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Research article

On some extrapolation in generalized grand Morrey spaces with applications to PDEs

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Abstract: Rubio de Francia's extrapolation in generalized grand Morrey spaces is derived. This result is applied to the investigation of the regularity of solutions for the second order partial differential equations with discontinuous coefficients in the framework of generalized grand Morrey spaces under the Muckenhoupt condition on weights. Density properties for these spaces are also investigated.

Keywords: Grand Morrey spaces; weighted extrapolation; weighted estimates; regularity of partial differential equations

1. Introduction

Let (X, ρ, μ) be a quasi–metric measure space with a quasi–metric ρ and a finite doubling measure μ. We deal with Rubio de Francia's extrapolation in generalized weighted grand Morrey spaces $M_w^{p),q,\varphi(\cdot)}(X)$ defined on (X, ρ, μ), where w is a weight function on X. p, q and $\varphi(\cdot)$ are appropriate parameters of the space, and the "grandification" of the space is taken with respect to p.

Morrey spaces, introduced by Morrey in [1], describe the regularity of solutions of elliptic partial differential equations (*PDE*s) more precisely than Lebesgue spaces.

Let w be a weight function on X, i.e., w is a μ- a.e. positive integrable function on X. Let $M_w^{p,q}(X)$ be the weighted Morrey space defined with respect to the norm [2]:

$$\|f\|_{M_w^{p,q}(X)} := \sup_B \frac{1}{(w(B))^{\frac{1}{p}-\frac{1}{q}}} \|f\|_{L_w^p(B)} := \sup_B \frac{1}{(w(B))^{\frac{1}{p}-\frac{1}{q}}} \left(\int_B |f(x)|^p w(x) d\mu(x) \right)^{\frac{1}{p}},$$

where $1 < p \le q$, and the supremum is taken over all balls B in X. It is easy to notice that if $p = q$, then we have the weighted Lebesgue space denoted by $L_w^p(X)$. For definitions and essential properties of the classical Morrey spaces, we refer, e.g., to the recent monograph [3].

In 1992 Iwaniec and Sbordone [4] introduced new function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. That investigation was related to the integrability problem of the Jacobian on a bounded open set Ω. More general spaces of $L^{p)}(\Omega)$, denoted by $L^{p),\theta}(\Omega)$, appeared first in the work by Greco et al. [5] in 1997 as the appropriate ambient spaces in which some nonlinear *PDE*s have to be considered.

Problems related to Harmonic Analysis in grand Lebesgue spaces and their associate spaces (called *small Lebesgue spaces*), were intensively studied during the last two decades along with various applications. The reader is referred, e.g., to the monographs [6] and references therein.

Denote by Φ_p the class of non-decreasing functions $\varphi(\cdot)$ on $(0, p - 1)$ such that $\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0$.

Let w be a weight function on X, i.e., w is a $\mu-$ a.e. positive integrable function on X. We consider the weighted grand Morrey space $M_w^{p),q,\varphi(\cdot)}(X)$ defined by the finite norm:

$$\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)} := \sup_{0 < \varepsilon < p-1} \sup_B \frac{\varphi(\varepsilon)}{(w(B))^{\frac{1}{p-\varepsilon} - \frac{1}{q}}} \|f\|_{L_w^{p-\varepsilon}(B)}$$

$$:= \sup_{0 < \varepsilon < p-1} \varphi(\varepsilon) \|f\|_{M_w^{p-\varepsilon,q}(X)},$$

where $1 < p \le q$, and $\varphi(\cdot) \in \Phi_p$. If $\varphi(t) = t^\theta$, where $\theta > 0$, then we use the notation $M_w^{p),q,\theta}(X)$ for $M_w^{p),q,\varphi(\cdot)}(X)$.

One of our motivations to study the extrapolation problem in $M_w^{p),q,\varphi(\cdot)}(X)$ is related to the investigations carried out in [7] and [8], where the same problem was investigated in $M_w^{p,q}(\mathbb{R}^n)$ and $M_w^{p),q,\theta}(X)$, respectively. Komori and Shirai [2] obtained pioneering results regarding the one-weight problem for Harmonic Analysis operators in weighted classical Morrey spaces with Muckenhoupt A_p weights defined on $\mathbb{R}^n$. Similar problems for sublinear operators involving, for example, maximal, fractional, Calderón-Zygmund integral operators in the spaces $M_w^{p,q}(\mathbb{R}^n)$ with A_p weights were explored in [7, 9–14].

We emphasize that the one-weight estimates for sublinear operators including their commutators in grand Morrey spaces were investigated in [15] and [16]. Extrapolation results in weighted grand Lebesgue spaces were derived in [17].

Historically, unweighted grand Morrey spaces $L^{p),\lambda}(X)$ were introduced and studied in [18]. Later, these spaces were generalized in [19] by introducing grand grand Morrey spaces having the "grandification" not only for p, but also for λ.

2. Preliminaries

Let (X, ρ, μ) be a quasi-metric measure space ($QMMS$, briefly), where X is an abstract set, ρ is a quasi-metric on X, and μ is a measure defined on a σ- algebra of subsets of X. Quasi-metric ρ on X is a non-negative function on $X \times X$ satisfying the following conditions: (a) $\rho(x, y) = 0$ if and only if $x = y$; (b) $\rho(x, y) = \rho(y, x)$, $\forall x, y \in X$; (c) there exists a constant $\kappa \ge 1$ such that $\rho(x, y) \le \kappa[\rho(x, z) + \rho(z, y)]$, $\forall x, y, z \in X$. Denote by $B(x, R)$ the ball with center x and radius R, i.e., $B(x, R) := \{y \in X : \rho(x, y) \le R\}$. We say that a measure μ satisfies the doubling condition if there exists a positive

constant C_{dc} such that for all $x \in X$ and $r > 0$, $\mu B(x, 2r) \leq C_{dc} \, \mu B(x, r)$. We will deal with a *QMMS* with doubling measure. Such a *QMMS* is called a space of homogeneous type (*SHT*, briefly).

There are many important examples of an *SHT*. We list some of them:

- Carleson (regular) curves on $\mathbb{C}$ with arc-length measure $d\nu$ and Euclidean distance on $\mathbb{C}$;
- nilpotent Lie groups with Haar measure and homogeneous norm (homogeneous groups);
- the triple (Ω, ρ, dx), where Ω is a domain in $\mathbb{R}^n$, ρ is the Euclidean metric, and dx is the Lebesgue measure induced to Ω satisfying the $\mathcal{A}$ condition [20], i.e., there exists a constant $C > 0$ such that for all $x \in \overline{\Omega}$ and $R \in (0, diam(\Omega))$,

$$\mu(\widetilde{B}(x, R)) \geq CR^n, \tag{2.1}$$

where

$$\widetilde{B}(x, R) := \Omega \cap B(x, R). \tag{2.2}$$

Other properties and examples of *SHT*s can be found, e.g., in [21, 22].

Let $1 < s < \infty$. We say that a weight w belongs to the class $A_s(X)$ (Muckenhoupt class of weights) if

$$[w]_{A_s} := \sup_B \left(\mu(B)^{-1} \int_B w(x) d\mu(x) \right) \left(\mu(B)^{-1} \int_B w^{1-s'}(x) d\mu(x) \right)^{s-1} < \infty, \quad s' = \frac{s}{s-1},$$

where the least upper bound is taken over all balls $B \subset X$. In the literature, $[w]_{A_s}$ is called A_s characteristic of the weight w.

Furthermore, a weight function w is in the class $A_1(X)$ if $Mw(x) \leq Cw(x)$ a.e., where Mw is the Hardy–Littlewood maximal function of w:

$$Mw(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B w(y) d\mu(y) \quad (B \text{ is a ball in } X).$$

In this case, it is assumed that $[w]_{A_1(X)}$ is determined as the essential supremum of Mw/w.

Furthermore, the following monotonicity property holds for Muckenhoupt classes:

$$A_r(X) \subset A_s(X), \quad 1 \leq r < s < \infty.$$

Let us recall that the class of weights $A_\infty(X)$ is defined as follows: $A_\infty(X) = \cup_{\ell \geq 1} A_\ell(X)$.

3. Density in $M_w^{p),q,\varphi(\cdot)}(X)$

Let E be a Banach space and F be its subset. Let us denote by $[F]_E$ the closure of F in E. We are interested in density in $M_w^{p),q,\varphi(\cdot)}(X)$ spaces. In particular we have the following statement [6, 23].

Proposition 3.1. *Let $1 < p \leq q$ and let $\varphi(\cdot) \in \Phi_p$. Suppose that w is a weight function on X. Then,*

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) \|f\|_{M_w^{p-\varepsilon,q}(X)} = 0 \tag{3.1}$$

for $f \in \left[M_w^{p,q}(X) \right]_{M_w^{p),q,\varphi(\cdot)}(X)}$.

Proof. Let $f \in \left[M_w^{p,q}(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ and $\varepsilon_0 > 0$. Then, there is a function $f_{n_0} \in M_w^{p,q}(X)$ such that $\|f - f_{n_0}\|_{M_w^{p),q,\varphi(\cdot)}(X)} < \varepsilon_0$.

Consequently, for such f_{n_0} and ε_0, in view of the condition $\varphi(\cdot) \in \Phi_p$, we have that for sufficiently small ε, $\varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq \varepsilon_0$. Hence,

$$\varphi(\varepsilon)\|f\|_{M_w^{p-\varepsilon,q}(X)} = \varphi(\varepsilon) \sup_B w(B)^{-\frac{1}{p-\varepsilon}+\frac{1}{q}} \|f\|_{L_w^{p-\varepsilon}(B)}$$

$$\leq \varphi(\varepsilon)\|f - f_{n_0}\|_{M_w^{p-\varepsilon,q}(X)} + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq \|f - f_{n_0}\|_{M_w^{p),q,\varphi(\cdot)}(X)} + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)}$$

$$\leq \varepsilon_0 + \varphi(\varepsilon)\|f_{n_0}\|_{M_w^{p,q}(X)} \leq [1 + C_{\varphi,p}]\varepsilon_0$$

for sufficiently small ε, where the constant $C_{\varphi,p}$ depends only on φ and p. Here, we used the embedding

$$M_w^{s,q}(X) \hookrightarrow M_w^{p,q}(X), \quad 1 \leq p \leq s \leq q,$$

which follows from the Hölder inequality and the definition of the weighted Morrey norm. $\qquad\square$

Proposition 3.2. *Let $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$. Suppose that w is a weight function on X. Then,*

$$\left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)} = \left\{ f \in M_w^{p),q,\varphi(\cdot)}(X) : \lim_{N\to\infty} \left\|\chi_{\{|f|>N\}}f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} = 0 \right\}. \tag{3.2}$$

Proof. We use arguments from [24]. Initially observe that if $\lim_{N\to\infty} \left\|\chi_{\{|f|>N\}}f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} = 0$, then $f \in \left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ because $f = \chi_{\{|f|>N\}}f + \chi_{\{|f|\leq N\}}f$, where $\chi_{\{|f|\leq N\}}f \in L^\infty(X)$.

Let us now take $f \in \left[L^\infty(X)\right]_{M_w^{p),q,\varphi(\cdot)}(X)}$ and let $\varepsilon_0 > 0$. We choose $g \in L^\infty(X)$ such that $\|f - g\|_{M_w^{p),q,\varphi(\cdot)}(X)} < \varepsilon_0$. In view of the representation $\left|\chi_{\{|f|>N\}}f\right| \leq |f - g| + \left|\chi_{\{|f|>N\}\cap\{|g|\leq\frac{N}{2C_p}\}}g\right| + \left|\chi_{\{|g|>\frac{N}{2C_p}\}}g\right|$, where $N \in \mathbb{N}$ and $C_p = \varphi(p-1)$, we have

$$|g| \leq \frac{N}{2C_p} < \frac{|f|}{2C_p} \leq \frac{|f - g|}{2C_p} + \frac{|g|}{2C_p}, \text{ on the set } \{|f| > N\} \cap \{2C_p|g| \leq N\}.$$

Hence, $|g| \leq C|f - g|$, where the positive constant C is independent of f and g. Therefore, if $N > 2C_p\|g\|_{L^\infty(X)}$, we have $\left\|\chi_{\{|f|>N\}}f\right\|_{M_w^{p),q,\varphi(\cdot)}(X)} \leq c\|f - g\|_{M_w^{p),q,\varphi(\cdot)}(X)} < c\varepsilon_0$. Finally, we are done. $\qquad\square$

4. Weighted extrapolation

The main result regarding the extrapolation reads as follows:

Theorem 4.1. *Assume that $1 \leq p_0 < \infty$ and that $\mathcal{F}(X)$ is a family of pairs of non-negative measurable functions defined on X. Let, for all $(f, g) \in \mathcal{F}(X)$ and $w \in A_{p_0}(X)$, the inequality*

$$\|g\|_{L_w^{p_0}(X)} \leq N(p_0, [w]_{A_{p_0}(X)})\|f\|_{L_w^{p_0}(X)} \tag{4.1}$$

hold, where $N(p_0, [w]_{A_{p_0}(X)})$ is the positive constant depending only on p_0 and $[w]_{A_{p_0}(X)}$ such that the mapping $\cdot \mapsto N(p_0, \cdot)$ is a non-decreasing for a fixed p_0. Then, for every $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$ and $w \in A_p(X)$, the estimate

$$\|g\|_{M_w^{p),q,\varphi(\cdot)}(X)} \leq C\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}, \quad (f, g) \in \mathcal{F}(X),$$

holds, where the constant C is independent of (f, g).

Extrapolation result for A_∞ weights is given by the next statement:

Theorem 4.2. *Suppose that $\mathcal{F}(X)$ is a class of pairs of functions (f, g), where f and g are μ-measurable functions on X. Let $p_0 \in (0, \infty)$ and $l \geq 1$ be fixed parameters. Suppose that there is a function $N : (0, \infty) \times (0, \infty) \mapsto (0, \infty)$, which is non-decreasing with respect to the second variable, such that the inequality*

$$\|g\|_{L_w^{p_0}(X)} \leq N(p_0, [w]_{A_l(X)}) \|f\|_{L_w^{p_0}(X)} \tag{4.2}$$

holds for all $(f, g) \in \mathcal{F}(X)$ and $w \in A_l(X)$. Then, for every $1 < p \leq q$, $\varphi(\cdot) \in \Phi_p$ and all $w \in A_\infty(X)$ the estimate

$$\|g\|_{M_w^{p),q,\varphi(\cdot)}(X)} \leq C\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}, \quad (f, g) \in \mathcal{F}(X), \tag{4.3}$$

is valid, where the constant C does not depend on (f, g).

These statements for $\varphi(t) = t^\theta$, $t > 0$ were proved in [8].

Remark 4.1. *According to Theorem 4.1 and the fact that the Muckenhoupt condition $w \in A_{p_0}(X)$ guarantees the boundedness of Harmonic Analysis operators such as Calderón–Zygmund singular integrals, commutators of singular integrals, fractional integrals and commutators of fractional integrals in $L_w^{p_0}(X)$ spaces [21, 25], we have appropriate one-weight norm estimates for those operators in grand Morrey spaces $M_w^{p),q,\varphi(\cdot)}(X)$ for $w \in A_p(X)$.*

To prove Theorem 4.2 we need some auxiliary statements from [7, 8]:

Lemma 4.1. *Let $0 < \gamma < 1$ and let f be a μ-locally integrable function on X. Then, $(Mf)^\gamma \in A_1(X)$. Moreover,*

$$\left[(Mf)^\gamma\right]_{A_1} \leq \frac{C_{\kappa,\mu}}{1 - \gamma},$$

where $C_{\kappa,\mu}$ is a structural constant.

Lemma 4.2. *Let $1 \leq \gamma < p < \infty$ and let $w \in A_{p/\gamma}(X)$. Suppose that $p \leq q$. Then, there is $q_0 \in (\gamma, p)$ such that for all $r \in [\gamma, q_0]$, all $s \in (1, s_0(r, w))$, where $s_0(r, w)$ is the constant depending on r and w, all balls B, sufficiently small numbers ε, and all $h \in L_w^{(p/r)'}(B)$ with $\|h\|_{L_w^{(p/r)'}(B)} = 1$, the inequality*

$$\|f\|_{L^r_{(HW)_{s,B}}(X)} \leq C(w(B))^{\frac{1}{p-\varepsilon} - \frac{1}{q}} \|f\|_{M_w^{p-\varepsilon,q}(X)} \tag{4.4}$$

holds, where

$$(HW)_{s,B} := M(h^s w^s \chi_B)^{\frac{1}{s}}, \tag{4.5}$$

and the constant C does not depend on f, B and ε.

Proof of Theorem 4.1. Following [7, 8], initially observe that in view of the Hölder inequality we have for $\sigma < \varepsilon < p - 1$,

$$\frac{1}{w(B)^{\frac{1}{p-\varepsilon} - \frac{1}{q}}} \left(\int_B g^{p-\varepsilon} w d\mu \right)^{1/(p-\varepsilon)} \leq \frac{1}{w(B)^{\frac{1}{p-\sigma} - \frac{1}{q}}} \left(\int_B g^{p-\sigma} w d\mu \right)^{1/(p-\sigma)}, \quad g \geq 0. \tag{4.6}$$

So, it is enough to show that there is a positive constant $C_{\mu,\sigma,w}$ depending only on μ, σ, w such that

$$\sup_{0<\varepsilon<\sigma} \frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}} \left(\int_B g^{p-\varepsilon}wd\mu \right)^{1/(p-\varepsilon)} \leq C_{\mu,\sigma,w}\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}$$

for some sufficiently small positive number σ.

Let $1 < p < \infty$. A classical extrapolation result [17, 26] yields that

$$\|g\|_{L_w^p(X)} \leq C\psi([w]_{A_p(X)})\|f\|_{L_w^p(X)}, \qquad w \in A_p(X), \tag{4.7}$$

for all $(f,g) \in \mathcal{F}(X)$, where C is the constant independent of (f,g) and w, and the mapping $\cdot \mapsto \psi(\cdot)$ is non-decreasing. Furthermore, take $w \in A_p(X)$ and choose $s > 1$ and $r \in (1, p)$ so that inequality (4.4) holds. Introducing the notation $p_\varepsilon := \frac{p-\varepsilon}{r}$, for a ball $B \subset X$, we find that

$$\left(\int_B g^{p-\varepsilon}wd\mu \right)^{\frac{1}{p-\varepsilon}} = \left(\int_B g^{p_\varepsilon r}wd\mu \right)^{\frac{1}{p_\varepsilon r}} = \sup_{\|h\|_{L_w^{p'_\varepsilon}(X)}=1} \left(\int_B g^r hwd\mu \right)^{\frac{1}{r}}.$$

For such an h, in view of Lemma 4.1 we see that $[(HW)_{s,B}]_{A_q} \leq [(HW)_{s,B}]_{A_1} \leq \frac{C_\mu}{1-s^{-1}}$. Furthermore, observe that (4.7) implies that

$$\left(\int_X g^r wd\mu \right)^{\frac{1}{r}} \leq C_\mu \psi([w]_{A_r(X)}) \left(\int_X f^r wd\mu \right)^{\frac{1}{r}}$$

for all $w \in A_r(X)$ and all $(f,g) \in \mathcal{F}(X)$, where the mapping $\cdot \mapsto \varphi(\cdot)$ is non-decreasing. Therefore, in view of Lemmas 4.2 and 4.1, we get

$$\left(\int_X g^r hw\chi_B d\mu \right)^{\frac{1}{r}} \leq \left(\int_X g^r (HW)_{s,B}d\mu \right)^{\frac{1}{r}}$$
$$\leq C\psi\left([(HW)_{s,B}]_{A_r(X)}\right) \left(\int_X f^r (HW)_{s,B}d\mu \right)^{\frac{1}{r}}$$
$$\leq C\widetilde{C}\varphi\left(\left[(HW)_{s,B}\right]_{A_r(X)}\right) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \|f\|_{M_w^{p-\varepsilon,q}(X)}$$
$$\leq C\widetilde{C}\psi\left(\left[(HW)_{s,B}\right]_{A_1(X)}\right) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \varphi(\varepsilon)^{-1} \|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}$$
$$\leq C\widetilde{C}\psi\left(\frac{C_\mu}{1-s^{-1}}\right) w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}} \varphi(\varepsilon)^{-1} \|f\|_{M_w^{p),q,\varphi(\cdot)}(X)},$$

where $\widetilde{C}$ is the constant depending only on p, σ, w.

Finally we deduce

$$\frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}} \left(\int_B g^{p-\varepsilon}wd\mu \right)^{\frac{1}{p-\varepsilon}} \leq C\|f\|_{M_w^{p),q,\varphi(\cdot)}(X)}$$

for sufficiently small ε. Since (see (4.6))

$$\|g\|_{M_w^{p,\varphi(\cdot)}(X)} \sim \sup_{0<\varepsilon<\sigma} \frac{\varphi(\varepsilon)}{w(B)^{\frac{1}{p-\varepsilon}-\frac{1}{q}}} \left(\int_B g^{p-\varepsilon}wd\mu \right)^{\frac{1}{p-\varepsilon}},$$

where $\sigma \in (0, p - 1)$, we are done.

$\square$

Proof of Theorem 4.2. Let (4.2) hold for some $p_0 > 0$. Then, the classical A_∞ extrapolation [27, 28] gives

$$\|g\|_{L_w^p(X)} \leq C_p \psi\big([w]_{A_p}\big) \|f\|_{L_w^p(X)} \tag{4.8}$$

for all $1 < p < \infty$ and $w \in A_p$, where C_p is the positive constant depending on p, and $\psi(\cdot)$ is a non-decreasing mapping.

Let $1 < p < \infty$ and let $w \in A_\infty$. Now, we will show that (4.3) holds for such a weight w and all $(f, g) \in \mathcal{F}(X)$. If $p \geq r$, then $A_r \subset A_p$, and by (4.8) and Theorem 4.1, we get that (4.3) holds for that w and all $(f, g) \in \mathcal{F}(X)$.

Suppose now that $p < r$. Since $w \in A_r$, by the openness property of Muckenhoupt classes [21] we have that $w \in A_{r-\sigma}$ for some small positive σ (the exact value of σ can be found in [29]). Consequently, by the monotonicity property of Muckenhoupt classes, $w \in A_{r-\eta}$ for all η satisfying $0 < \eta < \sigma$. Hence, in view of (4.8), we find that

$$\left\| |g|^{\frac{p-\varepsilon}{r-\eta}} \right\|_{M^{r-\eta, \frac{r(p-\varepsilon)}{r-\eta}}(X)} \leq C_{p,r,\varepsilon,\eta} \psi\big([w]_{A_{r-\eta}}\big) \left\| |f|^{\frac{p-\varepsilon}{r-\eta}} \right\|_{M^{r-\eta, \frac{r(p-\varepsilon)}{r-\eta}}(X)}, \tag{4.9}$$

were $C_{p,r,\varepsilon,\eta}$ is the positive constant depending only on p, r, ε, η, and ψ is a non-decreasing function. Since $[w]_{A_{r-\eta}} \leq [w]_{A_{r-\sigma}}$ and $\sup_{\varepsilon,\eta} C_{p,r,\varepsilon,\eta} < \infty$ (see also the proof of Theorem 4.1 for this fact), we have that

$$\sup_{\varepsilon,\eta} C_{p,r,\varepsilon,\eta} \psi\big([w]_{A_{r-\eta}}\big) < \infty,$$

where the least upper bound is taken over all sufficiently small η and ε. Due to (4.9) we see that

$$\|g\|_{M^{p-\varepsilon,q}(X)}^{\frac{p-\varepsilon}{r-\eta}} \leq C_{p,r,\varepsilon,\eta} \psi\big([w]_{A_{r-\eta}}\big) \|f\|_{M^{p-\varepsilon,q}(X)}^{\frac{p-\varepsilon}{r-\eta}}. \tag{4.10}$$

Raising both sides of (4.10) to the power $\frac{r-\eta}{p-\varepsilon}$, multiplying them by $\varphi(\varepsilon)$ and taking the supremum with respect to ε, we are done.

$\square$

5. Applications to *PDE*s

During the last three decades a quite large number of papers explored local and global regularity problems for strong solutions to elliptic *PDE*s with discontinuous coefficients. To be evident, we take the second order *PDE*

$$\mathcal{L}u(x) \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i x_j} u(x) = f(x) \quad \text{for almost all } x \in \Omega, \tag{5.1}$$

where $\mathcal{L}$ denotes a uniformly elliptic operator on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$.

Suppose that Ω is a domain in $\mathbb{R}^n$. As we know, the triple (Ω, ρ, dx) satisfying the condition $\mathcal{A}$ (see (2.1) for this condition), where ρ is the Euclidean metric, and dx is the Lebesgue measure induced to Ω, is an example of an *SHT*. Hence, the previous statements are true for such domains.

The regularizing property of $\mathcal{L}$ in Hölder spaces (i.e., $\mathcal{L}u \in C^\alpha(\bar{\Omega})$ implies $u \in C^{2+\alpha}(\bar{\Omega})$) has been intensively investigated for the case of Hölder continuous coefficients a_{ij}. Also, we emphasize that unique classical solvability of the Dirichlet problem for (5.1) has been obtained in this case (we refer to [30] and references therein). For uniformly continuous coefficients a_{ij}, an L^p-Schauder theory has been elaborated for the operator $\mathcal{L}$ [30, 31]. In particular, $\mathcal{L}u \in L^p(\Omega)$ implies that the strong solution to (5.1) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1, \infty)$. However, the situation becomes more complicated if we try to allow discontinuity at the principal coefficients of $\mathcal{L}$. In general, it is known (cf. [32]) that discontinuity of the coefficients a_{ij} implies that the L^p-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for (5.1) fail. A considerable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). Talenti [33] proved that the solely condition on measurability and boundedness of the a_{ij}'s guarantees isomorphic properties for $\mathcal{L}$ as a function from $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ to $L^2(\Omega)$. For the multidimensional case, i.e., when $n \geq 3$, except the uniform ellipticity, some additional properties on the coefficients a_{ij} are assumed in order to ensure that $\mathcal{L}$ possesses the regularizing property in Sobolev functional scales. In particular, if a_{ij} belong to $W^{1,n}(\Omega)$ (cf. [34]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}\}$ is sufficiently small (the Cordes condition), then $\mathcal{L}u \in L^2(\Omega)$ yields $u \in W^{2,2}(\Omega)$, and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2 - \varepsilon, 2 + \varepsilon)$ with sufficiently small ε.

Later, the Sarason class of functions with vanishing mean oscillation (denoted by VMO) was applied in the investigation of local and global Sobolev regularity of the strong solutions for (5.1).

Furthermore, let us define the space BMO of functions of bounded mean oscillation, and the smaller class of functions of vanishing mean oscillation denoted by VMO, where we consider coefficients a_{ij} and later that one where we consider the known term f.

In the sequel, we will assume that Ω is an open bounded set in $\mathbb{R}^n$.

Definition 5.1. *For $f \in L_{\text{loc}}^1(\Omega)$, define the integral mean $f_{x,R}$ by the formula*

$$f_{x,R} := |B(x,R)|^{-1} \int_{\widetilde{B}(x,R)} f(y)dy,$$

where $\widetilde{B}(x,R)$ is defined by (2.2).

If there is no need to specify the center, we just use the symbol B_R for $B(x,R)$.

We now recall the definition of the class of functions with bounded mean oscillation functions (denoted by BMO) that appeared for the first time in the publication by John and Nirenberg [35].

Definition 5.2. *For $f \in L_{\text{loc}}^1(\Omega)$, we say that f belongs to $BMO(\Omega)$ if $\|f\|_* < \infty$, where*

$$\|f\|_* := \sup_{B(x,R)} |B(x,R)|^{-1} \int_{\widetilde{B}(x,R)} |f(y) - f_{x,R}|dy.$$

Next, we consider the class of functions with Vanishing Mean Oscillation (VMO), introduced by Sarason [36].

Definition 5.3. *Let $f \in BMO(\Omega)$ and define*

$$\eta(f, R) := \sup_{\rho \leq R} |B_\rho|^{-1} \int_{\widetilde{B}_\rho} |f(y) - f_\rho|dy.$$

Furthermore, a function f belongs to the class $VMO(\Omega)$ if $\lim_{R \to 0} \eta(f, R) = 0$.

In fact, the class *VMO* is the subspace of *BMO* whose *BMO* norm over a ball vanishes when the radius of balls goes to zero. From this property it follows that a number of good features of functions from *VMO* are not shared by *BMO* functions; for example, functions from this class can be approximated by smooth functions. The *VMO* class was studied by various authors from different viewpoints. It is worth mentioning the work by Chiarenza et al. [37], in which the authors answer a question that arose thirty years before Miranda [34]. In the latter work the author considered linear elliptic *PDE*s, in which the coefficients a_{ij} with the higher order derivatives belong to the class $W^{1,n}(\Omega)$, and, moreover, he asked whether the gradient of the solution is bounded, if $p > n$. In the work [37] the authors supposed that $a_{ij} \in VMO$ and proved that Du is Hölder continuous.

Furthermore, it is possible to see that bounded uniformly functions belong to the class *VMO* as well as functions belonging to fractional Sobolev spaces $W^{\theta, \frac{n}{\theta}}$, $\theta \in (0, 1)$.

The investigation of Sobolev regularity of strong solutions of (5.1) was initiated in 1991 by the pioneering work of Chiarenza et al. [38]. In that work it was proved that, if $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $\mathcal{L}u \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$ for *each* value of $p \in (1, \infty)$. Moreover, well-posedness of the Dirichlet problem for (5.1) in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ was obtained. As a consequence, if the exponent p is sufficiently large, then it follows Hölder continuity for the strong solution or for its gradient.

By virtue of the fundamental accessibility of the works [37, 39], many other authors have used *VMO* class to obtain regularity results for *PDE*s and systems with discontinuous coefficients.

It can be checked that Hölder continuity can be inferred for small p if one has more information on $\mathcal{L}u$, such as, for example, its belonging to suitable Morrey class $L^{p,\lambda}(\Omega)$.

We denote by $L^{p,\lambda}(\Omega)$ the Morrey space defined on a domain $\Omega \subset \mathbb{R}^n$ which is determined by the following norm:

$$\|f\|_{p,\lambda} := \sup_{\substack{x \in \Omega \\ 0 < R < diam\,(\Omega)}} \left(\frac{1}{R^\lambda} \int_{\widetilde{B}(x,R)} |f(y)|^p \, dy \right)^{1/p},$$

where $\widetilde{B}(x, R)$ is defined by (2.2).

The exponent λ can take values outside $(0, n)$ but, as usual, the unique case of real interest is that one for which $\lambda \in (0, n)$. Indeed, from the definition we easily see that $L^{p,\lambda}(\Omega) = L^p(\Omega)$, if $\lambda \leq 0$. It is also clear that $L^{p,0}(\Omega) = L^p(\Omega)$.
Moreover, if $\lambda = n$, by using the Lebesgue differentiation theorem, we find that

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{\widetilde{B}(x,\rho)} |f(y)|^p \, dy = \lim_{\rho \to 0^+} \rho^{-n} \int_{B(x,\rho)} |f(y)|^p dy = C|f(x)|^p$$

for every Lebesgue point $x \in \Omega$. Then, $f(x) \in L^{p,n}(\Omega)$ if and only if f is bounded. This means that $L^{p,n}(\Omega) = L^\infty(\Omega)$. Furthermore, if $\lambda > n$, then $L^{p,\lambda}(\Omega) = \{0\}$.

In view of the spaces defined above, a natural problem arises when one studies the regularizing properties of the operator $\mathcal{L}$ in Morrey spaces for the case of *VMO* principal coefficients. In [40] it was proved that each $W^{2,p}$-viscosity solution to (5.1) lies in $C^{1+\alpha}(\Omega)$ if f belongs to $L^{n,n\alpha}(\Omega)$ with $\alpha \in (0, 1)$.

One of the main results of this note is to obtain local regularity in generalized grand Morrey spaces $M_w^{p),q,\varphi(\cdot)}(X)$, for highest order derivatives of solutions of elliptic *PDE*s in non-divergence form with coefficients, which might be discontinuous.

We recall the work by Agmon et al. [31] in which the appropriate results were obtained for the case of continuous coefficients of the above kind of equation. Later, discontinuous coefficients were considered also by Campanato [41].

In this paper we continue the study of the L^p regularity of solutions of second order elliptic *PDE*s to the maximum order derivatives of the solutions to a certain class of linear elliptic *PDE*s in nondivergence form having discontinuous coefficients [8].

We consider the second order differential operator

$$\mathcal{L} \equiv \sum_{i,j=1}^n a_{ij}D_{ij}, \qquad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

Here, we have adopted the usual summation convention on repeated indices.

We will also need the following regularity and ellipticity assumptions for the coefficients of $\mathcal{L}$, $\forall i, j = 1 \dots n$:

$$\begin{cases} a_{ij} \in L^\infty(\Omega) \cap VMO, \\ a_{ij}(x) = a_{ji}(x), \quad \text{for a.e. } x \in \Omega, \\ \exists \kappa > 0 : \frac{1}{\kappa}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega. \end{cases} \tag{5.2}$$

Set η_{ij} for the *VMO*-modulus of the function a_{ij} and suppose that $\eta = \left(\sum_{i,j=1}^n \eta_{ij}^2\right)^{1/2}$. In this case the normalized fundamental solution is given by the formula

$$\Gamma(x,\xi) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \left(\sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j\right)^{(2-n)/2}, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \text{ and a.e. } x,$$

where $A_{ij}(x)$ are the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the volume of the unit ball in $\mathbb{R}^n$. We set

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \quad \Gamma_{ij}(x,\xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi),$$

$$M = \max_{i,j=1,\dots,n} \max_{|\alpha|\leq 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(\cdot,\xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega\times\Sigma)}.$$

It is well known that $\Gamma_{ij}(x,\xi)$ are Calderón–Zygmund kernels with respect to the variable ξ.

Recall that since the condition $\mathcal{A}$ for Ω is satisfied, Ω with the Euclidean distance and the Lebesgue measure induced on Ω is a special case of SHT.

Theorem 5.1. *Suppose that (5.2) holds, $1 < p \leq q < \infty$, $\varphi(\cdot) \in \Phi_p$. Let Ω be a domain satisfying $\mathcal{A}$ condition (see (2.1)) and let w be a weight on Ω such that $w \in A_p(\Omega)$. Then, for every ball $B_\rho \subset\subset \Omega$, and every $u \in W_0^{2,p}(B_\rho)$ with $\mathcal{L}u \in M_w^{p),q,\varphi(\cdot)}(B_\rho)$, we have $D_{ij}u \in M_w^{p),q,\varphi(\cdot)}(B_\rho)$, and moreover, there exist positive constants $c = c(n, \kappa, p, q, \varphi(\cdot), M, w)$ such that the estimate*

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \leq c\|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)}, \quad \forall i, j = 1, \dots, n \tag{5.3}$$

holds.

Proof. Initially observe that the representation for the second order derivatives of functions in $W_0^{2,p}(B)$, where B is an open ball in $\mathbb{R}^n$, is given by the formula: [38]:

$$D_{ij}u(x) = \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \sum_{h,k=1}^n (a_{hk}(x) - a_{hk}(y)) D_{hk}u(y) dy \tag{5.4}$$

$$+ \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \mathcal{L}u(y) dy + \mathcal{L}u(x) \int_{|\xi|=1} \Gamma_i(x, \xi) \xi_j d\sigma_\xi.$$

Let us remark that

i) The first and the second integrals appearing in (5.4) are Principal Value ones (In fact, they are commutators of the Calderón–Zygmund singular integrals. The reader is referred, e.g., to [42], Ch. 7, [25], and references therein for appropriate weighted inequalities), and we can use Theorem 4.1 together with Remark 4.1 and Condition (2.1) to obtain the appropriate weighted inequality in $M_w^{p),q,\varphi(\cdot)}(\Omega)$, where w is the Muckenhoupt weight.

ii) $\int_{|\xi|=1} \Gamma_i(\cdot, \xi) \xi_j d\sigma_\xi \in L^\infty(B_\rho)$ with a bound independent of ρ.

Now, taking the $M_w^{p),q,\varphi(\cdot)}(B_\rho)$ norms of both sides in (5.4), applying Theorem 4.1 and taking into account Remark 4.1 and Condition (2.1), we get

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \leq c \left(\eta(\rho) \|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} + \|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \right).$$

This way, in view of the *VMO* assumption on the coefficients $a_{ij}(x)$, it is possible to choose ρ_0 so small that $c\eta(\rho_0) \leq 1/2$ and then

$$\|D_{ij}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \leq c \|\mathcal{L}u\|_{M_w^{p),q,\varphi(\cdot)}(B_\rho)} \qquad \text{for each } \rho < \rho_0.$$

$\square$

6. Conclusions

The authors obtained regularity results for solutions of second order *PDE*s having discontinuous coefficients in the framework of generalized grand Morrey spaces under the Muckenhoupt condition on weights. In the future it will be possible to extend the obtained properties to other kinds of equations, making use of density properties and extrapolation in generalized weighted grand Morrey spaces, that are proved in the present paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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