I) \subseteq I$, for all endomorphism φ of $F\langle X\rangle$, it is easy to check that $\mathrm{Id}(A)$ is a T-ideal of $F\langle X\rangle$. Moreover, given a T-ideal I, we have that $\mathrm{Id}(F\langle X\rangle/I) = I$. Hence every T-ideal of $F\langle X\rangle$ is actually the ideal of polynomial identities of a suitable algebra A.

Since many algebras may correspond to the same set of polynomial identities (or T-ideal) we need to introduce the notion of variety of algebras.

Definition 1.1.3. *Given a non-empty set $S \subseteq F\langle X\rangle$, the class of all algebras A such that $f \equiv 0$ on A, for all $f \in S$, is called the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S.*

A variety $\mathcal{V}$ is called non-trivial if $S \neq 0$ and $\mathcal{V}$ is proper if it is non-trivial and contains a non-zero algebra.

Example 1.1.5. *The class of all commutative algebras is a proper variety with $S = \{[x_1, x_2]\}$.*

Example 1.1.6. *The class of all nil algebras of exponent bounded by n is a variety with $S = \{x^n\}$.*

Observe that a variety $\mathcal{V}$ is closed under taking homomorphic images, subalgebras and direct products. As a matter of fact, a theorem of Birkhoff (see, for instance, [14, Theorem 2.3.2]) shows that these properties characterize the varieties of algebras.

There is a close correspondence between T-ideals and varieties of algebras. In fact, if $\mathcal{V}$ is the variety determined by the set S and $\langle S\rangle_T$ is the T-ideal of $F\langle X\rangle$ generated by S, then $\mathcal{V}(S) = \mathcal{V}(\langle S\rangle_T)$ and $\langle S\rangle_T = \bigcap_{A\in\mathcal{V}} \mathrm{Id}(A)$. We write $\langle S\rangle_T = \mathrm{Id}(\mathcal{V})$. Thus to each variety corresponds a T-ideal of $F\langle X\rangle$. Actually, the converse is also true. In order to show the converse, we introduce the concept of relatively free algebra.

Definition 1.1.4. *Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$ an algebra and $Y \subseteq A$ a subset of A. We say that A is relatively free on Y (with respect to $\mathcal{V}$), if for any algebra $B \in \mathcal{V}$ and for every function $\alpha : Y \to B$, there exists a unique homomorphism $\beta : A \to B$ extending α.*

When $\mathcal{V}$ is the variety of all algebras, this is just the definition of a free algebra on Y. The cardinality of Y is called the rank of A.

Relatively free algebras are easily described in terms of free algebras (see, for instance, [24, Theorem 1.2.4]).

Theorem 1.1.1. *Let X be a non-empty set, $F\langle X\rangle$ a free algebra on X and $\mathcal{V}$ a variety with corresponding ideal $Id(\mathcal{V}) \subseteq F\langle X\rangle$. Then $F\langle X\rangle/\, Id(\mathcal{V})$ is a relatively free algebra on the set $\bar{X} = \{x + Id(\mathcal{V}) \mid x \in X\}$. Moreover, any two relatively free algebras with respect to $\mathcal{V}$ of the same rank are isomorphic.*

Thus the correspondence between T-ideals and varieties is well understood (see, for instance, [14, Theorem 2.2.7]).

Theorem 1.1.2. *There is a one-to-one correspondence between T-ideals of $F\langle X\rangle$ and varieties of algebras. More precisely, a variety $\mathcal{V}$ corresponds to the T-ideal of identities $Id(\mathcal{V})$ and a T-ideal I corresponds to the variety of algebras satisfying all the identities of I.*

If $\mathcal{V}$ is a variety and A is an algebra such that $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$, then we say that $\mathcal{V}$ is the variety generated by A and we write $\mathcal{V} = \mathrm{var}(A)$. Also, we shall refer to $F\langle X\rangle/\, \mathrm{Id}(\mathcal{V})$ as the relatively free algebra of the variety $\mathcal{V}$ of rank $|X|$.

The study of polynomial identities of an algebra A over a field F can be reduced to the study of the homogeneous or multilinear polynomials, if the ground field F is infinite. This reduction is very useful because this kind of polynomials is easier to deal with.

Let $F_n = F\langle x_1, \dots, x_n\rangle$ be the free algebra of rank $n \geq 1$ over F. This algebra can be naturally decomposed as

$$F_n = F_n^{(1)} \oplus F_n^{(2)} \oplus \cdots$$

where, for every $k \geq 1$, $F_n^{(k)}$ is the subspace spanned by all monomials of total degree k. The $F_n^{(i)}$s are called the homogeneous components of F_n. This decomposition can be further refined as follows: for every $k \geq 1$, write

$$F_n^{(k)} = \bigoplus_{i_1+\cdots+i_n=k} F_n^{(i_1,\dots,i_n)}$$

where $F_n^{(i_1,\dots,i_n)}$ is the subspace spanned by all monomials of degree i_1 in x_1, …, i_n in x_n.

Definition 1.1.5. *A polynomial $f \in F_n^{(k)}$, for some $k \geq 1$, is called homogeneous of degree k. Any $f \in F_n^{(i_1,\dots,i_n)}$ will be called multihomogeneous of multidegree $(i_1, \dots, i_n)$. We also say that a polynomial f is homogeneous in the variable x_i if x_i appears with the same degree in every monomial of f.*

If $f(x_1,\ldots,x_n) \in F\langle X\rangle$ is a polynomial, then f can be decomposed into a sum of multihomogeneous polynomials. In fact, it can be written as:

$$f = \sum_{k_1\geq 0,\ldots,k_n\geq 0} f^{(k_1,\ldots,k_n)}$$

where $f^{(k_1,\ldots,k_n)} \in F_n^{(i_1,\ldots,i_n)}$. The polynomial $f^{(k_1,\ldots,k_n)}$ which are non-zero are called the multihomogeneous components of f.

The importance of the multihomogeneous polynomials turns out from the following theorem (see, for instance, [14, Proposition 4.2.3]).

Theorem 1.1.3. *Let F be an infinite field. If $f \equiv 0$ is a polynomial identity of the algebra A, then every multihomogeneous component of f is still a polynomial identity of A.*

One of the most important consequences of the previous theorem is that over an infinite field every T-ideal is generated by its multihomogeneous polynomials.

Among multihomogeneous polynomials a special role is played by the multilinear ones.

Definition 1.1.6. *A polynomial $f \in F\langle X\rangle$ is called linear in the variable x_i if x_i occurs with degree 1 in every monomial of f. Moreover f is called multilinear if f is linear in each of its variables (multihomogeneous of multidegree $(1,\ldots,1)$).*

One of the most interesting features of the multilinear polynomials is given by the following remark.

Remark 1.1.1. *Let A be an algebra over F. If a multilinear polynomial f vanishes on a basis of A, then f is a polynomial identity of A.*

It is always possible to reduce an arbitrary polynomial to a multilinear one. This process, called process of multilinearization, can be found, for instance, in [24, Theorem 1.3.7].

Definition 1.1.7. *Let S be a set of polynomials in $F\langle X\rangle$ and $f \in F\langle X\rangle$. We say that f is a consequence of the polynomials in S (or follows from the polynomials in S) if $f \in \langle S\rangle_T$, the T-ideal generated by the set S.*

Definition 1.1.8. *Two sets of polynomials are equivalent if they generate the same T-ideal.*

One of the main consequences of the process of multilinearization is the following.

Theorem 1.1.4. *If* $charF = 0$, *every non-zero polynomial* $f \in F\langle x\rangle$ *is equivalent to a finite set of multilinear polynomials.*

In the language of T-ideals the previous result takes the following form.

Theorem 1.1.5. *If* $charF = 0$, *every* T*-ideal is generated, as a* T*-ideal, by the multilinear polynomials it contains.*

1.2 Representations of finite groups and S_n-representations

As a necessary background, we shall briefly describe the representation theory of finite groups and in particular that of the symmetric group (see, for instance, [24, Chapter 2]).

Let V be a vector space over a field F and let $GL(V)$ be the group of invertible endomorphisms of V. We recall the following.

Definition 1.2.1. *A representation of a group* G *on* V *is a homomorphism of groups* $\rho : G \to GL(V)$.

Let us denote by $End(V)$ the algebra of F-endomorphisms of V. If FG is the group algebra of G over F and ρ is a representation of G on V, it is clear that ρ induces a homomorphism of F-algebras $\rho' : FG \to End(V)$ such that $\rho'(1_{FG}) = 1$.

Throughout we shall be dealing only with the case of finite dimensional representations. In this case, $n = \dim_F V$ is called the dimension or the degree of the representation ρ.

There is a one-to-one correspondence between the representations of a group G on a finite dimensional vector space and the finite dimensional FG-modules (or G-modules). In fact, if $\rho : G \to GL(V)$ is a representation of G, V becomes a (left) G-module by defining $gv = \rho(g)(v)$, for all $g \in G$ and $v \in V$. It is also clear that if M is a G-module which is finite dimensional as a vector space over F, then $\rho : G \to GL(M)$, such that $\rho(g)(l) = gl$, for $g \in G$ and $l \in M$, defines a representation of G on M.

Definition 1.2.2. *If* $\rho : G \to GL(V)$ *and* $\rho' : G \to GL(W)$ *are two representations of a group* G, *we say that* ρ *and* ρ' *are equivalent, and we write* $\rho \sim \rho'$, *if* V *and* W *are isomorphic as* G*-modules.*

Definition 1.2.3. *A representation $\rho : G \to GL(V)$ is irreducible if V is an irreducible G-module and ρ is completely reducible if V is the direct sum of its irreducible submodules.*

Recall that an algebra A is simple if $A^2 \neq 0$ and it does not contain non-trivial two-side ideals, A is semisimple if $J(A) = 0$, where $J(A)$ is the Jacobson radical of A.

We now state the two important structure theorems of Wedderburn and Wedderburn-Artin concerning simple and semisimple artinian rings. Recall that a ring R is left artinian if it satisfies the descending chain condition on left ideals (i.e., if every strictly descending sequence of left ideals eventually terminates) (see, for instance, [29, Chapter 1]).

Theorem 1.2.1 (Wedderburn, Wedderburn-Artin)**.** *Let R be a ring. Then*

1. *R is simple left artinian if and only if $R \cong M_k(D)$, the ring of $k \times k$ matrices over a division ring D, $k \geq 1$.*
2. *R is semisimple left artinian if and only if $R = I_1 \oplus \cdots \oplus I_n$, where $I_1, \ldots, I_n$ are simple left artinian rings and they are all the minimal two-sided ideals of R.*

The basic tool for studying the representations of a finite group in case $\operatorname{char} F = 0$, is Maschke's theorem.

Theorem 1.2.2 (Maschke)**.** *Let G be a finite group and let $charF = 0$ or $charF = p > 0$ and $p \nmid |G|$. Then the group algebra FG is semisimple.*

As a consequence of the theorems of Wedderburn and Wedderburn-Artin, it follows that, under the hypothesis of Maschke's theorem,

$$FG \cong M_{n_1}(D^{(1)}) \oplus \cdots \oplus M_{n_k}(D^{(k)}),$$

where $D^{(1)}, \ldots, D^{(k)}$ are finite dimensional division algebras over F.

It can also be deduced that every G-module V is completely reducible. Hence if $\dim_F V < \infty$, V is the direct sum of a finite number of irreducible G-modules. We record this fact in the following.

Corollary 1.2.1. *Let G be a finite group and let $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 0$ and $p \nmid |G|$. Then every representation of G is completely reducible and the number of non-equivalent irreducible representations of G equals the number of simple components in the Wedderburn decomposition of the group algebra FG.*

In representation theory, the theory of characters represents a fundamental tool. Throughout we shall assume that $\operatorname{char} F = 0$ and let $\operatorname{tr} : End(V) \to F$ be the trace function on $End(V)$.

Definition 1.2.4. *Let $\rho : G \to GL(V)$ be a representation of G. Then the map $\chi_\rho : G \to F$ such that $\chi_\rho(g) = tr(\rho(g))$ is called the character of the representation ρ. Moreover, $\dim_F V = \deg \chi_\rho$ is called the degree of the character χ_ρ. We say that the character χ_ρ is irreducible if ρ is irreducible.*

Notice that χ_ρ is constant on the conjugacy classes of G, i.e., χ_ρ is a class function of G, and also that $\chi_\rho(1) = \deg \chi_\rho$.

The following theorem shows that the knowledge of the character gives a lot of information for the representation and the number of the non-isomorphic irreducible representations (G-modules) is determined by a purely group property of the group.

Theorem 1.2.3. *Let G be a finite group and let the field F be algebraically closed.*

1. *Every finite dimensional representation of G is determined, up to isomorphism, by its character.*

2. *The number of non isomorphic irreducible representations (G-modules) is equal to the number of conjugacy classes of G.*

Next we introduce the necessary background on the representation theory of the symmetric group S_n, $n \geq 1$.

Definition 1.2.5. *Let $n \geq 1$ be an integer. A partition λ of n is a finite sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$ and $\sum_{i=1}^{r} \lambda_i = n$. In this case we write $\lambda \vdash n$ or $|\lambda| = n$.*

It is well known that there is a one-to-one correspondence between partitions of n and conjugacy classes of S_n. Hence, by Theorem 1.2.3, all the irreducible non-isomorphic S_n-modules are indexed by partitions of n. Thus let us denote by χ_λ the irreducible S_n-character corresponding to $\lambda \vdash n$. Therefore we state the following result.

Proposition 1.2.1. *Let F be a field of characteristic zero and $n \geq 1$. There is a one-to-one correspondence between irreducible S_n-characters and partitions of n. Let $\{\chi_\lambda \mid \lambda \vdash n\}$ be a complete set of irreducible characters of S_n and let $d_\lambda = \chi_\lambda(1)$ be the degree of χ_λ, $\lambda \vdash n$. Then*

$$FS_n = \bigoplus_{\lambda \vdash n} I_\lambda \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(F),$$

where $I_\lambda = e_\lambda F S_n$ *and* $e_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)\sigma$ *is, up to a scalar, the unit element of* I_λ.

It is always possible to associate to $\lambda \vdash n$ a diagram.

Definition 1.2.6. *Let* $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. *The Young diagram associated to* λ *is the finite subset of* $\mathbb{Z} \times \mathbb{Z}$ *defined as* $D_\lambda = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i = 1, \ldots, r,\ j = 1, \ldots, \lambda_i\}$.

A Young diagram is denoted as an array of boxes with the convention that the first coordinate i (the row index) increases from top to bottom and the second coordinate j (the column index) increases left to right.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ we shall denote by $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ the conjugate partition of λ in which $\lambda'_1, \ldots, \lambda'_s$ are the lengths of the columns of D_λ. Hence $D_{\lambda'}$ is obtained from D_λ by flipping D_λ along its main diagonal.

Definition 1.2.7. *Let* $\lambda \vdash n$. *A Young tableau* T_λ *of the diagram* D_λ *is a filling of the boxes of* D_λ *with the integers* $1, 2, \ldots, n$. *We shall say that* T_λ *is a tableau of shape* λ.

Of course there are $n!$ distinct tableaux. Among these a prominent role is played by the so-called standard tableaux.

Definition 1.2.8. *A tableau* T_λ *of shape* λ *is standard if the integers in each row and in each column of* T_λ *increase from left to right and from top to bottom, respectively.*

Given a diagram D_λ, $\lambda \vdash n$, we identify a box of D_λ with the corresponding point (i,j).

Definition 1.2.9. *For any box* $(i,j) \in D_\lambda$, *we define the hook number of* (i,j) *as*

$$h_{ij} = \lambda_i + \lambda'_j - i - j + 1,$$

where λ' *is the conjugate partition of* λ.

Note that h_{ij} counts the number of boxes in the "hook" with edge in (i,j), i.e., the boxes to the right and below (i,j).

Next we give a formula to compute de degree d_λ of the irreducible character χ_λ (see, for instance, [31]).

Proposition 1.2.2 (The hook formula)**.** *The number of standard tableaux of shape* $\lambda \vdash n$ *is*

$$d_\lambda = \frac{n!}{\prod_{i,j} h_{ij}},$$

where the product runs over all boxes of D_λ.

It is possible calculate the number of standard tableau of shape λ. In fact, there is a strict connection between standard tableaux and degrees of the irreducible S_n-characters.

Theorem 1.2.4. *Given a partition $\lambda \vdash n$, the number of standard tableaux of shape λ equals d_λ, the degree of χ_λ, the irreducible character corresponding to λ.*

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we denote by $T_\lambda = D_\lambda(a_{ij})$ the tableau T_λ of shape λ in which a_{ij} is the integer in the box (i, j). Then we can give the following definitions.

Definition 1.2.10. *The row-stabilizer of T_λ is*

$$R_{T_\lambda} = S_{\lambda_1}(a_{11}, a_{12}, \ldots, a_{1\lambda_1}) \times \cdots \times S_{\lambda_r}(a_{r1}, a_{r2}, \ldots, a_{r\lambda_r})$$

where $S_{\lambda_i}(a_{i1}, a_{i2}, \ldots, a_{i\lambda_i})$ denotes the symmetric group acting on the integers a_{i1}, a_{i2}, $\ldots, a_{i\lambda_i}$.

Definition 1.2.11. *The column-stabilizer of T_λ is*

$$C_{T_\lambda} = S_{\lambda'_1}(a_{11}, a_{21}, \ldots, a_{\lambda'_1 1}) \times \cdots \times S_{\lambda'_r}(a_{1\lambda_1}, a_{2\lambda_1}, \ldots, a_{\lambda'_s \lambda_1})$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ is the conjugate partition of λ.

Hence R_{T_λ} (C_{T_λ} resp.) are the subgroups of S_n consisting of all permutations stabilizing the rows (columns resp.) of T_λ.

Definition 1.2.12. *For a given tableau T_λ we define*

$$e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (sgn\, \tau)\sigma\tau,$$

where $sgn\,\tau$, the sign of the permutation τ, is equal to 1 or -1 according as τ is an even or an odd permutation, respectively.

It can be shown that $e^2_{T_\lambda} = a e_{T_\lambda}$, where $a = \frac{n!}{d_\lambda}$ is a non-zero integer, i.e., e_{T_λ} is an essential idempotent of FS_n.

We conclude this section with the following result that record the most important fact about e_{T_λ}.

Proposition 1.2.3. *For every Young tableau T_λ of shape $\lambda \vdash n$, the element e_{T_λ} is a minimal essential idempotent of FS_n and $FS_n e_{T_\lambda}$ is a minimal left ideal of FS_n, with character χ_λ. Moreover, if T_λ and T'_λ are Young tableaux of the same shape, then e_{T_λ} and $e_{T'_\lambda}$ are conjugated in FS_n through some $\sigma \in S_n$.*

Remark 1.2.1. *For any two tableaux T_λ and T'_λ of the same shape $\lambda \vdash n$, $FS_n e_{T_\lambda} \cong FS_n e_{T'_\lambda}$, as S_n-modules.*

1.3 Invariants of T-ideals and Codimensions growth

In this section we shall describe some numerical invariants of a T-ideal of $F\langle X\rangle$. In order to define these invariants, we shall introduce an action of the symmetric group S_n on the space of multilinear polynomials of degree n.

We state the following lemmas concerning arbitrary irreducible S_n-modules (see [24, Lemma 2.4.1, Lemma 2.4.2]).

Lemma 1.3.1. *Let M be a left S_n-module.*

1. *If M is irreducible with character $\chi(M) = \chi_\lambda$, $\lambda \vdash n$., then M can be generated, as an S_n-module, by an element of the form $e_{T_\lambda} f$, for some $f \in M$ and some Young tableau T_λ. And also for any Young tableau T'_λ of shape λ, there exists $f' \in M$ such that $M = FS_n e_{T'_\lambda} f'$.*
2. *If $M = M_1 \oplus \cdots \oplus M_k$, where $M_1, \ldots, M_k$ are irreducible S_n-submodules with character χ_λ, then k is equal to the number of linearly independent elements $g \in M$ such that $\sigma g = g$, for all $\sigma \in R_{T_\lambda}$.*

Let now A be a PI-algebra over an infinite field F and $\mathrm{Id}(A)$ its T-ideal. We introduce

$$P_n = \mathrm{span}\left\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\right\}$$

the vector space of multilinear polynomials in the variables $x_1, \ldots, x_n$ in the free algebra $F\langle X\rangle$.

Definition 1.3.1. *The non-negative integer*

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap Id(A)}$$

is called the nth codimension of the algebra A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the codimension sequence of A.

If $\mathrm{char}\, F = 0$, then $\mathrm{Id}(A)$ is determined by its multilinear polynomials (Theorem 1.1.5). Hence, in this case, it suffices to study the multilinear identities of A, that is $\{P_n \cap \mathrm{Id}(A)\}_{n\geq 1}$. It is clear that the codimension sequence of an algebra A gives us, in some sense, the growth of the identities of A. Notice that $\dim(P_n \cap \mathrm{Id}(A)) = n! - c_n(A)$. Also A is a PI-algebra if and only if $c_n(A) < n!$ for some $n \geq 1$.

If $\mathcal{V}$ is a variety of algebras and $\mathcal{V} = \mathrm{var}(A)$ then we define $c_n(\mathcal{V}) = c_n(A)$.

Example 1.3.1. *If A is a nilpotent algebra, i.e., $A^k = 0$, then $c_n(A) = 0$ for all $n \geq k$.*

Example 1.3.2. *If A is commutative then $c_n(A) \leq 1$, for all $n \geq 1$.*

We next present two more examples in which we can compute explicitly the sequence of codimensions.

Let UT_2 be the algebra of 2×2 upper triangular matrices over F and let G be the infinite dimensional Grassmann algebra over F, i.e., the algebra generated by a countable set of elements $\{e_1, e_2, \ldots\}$ satisfying the condition $e_i e_j = -e_j e_i$.

In the following theorems we collect the results of [17], [37] and [39] concerning the T-ideals and the codimension sequences of these two algebras. Recall that $\langle f_1, \ldots, f_n\rangle_T$ denotes the T-ideal generated by the polynomials $f_1, \ldots, f_n \in F\langle X\rangle$.

Theorem 1.3.1. *Let $UT_2(F)$ be the algebra of 2×2 upper triangular matrices over a field F of characteristic zero. Then*

1. *$Id(UT_2) = \langle [x_1, x_2][x_3, x_4]\rangle_T$.*
2. *$\{x_{i_1} \ldots x_{i_m}[x_k, x_{j_1}, \ldots, x_{j_{n-m-1}}] : i_1 < \cdots < i_m, k > j_1 < \cdots < j_{n-m-1}, m \neq n-1\}$ is a basis of P_n mod. $P_n \cap Id(UT_2)$.*
3. *$c_n(UT_2) = 2^{n-1}(n-2) + 2$.*

Theorem 1.3.2. *Let G be the infinite dimensional Grassmann algebra over a field F of characteristic $p \neq 2$. Then*

1. *$Id(G) = \langle [x_1, x_2, x_3]\rangle_T$.*
2. *$\{x_{i_1} \ldots x_{i_m}[x_{j_1}, x_{j_2}] \ldots [x_{j_{2q-1}}, x_{2q}] : i_1 < \cdots < i_m, j_1 < \cdots < j_{2q}, 2q + m = n\}$ is a basis of P_n mod. $P_n \cap Id(G)$.*
3. *$c_n(G) = 2^{n-1}$.*

It is possible to define an action of the symmetric group S_n on P_n. If $\sigma \in S_n$ and $f(x_1, \ldots, x_n) \in P_n$, then σ acts on $f(x_1, \ldots, x_n)$ by permuting the variables in the following way:

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Since T-ideals are invariant under permutation of the variables, we obtain that the subspace $P_n \cap \mathrm{Id}(A)$ is invariant under this action, that is $P_n \cap \mathrm{Id}(A)$ is a left S_n-submodule of P_n. Hence

$$P_n(A) = \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

has an induced structure of left S_n-module.

If F is a field of characteristic zero, then we can consider the character of $P_n^H(A)$ and we give the following definition.

Definition 1.3.2. *For $n \geq 1$, the S_n-character of $P_n(A) = P_n/(P_n \cap Id(A))$, denoted by $\chi_n(A)$, is called the nth cocharacter of A.*

If A is an algebra over a field of characteristic zero, we can decompose the nth cocharacter into irreducibles as follows:

$$\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where χ_λ is the irreducible S_n-character associated to the partition $\lambda \vdash n$ and $m_\lambda \geq 0$ is the corresponding multiplicity.

By the proof of Lemma 3.5 in [8] we have the following.

Theorem 1.3.3. *Let $UT_2(F)$ be the algebra of 2×2 upper triangular matrices over a field F of characteristic zero. If $\chi_n(UT_2) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ is the nth cocharacter of UT_2, then we have:*

1. *$m_{(n)} = 1$;*
2. *$m_\lambda = q + 1$ if $\lambda = (p + q, p)$ or $\lambda = (p + q, p, 1)$;*
3. *$m_\lambda = 0$ in all other cases.*

Given integers $d, l \geq 0$, we define a hook shaped part of the plane of arm d and leg l as

$$H(d, l) = \{\lambda = (\lambda_1, \lambda_2, \dots) \vdash n \geq 1 \mid \lambda_{d+1} \leq l\}.$$

In particular, if λ is a partition of $n \geq 1$, then $\lambda \subset H(1, 1)$ if

$$\lambda = (p, 1, \dots, 1) = (p, 1^{n-p}), \quad p \geq 1.$$

Let $\chi_n(G) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be the nth cocharacter of G. Then we have the following theorem (see [43]).

Theorem 1.3.4. *If G is the infinite dimensional Grassmann algebra over a field F of characteristic zero, then $\chi_n(G) = \sum_{\substack{\lambda \vdash n \\ \lambda \subset H(1,1)}} \chi_\lambda$.*

One of the most interesting and challenging problem in PI-theory is to compute the growth of the codimension sequence of an algebra.

Let A be a PI-algebra over a field F of characteristic zero and let $\{c_n(A)\}_{n\geq 1}$ be its codimension sequence. The starting point for understanding the asymptotic behaviour of the sequence of codimensions is the following result due to Regev (see [47]).

Theorem 1.3.5 (Regev)**.** *If A is a PI-algebra then $\{c_n(A)\}_{n\geq 1}$ is exponentially bounded.*

Kemer in [35] proved another fundamental result about the growth of the sequence of codimensions.

Theorem 1.3.6 (Kemer)**.** *Let A be a PI-algebra. Then $\{c_n(A)\}_{n\geq 1}$ is polynomially bounded or grows exponentially.*

In the 80's, Amitsur conjectured that the exponential rate of growth of an PI-algebra is a non-negative integer. This conjecture was proved in 1999 by Giambruno and Zaicev (see [22] and [23]). We record this in the following.

Theorem 1.3.7 (Giambruno and Zaicev)**.** *For any PI-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer.*

At the light of the previous theorem we can define the exponent of A.

Definition 1.3.3. *Let A be a PI-algebra. The integer*

$$exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

is called the exponent (or PI-exponent) of A.

Example 1.3.3. *$exp(A) = 0$ if and only if A is nilpotent*

Example 1.3.4. *$exp(UT_2(F)) = 2$*

Example 1.3.5. *$exp(G) = 2$*

If $\mathcal{V}$ is a variety of algebras, then the growth of $\mathcal{V}$ is defined as the growth of the sequence of codimensions of any algebra A generating $\mathcal{V}$, i.e., $\mathcal{V} = \mathrm{var}(A)$. One of the main advantages of the exponent is that now we have an integer scale allowing us to measure the growth of any non-trivial variety.

As a consequence of what we said before we give the following.

Definition 1.3.4. *A variety $\mathcal{V}$ has polynomial growth if its sequence of codimensions $\{c_n(\mathcal{V})\}_{n\geq 1}$ is polynomially bounded, i.e., $c_n(\mathcal{V}) \leq an^b$, for some constants a and b. We say that $\mathcal{V}$ has almost polynomial growth if $\{c_n(\mathcal{V})\}_{n\geq 1}$ is not polynomially bounded but any proper subvariety of $\mathcal{V}$ has polynomial growth.*

Next we state a celebrated result of Kemer (see [32] and [33]) that give a characterization of varieties of polynomial growth.

Theorem 1.3.8 (Kemer)**.** *A variety of algebras $\mathcal{V}$ has polynomial growth if and only if G, $UT_2 \notin \mathcal{V}$.*

Corollary 1.3.1. *The varieties $var(G)$ and $var(UT_2)$ are the only varieties of almost polynomial growth.*

Chapter 2

Algebras with a generalized Hopf algebra action

In this chapter we extend the concepts developed in the previous chapter to algebras with a generalized Hopf algebra action.

The first section is devoted to the study of algebras with an action of a Hopf algebra and their identities. Since the universal enveloping algebra of a Lie algebra is a Hopf algebra, we are able to define algebras with derivations, i.e., algebras on which a Lie algebra acts as derivations.

In the last two sections we generalize the concepts of the first section in order to include the case of algebras with involution $*$, and then we present some relevant results concerning $*$-varieties of almost polynomial growth.

2.1 Action of Hopf algebras on algebras and H-identities

Let H be a Hopf algebra over a field F with comultiplication $\Delta : H \to H \otimes H$, counit $\epsilon : F \to H$, and antipode $S : H \to H$.

Definition 2.1.1. *An associative algebra A is a (left) H-module algebra or an algebra with a H-action, if A is a left H-module with action $h \otimes a \to ha$ for all $h \in H, a \in A$, such that*

$$h(ab) = (h_{(1)}a)(h_{(2)}b) \quad \forall h \in H, a, b \in A,$$

where $\Delta h = h_{(1)} \otimes h_{(2)}$ (Sweedler's notation).

We refer the reader to [12, 42, 49] for an account of Hopf algebras and algebras with Hopf algebra actions.

Next we give some relevant examples of algebras with a Hopf action.

Example 2.1.1. *Every algebra A is an H-module algebra for $H = F$.*

Definition 2.1.2. *Let A be an associative algebra over F. A derivation of A is a linear map $\partial : A \to A$ such that*

$$\partial(ab) = \partial(a)b + a\partial(b), \qquad \forall a, b \in A.$$

In particular an inner derivation induced by $x \in A$ is the derivation $\operatorname{ad} x : A \to A$ of A define by

$$(\operatorname{ad} x)(y) = [x, y], \qquad \forall y \in A.$$

The set of all derivations of A is a Lie algebra denoted by $Der(A)$, and the set $\operatorname{ad}(A)$ of all inner derivations of A is a Lie subalgebra of $Der(A)$.

Example 2.1.2 (Algebras with derivations). *Let L be a Lie algebra and A an associative algebra such that L acts on A as derivations. Then the universal enveloping algebra $U(L)$ of L is a Hopf algebra with comultiplication Δ defined by $\Delta(m) = 1 \otimes m + m \otimes 1$, counit ϵ defined by $\epsilon(m) = 0$, and antipode S defined by $S(m) = -m$, for all $m \in L$. Since the L-action on A can be naturally extended to the $U(L)$-action, A is a $U(L)$-module algebra.*

Example 2.1.3 (Algebras with group-action). *Let G be a finite group acting as automorphisms on an F-algebra A. Then the group algebra FG is a Hopf algebra with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$, for all $g \in G$. It is easy to see that A is a FG-module algebra.*

Let A be an F-algebra and G any group. Recall the following.

Definition 2.1.3. *The algebra A is a G-graded algebra if A can be written as the direct sum of subspaces $A = \bigoplus_{g \in G} A_g$ such that for all $g, h \in G$, $A_g A_h \subseteq A_{gh}$.*

It is clear that any $a \in A$ can be uniquely written as a finite sum $a = \sum_{g \in G} a_g$ with $a_g \in A_g$. The subspaces A_g are called the homogeneous components of A.

Example 2.1.4 (Group-graded algebras). *Let A be an algebra graded by a finite group G and let $(FG)^*$ be the dual algebra of FG. If $B = \{h_g \mid g \in G\}$ is a basis of $(FG)^*$ dual to the basis $\{g \mid g \in G\}$ of FG, i.e., $h_{g_1}(g_2) = \delta_{g_1, g_2}$ where δ_{g_1, g_2} is the Kronecker symbol, then $(FG)^*$ is a Hopf algebra with $\Delta(h_g) = \sum_{g \in G} h_{g_1 g_2^{-1}} \otimes h_{g_2}$, $\epsilon(h_g) = \delta_{g,1}$, $S(h_g) = h_{g^{-1}}$, for all $h_g \in B$.*

Therefore A is an $(FG)^*$-module algebra where $h_g a = a_g$, for all $h_g \in B$ and $a \in A$.

Let H be a Hopf algebra over an infinite field F and A an H-modulo algebra. Next we shall study the H-identities of A. To this end we introduce a universal object called the free H-module algebra.

Let V be a vector space with countable basis $X = \{x_1, x_2, \dots\}$. The tensor algebra $T(V \otimes H) = F\langle X|H\rangle$ is the free associative algebra over F with free formal generators $\gamma_i \otimes x_j = x_j^{\gamma_i}$, $j \geq 1$, $i \in I$, where $\{\gamma_i | i \in I\}$ is a basis of H. We write $x_i = x_i^1$, $1 \in H$.

Notice that $F\langle X|H\rangle$ has a structure of left H-module via the following:

$$h(x_{j_1}^{\gamma_{i_1}} x_{j_2}^{\gamma_{i_2}} \dots x_{j_n}^{\gamma_{i_n}}) := x_{j_1}^{h_{(1)}\gamma_{i_1}} x_{j_2}^{h_{(2)}\gamma_{i_2}} \dots x_{j_n}^{h_{(n)}\gamma_{i_n}}$$

for $h \in H$,$i_1, i_2, \dots, i_n \in I$, where $h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n)}$ is the image of h under the comultiplication Δ applied $(n-1)$ times.

Definition 2.1.4. *$F\langle X|H\rangle$ is called the free associative H-module algebra on the countable set X and its elements are called H-polynomials.*

$F\langle X|H\rangle$ satisfies the following universal property: any map $\varphi : X \to A$ extends uniquely to an algebra homomorphism $\overline{\varphi} : F\langle X|H\rangle \to A$ such that $\overline{\varphi}(f^h) = h\overline{\varphi}(f)$, for any $f \in F\langle X|H\rangle$ and $h \in H$.

Definition 2.1.5. *An H-polynomial $f = f(x_1, \dots, x_n) \in F\langle X|H\rangle$ is an H-polynomial identity for A (or H-identity) if $\overline{\varphi}(f) = 0$ for any map $\varphi : X \to A$. In other words, f is an H-identity of A if and only if $f(a_1, \dots, a_n) = 0$ for any $a_i \in A$, and we write $f \equiv 0$.*

The set

$$\mathrm{Id}^H(A) = \{f \in F\langle X|H\rangle | f \equiv 0 \text{ on } A\}$$

is a T_H-ideal of $F\langle X|H\rangle$, i.e., an ideal of $F\langle X|H\rangle$ invariant under the H-action.

The H-identities satisfy many of the same general properties as ordinary polynomial identities. Next we indicate some of them.

By naturally extending the ordinary case ($H = F$), the degree of a monomial M in a variable $x \in X$, is defined as the number of times the variables x^h appear in M (regardless of the exponent $h \in H$). Thus it is possible to define in a natural way the homogeneous H-polynomials and the multilinear H-polynomials.

Since F is an infinite field, we can state the following results (see [7]).

Lemma 2.1.1. *Let $f \in F\langle X|H\rangle$ be an H-identity for A. Then each homogeneous components of f is also an H-identity for A.*

Lemma 2.1.2. *Let A and B be algebras with H-action such that, for all $n \geq 1$, every multilinear H-identity of degree n for A is also an identity for B. Then every H-identity for A is an identity for B.*

In light of the above, it is reasonable to pay particular attention to the multilinear H-identities. We denote by

$$P_n^H = span\{x_{\sigma(1)}^{h_1} \dots x_{\sigma(n)}^{h_n} | \sigma \in S_n, h_i \in H\}$$

the space of multilinear H-polynomials in $x_1, \dots, x_n$, $n \geq 1$.

Definition 2.1.6. *The non-negative integer*

$$c_n^H(A) = \dim_F \frac{P_n^H}{P_n^H \cap Id^H(A)}$$

is called the nth H-codimension of the algebra A.

Notice that P_n^H has a natural structure of left S_n-module induced by defining for $\sigma \in S_n$,

$$\sigma(x_i^h) = x_{\sigma(i)}^h.$$

Since $P_n^H \cap Id^H(A)$ is invariant under the S_n action, the space

$$P_n^H(A) = \frac{P_n^H}{P_n^H \cap Id^H(A)}$$

has a structure of left S_n-module.

If F is a field of characteristic zero, then $Id^H(A)$ is generated by is multilinear differential polynomials. In this case, i.e., $\operatorname{char} F = 0$, we can consider the character of $P_n^H(A)$ and we give the following definition.

Definition 2.1.7. *For $n \geq 1$, the S_n-character of $P_n^H(A)$, denoted by $\chi_n^H(A)$, is called nth H-cocharacter of A.*

Thus, if we assume that F is a field of characteristic zero, we decompose the nth differential cocharacter into irreducibles as follows:

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda,$$

where $m_\lambda^H \geq 0$ is the multiplicity of χ_λ in $\chi_n^H(A)$.

We now want to compare the sequence of H-codimensions and the sequence of ordinary codimensions of an algebra. Since $F = F1_H$ is a Hopf subalgebras of H, we can identify in a natural way P_n with a subspace of P_n^H. Thus we have $P_n \subseteq P_n^H$ and

$$P_n \cap \mathrm{Id}(A) = P_n \cap \mathrm{Id}^H(A).$$

Next lemma holds for any algebra A.

Lemma 2.1.3. *For all $n \geq 1$, $c_n(A) \leq c_n^H(A)$.*

In the case of PI-algebras the H-codimensions can be bounded from above (see [26, Lemma 5]).

Lemma 2.1.4. *Let A be an associative algebra with a H-action over any field F, and let H be a Hopf algebra. Then*

$$c_n(A) \leq c_n^H(A) \leq (\dim H)^n c_n(A)$$

for all $n \geq 1$.

In this context (as in the ordinary case) it is often convenient to use the language of varieties of algebras.

Let $\mathcal{V}$ be a variety of H-modulo algebras. We write $\mathcal{V} = \mathrm{var}^H(A)$ in case $\mathcal{V}$ is generated by an algebra A with H-action. As in the ordinary case, we write $c_n^H(\mathcal{V}) = c_n^H(A)$ and the growth of $\mathcal{V}$ is the growth of the sequence $c_n^H(\mathcal{V}), n \geq 1$. Recall that we say that $\mathcal{V}$ has polynomial growth if $c_n^H(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^H(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

2.2 Differential identities

In this section we let $H = U(L)$ be the universal enveloping algebra of a Lie algebra L and A an associative algebra such that L acts on A as derivations. Since A is a $U(L)$-module algebra (see the Example 2.1.2), then we define the polynomial identities with derivation of A as a particular case of H-identities (see [28, 36]).

We denote by $\mathrm{var}^L(A)$ the variety of algebras with derivations generated by A, by $Id^L(A)$ the ideal of $F\langle X|L\rangle$ of differential identities of A, and also we use the following notation:

- P_n^L is the space of multilinear differential polynomials in the variables $x_1, \ldots, x_n$, $n \geq 1$;

- $c_n^L(A)$ is the nth differential codimension of A;

- if $\operatorname{char} F = 0$, $\chi_n^L(A)$ is the nth differential cocharacter of A.

Recall that, if F is a field of characteristic zero, the nth differential cocharacter of A is the character of the S_n-module $P_n^L(A) = P_n^L/(P_n^L \cap \mathrm{Id}^L(A))$ and has the decomposition

$$\chi_n^L(A) = \sum_{\lambda \vdash n} m_\lambda^L \chi_\lambda.$$

Remark 2.2.1. *If L acts on A as inner derivations, then $F\langle X|L\rangle$ is the free associative algebra with inner derivations on X.*

If A is a finite dimensional associative algebra with an action of its Lie algebra L of derivations satisfying a non trivial differential identity, then the sequence of differential codimensions $c_n^L(A)$ is exponentially bounded (see [27, Theorem 3]). We record this in the following.

Theorem 2.2.1. *Let A be a finite dimensional algebra over a field F of characteristic zero with an action of a Lie algebra L by derivations. Then there exist constants $d \in \mathbb{N}$, $C_1, C_2 > 0$, $r_1, r_2 \in \mathbb{R}$ such that*

$$C_1 n^{r_1} d^n \leq c_n^L(A) \leq C_2 n^{r_2} d^n \qquad \textit{for all } n \in \mathbb{N}.$$

Consequently, in this case the PI-exponent

$$\exp^L(A) := \lim_{n\to\infty} (c_n^L(A))^{\frac{1}{n}} \in \mathbb{Z}_+$$

exists.

2.3 Generalized Hopf algebra action

In order to embrace the case when a group acts by anti-automorphisms as well as automorphisms, we consider the following generalized H-action (see [7, 26]).

Throughout this section H will be an associative algebra with unit over a field F of characteristic zero.

Definition 2.3.1. *An associative algebra A is an algebra with a generalized H-action if A is a left H-module with action $h \otimes a \to ha$ for all $h \in H, a \in A$, such that for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ such that*

$$h(ab) = \sum_i ((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a))$$

for all $a, b \in A$.

As in the first section, given a basis $B = \{\eta_i | i \in I\}$ of H, we let $F\langle X|H\rangle$ be the free associative algebra over F with free formal generators $x_j^{\eta_i}$, $i \in I$, $j \in \mathbb{N}$. If $h = \sum_{i \in I} \alpha_i \eta_i$, $\alpha_i \in F$, where only a finite number of α_i are nonzero, then we put $x^h := \sum_{i \in I} \alpha_i x^{\eta_i}$. We also write $x_i = x_i^1$, $1 \in H$, and then we set $X = \{x_1, x_2, \dots\}$. We refer to the elements of $F\langle X|H\rangle$ as H-polynomials. Note that here we do not consider any H-action on $F\langle X|H\rangle$. We also denote by

$$P_n^H = \operatorname{span}\{x_{\sigma(1)}^{h_1} \dots x_{\sigma(n)}^{h_n} | \sigma \in S_n, h_i \in H\}$$

the space of multilinear H-polynomials in $x_1, \dots, x_n$, $n \geq 1$. As in the previous section P_n^H has a natural structure of left S_n-module.

A polynomial $f(x_1, \dots, x_n) \in F\langle X|H\rangle$ is an H-polynomial identity (or H-identity) of A if $f(a_1, \dots, a_n) = 0$ for any $a_i \in A$, and we write $f \equiv 0$. The set

$$\mathrm{Id}^H(A) = \{f \in F\langle X|L\rangle | f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle X|H\rangle$. Note that this definition of $F\langle X|H\rangle$ depends on the choice of the basis in H. However such algebras can be identified in a natural way and $\mathrm{Id}^H(A)$.

As above, since $P_n^H \cap Id^H(A)$ is invariant under the S_n action, the space $P_n^H / P_n^H \cap Id^H(A) = P_n^H(A)$ has a structure of left S_n-module and its dimension, $c_n^H(A)$, is called the nth H-codimension of A. By complete reducibility the character $\chi_n^H(A)$, called the nth H-cocharacter of A, decomposes as

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda.$$

Next we introduce a refining of the nth H-cocharacter of an algebra A in case H is a finite dimensional semisimple algebra and F is algebraically closed.

Let H be a semisimple associative algebra over an algebraically closed field F and let H^{op} be its opposite algebra. Recall that the wreath product of H^{op} and S_n is the group defined by

$$H^{\mathrm{op}} \wr S_n = \{(h_1, \dots, h_n; \sigma) \mid h_1, \dots, h_n \in H^{\mathrm{op}}, \sigma \in S_n\}$$

with multiplication given by

$$(h_1, \dots, h_n; \sigma)(h'_1, \dots, h'_n; \tau) = (h_1 h'_{\sigma^{-1}(1)}, \dots, h_n h'_{\sigma^{-1}(n)}; \sigma\tau).$$

Next we define an action of $H^{\mathrm{op}} \wr S_n$ on P_n^H preserving the ideals of H-identities. Following [7], we can identify $H^{\mathrm{op}} \wr S_n$ and P_n^H via the linear isomorphism $\psi : H^{\mathrm{op}} \wr S_n \to P_n^H$ defined by

$$\psi((h_1, \dots, h_n; \sigma)) = x_{\sigma(1)}^{h_{\sigma(1)}} \dots x_{\sigma(n)}^{h_{\sigma(n)}},$$

for all $(h_1, \dots, h_n; \sigma) \in H^{\mathrm{op}} \wr S_n$. This defines a left action of $H^{\mathrm{op}} \wr S_n$ on P_n^H given by $a\psi(b) = \psi(ab)$, for $a, b \in H^{\mathrm{op}} \wr S_n$. As a consequence we obtain that the ideal $\mathrm{Id}^H(A)$ of H-identities of the algebra A is left invariant under the $H^{\mathrm{op}} \wr S_n$-action (see [7, Lemma 16]). This makes $P_n^H(A)$ a left $H^{\mathrm{op}} \wr S_n$-module and we define $\chi_{H \wr S_n}(A)$ to be its $H^{\mathrm{op}} \wr S_n$-character.

Definition 2.3.2. *A multipartition $\langle\lambda\rangle$ of n is a finite sequence of partitions $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(t))$, such that $\lambda(1) \vdash n_1 \geq 0, \dots, \lambda(t) \vdash n_t \geq 0$ and $n = n_1 + \dots + n_t$.*

Since H^{op} is a finite dimensional semisimple algebra, there is a one-to-one correspondence between multipartitions $\langle\lambda\rangle$ of n and non-isomorphic irreducible representations $N_{\langle\lambda\rangle}$ of $H^{\mathrm{op}} \wr S_n$ (see [7, Theorem 21]).

For any multipartition $\langle\lambda\rangle$ of n, let us denote by $\chi_{\langle\lambda\rangle}$ the irreducible $H^{\mathrm{op}} \wr S_n$-character corresponding to $\langle\lambda\rangle$. Since $\operatorname{char} F = 0$, we can write $\chi_{H \wr S_n}(A)$ as a sum of irreducible characters

$$\chi_{H \wr S_n}(A) = \sum_{\langle\lambda\rangle \vdash n} m^H_{\langle\lambda\rangle} \chi_{\langle\lambda\rangle},$$

where $m^H_{\langle\lambda\rangle} \geq 0$ denotes the corresponding multiplicity.

2.4 Algebras with involution

Let A be an associative algebra over a field F of characteristic zero.

Definition 2.4.1. *An involution on A is a linear map $* : A \to A$ of order two ($(a^*)^* = a$, for all $a \in A$) such that, for all $a, b \in A$,*

$$(ab)^* = b^* a^*.$$

Let A be an algebra with involution $*$. We write $A = A^+ \oplus A^-$, where $A^+ = \{a \in A \mid a^* = a\}$ and $A^- \{a \in A \mid a^* = -a\}$ denote the sets of symmetric and skew elements of A, respectively.

Since $*$ is an anti-automorphism of order two, then A is an algebra with a generalized $F\mathbb{Z}_2$-action. We denote by $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \dots\rangle$ the free algebra with involution on the countable set $X = \{x_1, x_2, \dots\}$ over F and by $\mathrm{var}^*(A)$ the $*$-variety generated by A.

Since $F\mathbb{Z}_2$ is a finite dimensional semisimple algebra, then we can regard $F\langle X, *\rangle$ as generated by the symmetric variables $x_i^+ = x_i + x_i^*$ and by the skew variables $x_i^- = x_i - x_i^*$, i.e. $F\langle X, *\rangle = F\langle x_1^+, x_1^-, x_2^+, x_2^-, \dots\rangle$. We also define P_n^* as the space of multilinear polynomials of degree n in $x_1^+, x_1^-, \dots, x_n^+, x_n^-$; hence for every $i = 1, 2, \dots, n$ either x_i^+ or x_i^- appears in every monomial of P_n^* at degree 1 (but not both), for any $i = 1, \dots, n$. Thus a polynomial $f(x_1^+, \dots, x_n^+, x_1^-, \dots, x_m^-) \in F\langle X, *\rangle$ is a $*$-identity of A if $f(s_1, \dots, s_n, k_1, \dots, k_m) = 0$ for all $s_1, \dots, s_n \in A^+$, $k_1, \dots, k_m \in A^-$. We denote by $\mathrm{Id}^*(A)$ the ideal $F\langle X, *\rangle$ of $*$-identities of A and by $c_n^*(A)$ the nth $*$-codimension of A.

Since in this case we are interested in the study of $P_n^*/P_n^* \cap Id^*(A) = P_n^*(A)$ as $F\mathbb{Z}_2 \wr S_n$-module, we denote by $\chi_n^*(A) = \chi_{F\mathbb{Z}_2 \wr S_n}(A)$ its $F\mathbb{Z}_2 \wr S_n$-character and without lead to confusion we can call $\chi_n^*(A)$ the nth $*$-cocharacter of A.

Following the previous section, $\chi_n^*(A)$ decomposes as

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}, \tag{2.1}$$

where $\chi_{\lambda,\mu}$ is the irreducible $F\mathbb{Z}_2 \wr S_n$-character associated to the multipartition (λ, μ), $m_{\lambda,\mu} \geq 0$ is the corresponding multiplicity and $|\lambda|+|\mu| = n$ indicates $\lambda \vdash r$ and $\mu \vdash n-r$, for all $r = 0, 1, \dots, n$.

For fixed $0 \leq r \leq n$, let $P_{r,n-r}^*$ denote the space of multilinear $*$-polynomials in the variables $x_1^+, \dots, x_r^+$, $x_{r+1}^-, \dots, x_n^-$. It is clear that in order to study $P_n^* \cap Id^*(A)$ it is enough to study $P_{r,n-r}^* \cap Id^*(A)$ for all $r \geq 0$, and this can be done through the representation theory of $S_r \times S_{n-r}$. If we let S_r act on the symmetric variables $x_1^+, \dots, x_r^+$ and S_{n-r} on the skew variables $x_{r+1}^-, \dots, x_n^-$, then we obtain an action of $S_r \times S_{n-r}$ on $P_{r,n-r}^*$. Since T^*-ideal are invariant under this action, we get that $P_{r,n-r}^*(A) = P_{r,n-r}^*/(P_{r,n-r}^* \cap Id^*(A))$ has an induced structure of left $S_r \times S_{n-r}$-module and we write $\chi_{r,n-r}^*(A)$ for its character. By complete reducibility we have

$$\chi_{r,n-r}^*(A) = \sum_{|\lambda|+|\mu|=n} \overline{m}_{\lambda,\mu}(\chi_\lambda \otimes \chi_\mu), \tag{2.2}$$

where χ_λ (respectively χ_μ) is the ordinary S_r -character corresponding to $\lambda \vdash r$ (respectively S_{n-r}-character corresponding to $\mu \vdash n-r$), $\chi_\lambda \otimes \chi_\mu$ is the irreducible $S_r \times S_{n-r}$-character associated to the pair (λ, μ) and $\overline{m}_{\lambda,\mu} \geq 0$ is the corresponding multiplicity.

There is a well-understood duality between $F\mathbb{Z}_2 \wr S_n$-characters and $S_r \times S_{n-r}$-characters given by Drensky and Giambruno as follows.

Theorem 2.4.1 ([15], Theorem 1.3)**.** *Let A be an algebra with involution. If the nth $*$-cocharacter has the decomposition given in* (2.1) *and the $S_r \times S_{n-r}$-character of $P^*_{r,n-r}$ has the decomposition given in* (2.2)*, then*

$$m_{\lambda,\mu} = \overline{m}_{\lambda,\mu}$$

for all $\lambda \vdash r$ and $\mu \vdash n-r$.

Next we recall some basic results concerning the sequence of cocharacters.

Lemma 2.4.1. *Let A and B be two algebras with involution such that*

$$\chi^*_n(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

and

$$\chi^*_n(B) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}\chi_{\lambda,\mu}.$$

Then:

1. *If $B \in var^*(A)$, then $m'_{\lambda,\mu} \leq m_{\lambda,\mu}$, for all pairs of partitions (λ,μ) such that $|\lambda| + |\mu| = n$.*

2. *The direct sum $A \oplus B$ is also an algebra with involution induced by the involutions defined on A and B. Moreover, if*

$$\chi^*_n(A \oplus B) = \sum_{|\lambda|+|\mu|=n} \overline{m}_{\lambda,\mu}\chi_{\lambda,\mu}$$

   is the decomposition of the nth $*$-cocharacter of $A \oplus B$, then $\overline{m}_{\lambda,\mu} \leq m_{\lambda,\mu} + m'_{\lambda,\mu}$, for all pairs of partitions (λ,μ) such that $|\lambda| + |\mu| = n$.

Next we introduce two algebras with involution generating $*$-varieties with almost polynomial growth. By analogy with the ordinary case we make the following.

Definition 2.4.2. *If a variety $\mathcal{V}$ of algebras with involution has sequence of $*$-codimensions polynomially bounded, we say that $\mathcal{V}$ has polynomial growth. We say that $\mathcal{V}$ has almost polynomial growth if $\mathcal{V}$ does not have polynomial growth, but every proper subvariety of $\mathcal{V}$ has polynomial growth.*

Let $D = F \oplus F$ be the two dimensional algebra with exchange involution ex given by $(a, b)^{ex} = (b, a)$, for all $(a, b) \in D$. Giambruno and Mishchenko proved in [18] that such an algebra generates a $*$-variety of almost polynomial growth and

$$\chi_n^*(D) = \sum_{j=0}^{n} \chi_{(n-j),(j)}.$$

Let now $UT_4(F)$ be the algebra of 4×4 upper triangular matrices, and let $M \subset UT_4(F)$ be the algebra

$$M = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33}) \oplus Fe_{12} \oplus Fe_{34},$$

where the e_{ij}'s are the usual matrix units, endowed with the involution ρ obtained by reflecting a matrix along its secondary diagonal,

$$\begin{pmatrix} a & b & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & a \end{pmatrix}^{\rho} = \begin{pmatrix} a & d & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

for same $a, b, c, d \in F$. In [41] Mishchenko and Valenti proved that M generates a variety of almost polynomial growth with T^*-ideal $Id^* = \langle x_1^- x_2^- \rangle$.

Theorem 2.4.2 ([41], Theorem 1)**.** *If $\chi_n^*(M) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$ is the nth $*$-cocharacter of M, we have:*

(1) $m_{\lambda,\mu} = 1$, if $\lambda = (n)$ and $\mu = \emptyset$;

(2) $m_{\lambda,\mu} = q + 1$, if $\lambda = (p + q, p)$ and $\mu = (1)$, for all $p \geq 0$, $q \geq 0$;

(3) $m_{\lambda,\mu} = q + 1$, if $\lambda = (p + q, p)$ and $\mu = \emptyset$, for all $p \geq 1$, $q \geq 0$;

(4) $m_{\lambda,\mu} = q + 1$, if $\lambda = (p + q, p, 1)$ and $\mu = \emptyset$, for all $p \geq 1$, $q \geq 0$;

(5) $m_{\lambda,\mu} = 0$ in all other case.

The above algebras characterize the $*$-variety of polynomial growth.

Theorem 2.4.3 ([18], Theorem 4.7)**.** *Let $\mathcal{V}$ be a $*$-variety. Then $\mathcal{V}$ has polynomial growth if and only if $D, M \notin \mathcal{V}$.*

Chapter 3

2×2 Upper triangular matrices and its differential identities

In this chapter we study the differential identities of the algebra UT_2 of 2×2 upper triangular matrices over a field of characteristic zero (see [21]).

We let the Lie algebra $L = \mathrm{Der}(UT_2)$ of derivations of UT_2 (and its universal enveloping algebra) act on it. We study the space of multilinear differential identities in n variables as a module for the symmetric group S_n and we determine the decomposition of the corresponding character into irreducibles.

If $\mathcal{V}$ is the variety of differential algebras generated by UT_2, we prove that unlike the other cases (ordinary identities, group graded identities) $\mathcal{V}$ does not have almost polynomial growth. Nevertheless we exhibit a subvariety $\mathcal{U}$ of $\mathcal{V}$ having almost polynomial growth.

3.1 Preliminaries

Let UT_2 be the algebra of 2×2 upper triangular matrices over a field F of characteristic zero. The description of its derivations is as follows.

Let the e_{ij}'s be the usual matrix units and consider the basis $\{e_{11}+e_{22}, e_{11}-e_{22}, e_{12}\}$ of UT_2. Let ε be the inner derivation induced by $2^{-1}(e_{11} - e_{22})$, i.e.,

$$\varepsilon(a) = 2^{-1}[e_{11} - e_{22}, a],$$

for all $a \in UT_2$, and let δ be the inner derivation induced by $2^{-1}e_{12}$, i.e.,

$$\delta(a) = 2^{-1}[e_{12}, a].$$

Then, for $a = \alpha(e_{11} + e_{22}) + \beta(e_{11} - e_{22}) + \gamma e_{12} \in UT_2$, we have

$$\varepsilon(a) = \gamma e_{12}$$

and

$$\delta(a) = -\beta e_{12}.$$

We shall study the differential identities of the algebra UT_2 when two distinct Lie algebras of derivations act on it. Namely first we shall consider $L = F\varepsilon$, the one dimensional Lie algebra with basis $\{\varepsilon\}$. We shall denote by UT_2^ε the algebra UT_2 with the $F\varepsilon$-action. The elements of $\mathrm{Id}^\varepsilon(UT_2) = \mathrm{Id}^L(UT_2^\varepsilon)$ will be called differential ε-polynomial identities (or differential ε-identities) of UT_2^ε. In this case we shall denote by P_n^ε the space of multilinear differential ε-polynomials in $x_1, \ldots, x_n$. Also we write $c_n^\varepsilon(UT_2) = c_n^L(UT_2^\varepsilon)$ for the nth differential ε-codimension of UT_2^ε and $\chi_n^\varepsilon(UT_2) = \chi_n^L(UT_2^\varepsilon)$ for the nth differential ε-cocharacter of UT_2^ε.

Next we shall consider $L = \mathrm{Der}(UT_2)$, the Lie algebra of all derivations of UT_2. Notice that since any derivation of UT_2 is inner (see [10]), L is the 2 dimensional metabelian Lie algebra with basis $\{\varepsilon, \delta\}$ such that $[\varepsilon, \delta] = \delta$. We shall denote by UT_2^D the algebra UT_2 with the $\mathrm{Der}(UT_2)$-action. Also P_n^D will be the space of multilinear differential polynomials in $x_1, \ldots, x_n$ and $\mathrm{Id}^D(UT_2) = \mathrm{Id}^L(UT_2^D)$ will be the T_L-ideal of identities with derivations of UT_2. Recall that $c_n^D(UT_2)$ is the nth differential codimension of UT_2, and $\chi_n^D(UT_2)$ is the nth differential cocharacter of UT_2. Notice that in both cases L is a Lie algebra of inner derivations of UT_2, then $F\langle X|L\rangle$ is the free associative free algebra with inner derivations.

Notice that in both cases L is a Lie algebra of inner derivations of UT_2, then throughout this section $F\langle X|L\rangle$ will be the free associative algebra with inner derivations on X.

3.2 Generators of the ideal of differential identities of UT_2^ε and its codimensions

We start by describing the differential identities of UT_2^ε.

It is easy to check that $[x, y]^\varepsilon - [x, y] \equiv 0$ and $x^\varepsilon y^\varepsilon \equiv 0$ are differential ε-identities of $\mathrm{Id}^\varepsilon(UT_2)$. Next we show that these identities generate $\mathrm{Id}^\varepsilon(UT_2)$ as T_L-ideal. The proof of the following remarks follow from easy computations.

Remark 3.2.1. *Since $L = F\varepsilon$ ia a Lie algebra of inner derivations of UT_2, then $x^{\varepsilon^2} - x^\varepsilon \equiv 0$ is a consequence of $[x, y]^\varepsilon - [x, y] \equiv 0$ on UT_2^ε.*

Remark 3.2.2. *1.* $x^\varepsilon y[w,z], [x,w]yz^\varepsilon, [x^{\alpha_1}, y^{\alpha_2}][z^{\alpha_3}, w^{\alpha_4}] \in \langle [x,y]^\varepsilon - [x,y], x^\varepsilon y^\varepsilon\rangle_{T_L}$, *with* $\alpha_i \in \{1, \varepsilon\}$, $i = 1,2,3,4$.

2. $x^\varepsilon y z^\varepsilon \in \langle x^\varepsilon y^\varepsilon \rangle_{T_L}$.

Remark 3.2.3. *For any* $1 \leq t, p \leq n$, *and for any permutations* $\sigma \in S_t$, $\tau \in S_p$, *we have*

$$x_{\sigma(1)} \dots x_{\sigma(t)} y^\varepsilon z_{\tau(1)} \dots z_{\tau(p)} \equiv x_1 \dots x_t y^\varepsilon z_1 \dots z_p \pmod{\langle x^\varepsilon y[w,z], [x,w]yz^\varepsilon\rangle_{T_L}}.$$

Proof. Let u_1, u_2, u_3 be monomials. We consider $w = u_1 x_i x_j u_2 y^\varepsilon u_3$. Since $x_i x_j = x_j x_i + [x_i, x_j]$, it follows that $w \equiv u_1 x_j x_i u_2 y^\varepsilon u_3 \pmod{\langle [x,w]yz^\varepsilon\rangle_{T_L}}$. In the same way we can show that $u_1 y^\varepsilon u_2 z_i z_j u_3 \equiv u_1 y^\varepsilon u_2 z_j z_i u_3 \pmod{\langle x^\varepsilon y[w,z]\rangle_{T_L}}$. Hence in every monomial

$$x_{i_1} \dots x_{i_t} y^\varepsilon z_{j_1} \dots z_{j_p}$$

we can reorder the variables to the left and to the right of y^ε as claimed. □

Next we prove the main result of this section.

Theorem 3.2.1. *Let* $UT_2^\varepsilon(F)$ *be the algebra of* 2×2 *upper triangular matrices over* F *with* $L = F\varepsilon$*-action. Then*

1. $Id^\varepsilon(UT_2) = \langle [x,y]^\varepsilon - [x,y], x^\varepsilon y^\varepsilon\rangle_{T_L}$.

2. $c_n^\varepsilon(UT_2) = 2^{n-1}n + 1$.

Proof. Let $Q = \langle [x,y]^\varepsilon - [x,y], x^\varepsilon y^\varepsilon\rangle_{T_L}$. It is clear that $Q \subseteq \mathrm{Id}^\varepsilon(UT_2)$. By the Poincaré-Birkhoff-Witt Theorem (see [45]) every differential multilinear polynomial in $x_1, \dots, x_n$ can be written as a linear combination of products of the type

$$x_{i_1}^{\alpha_1} \dots x_{i_k}^{\alpha_k} w_1 \dots w_m \tag{3.1}$$

where $\alpha_1, \dots, \alpha_k \in U(L)$, $w_1 \dots, w_m$ are left normed commutators in the $x_i^{\alpha_j}$s, $\alpha_j \in U(L)$, and $i_1 < \dots < i_k$. By Remark 3.2.1 $x^{\varepsilon^2} - x^\varepsilon \in Q$, hence, modulo $\langle x^{\varepsilon^2} - x^\varepsilon\rangle_{T_L}$, $\alpha_j \in \{1, \varepsilon\}$. Also, since $[x_1^{\alpha_1}, x_2^{\alpha_2}][x_3^{\alpha_3}, x_4^{\alpha_4}] \in Q$, with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \{1, \varepsilon\}$, then, modulo $[x_1^{\alpha_1}, x_2^{\alpha_2}][x_3^{\alpha_3}, x_4^{\alpha_4}]$, in (3.1) we have $m \leq 1$, so, only at most one commutator can appear in (3.1).

Now observe that $x^\varepsilon y z^\varepsilon \in Q$, hence, modulo Q, only one ε can appear as exponent of a variable in the monomials of (3.1). Moreover, since $x^\varepsilon y[w,z], [x,w]yz^\varepsilon \in Q$, every multilinear monomial in P_n^ε can be written, modulo Q, as linear combination of the elements of the type

$$x_1 \dots x_n, \quad x_{h_1} \dots x_{h_{n-1}} x_j^\varepsilon, \quad x_{i_1} \dots x_{i_k}[x_{j_1}^\gamma, x_{j_2}, \dots, x_{j_m}],$$

where $h_1 < \dots < h_{n-1}$, $i_1 < \dots < i_k$, $m + k = n$, $m \geq 2$, $\gamma \in \{1, \varepsilon\}$.

Let us now consider the left normed commutators $[x_{j_1}^\gamma, x_{j_2}, \dots, x_{j_m}]$ and suppose first that $\gamma = 1$. Since $[x_1, x_2][x_3, x_4] \in Q$, then by Theorem 1.3.1

$$[x_{j_1}, x_{j_2}, \dots, x_{j_m}] \equiv [x_k, x_{h_1}, \dots, x_{h_{m-1}}] \pmod{Q},$$

where $k > h_1 < \dots < h_{m-1}$.

Suppose now $\gamma = \varepsilon$. By Remark 3.2.3 any left normed commutator $[x_{j_1}^\varepsilon, x_{j_2}, \dots, x_{j_m}]$ satisfies the relation

$$[x_{j_1}^\varepsilon, x_{j_2}, \dots, x_{j_m}] \equiv [x_k^\varepsilon, x_{h_1}, \dots, x_{h_{m-1}}] \pmod{\langle x^\varepsilon y[w, z], [x, w]yz^\varepsilon \rangle_{T_L}}$$

with $h_1 < \dots < h_{m-1}$. If $k > h_1$, since $[x, y]^\varepsilon - [x, y] = [x^\varepsilon, y] - [y^\varepsilon, x] - [x, y]$, we have

$$\begin{aligned}[x_k^\varepsilon, x_{h_1}, \dots, x_{h_{m-1}}] \equiv ([&x_k, x_{h_1}, \dots, x_{h_{m-1}}] \\ &+ [x_{l_1}^\varepsilon, x_{l_2}, \dots, x_{l_m}]) \pmod{\langle [x, y]^\varepsilon - [x, y], x^\varepsilon y[w, z], [x, w]yz^\varepsilon \rangle_{T_L}}\end{aligned}$$

with $h_1 < \dots < h_{m-1}$, $k > h_1$ and $l_1 < l_2 < \dots < l_m$. It follows that P_n^ε is generated $(\bmod\ P_n^\varepsilon \cap Q)$ by the polynomials

$$\begin{gathered} x_1 \dots x_n, \quad x_{i_1} \dots x_{i_m}[x_k, x_{j_1}, \dots, x_{j_{n-m-1}}], \\ x_{h_1} \dots x_{h_{n-1}} x_r^\varepsilon, \quad x_{i_1} \dots x_{i_m}[x_{l_1}^\varepsilon, x_{l_2}, \dots, x_{l_{n-m}}], \end{gathered} \tag{3.2}$$

where $i_1 < \dots < i_m$, $k > j_1 < \dots < j_{n-m-1}$, $h_1 < \dots < h_{n-1}$, $l_1 < \dots < l_{n-m}$, $m \neq n-1, n$.

Next we prove that these elements are linearly independent modulo $\mathrm{Id}^\varepsilon(UT_2)$. Let $I = \{i_1, \dots, i_m\}$, $J = \{j_1, \dots, j_{n-m-1}\}$ be disjoint subsets of $\{1, \dots, n\}$ and set

$$X_{I,J} = x_{i_1} \dots x_{i_m}[x_k, x_{j_1}, \dots, x_{j_{n-m-1}}].$$

Also for $I' = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, set

$$X_{I'}^\varepsilon = x_{i_1} \dots x_{i_m}[x_{l_1}^\varepsilon, x_{l_2}, \dots, x_{l_{n-m}}]$$

and suppose that

$$f = \sum_{I,J} \alpha_{I,J} X_{I,J} + \sum_{I'} \alpha^{\varepsilon}_{I'} X^{\varepsilon}_{I'} + \sum_{k=1}^{n} \alpha^{\varepsilon}_{r} x_{h_1} \dots x_{h_{n-1}} x^{\varepsilon}_{r} + \beta x_1 \dots x_n \equiv 0 \pmod{P^{\varepsilon}_n \cap \mathrm{Id}^{\varepsilon}(UT_2)}.$$

In order to show that all coefficients $\alpha_{I,J}$, $\alpha^{\varepsilon}_{I'}$, α^{ε}_{r}, β are zero we will make some evaluations. If we evaluate $x_1 = \cdots = x_n = e_{11} + e_{22}$ we get $\beta = 0$. Also, for a fixed r, the evaluations $x_{h_1} = \cdots = x_{h_{n-1}} = e_{11} + e_{22}$ and $x_r = e_{12}$ gives $\alpha^{\varepsilon}_{r} = 0$. For fixed $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_{n-m-1}\}$, from the substitutions $x_{i_1} = \cdots = x_{i_m} = e_{11} + e_{22}$, $x_k = e_{12}$, $x_{j_1} = \cdots = x_{j_{n-m-1}} = e_{22}$ we get $I' = I$ and, by the structure of the polynomials in (3.2), it follows that $\alpha_{I,J} = 0$. Finally, for a fixed $I' = \{i_1, \dots, i_m\}$, by evaluating $x_{i_1} = \cdots = x_{i_m} = e_{11} + e_{22}$, $x_{l_1} = e_{12}$, $x_{l_2} = \cdots = x_{l_{n-m}} = e_{22}$ we obtain $\alpha^{\varepsilon}_{I} = 0$.

Thus the elements (3.2) are linearly independent modulo $\mathrm{Id}^{\varepsilon}(UT_2)$. Since $P^{\varepsilon}_n \cap Q \subseteq P^{\varepsilon}_n \cap \mathrm{Id}^{\varepsilon}(UT_2)$, this proves that $Id^{\varepsilon}(UT_2) = Q$ and the elements (3.2) are a basis of P^{ε}_n modulo $P^{\varepsilon}_n \cap \mathrm{Id}^{\varepsilon}(UT_2)$. Hence, using Theorem 1.3.1 and by counting the elements in (3.2), we get

$$c^{\varepsilon}_n(UT_2) = \dim_F \frac{P^{\varepsilon}_n}{P^{\varepsilon}_n \cap \mathrm{Id}^{\varepsilon}(UT_2)} = c_n(UT_2) + 2^n - 1 = 2^{n-1} n + 1.$$

□

As a consequence of Theorem 3.2.1 we have the following result.

Corollary 3.2.1. $P_n(UT_2) = \dfrac{P_n}{P_n \cap Id(UT_2)}$ *is isomorphic to an* S_n*-submodule of* $P^{\varepsilon}_n(UT_2) = \dfrac{P^{\varepsilon}_n}{P^{\varepsilon}_n \cap Id^{\varepsilon}(UT_2)}$.

We shall write $\exp^{L}(UT^{\varepsilon}_2) := \exp^{\varepsilon}(UT_2)$. Thus, as a consequence of Theorems 3.2.1 and 2.2.1, we have the following.

Corollary 3.2.2. $exp^{\varepsilon}(UT_2) = 2$.

3.3 Differential cocharacter of UT^{ε}_2

In this section we determine the differential cocharacter of UT^{ε}_2.

Let $\chi^{\varepsilon}_n(UT_2) = \sum_{\lambda \vdash n} m^{\varepsilon}_{\lambda} \chi_{\lambda}$ be the nth differential ε-cocharacter of UT^{ε}_2 and $\chi_n(UT_2) = \sum_{\lambda \vdash n} m_{\lambda} \chi_{\lambda}$ the nth (ordinary) cocharacter of UT_2.

Next we prove some technical lemmas which give us a lower bound for the multiplicities m_λ^ε in

$$\chi_n^\varepsilon(UT_2) = \sum_{\lambda \vdash n} m_\lambda^\varepsilon \chi_\lambda. \tag{3.3}$$

Lemma 3.3.1. $m_{(n)}^\varepsilon \geq n+1$.

Proof. We consider the following tableau:

$$T_{(n)} = \boxed{1}\boxed{2}\boxed{\ \dots\ }\boxed{n}\,.$$

We associate to $T_{(n)}$ the monomials

$$a(x) = x^n, \tag{3.4}$$

$$a_k^{(\varepsilon)}(x) = x^{k-1}x^\varepsilon x^{n-k}, \tag{3.5}$$

for all $k = 1, \dots, n$. These monomials are obtained from the essential idempotents corresponding to the tableau $T_{(n)}$ by identifying all the elements in the row. It is easily checked that $a(x)$, $a_k^{(\varepsilon)}(x)$, $k = 1, \dots, n$, do not vanish in UT_2^ε.

We shall prove that the $n+1$ monomials $a(x)$, $a_k^{(\varepsilon)}(x)$, $k = 1, \dots, n$, are linearly independent modulo $\mathrm{Id}^\varepsilon(UT_2)$.

In fact, suppose that

$$\alpha a(x) + \sum_{k=1}^{n} \alpha_k^\varepsilon a_k^{(\varepsilon)}(x) \equiv 0 \pmod{\mathrm{Id}^\varepsilon(UT_2)}.$$

The evaluation $x = e_{11} + e_{22}$ gives $\alpha = 0$. Moreover, we consider the substitution $x = \beta e_{11} + e_{12} + e_{22}$, where $\beta \in F$, $\beta \neq 0$. Then we get

$$\sum_{k=1}^{n} \beta^{k-1} \alpha_k^\varepsilon = 0 \tag{3.6}$$

Since $|F| = \infty$, we can choose $\beta_1, \dots, \beta_n \in F$, where $\beta_i \neq 0$, $\beta_i \neq \beta_j$, for all $1 \leq i \neq j \leq n$, then from (3.6) we get a homogeneous linear system of n equations in the n variables α_k^ε. Since the matrix associated to this system is a Vandermonde matrix whose determinant is nonzero, it follows that $\alpha_k^\varepsilon = 0$, for all $k = 1, \dots, n$. Hence $a(x)$, $a_k^{(\varepsilon)}(x)$, $k = 1, \dots, n$, are linearly independent $(\mathrm{mod}\ \mathrm{Id}^\varepsilon(UT_2))$.

Notice that the complete linearization of $a(x)$ and $a_k^{(\varepsilon)}(x)$ are the polynomials

$$e_{T_{(n)}}(x_1, \dots, x_n) = e_{T_{(n)}}(x_1 \dots x_n)$$

and

$$e^{\varepsilon,k}_{T_{(n)}}(x_1,\dots,x_n) = e_{T_{(n)}}(x_1 \dots x_k^{\varepsilon} \dots x_n),$$

respectively. It follows that the polynomials $e_{T_{(n)}}$ and $e^{\varepsilon,k}_{T_{(n)}}$ are linearly independent modulo $\mathrm{Id}^{\varepsilon}(UT_2)$. This implies that $m^{\varepsilon}_{(n)} \geq n+1$. $\square$

Lemma 3.3.2. *Let $p \geq 1$ and $q \geq 0$. If $\lambda = (p+q,p)$ then in (3.3) we have $m^{\varepsilon}_{\lambda} \geq 2(q+1)$.*

Proof. For every $i = 0,\dots,q$ we define $T^{(i)}_{\lambda}$ to be the tableau

$i+1$	$i+2$	$\dots$	$i+p-1$	$i+p$	1	$\dots$	i	$i+2p+1$	$\dots$	n
$i+p+2$	$i+p+3$	$\dots$	$i+2p$	$i+p+1$						

.

We associate to $T^{(i)}_{\lambda}$ the polynomials

$$b_i^{(p,q)}(x,y) = x^i \underbrace{\bar{x}\dots\tilde{x}}_{p-1}[x,y]\underbrace{\bar{y}\dots\tilde{y}}_{p-1}x^{q-i}, \tag{3.7}$$

$$b_i^{(p,q,\varepsilon)}(x,y) = x^i \underbrace{\bar{x}\dots\tilde{x}}_{p-1}(x^{\varepsilon}y - y^{\varepsilon}x)\underbrace{\bar{y}\dots\tilde{y}}_{p-1}x^{q-i}, \tag{3.8}$$

where the symbols $-$ or $\sim$ means alternation on the corresponding variables. The polynomials $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$ are obtained from the essential idempotents corresponding to the tableau $T^{(i)}_{\lambda}$ by identifying all the elements in each row of the tableau. It is clear that $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $i = 0,\dots,q$, are not differential identities of UT_2^{ε}. We shall prove that the polynomials $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $i = 0,\dots,q$, are linearly independent modulo $\mathrm{Id}^{\varepsilon}(UT_2)$. Suppose that

$$\sum_{i=0}^{q} \alpha_i b_i^{(p,q)} + \sum_{i=0}^{q} \alpha_i^{\varepsilon} b_i^{(p,q,\varepsilon)} \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(UT_2)}.$$

If we evaluate $x = \beta e_{11} + e_{12} + e_{22}$, where $\beta \in F$, $\beta \neq 0$, and $y = e_{11}$, then we get

$$\sum_{i=0}^{q} (-1)^{p-1}\beta^i \alpha_i = 0. \tag{3.9}$$

Since $|F| = \infty$, we choose $\beta_1,\dots,\beta_{q+1} \in F$, where $\beta_j \neq 0$, $\beta_j \neq \beta_k$, for all $1 \leq j \neq k \leq q+1$. Then from (3.9) we obtain a homogeneous linear system of $q+1$ equations in the $q+1$ variables α_i, $i = 0,\dots,q$, equivalent to the linear system

$$\sum_{i=0}^{q} \beta_j^i \alpha_i = 0, \quad j = 1,\dots,q+1. \tag{3.10}$$

Since the matrix associated to this system is a Vandermonde matrix whose determinant is nonzero, it follows that $\alpha_i = 0$, for all $i = 0, \dots, q$.

Next we consider

$$\sum_{i=0}^{q} \alpha_i^{\varepsilon} b_i^{(p,q,\varepsilon)} \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(UT_2)}.$$

If we substitute $x = \beta e_{11} + e_{12} + e_{22}$, where $\beta \in F$, $\beta \neq 0$, and $y = e_{22}$, then we obtain

$$\sum_{i=0}^{q} \beta^i \alpha_i^{\varepsilon} = 0. \tag{3.11}$$

Since $|F| = \infty$, we now consider $\beta_1, \dots, \beta_{q+1} \in F$, where $\beta_j \neq 0$, $\beta_j \neq \beta_k$, for all $1 \leq j \neq k \leq q+1$, then from (3.11) we get an homogeneous linear system of $q+1$ equations in the $q+1$ variables α_i^{ε}, $i = 0, \dots, q$, equivalent to the system (3.10). Therefore $\alpha_i^{\varepsilon} = 0$, for all $i = 0, \dots, q$. Hence the polynomials $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $i = 0, \dots, q$, are linearly independent $(\bmod\ \mathrm{Id}^{\varepsilon}(UT_2))$ and this implies that $m_\lambda^{\varepsilon} \geq 2(q+1)$. □

As an immediate consequence of Corollary 3.2.1 and Theorem 1.3.3 we have the following.

Lemma 3.3.3. *Let $p \geq 1$ and $q \geq 0$. If $\lambda = (p+q, p, 1)$ then in (3.3) we have $m_\lambda^{\varepsilon} \geq q+1$.*

Now we are ready to prove the following theorem.

Theorem 3.3.1. *Let $\chi_n^{\varepsilon}(UT_2) = \sum_{\lambda \vdash n} m_\lambda^{\varepsilon} \chi_\lambda$ be the nth differential ε-cocharacter of UT_2^{ε}. Then we have:*

1. *$m_{(n)}^{\varepsilon} = n + 1$;*
2. *$m_\lambda^{\varepsilon} = 2(q+1)$, if $\lambda = (p+q, p)$;*
3. *$m_\lambda^{\varepsilon} = q+1$, if $\lambda = (p+q, p, 1)$;*
4. *$m_\lambda^{\varepsilon} = 0$ in all other cases.*

Proof. By computing the degrees $\chi_\lambda(1)$ through the hook formula (Proposition 1.2.2) and by using the results of Lemmas 3.3.1, 3.3.2 and 3.3.3 we shall be able to compute the multiplicities m_λ^{ε}.

By Lemmas 3.3.1, 3.3.2 and 3.3.3, we have

$$(n+1)\chi_{(n)}(1) + \sum_{\substack{p>0 \\ q\geqslant 0}} 2(q+1)\chi_{(p+q,p)}(1) + \sum_{\substack{p>0 \\ q\geqslant 0}} (q+1)\chi_{(p+q,p,1)}(1) \leq c_n^{\varepsilon}(UT_2). \tag{3.12}$$

We shall prove the other inequality. Since $c_n^\varepsilon(UT_2) = \sum_{\lambda \vdash n} m_\lambda^\varepsilon \chi_\lambda(1)$, this will complete the proof. By Theorem 1.3.3, we have

$$c_n(UT_2) = \chi_{(n)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p,1)}(1).$$

Then we can rewrite the left hand side of (3.12) as

$$(n+1)\chi_{(n)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} 2(q+1)\chi_{(p+q,p)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p,1)}(1) =$$

$$c_n(UT_2) + n\chi_{(n)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1).$$

On the other hand, by Theorems 1.3.1 and 3.2.1, $c_n^\varepsilon(UT_2) = c_n(UT_2) + 2^n - 1$. Hence in order to get the equality in (3.12) we need to prove that

$$n\chi_{(n)}(1) + \sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1) \geq 2^n - 1.$$

Since $\chi_{(n)}(1) = 1$, we need to check that $\sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1) \geq 2^n - n - 1$. Now, since $q = n - 2p$ and by the hook formula $\chi_{(p+q,p)}(1) = \binom{n}{p}\frac{n-2p+1}{n-p+1}$, it is easily checked that

$$\begin{aligned}
\sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1) &= (n+1)\sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} - 3\sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} p + \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} \frac{p^2}{n-p+1} \\
&= (n+1)\Big(\sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} + \sum_{p=n-\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n}{p}\Big) \\
&\quad - \Big(\sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} p + \sum_{p=n-\lfloor \frac{n}{2} \rfloor+1}^{n} \binom{n}{p} p\Big) - 2\sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{p} p,
\end{aligned}$$

where in the last equality we use that $\binom{k}{k-j+1} = \binom{k}{j}\frac{j}{k-j+1}$. Now recall that $k\binom{k-1}{j-1} = j\binom{k}{j}$ and $\sum_{j=0}^{k}\binom{k}{j} = 2^k$. Hence, if $n = 2m$,

$$\begin{aligned}
\sum_{\substack{p>0\\q\geqslant 0}} (q+1)\chi_{(p+q,p)}(1) &= (2m+1)(2^{2m} - 1) - m2^{2m} - 4m\sum_{p=1}^{m} \binom{2m-1}{p-1} \\
&= 2^{2m} - 2m - 1.
\end{aligned}$$

In case $n = 2m + 1$,

$$\begin{aligned}\sum_{\substack{p>0\\ q\geqslant 0}}(q+1)\chi_{(p+q,p)}(1) =&(2m+2)(2^{2m+1}-1-\binom{2m+1}{m+1})\\ &-((2m+1)2^{2m}-(m+1)\binom{2m+1}{m+1})\\ &-2(2m+1)\sum_{p=1}^{m}\binom{2m}{p-1}\\ =&2^{2m+1}-2m-2.\end{aligned}$$

Thus

$$\sum_{\substack{p>0\\ q\geqslant 0}}(q+1)\chi_{(p+q,p)}(1) = 2^n - n - 1,$$

and this completes the proof of the theorem. □

3.4 Computing the growth of the differential codimensions of UT_2^ε

In this section we shall deal with algebras with derivations and the growth of the corresponding codimensions.

Recall that if $\mathcal{V} = \mathrm{var}^L(A)$ is a variety of algebras with derivations generated by an algebra A with derivations (the Lie algebra L acts on A as derivations). We say that $c_n^L(\mathcal{V}) = c_n^L(A)$ has polynomial growth if $c_n^L(\mathcal{V})$ is polynomially bounded and $\mathcal{V}$ has almost polynomial growth if $c_n^L(\mathcal{V})$ is not polynomially bounded but every proper subvariety of $\mathcal{V}$ has polynomial growth.

We shall prove that $\mathrm{var}^L(UT_2^\varepsilon)$ is a variety with almost polynomial growth. To this end, we follow closely the proof of [41] (or [50]), taking into account the due changes.

Lemma 3.4.1. *Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V} = var^L(UT_2^\varepsilon)$. Then there exist constants $M < N$ such that*

$$x^M y^\tau x^{N-M} \equiv \sum_{i<M} \mu_i x^i y^\tau x^{N-i} \pmod{Id^\varepsilon(\mathcal{U})},$$

where $\mu_i \in F$ and $\tau \in \{1, \varepsilon\}$.

Proof. Let $a, a_k^{(\varepsilon)}$, $k = 0, \dots, n$, and $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $i = 0, \dots, q$, be the polynomials introduced in Lemma 3.3.1 and Lemma 3.3.2, respectively. It is easy to check that, if

$\lambda = (p+q, p, 1)$, then the $q+1$ polynomials

$$c_i^{(p,q)}(x,y,z) = x^i\,\widehat{x}\dots\widetilde{x}\,\overline{x}\,\overline{y}\,\overline{z}\,\widehat{y}\dots\widetilde{y}\,x^{q-i}, \quad i = 0,\dots,q,$$

are linearly independent modulo $\mathrm{Id}^\varepsilon(UT_2)$. Since $\mathcal{U} \subsetneq \mathcal{V}$, then there exists $\lambda \vdash n$ such that $m_\lambda^\varepsilon(\mathcal{U}) < m_\lambda^\varepsilon(\mathcal{V})$. It follows that either

$$\alpha a + \sum_k \alpha_k^\varepsilon a_k^{(\varepsilon)}(x) \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}, \quad \text{with } \alpha, \alpha_j^\varepsilon \text{ not all zero,} \tag{3.13}$$

or

$$\sum_i \beta_i b_i^{(p,q)} + \sum_i \beta_i^\varepsilon b_i^{(p,q,\varepsilon)} \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}, \quad \text{with } \beta_j, \beta_j^\varepsilon \text{ not all zero,} \tag{3.14}$$

or

$$\sum_i \gamma_i c_i^{(p,q)} \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}, \quad \text{with } \gamma_j \text{ not all zero.} \tag{3.15}$$

Suppose that (3.14) holds. Hence

$$\begin{aligned} f(x,y) = &\sum_i \beta_i x^i \overline{x}\dots\widetilde{x}[x,y]\overline{y}\dots\widetilde{y}x^{q-i} \\ &+ \sum_i \beta_i^\varepsilon x^i \overline{x}\dots\widetilde{x}(x^\varepsilon y - y^\varepsilon x)\overline{y}\dots\widetilde{y}x^{q-i} \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}. \end{aligned}$$

Since $[x,y]^\varepsilon - [x,y] \in \mathrm{Id}^\varepsilon(\mathcal{V}) \subseteq \mathrm{Id}^\varepsilon(\mathcal{U})$, from $f(x,y) \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}$ we get

$$\begin{aligned} f'(x,y) = &\sum_i \beta_i x^i \overline{x}\dots\widetilde{x}(xy^\varepsilon - yx^\varepsilon)\overline{y}\dots\widetilde{y}x^{q-i} \\ &+ \sum_i \beta_i' x^i \overline{x}\dots\widetilde{x}(x^\varepsilon y - y^\varepsilon x)\overline{y}\dots\widetilde{y}x^{q-i} \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}. \end{aligned}$$

By substituting in $f'(x,y)$ the variable y with $y_1 + y_2$, we obtain

$$\begin{aligned} &\sum_i \beta_i x^i \overline{x}\dots\widetilde{x}(x(y_1^\varepsilon + y_2^\varepsilon) - (y_1+y_2)x^\varepsilon)\overline{(y_1+y_2)}\dots\widetilde{(y_1+y_2)}x^{q-i} \\ &+ \sum_i \beta_i' x^i \overline{x}\dots\widetilde{x}(x^\varepsilon(y_1+y_2) - (y_1^\varepsilon + y_2^\varepsilon)x)\overline{(y_1+y_2)}\dots\widetilde{(y_1+y_2)}x^{q-i} \\ &\equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}. \end{aligned}$$

In the last polynomial we consider the homogeneous component g of degree 1 in y_2. By substituting in g the variable y_1 with x^2 and y_2 with $[x, y^\tau]$, $\tau \in \{1, \varepsilon\}$, we obtain

$$\begin{aligned} h(x,y) = &\sum_i \beta_i x^i \overline{x}\dots\widetilde{x}x[x,y^\tau]\overline{y}\dots\widetilde{y}x^{q-i} \\ &+ \sum_i \beta_i' x^i \overline{x}\dots\widetilde{x}[x,y^\tau]x\,\overline{y}\dots\widetilde{y}x^{q-i} \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}. \end{aligned}$$

Let $t = \{i : \beta_i \neq 0\}$ and $\overline{N} = 3p + q = \deg h(x, y)$. Since $h(x, y) \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}$ is a differential ε-identity of $\mathcal{U}$, we can write

$$\beta_t x^{t+2p-1}[x, y^{\tau}]x^{\overline{N}-2p-t-1} \equiv \sum_{i<t+2p-1} \beta_i'' x^i [x, y^{\tau}] x^{\overline{N}-i-2} \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}.$$

Since $\beta_t \neq 0$, we get

$$x^{t+2p} y^{\tau} x^{\overline{N}-2p-t-1} \equiv \sum_{i<t+2p} \mu_i x^i y^{\tau} x^{\overline{N}-i-1} \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}.$$

If we set $N = \overline{N} - 1$ and $M = t + 2p$, then it follows that

$$x^M y^{\tau} x^{N-M} \equiv \sum_{i<M} \mu_i x^i y^{\tau} x^{N-i} \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}, \tag{3.16}$$

for same $\mu_i \in F$ and $\tau \in \{1, \varepsilon\}$.

Suppose now that (3.13) holds, then

$$\alpha x^n + \sum_k \alpha_k^{\varepsilon} x^{k-1} x^{\varepsilon} x^{n-k} \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}.$$

We substitute x with $x_1 + x_2$, and we consider the homogeneous component of degree 1 in x_2. We substitute in this homogeneous component x_1 with x and x_2 with $[x, y^{\tau}]$, $\tau \in \{1, \varepsilon\}$. As in the previous case, one deduces that for $N = n$ and suitable $M < N$, the relation (3.16) holds.

Finally suppose that (3.15) holds in $\mathcal{U}$. By substituting in $c_i^{(p,q)}$ the variable z with x^2, we obtain (3.14), and the proof is complete. □

Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V}$. Then for every $n \geq 1$ we write

$$\chi_n^{\varepsilon}(\mathcal{U}) = \sum_{\lambda \vdash n} m_{\lambda}^{\varepsilon}(\mathcal{U}) \chi_{\lambda},$$

where $m_{\lambda}^{\varepsilon}(\mathcal{U})$ is the multiplicity of χ_{λ} in $\chi_n^{\varepsilon}(\mathcal{U})$.

Proposition 3.4.1. *Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V} = var^L(UT_2^{\varepsilon})$. Then there exists a constant N' such that for all $n \geq 1$ and $\lambda \vdash n$ we have that $m_{\lambda}^{\varepsilon}(\mathcal{U}) \leq N'$.*

Proof. By Lemma 3.4.1, there exists N such that

$$x^M y x^{N-M} \equiv \sum_{i<M} \mu_i x^i y x^{N-i} \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}, \tag{3.17}$$

for some $\mu_i \in F$ and a suitable $M < N$. We shall prove that for all $\lambda \vdash n$, $m_{\lambda}^{\varepsilon}(\mathcal{U}) \leq 2N$. By Theorem 3.3.1 it is enough to consider the cases when either $\lambda = (n)$, or $\lambda = (p+q, p)$, or $\lambda = (p+q,p,1)$. Suppose first that $\lambda = (p+q,p)$ and $q \geq N$. By setting either $y = \overline{x}\dots\widetilde{x}[x,y]\overline{y}\dots\widetilde{y}$ or $y = \overline{x}\dots\widetilde{x}(x^\varepsilon y - y^\varepsilon x)\overline{y}\dots\widetilde{y}$, we can apply the relation (3.17) to any polynomial $b_i^{(p,q)}(x,y)$ or $b_i^{(p,q,\varepsilon)}(x,y)$ such that $i \geq M$. Hence we obtain

$$b_i^{(p,q)} = \sum_{j<M} b_j^{(p,q)},$$

$$b_i^{(p,q,\varepsilon)} = \sum_{j<M} b_j^{(p,q,\varepsilon)},$$

and $m_\lambda^\varepsilon(\mathcal{U}) \leq 2M \leq 2N$ follows. With a similar argument, we prove the statement in case $\lambda = (p+q,p,1)$ with $q \geq 2N$, and $\lambda = (n)$ with $n \geq 2N$. $\square$

Theorem 3.4.1. *The variety of algebras with derivations generated by the algebra UT_2^ε has almost polynomial growth.*

Proof. Let $\mathcal{U}$ be a proper subvariety of $\mathcal{V} = \mathrm{var}^L(UT_2^\varepsilon)$. We shall prove that $\mathcal{U}$ has polynomial growth. By Lemma 3.4.1 there exists N such that

$$x^M y^\varepsilon x^{N-M} \equiv \sum_{i<M} \mu_i x^i y^\varepsilon x^{N-i} \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}, \tag{3.18}$$

for some $\mu_i \in F$ and a suitable $M < N$. By proceeding as in the proof of [41, Theorem 3] (or [50]) we multilinearize the relation (3.18) and we obtain

$$\sum_{\sigma\in S_N} x_{\sigma(1)}\dots x_{\sigma(M)} y^\varepsilon x_{\sigma(M+1)}\dots x_{\sigma(N)}$$
$$\equiv \sum_{i<M}\sum_{\sigma\in S_N} \mu_i x_{\sigma(1)}\dots x_{\sigma(i)} y^\varepsilon x_{\sigma(i+1)}\dots x_{\sigma(N)} \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})},$$

where the x_i's are new variables.

We multiply the above expression on the right by $z_1,\dots,z_M$ and we alternate x_i with z_i for $i = 1,\dots,M$. Since any variable to the right (and to the left) of y^ε can be reordered, we get

$$\overline{x}_1\dots\widetilde{x}_M y^\varepsilon \overline{z}_1\dots\widetilde{z}_M x_{M+1}\dots x_N \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}.$$

Now we multiply this relation on the left by $z_{M+1},\dots,z_N$ and then we alternate x_j with z_j for $j = M+1,\dots,N$. We obtain

$$\overline{x}_1\dots\widetilde{x}_N y^\varepsilon \overline{z}_1\dots\widetilde{z}_N \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}.$$

If we identify y with x_{N+1}, it follows that

$$\overline{x}_1\dots\widetilde{x}_N x_{N+1}^\varepsilon \overline{z}_1\dots\widetilde{z}_N \equiv 0 \pmod{\mathrm{Id}^\varepsilon(\mathcal{U})}. \tag{3.19}$$

If we multiply this expression on the right by z_{N+1} and alternate x_{N+1} with z_{N+1}, we get

$$\overline{x}_1 \ldots \widetilde{x}_N (x^{\varepsilon}_{N+1} z_{N+1} - z^{\varepsilon}_{N+1} x_{N+1}) \overline{z}_1 \ldots \widetilde{z}_N \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}. \tag{3.20}$$

On the other hand, we substitute in (3.19) x_{N+1} with $[x_{N+1}, z_{N+1}]$. Thus we obtain

$$\overline{x}_1 \ldots \widetilde{x}_N [x_{N+1}, z_{N+1}] \overline{z}_1 \ldots \widetilde{z}_N \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}, \tag{3.21}$$

since $[x_{N+1}, z_{N+1}]^{\varepsilon} \equiv [x_{N+1}, z_{N+1}] \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}$.

The relations (3.20) and (3.21) show that the irreducible $S_{(N+1,N+1)}$-character corresponding to the partition $\lambda = (N+1, N+1)$ participates into the $2(N+1)$th differential ε-cocharacter of $\mathcal{U}$ with zero multiplicity, i.e., $m^{\varepsilon}_{(N+1,N+1)}(\mathcal{U}) = 0$.

Next we multiply the relation (3.21) on the right by y_{N+1} and alternate x_{N+1}, z_{N+1} and y_{N+1}. We obtain

$$\overline{x}_1 \ldots \widetilde{x}_N \widehat{x}_{N+1} \widehat{y}_{N+1} \widehat{z}_{N+1} \overline{z}_1 \ldots \widetilde{z}_N \equiv 0 \pmod{\mathrm{Id}^{\varepsilon}(\mathcal{U})}.$$

As before we get $m^{\varepsilon}_{(N+1,N+1,1)}(\mathcal{U}) = 0$.

Hence if λ is a partition of n such that $\lambda_2 \geq N+2$ then $m^{\varepsilon}_{\lambda}(\mathcal{U}) = 0$. By Theorem 3.3.1, it follows that if χ_{λ} appears in the differential ε-cocharacter with nonzero multiplicity then λ must contain at most $N+1$ boxes below the first row. Thus

$$\chi^{\varepsilon}(\mathcal{U}) = \sum_{\substack{\lambda \vdash n \\ |\lambda| - \lambda_1 \leq N+1}} m^{\varepsilon}_{\lambda}(\mathcal{U}) \chi_{\lambda}.$$

Since $|\lambda| - \lambda_1 \leq N+1$, then $\lambda_1 \geq n - (N+1)$ and by the hook formula we immediately get

$$\chi_{\lambda}(1) \leq \frac{n!}{(n-(N+1))!} \leq n^{N+1}.$$

Recall that, in the ordinary case, $m_{\lambda}(\mathcal{V}) \leq \chi_{\lambda}(1)$ (see [31]). Then by Theorem 1.3.3 and Theorem 3.3.1, $m^{\varepsilon}_{\lambda}(\mathcal{V}) \leq (n+1) m_{\lambda}(\mathcal{V})$. Hence $m^{\varepsilon}_{\lambda}(\mathcal{U}) \leq m^{\varepsilon}_{\lambda}(\mathcal{V}) \leq (n+1) n^{N+1}$ in $\chi^{\varepsilon}_{n}(\mathcal{U})$ and

$$c^{\varepsilon}_{n}(\mathcal{U}) = \sum_{\lambda \vdash n} m^{\varepsilon}_{\lambda}(\mathcal{U}) \chi_{\lambda}(1) \leq (N+1)^2 (n+1) n^{2(N+1)}.$$

From this relation it follows that $\mathcal{U}$ has polynomial growth. □

3.5 The algebra UT_2^D and its invariants

In this section we shall be concerned with the differential identities of the algebra UT_2^D, i.e., the algebra UT_2 with the action of its Lie algebra of derivations

$L = \mathrm{Der}_F(UT_2)$. We start by describing a basis of $\mathrm{Id}^D(UT_2)$ and the decomposition of the differential cocharacter of UT_2^D into irreducibles.

The following remarks are easily verified.

Remark 3.5.1. *$[x,y][z,w] \equiv 0$, $[x,y]^\varepsilon - [x,y] \equiv 0$ and $[x,y]^\delta \equiv 0$ are differential identities of UT_2^D.*

Remark 3.5.2. *Since $L = Der_F(UT_2)$ ia a Lie algebra of inner derivations of UT_2, then:*

1. *$x^{\varepsilon^2} - x^\varepsilon, x^{\delta^2}, x^{\delta\varepsilon}, x^{\varepsilon\delta} - x^\delta \in \langle [x,y]^\varepsilon - [x,y], [x,y]^\delta \rangle_{T_L}$.*
2. *$x^\alpha y[z,w], [x,y]zw^\alpha, x^\alpha yz^\beta \in \langle [x,y][z,w] \rangle_{T_L}$, where $\alpha, \beta \in \{\varepsilon, \delta\}$.*

Remark 3.5.3. *$[x^{\alpha_1}, y^{\alpha_2}][z^{\alpha_3}, w^{\alpha_4}] \equiv 0$, with $\alpha_i \in \{\varepsilon, \delta\}$, $i = 1,2,3,4$, is a consequence of $[x,y][z,w] \equiv 0$.*

Remark 3.5.4. *For any permutations $\sigma \in S_t$, we have*

$$[x^\delta_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(t)}] \equiv [x^\delta_1, x_2, \dots, x_t] \pmod{\langle x^\delta y[z,w], [x,y]zw^\delta, [x,y]^\delta \rangle_{T_L}}.$$

Proof. Proceeding as in the proof of Remark 3.2.3, we obtain that

$$x_{\rho(1)} \dots x_{\rho(p)} y^\delta z_{\tau(1)} \dots z_{\tau(q)} \equiv x_1 \dots x_p y^\delta z_1 \dots z_q \pmod{\langle x^\delta y[z,w], [x,y]zw^\delta \rangle_{T_L}},$$

for all $1 \leq p,q \leq n$, and for all $\rho \in S_p$, $\tau \in S_q$. Thus, since $[x,y]^\delta = [x^\delta, y] - [y^\delta, x]$, we can reorder all the variables in any commutator $[x^\delta_{i_1}, x_{i_2}, \dots, x_{i_t}]$ as claimed. $\square$

Theorem 3.5.1. *Let $UT_2^D(F)$ be the algebra of 2×2 upper triangular matrices over F with $L = Der(UT_2)$-action. Then*

1. *$Id^D(UT_2) = \langle [x,y][z,w], [x,y]^\varepsilon - [x,y], [x,y]^\delta \rangle_{T_L}$.*
2. *$c_n^D(UT_2) = 2^{n-1}(n+2)$.*

Proof. We prove the theorem using the strategy of the proof of Theorem 3.2.1. Let $Q = \langle [x,y][z,w], [x,y]^\varepsilon - [x,y], [x,y]^\delta \rangle_{T_L}$. By Remark 3.5.1, $Q \subseteq \mathrm{Id}^D(UT_2)$. Also since $[x,y]^\varepsilon - [x,y], x^\varepsilon y^\varepsilon \in Q$, we have that $\mathrm{Id}^\varepsilon(UT_2) \subseteq Q$.

Let $f \in P_n^D$ be a multilinear polynomial with derivations of degree n. Then, by Theorem 3.2.1 and Remarks 3.5.3, 3.5.2 and 3.5.4, we may write f, modulo Q, as a linear combination of the polynomials

$$\begin{aligned} &x_1 \dots x_n, \quad x_{i_1} \dots x_{i_m}[x_k, x_{j_1}, \dots, x_{j_{n-m-1}}], \\ &x_{h_1} \dots x_{h_{n-1}} x_r^{\varepsilon}, \quad x_{i_1} \dots x_{i_m}[x_{l_1}^{\varepsilon}, x_{l_2}, \dots, x_{l_{n-m}}], \\ &x_{h_1} \dots x_{h_{n-1}} x_s^{\delta}, \quad x_{i_1} \dots x_{i_m}[x_{l_1}^{\delta}, x_{l_2}, \dots, x_{l_{n-m}}], \end{aligned} \tag{3.22}$$

where $i_1 < \dots < i_m$, $k > j_1 < \dots < j_{n-m-1}$, $h_1 < \dots < h_{n-1}$, $l_1 < \dots < l_{n-m}$, $m \neq n-1, n$.

Next we show that these polynomials are linearly independent modulo $\mathrm{Id}^D(UT_2)$. For $I = \{i_1, \dots, i_m\}$ and $J = \{j_1, \dots, j_{n-m-1}\}$ disjoint subsets of $\{1, \dots, n\}$, we set $X_{I,J} = x_{i_1} \dots x_{i_m}[x_k, x_{j_1}, \dots, x_{j_{n-m-1}}]$. Also for $I' = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, we put $X_{I'}^{\varepsilon} = x_{i_1} \dots x_{i_m}[x_{l_1}^{\varepsilon}, x_{l_2}, \dots, x_{l_{n-m}}]$, and for $I'' = \{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$, we set $X_{I''}^{\delta} = x_{i_1} \dots x_{i_m}[x_{l_1}^{\delta}, x_{l_2}, \dots, x_{l_{n-m}}]$. Hence we consider a linear combination of elements in (3.22) and suppose that

$$\begin{aligned} \sum_{I,J} \alpha_{I,J} X_{I,J} + \sum_{I'} \alpha_{I'}^{\varepsilon} X_{I'}^{\varepsilon} + \sum_{I''} \alpha_{I''}^{\delta} X_{I''}^{\delta} + \sum_{r=1}^{n} \alpha_r^{\varepsilon} x_{i_1} \dots x_{i_{n-1}} x_r^{\varepsilon} \\ + \sum_{s=1}^{n} \alpha_s^{\delta} x_{i_1} \dots x_{i_{n-1}} x_s^{\delta} + \beta x_1 \dots x_n \equiv 0 \pmod{P_n^D \cap \mathrm{Id}^D(UT_2)}. \end{aligned}$$

We will show that all coefficients $\alpha_{I,J}$, $\alpha_{I'}^{\varepsilon}$, $\alpha_{I''}^{\delta}$, α_r^{ε}, α_s^{δ}, β are zero by making suitable evaluations.

If we evaluate $x_1 = \dots = x_n = e_{11} + e_{22}$ we get $\beta = 0$. For a fixed s, by setting $x_{i_1} = \dots = x_{i_{n-1}} = e_{11} + e_{22}$ and $x_s = e_{22}$ we get $\alpha_s^{\delta} = 0$. Also, for a fixed $I'' = \{i_1, \dots, i_m\}$, by making the evaluations $x_{i_1} = \dots = x_{i_m} = e_{11} + e_{22}$, $x_{l_1} = \dots = x_{l_{n-m}} = e_{22}$ we obtain $\alpha_{I''}^{\delta} = 0$. Moreover, by making the same evaluations as in the proof of Theorem 3.2.1, we get $\alpha_r^{\varepsilon} = 0$, $\alpha_{I'}^{\varepsilon} = 0$ and $\alpha_{I,J} = 0$, for any r, I', I, J.

We have proved that $\mathrm{Id}^D(UT_2) = Q$ and the elements in (3.22) are a basis of P_n^D modulo $P_n^D \cap \mathrm{Id}^D(UT_2)$. By using Theorem 3.2.1 and by counting the elements in (3.22), we obtain $c_n^D(UT_2) = c_n^{\varepsilon}(UT_2) + 2^n - 1 = 2^{n-1}(n+2)$. □

Corollary 3.5.1. *$P_n^{\varepsilon}(UT_2)$ is isomorphic to an S_n-submodule of $P_n^D(UT_2) = \dfrac{P_n^D}{P_n^D \cap Id^D(UT_2)}$.*

Corollary 3.5.2. *$exp^L(UT_2^D) = 2$.*

Next we compute the nth differential cocharacter $\chi_n^D(UT_2)$ of UT_2^D.

We write

$$\chi_n^D(UT_2) = \sum_{\lambda \vdash n} m_\lambda^D \chi_\lambda. \tag{3.23}$$

The following remark is an immediate consequence of Corollary 3.5.1.

Remark 3.5.5. *For any partition $\lambda \vdash n$, $m_\lambda^\varepsilon \leq m_\lambda^D$.*

Lemma 3.5.1. $m_{(n)}^D \geq 2n + 1$.

Proof. We consider the tableau $T_{(n)}$ define in Lemma 3.3.1 and let $a(x)$ and $a_k^{(\varepsilon)}(x)$ be the corresponding monomials in (3.4) and (3.5). Let also

$$a_k^{(\delta)}(x) = x^{k-1} x^\delta x^{n-k},$$

for all $k = 1, \ldots, n$. It is easily checked that $a(x)$, $a_k^{(\varepsilon)}(x)$, $a_k^{(\delta)}(x)$, $k = 1, \ldots, n$, do not vanish in UT_2^D.

As in the proof of Lemma 3.3.1, next we shall prove that the $2n + 1$ monomials $a(x)$, $a_k^{(\varepsilon)}(x)$, $a_k^{(\delta)}(x)$, $k = 1, \ldots, n$, are linearly independent modulo $\mathrm{Id}^D(UT_2)$. In fact, suppose that

$$\alpha a(x) + \sum_{k=1}^n \alpha_k^\varepsilon a_k^{(\varepsilon)}(x) + \sum_{k=1}^n \alpha_k^\delta a_k^{(\delta)}(x) \equiv 0 \pmod{\mathrm{Id}^D(UT_2)}.$$

By setting $x = e_{11} + e_{22}$ it follows that $\alpha = 0$. Moreover, if we substitute $x = \beta e_{11} + e_{22}$ where $\beta \in F$, $\beta \neq 0$, we get $\sum_{k=1}^n (1 - \beta)\beta^{k-1} \alpha_k^\delta = 0$. Since $|F| = \infty$, we can choose $\beta_1, \ldots, \beta_n \in F$, where $\beta_i \neq 0$ and $\beta_i \neq \beta_j$, for all $1 \leq i \neq j \leq n$. Then we get the following homogeneous linear system of n equations in the n variables α_k^δ, $k = 1, \ldots, n$,

$$\sum_{k=1}^n \beta_i^{k-1} \alpha_k^\delta = 0, \quad i = 1, \ldots, n. \tag{3.24}$$

Since the matrix associated to the system (3.24) is a Vandermonde matrix, it follows that $\alpha_k^\delta = 0$, for all $k = 1, \ldots, n$. Hence we may assume that

$$\sum_{k=1}^n \alpha_k^\varepsilon a_k^{(\varepsilon)}(x) \equiv 0 \pmod{\mathrm{Id}^D(UT_2)}.$$

As in the proof of Lemma 3.3.1, it follows that $\alpha_k^\varepsilon = 0$, for all $k = 1, \ldots, n$. Thus the monomials $a(x)$, $a_k^{(\varepsilon)}(x)$, $a_k^{(\delta)}(x)$, $k = 1, \ldots, n$, are linearly independent modulo $\mathrm{Id}^D(UT_2)$. This says that $m_{(n)}^D \geq 2n + 1$. □

Lemma 3.5.2. *Let $p \geq 1$ and $q \geq 0$. If $\lambda = (p + q, p)$ then in* (3.23) *we have $m_\lambda^D \geq 3(q + 1)$.*

Proof. For every $i = 0, \ldots, q$ we define $T_\lambda^{(i)}$ to be the tableau define in Lemma 3.3.2 and let $b_i^{(p,q)}(x, y)$ and $b_i^{(p,q,\varepsilon)}(x, y)$ be the corresponding polynomials defined in (3.7) and (3.8). Let also

$$b_i^{(p,q,\delta)}(x, y) = x^i \underbrace{\overline{x} \ldots \widetilde{x}}_{p-1}(x^\delta y - y^\delta x) \underbrace{\overline{y} \ldots \widetilde{y}}_{p-1} x^{q-i}.$$

It is clear that $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $b_i^{(p,q,\delta)}$ are not differential identities of UT_2^D. We shall prove that the above $3(q+1)$ polynomials are linearly independent modulo $\mathrm{Id}^D(UT_2)$. Suppose that

$$\sum_{i=0}^{q} \alpha_i b_i^{(p,q)} + \sum_{i=0}^{q} \alpha_i^\varepsilon b_i^{(p,q,\varepsilon)} + \sum_{i=0}^{q} \alpha_i^\delta b_i^{(p,q,\delta)} \equiv 0 \pmod{\mathrm{Id}^D(UT_2)}.$$

If we set $x = \beta e_{11} + e_{22}$, with $\beta \in F$, $\beta \neq 0$, and $y = e_{11}$, we obtain

$$\sum_{i=0}^{q} (-1)^{p-1} \beta^i \alpha_i^\delta = 0.$$

Since $|F| = \infty$, we can take $\beta_1, \ldots, \beta_{q+1} \in F$, where $\beta_j \neq 0$, $\beta_j \neq \beta_k$, for all $1 \leq j \neq k \leq q+1$. Then we obtain the following homogeneous linear system of $q+1$ equations in the $q+1$ variables α_i^δ, $i = 0, \ldots, q$,

$$\sum_{i=0}^{q} \beta_j^i \alpha_i^\delta = 0, \quad j = 1, \ldots, q+1.$$

Since the matrix of this system is a Vandermonde matrix, it follows that $\alpha_i^\delta = 0$, for all $i = 0, \ldots, q$. Hence we may assume that the following identity holds

$$\sum_{i=0}^{q} \alpha_i b_i^{(p,q)} + \sum_{i=0}^{q} \alpha_i^\varepsilon b_i^{(p,q,\varepsilon)} \equiv 0 \pmod{\mathrm{Id}^D(UT_2)}.$$

As in the proof of Lemma 3.3.2, it follows that $\alpha_i = 0$, $\alpha_i^\varepsilon = 0$, for all $i = 0, \ldots, q$. Therefore the polynomials $b_i^{(p,q)}$, $b_i^{(p,q,\varepsilon)}$, $b_i^{(p,q,\delta)}$, $i = 0, \ldots, q$, are linearly independent modulo $\mathrm{Id}^D(UT_2)$ and, so, $m_\lambda^D \geq 3(q+1)$. □

As a consequence of Lemma 3.5.1, Lemma 3.5.2, Remark 3.5.5 and by following verbatim the proof of Theorem 3.3.1 we get the following theorem which gives the decomposition into irreducible characters of $\chi_n^D(UT_2)$.

Theorem 3.5.2. *Let* $\chi_n^D(UT_2) = \sum_{\lambda \vdash n} m_\lambda^D \chi_\lambda$ *be the nth differential cocharacter of* UT_2^D*. Then we have:*

1. $m_{(n)}^D = 2n + 1$;

2. $m_\lambda^D = 3(q+1)$, *if* $\lambda = (p+q, p)$;

3. $m_\lambda^D = q+1$, *if* $\lambda = (p+q, p, 1)$;

4. $m_\lambda^D = 0$ *in all other cases.*

We remark that $\mathrm{var}^D(UT_2) = \mathrm{var}^L(UT_2^D)$ does not have almost polynomial growth. In fact since δ acts trivially on UT_2^ε, i.e., $x^\delta \equiv 0$ is a differential identity of UT_2^ε, it follows that $UT_2^\varepsilon \in \mathrm{var}^D(UT_2)$ and we have see that $\exp^L(UT_2^\varepsilon) = \exp^L(UT_2^D) = 2$. We state this fact in the following theorem.

Theorem 3.5.3. *$var^D(UT_2)$ has no almost polynomial growth.*

Chapter 4

The Grassmann algebra and its differential identities

Let G be the infinite dimensional Grassmann algebra over an infinite field F of characteristic $p \neq 2$. In this chapter we study the differential identities of G with respect to the action of a finite dimensional Lie algebra L of inner derivations (see [48]).

In the first section we explicitly determine a set of generators of the ideal of differential identities of G. Moreover we prove that unlike the ordinary case the variety of differential algebras with L action generated by G has no almost polynomial growth.

In the second section we assume that F is of characteristic zero and we study the space of multilinear differential identities in n variables as a module for the symmetric group S_n and we compute the decomposition of the corresponding character into irreducibles.

4.1 The ideal of differential identities of G and its codimensions

Let us consider the infinite dimensional Grassmann algebra G over an infinite field F of characteristic $p \neq 2$.

Recall that if $g = e_{i_1} \dots e_{i_n} \in G$, the set $\mathrm{Supp}\{g\} = \{e_{i_1}, \dots, e_{i_n}\}$ is called the support of g. Let now $g_1, \dots, g_t \in G_1$ be such that $\mathrm{Supp}\{g_i\} \cap \mathrm{Supp}\{g_j\} = \emptyset$, for all $i, j \in \{1, \dots, t\}$. Since $\operatorname{char} F \neq 2$, we set

$$\delta_i = 2^{-1} \operatorname{ad} g_i, \quad i = 1, \dots, t.$$

Then for all $g \in G$ we have

$$\delta_i(g) = \begin{cases} 0, & \text{if } g \in G_0 \\ g_i g, & \text{if } g \in G_1 \end{cases}, \quad i = 1, \ldots, t.$$

We shall consider $L = \mathrm{span}_F\{\delta_1, \ldots, \delta_t\} \subset \mathrm{ad}(G)$. Since for all $g \in G$, $[\delta_i, \delta_j](g) = 0$, $i, j \in \{1, \ldots, t\}$, L is a t dimensional abelian Lie algebra of inner derivations of G. We shall denote by $\widetilde{G}$ the algebra G with this L-action. Also throughout this section $F\langle X|L\rangle$ will be the free associative algebra with inner derivations on X.

We start by describing the differential identities of $\widetilde{G}$.

Remark 4.1.1. *It can be checked that*

$$[x_1, x_2][x_1, x_2] \equiv 0 \tag{4.1}$$

is a consequence of $[x_1, x_2, x_3] \equiv 0$ *in* G *(see for example [24]). Since* $[x_1, x_2, x_3] \equiv 0$ *is also a differential identity on* $\widetilde{G}$*, then the linearization of* (4.1) *leads to the identity* $[x_1, x_2][x_3, x_4] \equiv -[x_3, x_2][x_1, x_4]$ *on* $\widetilde{G}$*. Notice that the linearization is harmless because* $\mathrm{char}\, F \neq 2$ *and the degree of* x_1 *is equal to* 2*.*

Remark 4.1.2. *Since* $L = \mathrm{span}_F\{\delta_1, \ldots, \delta_t\}$ *is a Lie algebra of inner derivations of* G*, then* $[x_1^{\delta_i}, x_2] \equiv 0$ *and* $x^{\delta_i \delta_j} \equiv 0$*, for* $i, j \in \{1, \ldots, t\}$*, are consequences of* $[x_1, x_2, x_3] \equiv 0$ *in* $\widetilde{G}$*.*

Next we prove the main result of this section. Recall that for a real number x we denote by $\lfloor x \rfloor$ its integer part.

Theorem 4.1.1. *Let* F *be an infinite field of characteristic* $p \neq 2$ *and* $\widetilde{G}$ *be the infinite dimensional Grassmann algebra over* F *with* $L = \mathrm{span}_F\{\delta_1, \ldots, \delta_t\}$*-action. Then*

1. $\mathrm{Id}^L(\widetilde{G}) = \langle [x_1, x_2, x_3] \rangle_{T_L}$.
2. $c_n^L(\widetilde{G}) = 2^t 2^{n-1} - \sum_{j=1}^{\lfloor t/2 \rfloor} \sum_{i=2j}^{t} \binom{t}{i} \binom{n}{i-2j}$.

Proof. Let $Q = \langle [x_1, x_2, x_3] \rangle_{T_L}$. It is readily checked that $Q \subseteq \mathrm{Id}^L(\widetilde{G})$. Let $f \in F\langle X|L\rangle$ be a differential polynomial in $x_1, \ldots, x_n$. Since $1 \in \widetilde{G}$, f can be written as a linear combination of products of the type

$$x_{i_1}^{\alpha_1} \ldots x_{i_k}^{\alpha_k} w_1 \ldots w_m \tag{4.2}$$

where $\alpha_i \in U(L)$, $\alpha_i \neq 1$, for $1 \leq i \leq k$, and $w_1 \ldots, w_m$ are left normed commutators in the $x_j^{\beta_h}$s, $\beta_h \in U(L)$. Notice that $[x_1^{\gamma_1}, x_2^{\gamma_2}, x_3^{\gamma_3}] \equiv 0$ and $[x_1^{\gamma_1}, x_2] \equiv 0$ with $\gamma_i \in U(L)$, for $1 \leq i \leq 3$, are consequence of $[x_1, x_2, x_3] \equiv 0$. Then, modulo Q, in (4.2) we have $w_j = [x_{j_h}, x_{j_k}]$, for all $j = 1, \ldots, m$, and they are central. Also since $x^{\delta_i \delta_j} \in Q$, for all $i, j \in \{1, \ldots, t\}$, it follows that in (4.2) $\alpha_i \in \{\delta_1, \ldots, \delta_t\}$ modulo Q. Moreover it is clear that $x^{\delta_i} x^{\delta_j} \equiv 0$ is a consequence of $[x_1, x_2][x_1, x_3] \equiv 0$ and by Remark 4.1.1 $[x_1, x_2][x_3, x_4] + [x_3, x_2][x_1, x_4] \in Q$. Then we may assume that f is multilinear. Now observe that $x_1^{\delta_i} x_2^{\delta_j} \equiv -x_2^{\delta_i} x_1^{\delta_j}$ and $x_1^{\delta_i}[x_2, x_3] \equiv -x_3^{\delta_i}[x_2, x_1]$ are consequences of $[x_1, x_2][x_3, x_4] \equiv -[x_3, x_2][x_1, x_4]$. Then f can be written, modulo Q, as a linear combination of elements of the type

$$x_1^{\delta_{h_1}} \ldots x_k^{\delta_{h_k}} [x_{k+1}, x_{k+2}] \ldots [x_{k+2q-1}, x_{k+2q}], \tag{4.3}$$

with

$$h_1 < \cdots < h_k, \quad k + 2q = n, \quad 0 \leq k \leq t. \tag{4.4}$$

Next we prove that these elements are linearly independent modulo $\mathrm{Id}^L(\widetilde{G})$.

For any $0 \leq k \leq t$, consider $\Delta_k = \{\delta_{h_1}, \ldots, \delta_{h_k}\} \subseteq \{\delta_1, \ldots, \delta_t\}$, set

$$X_{\Delta_k} = x_1^{\delta_{h_1}} \ldots x_k^{\delta_{h_k}} [x_{k+1}, x_{k+2}] \ldots [x_{k+2q-1}, x_{k+2q}]$$

and suppose that

$$f = \sum_{\Delta_k} \alpha_{\Delta_k} X_{\Delta_k} \in \mathrm{Id}^L(\widetilde{G}).$$

In order to show that all coefficients α_{Δ_k} are zero we consider the following evaluations: for any $\Delta_k = \{\delta_{h_1}, \ldots, \delta_{h_k}\}$ we choose $x_1 = g_1', \ldots, x_{k+2q} = g_{k+2q}'$ where $g_i' \in G_1$, $1 \leq i \leq k + 2q$, and for all $r \in \{1, \ldots, t\} \setminus \{h_1, \ldots, h_k\}$, there exists $s \in \{1, \ldots, k + 2q\}$ such that $\mathrm{Supp}\{g_s'\} \cap \mathrm{Supp}\{g_r\} \neq \emptyset$. Then if we make these evaluations for increasing value of k ($0 \leq k \leq t$), by the properties of the polynomial in (4.3), it follows that $\alpha_{\Delta_k} = 0$ for any Δ_k. Thus the elements (4.3) are linearly independent modulo $\mathrm{Id}^L(\widetilde{G})$, and this proves that $\mathrm{Id}^L(\widetilde{G}) = Q$.

Notice that if we consider the multilinear differential polynomials, then the elements

$$x_{i_1} \ldots x_{i_m} x_{j_1}^{\delta_{h_1}} \ldots x_{j_k}^{\delta_{h_k}} [x_{j_{k+1}}, x_{j_{k+2}}] \ldots [x_{j_{k+2q-1}}, x_{j_{k+2q}}], \tag{4.5}$$

with

$$i_1 < \cdots < i_m, \ j_1 < \cdots < j_{k+2q}, \ h_1 < \cdots < h_k, \ m + k + 2q = n, \ 0 \leq k \leq t, \tag{4.6}$$

are a basis of P_n^L modulo $P_n^L \cap \mathrm{Id}^L(\widetilde{G})$. Thus we count for any fixed n, the total number of elements in (4.5) subject to the conditions (4.6), i.e. the nth differential codimension

$c_n^L(G)$. If $0 \leq k \leq t$, then this number is equal to

$$s_k = \binom{t}{k} \sum_{q=0}^{\lfloor (n-k)/2 \rfloor} \binom{n}{k+2q}.$$

Notice that $s_0 = 2^{n-1}$ and $s_1 = \binom{t}{1} 2^{n-1}$. Moreover, if $k = 2l$ with $l \geq 1$,

$$s_{2l} = \binom{t}{2l} \left(\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} - \sum_{p=0}^{l-1} \binom{n}{2p} \right) = \binom{t}{2l} \left(2^{n-1} - \sum_{p=0}^{l-1} \binom{n}{2p} \right).$$

Finally, in case $k = 2l+1$ with $l \geq 1$,

$$\begin{aligned} s_{2l+1} &= \binom{t}{2l+1} \left(\sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2r+1} - \sum_{p=0}^{l-1} \binom{n}{2p+1} \right) \\ &= \binom{t}{2l+1} \left(2^{n-1} - \sum_{p=0}^{l-1} \binom{n}{2p+1} \right). \end{aligned}$$

Thus

$$\begin{aligned} c_n^L(\widetilde{G}) =& \sum_{k=0}^{t} s_k = 2^{n-1} + \binom{t}{1} 2^{n-1} + \sum_{l=1}^{\lfloor t/2 \rfloor} \binom{t}{2l} \left(2^{n-1} - \sum_{p=0}^{l-1} \binom{n}{2p} \right) \\ &+ \sum_{l=1}^{\lfloor (t-1)/2 \rfloor} \binom{t}{2l+1} \left(2^{n-1} - \sum_{p=0}^{l-1} \binom{n}{2p+1} \right) \\ =& 2^t 2^{n-1} - \sum_{l=1}^{\lfloor t/2 \rfloor} \binom{t}{2l} \sum_{p=0}^{l-1} \binom{n}{2p} - \sum_{l=1}^{\lfloor (t-1)/2 \rfloor} \binom{t}{2l+1} \sum_{p=0}^{l-1} \binom{n}{2p+1} \\ =& 2^t 2^{n-1} - \sum_{i=2}^{t} \binom{t}{i} \binom{n}{i-2} - \sum_{l=2}^{\lfloor t/2 \rfloor} \binom{t}{2l} \sum_{p=0}^{l-2} \binom{n}{2p} \\ &- \sum_{l=2}^{\lfloor (t-1)/2 \rfloor} \binom{t}{2l+1} \sum_{p=0}^{l-2} \binom{n}{2p+1} = \dots \\ =& 2^t 2^{n-1} - \sum_{j=1}^{\lfloor t/2 \rfloor} \sum_{i=2j}^{t} \binom{t}{i} \binom{n}{i-2j}. \end{aligned}$$

□

Recall that two functions $\varphi_1(n)$ and $\varphi_2(n)$ are asymptotically equal and we write $\varphi_1(n) \approx \varphi_2(n)$ if $\lim_{n \to \infty} \varphi_1(n)/\varphi_2(n) = 1$. Then the following corollary is an obvious consequence of the previous theorem.

Corollary 4.1.1. $c_n^L(\widetilde{G}) \approx 2^t 2^{n-1}$.

The proof of Theorem 4.1.1 suggests a convenient decomposition of $P_n^L(\widetilde{G})$. For any $n \geq 1$ and for all $\gamma_1, \ldots, \gamma_k \in L$ distinct, we set

$$\Phi_{\gamma_1,\ldots,\gamma_k} = \{\gamma_1, \ldots, \gamma_k, \underbrace{1, \ldots, 1}_{n-k}\}.$$

We define

$$P_n^{\Phi_{\gamma_1,\ldots,\gamma_k}} = \mathrm{span}_F\{x_{\sigma(1)}^{\varepsilon_1} \ldots x_{\sigma(n)}^{\varepsilon_n} \mid \sigma \in S_n,\ \varepsilon_i \in \Phi_{\gamma_1,\ldots,\gamma_k}\},$$

a S_n-submodule of P_n^L. Since for all $\gamma_1, \ldots, \gamma_k, \beta_1, \ldots, \beta_k \in L$, $P_n^{\Phi_{\gamma_1,\ldots,\gamma_k}}$ and $P_n^{\Phi_{\beta_1,\ldots,\beta_k}}$ are isomorphic as S_n-modules, we introduce the notation

$$P_{n,k}^L = P_n^{\Phi_{\delta_1,\ldots,\delta_k}}.$$

In particular, for $k = 0$ we have $P_{n,0}^L = P_n$. Hence for any $0 \leq k \leq t$, we set

$$P_{n,k}^L(\widetilde{G}) = \frac{P_{n,k}^L}{P_{n,k}^L \cap \mathrm{Id}^L(\widetilde{G})}$$

and

$$c_{n,k}^L(\widetilde{G}) = \dim_F P_{n,k}^L(\widetilde{G}).$$

As consequence of proof of the Theorem 4.1.1 we have the following.

Corollary 4.1.2. $c_n^L(\widetilde{G}) = \sum_{k=0}^t \binom{t}{k} c_{n,k}^L(\widetilde{G})$, *where*

$$c_{n,k}^L(\widetilde{G}) = \begin{cases} 2^{n-1}, & \text{if } k = 0, 1 \\ 2^{n-1} - \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \binom{n}{2j}, & \text{if } k \geq 2 \text{ is even} \\ 2^{n-1} - \sum_{j=0}^{\lfloor k/2 \rfloor - 1} \binom{n}{2j+1}, & \text{if } k \geq 3 \text{ is odd} \end{cases}.$$

Next we shall be concerned with the growth of the differential codimension of $\widetilde{G}$.

Notice that by Corollary 4.1.1 $\mathrm{var}^L(\widetilde{G})$ has exponential growth, nevertheless it has no almost polynomial growth. In fact, the Grassmann algebra G (ordinary case) is an algebra with L-action where δ_i, $i = 1, \ldots, t$, acts trivially on G, i.e., $x^{\delta_i} \equiv 0$, $i = 1, \ldots, t$, are differential identities of G. Then it follows that $G \in \mathrm{var}^L(\widetilde{G})$, but by Theorem 1.3.1 $c_n(G) = 2^{n-1}$. Thus we have the following result.

Theorem 4.1.2. *$\mathrm{var}^L(\widetilde{G})$ has no almost polynomial growth.*

4.2 Differential cocharacter of G

Throughout this section F will be a field of characteristic zero.

Let $\chi^L_{n,k}(\widetilde{G})$ be the character of the S_n-module $P^L_{n,k}(\widetilde{G})$. Then we can write

$$\chi^L_{n,k}(\widetilde{G}) = \sum_{\lambda \vdash n} m^L_{\lambda,k}\chi_\lambda, \tag{4.7}$$

where $m^L_{\lambda,k} \geq 0$ is the multiplicity corresponding to the irreducible character χ_λ.

Next we shall compute the multiplicities $m^L_{\lambda,k}$ in (4.7).

Lemma 4.2.1. *Let $\chi^L_{n,k}(\widetilde{G}) = \sum_{\lambda\vdash n} m^L_{\lambda,k}\chi_\lambda$ be the character of $P^L_{n,k}(\widetilde{G})$. Then we have:*

1. *$m^L_{\lambda,k} = 1$, if $\lambda = (n-r+1, 1^{r-1})$ and $r \geq k$, $r \neq 0$;*

2. *$m^L_{\lambda,k} = 0$ in all other cases.*

Proof. If $k = 0$, we have $P^L_{n,0} = P_n$ and $\chi^L_{n,0}(\widetilde{G}) = \chi_n(G)$. Then by Theorem 1.3.4 the theorem is proved in case $k = 0$.

Suppose that $k \geq 1$. Assume that $\delta_1, \dots, \delta_k$, act on $P^L_{n,k}(\widetilde{G})$. If $\lambda = (n-r+1, 1^{r-1})$ and $r \geq k$, we define T_λ to be the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & r+1 & \dots & n \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{r} \\ \cline{1-1} \end{array}\ .$$

Then $R_{T_\lambda} = S_{n-r+1}\{1, r+1, \dots, n\}$ and $C_{T_\lambda} = S_r$, where $S_{n-r+1}\{1, r+1, \dots, n\}$ denotes the symmetric group acting on the set $\{1, r+1, \dots, n\}$. We associate to T_λ the polynomial

$$w_r^{\delta_1\dots\delta_k} = e_{T_\lambda}(x_1^{\delta_1}\dots x_k^{\delta_k}x_{k+1}\dots x_n)$$
$$= \left(\sum_{\sigma\in S_{n-r+1}\{1,r+1,\dots,n\}} \sigma\right)\left(\sum_{\tau\in S_r}(\operatorname{sgn}\tau)x^{\delta_1}_{\tau(1)}\dots x^{\delta_k}_{\tau(k)}x_{\tau(k+1)}\dots x_{\tau(r)}\right)x_{r+1}\dots x_n.$$

We claim that $w_r^{\delta_1\dots\delta_k}$, $r \geq k$, is not an identity of $\widetilde{G}$. In fact, we consider the evaluation $\varphi : F\langle X|L\rangle \to G$ such that

$$\varphi(x_i) = e_i, \quad 1 \leq i \leq r$$

and

$$\varphi(x_{r+1}) = e_{r+1}e_{r+2}, \dots, \varphi(x_n) = e_{2n-r-1}e_{2n-r}$$

such that for all $i \in \{1,\ldots,2n-r\}$, $e_i \notin \mathrm{Supp}\{g_j\}$, for all $j \in \{1,\ldots,k\}$. Then, since for all $i \in \{1,\ldots,r\}$ and $j \in \{1,\ldots,k\}$, $\varphi(x_i^{\delta_j}) = g_j\varphi(x_i)$, we obtain

$$\begin{aligned}
&\varphi(\sum_{\tau\in S_r}(\operatorname{sgn}\tau)x^{\delta_1}_{\tau(1)}\ldots x^{\delta_k}_{\tau(k)}x_{\tau(k+1)}\ldots x_{\tau(r)})\\
&\qquad=\sum_{\tau\in S_r}(\operatorname{sgn}\tau)g_1\varphi(x_{\tau(1)})\ldots g_k\varphi(x_{\tau(k)})\varphi(x_{\tau(k+1)})\ldots\varphi(x_{\tau(r)})\\
&\qquad=(\pm g_1\ldots g_k)\sum_{\tau\in S_r}(\operatorname{sgn}\tau)e_{\tau(1)}\ldots e_{\tau(r)}=\pm(r!)g_1\ldots g_ke_1\ldots e_r\neq 0.
\end{aligned}$$

Thus, since $\varphi(x_{r+1}),\ldots,\varphi(x_n)$ are central in G,

$$\varphi(w_r^{\delta_1\ldots\delta_k}) = \pm(r!)(n-r+1)!g_1\ldots g_ke_1\ldots e_{2n-r}\neq 0.$$

We have proved that $w_r^{\delta_1\ldots\delta_k}$ is not an identity of $\widetilde{G}$. Hence this implies that $m^L_{\lambda,k}\geq 1$, if $\lambda=(n-r+1,1^{r-1})$ and $r\geq k$. Then, since $c^L_{n,k}(\widetilde{G})=\sum_{\lambda\vdash n}m^L_{\lambda,k}\chi_\lambda(1)$, we have

$$\sum_{r=k}^{n}\chi_{(n-r+1,1^{r-1})}(1)\leq c^L_{n,k}(\widetilde{G}). \tag{4.8}$$

By the hook formula (Proposition 1.2.2) $\chi_{(n-r+1,1^{r-1})}(1) = \binom{n-1}{r-1}$, then, if $k=1$, we have

$$\sum_{r=1}^{n}\chi_{(n-r+1,1^{r-1})}(1)=\sum_{r=1}^{n}\binom{n-1}{r-1}=2^{n-1}.$$

On the other hand, by Corollary 4.1.2, $c^L_{n,1}(\widetilde{G}) = 2^{n-1}$. Then, if $k=1$ we get the equality in (4.8), and in this case the theorem is proved. Suppose then $k\geq 2$,

$$\sum_{r=k}^{n}\chi_{(n-r+1,1^{r-1})}(1)=\sum_{r=1}^{n}\binom{n-1}{r-1}-\sum_{r=1}^{k-1}\binom{n-1}{r-1}=2^{n-1}-\sum_{r=1}^{k-1}\binom{n-1}{r-1}.$$

Hence in order to get the equality in (4.8) we need to prove that

$$2^{n-1}-\sum_{r=1}^{k-1}\binom{n-1}{r-1}\geq c^L_{n,k}(\widetilde{G}).$$

Thus, if $k=2l$ with $l\geq 1$, by Corollary 4.1.2 we need to check that

$$2^{n-1}-\sum_{r=1}^{2l-1}\binom{n-1}{r-1}\geq 2^{n-1}-\sum_{j=0}^{l-1}\binom{n}{2j}.$$

But by induction on $l\geq 1$, it is easy to verify that $\sum_{r=1}^{2l-1}\binom{n-1}{r-1}=\sum_{j=0}^{l-1}\binom{n}{2j}$ and also in this case the theorem is proved. Suppose finally that $k=2l+1$ with $l\geq 1$. Since $\sum_{r=1}^{2l-1}\binom{n-1}{r-1}=\sum_{j=0}^{l-1}\binom{n}{2j+1}$, by Corollary 4.1.2 we get the equality in (4.8) and the theorem is proved. □

Theorem 4.2.1. *Let F be a field of characteristic zero and $\widetilde{G}$ be the infinite dimensional Grassmann algebra over F with $L = span_F\{\delta_1, \dots, \delta_t\}$-action. If $\chi_n^L(\widetilde{G}) = \sum_{\lambda \vdash n} m_\lambda^L \chi_\lambda$ is the nth differential cocharacter of $\widetilde{G}$, then we have:*

1. $m_\lambda^L = \begin{cases} \sum_{i=0}^r \binom{t}{i}, & r < t \\ 2^t, & r \geq t \end{cases}$, *if* $\lambda = (n - r + 1, 1^{r-1})$;

2. $m_\lambda^L = 0$ *in all other cases.*

Proof. By Corollary 4.1.2, $m_\lambda^L = \sum_{k=0}^t \binom{t}{k} m_{\lambda,k}^L$. Then by using Lemma 4.2.1 we get the proof of the theorem. □

Chapter 5

Algebras with involution and multiplicities bounded by a constant

In this chapter we characterize algebras with involution, satisfying a non-trivial identity, whose multiplicities of the cocharacter are bounded by a constant.

Throughout this chapter F will be a field of characteristic zero.

5.1 Grassmann envelope and superalgebras with superinvolution

Recall that the Grassmann algebra G over F has a natural $\mathbb{Z}_2$-grading $G = G_0 \oplus G_1$ where G_0 is the subspace of G spanned ba all monomials of even length and G_1 is the subspace of spanned by all monomials of odd length.

Given any $\mathbb{Z}_2$-graded algebra A one can form a new superalgebra with the help of G.

Definition 5.1.1. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded algebra (or superalgebra). The algebra*

$$G(A) = (G_0 \otimes A_0) \oplus (G_1 \otimes A_1)$$

is called the Grassmann envelope of A.

Clearly the Grassmann envelope $G(A)$ has a natural $\mathbb{Z}_2$-grading, $G(A) = G(A)_0 \oplus G(A)_1$, where $G(A)_0 = A_0 \otimes G_0$, $G(A)_1 = A_1 \otimes G_1$.

Next we introduce the concept of superinvolution in order to define such kind of map on G.

Definition 5.1.2. *Let $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$ be two superalgebras. A linear map $\varphi : A \to B$ is said to be graded if $\varphi(A_i) \subseteq B_i$, $i = 0, 1$.*

Definition 5.1.3. *Let $A = A_0 \oplus A_1$ be a superalgebra. We say that A is a superbalgebra with superinvolution $\sharp$ if it is endowed with a graded linear map $\sharp : A \to A$ with the following properties:*

1. *$(a^\sharp)^\sharp = a$, for all $a \in A$,*
2. *$(ab)^\sharp = (-1)^{|a||b|} b^\sharp a^\sharp$, for any homogeneous elements $a, b \in A_0 \cup A_1$ of homogeneous degree $|a|$ and $|b|$, respectively.*

Let $A = A_0 \oplus A_1$ be a superalgebra with superinvolution $\sharp$. Since $\operatorname{char} F = 0$, we can write $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$, where for $i = 0, 1$, $A_i^+ = \{a \in A_i | a^* = a\}$ and $A_i^- = \{a \in A_i | a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively.

We define a superinvolution on G, that we denote $\star$, by requiring that

$$e_i^\star = -e_i,$$

for $i \geq 1$. A basic property of this superinvolution is that $G^+ = G_0$ and $G^- = G_1$.

A fundamental property of the superinvolution of the Grassmann algebra defined above is that of allowing to bridge between involutions and superinvolutions of a superalgebra and its Grassmann envelope.

Notice that if A is a superalgebra with superinvolution $\sharp$, we can write $A = A_0 \oplus A_1$ where $A_0 = A_0^+ \oplus A_0^-$ and $A_1 = A_1^+ \oplus A_1^-$. Hence the Grassmann envelope $G(A)$ can be regarded as an algebra with the involution $* : G(A) \to G(A)$ such that

$$(a \otimes g)^* = a^\sharp \otimes g^\star.$$

In [1] Aljadeff, Giambruno and Karasik proved a very useful theorem.

Theorem 5.1.1 ([1],Theorem 4)**.** *If A is an algebra with involution satisfying a non-trivial $*$-identity, then there exists a finite dimensional superalgebra with superinvolution B such that $Id^*(A) = Id^*(G(B))$.*

Next we want to define a map whose properties will be late use (see [1]). In order to do this we first recall some notation.

Let $F\langle X, s, \sharp\rangle$ be the free super algebra with superinvolution on a countable set X over F. We write X as the disjoint union $Y^+ \cup Z^+ \cup Y^- \cup Z^-$, where

$$Y^+ = \{y_i^+ | i \geq 1\},\ Z^+ = \{z_i^+ | i \geq 1\},\ Y^- = \{y_i^- | i \geq 1\},\ Z^- = \{z_i^- | i \geq 1\}$$

are countable sets such that the variables in Y^+ are even and symmetric, the variables in Y^- are even skew, those in Z^+ are odd symmetric and those in Z^- are odd skew.

It is clear that $F\langle X, *\rangle$ is embedded into $F\langle X, s, \sharp\rangle$ by identifying x_i^+ with $y_i^+ + z_i^+$ and x_i^- with $y_i^- + z_i^-$, $i \geq 1$.

If $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ is a superalgebra with superinvolution, then a polynomial $f(y_1^+, \dots, y_m^+, y_1^-, \dots, y_n^-, z_1^+, \dots, z_p^+, z_1^-, \dots, z_q^-) \in F\langle X, s, \sharp\rangle$ is a $\mathbb{Z}_2$-graded polynomial identity with superinvolution of A (or simply a superidentity with superinvolution), and we write $f \equiv 0$, if $f(s_1, \dots, s_m, r_1, \dots, r_n, k_1, \dots, k_p, h_1, \dots, h_q) = 0$ for all $s_1, \dots, s_m \in A_0^+$, $r_1, \dots, r_n \in A_0^-$ $k_1, \dots, k_p \in A_1^-$, $h_1, \dots, h_q \in A_1^-$. We shall denote by $Id^\sharp(A)$ the ideal of $\mathbb{Z}_2$-graded identities with superinvolution of A.

Let $m, n, p, q \geq 0$ be integers and $P^\sharp_{m,n,p,q}$ the space of multilinear polynomials of $F\langle X, s, \sharp\rangle$ in the variables $y_1^+, \dots, y_m^+, y_1^-, \dots, y_n^-, z_1^+, \dots, z_p^+, z_1^-, \dots, z_q^-$. If $w \in P^\sharp_{m,n,p,q}$, we write

$$w = w_1 z_{\sigma(1)}^{\varepsilon_{\sigma(1)}} \dots z_{\sigma(i_1)}^{\varepsilon_{\sigma(i_1)}} w_2 z_{\sigma(i_1+1)}^{\varepsilon_{\sigma(i_1+1)}} \dots z_{\sigma(i_2)}^{\varepsilon_{\sigma(i_2)}} w_3 \dots w_{r+1},$$

where $\sigma \in S_{p+q}$, $\varepsilon_{i_j} \in \{+, -\}$ and the w_i's are (eventually empty) monomials in even variables.

Then we consider the linear map

$$\tilde{}: P^\sharp_{m,n,p,q} \to P^\sharp_{m,n,q,p}$$

define by

$$\tilde{w} = (sgn\sigma) w_1 z_{\sigma(1)}^{\eta_{\sigma(1)}} \dots z_{\sigma(i_1)}^{\eta_{\sigma(i_1)}} w_2 z_{\sigma(i_1+1)}^{\eta_{\sigma(i_1+1)}} \dots z_{\sigma(i_2)}^{\eta_{\sigma(i_2)}} w_3 \dots w_{r+1},$$

where $\eta_i = -\varepsilon_i$ for all i.

In [1], the authors gave the following basic properties of the map $\tilde{}$.

Lemma 5.1.1 ([1], Lemma 2)**.** *The map* $\tilde{}: P^\sharp_{m,n,p,q} \to P^\sharp_{m,n,q,p}$ *has the following properties.*

1. *If* $f \in P^\sharp_{m,n,p,q}$*, then* $\tilde{\tilde{f}} = f$.

2. *If A is a superalgebra with superinvolution, then $f \in Id^\sharp(A)$ if and only if $\tilde{f} \in Id^*(G(A))$.*

Another basic result we shall need in what follows is the Wedderburn-Malcev theorem for finite dimensional superalgebras with superinvolution. First we recall some definitions.

Definition 5.1.4. *An ideal (subalgebra) I of a superalgebra A with superinvolution $\sharp$ is a $\sharp$-superideal ($\sharp$-superalgebra) if it is a graded ideal(subalgebra) and $I^\sharp = I$.*

Definition 5.1.5. *An algebra A is a simple $\sharp$-superalgebra if $A^2 \neq 0$ and A has non-trivial $\sharp$-superideals.*

Theorem 5.1.2 ([16], Theorem 4.1)**.** *Let A a finite dimensional superalgebra with superinvolution over a field F of characteristic zero. Then there exists a semisimple $\sharp$-superalgebra $B \subset A$ such that*

$$A = B + J(A)$$

and $J(A)$ is a $\sharp$-superideal.

We shall present the classification of the finite dimensional simple $\sharp$-superalgebras over an algebracally close field F (see [5, 25, 46]). In order to describe such a result we first recall same important facts.

It is well known (see [24],Theorem 3.5.3) that if F is algebraically closed, a simple superalgebra A is of one the following types:

(i) Given $k + l \geq 1$, $k \geq l \geq 0$,

$$\begin{aligned} M_{k,l}(F) &= \left\{ \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} | X \in M_k(F), Y \in M_{k\times l}(F), Z \in M_{l\times k}(F), T \in M_l(F) \right\} \\ &= (M_{k,l}(F))_0 \oplus (M_{k,l}(F))_1 \end{aligned}$$

where $(M_{k,l}(F))_0 = \left\{ \begin{pmatrix} X & 0 \\ 0 & T \end{pmatrix} \right\}$ and $(M_{k,l}(F))_1 = \left\{ \begin{pmatrix} 0 & Y \\ Z & 0 \end{pmatrix} \right\}$;

(ii) $Q(n) = M_n(F \oplus cF) = Q(n)_0 \oplus Q(n)_1$, where $Q(n)_0 = M_n(F)$ and $Q(n)_1 = cM_n(F)$ with $c^2 = 1$.

If A is a superalgebra, we denote by A^{sop} the superalgebra which has the same graded vector space structure as A but the product in A^{sop} is given on homogeneous elements a, b by

$$a \circ b = (-1)^{(\deg a)(\deg b)} ba.$$

The direct sum $R = A \oplus A^{\mathrm{sop}}$ is a superalgebra $R_0 = A_0 \oplus A_0^{\mathrm{sop}}$, $R_1 = A_1 \oplus A_1^{\mathrm{sop}}$ and R is endowed with the exchange superinvolution

$$(a, b)^{\mathrm{ex}} = (b, a).$$

Recall that A and B are two algebras (superalgebras) endowed with an involution (superinvolution) $*$ and $\star$, respectively, then $(A, *)$ and $(B, \star)$ are isomorphic, as algebras (superalgebras) endowed with involution (superinvolution), if there exist an isomorphism of algebras (superalgebras) $\phi : A \to B$ such that $\phi(x^*) = \phi(x)^\star$, for all $x \in A$.

Theorem 5.1.3. *Let A be a finite dimensional $\sharp$-simple superalgebra over an algebraically closed field F of characteristic different from 2. Then A is isomorphic to one of the following:*

1. $M_{k,2s}(F)$ *with the orthosymplectic superinvolution* osp *define by*

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{\mathrm{osp}} = \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} X & -Y \\ Z & T \end{pmatrix}^{t} \begin{pmatrix} I_k & 0 \\ 0 & Q \end{pmatrix}$$

   where t denotes the usual matrix transpose, $Q = \begin{pmatrix} 0 & I_s \\ -I_s & 0 \end{pmatrix}$ *and I_k, I_s are the identity matrices of orders k and s, respectively;*

2. $M_{k,k}(F)$ *with the transpose superinvolution* trp *define by*

$$\begin{pmatrix} X & Y \\ Z & T \end{pmatrix}^{\mathrm{trp}} = \begin{pmatrix} T^t & -Y^t \\ Z^t & X^t \end{pmatrix};$$

3. $M_{k,l}(F) \oplus M_{k,l}(F)^{\mathrm{sop}}$ *with the exchange superinvolution;*

4. $Q(n) \oplus Q(n)^{\mathrm{sop}}$ *with the exchange superinvolution.*

Let A be an algebra with involution over a field F. We may assume that F is algebraically closed. In fact, if $\overline{F}$ is the algebraic closure of F, then A can be naturally embedded in the $*$-algebra $A \otimes_F \overline{F}$ and $Id^*(A) \otimes_F \overline{F} = Id^*(A \otimes_F \overline{F})$.

By this argument, if B is a finite dimensional superalgebra with superinvolution such that $Id^*(A) = Id^*(G(B))$, then we can assume that B has a Wedderburn-Malcev decomposition $B_1 + \cdots + B_m + J$ where B_i are $\sharp$-simple superalgebras specified in Theorem 5.1.3.

We finish the section with the following results which will be useful later.

Let now consider $M = F(e_{11}+e_{44}) \oplus F(e_{22}+e_{33}) \oplus Fe_{12} \oplus Fe_{34} \subset UT_4(F)$ endowed with the reflection involution ρ, i.e. the involution obtained by reflecting a matrix along its secondary diagonal. If M^{sup} is the algebra M with grading $M_0 = F(e_{11} + e_{44}) \oplus F(e_{22} + e_{33})$ and $M_1 = Fe_{12} \oplus Fe_{34}$.

Since $M_1^2 = 0$, the reflection involution ρ is a graded involution, i.e. the homogeneous components are stable under ρ: $M_0^\rho \subseteq M_0$ and $M_1^\rho \subseteq M_1$. Hence M^{sup} can be viewed as algebra with superinvolution.

Lemma 5.1.2. *The Grassmann envelope $G(M^{\text{sup}})$ of M^{sup} is $*$-PI-equivalent to M, i.e. $Id^*(G(M^{\text{sup}})) = Id^*(M)$.*

Proof. Notice that the Grassmann envelope of M^{sup} is

$$G(M^{\text{sup}}) \cong \left\{ \begin{pmatrix} g_1^0 & g_1^1 & 0 & 0 \\ 0 & g_2^0 & 0 & 0 \\ 0 & 0 & g_2^0 & g_2^1 \\ 0 & 0 & 0 & g_1^0 \end{pmatrix} | g_1^0, g_2^0 \in G_0, g_1^1, g_2^1 \in G_1 \right\}$$

with involution

$$\begin{pmatrix} g_1^0 & g_1^1 & 0 & 0 \\ 0 & g_2^0 & 0 & 0 \\ 0 & 0 & g_2^0 & g_2^1 \\ 0 & 0 & 0 & g_1^0 \end{pmatrix}^* = \begin{pmatrix} g_1^0 & -g_2^1 & 0 & 0 \\ 0 & g_2^0 & 0 & 0 \\ 0 & 0 & g_2^0 & -g_1^1 \\ 0 & 0 & 0 & g_1^0 \end{pmatrix}.$$

Clearly, $G(M^{\text{sup}})$ satisfies $z_1 z_2 \equiv 0$. Conversely, let $g \in G_1$, $g \neq 0$, and

$$C_g = \operatorname{span}_F\{\underbrace{e_{11}+e_{44}}_{a}, \underbrace{e_{22}+e_{33}}_{b}, \underbrace{ge_{12}}_{c}, \underbrace{-ge_{34}}_{c^*}\} \subset G(M^{\text{sup}})$$

with induced involution. Then the application $\phi : M \to C_g$ given by

$$\phi(e_{11}+e_{44}) = a, \quad \phi(e_{22}+e_{33}) = b, \quad \phi(e_{12}) = c, \quad \phi(e_{34}) = c^*,$$

is an isomorphism such that $\phi(X^\rho) = X^*$ for all $X \in M$. It follows that $G(M^{\text{sup}})$ is PI $*$-equivalent to M. □

Remark 5.1.1. *The algebras $(M_{1,1}(F), \text{trp})$ and $(Q(1) \oplus Q(1)^{\text{sop}}, \text{ex})$ contain a subalgebra with induced superinvolution isomorphic to $F \oplus F$ with exchange superinvolution. In fact, if we consider the subalgebra $C_1 = Fe_{11} + F_{22}$ of $M_{1,1}(F)$ and the subalgebra $C_2 = (Q(1) \oplus Q(1)^{\text{sop}})_0$ of $Q(1) \oplus Q(1)^{\text{sop}}$, it is not difficult to see that (C_1, trp) and (C_2, ex) are isomorphic to $(F \oplus F, \text{ex})$.*

5.2 Some lemmas

In this section, we shall study the structure of a generating algebra with involution of a variety $\mathcal{V}$ not containing the algebra (M, ρ).

Lemma 5.2.1. *Let A be an algebra with involution. If $(M, \rho) \notin var^*(A)$, then*

$$(G(M_{k,2l}(F)), \text{osp}) \notin var^*(A),$$

for any $k \geq 2$ or $l \geq 2$.

Proof. Suppose that $(G(M_{k,2l}(F)), \text{osp}) \in \text{var}^*(A)$. Let us consider first $k \geq 0$ and $l \geq 2$. We can consider the elements $a = e_{k+1,k+1} + e_{k+l+1,k+l+1}$, $b = e_{k+2,k+2} + e_{k+l+2,k+l+2}$, $c = e_{1,2}$ and $c^{\text{osp}} = e_{k+l+2,k+l+1}$. Let $C \cong \text{span}_F\{a, b, c, c^{\text{osp}}\}$ be a subalgebra of $(G(M_{k,2l}(F)), \text{osp})$ with induced involution. Then the application $\phi : M \to C$ given by

$$\phi(e_{11} + e_{44}) = a, \quad \phi(e_{22} + e_{33}) = b, \quad \phi(e_{12}) = c, \quad \phi(e_{34}) = c^{\text{ex}},$$

is an isomorphism such that $\phi(X^\rho) = X^{\text{osp}}$ for all $X \in M$. Hence C is an algebra with involution isomorphic to (M, ρ) and $C \in \text{var}^*(A)$, a contradiction.

Let now $k \geq 2$ and $l \in \{0, 1\}$. Let $C \cong \text{span}_F\{e_{11}, e_{12}, e_{21}, e_{22}\}$ be a subalgebra of $(G(M_{k,2l}(F)), \text{osp})$ with induced involution. Clearly C is isomorphic to the matrix algebra of order 2 with the transpose involution t, $(M_2(F), t)$. Since $(M, \rho) \in \text{var}^*(M_2(F), t)$ (see [51] Remark 3.2), it follows $(M, \rho) \in \text{var}^*(M_2(F), t) = \text{var}^*(C) \subseteq \text{var}^*(G(M_{k,2l}(F)), \text{osp}) \subseteq \text{var}^*(A)$, a contradiction. □

Lemma 5.2.2. *Let A be an algebra with involution. If $(M, \rho) \notin var^*(A)$, then*

$$(G(M_{1,2}(F)), \text{osp}) \notin var^*(A).$$

Proof. First notice that

$$Id^*(G(M_{1,2}(F))) \subseteq Id^*(G(M^{\text{sup}})). \tag{5.1}$$

In fact, by Lemma 4.3 in [30], if $f \in Id^\sharp(M_{1,2}(F))$, then $f \in Id^\sharp(M^{\text{sup}})$. Hence, by lemma 5.1.1, if $\tilde{f} \in Id^*(G(M_{1,2}(F)))$, then $\tilde{f} \in Id^*(G(M^{\text{sup}}))$.

Since $M \notin \text{var}^*(A)$, by lemma 5.1.2, follows $G(M^{\text{sup}}) \notin \text{var}^*(A)$. Thus, by (5.1), $(G(M_{1,2}(F)), \text{osp}) \notin \text{var}^*(A)$. □

Lemma 5.2.3. *Let A be an algebra with involution. If $(M, \rho) \notin var^*(A)$, then*

$$(G(M_{k,k}(F)), \text{trp}) \notin var^*(A),$$

for any $k \geq 2$.

Proof. Suppose that $(G(M_{k,k}(F)), \mathrm{trp}) \in \mathrm{var}^*(A)$ with $k \geq 2$. We can consider its subalgebra

$$C \cong \mathrm{span}_F\{\underbrace{e_{11} + e_{k+1,k+1}}_{a}, \underbrace{e_{22} + e_{k+2,k+2}}_{b}, \underbrace{e_{12}}_{c}, \underbrace{e_{k+2,k+1}}_{c^{\mathrm{trp}}}\}$$

with induced involution trp. Then the application $\phi : M \to C$ given by

$$\phi(e_{11} + e_{44}) = a, \quad \phi(e_{22} + e_{33}) = b, \quad \phi(e_{12}) = c, \quad \phi(e_{34}) = c^{\mathrm{trp}},$$

is an isomorphism such that $\phi(X^\rho) = X^{\mathrm{trp}}$ for all $X \in M$. Hence C is an algebra with involution isomorphic to (M, ρ) and $C \in \mathrm{var}^*((G(M_{k,k}(F)), \mathrm{trp})) \subseteq \mathrm{var}^*(A)$, a contradiction. □

Lemma 5.2.4. *Let A be an algebra with involution. If $(M, \rho) \notin var^*(A)$, then*

$$(G(M_{k,l}(F) \oplus M_{k,l}(F)^{\mathrm{sop}}), \mathrm{ex}) \notin var^*(A),$$

for any k, l such that $k + l \geq 1$.

Proof. Suppose that $(G(M_{k,l}(F) \oplus M_{k,l}(F)^{\mathrm{sop}}), \mathrm{ex}) \in \mathrm{var}^*(A)$ with $k + l \geq 1$ and let us consider the elements $a = (e_{11}, e_{11})$, $b = (e_{k+1,k+1}, e_{k+1,k+1})$, $c = (ge_{1,k+1}, 0)$ and $c^{\mathrm{ex}} = (0, -ge_{1,k+1})$, where $g \in G_1$, $g \neq 0$. Let $C \cong \mathrm{span}_F\{a, b, c, c^{\mathrm{ex}}\}$ be a subalgebra of $(G(M_{k,l}(F) \oplus M_{k,l}(F)^{\mathrm{sop}}), \mathrm{ex})$ with induced involution. Then the application $\phi : M \to C$ given by

$$\phi(e_{11} + e_{44}) = a, \quad \phi(e_{22} + e_{33}) = b, \quad \phi(e_{12}) = c, \quad \phi(e_{34}) = c^{\mathrm{ex}},$$

is an isomorphism such that $\phi(X^\rho) = X^{\mathrm{ex}}$ for all $X \in M$. Hence C is an algebra with involution isomorphic to (M, ρ) and $C \in \mathrm{var}^*(A)$, a contradiction. □

Lemma 5.2.5. *Let A be an algebra with involution. If $(M, \rho) \notin var^*(A)$, then*

$$(G(Q(n) \oplus Q(n)^{\mathrm{sop}}), \mathrm{ex}) \notin var^*(A),$$

for any $n \geq 2$.

Proof. Suppose by contradiction that $(G(Q(n) \oplus Q(n)^{\mathrm{sop}}), \mathrm{ex}) \in \mathrm{var}^*(A)$ with $n \geq 2$.

Notice that $((Q(n) \oplus Q(n)^{\mathrm{sop}})_0, \mathrm{ex})$ is an algebra with involution equal to the direct sum of the full matrix algebra of order n and its opposite algebra with the exchange involution, $(M_n(F) \oplus M_n(F)^{op}, \mathrm{ex})$. Then $(M_n(F) \oplus M_n(F)^{op}, \mathrm{ex})$ is isomorphic to a subalgebra $C \cong (Q(n) \oplus Q(n)^{\mathrm{sop}})_0$ of $(G(Q(n) \oplus Q(n)^{\mathrm{sop}}), \mathrm{ex})$ with induced involution. Hence $(M_n(F) \oplus M_n(F)^{op}, \mathrm{ex}) \in \mathrm{var}^*(C) \subseteq \mathrm{var}^*(G(Q(n) \oplus Q(n)^{\mathrm{sop}}), \mathrm{ex}) \subseteq \mathrm{var}^*(A)$, contradiction to the lemma 4.4 of [51]. □

Lemma 5.2.6. *Let $\mathcal{V}$ be a $*$-variety such that $(M, \rho) \notin \mathcal{V}$. Then*

$$\mathcal{V} = var^*(G(A_1) \oplus \cdots \oplus G(A_n)),$$

where for every $i \in \{1, \ldots, n\}$, A_i is a finite dimensional superalgebra with superinvolution isomorphic to one of the following algebras:

1. *$F + J_i$, with trivial superinvolution on F,*
2. *$F \oplus F + J_i$, with exchange superinvolution on $F \oplus F$,*
3. *$M_{0,2}(F) + J_i$, with orthosymplectic superinvolution on $M_{0,2}(F)$,*
4. *$M_{1,1}(F) + J_i$, with transpose superinvolution on $M_{1,1}(F)$,*
5. *$Q(1) \oplus Q(1)^{\mathrm{sop}} + J_i$, with exchange superinvolution on $Q(1) \oplus Q(1)^{\mathrm{sop}}$,*

and J_i is the Jacobson radical of A_i.

Proof. By theorem 5.1.1, we can write $\mathcal{V} = \mathrm{var}^*(G(A))$ where $G(A)$ is the Grassmann envelope of a finite dimensional superalgebra with superinvolution A.

Let $A = B + J$ be the Wedderburn-Malcev decomposition of A, where $J = J(A)$ is the Jacobson radical and B is a maximal semisimple subalgebra of A. It is well known that B is a $\sharp$-superalgebra and J is a $\sharp$-superideal of A. Moreover, we can write

$$B = B_1 \oplus \cdots \oplus B_m$$

where $B_1, \ldots, B_m$ are simple $\sharp$-superalgebras. Now, by previous lemmas, for each $i = 1, \ldots, m$, either $B_i \cong F$ with trivial superinvolution or $B_i \cong (F \oplus F, \mathrm{ex})$ or $B_i \cong (M_{0,2}(F), \mathrm{osp})$ or $B_i \cong (M_{1,1}(F), \mathrm{trp})$ or $B_i \cong (Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$.

Suppose now that $B_i J B_j \neq 0$ for some $i \neq j$. Since by [51, Remark 3.3] and Remark 5.1.1 the algebras $(M_{0,2}(F), \mathrm{osp})$, $(M_{1,1}(F), \mathrm{trp})$ and $(Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$ contain a subalgebra $C \cong F \oplus F$ with exchange superinvolution, then we can apply the same technique of [51, Lemma4.6] for the superinvolution case to reach a contradiction.

Thus we have

$$B_i J B_j = B_i B_j = 0$$

for all $i \neq j$. Clearly, these relations imply that

$$G(B_i)G(J)G(B_j) = G(B_i)G(B_j) = 0, \quad \text{for all } i \neq j \tag{5.2}$$

Set $A_i = B_i + J$, $i = 1, \ldots, m$. Then $A = B_1 \oplus \cdots \oplus B_m + J = (B_1 + J) + \cdots + (B_m + J) = A_1 + \cdots + A_m$. Moreover for each $i = 1, \ldots, m$, $J \subseteq A_i$ is the Jacobson radical of A_i, and $A_i/J \cong B_i$. So, each A_i is isomorphic to one of the algebras (i),(ii), (iii), (iv) or (v).

Now we claim that $Id^*(G(A_1)+\cdots+G(A_m)) = Id^*(G(A_1))\cap\cdots\cap Id^*(G(A_m))$.

In fact, if $f = f(y_1,\ldots,y_r,z_1,\ldots,z_{n-r}) \in Id^*(G(A_1))\cap\cdots\cap Id^*(G(A_m))$ is multilinear, we shall prove that $f \equiv 0$ on $Id^*(G(A_1)+\cdots+G(A_m))$. In order to do so, let us consider an evaluation in $G(A_1)\cup\cdots\cup G(A_m)$ such that $y_i \to \overline{y}_i \in G(A_1)^+\cup\cdots\cup G(A_m)^+$ and $z_j \to \overline{z}_j \in G(A_1)^-\cup\cdots\cup G(A_m)^-$. Now if $\overline{y}_1,\ldots,\overline{y}_r,\overline{z}_1,\ldots,\overline{z}_{n-r} \in G(A_k)$, for some k, then $f(\overline{y}_1,\ldots,\overline{y}_r,\overline{z}_1,\ldots,\overline{z}_{n-r}) = 0$, since $f \in Id^*(G(A_k))$. Otherwise, by observing that $G(A_i) = G(B_i+J)$ for all i, there exist k,l with $k\neq l$ such that one of the following occurs: either $\overline{y}_k \in G(A_k)^+$ and $\overline{y}_l \in G(A_l)^+$, or $\overline{y}_k \in G(A_k)^+$ and $\overline{z}_l \in G(A_l)^-$, or $\overline{z}_k \in G(A_k)^-$ and $\overline{z}_l \in G(A_l)^-$. In all these cases, by 5.2 we obtain $\overline{w}_{\sigma(1)}\ldots\overline{w}_{\sigma(n)} = 0$, for the corresponding evaluation of any monomial. Thus $f \in Id^*(G(A_1)\oplus\cdots\oplus G(A_m))$. Since the other inclusion is obvious we get the equality. Since $A = A_1+\cdots+A_m$ and $Id^*(G(A_1))\cap\cdots\cap Id^*(G(A_m)) = Id^*(G(A_1)\oplus\cdots\oplus G(A_n))$, it follows that $\mathrm{var}^*(G(A)) = \mathrm{var}^*(G(A_1)\oplus\cdots\oplus G(A_n))$. $\square$

5.3 The main result

In this section we study the nth cocharacter of a variety $\mathcal{V}$ not containing the algebra (M,ρ).

We start by recalling some notation. For integers $k,l\geq 0$, we define a hook shaped part of the plane of arm d and leg l,

$$H(d,l) = \{\lambda = (\lambda_1,\lambda_2,\ldots)\vdash n \geq \ |\lambda_{d+1}\leq l\}.$$

Also given an algebra A with involution $*$ satisfying a non-trivial $*$-polynomial identities we say that its nth $*$-cocharacter $\chi_n^*(A)$ lies in two hooks $H(k_1,l_1)$ and $H(k_2,l_2)$ if in the decomposition (2.1) of $\chi_n^*(A)$, $m_{\lambda,\mu}\neq 0$ implies that $\lambda\in H(k_1,l_1)$ and $\mu\in H(k_2,l_2)$. We write

$$\chi_n^*(A) \subseteq (H(k_1,l_1),H(k_2,l_2)).$$

There is a close relation between algebras with involution satisfying a non-trivial $*$-polynomial identity and infinite hooks in the following sense.

Theorem 5.3.1 ([20],Theorem 5.9)**.** *Let A be an algebra with involution satisfying a non-trivial $*$-polynomial identity. Then there exist integer $k_1,l_1,k_2,l_2\geq 0$ such that*

$$\chi_n^*(A) \subseteq (H(k_1,l_1),H(k_2,l_2)),$$

for all $n \geq 1$.

For any partition $\lambda \vdash r$ let T_λ be a Young tableau of shape λ and e_{T_λ} the corresponding minimal essential idempotent of the group algebra FS_n. Recall that $e_{T_\lambda} = \sum_{\substack{\sigma \in R_{T_\lambda} \\ \tau \in C_{T_\lambda}}} (\operatorname{sgn} \tau)\sigma\tau$ where R_{T_λ} and C_{T_λ} are the subgroups of row and column permutations of T_λ, respectively.

Let T_λ and T_μ be tableaux of shape $\lambda \vdash r$ and $\mu \vdash n-r$, respectively. In what follows whenever we write $e_{T_\lambda} e_{T_\mu} g(x_1^+, \ldots, x_r^+, x_1^-, \ldots, x_{n-r}^-)$, for same polynomial $g \in P^*_{r,n-r}$, we understand that e_{T_λ} acts on the symmetric variables $x_1^+, \ldots, x_r^+$ and e_{T_μ} acts on the skew variables $x_1^-, \ldots, x_{n-r}^-$.

Next we recall the following useful result.

Lemma 5.3.1 ([1],Lemma 7)**.** *Let $\lambda \vdash r$, $\mu \vdash n-r$ be such that $\lambda \in H(k_1, l_1)$ and $\mu \in H(k_2, l_2)$. Suppose that for some tableaux T_λ and T_μ and some polynomial $g \in P^*_{r,n-r}$, we have that $e_{T_\lambda} e_{T_\mu} g \neq 0$. Then there exist a polynomial $f \in P^*_{r,n-r}$ such that $F(S_r \times S_{n-r}) f = F(S_r \times S_{n-r}) e_{T_\lambda} e_{T_\mu} g$ and two decompositions into disjoint sets*

$$\{x_1^+, \ldots, x_r^+\} = X_1^+ \cup \cdots \cup X_{k_1'}^+ \cup T_1^+ \cup \cdots \cup T_{l_1'}^+,$$

$$\{x_1^-, \ldots, x_{n-r}^-\} = X_1^- \cup \cdots \cup X_{k_2'}^- \cup T_1^- \cup \cdots \cup T_{l_2'}^-,$$

where $k_i' \leq k_i$, $l_i' \leq l_i$ $(i = 1, 2)$, such that f is symmetric in the variables of each of the sets X_i^+, X_j^-, $1 \leq i \leq k_1'$, $1 \leq j \leq k_2'$, and alternating on each of the sets T_i^+, T_j^-, $1 \leq i \leq l_1'$, $1 \leq j \leq l_2'$.

Lemma 5.3.2. *Let $A = C + J$ be a finite dimensional superalgebra with superinvolution, where $J = J(A)$ is its Jacobson radical and C is a $\sharp$-simple subalgebra of A which is isomorphic to either F with trivial superinvolution or $(F \oplus F, \mathrm{ex})$ or $(M_{1,1}(F), \mathrm{trp})$ or $(Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$. If $\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu} \chi_{\lambda,\mu}$, then there exists a constant K such that $m_{\lambda,\mu} \leq K$, for all $n \geq 1$ and $|\lambda| + |\mu| = n$.*

Proof. Since $G(A) \subseteq A \otimes G$, $G(A)$ satisfies a non-trivial $*$-polynomial identity. Then by Theorem 5.3.1, the $*$-cocharacter of $G(A)$ lies in two hooks $H(d_1, d_1)$ and $H(d_2, d_2)$ for some integers d_1, d_2, i.e.,

$$\chi_n^*(G(A)) \subseteq (H(d_1, d_1), H(d_2, d_2)).$$

Suppose that C is isomorphic to $(M_{1,1}(F), \mathrm{trp})$, then one can choose $\{a_0^+, \ldots, a_{s-1}^+\}$, $\{b_0^+, \ldots, b_{t-1}^+\}$, $\{a_0^-, \ldots, a_{u-1}^-\}$, $\{b_0^-, \ldots, b_{v-1}^-\}$ basis of A_0^+, A_1^+, A_0^- and A_1^-, respectively, such that $a_0^+ \in C_0^+$, $b_0^+ \in C_1^+$, $a_0^- \in C_0^-$, $b_0^- \in C_1^-$ and $a_1^+, \dots, a_{s-1}^+$, $b_1^+, \dots, b_{t-1}^+$, $a_1^-, \dots, a_{u-1}^-$, $b_1^-, \dots, b_{v-1}^- \in J$.

Let q be the least positive integer such that $J^q = 0$ and set $N_0 = ((2q)^d)^{4d_1d_2}$ where $d = s + t + u + v$ is the dimension of A. We shall prove that any multiplicity $m_{\lambda,\mu}$ in $\chi_n^*(G(A))$ is bounded by dN_0.

To this end let $\lambda \vdash r$ and $\mu \vdash n - r$ be two partitions such that $\lambda \subset H(d_1, d_1)$ and $\mu \subset H(d_2, d_2)$, and consider the corresponding pair of Young tableaux (T_λ, T_μ). Let e_{T_λ} and e_{T_μ} be the essential idempotents corresponding to T_λ and T_μ, respectively. Hence the element $e = e_{T_\lambda} e_{T_\mu}$ is an essential idempotent in the group algebra $F(S_r \times S_{n-r})$. Clearly, there exists a multilinear polynomial $g \in F\langle X, *\rangle$, such that $eg = e_{T_\lambda} e_{T_\mu} g \neq 0$ in $F\langle X, *\rangle$.

By Lemma 5.3.1, there exists $f \in P^*_{r,n-r}$, $f \neq 0$, such that $F(S_r \times S_{n-r})f = F(S_r \times S_{n-r}) e_{T_\lambda} e_{T_\mu} g$ and the variables of the polynomial f are partitioned into $2(d_1 + d_2)$ disjoint subsets

$$X_1^+ \cup \dots \cup X_{d_1}^+ \cup T_1^+ \cup \dots \cup T_{d_1}^+ \cup X_1^- \cup \dots \cup X_{d_2}^- \cup T_1^- \cup \dots \cup T_{d_2}^-,$$

such that f is symmetric in the variables of each sets X_i^+, X_j^-, $1 \leq i \leq d_1$, $1 \leq j \leq d_2$, and alternating on each of the sets T_i^+, T_j^-, $1 \leq i \leq d_1$, $1 \leq j \leq d_2$. Notice that if $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, then X_i^+ is empty if $\lambda_i \leq d_1$ and X_j^- is empty if $\mu_j \leq d_2$. On the other hand, if $\lambda_i > d_1$ and $\mu_j > d_2$, then $|X_i^+| = \lambda_i - d_1$ and $|X_j^-| = \mu_j - d_2$. Moreover, $|T_i^+| = \lambda_i^{'}$ and $|T_j^-| = \mu_j^{'}$ where $\lambda^{'} = (\lambda_1^{'}, \lambda_2^{'}, \dots)$ and $\mu^{'} = (\mu_1^{'}, \mu_2^{'}, \dots)$ are the conjugate partitions of λ and μ, respectively.

Notice that for any $\sigma_1 \in S_r$ and any $\sigma_2 \in S_{n-r}$ we have $\sigma_1 e_{T_\lambda} \neq 0$ and $\sigma_2 e_{T_\mu} \neq 0$ and so for $\rho = (\sigma_1, \sigma_2) \in S_r \times S_{n-r}$ we have $\rho e \neq 0$. It follows that if $f \in P^*_{r,n-r}$ is a $*$-polynomial such that $ef \neq 0$, then the polynomials ef and $g^{'} = \rho e f$ generate the same irreducible $S_r \times S_{n-r}$-module. Now we choose σ_1 and σ_2, in such a way that $\sigma_1 e_{T_\lambda} f$ is symmetric on the first $\lambda_1 - d_1$ variables, on the next $\lambda_2 - d_1$ variables and so on. A similar condition holds for the alternating sets of variables T_j^+, $1 \leq j \leq d_1$. In the same way we choose $\sigma_2 e_{T_\mu} f$ is symmetric on the first $\mu_1 - d_2$ variables, on the next $\mu_2 - d_2$ variables and so on. A similar condition holds for the alternating sets T_j^-, $1 \leq j \leq d_2$.

Let $f_1, \dots, f_K$ be multilinear $*$-polynomials generating in $P^*_{r,n-r}$ different but isomorphic $S_r \times S_{n-r}$-modules corresponding to the same pair of partitions (λ, μ). By the above remark, we can choose permutations $\rho_1, \dots, \rho_K \in S_r \times S_{n-r}$ and a decomposition $X = X^+ \cup X^- \cup T^+ \cup T^-$, where $X^+ = X_1^+ \cup \dots \cup X_{d_1}^+$, $X^- = X_1^- \cup \dots \cup X_{d_2}^-$, $T^+ = T_1^+ \cup \dots \cup T_{d_1}^+$ and $T^- = T_1^- \cup \dots \cup T_{d_2}^-$ and $\rho_1 f_1, \dots, \rho_K f_K$ are simultaneously symmetric on X_i^+ and X_j^- and alternating on T_i^+ and T_j^- for all $i = 1, \dots, d_1$, $j = 1, \dots, d_2$. Thus without loss of generality, we may assume that $f_1, \dots, f_K$ satisfy this condition.

Let us now assume by contradiction that $m_{\lambda,\mu} = K > d((2q)^d)^{4d_1d_2}$. We shall prove that $G(A)$ satisfies a $*$-identity of the type

$$f = \gamma_1 f_1 + \cdots + \gamma_K f_K, \tag{5.3}$$

where $\gamma_1, \dots, \gamma_K \in F$ are not all zero. Then we shall reach a contradiction because this will say that $f_1, \dots, f_M$ are linearly dependent modulo $\mathrm{Id}^*(A)$.

It is sufficient to verify that f has only zero values on elements of the form $a_i^+ \otimes g_1$, $b_l^- \otimes g_2^{'}$, $a_j^- \otimes g_2$ and $b_k^+ \otimes g_1^{'}$, where $g_1, g_2 \in G_0$ and $g_1^{'}, g_2^{'} \in G_1$. First we define a substitution of special kind.

Let $0 \leq \alpha_{j0}^+, \dots, \alpha_{j(s-1)}^+, \beta_{j0}^+, \dots, \beta_{j(t-1)}^+, \alpha_{k0}^-, \dots, \alpha_{k(u-1)}^-, \beta_{k0}^-, \dots, \beta_{k(v-1)}^-$ be integers such that

$$\sum_{i=0}^{s-1} \alpha_{ji}^+ + \sum_{i=0}^{t-1} \beta_{ji}^+ = |X_j^+|, \quad 1 \leq j \leq d_1,$$

$$\sum_{i=0}^{s-1} \alpha_{ji}^+ + \sum_{i=0}^{t-1} \beta_{ji}^+ = |T_{j-d_1}^+|, \quad d_1 + 1 \leq j \leq 2d_1,$$

$$\sum_{i=0}^{u-1} \alpha_{ki}^- + \sum_{i=0}^{v-1} \beta_{ki}^- = |X_k^-|, \quad 1 \leq k \leq d_2,$$

$$\sum_{i=0}^{u-1} \alpha_{ki}^- + \sum_{i=0}^{v-1} \beta_{ki}^- = |T_{k-d_2}^-|, \quad d_2 + 1 \leq k \leq 2d_2.$$

We say that an evaluation φ has type

$$(\alpha_{j0}^+, \dots, \alpha_{j(s-1)}^+, \beta_{j0}^+, \dots, \beta_{j(t-1)}^+, \alpha_{k0}^-, \dots, \alpha_{k(u-1)}^-, \beta_{k0}^-, \dots, \beta_{k(v-1)}^-),$$

$1 \leq j \leq 2d_1$, $1 \leq k \leq 2d_2$, if we replace the variables in the following way: for fixed j, $1 \leq j \leq d_1$, we replace the first α_{j0}^+ variables from X_j^+ by elements $a_0^+ \otimes g$ (with distinct elements g for distinct $x \in X_j^+$), the next α_{j1}^+ variables by elements $a_1^+ \otimes g$, and so on up to the last $\alpha_{j(s-1)}^+$ symmetric variables by elements $a_{s-1}^+ \otimes g$, where all elements g lie in G_0. Now we evaluate the next β_{j1}^+ variables from X_j^+ in elements $b_0^+ \otimes g^{'}$, the next β_{j1}^+ variables in elements $b_1^+ \otimes g^{'}$, and so on up to the last $\beta_{j(t-1)}^+$ symmetric variables in elements $b_{t-1}^+ \otimes g^{'}$, where all elements $g^{'}$ lie in G_1. For $j = d_1, \dots, 2d_1$ we apply the same procedure in order to replace variables in $T_{j-d_1}^+$ by elements of type $a_h^+ \otimes g$ and $b_h^+ \otimes g^{'}$.

For fixed k an analogous evaluation will be made in order to replace the skew variables from X_k^- and $T_{k-d_2}^-$ by elements of type $a_{h_1}^- \otimes g$, $1 \leq h_1 \leq u-1$, and $b_{h_2}^- \otimes g^{'}$, $1 \leq h_2 \leq v-1$, where all elements g lie in G_0 and all elements $g^{'}$ lie in G_1.

In order to give a non-zero value in (5.3) any substitution should satisfy the following restrictions. If $1 \leq j \leq d_1$ and $1 \leq k \leq d_2$, then:

1. $\beta_{ji}^+ \leq 1$, $0 \leq i \leq t-1$, and $\beta_{kh}^- \leq 1$, $0 \leq h \leq v-1$;
2. $\sum_{i=1}^{s-1} \alpha_{ji}^+ \leq q-1$ and $\sum_{i=1}^{u-1} \alpha_{ki}^- \leq q-1$;
3. $\alpha_{j0}^+ = |X_j^+| - (\alpha_{j1}^+ + \cdots + \alpha_{j(s-1)}^+ + \beta_{j0}^+ + \cdots + \beta_{j(t-1)}^+)$ and $\alpha_{k0}^- = |X_k^-| - (\alpha_{k1}^- + \cdots + \alpha_{k(u-1)}^- + \beta_{k0}^- + \cdots + \beta_{k(v-1)}^-)$.

The first property follows since f is symmetric on X_j^+ and X_k^- ; for example, f becomes zero when we evaluate two variables of X_j^+ in $b_h^+ \otimes g'$, $b_h^+ \otimes g''$, for some $g', g'' \in G_1$. The second property follows since $J^q = 0$. Similarly, if $d_1+1 \leq j \leq 2d_1$ and $d_2+1 \leq k \leq 2d_2$, it follows that:

1. $\alpha_{ji}^+ \leq 1$, $0 \leq i \leq s-1$, and $\alpha_{kh}^- \leq 1$, $0 \leq h \leq u-1$;
2. $\sum_{i=1}^{t-1} \beta_{ji}^+ \leq q-1$ and $\sum_{i=1}^{v-1} \beta_{ki}^- \leq q-1$;
3. $\beta_{j0}^+ = |T_{j-d_1}^+| - (\beta_{j1}^+ + \cdots + \beta_{j(t-1)}^+ + \alpha_{j0}^+ + \cdots + \alpha_{j(s-1)}^+)$ and $\beta_{k0}^- = |T_{k-d_2}^-| - (\beta_{k1}^- + \cdots + \beta_{k(v-1)}^- + \alpha_{k0}^- + \cdots + \alpha_{k(u-1)}^-)$.

Let $1 \leq j \leq d_1$. Then the number of distinct t-tuples $(\beta_{j0}^+, \ldots, \beta_{j(t-1)}^+)$ is less then 2^t and the number of distinct s-tuples $(\alpha_{j0}^+, \ldots, \alpha_{j(s-1)}^+)$ is at most q^s. Thus the total number of distinct $t+s$-tuples $(\alpha_{j0}^+, \ldots \beta_{j(t-1)}^+)$ is bounded by $(2q)^{s+t}$. Similarly, for $d_1+1 \leq j \leq 2d_1$, the number of distinct s-tuples $(\alpha_{j0}^+, \ldots, \alpha_{j(s-1)}^+)$ is less then 2^s and the number of distinct t-tuples $(\beta_{j0}^+, \ldots, \beta_{j(t-1)}^+)$ is at most q^t. Hence $(2q)^{s+t}$ is an upper bound of the total number of distinct $t+s$-tuples $(\alpha_{j0}^+, \ldots \beta_{j(t-1)}^+)$, for each $1 \leq j \leq 2d_1$. In the same way from the conditions (1)-(3) above we get that the total number of distinct $u+v$-tuples $(\alpha_{k0}^-, \ldots, \beta_{k(v-1)}^-)$ is bounded by $(2q)^{u+v}$, for each $1 \leq k \leq 2d_2$.

Thus, for given $1 \leq j \leq 2d_1$, $1 \leq k \leq 2d_2$ the total number of different special substitutions is less than $(2q)^{s+t+u+v} = (2q)^d$. Since the number of pairs (j,k) is $4d_1d_2$, it follows that the total number N of distinct types of substitutions is less than $((2q)^d)^{4d_1d_2}) = N_0$.

Notice that if φ, φ' are two substitutions of the same type and $\varphi(z) = r \otimes p$ for some $z \in X$, $r \in A$, $p \in G$, then $\varphi'(z) = r \otimes p'$ with the same grading of the elements p, p'. Hence if $X = \{z_1, \ldots, z_n\}$, $\varphi(z_i) = r_i \otimes p_i$, $\varphi'(z_i) = r_i \otimes p_i'$, then

$$\varphi(f) = f(r_1 \otimes p_1, \ldots, r_n \otimes p_n) = w \otimes p_1 \ldots p_n,$$

$$\varphi'(f) = f(r_1 \otimes p'_1, \ldots, r_n \otimes p'_n) = w \otimes p'_1 \ldots p'_n,$$

with the same w. In this case we say that φ and φ' are similar.

Now let $\varphi_1, \ldots, \varphi_N$ be substitutions chosen one from each similarity class of distinct types. If φ is one of these substitutions, and h_1, h_2 are two multilinear polynomials of degree n, then by multilinearity and supercommutativity $\varphi(h_1) = q_1 \otimes p_1 \ldots p_n$ and $\varphi(h_2) = q_2 \otimes p_1 \ldots p_n$, where $p_1, \ldots, p_n \in G$ and $q_1, q_2 \in A$.

Now consider all these N substitutions of distinct type $\varphi_1, \ldots, \varphi_N$. Then, for each $j = 1, \ldots, N$ and $i = 1, \ldots, L$ we get

$$\varphi_j(f_i) = a_{ij} \otimes p_{j1} \ldots p_{jn}, \tag{5.4}$$

where $a_{ij} \in A$ and $p_{j1}, \ldots, p_{jn}$ depend on φ_j only.

The matrix (a_{ij}), $1 \leq i \leq K$, $1 \leq j \leq N$, has K rows and N columns of elements from A. Since $K \geq d((2q)^d)^{4d_1d_2}$, where $dimA = d$, the rows of (a_{ij}) are linearly dependent. Hence there exist $\gamma_1, \ldots, \gamma_K \in F$ not all zero, such that

$$\sum_{i=1}^{K} \gamma_i a_{ij} = 0, \quad 1 \leq j \leq N.$$

This, together with (5.4), implies that $\varphi_j(\sum_{i=1}^{K} \gamma_i f_i) = 0$, $1 \leq j \leq N$.

We claim that this implies that $f = \sum_{i=1}^{K} \gamma_i f_i$ is an identity of $G(A)$. In fact, by multilinearity of f, it is enough to check only substitutions φ^* where the variables are evaluated into elements of the type $r \otimes p$, where $r = a_i^+$ or b_i^+ or a_i^- or b_i^-, for some i, and $p \in G_0 \cup G_1$.

Now, given such φ^*, there exists a permutation $\sigma \in S_n$ of the variables (preserving the involution) such that $\varphi^*\sigma = \varphi'$ is similar to some φ_j, $1 \leq j \leq N$. Thus $\varphi'(f_i) = a_{ij} \otimes p'_{j1} \ldots p'_{jn}$ and, so, $\varphi'(f_i) = 0$. We remark that the above σ satisfies $\sigma(X_i^+) = X_i^+$, $\sigma(T_i^+) = T_i^+$, $\sigma(X_j^-) = X_j^-$, $\sigma(T_j^-) = T_j^-$, $1 \leq i \leq d_1$, $1 \leq j \leq d_2$. Since f is symmetric on X_i^+, X_j^- and alternating on T_i^+, T_j^-, therefore $\varphi'(f) = \varphi^*\sigma(f) = \varphi^*(\sigma f) = \pm\varphi(f)$. Thus $\varphi^*(f) = 0$.

This show that modulo the identities of $G(A)$, any K polynomials corresponding to the same pair of tableau are linearly dependent and this is equivalent to say that $m_{\lambda,\mu} \leq K$ for any pair $(\lambda, \mu)$and the proof of the lemma is complete in case C is isomorphic to $(M_{1,1}(F), \mathrm{trp})$.

A similar proof holds also in case C is isomorphic to either F with $F^* = F$ or $(F \oplus F, \mathrm{ex})$ or $(Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$, so it will be omitted. $\square$

Recall that $(M_2(F)), s)$ is the 2×2 matrix algebra with the symplectic involution. Then we have the following.

Theorem 5.3.2. *Let A be an algebra with involution $*$ satisfying a nontrivial identity, and*

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

its nth cocharacter. If there exist a constant K such that for all $n \geq 1$ and $|\lambda| + |\mu| = n$ the inequality $m_{\lambda,\mu} \leq K$ holds, then $(M, \rho) \notin var^*(A)$.

Conversely, if $(M, \rho), (M_2(F)), s) \notin var^*(A)$, then there exist a constant K such that for all $n \geq 1$ and $|\lambda| + |\mu| = n$ the inequality $m_{\lambda,\mu} \leq K$ holds.

Proof. Suppose by contradiction that $(M, \rho) \in \mathrm{var}^*(A)$. By Theorem 2.4.2 the multiplicities in $\chi^*(M)$ are not bounded by a constant. Thus by Lemma 2.4.1 we get an absurd and the first statement is proved.

Conversely, if $(M, \rho), (M_2(F)), s) \notin \mathrm{var}^*(A)$, then the proof follows from Lemmas 2.4.1, 5.2.6 and 5.3.2. $\square$

Lemma 5.3.3. *Let $A = C + J$ be a finite dimensional superalgebra with superinvolution, where $J = J(A)$ is its Jacobson radical and C is a $\sharp$-simple subalgebra of A which is isomorphic to either F with trivial involution or $(F \oplus F, \mathrm{ex})$ or $(M_{0,2}(F), \mathrm{osp})$ or $(M_{1,1}(F), \mathrm{trp})$ or $(Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$. If $\chi_n^*(G(A)) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$, then there exists a constant N such that*

$$|\lambda| - \lambda_1 - \lambda_1' \leq N \quad \textit{and} \quad |\mu| - \mu_1 - \mu_2 - \mu_3 - \mu_1' \leq N.$$

Proof. Let (λ, μ) be a pair of partitions with $|\lambda| + |\mu| = n$ and let q be the index of nilpotence of the Jacobson radical J of A. We claim that if $m_{\lambda,\mu} \neq 0$, then $\lambda_2 \leq q + 1$ and $\mu_4 \leq q + 3$.

In fact, suppose by contradiction that $\mu_4 \geq q + 4$ and $m_{\lambda,\mu} \neq 0$ (the proof is similar if $\lambda_2 \geq q + 2$). Then there exists a pair of Young tableau (T_λ, T_μ), a corresponding essential idempotent $e_{T_\mu}e_{T_\lambda}$ and a polynomial $f \in P_n^*$ such that $e_{T_\mu}e_{T_\lambda}f \notin Id^*(G(A))$. Hence there exists $\tau \in R_{T_\mu}$ such that $g = \tau C_{T_\mu}^- e_{T_\mu}e_{T_\lambda}f \notin Id^*(G(A))$, where $C_{T_\mu}^- = \sum_{\sigma \in C_{T_\mu}} (\mathrm{sgn}\,\sigma)\sigma$. Let T_μ contain the integers $i_1, \ldots, i_{q+2}$ in the first $q + 4$ boxes of the first row, $j_1, \ldots, j_{q+2}$ in the first $q + 4$ boxes of the second row, $k_1, \ldots, k_{q+2}$ in the first

$q+4$ boxes of the third row and $h_1, \dots, h_{q+2}$ in the first $q+4$ boxes of the fourth row. Then the polynomial $g = \tau C^-_{T_\mu} e_{T_\lambda} f$ is alternating on each of the following sets

$$\{x^-_{\tau(i_1)}, x^-_{\tau(j_1)}, x^-_{\tau(k_1)}, x^-_{\tau(h_1)}\}, \dots, \{x^-_{\tau(i_{q+4})}, x^-_{\tau(j_{q+4})}, x^-_{\tau(k_{q+4})}, x^-_{\tau(h_{q+4})}\}.$$

Notice that the variables are evaluated in

$$G(A)^- = (C^-_0 \otimes G_0) \oplus (J^-_0 \otimes G_0) \oplus (C^+_1 \otimes G_1) \oplus (J^+_1 \otimes G_1)$$

and since C is isomorphic to either F with $F^* = F$ or $(F \oplus F, \mathrm{ex})$ or $(M_{0,2}(F), \mathrm{osp})$ or $(M_{1,1}(F), \mathrm{trp})$ or $(Q(1) \oplus Q(1)^{\mathrm{sop}}, \mathrm{ex})$, then C^-_0 is at most 3-dimensional and C^+_1 is at most 1-dimensional. If at least q sets $\{x^-_{\tau(i_s)}, x^-_{\tau(j_s)}, x^-_{\tau(k_s)}, x^-_{\tau(h_s)}\}$ are evaluated in $J^-_0 \otimes G_0 \cup J^+_1 \otimes G_1$, then we get $g \equiv 0$ on $G(A)$ since $J^q = 0$. Hence we have at least five sets $\{x^-_{\tau(i_s)}, x^-_{\tau(j_s)}, x^-_{\tau(k_s)}, x^-_{\tau(h_s)}\}$ that are evaluated in $C^-_0 \otimes G_0 \cup C^+_1 \otimes G_1$. If one of these sets, say $\{x^-_{\tau(i_1)}, x^-_{\tau(j_1)}, x^-_{\tau(k_1)}, x^-_{\tau(h_1)}\}$, is evaluated in the algebra $C^-_0 \otimes G_0$, then we will get that g vanishes in $G(A)$, since g is alternating on $x^-_{\tau(i_1)}$, $x^-_{\tau(j_1)}$, $x^-_{\tau(k_1)}$, $x^-_{\tau(h_1)}$ and $\dim C^-_0 \leq 3$. Then we deduce that there are at least two variables corresponding to indices in the same row of T_μ, say $x^-_{\tau(i_1)}$ and $x^-_{\tau(i_2)}$, that are evaluated in $C^+_1 \otimes G_1$. But the polynomial $e_{T_\mu} e_{T_\lambda} f$ is symmetric on $x^-_{i_1}, \dots, x^-_{i_{q+4}}$. Since $\tau \in R_{T_\mu}$, $e_{T_\mu} e_{T_\lambda} f$ is also symmetric on $x^-_{\tau(i_1)}, \dots, x^-_{\tau(i_{q+4})}$. Since the variables $x^-_{\tau(i_1)}$ and $x^-_{\tau(i_2)}$ are evaluated on $C^+_1 \otimes G_1$, which is anticommutative, we get that $e_{T_\mu} e_{T_\lambda} f \in Id^*(G(A))$, a contradiction. Hence the claim is proved.

Next we claim that if $m_{\lambda,\mu} \neq 0$, then $\lambda'_2 \leq 2q$ and $\mu'_2 \leq 2q+4$.

In fact, suppose to the contrary that $\mu'_2 \geq 2q+5$ and $m_{\lambda,\mu} \neq 0$ (the proof is similar if $\lambda'_2 \geq 2q+1$). As before, there exists a pair of Young tableau (T_λ, T_μ), a corresponding essential idempotent $e_{T_\mu} e_{T_\lambda}$ and a polynomial $f \in P^*_n$ such that $e_{T_\mu} e_{T_\lambda} f \notin Id^*(G(A))$. Hence there exists $\tau \in R_{T_\mu}$ such that $g = \tau C^-_{T_\mu} e_{T_\mu} e_{T_\lambda} f \notin Id^*(G(A))$. Let $i_1, \dots, i_{2q+5}$ denote the integers in the first $2q+5$ boxes of the first column of the T_μ. Similarly, let $j_1, \dots, j_{2q+5}$ be the integers in the first $2q+5$ positions of the second column of T_μ. Then g is alternating on $\{x^-_{\tau(i_1)}, \dots, x^-_{\tau(i_{2q+5})}\}$ and on $\{x^-_{\tau(j_1)}, \dots, x^-_{\tau(j_{2q+5})}\}$.

In order to get a non zero value of g, we have to evaluate at most $q-1$ variables of each set into $J^-_0 \otimes G_0 \cup J^+_1 \otimes G_1$. Moreover, since C^-_0 is at most 3-dimensional and G_0 is commutative, we evaluate at most three variables of each set on $C^-_0 \otimes G_0$. It follows that two variables corresponding to the same row, say $x^+_{\tau(i_1)}$ and $x^+_{\tau(j_1)}$, are evaluate on $C^+_1 \otimes G_1$. Since g is symmetric on these two variables and $C^+_1 \otimes G_1$ is anticommutative, we get that g vanishes in $G(A)$ and the claim is proved.

Thus $\lambda_2 \leq q+1$, $\lambda'_2 \leq 2q$, $\mu_4 \leq q+3$ and $\mu'_2 \leq 2q+4$, if $m_{\lambda,\mu} \neq 0$. It follows that the diagram of λ out of the first row and the first column contains at most $q(2q-1)$

boxes and the diagram of μ out of the first three row and the first column contains at most $(q+2)(2q+1)$ boxes. Hence, $m_{\lambda,\mu}$ may be non zero only if $|\lambda| - \lambda_1 - \lambda_1' \leq K$ and $|\mu| - \mu_1 - \mu_2 - \mu_3 - \mu_1' \leq K'$ where $K = q(2q-1)$ and $K' = (q+2)(2q+1)$. Therefore $N = K'$ is the desired constant, and the proof is complete. □

Theorem 5.3.3. *Let A be an algebra with involution $*$ satisfying a nontrivial identity, and*

$$\chi_n^*(A) = \sum_{|\lambda|+|\mu|=n} m_{\lambda,\mu}\chi_{\lambda,\mu}$$

its nth cocharacter. Then the following conditions are equivalent.

1. *$(M, \rho) \notin var^*(A)$.*
2. *There exists a constant N such that for all $n \geq 1$ and $|\lambda| + |\mu| = n$ the inequalities*

$$|\lambda| - \lambda_1 - \lambda_1' \leq N, \quad |\mu| - \mu_1 - \mu_2 - \mu_3 - \mu_1' \leq N$$

   hold whenever $m_{\lambda,\mu} \neq 0$.

Proof. By Lemmas 5.2.6 and 5.3.3 it follows that (1) implies (2).

Conversely, suppose by contradiction that $(M, \rho) \in \mathrm{var}^*(A)$. If

$$\chi_n^*(M) = \sum_{|\lambda|+|\mu|=n} m'_{\lambda,\mu}\chi_{\lambda,\mu},$$

then by Theorem 2.4.2 for $\lambda = (\lambda_1, \lambda_2, 1)$ and $\mu = \emptyset$ we have $m'_{\lambda,\mu} = \lambda_1 - \lambda_2 - 1 > 0$. Thus $m'_{\lambda,\mu} \neq 0$ for any pair of partitions (λ, μ) with $\mu = \emptyset$ and $|\lambda| - \lambda_1$ arbitrary large. Hence A does not satisfy condition (2) and the proof is complete. □

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