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Dipartimento di Fisica e Astronomia "Ettore Majorana"

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Arcangelo Pernace

UV-SENSITIVITY IN QUANTUM GRAVITY AND PATH INTEGRAL MEASURE

TESI DI DOTTORATO

Relatore:
Chiar.mo Prof. Vincenzo Branchina

XXXVIII CICLO

Alla mia famiglia

Abstract

Usual calculations of the (Euclidean) effective action in quantum gravity, typically performed within the heat-kernel formalism, give rise to quartically and quadratically UV-sensitive contributions (Planck scale) to the vacuum energy $\rho_{\rm vac} = \Lambda_{\rm cc}/8\pi G$, where $\Lambda_{\rm cc}$ and G are the cosmological and the Newton constant, respectively. The comparison with the observed value of $\rho_{\rm vac}$ unveils a severe naturalness problem, the strongest facet to the long-standing cosmological constant problem. Several attempts have been put forward to dispose of the aforementioned power-like UV-sensitive contributions. Among them, models with compact extra dimensions received particular attention. Usual calculations of the vacuum energy in these models lead to the automatically finite (no fine-tuning) result $\rho_{\rm vac} \sim m_{\rm KK}^4$, with $m_{\rm KK}^{-1}$ the size of the compact extra dimension. In the present thesis, we show that such a result comes from a mistreatment of the asymptotics of the five-dimensional loop momentum, and as such it is incorrect. When a proper calculation is performed, previously missed UV-sensitive terms arise that do not cancel even in the presence of supersymmetry. We then discuss the relevance of these results, also in connection with the recent dark dimension (DD) proposal. Formulated in the "swampland" framework, the latter suggests that we might live in a universe with a compact extra dimension of micrometer size dictated by the observed value of the vacuum energy. The DD scenario is based on swampland conjectures in quantum gravity and phenomenological bounds, that lead to the relation $\rho_{\rm swamp} \sim m_{\rm KK}^4$, and on the corresponding result $\rho_{\rm EFT}$ from the effective field theory limit. The results of the present thesis show that the matching between $\rho_{\rm swamp}$ and $\rho_{\rm EFT}$ is a non-trivial issue, and further studies are needed to understand whether the DD scenario might really be a physical reality. After this opening analysis, we continue the investigation on the UV-sensitivity of the vacuum energy considering the calculation of the one-loop effective action in four-dimensional quantum gravity. We show that the appearance in usual calculations of the aforementioned quartically and quadratically UV-sensitive terms is due to an improper treatment of the path integral measure and of the UV physical cutoff Λ of the theory. When the diffeomorphism invariant measure proposed by Fradkin and Vilkovisky is used, and Λ is properly introduced, the radiative correction to $\rho_{\rm vac}$ turns out to depend only logarithmically on Λ. We then extend this analysis considering the renormalization group (RG) flow of the gravitational action. Taking for the latter the Einstein-Hilbert truncation, we derive the RG equations for the running cosmological and Newton constant. We show that, if the Fradkin-Vilkovisky (FV) measure is used and the running scale k properly introduced, the beta functions are profoundly different from those of previous literature. In particular, they do not possess the non-trivial UV-attractive fixed point of the so-called asymptotic safety scenario. We then consider the theory of an interacting scalar field on a non-trivial gravitational background. We find that the issues concerning the path integral measure and the UV physical cutoff have a further important implication on another long-standing problem in quantum field theory, the Higgs naturalness problem. We show that the use of the FV measure, together with a careful introduction of the UV cutoff Λ, ensure for the radiative correction δm^2 to the mass m^2 of the scalar field a result free of the well-known quadratically divergent contributions. Similarly to what we find for the vacuum energy, δm^2 depends only logarithmically on Λ. It is important to stress that the above results for the vacuum energy and for the mass of scalar particles have been obtained without resorting to a supersymmetric embedding of the theory, nor to regularization schemes where power-like divergences are absent by construction. Finally, in light of the important role that the FV measure plays in the derivation of the results of the present thesis, we thoroughly investigate on its transformation properties under diffeomorphisms. This measure is sometimes claimed not to be diffeomorphism invariant due to the presence in it of non-covariant g^{00} factors of the time-time component of the inverse metric. We show that such a claim is incorrect, and that, on the contrary, these g^{00} factors turn out to be crucial to ensure the diffeomorphism invariance of the path integral measure, and ultimately of the effective action.

Publications

Below is a list of my papers on which the original part of the present thesis (from chapter 2 to chapter 6) is based:

[1] C. Branchina, V. Branchina, F. Contino and A. Pernace, *Does the cosmological constant really indicate the existence of a dark dimension?*, Int. J. Geom. Meth. Mod. Phys. **22** (2025) no.04, 2450305, arXiv:2308.16548.

[2] C. Branchina, V. Branchina, F. Contino and A. Pernace, *Dark dimension and the effective field theory limit*, Int. J. Geom. Meth. Mod. Phys. **22** (2025) no.04, 2450303, arXiv:2404.10068.

[3] C. Branchina, V. Branchina, F. Contino and A. Pernace, *Path integral measure and the cosmological constant*, Phys. Rev. D **111** (2025) no.10, 105018, arXiv:2412.10194.

[4] C. Branchina, V. Branchina, F. Contino and A. Pernace, *Path integral measure and RG equations for gravity*, Phys. Rev. D **111** (2025) no.12, 125021, arXiv:2412.14108.

[5] C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *On the RG flow of the Newton and cosmological constant*, arXiv:2505.07628.

[6] C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *Diffeomorphism invariance of the effective gravitational action*, Phys. Rev. D **112** (2025) no.4, 045002, arXiv:2506.05100.

[7] C. Branchina, V. Branchina, F. Contino, R. Gandolfo and A. Pernace, *Gravity and the Higgs boson mass*, arXiv:2507.13832, submitted for publication to Phys. Rev. D.

Contents

Introduction

It has been known for a long time that our universe is going through a phase of accelerated expansion [1], a feature that might be explained with the introduction of a "tiny", positive, vacuum energy $\rho_{\text{vac}} \sim 2.7 \cdot 10^{-47}\,\text{GeV}^4$ [2] in Einstein's equations. Still, we have no explanation of why ρ_{vac} has such a small value. In fact, typical perturbative calculations of the vacuum energy in quantum field theory (QFT) give rise to contributions that are quartically and quadratically sensitive ("divergences") to the ultraviolet (UV) physical cutoff Λ of the theory (i.e. to the ultimate scale up to which the theory is valid). If Λ is identified with the Planck scale M_P ($\sim 1.22 \cdot 10^{19}\,\text{GeV}$), the quartically divergent contribution to ρ_{vac} is $\sim M_P^4$, and the comparison with the observed value reported above shows a discrepancy of approximately 123 orders of magnitude, that calls for an enormous and quite unnatural fine-tuning. This is the long-standing cosmological constant problem [3–5].

The most studied and well-known way to realize a physical cancellation of power-like UV-sensitive terms consists in performing a supersymmetric embedding of the theory. The latest LHC runs up to TeV scales, however, have not shown any evidence for supersymmetry (SUSY). If present, SUSY must be broken at higher energies. Therefore, though the injection of supersymmetry successfully cancels the M_P^4 and M_P^2 terms, contributions to ρ_{vac} proportional to the fourth power of particles masses (and/or vevs) are left. If the SUSY breaking scale M_{SUSY} is not hierarchically higher than the Fermi scale, the problem is sensibly alleviated, although still present at the level of about 50 orders of magnitude. A complete solution to the cosmological constant problem necessarily requires a mechanism that not only disposes of the large, power-like UV-sensitive terms, but also suppresses the $(\text{mass})^4$ contributions, driving the vacuum energy ρ_{vac} down to its measured value.

There have been several other attempts to obtain for the vacuum energy a result free of power-like "divergences". Among them, models with compact extra dimensions received particular attention. If one considers for instance a 5D supersymmetric QFT with one compact extra dimension in the shape of a circle of radius m_{KK}^{-1} (m_{KK} is the scale of a Kaluza-Klein (KK) tower) and Scherk-Schwarz SUSY breaking [6, 7], usual calculations of the vacuum energy ρ_{vac} in this model give the automatically finite (no fine-tuning) result $\rho_{\text{vac}} \sim m_{\text{KK}}^4$ [8, 9]. A recent surge of interest towards these models has followed the so-called "dark dimension" (DD) scenario [10]. This scenario suggests that we might live in a universe with a compact extra dimension of mesoscopic size dictated by the measured value of the vacuum energy. It is based on swampland conjectures[1] in string theory/quantum

[1] For the reader not familiar with the so-called "swampland program", the "swampland" is the set of low-energy effective theories which cannot be consistently embedded in a higher energy theory of quantum gravity (for instance string theory). In the opposite case, a low-energy effective theory belongs to the

gravity and phenomenological bounds, that lead to the relation $\rho_{\text{swamp}} \sim m_{\text{KK}}^4$ between the vacuum energy ρ_{swamp} and the size m_{KK}^{-1} of the compact dimension, and also on the corresponding result ρ_{EFT} for the vacuum energy in the effective field theory (EFT) limit.

In chapter 2, we retrace carefully the steps that lead to the finite result $\rho_{\text{EFT}} \sim m_{\text{KK}}^4$ mentioned above. We show that the latter arises from an improper calculation based on a mistreatment of the asymptotics of the five-dimensional loop momentum in the higher-dimensional field theory underlying the DD scenario. We then perform a proper EFT calculation, from which previously missed UV-sensitive terms are found in the radiative correction to ρ_{EFT}. These terms do not cancel even in a SUSY theory since they are proportional to powers of the boundary charges of the superpartners along the compact dimension, that have to be different to trigger the SUSY breaking. These results show that the matching between the swampland result ρ_{swamp} and the corresponding EFT one ρ_{EFT}, at the basis of the DD scenario, is a non-trivial issue, and raise doubts on the possibility that the DD scenario might be a physical reality. We also discuss the global renormalization group (RG) picture (for a general introduction to the Wilsonian RG see section 1.4) in which this scenario should be framed, along the lines of the approach developed in [11]. The results of chapter 2 have been published in [12, 13].

The rest of the present thesis is mainly devoted to the study of the UV-sensitivity of the vacuum energy in four-dimensional quantum gravity. In this framework, the spacetime metric $g_{\mu\nu}$ is a quantum field as any other matter field. If S is the classical action, the quantum theory is defined through the path integral (see section 1.1 for an introduction to the path integral formalism)

$$\int \mathrm{d}\mu \, e^{-S} \,, \tag{0.1}$$

where $\mathrm{d}\mu$ is the diffeomorphism invariant measure obtained from the integration over the conjugate momenta in the original Hamiltonian formulation of the theory.

In chapter 3, the calculation of the one-loop (Euclidean) effective action $\Gamma^{1l}_{\text{grav}}$ is re-examined. The results of this analysis have been published in [14]. We show that previous literature calculations, usually performed within the heat-kernel formalism [15–17], overlook two important points that prove to be fundamental in the derivation of $\Gamma^{1l}_{\text{grav}}$. One concerns the use of the correct, diffeomorphism invariant measure $\mathrm{d}\mu$, that is the one derived by Fradkin and Vilkovisky in [18, 19], the other concerns the way in which the UV physical cutoff Λ is introduced in gravitational theories. Paying due attention to both of these points, we find that the radiative correction $\delta\rho_{\text{vac}}$ to the vacuum energy ρ_{vac} (read from $\Gamma^{1l}_{\text{grav}}$) is only logarithmically sensitive to the UV scale[2] Λ. It is important to stress that such a result is obtained without resorting to any "physical" cancellation, as it would be the case for a supersymmetric embedding of the theory. Moreover, no regularization schemes are used where power-like divergences are absent by construction, as it is for instance the case for dimensional or zeta-function regularization [22].

The analysis of [14] has been extended in [23, 24], which are the subject of chapter 4. Taking for the running (pure) gravitational action S_k (k is the running scale) the Einstein-

"landscape". Swampland conjectures are criteria, formulated on the basis of black hole physics, universal patterns in string theory, that have to be fulfilled by a low-energy effective theory for it to belong to the landscape.

[2] Concerning quartic divergences, their absence for a massless scalar theory on a gravitational background was first pointed out in [20]. For pure gravity, the absence of these quartic divergences was already noted in [21].

Hilbert truncation, we derive RG equations for the Newton and cosmological constant, G_k and Λ_k, respectively.

In this respect, we recall that Einstein's theory is perturbatively non-renormalizable. This essentially suggests two possible scenarios [25]: either it is an EFT valid up to a certain scale (say the Planck scale M_P), above which it is replaced by a UV completion of different nature (possibly string theory), or it is non-perturbatively renormalizable through the existence of a UV-attractive fixed point of its RG flow with finite dimensional critical surface, a possibility that S. Weinberg dubbed asymptotic safety (AS) scenario [26].

The question of whether this latter scenario could be realized in quantum gravity was first examined by Weinberg himself. Resorting to dimensional regularization, he considered the theory in $d = 2 + \epsilon$ dimensions [26, 27] and noted that for small values of the Newton constant G the beta function is $\beta(G, \epsilon) = \epsilon G - b G^2 + \mathcal{O}(G^3)$ and that for $b > 0$ (and sufficiently small values of ϵ) a UV-attractive $\mathcal{O}(\epsilon)$ fixed point exists. Inspired by the work of Wilson and Fisher [28] (who calculated the critical exponents for the scalar theory in $d = 3$ dimensions by working in $4 - \epsilon$ dimensions and expanding the results around $\epsilon = 1$), he wondered on the possibility of getting results in $d = 4$ dimensions expanding those obtained in $d = 2+\epsilon$ around $\epsilon = 2$ [26]. However, Weinberg showed that this program cannot be pursued [27] since in $d = 4$ dimensions polar singularities appear that cannot be canceled by counterterms contained in the Lagrangian of the original $(2+\epsilon)$-dimensional theory. He then observed that the possibility of an asymptotic safety scenario should be investigated directly in $d = 4$ dimensions [27].

Investigations on the possibility of realizing AS directly in $d = 4$ dimensions were started in the late nineties [29–31] (see [32] for a popular introduction to the asymptotic safety scenario) where the RG approach to quantum gravity was implemented resorting to the effective average action formalism introduced in [33]. Considering the Einstein-Hilbert truncation, a non-trivial UV-attractive fixed point $g_{\text{FP}} > 0$, $\lambda_{\text{FP}} > 0$ was found in addition to the Gaussian one [30]. Here $g \equiv k^2 G_k$ and $\lambda \equiv \Lambda_k/k^2$ are the dimensionless running Newton and cosmological constant, corresponding to the dimensionful parameters G_k and Λ_k. Successively, the existence of this non-trivial fixed point was confirmed resorting to the proper-time formalism [34].

In chapter 4, we show that, when the Fradkin - Vilkovisky (FV) diffeomorphism invariant measure is used in the path integral that defines the RG flow of S_k, and the running scale k is introduced according to [14], the beta functions for G_k and Λ_k turn out to be profoundly different from those of previous literature. In particular, the beta functions for the corresponding dimensionless parameters g and λ do not show any sign of the non-trivial UV-attractive fixed point of the AS scenario; they possess only the Gaussian fixed point. We also explain why and how the AS fixed point is *artificially* generated in usual implementations of the RG flow in quantum gravity.

Chapter 5 is devoted to the analysis performed in [35]. Building on the results of [14, 23, 24], in this chapter we show that the delicate points mentioned above concerning the path integral measure and the UV physical cutoff have a crucial impact also on another long-standing issue of theoretical physics, the Higgs naturalness problem. It is a well-known fact that, when the calculation of the Higgs propagator and/or effective action is performed within the standard model (SM) both in flat and curved spacetime, the bare Higgs boson mass $m^2(\Lambda)$ receives contributions δm^2 proportional to Λ^2 due to unsuppressed quantum fluctuations (no symmetry protection). Here Λ is the ultimate scale up to which the SM provides an accurate description of particle physics. If Λ is hierarchically larger than the Fermi scale, the contribution $\delta m^2 \sim \Lambda^2$ requires the bare mass $m^2(\Lambda)$ to be *unnaturally* large ($\mathcal{O}(\Lambda^2)$) with respect to the experimental value $\sim (125\,\mathrm{GeV})^2$, and also very finely tuned to eventually have $m_{\mathrm{H}}^2 = m^2(\Lambda) + \delta m^2 \sim (125\,\mathrm{GeV})^2$.

In chapter 5, we show that, contrary to what is found with usual heat-kernel calculations in curved spacetime, when the one-loop Higgs effective action[3] Γ^{1l} is calculated using the diffeomorphism invariant FV measure and introducing the UV physical cutoff as in [14], no quadratic "divergences" are present in the radiative correction to the Higgs mass. It is important to stress that, as for the vacuum energy, this result is found without resorting to a supersymmetric embedding of the theory, nor to regularization schemes where power-like divergences are absent by construction.

The results of chapters 3, 4 and 5 were obtained using the path integral measure derived by Fradkin and Vilkovisky in [18]. This measure comes from the original Hamiltonian formulation of the theory after integration over the conjugate momenta of the fields, and contains non-covariant factors of the time-time component g^{00} of the inverse metric tensor. Considering the case of pure quantum gravity, Fradkin and Vilkoviksy proved that their path integral measure is diffeomorphism invariant despite the presence of these non-covariant g^{00} factors. In spite of that, in the literature some authors have made claims to the contrary, see for instance the recent [36].

To bring clarity on this important question, part of my PhD activity has been dedicated to a thorough investigation on the transformation properties of the path integral measure under diffeomorphisms. The results have been published in [37], and will be presented in chapter 6, the last chapter of the present thesis. Considering the contribution to the effective gravitational action due to a free scalar field in curved spacetime, a detailed calculation is performed along the lines of [18], which shows that the invariance of the FV path integral measure arises from a delicate balance between the different elements involved in the definition of the path integral. Among them, the necessity of introducing a time ordering parameter and a discretization (lattice) of spacetime. Under a general coordinate transformation, the time ordering parameter and the lattice both transform, and this induces the appearance of non-trivial terms from the path integral. The g^{00} factors ensure the cancellation of these non-trivial terms, and ultimately guarantee the diffeomorphism invariance of the FV measure.

[3] Actually, for the purposes of this analysis it is not necessary to take into account the full Higgs sector of the SM. It is rather sufficient to consider the simpler case of a single component scalar field on a non-trivial gravitational background, that is what is done in chapter 5.

Chapter 1

Some topics in QFT

This chapter is devoted to the introduction of some topics in quantum field theory that constitute the theoretical framework in which the results of my PhD activity [12–14, 23, 24, 35, 37] were derived. Section 1.1 is devoted to an introduction of the path integral formalism in QFT and to the definition of the generating functionals of Green functions. In section 1.1.1, in particular, we highlight all the elements involved in the definition of the path integral, among which the necessity to introduce a time ordering parameter and a discretization of spacetime (lattice). The content of this section will be useful to the analysis performed in chapter 6 (based on [37]), where we study how the path integral that defines the effective gravitational action transforms under diffeomorphisms. In section 1.2, the heat-kernel formalism for the evaluation of functional traces is presented. Section 1.3 concerns a brief introduction to gauge theories and to the Vilkovisky - DeWitt effective action. In section 1.4, the Wilsonian renormalization group approach is presented to pave the way to the analysis of chapter 4, where we derive the RG equations for the Newton and the cosmological constant of [23]. Finally, section 1.5 is devoted to the presentation of some results in field theories with compact extra dimensions, which are at the basis of the analysis performed in [12, 13] and presented in chapter 2.

1.1 Path integral formalism

There are two approaches to the problem of field quantization. One is the canonical method, where field operators are introduced and canonical commutation rules are postulated. The other is the path integral formalism. This latter method does not deal with operators. It is rather based on an integral over all possible classical field configurations, each contributing to the transition amplitude weighted by the exponential of the corresponding value of the classical action. From the path integral, all the properties of a system can be deduced by means of functional techniques. It leads to the same results as those obtained within the canonical method. The two methods are, in fact, equivalent, and it is just a matter of convenience to use one or the other. As said above, the results of the present thesis have been obtained within the path integral formalism, that will be now illustrated for the case of a scalar field theory. Clearly, all the techniques that will be presented below can be suitably adapted to the case of (nonabelian) gauge theories (see section 1.3), and in particular to the gravitational theories considered in the next chapters.

1.1.1 Path integral construction and generating functionals

Let $\phi(x) \equiv \phi(\vec{x}, t)$ be a real scalar field whose dynamics is described by a Lagrange function ($\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$)

$$L(t) = \int \mathrm{d}^3\vec{x}\ \mathcal{L}(\phi(x), \partial_\mu\phi(x))\,, \tag{1.1}$$

with $\mathcal{L}(\phi(x), \partial_\mu\phi(x))$ the Lagrangian density. To move from the Lagrangian to the Hamiltonian formalism, we introduce the momenta $\pi(x)$ conjugate to $\phi(x)$ through

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\,. \tag{1.2}$$

The Hamiltonian H corresponding to the Lagrangian L is given by the Legendre transform

$$H = \int \mathrm{d}^3\vec{x}\ \mathcal{H}(\pi, \phi) \equiv \int \mathrm{d}^3\vec{x}\left[\pi\,\partial_0\phi - \mathcal{L}(\phi(x), \partial_\mu\phi(x))\right]_{\partial_0\phi\to\pi}\,. \tag{1.3}$$

In the canonical quantization, $\phi(x)$ and $\pi(x)$ are promoted to operators $\hat{\phi}(x)$ and $\hat{\pi}(x)$ satisfying the equal time commutation relations ($\hbar = 1$)

$$\left[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\right] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad ; \quad \left[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{y}, t)\right] = \left[\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{y}, t)\right] = 0\,. \tag{1.4}$$

The field operator $\hat{\phi}(\vec{x}, t)$ obeys the equation (Heisenberg picture; as usual $\dot{\hat{\phi}}(\vec{x}, t) \equiv \partial_0\,\hat{\phi}(\vec{x}, t)$ and $\hat{H} \equiv \int \mathrm{d}^3\vec{x}\ \mathcal{H}(\hat{\pi}, \hat{\phi})$)

$$-i\dot{\hat{\phi}}(\vec{x}, t) = \left[\hat{H}, \hat{\phi}(\vec{x}, t)\right]\,, \tag{1.5}$$

which is formally solved by $\hat{\phi}(\vec{x}, t) = e^{i\hat{H}t}\,\hat{\phi}(\vec{x}, 0)\,e^{-i\hat{H}t}$, and has a set of eigenstates $|\phi, t\rangle$ that satisfy (note that these states have a functional dependence on $\phi(\vec{x}, t)$)

$$\hat{\phi}(\vec{x}, t)\,|\phi, t\rangle = \phi(\vec{x}, t)\,|\phi, t\rangle \qquad ; \qquad |\phi, t\rangle = e^{i\hat{H}t}\,|\phi, 0\rangle \equiv e^{i\hat{H}t}\,|\phi\rangle\ . \tag{1.6}$$

The starting point of the path integral formalism is the transition amplitude $\langle\phi'', t''|\phi', t'\rangle$ between two state vectors at different times. It is given by (see (1.6))

$$\langle\phi'', t''|\phi', t'\rangle = \langle\phi''|\,e^{-i(t''-t')\hat{H}}\,|\phi'\rangle\ , \tag{1.7}$$

and it is nothing but the probability amplitude to make a transition from the field configuration ϕ' at time t' to the field configuration ϕ'' at time t'' ($\phi(\vec{x}, t') = \phi'(\vec{x})$ and $\phi(\vec{x}, t'') = \phi''(\vec{x})$). The path integral formalism allows to write this amplitude in terms of the classical Hamiltonian only, without any explicit reference to the operators $\hat{\phi}$ and $\hat{\pi}$ or to Hilbert space vectors. This is done as follows. We begin by introducing a discretization of spacetime. After dividing the time interval $[t', t'']$ into N segments of size ϵ ($t_n = t' + \epsilon\, n$, with $t'' - t' \equiv \epsilon N$), each space-like hypersurface $t = t_n = \text{const}$ (volume V) is sliced into M cells of volume ΔV ($M\Delta V \equiv V$), each of them centered at a point of coordinates $\vec{x}_l$, $l = 1, \ldots, M$ [38–40]. In this way, from the continuous functions $\phi(\vec{x}, t)$ and $\pi(\vec{x}, t)$ we get the finite-dimensional vectors $\phi_l(t) \equiv \phi(\vec{x}_l, t)$ and $\pi_l(t) \equiv \pi(\vec{x}_l, t)$. For each of the cells, a basis of eigenstates of the Heisenberg field operator $\hat{\phi}_{ln} \equiv \hat{\phi}(\vec{x}_l, t_n)$ can be constructed that satisfies the completeness relation

$$\int \mathrm{d}\phi_{ln} \, |\phi_{ln}, t_n\rangle \, \langle\phi_{ln}, t_n| = 1 \,. \tag{1.8}$$

The transition amplitude $\langle\phi'', t''|\phi', t'\rangle$ is then approximated by $(\{\phi_l(t)\} \equiv \phi_1(t), \ldots, \phi_M(t))$

$$\begin{aligned}
\langle\phi'', t''|\phi', t'\rangle \;&\simeq\; \langle\{\phi_l''\}, t''|\{\phi_l'\}, t'\rangle \\
&= \prod_{l=1}^{M} \int \mathrm{d}\phi_{l\,N-1} \cdots \int \mathrm{d}\phi_{l1} \, \langle\phi_l'', t''|\phi_{lN-1}, t_{N-1}\rangle \ldots \langle\phi_{l2}, t_2|\phi_{l1}, t_1\rangle \, \langle\phi_{l1}, t_1|\phi_l', t'\rangle \;.
\end{aligned} \tag{1.9}$$

If $N \gg 1$, the amplitudes $\langle\phi_{ln+1}, t_{n+1}|\phi_{ln}, t_n\rangle = \langle\phi_{ln+1}| \, e^{-i\epsilon\hat{H}} \, |\phi_{ln}\rangle = \langle\phi_{ln+1}| \, 1 - i\epsilon\hat{H} \, |\phi_{ln}\rangle + \mathcal{O}(\epsilon^2)$ in (1.9) are infinitesimal, and they can be evaluated considering the complete set of eigenstates of the (Schrödinger) conjugate field operator $\hat{\pi}_l \equiv \hat{\pi}(\vec{x}_l, 0)$

$$\hat{\pi}_l \, |\pi_{ln}\rangle = \pi_{ln} \, |\pi_{ln}\rangle \quad ; \quad \int \frac{\Delta V \, \mathrm{d}\pi_{ln}}{2\pi} \, |\pi_{ln}\rangle \, \langle\pi_{ln}| = 1 \quad ; \quad \langle\phi_{ln}|\pi_{ln}\rangle = e^{i\Delta V \pi_{ln}\phi_{ln}} \,. \tag{1.10}$$

We have $\left(\hat{H} = \sum_{l=1}^{M} \Delta V \, \mathcal{H}(\hat{\pi}_l, \hat{\phi}_l) \,, \; \mathcal{H}_{ln} \equiv \mathcal{H}(\pi_{ln}, \phi_{ln}) \right)$

$$\langle\phi_{ln+1}, t_{n+1}|\phi_{ln}, t_n\rangle = \int \frac{\Delta V \, \mathrm{d}\pi_{ln}}{2\pi} \, e^{i\Delta V \pi_{ln}(\phi_{ln+1} - \phi_{ln})} \, (1 - i\epsilon\Delta V \mathcal{H}_{ln}) + \mathcal{O}(\epsilon^2) \,. \tag{1.11}$$

Inserting (1.11) in (1.9) we get

$$\langle\phi'', t''|\phi', t'\rangle \;\simeq\; \prod_{l=1}^{M} \left(\prod_{n=1}^{N-1} \int \mathrm{d}\phi_{ln} \prod_{n=0}^{N-1} \int \frac{\Delta V \, \mathrm{d}\pi_{ln}}{2\pi} \right) e^{i \sum_{n=0}^{N-1} \epsilon \sum_{l=1}^{M} \Delta V \left(\pi_{ln} \frac{\phi_{ln+1} - \phi_{ln}}{\epsilon} \, - \mathcal{H}_{ln} \right)} \,, \tag{1.12}$$

and taking the continuum limit[1]

$$\begin{aligned}
\langle\phi'', t''|\phi', t'\rangle &= \lim_{V\to\infty} \lim_{M\to\infty} \lim_{N\to\infty} \prod_{l=1}^{M} \left(\prod_{n=1}^{N-1} \int \mathrm{d}\phi_{ln} \prod_{n=0}^{N-1} \int \frac{\Delta V \, \mathrm{d}\pi_{ln}}{2\pi} \right) e^{i \sum_{n=0}^{N-1} \epsilon \sum_{l=1}^{M} \Delta V \left(\pi_{ln} \frac{\phi_{ln+1} - \phi_{ln}}{\epsilon} \, - \mathcal{H}_{ln} \right)} \\
&\equiv \int \mathcal{D}\phi \int \mathcal{D}\pi \; \exp\left(i \int_{t'}^{t''} \mathrm{d}t \int \mathrm{d}^3\vec{x} \; \left[\, \pi \, \partial_0 \phi \; - \; \mathcal{H}(\pi, \phi) \, \right] \right) ,
\end{aligned} \tag{1.13}$$

that is the probability amplitude $\langle\phi'', t''|\phi', t'\rangle$ expressed in terms of the phase space path integral in the second line of the above equation.

If the Lagrangian is quadratic in the field derivatives, then the Hamiltonian will be quadratic in the conjugate momenta π and the corresponding functional integration can be performed exactly (Gaussian integration). This allows to go from the phase space path integral in (1.13), to the configuration space one (Feynman path integral)

$$\langle\phi'', t''|\phi', t'\rangle = \int \mathcal{D}\phi \, \mu[\phi] \; \exp\left(i \int_{t'}^{t''} \mathrm{d}t \int \mathrm{d}^3\vec{x} \; \mathcal{L}(\phi, \partial_\mu \phi) \right) , \tag{1.14}$$

[1]The continuum limit is performed in three steps: (i) limit $N \to \infty$ with fixed $\epsilon N = t'' - t'$; (ii) limit $M \to \infty$ with fixed $M\Delta V = V$; (iii) limit $V \to \infty$.

where $\mu[\phi]$ is the result of the Gaussian integration over the π. In general, $\mu[\phi]$ can depend on the field ϕ and cannot be neglected. It can be equivalently regarded as a non-trivial contribution to the path integral measure in configuration space, or (after exponentiation) as a shift of the classical Lagrangian.

As we will see in the next chapters, this is the case for quantum gravity, where non-trivial terms in the configuration space measure arise from the Gaussian integration over the conjugate momenta. One of the main results obtained during my PhD activity is that these terms have a crucial impact in the calculation of the one-loop effective action [14,35] and also on the renormalization group flow of the running action [23]. Moreover, we will see that they ensure the diffeomorphism invariance of the effective action [18, 19, 37, 41]. These results will be presented in chapters 3, 4, 5 and 6.

The path integral representation (1.13) of the amplitude $\langle\phi'', t''|\phi', t'\rangle$ can be used to evaluate transition matrix elements and expectation values. In particular, it is possible to evaluate matrix elements of the time ordered product $T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))$. Similar steps as those performed above for the amplitude $\langle\phi'', t''|\phi', t'\rangle$ lead to

$$\begin{aligned}&\langle\phi'', t''|\, T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))\,|\phi', t'\rangle\\ &= \int \mathcal{D}\phi \int \mathcal{D}\pi \;\phi(x_1)\cdots\phi(x_n)\, \exp\left(i\int_{t'}^{t''} \mathrm{d}t \int \mathrm{d}^3\vec{x}\,\left[\,\pi\,\partial_0\phi \;-\; \mathcal{H}(\pi,\phi)\,\right]\right). \end{aligned} \tag{1.15}$$

Actually, of major interest in QFT is the vacuum (ground state $|0\rangle$) expectation value (vev) of $T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))$, i.e. the so-called n-point Green functions $G^{(n)}(x_1,\ldots,x_n)$. Thanks to the Lehmann–Symanzik–Zimmermann reduction formula, in fact, transition (scatterings, decays, ...) amplitudes are written in terms of these fundamental objects [42]. It can be shown that the $G^{(n)}(x_1,\ldots,x_n)$ are obtained from the matrix element (1.15) through the limiting procedure[2] below (see for instance [38])

$$\begin{aligned}G^{(n)}(x_1,\ldots,x_n) \equiv \langle 0|\, T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))\,|0\rangle &= \lim_{\substack{t''\to+\infty\\ t'\to-\infty}} \frac{\langle\phi'', t''|\, T(\hat{\phi}(x_1)\cdots\hat{\phi}(x_n))\,|\phi', t'\rangle}{\langle\phi'', t''|\phi', t'\rangle}\\ &= \frac{\int\mathcal{D}\phi\int\mathcal{D}\pi\;\phi(x_1)\cdots\phi(x_n)\,\exp\left(i\int \mathrm{d}^4x\,\left[\,\pi\,\partial_0\phi \;-\; \mathcal{H}(\pi,\phi)\,\right]\right)}{\int\mathcal{D}\phi\int\mathcal{D}\pi\;\exp\left(i\int \mathrm{d}^4x\,\left[\,\pi\,\partial_0\phi \;-\; \mathcal{H}(\pi,\phi)\,\right]\right)}\,. \end{aligned} \tag{1.16}$$

Let us introduce now the so-called *generating functional* $Z[J]$ as the vacuum-vacuum amplitude $\langle 0|0\rangle_J$ in the presence of an external source $J(x)$ coupled to the field $\phi(x)$ through an additional term $J(x)\phi(x)$ in the Lagrangian density. It is possible to prove that [38]

$$\begin{aligned}Z[J] \equiv \langle 0|0\rangle_J &= \lim_{\substack{t''\to+\infty\\ t'\to-\infty}} \frac{\langle\phi'', t''|\phi', t'\rangle_J}{\langle\phi'', t''|\phi', t'\rangle}\\ &= \frac{\int\mathcal{D}\phi\int\mathcal{D}\pi\;\exp\left(i\int \mathrm{d}^4x\,\left[\,\pi\,\partial_0\phi \;-\; \mathcal{H}(\pi,\phi) + J(x)\phi(x)\,\right]\right)}{\int\mathcal{D}\phi\int\mathcal{D}\pi\;\exp\left(i\int \mathrm{d}^4x\,\left[\,\pi\,\partial_0\phi \;-\; \mathcal{H}(\pi,\phi)\,\right]\right)}\,. \end{aligned} \tag{1.17}$$

[2]The "large time" limits are intended to be performed on the imaginary axes after a Wick rotation to complex time $\tau = it$.

This quantity is of central importance in QFT since it can be used to calculate the n-point functions $G^{(n)}(x_1, \ldots, x_n)$. From (1.16) and (1.17), in fact, we have

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^n} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} . \tag{1.18}$$

Starting from $Z[J]$, we can also define

$$W[J] \equiv -i \log Z[J] . \tag{1.19}$$

The functional $W[J]$ is proved to be the generating functional for *connected* n-point Green functions $G_{\rm C}^{(n)}(x_1, \ldots, x_n)$ [38],

$$G_{\rm C}^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^{n-1}} \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} . \tag{1.20}$$

Finally, a generating functional for *one-particle irreducible* (1PI) n-point functions (or irreducible vertex functions) $\Gamma^{(n)}(x_1, \ldots, x_n)$ can be also introduced as follows. We define the *classical field* (known also as mean field or background field) $\phi_c(x)$ as (the last equality below is obtained using (1.17) and (1.19))

$$\phi_c(x) \equiv \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0 | \, \hat{\phi}(x) \, | 0 \rangle_J}{\langle 0 | 0 \rangle_J} , \tag{1.21}$$

and consider the Legendre transform of $W[J]$

$$\Gamma[\phi_c] = W[J_{\phi_c}] - \int \mathrm{d}^4 x \, J_{\phi_c}(x) \phi_c(x) , \tag{1.22}$$

where J_{ϕ_c} is implicitly defined as the solution to (1.21). The functional $\Gamma[\phi_c]$ is the so-called *effective action*, and it is proved to be the generating functional for the $\Gamma^{(n)}(x_1, \ldots, x_n)$,

$$\Gamma^{(n)}(x_1, \ldots, x_n) = i \left. \frac{\delta^n \Gamma[\phi_c]}{\delta \phi_c(x_1) \cdots \delta \phi_c(x_n)} \right|_{\phi_c=0} . \tag{1.23}$$

To understand the reason why this functional is called "effective action", it is useful to calculate $\Gamma[\phi_c]$ for the case of a free scalar theory, whose classical action is

$$S[\phi] = \frac{1}{2} \int \mathrm{d}^4 x \left[\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right] . \tag{1.24}$$

It is easy to find that in this case $\Gamma[\phi_c] = S[\phi_c]$ [38]. For interacting theories, instead, this fails to be true. Due to the piling up of quantum fluctuations, $\Gamma[\phi_c]$ will be in general a highly non-local functional of ϕ_c, different from the classical action, whose integrand depends on values of ϕ_c at many different spacetime points. In this respect, it is possible to introduce the so-called *gradient expansion*, i.e. an expansion in gradients of the classical field ϕ_c obtained expanding all the ϕ_c around a common spacetime point

$$\Gamma[\phi_c] = \int \mathrm{d}^4 x \left[- V_{\rm eff}(\phi_c) + \frac{1}{2} F(\phi_c) \partial_\mu \phi_c \, \partial^\mu \phi_c + Y(\phi_c) (\partial_\mu \phi_c \, \partial^\mu \phi_c)^2 + \ldots \right] , \tag{1.25}$$

where $V_{\text{eff}}(\phi_c)$, $F(\phi_c)$, $Y(\phi_c)$ are local functions of ϕ_c and the "$\ldots$" indicate higher derivative contributions. The term $V_{\text{eff}}(\phi_c)$ with no derivatives is called the effective potential. From the calculation of the effective action, and a comparison with the classical action S, we deduce how the latter is modified due to quantum fluctuations. This is explicitly shown in the next section, where we calculate the effective action resorting to a semiclassical expansion in powers of the reduced Planck constant $\hbar$.

Another important property of $\Gamma[\phi_c]$ can be deduced differentiating it with respect to ϕ_c. From (1.22) one has

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J_{\phi_c}(x)\,, \tag{1.26}$$

that, taking $J_{\phi_c} = 0$ and using (1.21), becomes (below the vacuum state $|0\rangle$ is assumed to be stable in the absence of the source J, i.e. we assume that $\langle 0|0\rangle = 1$)

$$\left.\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)}\right|_{\phi_c=\langle\phi\rangle} = 0\,. \tag{1.27}$$

Therefore, we see that the effective action plays for the quantum theory a role analogous to the one that the action functional $S[\phi]$ plays for the classical theory: the vev of the quantum field $\hat{\phi}(x)$ is the solution to the equations obtained from the first variation of $\Gamma[\phi_c]$ with respect to ϕ_c, in the same way as the field $\phi(x)$ is the solution to the equations derived from the first variation of $S[\phi]$.

1.1.2 One-loop effective action

For the case of interacting theories, an exact calculation of the effective action is not possible in general, and one searches for suitable approximations. A very useful one is obtained resorting to the steepest descent (or stationary phase) method to solve the path integral, that corresponds to a formal expansion of $\Gamma[\phi_c]$ in powers of the reduced Planck constant $\hbar \to 0$ (semiclassical expansion) [43]. In this section, we calculate the effective action for a real scalar field ϕ considering terms up to $\mathcal{O}(\hbar)$ in this expansion (one-loop). The same calculation will be performed in chapter 3 [14] for the case of pure quantum gravity in the Einstein-Hilbert truncation, and in chapter 5 [35] for the case of an interacting scalar field in curved spacetime.

The starting point is the generating functional[3] $Z[J]$ in Eq. (1.17) (the $\hbar$ factors are restored below)

$$Z[J] = \frac{\int \mathcal{D}\phi\, \mu[\phi]\, e^{\frac{i}{\hbar}\left(S[\phi]+\int \mathrm{d}^4x\, J(x)\phi(x)\right)}}{\int \mathcal{D}\phi\, \mu[\phi]\; e^{\frac{i}{\hbar}S[\phi]}} \equiv \mathcal{N}\int \mathcal{D}\phi\; e^{\frac{i}{\hbar}\left(S[\phi]+\int \mathrm{d}^4x\, J(x)\phi(x)+\frac{\hbar}{i}\log\mu[\phi]\right)}\,, \tag{1.28}$$

where $S[\phi] = \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi)$ is the classical action. As said above, for $\hbar \to 0$ we can calculate the functional integral resorting to the steepest descent method as follows. Let us consider the saddle-point equation

$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_0} + J(x) = 0\,, \tag{1.29}$$

[3] In this section, we consider scalar theories whose Hamiltonian density is quadratic in the conjugate field π, and indicate with $\mu[\phi]$ the result of the Gaussian integration over π.

where ϕ_0 is the saddle point, i.e. the (assumed to be) unique solution to (1.29). To evaluate the integral in (1.28), we now expand $S[\phi]$ around ϕ_0 up to quadratic terms in $\phi - \phi_0$. Writing

$$\phi(x) = \phi_0(x) + \hbar^{1/2}\,\eta(x)\,, \tag{1.30}$$

we have

$$S[\phi] + \int \mathrm{d}^4x\, J(x)\phi(x) = S[\phi_0] + \int \mathrm{d}^4x\, J(x)\phi_0(x) + \hbar^{1/2}\int \mathrm{d}^4x\, \Big[\,\frac{\delta S[\phi]}{\delta\phi(x)}\Big|_{\phi=\phi_0} + J(x)\Big]\eta(x)$$
$$+\,\frac{\hbar}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\;\eta(x)\,\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_0}\eta(y) + \mathcal{O}(\hbar^{3/2})\,. \tag{1.31}$$

Inserting (1.29) and (1.31) in (1.28), and using η as integration variable, we get[4] ($\hbar\log\mu[\phi] = \hbar\log\mu[\phi_0] + \mathcal{O}(\hbar^{3/2})$)

$$\begin{aligned}Z[J] &= \widetilde{\mathcal{N}} e^{\frac{i}{\hbar}\left(S[\phi_0]+\int \mathrm{d}^4x\,J(x)\phi_0(x)+\frac{\hbar}{i}\log\mu[\phi_0]\right)}\int \mathcal{D}\eta\; e^{\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\,\eta(x)\,\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\big|_{\phi=\phi_0}\eta(y)+\mathcal{O}(\hbar^{1/2})}\\ &= \widehat{\mathcal{N}} e^{\frac{i}{\hbar}\left(S[\phi_0]+\int \mathrm{d}^4x\,J(x)\phi_0(x)+\frac{\hbar}{i}\log\mu[\phi_0]\right)}\Big[\mathrm{Det}\Big(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_0}\Big)\Big]^{-1/2} + \mathcal{O}(\hbar^{1/2})\\ &= \widehat{\mathcal{N}} e^{\frac{i}{\hbar}\left(S[\phi_0]+\int \mathrm{d}^4x\,J(x)\phi_0(x)+\frac{\hbar}{i}\log\mu[\phi_0]-\frac{\hbar}{2i}\,\mathrm{Tr}\log\left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\big|_{\phi=\phi_0}\right)\right)} + \mathcal{O}(\hbar^{1/2})\,. \end{aligned}\tag{1.32}$$

From the above equation, we see that at leading order in $\hbar$

$$\log Z[J] \sim \log Z_0[J] \equiv \frac{i}{\hbar}\big(S[\phi_0] + \int \mathrm{d}^4x\; J(x)\phi_0(x)\big)\,, \tag{1.33}$$

while, at $\mathcal{O}(\hbar)$ (one-loop), $Z[J]$ receives the contribution

$$\delta\log Z[J] = \log\mu[\phi_0] - \frac{1}{2}\,\mathrm{Tr}\log\Big(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_0}\Big)\,. \tag{1.34}$$

Inserting (1.32) in (1.19), for the functional $W[J]$ we obtain (below the inessential J-independent constant $\widehat{\mathcal{N}}$ is discarded)

$$\begin{aligned}W[J] &\equiv -i\hbar\log Z[J]\\ &= S[\phi_0] + \int \mathrm{d}^4x\; J(x)\phi_0(x) - i\hbar\log\mu[\phi_0] + \frac{i\hbar}{2}\,\mathrm{Tr}\log\Big(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_0}\Big)\,.\end{aligned}\tag{1.35}$$

To get the one-loop ($\mathcal{O}(\hbar)$) effective action $\Gamma^{1l}[\phi_c]$, we have to consider now the classical field ϕ_c (see (1.21)) and perform the Legendre transform in Eq. (1.22). From (1.21) and (1.35) we find that

$$\phi_c(x) = \phi_0(x) + \mathcal{O}(\hbar)\,, \tag{1.36}$$

[4]In Eq. (1.32), $\widetilde{\mathcal{N}}$ differs from $\mathcal{N}$ for $\hbar$ factors coming from the Jacobian related to the change of variables $\phi \to \eta$, while $\widehat{\mathcal{N}}$ differs from $\widetilde{\mathcal{N}}$ for constant factors coming from the Gaussian integrations.

i.e. the saddle point configuration $\phi_0(x)$ coincides with the classical field $\phi_c(x)$ apart from $\mathcal{O}(\hbar)$ terms. Using (1.29) and (1.36), we then have

$$\begin{aligned} S[\phi_c] + \int \mathrm{d}^4x\, J(x)\phi_c(x) &= S[\phi_0] + \int \mathrm{d}^4x\, J(x)\phi_0(x) \\ &+ \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, \left(\phi_c(x) - \phi_0(x)\right) \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_0} \left(\phi_c(y) - \phi_0(y)\right) + \ldots \\ &= S[\phi_0] + \int \mathrm{d}^4x\, J(x)\phi_0(x) + \mathcal{O}(\hbar^2)\,. \end{aligned} \tag{1.37}$$

Finally, inserting (1.35), (1.36) and (1.37) in (1.22) we get[5]

$$\Gamma^{1l}[\phi_c] = S[\phi_c] - i\hbar \log \mu[\phi_c] + \frac{i\hbar}{2}\,\mathrm{Tr}\log\left(\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_c} \right). \tag{1.38}$$

Eq. (1.38) is a fundamental result. It says that the classical action S gets modified due to quantum fluctuations, and that at $\mathcal{O}(\hbar)$ the correction is given by the last two terms in the right hand side of this equation. When considering specific truncations for S, the modification of the latter is ultimately encoded in a modification of the couplings of the theory.

The next section is devoted to a brief review, in the case of a scalar theory, of the heat-kernel formalism for the calculation of functional traces as the one in (1.38). This is a very powerful technique due to Schwinger [44] and De Witt [17, 45], though, as shown in [14, 18, 20, 23] and discussed in the next chapters, peculiarities can arise when it is applied to the case of quantum gravity, and due attention should be paid not to miss important terms.

1.2 Heat-kernel expansion

The results of the previous section show that the one-loop effective action is obtained from the calculation of the functional trace in (1.38). This calculation requires to consider the Green functions of the differential operator $\frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_c}$. It has been shown [46] that their construction can be reduced to that of the Green functions of the "minimal" second order differential operator ($\delta(x,\hat{x})$ is the delta-function)

$$\mathcal{O}(x,\hat{x}) = (-g(x))^{1/2}(\Box - m^2 + Q(x))\,\delta(x,\hat{x})\,, \tag{1.39}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant D'Alambert operator for the spacetime metric $g_{\mu\nu}$ (∇_μ is the covariant derivative; $g(x) = \det g_{\mu\nu}(x)$), m is the mass parameter and $Q(x)$ is an arbitrary scalar function, whose form is known once $S[\phi]$ is specified.

The Green functions $G(x,\hat{x})$ of the operator $\mathcal{O}(x,\hat{x})$ are biscalars, i.e. they tranform as scalars both at the point x and $\hat{x}$ under diffeomorphisms, and obey to the equation [47]

$$(\Box - m^2 + Q(x))G(x,\hat{x}) = -(-g(x))^{-1/2}\,\delta(x,\hat{x})\,, \tag{1.40}$$

[5] As it is well-know, the trace in the right hand side of (1.38) is divergent, and counterterms must be introduced to dispose of the regularized divergences.

supplemented of appropriate boundary conditions. The Fock-Schwinger-DeWitt [17, 44, 45, 48] proper-time method is a way to construct the solutions to this equation. Here, this method is presented for the case of a scalar field theory in curved space-time, but it is easily generalized to fermion or gauge fields. The starting point is to write $G(x, \hat{x})$ in terms of a contour integral over a “proper-time” variable s (below $U(s) \equiv U(s, x, \hat{x})$),

$$G(x, \hat{x}) = \int_C i\, \mathrm{d}s\, e^{-ism^2} (-g(x))^{-1/4}\, U(s)\, (-g(\hat{x}))^{-1/4}\,, \tag{1.41}$$

where C is the contour (∂C is the boundary) and $U(s)$ is the “heat-kernel” that satisfies the equation [17, 44]

$$\frac{\partial}{\partial(is)} U(s) = (\Box + Q) U(s) \tag{1.42}$$

with boundary condition $U(s, x, \hat{x})\big|_{\partial C} = \delta(x, \hat{x})$. Eq. (1.41) gives the Feynman propagator if s is integrated between 0 and ∞, and a small negative imaginary part is added to m^2. From now on, the use of this contour C is understood.

It is useful to write $U(s)$ in a form that reproduces the initial condition as $s \to 0$. This is done isolating in it a rapidly oscillating factor as follows [17, 47]

$$U(s) = \frac{i\, [D(x, \hat{x})]^{1/2}}{(4\pi s)^2}\; e^{-\frac{\sigma(x,\hat{x})}{2is}}\; \Omega(s)\,, \tag{1.43}$$

where $\sigma(x, \hat{x})$ is half the square of the geodesic distance between the points x and $\hat{x}$,

$$\sigma = \frac{1}{2}\sigma_\mu \sigma^\mu \quad ; \quad \sigma_\mu = \partial_\mu \sigma \quad ; \quad \sigma^\mu = g^{\mu\nu}\sigma_\nu\,, \tag{1.44}$$

$D(x, \hat{x})$ is the Van Fleck-Morette determinant ($\hat{\partial}_\mu \equiv \frac{\partial}{\partial \hat{x}^\mu}$)

$$\begin{aligned} &D(x, \hat{x}) = -\det(-\hat{\partial}_\mu \partial_\nu \sigma(x, \hat{x})) \\ &\Delta(x, \hat{x}) \equiv (-g(x))^{-1/2} D(x, \hat{x}) (-g(\hat{x}))^{-1/2} \quad ; \quad \Delta^{-1}\nabla_\mu(\Delta\sigma^\mu) = 4\,, \end{aligned} \tag{1.45}$$

and $\Omega(s) \equiv \Omega(s, x, \hat{x})$ is called the “transfer function”. The latter is a biscalar regular in s at $s = 0$, i.e. we have

$$\Omega(0, x, \hat{x})|_{x \to \hat{x}} = 1 \tag{1.46}$$

independently of how x approaches $\hat{x}$. From (1.42) and (1.43), for $\Omega(s)$ we find

$$\left(\frac{\partial}{\partial(is)} + \frac{\sigma^\mu \nabla_\mu}{is} \right) \Omega(s) = \Delta^{-1/2}\, \Box\, (\Delta^{1/2}\Omega(s)) + Q\, \Omega(s)\,, \tag{1.47}$$

whose solution can be searched as a power series in s

$$\Omega(s, x, \hat{x}) = \sum_{n=0}^{\infty} (is)^n\, a_n(x, \hat{x})\,. \tag{1.48}$$

The coefficients $a_n(x, \hat{x})$ are known as the “heat-kernel coefficients” or “Hadamard-Minakshisundaram-DeWitt-Seeley coefficients” [49–52]. Inserting (1.48) in (1.46) and (1.47), we find that the a_n satisfy the recursion relations

$$\sigma^\mu \nabla_\mu a_0(x, \hat{x}) = 0 \quad ; \quad a_0(\hat{x}, \hat{x}) = 1 \tag{1.49}$$

$$(n + \sigma^\mu \nabla_\mu)\, a_n(x, \hat{x}) = (\, \Delta^{-1/2}\, \Box\, \Delta^{1/2} + Q\,)\, a_{n-1}(x, \hat{x})\,. \tag{1.50}$$

If we were able to calculate all the a_n, the transfer function $\Omega(s)$ would be known and the integration over s in (1.41) could be performed to finally obtain $G(x,\hat{x})$. When $x \neq \hat{x}$, this latter integral converges for $s \to 0$ because of the factor $\exp[i(\sigma + i0)/2s]$, and the singularities of $G(x,\hat{x})$ lie on the surface $\sigma(x,\hat{x}) = 0$ [17, 46].

As said above, the calculation of the one-loop effective action reduces to functional traces of the kind[6] (the second line is obtained using the proper-time representation of the "log" analogous to (1.41) [17])

$$
\begin{aligned}
& i\,\mathrm{Tr}\,\log\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right] \equiv i\int \mathrm{d}^4\hat{x}\,\log\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right]_{x\to\hat{x}} \\
&= -i\int \mathrm{d}^4\hat{x}\int_0^\infty \frac{\mathrm{d}s}{s}\,e^{-im^2 s}\,U(s,x,\hat{x})|_{x\to\hat{x}} \\
&= -i\int \mathrm{d}^4\hat{x}\int_0^\infty \frac{i\,\mathrm{d}s}{(4\pi i)^2 s^3}\,e^{-\frac{\sigma(x,\hat{x})}{2is}-im^2 s}\,[D(x,\hat{x})]^{1/2}\,\Omega(s,x,\hat{x})|_{x\to\hat{x}} \\
&= -\int \mathrm{d}^4\hat{x}\,(-g(\hat{x}))^{1/2}\int_0^\infty \frac{\mathrm{d}s}{(4\pi)^2 s^3}\,e^{-im^2 s}\left[\sum_{n=0}^{\infty}(is)^n\,a_n(x,\hat{x})\right]_{x\to\hat{x}}, \qquad (1.51)
\end{aligned}
$$

where the fourth line is obtained using (1.48) together with the coincidence limits

$$
\sigma|_{x\to\hat{x}} = \nabla_\mu\sigma|_{x\to\hat{x}} = 0 \quad ; \quad \nabla_\mu\nabla_\nu\sigma|_{x\to\hat{x}} = -\hat{\partial}_\mu\partial_\nu\sigma|_{x\to\hat{x}} = g_{\mu\nu} \quad ; \quad D|_{x\to\hat{x}} = -g\,. \qquad (1.52)
$$

The coefficients $a_n(x,\hat{x})|_{x\to\hat{x}}$ in (1.51) can be calculated taking repeated covariant derivatives of (1.44), (1.45) and (1.50) and using the coincidence limits (1.52). For instance, for a_1 and a_2 one has [46]

$$
a_1(x,\hat{x})|_{x\to\hat{x}} = \frac{1}{6}R + Q \qquad (1.53)
$$

$$
a_2(x,\hat{x})|_{x\to\hat{x}} = \frac{1}{180}\left(R_{\mu\nu\rho\gamma}R^{\mu\nu\rho\gamma} - R_{\mu\nu}R^{\mu\nu}\right) + \frac{1}{2}\left(Q + \frac{1}{6}R\right)^2 + \frac{1}{6}\Box\left(Q + \frac{1}{5}R\right). \qquad (1.54)
$$

As it is well-known, when $x = \hat{x}$ the integral over s in the last line of (1.51) does not converge for $s \to 0$, and has to be regularized. Following Schwinger [44], to isolate the divergences in (1.51) we begin by rotating the integration contour into the negative imaginary axis, which is equivalent to making the replacement $s = -iu$, thus getting

$$
\begin{aligned}
& i\,\mathrm{Tr}\,\log\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right] \\
&= \int \mathrm{d}^4\hat{x}\,(-g(\hat{x}))^{1/2}\int_0^\infty \frac{\mathrm{d}u}{(4\pi)^2 u^3}\,e^{-m^2 u}\left[\sum_{n=0}^{\infty}u^n\,a_n(x,\hat{x})\right]_{x\to\hat{x}}, \qquad (1.55)
\end{aligned}
$$

At this point, the proper-time integrals over u are regularized through the introduction of a UV cutoff $1/\Lambda^2$ as lower integration bound, $\int_0^\infty \mathrm{d}u \to \int_{1/\Lambda^2}^\infty \mathrm{d}u$. Integrating the series term by term, a simple counting of the powers of u shows that only the $n = 0, 1, 2$ terms are divergent for $u \to 0$. Within this approach, these terms give the one-loop divergences

[6]The $(-g(x))^{-1/4}$ terms in Eq. (1.51) come from the $(-g(x))^{1/4}$ factors in the configuration space path integral measure for a scalar field in a gravitational background $g_{\mu\nu}$ (see Eqs. (5.4), (5.6) and comments therein).

of the effective action. Inserting (1.49), (1.53) and (1.54) in (1.55), we finally get

$$
\begin{aligned}
& i\,\mathrm{Tr}\,\log\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right]_{\mathrm{div}} \\
&= \int \mathrm{d}^4\hat{x}\,(-g(\hat{x}))^{1/2}\Big[\frac{a_0}{32\pi^2}\Lambda^4 + \frac{a_1 - a_0 m^2}{16\pi^2}\Lambda^2 + \frac{(2a_2 - 2m^2 a_1 + a_0 m^4)}{32\pi^2}\log\frac{\Lambda^2}{m^2}\Big] \\
&= \frac{1}{32\pi^2}\int \mathrm{d}^4\hat{x}\,(-g(\hat{x}))^{1/2}\Big\{\Lambda^4 - 2m^2\Lambda^2 + m^4\log\frac{\Lambda^2}{m^2} + \frac{1}{3}\Big(R+6Q\Big)\Big(\Lambda^2 - m^2\log\frac{\Lambda^2}{m^2}\Big) \\
&+ \Big[\frac{1}{90}\big(R_{\mu\nu\rho\gamma}R^{\mu\nu\rho\gamma} - R_{\mu\nu}R^{\mu\nu}\big) + \frac{1}{36}\big(R+6Q\big)^2 + \frac{1}{15}\Box\big(R+5Q\big)\Big]\log\frac{\Lambda^2}{m^2}\Big\}\,. \qquad (1.56)
\end{aligned}
$$

Some comments are in order. From an inspection of (1.56), we immediately recognize quartically and quadratically "divergent" contributions to the vacuum energy (coefficient of $\int \mathrm{d}^4\hat{x}\,(-g(\hat{x}))^{1/2}$). These one-loop generated contributions $\delta\rho$ require the bare vacuum energy ρ_{bare} to be also $\sim \Lambda^4$ ($\sim M_P^4$ if Λ is the Planck mass M_P), enormous compared to its measured value $\rho_{\mathrm{meas}} \sim 2.7\cdot 10^{-47}\,\mathrm{GeV}^4$, with a coefficient extremely fine-tuned in such a way to cancel (quite exactly) these contributions, and to finally have $\rho_{\mathrm{bare}} + \delta\rho \sim \rho_{\mathrm{meas}}$. This is the strongest facet to the cosmological constant problem discussed in the Introduction. Moreover, if we consider the case of a ϕ^4 theory, whose action is

$$
S[\phi] = -\int \mathrm{d}^4x\,\sqrt{-g}\left[\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]\,, \qquad (1.57)
$$

and for which we have $Q = -\frac{\lambda}{2}\phi^2$, from the last term in the third line of (1.56) we see the appearance of the well-known quadratically "divergent" radiative correction to the mass m^2 of the scalar field ϕ. This term gives rise to the long-standing naturalness problem related to the masses of scalar particles.

As it will be discussed in detail in the next chapters, however, the way the "Tr log" has been calculated in this section, that is the usual calculation found in the literature, misses some delicate points that arise when considering non-flat gravitational backgrounds [14, 18, 20, 23, 35, 37].

A first important point concerns the introduction of the UV physical cutoff Λ of the theory. It has been shown [14, 23, 35] that the usual way in which the proper-time cutoff Λ is introduced (see (1.56) and comments above this equation) profoundly alters the dependence of the one-loop effective action on the background metric $g_{\mu\nu}$. This important point will be discussed in detail in chapters 3, 4 and 5, which contain the results of [14], [23], [24] and [35]. We will show that, when Λ is properly identified, the power-like divergences mentioned above are not present. Both the vacuum energy and the mass of scalar particles receive only mild logarithmic corrections.

Another delicate point is that in the case of gravitational theories, subtleties arise in the calculation of the $\log\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right]$. When the distributional nature of the Green functions of the operator $\left[(-g(x))^{-1/4}\mathcal{O}(x,\hat{x})(-g(\hat{x}))^{-1/4}\right]$ is carefully taken into account [18, 20], a non-trivial term contributing to the trace in (1.51) is found that is missed by the usual calculations presented above. This term is of crucial importance since it cancels local terms in the path integral measure that are not diffeomorphism invariant, eventually ensuring the invariance of the effective gravitational action [18, 37]. This point will be further discussed in chapter 6.

The next section is devoted to an introduction to gauge theories. We will discuss the Faddeev-Popov procedure to write the generating functional of Green functions, and we will also briefly introduce the gauge-invariant Vilkovisky-DeWitt effective action. General relativity (GR) can be regarded as a gauge theory, whose gauge transformations are diffeomorphisms, and the techniques presented in the next section will be used in the following chapters for its quantization.

1.3 Gauge theories

Let us consider the theory of a set of boson fields ϕ^i with classical action $S[\phi]$. In this section, the De Witt notation is used, according to which i is a collective index that includes discrete indices (Lorenz indices, indices of internal symmetry and so on) and also the spacetime argument x^μ of the fields. For instance, in the case of a Yang-Mills theory the field is $\phi^i \equiv A^a_\mu(x)$, or, in the case of gravity, $\phi^i \equiv g_{\mu\nu}(x)$. Moreover, if $F[\phi]$ is a functional of ϕ^i (below A is a collective index for all the discrete indices),

$$F_{,i}[\phi] \equiv \frac{\delta F[\phi]}{\delta \phi^A(x)} \quad ; \quad F_{,i}[\phi]\delta\phi^i \equiv \int \mathrm{d}^4x \, \frac{\delta F[\phi]}{\delta \phi^A(x)} \delta\phi^A(x) \, . \tag{1.58}$$

Consider now the following infinitesimal transformations of the field ϕ^i with parameters $\xi^\alpha \equiv \xi^a(x)$ and generators $R^i_\alpha[\phi]$,

$$\delta\phi^i = R^i_\alpha[\phi]\xi^\alpha \, . \tag{1.59}$$

Correspondingly, the action S transforms as

$$\delta S[\phi] = S_{,i} R^i_\alpha[\phi]\xi^\alpha \, . \tag{1.60}$$

If $S_{,i}R^i_\alpha[\phi] = 0$ holds true without using the equation of motion ($S_{,i}[\phi] = 0$), the theory is called a *gauge theory*, ϕ^i are the gauge fields and (1.59) are the gauge transformations [53]. For the case of quantum gravity, the latter are diffeomorphisms and the corresponding generators are

$$R^i_\alpha[\phi] \equiv R_{\mu\nu\sigma}(x,y) = [g_{\mu\sigma}(x)\partial_\nu + g_{\nu\sigma}(x)\partial_\mu + \partial_\sigma g_{\mu\nu}(x)] \, \delta^{(4)}(x-y) \, . \tag{1.61}$$

Physical observables in a gauge theory are real gauge-invariant functionals of the fields ϕ^i, that depend only on gauge-independent combinations of the latter. It is possible to fix the gauge-dependent components of the fields without changing the values of physical quantities. This can be done imposing some additional conditions on the ϕ^i (realizable with an appropriate choice of the parameters ξ^α),

$$\chi^\alpha[\phi] = 0 \, , \tag{1.62}$$

the so-called gauge-fixing condition.

Let M be the space of the gauge fields, G a group of gauge transformations and ϕ^i an element of M. The set $\{\phi^i\} = \{ {}^g\phi^i \, , \, g \in G\}$, obtained by the action of all the elements of G on ϕ^i, is called gauge group orbit of the representative ϕ^i in M and it is proved to be completely defined by the representative. The latter can be found using the gauge-fixing condition (1.62). The set of the different group orbits is called "coset" and is indicated with M/G ($\{\phi^i\} \in M/G$). It can be proved that different orbits do not cross each other. As said above, physical observables are real gauge-invariant functionals of the fields, $F[\phi] = F[\phi']$ if $\phi'^{\,i} = \phi^i + R^i_\alpha[\phi]\,\xi^\alpha$, and are then functionals of the gauge orbits, $F[\phi] = F[\{\phi\}]$. The same is true for the classical action, $S[\phi] = S[\{\phi\}]$. It is then natural to associate $\{\phi\}$ to the physical gauge field and to call M/G the set of the physical gauge fields [54].

Consider a physical quantity $F[\phi]$. Since $F[\phi] = F[\{\phi\}]$, the vacuum expectation value of the time ordered product of $F[\phi]$ is defined as[7] (see section (1.1.1))

$$\langle 0|\, T(F[\{\phi\}])\, |0\rangle = \int \mathcal{D}\{\phi\}\, F[\{\phi\}]\, e^{iS[\{\phi\}]}\,, \tag{1.63}$$

that contains integration only over physical gauge fields. It is possible to write the path integral (1.63) along the physical gauge fields (over the gauge orbits) in terms of a path integral over all fields. This was done by L.D. Faddeev and V.N. Popov, and independently by B.S. De Witt. Considering the gauge condition $\chi^\alpha[\phi] - f^\alpha = 0$ (f^α is field independent), it is possible to find that [54]

$$\langle 0|\, T(F[\phi])\, |0\rangle = \int \mathcal{D}\phi\, \mathrm{Det}(M^\alpha_\beta[\phi])\, \delta(\chi^\alpha[\phi] - f^\alpha)\, F[\phi]\, e^{iS[\phi]}\,, \tag{1.64}$$

where $M^\alpha_\beta[\phi]$ is the DeWitt-Faddeev-Popov matrix

$$M^\alpha_\beta[\phi] = \chi^\alpha_{,i}[\phi]\, R^i_\beta[\phi]\,. \tag{1.65}$$

It can be shown that the path integral in (1.64) does not depend on the choice of the gauge-fixing functions $\chi^\alpha[\phi] - f^\alpha$. At this point, according to (1.64) we can introduce the generating functional $Z[J]$ of Green functions as

$$Z[J] = \int \mathcal{D}\phi\, \mathrm{Det}(M^\alpha_\beta[\phi])\, \delta(\chi^\alpha[\phi] - f^\alpha)\, e^{i(S[\phi] + J_i\phi^i)}\,, \tag{1.66}$$

where J_i are the sources associated to the fields ϕ^i. Since the latter are not gauge invariant and the sources are arbitrary, $Z[J]$ is not gauge invariant.

Let us write the path integral (1.64) in a convenient form. First of all, the functional determinant $\mathrm{Det}(M^\alpha_\beta[\phi])$ can be represented as an integral over anticommuting fields $\bar{c}_\alpha$, c^β, called the "ghost fields", as

$$\mathrm{Det}(M^\alpha_\beta[\phi]) = \int \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{i\bar{c}_\alpha M^\alpha_\beta[\phi] c^\beta}\,. \tag{1.67}$$

Moreover, we can multiply both members of (1.64) by the factor $e^{\frac{i}{2} f^\alpha G_{\alpha\beta}[\phi] f^\beta}$ and integrate over f^α. Using the gauge invariance of the path integral, we arrive at

$$\langle 0|\, T(F[\phi])\, |0\rangle = \int \mathcal{D}\phi\, \mathcal{D}\bar{c}\, \mathcal{D}c\, F[\phi]\, e^{i(S[\phi] + \frac{1}{2}\chi^\alpha G_{\alpha\beta}[\phi]\chi^\beta + \bar{c}_\alpha M^\alpha_\beta[\phi] c^\beta)} [\mathrm{Det}(G_{\alpha\beta}[\phi])]^{1/2}\,. \tag{1.68}$$

[7]For simplicity, here we consider the case in which the integration over the conjugate momenta in phase space does not give a non-trivial field-dependent contribution to the configuration space measure. However, similar results as those presented in this section hold also in this latter case, and will be used in the next chapters to quantize GR.

The functional determinant $[\mathrm{Det}(G_{\alpha\beta}[\phi])]^{1/2}$ can be written introducing other auxiliary anticommuting fields b^α as

$$[\mathrm{Det}(G_{\alpha\beta}[\phi])]^{1/2} = \int \mathcal{D}b\, e^{ib^\alpha G_{\alpha\beta}[\phi] b^\beta}\,, \tag{1.69}$$

so that

$$\langle 0|\, T(F[\phi])\, |0\rangle = \int \mathcal{D}\phi\, \mathcal{D}\bar{c}\, \mathcal{D}c\, \mathcal{D}b\, F[\phi]\, e^{i(S[\phi]+\frac{1}{2}\chi^\alpha G_{\alpha\beta}[\phi]\chi^\beta + \bar{c}_\alpha M^\alpha_\beta[\phi] c^\beta + b^\alpha G_{\alpha\beta}[\phi] b^\beta)}\,. \tag{1.70}$$

In the case of constant $G_{\alpha\beta}$, the contribution from the integration over b^α is also constant and can be omitted. The generating functional for gauge theories is then [54]

$$Z[J] = \int \mathcal{D}\phi\, \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{i(S_{\mathrm{tot}}[\phi,\,\bar{c},\,c] + J_i\phi^i)}\,, \tag{1.71}$$

where

$$S_{\mathrm{tot}}[\phi, \bar{c}, c] = S[\phi] + S_{\mathrm{gf}}[\phi] + S_{\mathrm{gh}}[\phi, \bar{c}, c]\,, \tag{1.72}$$

with gauge-fixing and ghost action given by

$$S_{\mathrm{gf}}[\phi] = \frac{1}{2}\chi^\alpha G_{\alpha\beta}\chi^\beta \tag{1.73}$$

$$S_{\mathrm{gh}}[\phi, \bar{c}, c] = \bar{c}_\alpha M^\alpha_\beta[\phi] c^\beta\,. \tag{1.74}$$

Finally, starting from (1.71), passing to the functional $W[J]$ and considering its Legendre transform (see section 1.1.1), we find that the effective action satisfies the equation ($\Gamma_{,i}[\phi_c] \equiv \frac{\delta\Gamma[\phi_c]}{\delta\phi^i_c}$)

$$e^{i\Gamma[\phi_c]} = \int \mathcal{D}\phi\, \mathcal{D}\bar{c}\, \mathcal{D}c\, e^{i\,[S_{\mathrm{tot}}[\phi,\,\bar{c},\,c] \,-\, \Gamma_{,i}[\phi_c]\,(\phi^i - \phi^i_c)\,]}\,. \tag{1.75}$$

Here comes a delicate point in connection with the definition (1.75). Due to the source term $((\phi^i - \phi^i_c)\Gamma_{,i}[\phi_c])$, the result for the effective action $\Gamma[\phi_c]$ depends on the parametrization of the background field ϕ_c (which does not satisfy the on-shell condition $\Gamma_{,i}[\phi_c] = 0$). In the case of gauge theories, this leads to a dependence of $\Gamma[\phi_c]$ on the choice of the gauge used for the quantization [55]. In fact, the ϕ^i can be regarded as one coordinate system on the infinite dimensional field space M. Due to the presence of the "coordinate" difference $\phi^i - \phi^i_c$ in (1.75), $\Gamma[\phi_c]$ is not a "scalar field" on M (i.e. it is not invariant under a change of coordinates in M, that is a field reparametrization). In gauge theories, the gauge conditions used to remove the redundancy in the path integral provide a coordinate system within the gauge orbits on M [56], and this is at the origin of the relation between the issue of gauge dependence and the one of reparametrization variance. From the geometric point of view, then, the gauge-dependence of the standard effective action defined by (1.75) is due to the fact that the path integral in the right-hand side of this equation depends on the parametrization of the space of group orbits M (see above).

This suggests to give a new definition of effective action, introducing a G-invariant metric on M. Clearly, the new gauge-invariant and gauge-independent effective action has to lead to the same S matrix as the one obtained from the standard definition (1.75). This was done for the first time by Vilkovisky [55], and later further analyzed by De

Witt [57], who proposed some modifications of Vilkovisky's original formulation[8]. The reparametrization-invariant, gauge-invariant, and gauge-independent off-shell effective action $\widetilde{\Gamma}[\phi_c]$ is

$$e^{i\,\widetilde{\Gamma}[\phi_c]} = \int \mathcal{D}\phi\,\mathcal{D}\bar{c}\,\mathcal{D}c\, e^{i\,[S_{\rm tot}[\phi,\,\bar{c},\,c]\,+\sigma^i(\phi_c,\phi)\,\widetilde{\Gamma}_{,i}[\phi_c]\,]}\,, \tag{1.76}$$

where $S_{\rm tot}$ is given in (1.72) and $\sigma^i(\phi_c, \phi)$ is constructed as follows. Consider the geodesic that connects ϕ_c and ϕ, then $\sigma^i(\phi_c, \phi)$ is the tangent vector at ϕ_c that points toward ϕ [45]. Its length is equal to the geodesic distance between ϕ_c and ϕ, and it transforms as a vector at ϕ_c and as a scalar at ϕ. The derivation of the G-invariant metric in M for the case of gauge theories with which σ^i and covariant derivatives are built is in [55] (see also [56, 58]).

Let us make now two important observations concerning $\widetilde{\Gamma}$. First, we observe that Eq. (1.27), according to which the vev of the quantum field is solution of $\Gamma_{,i}[\phi_c] = 0$, with Γ the standard effective action, is now replaced by

$$\left\langle \sigma^i(\bar{\phi}_c, \phi)\right\rangle = 0\,, \tag{1.77}$$

where $\bar{\phi}_c$ is solution of $\widetilde{\Gamma}_{,i}[\phi_c] = 0$.

A second observation is that we can obtain a covariant version of the loop expansion (see section 1.1.2) resorting to the covariant Taylor series for the classical action [46, 59]

$$S[\phi] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\big(\nabla_{i_1}\ldots\nabla_{i_n}S[\phi]\big)\sigma^{i_1}(\phi_c,\phi)\ldots\sigma^{i_n}(\phi_c,\phi)\,, \tag{1.78}$$

where ∇_i is the derivative covariant with respect to the connection on M. In chapter 3 [14], the one-loop Vilkovisky - DeWitt (VdW) effective action $\widetilde{\Gamma}^{1l}$ is calculated for the case of pure quantum gravity in the Einstein-Hilbert truncation. The calculation is performed using a result found by Fradkin and Tseytlin in [58], according to which, taking a spherical gravitational background, the one-loop VdW effective action coincides with the standard one calculated in a specific gauge. We will then proceed to the calculation of the standard one-loop effective action under the prescriptions described in [58] (see (3.2) and comments therein).

As seen in the previous section, quantum fluctuations can give rise to divergent contributions to the parameters of a given QFT. The way to get rid of these divergences is the so-called renormalization procedure. When it was first developed by Feynman, Schwinger and Tomonaga, renormalization was just a technical device to dispose of divergences, and its physical meaning was quite obscure. A deeper understanding was achieved with the advent of the renormalization group approach, that will be introduced in the next section for the case of a scalar theory. In chapter 4, this approach will be applied to the case of pure gravity in the Einstein-Hilbert truncation, and we will derive the RG equations for the running Newton and cosmological constant of [23].

[8] DeWitt's definition and Vilkovisky's one coincide at the one-loop level. Since in chapter 3 we will not go beyond the one-loop approximation in the calculation of the effective gravitational action, the modifications due to De Witt are not discussed here.

1.4 Renormalization group approach

The appearance of divergences in perturbative calculations in QFT is a well-known fact. In the past, it was considered a big problem by many physicists, who thought that QFT was intrinsically "ill" and had then to be abandoned. A turning point arrived with the advent of renormalization, developed by Feynman, Schwinger and Tomonaga in 1949 for QED. In the renormalization procedure, divergences are absorbed into unobservable quantities, the bare parameters or counterterms, leaving one with finite results. Once this machinery was used, QFT turned out to give predictions that were in an incredible accordance with experiments. However, the physical meaning behind renormalization was not so clear at the beginning.

A more profound understanding was obtained with the renormalization group approach. Stueckelberg and Petermann [60] were the first ones who realized that the many possible ways in which finite results are dug out from divergent loop integrals (i.e. the many renormalization prescriptions) can all be related by transformations that form a semi-group, to which they gave the name of "renormalization group". The request of independence of the final results on the renormalization prescription gives rise to a set of equations, the so-called renormalization group equations.

An even more physical perspective to the RG is given by Gell-Mann and Low in [61]. The authors consider the Serber-Uehling potential, i.e. the electrostatic potential between two charges in the vacuum given by the Coulomb potential plus the modification due to vacuum polarization effects in QED at one-loop. They observe that the result can be written defining "running" charges $q(r)$ and $q'(r)$ as

$$V(r) = \frac{q(r)q'(r)}{4\pi r}\,, \tag{1.79}$$

where (below q and q' are the measured charges at some scale $\tilde{r}$, α is the fine structure constant, m the measured electron rest mass and γ the Euler-Mascheroni constant)

$$q(r)q'(r) \equiv qq' \left\{ 1 + \frac{2\alpha}{3\pi} \left(\ln\left(\frac{\hbar}{mcr}\right) - \frac{5}{6} - \ln\gamma + \mathcal{O}(\alpha^2) \right) \right\}. \tag{1.80}$$

This suggests an interesting physical interpretation of the result. When $r \to 0$, each particle feels the *bare* charge of the other one ($q_0 = q(r \to 0)$). When the distance increases, at a finite value of r each of them feels a "dressed" value $q(r)$ of the other: the "bare charges" are *screened* by vacuum polarization effects in QED. The charges $q(r)$ and $q'(r)$ are the relevant quantities at the distance scale r.

Another crucial observation is that the physical charge $q\,(= q(\tilde{r}))$ shows divergences only when it is written in terms of an expansion in the bare charge q_0. When we consider, instead, an expansion in terms of the generic charge $q(\hat{r})$ at another scale $\hat{r}$, q only shows finite contributions that go like $\ln(\hat{r}/r)$. This observation opens the door to a new and deep interpretation of the renormalization group. The latter describes the way the theory evolves with the observational (distance/energy) scale. This point of view is at the base of a new way of understanding QFTs, that in the '70s culminated in the Wilsonian renormalization group developed by K. G. Wilson in [62–64].

The crucial point in Wilson's work is to recognize that the appearance of divergences in QFT is due to the fact that, similarly to what happens for critical phenomena, one has to deal with infinitely many degrees of freedom and scales that contribute all at once. The Wilsonian RG approach gives a way to handle these issues. In a nutshell, the idea is that, for systems with infinitely many scales, the "averaging out" (that for a QFT means the inclusion of quantum fluctuations) should be performed progressively, in a step by step manner, starting from fluctuations on the smallest possible distance scale and going to fluctuations on larger and larger scales.

Ultraviolet divergences appear when one tries to extend the validity of QFTs up to infinite energy (zero distance) scales. They are a signal that the theory is not fundamental, and that, on the contrary, it is the left over at a certain energy scale Λ of a higher energy theory. For the QFT under consideration, Λ is the UV-cutoff, i.e. the maximal scale up to which it gives an accurate description of physics. Above this scale, it is replaced by its UV-completion.

In the next section, the Wilsonian RG strategy is illustrated for the case of a scalar theory considering the formulation put forward by Wegner and Houghton in [65].

1.4.1 RG equations for a scalar theory

Let us consider a single-component scalar field φ in a Euclidean (flat) D-dimensional space, and indicate with $S_\Lambda[\varphi]$ the action at the UV-cutoff Λ. The field $\varphi(x)$ contains all the Fourier modes from 0 up to Λ

$$\varphi(x) = \sum_{|q|=0}^{\Lambda} \varphi_q \, e^{iq\cdot x}. \tag{1.81}$$

The parameters (masses, coupling constants, ...) that appear in S_Λ depend on the scale Λ. To describe physical processes at scales lower than Λ, it is convenient to use the action and the corresponding values of the parameters at those scales. This is basically the idea put forward by Gell-Mann and Low in [61] (see above). Given S_Λ, the question is to find a way to determine the action S_k at a lower scale $k < \Lambda$. This can be done as follows.

We begin by splitting $\varphi(x)$ into a background field $\phi(x)$ and a fluctuation $\eta(x)$. The background field ϕ contains all the Fourier modes from 0 to k, while the fluctuation η contains the remaining ones from k to Λ

$$\varphi(x) = \sum_{|q|=0}^{\Lambda} \varphi_q \, e^{iq\cdot x} = \sum_{|q|=0}^{k} \varphi_q \, e^{iq\cdot x} + \sum_{|q|=k}^{\Lambda} \varphi_q \, e^{iq\cdot x} \equiv \phi(x) + \eta(x)\,. \tag{1.82}$$

The action $S_k[\phi]$ at the scale k is obtained integrating over the fluctuation η

$$e^{-\frac{1}{\hbar}S_k[\phi]} = \int [\mathcal{D}\eta] \, e^{-\frac{1}{\hbar}S_\Lambda[\phi+\eta]} = \int \Big[\textstyle\prod_{|q|\in]k,\Lambda]} \mathrm{d}\varphi_q \Big] e^{-\frac{1}{\hbar}S_\Lambda[\phi+\eta]}. \tag{1.83}$$

The problem with this equation is that it is not possible (in general) to evaluate the path integral over η exactly, and one has to resort to some approximation. For instance, it can be computed by means of a loop expansion in powers of $\hbar$ (see section 1.1.2). However, in this way the right-hand side of (1.83) typically develops power-like terms and logarithms of the kind $\Lambda^n - (\Lambda - k)^n$ and $\log(\Lambda/k)$, respectively. If $k \ll \Lambda$, these contributions are large and can spoil the validity of the expansion. This problem can be overcome resorting to Wilson's strategy.

Suppose to have successfully defined the theory at a scale $k < \Lambda$, and that we want to define it at a still lower scale $p < k$. To truly disentangle the scales, quantum fluctuations have to be included gradually, in infinitesimal shells of momentum from k down to p. Following the logic presented above, the action at a scale $k - \delta k$ infinitesimally lower than k is defined as

$$e^{-\frac{1}{\hbar}S_{k-\delta k}[\phi]} = \int [D\eta]\, e^{-\frac{1}{\hbar}S_k[\phi+\eta]}, \tag{1.84}$$

where ϕ now contains Fourier modes from 0 to $k - \delta k$, while η those in the infinitesimal shell $]k - \delta k, k]$. Once $S_{k-\delta k}$ is known, we can iterate the procedure and calculate $S_{k-2\delta k}$ from the integration of the Fourier modes in the shell $]k - 2\delta k,\, k - \delta k]$, and so on until reaching p. Resorting to the loop expansion for the evaluation of the path integral in the right-hand side of (1.84), the power and logarithmic terms mentioned above now will be $k^n - (k - \delta k)^n \sim k^n\, n\,(\delta k/k)$, $\log(k/(k - \delta k)) \sim \delta k/k$, that are terms of order δk. This is an important point: the fact that quantum fluctuations are included in infinitesimal shells of width δk gives rise to a new small expansion parameter, $\delta k/k$. It is possible to show that the contributions from higher orders (beyond one-loop) in the loop expansion of the right-hand side of (1.84) are of order $(\delta k)^2$, and thus vanish in the limit $\delta k \to 0$ [65] (see also [66] for a careful derivation). In this sense, the one-loop approximation of the path integral in (1.84) is said to be "exact".

Let us proceed, then, expanding $S_k[\phi + \eta]$ up to $\mathcal{O}(\eta^2)$

$$\begin{aligned} S_k[\phi+\eta] = S_k[\phi] + \hbar^{1/2} \int \mathrm{d}^D x\, \left.\frac{\delta S_k[\phi+\eta]}{\delta \eta(x)}\right|_{\eta(x)=0} \eta(x) \\ + \frac{\hbar}{2} \int \mathrm{d}^D x\, \mathrm{d}^D y\, \left.\frac{\delta^2 S_k[\phi+\eta]}{\delta\eta(x)\delta\eta(y)}\right|_{\eta(x)=0} \eta(x)\,\eta(y) + \mathcal{O}(\eta^3)\,. \end{aligned} \tag{1.85}$$

Inserting (1.85) in (1.84), performing the Gaussian integration over η, and taking the limit $\delta k \to 0$ we find[9]

$$\begin{aligned} \frac{\partial S_k[\phi]}{\partial k} = \frac{\hbar}{2\delta k} \mathrm{Tr}_{k-\delta k,k} \log \left.\frac{\delta^2 S_k[\phi+\eta]}{\delta\eta\delta\eta}\right|_{\eta=0} \\ - \frac{1}{2\delta k} \left.\frac{\delta S_k[\phi+\eta]}{\delta\eta}\right|_{\eta=0} \cdot \left(\left.\frac{\delta^2 S_k[\phi+\eta]}{\delta\eta\delta\eta}\right|_{\eta=0} \right)^{-1} \cdot \left.\frac{\delta S_k[\phi+\eta]}{\delta\eta}\right|_{\eta=0}, \end{aligned} \tag{1.86}$$

where both the trace and the "$\cdot$" operation are restricted to the subspace of Fourier modes with $k - \delta k < |q| < k$. Eq. (1.86) is the so-called Wegner - Houghton equation for the running action S_k. It is an integro-differential equation, that describes the evolution of the action with the scale k, and S_Λ serves as the boundary condition at $k = \Lambda$.

An important observation has to be done at this point. The second order approximation (1.85) for the running action $S_k[\phi + \eta]$ is applicable only if the saddle point, i.e. the

[9]Note that despite the appearance of δk, the right hand side of (1.86) is of order $(\delta k)^0$ since the trace and the "$\cdot$" operation are of order δk.

solution to the equation

$$\left.\frac{\delta S_k[\phi+\eta]}{\delta\eta}\right|_{\eta=\eta_s} = 0\,, \tag{1.87}$$

has an amplitude of order $\mathcal{O}(\hbar)$. This important point was noted for the first time in [67], much after the derivation of the Wegner-Houghton equation. In fact, there are cases in which there is a saddle point during the elimination of the modes and, at the same time, there is no guarantee that its amplitude remains $\mathcal{O}(\hbar)$ during the RG evolution. In such a situation, one has to derive a different RG equation that holds even releasing this latter condition. Though quite interesting, these cases will not be treated here since in the following we will consider only situations in which the above condition on the saddle point amplitude is verified.

Going back to the Wegner-Houghton equation (1.86), it is not possible to find its exact solution, and again one should search for some suitable approximation. A first useful step is to write the running action $S_k[\phi]$ in terms of an expansion in gradients of ϕ (see (1.25), where a similar expansion was introduced for the effective action)

$$S_k[\phi] = \int \mathrm{d}^D x \left[U_k(\phi) + \frac{1}{2}F_k(\phi)\partial_\mu\phi\,\partial^\mu\phi + Y_k(\phi)(\partial_\mu\phi\,\partial^\mu\phi)^2 + \dots\right]. \tag{1.88}$$

where the dots indicate higher derivative terms. Inserting (1.88) in (1.86), we get a system of coupled RG equations for the functions U_k, F_k, $Y_k, \dots$. As a further simplification, we consider here the so-called local potential approximation (LPA). It is the lowest order approximation to the derivative expansion, and is defined by $F_k = 1$, $Y_k, \dots = 0$,

$$S_k[\phi] = \int \mathrm{d}^D x \left[\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + U_k(\phi)\right], \tag{1.89}$$

Consistently with this approximation, we can consider a homogeneous background field $\phi = \text{const}$. For this choice of background, inserting (1.89) in (1.86) we obtain the following equation[10] for the running potential $U_k(\phi)$

$$\frac{U_k(\phi) - U_{k-\delta k}(\phi)}{\delta k} = -\frac{1}{2\delta k}\int' \frac{\mathrm{d}^D q}{(2\pi)^D}\log\left(\frac{q^2 + U_k''(\phi)}{q^2}\right), \tag{1.90}$$

where the symbol $\int'$ means that the integration over q is restricted to the shell $k - \delta k < |q| \leq k$, and $U_k''(\phi)$ is the second derivative of U_k with respect to ϕ. For $\delta k \to 0$, we get the differential equation

$$k\frac{\partial}{\partial k}U_k(\phi) = -\frac{N_D}{2}k^D \log\left(\frac{k^2 + U_k''(\phi)}{k^2}\right), \tag{1.91}$$

where N_D is the result of the angular integration in D dimensions, $N_D = 2\left[(4\pi)^{D/2}\Gamma(D/2)\right]^{-1}$. This is a non-perturbative evolution equation for the potential $U_k(\phi)$. The potential $U_\Lambda(\phi)$

[10] As usual, the field-independent quantity $\log(q^2)$ is subtracted to make the argument of the logarithm dimensionless. Clearly, this does not affect the ϕ-dependent part of $U_k(\phi)$. As we will see in chapters 3, 4 and 5, in the case of gravitational theories no such an arbitrary subtraction is needed since, thanks to the Fradkin - Vilkovisky path integral measure, the fluctuation determinants turn out to be automatically dimensionless.

at the scale Λ is the boundary condition for the RG equation (1.91), while at $k \sim 0$ the effective potential $V_{\text{eff}}(\phi)$ defined in section 1.1.1 (see Eq. (1.25)) is recovered.

Let us now translate the RG equation (1.91) into a set of differential RG equations for the couplings (mass, self-interaction coupling, ...). We begin by expanding the potential $U_k(\phi)$ in powers of ϕ as (we consider the case of discrete Z_2 symmetry under $\phi \to -\phi$)

$$U_k(\phi) = \Omega_k + \frac{1}{2} m_k^2 \phi^2 + \frac{\lambda_k}{4!} \phi^4 + \sum_{n \geq 3} \frac{\lambda_k^{(2n)}}{n!} \phi^{2n} \,, \tag{1.92}$$

the coefficients of this expansion being the couplings of the theory. Inserting (1.92) in (1.91), we obtain the following (infinite) set of coupled differential equations

$$k \frac{\partial \Omega_k}{\partial k} \equiv \beta_{\Omega_k} = -\frac{N_D k^D}{2} \log \left(\frac{k^2 + m_k^2}{k^2} \right) \tag{1.93}$$

$$k \frac{\partial m_k^2}{\partial k} \equiv \beta_{m_k^2} = -\frac{N_D k^D}{2} \frac{\lambda_k}{k^2 + m_k^2} \tag{1.94}$$

$$k \frac{\partial \lambda_k}{\partial k} \equiv \beta_{\lambda_k} = -\frac{N_D k^D}{2} \left(\frac{\lambda_k^{(6)}}{k^2 + m_k^2} - \frac{3\lambda_k^2}{(k^2 + m_k^2)^2} \right) \tag{1.95}$$

$$k \frac{\partial \lambda_k^{(6)}}{\partial k} \equiv \beta_{\lambda_k^{(6)}} = -\frac{N_D k^D}{2} \left(\frac{\lambda_k^{(8)}}{k^2 + m_k^2} - \frac{15\lambda_k \, \lambda_k^{(6)}}{(k^2 + m_k^2)^2} + \frac{30\lambda_k^3}{(k^2 + m_k^2)^3} \right) \tag{1.96}$$

$$\ldots, \tag{1.97}$$

where β_i are the so-called beta functions that determine the running of the couplings, i.e. how they evolve as functions of k. Equations (1.93)-(1.97) are the Wilsonian RG equations for the couplings in the LPA. In chapter 4, similar equations will be derived for the running cosmological and Newton constant (see Eqs. (4.26) and (4.27)).

From Eqs. (1.93)-(1.97), it is evident that all the couplings $\lambda^{(2n)}$ contribute to the evolution of U_k, and there is no a priori distinction between renormalizable and non-renormalizable couplings. In the RG framework, the classification of the operators is made in terms of the analysis of the relevant, marginal and irrelevant directions in the vicinity of a "fixed point" of the RG transformation in the parameter space (see below).

Fixed points are defined as follows. Starting from the dimensionful parameters in (1.92), we consider the corresponding dimensionless couplings $g^{(2)}(k) \equiv k^{-2} m_k^2$, $g^{(4)}(k) \equiv k^{D-4} \lambda_k$, $g^{(6)}(k) \equiv k^{2(D-3)} \lambda_k^{(6)}$, ... The RG equations to which they obey are easily obtained starting from (1.93)-(1.97). From now on, we truncate the parameter space to n couplings only, and use the vectorial notation $\vec{g} \equiv (g^{(2)}, g^{(4)}, \ldots, g^{(2n)}) = g^{(i)} \vec{e}_i$, with $\{\vec{e}_i\}$ the canonical basis. Indicating with R the RG transformation, a fixed point is defined as a point in the parameter space such that

$$R(\vec{g}_*) = \vec{g}_*. \tag{1.98}$$

In the vicinity of a fixed point, the RG equations can (often but not always) be linearized[11].

[11] Marginal couplings (see the definition below (1.104)) around the Gaussian fixed point are an example of a case where the linearization of the beta functions is not enough to study the behaviour of the RG flow around the fixed point since the linear term in the expansion vanishes and the leading non-vanishing contribution is provided by the quadratic term.

We have

$$k\frac{d}{dk}(\vec{g}(k)-\vec{g}_*) = R^{(l)}\Big|_{\vec{g}=\vec{g}_*}\cdot(\vec{g}(k)-\vec{g}_*)\,, \tag{1.99}$$

where $R^{(l)}_{ij}$ is the linearization matrix defined as

$$R^{(l)}_{ij} = \frac{\partial\widetilde{\beta}^i}{\partial g^{(j)}}\,, \tag{1.100}$$

with the $\widetilde{\beta}$ being the equivalent for the $g^{(n)}$ of the β functions defined in (1.93)-(1.97). From (1.99), we see that the behavior of the couplings in the vicinity of the fixed point $\vec{g}_*$ is determined by the eigenvalues of $R^{(l)}|_{\vec{g}=\vec{g}_*}$.

Let us indicate with M the matrix $R^{(l)}|_{\vec{g}=\vec{g}_*}$, with $\vec{u}_i$ its eigenvectors and with $y^{(i)}$ its eigenvalues (no sum over the index i in the equation below),

$$M\,\vec{u}_i = y^{(i)}\,\vec{u}_i\,. \tag{1.101}$$

The eigenvectors $\vec{u}_i$ form a basis, and we can expand the difference $\vec{g}(k)-\vec{g}_*$ as

$$\vec{g}(k)-\vec{g}_* = v^{(i)}(k)\,\vec{u}_i\,. \tag{1.102}$$

Inserting (1.102) in (1.99), we obtain a system of differential equations for the coefficients $v^{(i)}(k)$ (again, no sum over i in the equation below),

$$k\frac{d}{dk}v^{(i)}(k) = y^{(i)}v^{(i)}(k)\,, \tag{1.103}$$

whose solution is (k_0 is an arbitrary value of k)

$$v^{(i)}(k) = v^{(i)}(k_0)\left(\frac{k}{k_0}\right)^{y^{(i)}}\,,\quad i=1,2,...,n\,. \tag{1.104}$$

The $v^{(i)}(k)$ are the scaling parameters, and they allow to define relevant and irrelevant directions around the fixed point $\vec{g}_*$.

Following the flow toward the IR (i.e. for decreasing values of k, that is how the coarse graining is defined), the direction given by the eigenvector $\vec{u}_i$ is said to be (IR) relevant if the corresponding eigenvalue $y^{(i)}$ is negative. In this case, in fact, while moving toward the IR, $v^{(i)}(k)$ increases. If $y^{(i)}$ is positive, $v^{(i)}(k)$ decreases for $k\to 0$ and the corresponding eigendirection is called (IR) irrelevant. It is also possible that M possesses null eigenvalues. In this case, the corresponding eigenvectors are called marginal directions. To study the behavior of the RG flow along these directions, one has to go beyond the linear approximation. This allows to classify them as marginal (weakly) IR relevant or marginal (weakly) IR irrelevant directions.

Let S be the matrix that diagonalizes M, i.e. $S^{-1}MS = Y = diag\,(y^{(1)}, y^{(2)}, ..., y^{(n)})$. We then have $\vec{u}_i = S_{ij}\,\vec{e}_i$, and from (1.102) the original couplings $g^{(i)}(k)$ are obtained as a linear combination of the $v^{(i)}(k)$

$$g^{(i)}(k) = g^{(i)}_* + S_{ij}\,v^{(i)}(k)\,. \tag{1.105}$$

From the above equation we see that the IR ($k\to 0$) flow of the $g^{(i)}(k)$ in the vicinity of the fixed point $\vec{g}_*$ is dominated by the relevant (negative eigenvalues) eigendirections of M.

For the coupling $g_k^{(i)}$ not to explode while moving toward the IR, the $v^{(i)}(k)$ corresponding to such directions need to be fine-tuned. In other words, the IR flow of the couplings $g_k^{(i)}$ is sensible to the precise value of the $v^{(i)}(k)$ corresponding to relevant eigendirections. On the contrary, concerning the irrelevant eigendirections, the system is not really sensible to the precise value of the corresponding $v^{(i)}(k)$, since their contribution is more and more suppressed as k decreases.

The analysis presented so far is reverted if we consider the flow in the opposite direction, i.e. the flow toward the UV (increasing values of k). Following the RG flow in this direction, the role of positive and negative eigenvalues is opposite to the one seen above. In fact, eigendirections with negative eigenvalues are UV irrelevant, while those with positive eigenvalues are UV relevant.

From the geometrical point of view, given a fixed point $\vec{g}_*$ it is possible to define the IR critical surface $\Sigma_{\rm IR}$ as the locus of points in the parameter space $(g^{(1)}, g^{(2)}, ..., g^{(n)})$ that asymptotically reach $\vec{g}_*$ for $k \to 0$. Similarly, the UV critical surface $\Sigma_{\rm UV}$ is defined as the locus of points that asymptotically reach $\vec{g}_*$ for $k \to \infty$.

A final observation is that for the Gaussian fixed point (i.e. $\vec{g}_* = \vec{0}$) the classification of the eigendirections presented above coincides with the usual perturbative classification. Actually, it gives a stronger motivation for it [68]. The RG method, however, goes further and also captures other regimes where couplings enjoy different, non-perturbative behaviors.

As anticipated above, the RG strategy introduced in this section will be implemented in chapter 4 (where the results of [23] are presented) for the case of pure quantum gravity. Considering the Einstein-Hilbert truncation for the running gravitational action, we will derive the RG equations for the Newton and cosmological constant. We will pay due attention to two aspects often overlooked in the literature, namely the use of the appropriate path integral measure, and a careful introduction of the running scale k. This will lead us to beta functions that profoundly differ from those of previous literature and that, in particular, do not possess the non-trivial UV-attractive fixed point of the AS scenario.

The next section contains some notions on field theories with compact extra dimensions, that are at the basis of the analysis performed in chapter 2 (based on [12, 13]). There, we calculate the radiative correction to the vacuum energy in a QFT with one compact extra dimension and show that usual calculations miss divergent contributions that do not cancel even in the presence of supersymmetry.

1.5 Higher-dimensional QFTs

Let us consider a 5D supersymmetric theory with one compact dimension in the shape of a circle S^1 of radius R, and let $\widehat{\Phi}(x,z)$ and $\widehat{\psi}(x,z)$ be a 5D boson and fermion field respectively (x are the spacetime coordinates in $\mathcal{M}_4$, or $\mathbb{R}^4$ after Euclideanization, while z is the one along the compact dimension). It is possible to consider boundary conditions of the kind

$$\widehat{\Phi}(x, z+2\pi R) = e^{i2\pi R q_b}\, \widehat{\Phi}(x,z) \quad ; \quad \widehat{\psi}(x, z+2\pi R) = e^{i2\pi R q_f}\, \widehat{\psi}(x,z)\,, \tag{1.106}$$

where q_b and q_f are the so-called R-symmetry charges or the periodicities of the boson and of the fermion, respectively. The fields $\widehat{\Phi}$ and $\widehat{\psi}$ can be expanded in Fourier series as

$$\widehat{\Phi}(x,z) = \frac{1}{2\pi R} \sum_{n=-\infty}^{+\infty} \int \frac{d^4p}{(2\pi)^4}\, \widehat{\phi}_{n,p}\, e^{i\left(p\cdot x + \left(\frac{n}{R}+q_b\right)z\right)} \equiv \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \phi_n(x) e^{i(\frac{n}{R}+q_b)z} \quad (1.107)$$

$$\widehat{\psi}(x,z) = \frac{1}{2\pi R} \sum_{n=-\infty}^{+\infty} \int \frac{d^4p}{(2\pi)^4}\, \widehat{\psi}_{n,p}\, e^{i\left(p\cdot x + \left(\frac{n}{R}+q_f\right)z\right)} \equiv \frac{1}{\sqrt{2\pi R}} \sum_{n=-\infty}^{+\infty} \psi_n(x) e^{i(\frac{n}{R}+q_f)z}\,, \quad (1.108)$$

where $\phi_n(x)$ and $\psi_n(x)$ are 4D fields called the Kaluza-Klein fields. What is usually done at this point is to insert the above expansions in the 5D action and integrate over z to get an effective 4D action. In this way, the original 5D fields give rise to infinite towers of 4D fields ($\phi_n(x)$ and $\psi_n(x)$) whose masses are given by

$$m_b^2 = \left(\frac{n}{R}+q_b\right)^2 \qquad\qquad m_f^2 = \left(\frac{n}{R}+q_f\right)^2, \quad (1.109)$$

where q_b and q_f are the boundary charges introduced above. Four-dimensional fields, such as the Higgs boson, can be identified with the zero mode of these towers (see for instance [69, 70]). An important observation is that, according to the expansions (1.107) and (1.108), n/R is essentially the component of the 5D momentum along the compact dimension ($m_{\rm KK} \equiv R^{-1}$ is the scale of the KK tower) [71].

From (1.109), we see that taking different values for the boundary charges of the superpartners induces a breaking of supersymmetry at the level of the 4D theory. The masses of the KK fields arising from the 5D superpartners, in fact, are different in the corresponding 4D theory. This is the so-called Scherk-Schwarz breaking of supersymmetry [6, 7].

In the late '90s, the Higgs effective potential was calculated considering supersymmetric extensions of the Standard Model with one extra dimension compactified on the orbifold S_1/Z_2 (where S_1 is a circle of radius R) and Scherk-Schwarz SUSY breaking. The result turned out to be automatically finite [69,70]. This was considered a quite important outcome as it would have provided the solution to the long-standing Higgs naturalness problem. To understand how this finite result was obtained, in the next section we consider a simple model with two 5D scalar fields and illustrate how the 4D one-loop Higgs effective potential $V_{1l}(\phi)$ (the 4D field ϕ being identified with the zero KK mode of a 5D scalar field) is usually derived from the higher-dimensional theory.

In section 1.5.2, we consider the calculation of the 5D one-loop effective potential and discuss its relation to the 4D effective potential. This analysis shows that the automatically finite (no fine-tuning) result for $V_{1l}(\phi)$ often advocated in the literature comes from a mistreatment of the 5D loop momentum asymptotics [71]. When the latter are properly treated, UV-sensitive terms are present in $V_{1l}(\phi)$ that do not cancel even in SUSY models since they are proportional to the boundary charges of superpartners.

1.5.1 One-loop potential: usual calculation

Let us consider the (Euclidean) action in $4+1$ dimensions (the compact extra dimension is in the shape of a circle of radius R and $a = 1, \ldots, 5$),

$$\mathcal{S}_{(5)}[\widehat{\Phi},\widehat{\chi}] = \int d^4x\, dz \left(\frac{1}{2}\partial_a\widehat{\Phi}\,\partial^a\widehat{\Phi} + \partial_a\widehat{\chi}\,\partial^a\widehat{\chi}^\dagger + \widehat{M}^2(\widehat{\Phi})\widehat{\chi}\widehat{\chi}^\dagger\right), \quad (1.110)$$

where $\widehat{\Phi}(x,z)$ and $\widehat{\chi}(x,z)$ are 5-dimensional scalar fields (real and complex respectively), and $\widehat{M}^2(\widehat{\Phi}) \equiv \widehat{m}^2 + f(\widehat{\Phi})$, where $\widehat{m}$ is the mass of $\widehat{\chi}$ and $f(\widehat{\Phi})\,\widehat{\chi}\widehat{\chi}^\dagger$ the interaction between $\widehat{\Phi}$ and $\widehat{\chi}$. Since $\widehat{\Phi}$ is a real scalar field, along the compact dimension we have

$$\widehat{\Phi}(x, z + 2\pi R) = \widehat{\Phi}(x,z)\,. \tag{1.111}$$

For the complex field $\widehat{\chi}$, instead, it is possible to consider the non-trivial boundary condition

$$\widehat{\chi}(x, z + 2\pi R) = e^{i2\pi R q_\chi}\widehat{\chi}(x,z)\,. \tag{1.112}$$

Accordingly, the fields admit the following Fourier expansions (see also (1.107) and (1.108))

$$\begin{aligned}
\widehat{\Phi}(x,z) &= \frac{1}{2\pi R}\sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\widehat{\Phi}_{n,p}\,e^{i\left(p\cdot x + \frac{n}{R}z\right)} \equiv \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{+\infty}\phi_n(x)e^{i\frac{n}{R}z}\\
\widehat{\chi}(x,z) &= \frac{1}{2\pi R}\sum_{n=-\infty}^{+\infty}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\,\widehat{\chi}_{n,p}\,e^{i\left(p\cdot x + \left(\frac{n}{R}+q_\chi\right)z\right)} \equiv \frac{1}{\sqrt{2\pi R}}\sum_{n=-\infty}^{+\infty}\chi_n(x)e^{i\left(\frac{n}{R}+q_\chi\right)z}\,.
\end{aligned} \tag{1.113}$$

Clearly, in realistic phenomenological applications additional fermion and boson fields (with supersymmetry explicitly implemented) are considered. The 4D one-loop Higgs effective potential $V_{1l}(\phi)$ is usually calculated inserting the expansions (1.113) in the 5D action (1.110), and integrating over the compact dimension z. In this way, we get a dimensionally reduced 4D action for the KK fields $\phi_n(x)$ and $\chi_n(x)$, and the 4D one-loop Higgs effective potential is obtained summing up the loop contributions [72] from the infinitely many KK fields. When the Higgs field is identified, for instance, with the zero KK mode of a 5D scalar field, as $\widehat{\Phi}$ in (1.110) (see for instance [69,70]), for $V_{1l}(\phi)$ one has

$$V_{1l}(\phi) = \frac{1}{2}\sum_a\sum_{i_a}(-1)^{\delta_{i_a,f_a}}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\log\left(p^2 + M_a^2(\phi) + \left(\frac{n}{R}+q_{i_a}\right)^2\right)\,, \tag{1.114}$$

where the index a runs over the families of bosons (b) and fermions (f) of the considered model, $M_a^2(\phi)$ is the field dependent mass of each family that interacts with the field ϕ, and $i_a \equiv b_a, f_a$ indicates the boson or the fermion partner in the family.

The usual way to perform the calculation in (1.114) is to sum over the infinite tower of KK-modes, and perform the integration over the four-momentum p introducing a cutoff Λ. As shown below, in this way power-like divergences proportional to $M_a^2(\phi)$ arise, which all cancel in a supersymmetric theory leaving us with the aforementioned finite result for $V_{1l}(\phi)$ (and obviously with a finite result also for the Higgs mass $m_{\rm H}^2$).

To better appreciate this point, let us calculate explicitly $V_{1l}(\phi)$. From (1.114), we see that it is enough to consider a generic bosonic contribution $V_{1l}^b(\phi)$. To make the argument of the logarithm dimensionless, a normalization term $\log\,(p^2 + \frac{n^2}{R^2})$ is subtracted to $\log\,(p^2 + M_a^2 + (\frac{n}{R} + q_{b_a})^2)$ that does not change the ϕ-dependent part of the potential. Following the strategy outlined above, that is the one usually adopted in the literature, we perform the infinite sum over n and the integral over p in (1.114) separately, carrying out the former first. A simple inspection of (1.114) shows that this calculation is possible only if a cutoff Λ is introduced for each of the 4D loop integrals over p (whatever order is considered to perform the infinite sum and the integral). Resorting to the Schwinger identity for the logarithm, we can then write[12] (below we use the streamlined notation $M^2 \equiv M_a^2$ and $q_b \equiv q_{b_a}$):

$$V_{1l}^b(\phi) = -\frac{1}{2}\int^{\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4}\int_0^{\infty}\frac{ds}{s}\sum_{n=-\infty}^{\infty}\left[e^{-s(R^2(p^2+M^2)+(n+R\,q_b)^2)} - e^{-s\,(p^2R^2+n^2)}\right]. \quad (1.115)$$

With the help of the Poisson summation formula we obtain

$$V_{1l}^b(\phi) = -\frac{\sqrt{\pi}}{2}\int^{\Lambda}\frac{\mathrm{d}^4 p}{(2\pi)^4}\int_0^{\infty}\frac{ds}{s^{3/2}}\left[\vartheta_3\left(\pi R\,q_b, e^{-\frac{\pi^2}{s}}\right)e^{-sR^2(p^2+M^2)} - \vartheta_3\left(0, e^{-\frac{\pi^2}{s}}\right)e^{-sR^2p^2}\right], \quad (1.116)$$

where

$$\vartheta_3(x,y) = 1 + 2\sum_{k=1}^{\infty}\cos(2kx)y^{k^2}\,. \quad (1.117)$$

Performing the integrations over p and s, and expanding for large Λ (neglecting all the terms suppressed in Λ), for $V_{1l}^b(\phi)$ we get (the details of the calculation can be found in [71], see also [69, 70])

$$\begin{aligned} V_{1l}^b(\phi) &= R\left(\frac{\Lambda^3 M^2}{48\pi} - \frac{\Lambda M^4}{64\pi} + \frac{M^5}{60\pi}\right) + \frac{3\zeta(5)}{64\pi^6 R^4} - \sum_{k=1}^{\infty}\frac{e^{-2\pi k M R}(2\pi k M R(2\pi k M R+3)+3)\cos(2\pi k R q_b)}{64\pi^6 k^5 R^4} \\ &\equiv R\left(\frac{\Lambda^3 M^2}{48\pi} - \frac{\Lambda M^4}{64\pi} + \frac{M^5}{60\pi}\right) + \frac{3\zeta(5)}{64\pi^6 R^4} - \frac{U(r_b,x)}{128\pi^6 R^4} \end{aligned} \quad (1.118)$$

where

$$U(r_b, x) \equiv x^2 \mathrm{Li}_3\left(r_b e^{-x}\right) + 3x\mathrm{Li}_4\left(r_b e^{-x}\right) + 3\mathrm{Li}_5\left(r_b e^{-x}\right) + h.c.\,, \quad (1.119)$$

with

$$r_b \equiv e^{2\pi i R q_b} \qquad , \qquad x \equiv 2\pi R\sqrt{M^2(\phi)}, \quad (1.120)$$

and $\mathrm{Li}_i(x)$ the Polylogarithm functions.

The contribution to $V_{1l}(\phi)$ from the corresponding fermion superpartner is the same as (1.118), but with q_b replaced by q_f (and then r_b by r_f) and an overall minus sign. Combining these two contributions, the first and second term in the right-hand side of (1.118) cancel the corresponding ones from the fermion superpartner, and for each couple (b, f) we are left with two finite contributions to $V_{1l}(\phi)$. The potential (1.114) becomes

$$V_{1l}(\phi) = -\sum_a \frac{U(r_{b_a}, x) - U(r_{f_a}, x)}{128\pi^6 R^4}\,. \quad (1.121)$$

Eq. (1.121) is the well-known aforementioned UV-insensitive (finite) result for $V_{1l}(\phi)$ (see [69, 70]). An important point to stress for the successive discussions is that within

[12]Note that, since we introduced the cutoff Λ for the d^4p integrals, the lower extreme of the proper-time integral does not need to be replaced with a UV cutoff. It has to be kept equal to zero; Schwinger's parametrization is used here just as an identity.

the calculation strategy shown above the terms that contain the charges q_i give rise only to finite contributions, more precisely to oscillatory functions of the q_i (see (1.121)).

To summarize, the usual approach to calculate the one-loop Higgs potential $V_{1l}(\phi)$ from a higher-dimensional theory with a compact dimension consists in considering the decomposition of the 5D fields in terms of KK-modes, obtaining the dimensionally reduced 4D action after integration over the compact dimension z, and finally calculating $V_{1l}(\phi)$ summing up the Coleman-Weinberg contributions from the infinitely many fields of the KK-towers.

As pointed out in [71], however, this approach suffers of a theoretical issue: performing the infinite sum over the KK-modes while introducing a cutoff in the integration over the four momentum p mistreats the asymptotics of the five-dimensional loop momentum $p^{(5)}$. In fact, the KK masses introduced above are nothing but the fifth component of $p^{(5)}$ (see Eqs. (1.107) - (1.109) and comments therein), and the sum over n should then be cut coherently together with the integral over the four momentum p. To better appreciate this point, in the next section we consider a 5D theory with a real scalar field $\widehat{\Phi}(x,z)$ interacting with a complex scalar field $\widehat{\chi}(x,z)$, and calculate the 5D one-loop potential $\mathcal{V}^{(5D)}(\widehat{\Phi})$. Successively, considering the KK-decomposition of $\widehat{\Phi}(x,z)$ and $\widehat{\chi}(x,z)$, and choosing $\phi_0(x)$ as the 4D Higgs field, we calculate the 4D one-loop potential $V_{1l}(\phi_0)$ and establish its relation with $\mathcal{V}^{(5D)}(\widehat{\Phi})$.

1.5.2 Cutting the sum over the KK modes

Let $\widehat{\Phi}$ and $\widehat{\chi}$ be a real and a complex 5D scalar field respectively (the compact dimension is in the shape of a circle S^1 of radius R), with 5D (Euclidean) action ($a = 1, \ldots, 5$)

$$\mathcal{S}_{(5)}[\widehat{\Phi},\widehat{\chi}] = \int \mathrm{d}^4x\, \mathrm{d}z \left(\frac{1}{2}\,\partial_a\widehat{\Phi}\,\partial^a\widehat{\Phi} + \partial_a\widehat{\chi}\,\partial^a\widehat{\chi}^\dagger + \frac{m_\Phi^2}{2}\,\widehat{\Phi}^2 + m_\chi^2\,\widehat{\chi}\widehat{\chi}^\dagger + \frac{\widehat{\lambda}}{4!}\,\widehat{\Phi}^4 + \frac{\widehat{g}}{2}\,\widehat{\Phi}^2\widehat{\chi}\widehat{\chi}^\dagger \right). \tag{1.122}$$

Taking for $\widehat{\Phi}(x,z)$ and $\widehat{\chi}(x,z)$ the boundary conditions (see previous section)

$$\widehat{\Phi}(x, z + 2\pi R) = \widehat{\Phi}(x,z) \quad ; \quad \widehat{\chi}(x, z+2\pi R) = e^{2i\pi R\, q}\,\widehat{\chi}(x,z)\,, \tag{1.123}$$

we consider their Fourier expansions. Since the fifth dimension is compact, and its size R is much smaller than the size of the other four dimensions, the 5D momentum $p^{(5)}$ is [13] (n integer) [71]

$$p^{(5)} \equiv (p_1, p_2, p_3, p_4, n/R) \equiv (p, n/R)\,. \tag{1.124}$$

To investigate on the UV-sensitivity of the dimensionally reduced Higgs one-loop potential $V_{1l}(\phi)$, we consider first the 5D one-loop potential $\mathcal{V}^{(5D)}(\widehat{\Phi})$ (a constant background $\widehat{\Phi} = \text{const}$ is taken for the field $\widehat{\Phi}$)

$$\begin{aligned}\mathcal{V}_{1l}^{(5D)}(\widehat{\Phi}) &= \frac{1}{2}\mathrm{Tr}_5 \log \frac{p^2 + \frac{n^2}{R^2} + m_\Phi^2 + \frac{\widehat{\lambda}}{2}\,\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}} + \frac{1}{2}\mathrm{Tr}_5 \log \frac{p^2 + \left(\frac{n}{R}+q\right)^2 + m_\chi^2 + \frac{\widehat{g}}{2}\,\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}} \\ &= \frac{1}{4\pi R}\sum_n \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\log \frac{p^2 + \frac{n^2}{R^2} + m_\Phi^2 + \frac{\widehat{\lambda}}{2}\,\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}} + \log \frac{p^2 + \left(\frac{n}{R}+q\right)^2 + m_\chi^2 + \frac{\widehat{g}}{2}\,\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}} \right), \end{aligned} \tag{1.125}$$

[13] In the case $q \neq 0$, the fifth component of $p^{(5)}$ is $\left(\frac{n}{R} + q\right)$.

where the subscript "5" indicates that the trace is calculated in the 5D Fourier space.

Some comments are in order. As said above, $p^{(5)} = (p, n/R)$ that appears in (1.125) is the 5D loop momentum. Therefore, the sum over n and the integral over p are intrinsically intertwined. This is a crucial point: in (1.125) one cannot take the asymptotics of one component of $p^{(5)}$ without considering also the asymptotics of the other components. In particular, n cannot be sent to infinity independently[14] of p, and the regularization in the right-hand side of (1.125) has to be implemented over the full momentum $p^{(5)}$. From the physical point of view, this means that the model under consideration is a 5D effective theory valid up to a maximal energy scale Λ, and the modulus of $p^{(5)}$ should not exceed this scale,

$$(p^{(5)})^2 = p^2 + n^2/R^2 \leq (p^{(5)}_{\max})^2 \equiv \Lambda^2 \,. \tag{1.126}$$

Accordingly, the sum over n and the integral over p in (1.125) (that are originally infinite) have to be replaced by [71]

$$\sum_n \int \frac{\mathrm{d}^4 p}{(2\pi)^5 R} \to \left(\sum_n \int \frac{\mathrm{d}^4 p}{(2\pi)^5 R} \right)' \equiv \frac{1}{2\pi R} \sum_{n=-[R\Lambda]}^{[R\Lambda]} \int^{C^n_\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \,, \tag{1.127}$$

where

$$C^n_\Lambda = \sqrt{\Lambda^2 - \frac{n^2}{R^2}} \,, \tag{1.128}$$

and $[R\Lambda]$ is the integer part of $R\Lambda$.

These considerations are better formulated in the Wilsonian framework introduced in section 1.4, that clearly applies also to theories with compact extra dimensions. Since (as said above) the 5D model under consideration is not a UV-complete theory, but it is rather an effective theory valid up to a certain UV scale Λ, only the Fourier components such that $p^{(5)}$ obeys the condition (1.126) must be included in the expansions of the 5D fields $\widehat{\Phi}(x, z)$ and $\widehat{\chi}(x, z)$,

$$\begin{aligned}
\widehat{\Phi}(x, z) &= \left(\sum_n \int \frac{\mathrm{d}^4 p}{(2\pi)^5 R} \right)' \widehat{\Phi}_{n,p} \, e^{i\left(p\cdot x + \frac{n}{R} z\right)} = \frac{1}{\sqrt{2\pi R}} \sum_{n=-R\Lambda}^{R\Lambda} \phi^\Lambda_n(x) \, e^{i \frac{n}{R} z} \\
\widehat{\chi}(x, z) &= \left(\sum_n \int \frac{\mathrm{d}^4 p}{(2\pi)^5 R} \right)' \widehat{\chi}_{n,p} \, e^{i\left(p\cdot x + \left(\frac{n}{R} + q\right) z\right)} = \frac{1}{\sqrt{2\pi R}} \sum_{n=-R\Lambda}^{R\Lambda} \chi^\Lambda_n(x) \, e^{i\left(\frac{n}{R} + q\right) z} \,,
\end{aligned} \tag{1.129}$$

where $\phi^\Lambda_n(x)$ and $\chi^\Lambda_n(x)$ are defined through the relations

$$\phi^\Lambda_n(x) \equiv \frac{1}{\sqrt{2\pi R}} \int^{C^n_\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \widehat{\Phi}_{n,p} \, e^{ip\cdot x} \quad ; \quad \chi^\Lambda_n(x) \equiv \frac{1}{\sqrt{2\pi R}} \int^{C^n_\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \widehat{\chi}_{n,p} \, e^{ip\cdot x} \,. \tag{1.130}$$

Therefore, in the calculation of the fluctuation determinants that appear in the one-loop potential $\mathcal{V}^{(5D)}(\widehat{\Phi})$, only eigenvalues of the fluctuation operator that satisfy (1.126) have to be considered, i.e. we have to make in (1.125) the replacement (1.127).

Let us consider now the calculation of the 4D effective potential. According to what said above, and differently to what is done in usual calculations, we have to insert in

[14] Note that this is the procedure followed in the usual calculation of the four-dimensional $V_{1l}(\phi)$ (see section 1.5.1).

the action (1.122) the cut Fourier expansions (1.129), rather than infinite expansions like (1.113). Choosing (similarly to what is done in section 1.5.1) $\phi_0^\Lambda(x)$ as the 4D Higgs field, and integrating over the compact variable z, we get a 4D action that contains a *finite* number of Kaluza-Klein fields $\phi_n^\Lambda(x)$ and $\chi_n^\Lambda(x)$. Taking a constant value ϕ of $\phi_0^\Lambda(x)$, for $V_{1l}(\phi)$ we obtain

$$V_{1l}(\phi) = \frac{1}{2}\sum_{n=-R\Lambda}^{R\Lambda}\int^{C_\Lambda^n}\frac{\mathrm{d}^4p}{(2\pi)^4}\log\frac{\left(p^2+\frac{n^2}{R^2}+m_\phi^2+\frac{\lambda}{2}\phi^2\right)\left(p^2+\left(\frac{n}{R}+q\right)^2+m_\chi^2+\frac{g}{2}\phi^2\right)}{\left(p^2+\frac{n^2}{R^2}\right)^2}\,, \tag{1.131}$$

where the 4D couplings λ and g are defined in terms of the 5D couplings $\widehat{\lambda}$ and $\widehat{g}$ (see (1.125)) through the relations

$$\lambda \equiv \frac{\widehat{\lambda}}{2\pi R} \quad ; \quad g \equiv \frac{\widehat{g}}{2\pi R}\,, \tag{1.132}$$

and the combinations

$$m_{\phi,n}^2 \equiv m_\phi^2 + \frac{n^2}{R^2} \quad ; \quad m_{\chi,n}^2 \equiv m_\chi^2 + \left(\frac{n}{R}+q\right)^2 \tag{1.133}$$

are the so called KK-masses of the 4D fields $\phi_n^\Lambda(x)$ and $\chi_n^\Lambda(x)$. Moreover, from (1.129) and (1.130) we have that the constant value $\widehat{\Phi}$ of the 5D field $\widehat{\Phi}(x,z)$ in (1.125) is related to the constant value ϕ of the 4D field $\phi_0^\Lambda(x)$ in (1.131) as

$$\widehat{\Phi} = \frac{1}{\sqrt{2\pi R}}\,\phi\,. \tag{1.134}$$

This shows that the 4D one-loop Higgs potential $V_{1l}(\phi)$ (1.131) (and not (1.114)) is obtained from the one-loop potential $\mathcal{V}_{1l}^{(5D)}(\widehat{\Phi})$ (1.125) of the original 5D theory multiplying the latter by a factor of $2\pi R$,

$$V_{1l}(\phi) = 2\pi R\,\mathcal{V}_{1l}^{(5D)}(\widehat{\Phi})\,. \tag{1.135}$$

The explicit calculation of (1.131) has been performed in [71]. For a bosonic contribution we have (see (1.119) and (1.120) for the definition of $U(r_b,x)$)

$$\begin{aligned} V_{1l}(\phi) = {} & \frac{5M^2(\phi)+3q_b^2}{180\pi^2}R\Lambda^3 - \frac{35M^4(\phi)+14M^2(\phi)q_b^2+3q_b^4}{840\pi^2}R\Lambda + \frac{M^5(\phi)R}{60\pi} \\ & + \frac{3\zeta(5)}{64\pi^6R^4} - \frac{U(r_b,x)}{128\pi^6R^4} + \mathcal{O}(\Lambda^{-1})\,. \end{aligned} \tag{1.136}$$

A comparison between (1.136) and (1.118) shows that (as anticipated above), when all the components of the five-dimensional loop momentum are coherently cut, new UV-sensitive terms arise in $V_{1l}(\phi)$ in addition to the divergent contributions already present in (1.118). From the first line of the above equation we see that these new terms are all proportional to powers of the boundary charge q_b. It is important to stress that the presence of these terms is entirely due to the non-trivial topology of the manifold $\mathbb{R}^4\times S^1$ on which the fields are defined, that is not simply connected. In fact, this opens the door to the possibility of considering non-trivial boundary conditions for the fields as those in (1.112). This holds true even when the size of the 4D box and the radius R are comparable, in which case the sum over the fifth component of the momentum can be replaced with an integral (as for the other four components). Therefore, these previously missed q-dependent UV-sensitive terms are a consequence of the non-trivial topology of spacetime, and are always present when non-trivial boundary conditions are realized.

Consider now a SUSY theory with Scherk-Schwarz SUSY breaking as in the previous section. Summing up the contributions from the different families of superpartners one has (below Λ-suppressed terms are discarded)

$$V_{1l}(\phi) = \sum_a \left[\frac{q_{b_a}^2 - q_{f_a}^2}{60\pi^2} R\Lambda^3 - \frac{q_{b_a}^4 - q_{f_a}^4}{280\pi^2} R\Lambda - \frac{q_{b_a}^2 - q_{f_a}^2}{60\pi^2} M^2(\phi)\, R\Lambda - \frac{U(r_{b_a}, x) - U(r_{f_a}, x)}{128\pi^6 R^4} \right]. \quad (1.137)$$

From the above equation we see that, besides the first two ϕ-independent terms, $V_{1l}(\phi)$ is given by the sum of the usual finite result (the last term of (1.137), to be compared with (1.121)) and a new UV-sensitive term proportional to the difference $q_{b_a}^2 - q_{f_a}^2$ between the squared boundary charges of the superpartners of a given family, that does not vanish since we must have $q_{b_a} \neq q_{f_a}$ to trigger the Scherk-Schwarz SUSY breaking (cases with $q_{b_a} = q_{f_a}$, i.e. unbroken SUSY, are not physically relevant). Therefore, while the UV-sensitive terms in (1.136) proportional to powers of $M^2(\phi)$ and independent of the q_i are canceled between superpartners, the same does not hold true for the UV-sensitive terms of the kind $q_i^2 M^2(\phi) R\Lambda$.

We have seen that the absence of the latter terms in the usual UV-insensitive result (1.121) for $V_{1l}(\phi)$ is due to an inconsistent limiting procedure for the components of the 5D loop momentum. The usual calculation illustrated in the previous section builds on different physical interpretations for p and n/R. The former is seen as the four-momentum of the KK-particles in the 4D theory, while the integer n as a counter of the *infinitely many* KK-fields, that gives their masses once it is divided by the size R of the compact extra dimension. From this 4D perspective, then, it seems that to calculate $V_{1l}(\phi)$ one has to sum up the infinitely many Coleman-Weinberg one-loop contributions from the KK-fields. However, from what we said in the present section, this amounts to erroneously consider the asymptotics of the fifth component p_5 of the 5D loop momentum independently of the asymptotics of p. In this respect, it is worth to note also that the interpretation of n/R as the mass of a KK particle has to be taken with a grain of salt [71] since, in truth, it is related to the fifth component of the loop momentum in the original 5D theory.

The results and considerations of the present section have been the starting point of the calculation of the vacuum energy performed in [12, 13] and presented in the next chapter. As said in the Introduction (and similarly to what we have seen above for the one-loop Higgs potential), usual calculations in SUSY theories with compact extra dimensions and Scherk-Schwarz SUSY breaking give the automatically finite result $\rho_{\text{vac}} \sim m_{\text{KK}}^4$, establishing that the vacuum energy is proportional to the fourth power of the KK scale m_{KK}. In the next chapter, we show that again such a finite result is due to a mistreatment of the asymptotics of the 5D loop momentum, and that, when the latter are correctly treated, previously missed (power-like) UV-sensitive terms proportional to the boundary charges of the fields appear in the radiative correction to ρ_{vac}. We will also discuss the impact of these results on the recent dark dimension proposal [10].

Chapter 2

Does the cosmological constant really indicate the existence of a dark dimension?

Theories with large compact extra dimensions were extensively explored in the nineties in search for a solution to the electroweak naturalness/hierarchy problem [73–75] (see also section 1.5). A recent surge of interest towards the physics of 5D effective field theories with one compact extra dimension of mesoscopic size has followed the "dark dimension" scenario [10] (see next section). Born in a string framework, it suggests that we might live in a universe with a single compact extra dimension, whose mesoscopic size is dictated by the measured value of the cosmological constant. This scenario is based on swampland conjectures (see footnote 1) and phenomenological bounds, which lead to the relation $\rho_{\rm swamp} \sim m_{\rm KK}^4$ between the vacuum energy $\rho_{\rm swamp}$ and the size of the extra dimension $m_{\rm KK}^{-1}$ ($m_{\rm KK}$ is the scale of a KK tower, see section 1.5), and also on the corresponding result $\rho_{\rm EFT}$ from the EFT limit.

Usual calculations of $\rho_{\rm EFT}$ lead to the automatically finite (no fine-tuning) result $\rho_{\rm EFT} \sim m_{\rm KK}^4$, giving the impression that the matching between $\rho_{\rm swamp}$ and $\rho_{\rm EFT}$, at the basis of the DD scenario, is trivially realized. In the present chapter, we show that this result for $\rho_{\rm EFT}$ comes from a mistreatment of the asymptotics of the five-dimensional loop momentum in the EFT limit. Performing a proper EFT calculation, we find that $\rho_{\rm EFT}$ contains previously missed power-like UV-sensitive terms that do not cancel even in SUSY theories. Their presence renders the matching between $\rho_{\rm swamp}$ and $\rho_{\rm EFT}$ a non-trivial issue, and a physical mechanism should be found to dispose of these additional strongly UV-sensitive contributions. We argue that such a mechanism should be provided by the piling up of quantum fluctuations operated by the UV → IR renormalization group flow that emanates from the ultimate theory of quantum gravity (maybe a string theory). This RG flow should connect the boundary (UV) value of the vacuum energy to its measured value in the IR.

The results of the present chapter have been published in [12, 13].

2.1 The dark dimension scenario

A generic feature of string theories is the existence of towers with an infinite number of states, whose masses are given in terms of a scale μ_{tow}. According to the swampland distance conjecture [76], at large distance in the moduli field space ϕ one of the tower scales becomes exponentially small, $\mu_{tow} \sim e^{-\alpha|\phi|}$ (α positive $\mathcal{O}(1)$ constant). The DD proposal is related to this asymptotic regime. As stressed in [10], in these regions only two cases seem to arise in string compactifications: a tower of string excitation modes or a tower of Kaluza-Klein (KK) states, i.e. $\mu_{tow} \sim M_s, m_{\rm KK}$ (Emergent String Conjecture [77, 78]). In general, in d spacetime non-compact dimensions an infinite tower of states contributes to the vacuum energy ρ_d an amount $\rho_d \sim \mu_{tow}^d$ [79, 80]. As said in the Introduction, a similar result seems to hold even in the framework of higher-dimensional field theories with compact extra dimensions. This is for instance the case for supersymmetric theories with Scherk-Schwarz or brane-localized SUSY breaking [8, 9]. Only KK modes are present and the usual calculation gives $\rho_d \sim m_{\rm KK}^d$.

Going back to the string (quantum gravity) framework, when the distance conjecture is implemented in AdS spaces [81]

$$\mu_{tow} \sim |\widetilde{\Lambda}_{\rm cc}|^{\gamma} \,, \tag{2.1}$$

where $\widetilde{\Lambda}_{\rm cc}$ is the cosmological constant times the squared Planck mass M_P^2. Even though there is much wider support in AdS, the conjecture is nonetheless extended also to dS spaces, where it forms the basis for the dark dimension proposal [10]. Restricting to the $d = 4$ case, the one-loop string calculation of ρ_4 gives

$$\rho_4 \sim \mu_{tow}^4 \,. \tag{2.2}$$

The authors of [10] note that higher loops might only contribute with higher powers of μ_{tow}, so that (barring cancellation of the μ_{tow}^4 term) the comparison of (2.2) with (2.1) gives[1]

$$\gamma \geq \frac{1}{4} \,. \tag{2.3}$$

They assume (2.3) as starting point for their proposal. Moreover, observing that the experimental bounds on possible violations of the $1/r^2$ Newton's law [84] give $\mu_{tow} \geq 6.6$ meV, and that the energy scale associated to the measured value of $\widetilde{\Lambda}_{\rm cc}$ [85] is of the same order, $\widetilde{\Lambda}_{\rm cc}^{1/4} \sim 2.31$ meV, they infer that (2.1) is saturated with $\gamma = 1/4$, and accordingly the "experimental value" of μ_{tow} is

$$\mu_{tow}^{exp} \sim 2.31\,\text{meV} \tag{2.4}$$

(order the neutrino scale). Finally they observe that, although it is in principle possible that $\mu_{tow} = M_s$, Eq. (2.4) indicates that this option is "ruled out by experiments" since we know that physics above the neutrino scale is well described by effective field theories, and no sign of string excitations is observed at these scales. They then conclude that the only possibility left is an EFT decompactification scenario, with a Kaluza-Klein mass $m_{\rm KK} \sim \mu_{tow}^{exp} \sim 2.31\,\text{meV}$.

This conclusion takes us from the string theory realm to the EFT terrain, and is crucial to the formulation of the DD proposal. Typically, when physics is described in terms of a string KK tower, the original string theory is replaced by the corresponding higher-dimensional EFT with compact extra dimensions. A thorough analysis of this delicate step is one of the goals of the present chapter.

[1] They also refer to [82, 83] to further support the bound (2.3).

According to [86], the strongest bounds for the compactification scale $m_{_{KK}}^{-1}$ come from the heating of neutron stars due to the surrounding cloud of trapped KK gravitons [86,87], which yields to the upper bounds: $m_{_{\rm KK}}^{-1} < 44\,\mu$m for $n=1$, $m_{_{\rm KK}}^{-1} < 1.6\times 10^{-4}\,\mu$m for $n=2$, with more stringent bounds for $n>2$. The authors of [10] then conclude that $n\geq 2$ is excluded since it is not compatible with $m_{_{\rm KK}} \sim \widetilde{\Lambda}_{\rm cc}^{1/4}$, and that there should be a *single* extra dimension, they call it *dark dimension*, of size $\sim 1-100\,\mu$m.

In string theory the finite result $\rho_d \sim \mu_{tow}^d$ arises from modular invariance, that requires to sum over the infinite tower of states. In higher-dimensional field theories with compact extra dimensions such an UV-insensitive result for ρ_d is obtained performing the calculation in a similar manner, i.e. summing over the infinite number of KK-states (the same is done for the calculation of the 4D Higgs effective potential and Higgs boson mass, see section 1.5.1). Differently from the string theory case, however, in this EFT framework such a way of performing the calculation is less obvious to justify [88]. In fact, this question was at the centre of a heated debate in the early 2000's. Several authors tried to support this way of operating with different arguments [89,90], and even nowadays there are attempts at justifying it from the string theory side [91].

This issue, and more generally the question of developing a well-founded EFT approach to field theories with compact extra dimensions, was recently re-analysed in [71] (see also section 1.5), where the focus was on the problem of the UV-(in)sensitivity of the one-loop Higgs effective potential and Higgs mass. It was shown that within the usual calculations the asymptotics of the loop momenta are mistreated, and that this results in an artificial washout of UV-sensitive terms of topological origin. The latter stem from the boundary conditions that must necessarily be given to define the theory on a multiply connected manifold. Their presence was first pointed out in [71].

In the next section we show that, when a proper EFT calculation of the vacuum energy ρ_4 is performed, UV-sensitive terms arise. Moreover, it is discussed how the EFT logic can and must be consistently applied to theories with compact extra dimensions, although it has been recently argued that no controlled approximation can be obtained cutting a KK tower at a finite value [92].

2.2 Vacuum energy

For concreteness, in the following we stick to the case (sufficient for our scopes) of a 5D EFT coupled to gravity, where the compact space dimension is in the shape of a circle of radius R. We take the 5D action to be

$$\mathcal{S}^{(4+1)} = \mathcal{S}^{(4+1)}_{\rm grav} + \mathcal{S}^{(4+1)}_{\rm matter}, \tag{2.5}$$

where

$$\mathcal{S}^{(4+1)}_{\rm grav} = \frac{1}{2\hat{\kappa}^2}\int {\rm d}^4x\,{\rm d}z\,\sqrt{\hat{g}}\,\left(\hat{\mathcal{R}} - 2\hat{\Lambda}_{cc}\right) \tag{2.6}$$

is the Einstein-Hilbert action in $(4+1)$ dimensions and $\mathcal{S}^{(4+1)}_{\rm matter}$ the matter action that contains the bosonic and fermionic fields of the theory. We indicate with x the 4D coordinates and with z the coordinate along the compact dimension. Using the signature $(+,-,-,-,-)$, the $(4+1)$D metric is parametrized as

$$\hat{g}_{_{MN}} = \begin{pmatrix} e^{2\alpha\phi}g_{\mu\nu} - e^{2\beta\phi}A_\mu A_\nu & e^{2\beta\phi}A_\mu \\ e^{2\beta\phi}A_\nu & -e^{2\beta\phi} \end{pmatrix}, \tag{2.7}$$

where A_μ is the so called graviphoton and ϕ the radion field. Considering only zero modes for $\hat{g}_{MN}$, i.e. taking $g_{\mu\nu}$, A_μ and ϕ in (2.7) depending only on x, the integration over z leads to the following 4D gravitational action $\mathcal{S}^{(4)}_{\rm grav}$ [93]

$$\mathcal{S}^{(4)}_{\rm grav} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\left[\mathcal{R} - 2e^{2\alpha\phi}\hat{\Lambda}_{cc} + 2\alpha\Box\phi + \frac{(\partial\phi)^2}{2} - \frac{e^{-6\alpha\phi}}{4}F^2\right], \qquad (2.8)$$

where the 4D constant κ is related to the 5D $\hat{\kappa}$ by

$$\kappa^2 = \frac{\hat{\kappa}^2}{2\pi R}. \qquad (2.9)$$

The constants α and β satisfy the relation

$$2\alpha + \beta = 0 \qquad (2.10)$$

and the canonical radion kinetic term fixes

$$\alpha = \frac{1}{\sqrt{12}}. \qquad (2.11)$$

For completeness we recall that the Newton constant κ can be used to write (2.8) in terms of dimensionful ϕ and A_μ fields through the redefinition

$$\phi \to \frac{\phi}{\sqrt{2}\kappa} \quad , \quad A_\mu \to \frac{A_\mu}{\sqrt{2}\kappa}. \qquad (2.12)$$

Let us consider the case of a complex 5D scalar field $\hat{\Phi}$ with action

$$\mathcal{S}^{(4+1)}_{\Phi} = \int \mathrm{d}^4x\,\mathrm{d}z\ \sqrt{\hat{g}}\left(\hat{g}^{MN}\partial_M\hat{\Phi}^*\partial_N\hat{\Phi} - m^2|\hat{\Phi}|^2\right), \qquad (2.13)$$

that along the compact dimension obeys the non-trivial boundary condition

$$\hat{\Phi}(x, z + 2\pi R) = e^{2\pi i\delta}\,\hat{\Phi}(x, z), \qquad (2.14)$$

where δ is a generic phase. Defining now ($[\delta]$ is the integer part of δ)

$$q \equiv \delta - [\delta]\,, \qquad (2.15)$$

(2.14) can be rewritten as

$$\hat{\Phi}(x, z + 2\pi R) = e^{2\pi i q}\,\hat{\Phi}(x, z)\,. \qquad (2.16)$$

It is then clear that the physical parameter to which we have access is not δ, but rather q. In other words, to implement the non-trivial boundary condition we have to refer to (2.16) rather than to (2.14). Consequently, in physical quantities q rather than δ will appear.

For the corresponding 4D action we have

$$\mathcal{S}^{(4)}_{\Phi} = \int d^4x\sqrt{-g}\,\sum_n\left[|D\varphi_n|^2 - \left(e^{2\alpha\phi}m^2 + e^{6\alpha\phi}\frac{(n+q)^2}{R^2}\right)|\varphi_n|^2\right], \qquad (2.17)$$

where

$$D_\mu \equiv \partial_\mu - i\left(\frac{n+q}{R}\right)A_\mu \tag{2.18}$$

and $\varphi_n(x)$ are the KK modes of $\hat{\Phi}(x,z)$ (see section 1.5). Taking a constant background for the radion (that for notational simplicity we continue to call ϕ) and the trivial background for A_μ,

$$\hat{g}^0_{MN} = \begin{pmatrix} e^{2\alpha\phi}\eta_{\mu\nu} & 0 \\ 0 & -e^{2\beta\phi} \end{pmatrix}, \tag{2.19}$$

from (2.17) we can define the ϕ-dependent radius $R_\phi \equiv R\, e^{-3\alpha\phi}$ ($R\, e^{(\beta-\alpha)\phi}$ before using (2.10)) and the ϕ-dependent mass $m_\phi^2 \equiv m^2 e^{2\alpha\phi}$, so that the KK masses are

$$m_n^2 \equiv m_\phi^2 + \frac{(n+q)^2}{R_\phi^2}. \tag{2.20}$$

Going to Euclidean space and considering the general case, the one-loop contribution[2] ρ^{1l} of a single bosonic or fermionic tower of mass m and boundary charge q to the 4D vacuum energy ρ_4 is then

$$\rho^{1l} \sim (-1)^{\delta_{if}} \sum_n \int \frac{d^4p}{(2\pi)^4} \log \frac{p^2 + \frac{(n+q)^2}{R_\phi^2} + m_\phi^2}{\mu^2}, \tag{2.21}$$

where μ is a subtraction scale, and $i = b, f$ for bosons and fermions respectively.

The right hand side of (2.21) is calculated according to different strategies. One of them consists in performing the sum over n all the way up to infinity, and the integral in d^4p with the help[3] of a cutoff Λ [8, 9, 70]. Other methods are related to the implementation of the proper-time [94], Pauli-Villars [89], thick brane [90], and dimensional regularizations [95]. They all give the same result. For the time being we focus on the first of them; we will comment on the others later. Crucial in getting the UV-insensitive result $\rho^{1l} \sim R_\phi^{-4} = m^4_{\rm KK}$ in [8, 9, 70] is that n is sent to infinity while Λ is kept fixed[4] (see the discussion in section 1.5.1). As explained in section 1.5.2, however, this way of performing the calculation mistreats the asymptotics of the 5D loop momentum of the original theory. In fact, n/R is the fifth component p_5 of the 5D loop momentum $\hat{p} \equiv (p, n/R)$. Sending $p_5 \to \infty$ while keeping Λ fixed means that in the loop corrections we are (improperly) including first the asymptotics of the fifth component (n/R) of the momentum and only later those of the other four components p_1, p_2, p_3 and p_4. As shown in [71], however, a necessary and physical requirement, overlooked in previous literature, is that the asymptotics of all the five components of $\hat{p}$ have to be treated on an equal footing. This can be realized considering in (2.21) a 5D cutoff, $\hat{g}_0^{MN}\hat{p}_M\hat{p}_N = e^{-2\alpha\phi}p^2 + e^{-2\beta\phi}n^2/R^2 \equiv \widetilde{p}^2 + n^2/\widetilde{R}^2 \leq \Lambda^2$, or equivalently through the insertion of a multiplicative smooth cutoff function $e^{-(\widetilde{p}^2+n^2/\widetilde{R}^2)/\Lambda^2}$ ($\hat{g}_0^{MN}$ is the inverse of the

[2] Note that the result for the vacuum energy ρ_4 is obtained summing ρ^{1l} to the bare vacuum energy $\rho_{\rm bare}$, $\rho_4 = \rho_{\rm bare} + \rho^{1l}$.

[3] We note that the introduction of Λ is necessary, otherwise each of the integrals in (2.21) would be divergent.

[4] Strictly speaking, this is true only in a supersymmetric theory. Moreover, sending $\Lambda \to \infty$ at the end of the calculation is harmless, as the only Λ-dependent terms all vanish in this limit.

Euclidean flat background 5D metric in (2.19)). Sticking to the first of these two options, defining $\Lambda_\phi \equiv \Lambda e^{\alpha\phi}$, and performing the integration over p, we get (below we write only the contribution of a bosonic tower)

$$\rho^{1l} = \frac{1}{64\pi^2} \sum_{n=-[R_\phi\Lambda_\phi]}^{[R_\phi\Lambda_\phi]} \Big\{ \Big(\Lambda_\phi^2 - \frac{n^2}{R_\phi^2}\Big)\Big(m_\phi^2 + \Big(\frac{n+q}{R_\phi}\Big)^2\Big) + \Big(\Lambda_\phi^2 - \frac{n^2}{R_\phi^2}\Big)^2 \log \frac{\Lambda_\phi^2 + m_\phi^2 - \frac{n^2}{R_\phi^2} + \big(\frac{n+q}{R_\phi}\big)^2}{\mu^2}$$

$$+ \Big(m_\phi^2 + \Big(\frac{n+q}{R_\phi}\Big)^2\Big)^2 \log \frac{m_\phi^2 + \big(\frac{n+q}{R_\phi}\big)^2}{\Lambda_\phi^2 + m_\phi^2 - \frac{n^2}{R_\phi^2} + \big(\frac{n+q}{R_\phi}\big)^2} - \frac{\Big(\Lambda_\phi^2 - \frac{n^2}{R_\phi^2}\Big)^2}{2} \Big\} \equiv \sum_{n=-[R_\phi\Lambda_\phi]}^{[R_\phi\Lambda_\phi]} F(n) \,, \quad (2.22)$$

where the brackets [...] indicate "integer part" (to simplify the notation, but without loss of generality, we take Λ such that $R_\phi\Lambda_\phi$ is an integer). The sum can be performed using the Euler-McLaurin (EML) formula,

$$\rho^{1l} = \int_{-R_\phi\Lambda_\phi}^{R_\phi\Lambda_\phi} \mathrm{d}x \; F(x) + \frac{F(R_\phi\Lambda_\phi) + F(-R_\phi\Lambda_\phi)}{2}$$

$$+ \sum_{j=1}^{r} \frac{B_{2j}}{(2j)!} \left(F^{(2j-1)}(R_\phi\Lambda_\phi) - F^{(2j-1)}(-R_\phi\Lambda_\phi)\right) + R_{2r} \,, \quad (2.23)$$

where r is an integer, B_n are the Bernoulli numbers, and the rest R_{2r} is given by

$$R_{2r} = \sum_{k=r+1}^{\infty} \frac{B_{2j}}{(2j)!} \left(F^{(2j-1)}(R_\phi\Lambda_\phi) - F^{(2j-1)}(-R_\phi\Lambda_\phi)\right)$$

$$= \frac{(-1)^{2r+1}}{(2r)!} \int_{-R_\phi\Lambda_\phi}^{R_\phi\Lambda_\phi} dx \, F^{(2r)}(x) B_{2r}(x - [x]), \quad (2.24)$$

with $B_n(x)$ the Bernoulli polynomials. Expanding for m_ϕ/Λ_ϕ, $q/\Lambda_\phi \ll 1$, we finally get

$$\rho^{1l} = \frac{5\log\frac{\Lambda^2 e^{2\alpha\phi}}{\mu^2} - 2}{300\pi^2} e^{2\alpha\phi} R\Lambda^5 + \frac{5m^2 + 3\frac{q^2 e^{4\alpha\phi}}{R^2}}{180\pi^2} e^{2\alpha\phi} R\Lambda^3 - \frac{35m^4 + 14m^2\frac{q^2 e^{4\alpha\phi}}{R^2} + 3\frac{q^4 e^{8\alpha\phi}}{R^4}}{840\pi^2} e^{2\alpha\phi} R\Lambda$$

$$+ \frac{m^5}{60\pi} e^{2\alpha\phi} R + \frac{3\log\frac{\Lambda^2 e^{2\alpha\phi}}{\mu^2} + 2}{2880\pi^2 R^4} e^{10\alpha\phi} R\Lambda + R_4 + \mathcal{O}(\Lambda^{-1}) \,, \quad (2.25)$$

where the rest R_4 is the UV-insensitive term (suppressed $\mathcal{O}\left(\Lambda^{-1}\right)$ terms are discarded below)

$$R_4 = -\frac{x^2 \mathrm{Li}_3\left(r_b e^{-x}\right) + 3x\mathrm{Li}_4\left(r_b e^{-x}\right) + 3\mathrm{Li}_5\left(r_b e^{-x}\right)}{128\pi^6 R^4} e^{12\alpha\phi} + h.c. + \frac{3\zeta(5)}{64\pi^6 R^4} e^{12\alpha\phi} \quad (2.26)$$

with

$$r \equiv e^{2\pi i q} \quad , \quad x \equiv 2\pi e^{-2\alpha\phi} R\sqrt{m^2} \,. \quad (2.27)$$

Eqs. (2.25) and (2.26) are re-written in terms of the original R, Λ and m (rather than R_ϕ, Λ_ϕ and m_ϕ) to explicitly show the ϕ-dependence. It is worth to note that the 4D one-loop contribution to the vacuum energy ρ^{1l} is related to the corresponding 5D one

ρ^{1l}_{4+1} through the relation $\rho^{1l} = 2\pi R e^{2\alpha\phi}\, \rho^{1l}_{4+1}$ (we have seen that this relation holds more in general for the effective potential, see Eq. (1.135)).

Several comments are in order. First of all we observe that, had we made the calculation in the usual way [8, 9, 70], all the q-dependent UV-sensitive terms in Eq. (2.25) (i.e. all the q-dependent terms except those contained in R_4) would be absent, while the other UV-sensitive terms are cancelled by SUSY. In fact, while the higher-dimensional SUSY imposes $m_b = m_f$ for the superpartners, q_b and q_f are necessarily different to have a broken SUSY spectrum at low energies. With the usual calculation we would then get the well-known result $\rho^{1l} \sim R_4^b - R_4^f \sim m^4_{\text{KK}}$. However, as we explain below, such a result comes from the fact that the UV-sensitive terms proportional to powers of q are artificially washed out due to an improper way of treating the asymptotics of the loop momentum.

In this respect, we now show that the interpretation of the 5D theory as a 4D one with an *infinite* tower of states (if pushed too far) is misleading. Within this interpretational framework, in fact, it is natural to consider that the correct thing to do is to sum the infinitely many ($n \to \infty$) Coleman-Weinberg one-loop contributions brought by each of the towers. Any reference to the original 5D loop momentum $\hat{p}$ (and a fortiori to the physical meaning of n) is lost. If on the contrary we correctly focus on the dynamical origin of the KK states, and recognize them as different momentum eigenstates that appear in the Fourier expansion of the 5D field $\hat{\Phi}(x, z)$, it is clear that sending $n \to \infty$ while keeping the modulus p of the other four components fixed is unphysical. If our universe has compact extra dimensions, low-energy 4D physical observables emerge from the piling up of quantum fluctuations above the compactification scale. In implementing such a dressing, it is clear that the components of the loop momenta must be treated in a consistent way, actually on an equal footing. We will further comment on this point later.

To better read the result (2.25), we stress that it contains four kinds of terms: (i) m- and q-independent UV-sensitive terms; (ii) UV-sensitive terms that depend only on m; (iii) q-dependent UV-sensitive terms; (iv) UV-insensitive terms. As stressed above, in SUSY theories boson and fermion superpartners have the same mass m, while the boundary charges q are necessarily different to trigger the Scherk-Schwarz mechanism (introduced in section 1.5). Therefore:
(a) in SUSY theories supersymmetry enforces cancellations between superpartners of all but the q-dependent terms in (2.25), so that for each supermultiplet the dominant contribution to ρ^{1l} is controlled by the SUSY breaking parameter $q_b^2 - q_f^2$, and is

$$\rho^{1l} \sim \frac{(q_b^2 - q_f^2)}{R^2}\, e^{6\alpha\phi} R\Lambda^3 = (q_b^2 - q_f^2)\, m^2_{\text{KK}} R\Lambda^3; \tag{2.28}$$

(b) in non-supersymmetric theories, each of the higher-dimensional fields (each tower in 4D language) gives to the vacuum energy the dominant (uncancelled) contribution

$$\rho^{1l} \sim e^{2\alpha\phi} R\Lambda^5 \log \frac{\Lambda^2 e^{2\alpha\phi}}{\mu^2} = m^{2/3}_{\text{KK}} R^{5/3}\Lambda^5 \log \frac{(m_{\text{KK}} R)^{2/3}\Lambda^2}{\mu^2}. \tag{2.29}$$

Therefore, even in the light tower limit $m_{\text{KK}} \to 0$ (large negative values of ϕ), by no means the UV-insensitive $R_4 \sim m^4_{\text{KK}}$ term in (2.25) can overthrow these dominating contributions.

The main point that emerges from our result (2.25) is that q-dependent UV-sensitive terms *always arise* when non-trivial boundary conditions on multiply-connected manifolds are realized (necessary for instance to implement the Scherk-Schwarz mechanism in SUSY theories), *independently* of the size of the extra dimensions. On the contrary, the UV-insensitive terms in the rest R_4 originate from the discreteness of the momentum along the circle, generated in the present case by the hierarchy between the size of the 4D box and the radius of the circle.

2.3 Vacuum Energy and Dark Dimension

As already stressed, taking a 5-dimensional supersymmetric EFT with one compact dimension in the shape of a circle of radius R, and performing the calculation in the usual manner, for the vacuum energy ρ^{1l} at the one-loop level we have

$$(\rho^{1l})^{\frac{1}{4}} = \Big(\sum_i R_4^{(i)}\Big)^{\frac{1}{4}} \sim \mathcal{C}\, m_{_{\rm KK}}\,, \tag{2.30}$$

where $R_4^{(i)}$ is of the kind (2.26) (see also (1.120)). The sum over i includes all the bosonic and fermionic contributions, and $\mathcal{C}$ is an $\mathcal{O}(1)$ coefficient[5]. The DD proposal [10] is based on the assumption that in the asymptotic region of the string moduli space where the EFT emerges, the vacuum energy ρ^{1l} goes as $m_{_{\rm KK}}^4$, where $m_{_{\rm KK}}$ is a Kaluza-Klein mass (of order the neutrino scale) given by the cosmological constant. The authors argue that the EFT result (2.30) supports their proposal.

However, we have shown that (2.30) results from an improper treatment of the asymptotics of the loop momentum in the original 5D theory, and that the correct result for ρ^{1l} is given by (2.25) (with q-independent terms cancelled by SUSY). From this latter equation we see that for a supersymmetric theory with SUSY breaking parameter $q_b^2 - q_f^2$ the dominant contribution to ρ^{1l} is (2.28), i.e. it goes as $m_{_{\rm KK}}^2 R\Lambda^3$. Far from being UV-finite as $m_{_{\rm KK}}^4$, this term is strongly UV-sensitive. We also see that, for a non-supersymmetric theory the term that dominates ρ^{1l} is (2.29): it scales with $m_{_{\rm KK}}$ as $m_{_{\rm KK}}^{2/3}\log m_{_{\rm KK}}$ and is UV-sensitive as $\Lambda^5\log\Lambda$. In both cases the expected $m_{_{\rm KK}}^4$ result is not recovered.

Therefore, the only possibility for the DD scenario to be considered a physical reality is to accept that a fine tuning is needed for the relation $\rho_4 \sim m_{_{\rm KK}}^4$ to hold also in the "EFT limit" of string theory at the basis of this scenario. In fact, a physical mechanism responsible for the suppression of the strongly UV-sensitive terms in (2.25) is needed. One might think that a possible way to escape from such a conclusion is to admit that the aforementioned EFT limit of string theory, framework in which the DD proposal is formulated, actually gives rise to a new type of Effective Field Theory, far from what is usually intended by the community. To support this claim, the calculation of ρ^{1l} in KK theories is sometimes compared with examples where finite results are obtained also in field theory, as for instance the calculation of the one-loop free energy F^{1l} in finite temperature field theory [96]. Also in this case, in fact, an infinite sum over an integer n appears in the expression of F^{1l}. As we will discuss in section 2.4, and contrary to the case

[5] A simple inspection of (2.26) shows that, strictly speaking, (2.30) rigorously holds: (i) for massless theories (with or without compactified higher-dimensional gravity); (ii) for theories with $\phi = 0$ (Minkowski).

of KK theories, however, in the case of finite temperature field theory the infinite sum has a precise physical justification, and must be implemented to obtain the correct result for F^{1l}. Concerning KK theories, instead, at present there is no hint for that, and, as stressed by the authors of [10] themselves, we all know that around and above the neutrino scale physics is well described by the EFT paradigm in the usual and well-known sense.

In this respect we also observe that to calculate the contribution of a KK tower to ρ^{1l}, in particular to study its UV (in)sensitivity, we could resort to the species scale cutoff $\Lambda_{\rm sp}$ [97,98], as it is sometimes done within the swampland program (see for instance [99–103]). Referring to our previous examples, we consider the case of a massless 5D field, where the masses of the tower states are given by (2.20) with $m_\phi^2 = 0$. The number N of states with mass below $\Lambda_{\rm sp}$ is

$$N \equiv n_{\rm max} + |n_{\rm min}| + 1 \,, \tag{2.31}$$

with $n_{\rm max}$ and $n_{\rm min}$ solutions of $(n+q)^2/R_\phi^2 = \Lambda_{\rm sp}^2$, i.e.

$$n_{\rm max} = R_\phi \Lambda_{\rm sp} - q \qquad ; \qquad n_{\rm min} = -R_\phi \Lambda_{\rm sp} - q \,. \tag{2.32}$$

The explicit calculation is performed in Appendix A.

We stress that when $q \neq 0$, cutting the sum over n in (2.21) with (2.31) and (2.32) and the integral over the four-momentum p with $\Lambda_{\rm sp}$ *is not* equivalent to the introduction of a cut on the 5D loop momentum $\hat{p}$. As repeatedly underlined in the present work, the latter is the (physically) correct cut to apply. From the calculations in Appendix A we see that, when the cut is imposed on the combination $(n+q)^2/R_\phi^2$ rather than[6] n^2/R_ϕ^2, an artificial washout of q-dependent UV-sensitive terms is operated. This again comes from a mistreatment of the 5D loop momentum asymptotics. Similarly to what we have already seen, the application of the $\Lambda_{\rm sp}$ cut pushes too far the interpretation of the KK modes as massive states of the 4D theory, losing sight of the original physical meaning of n.

These issues were also discussed in [71], where it was shown in full generality that the inclusion of the boundary charge q in the cut (whatever kind of cut) is at the origin of the artificial washout of the q-dependent UV-sensitive terms. In this respect, it is worth to stress that performing the infinite sum while keeping Λ fixed is equivalent to include q in the cut. As explained in the present work, both are physically illegitimate operations. The proper-time [94], Pauli-Villars [89], and thick brane [90] regularizations all implement the insertion of q in the cut over n, thus realizing the artificial washout mentioned above. It is worth to point out that the use of dimensional regularization (DR), as done for instance in [92, 95], does not help to cope with this kind of issues. By construction, in fact, DR does not detect the full UV-sensitivity of a theory, since in this regularization power "divergences" are automatically canceled (see [22] for a careful analysis of DR in comparison with other regularizations). We also note that DR totally masks the presence of UV-sensitive terms in odd dimensions, leading sometimes to the impression that no "divergences" appear in that case.

[6]Except for the $e^{2\alpha\phi}$ rescaling factor, n^2/R_ϕ^2 coincides with p_5^2, the square of the fifth component of $\hat{p}$.

2.4 Casimir energy and finite temperature field theory

As anticipated in the previous section, we now comment on two examples that have been sometimes invoked in the literature (see for instance [96]) to support the inclusion of the infinite KK tower in the calculation of ρ^{1l}, and to substantiate the widely spread belief that for field theories with compact extra dimensions the vacuum energy turns out to be automatically finite. Some authors refer to (i) the calculation of the one-loop free energy F^{1l} in finite temperature QFT, where the T-dependent contribution is finite and goes like $F_T^{1l} \sim T^4$, and (ii) the calculation of the Casimir energy.

Let us begin with the finite temperature field theory case. As stressed in [71], despite their apparent similarity, there is a profound difference between the sum over the integer n that appears in F^{1l} and the analogous sum in the calculation of the vacuum energy ρ^{1l} in KK theories. In fact, both for ρ^{1l} and F^{1l} we have an expression of the kind (below $d = 4$ for the $(4+1)$D KK theory and $d = 3$ for finite temperature field theory)

$$\rho^{1l}\ ,\ F^{1l} \sim \frac{1}{2}\sum_n \int \mathrm{d}^d p\, \log\left(p^2 + m^2 + f_n\right). \tag{2.33}$$

For the case of KK theories (ρ^{1l}) the integer n is related to the fifth component of the $(4+1)$D loop momentum $\hat{p}$, that is cut at Λ. Therefore n and p are intertwined between one another and are to be cut accordingly, see comments below Eq. (2.21) and Eq. (2.27) (see also [71]). This in particular implies an upper bound on the integer n. As a result, our Eq. (2.25) is obtained. For the finite temperature field theory case (F^{1l}), on the contrary, the integer n and the $3D$ momentum p are not intertwined. The $\mathrm{d}^3 p$ integral provides the trace over quantum fluctuations (and is cut at a certain momentum cutoff), while the $\sum_n$ gives the statistical average (mixed states), and as such has to be performed all the way up to infinity to implement ergodicity. This infinite sum is at the origin of the finiteness[7] of the T-dependent contribution F_T^{1l} to F^{1l}. In fact,

$$\begin{aligned} F_T^{1l} = \frac{T}{2}\sum_{n=-\infty}^{\infty} \int \mathrm{d}^3 p\, \log\left(p^2 + m^2 + f_n\right) \\ - \frac{1}{2}\int \mathrm{d}^3 p\, \sqrt{p^2 + m^2} \sim T^4 = \text{finite}. \end{aligned} \tag{2.34}$$

As discussed at length above, performing also in the KK case the sum over n up to infinity (while keeping p finite) mistreats the loop momentum. This is what causes the (artificial) disappearance of the q-dependent UV-sensitive terms in (2.25), giving the impression that there is a cancellation similar to what happens in the finite temperature case.

Moving now to the example of the Casimir energy $\mathcal{E}_C$, we observe that the latter is nothing but the difference between the vacuum energy ρ_R calculated in the KK theory with one compact dimension (that for our illustrative purposes we took as a circle of radius R) and the corresponding ρ_∞ calculated in the decompactification limit (i.e. in the theory with all non-compact dimensions): $\mathcal{E}_C = \rho_R - \rho_\infty$. Now, if we perform the one-loop

[7] It is worth to note that the full result for F^{1l} is finite only if the theory is supersymmetric.

calculation of ρ^{1l} according to previous literature, meaning that in (2.33) we perform the sum over n up to infinity, $\mathcal{E}_C^{1l}$ takes a form in all similar to (2.34), with T^4 replaced by $m_{\rm KK}^4$. However, as shown in the previous sections of the present chapter, (contrary to what is found in previous literature) the subtraction $\rho_{_R} - \rho_\infty$ is not in general sufficient to ensure the disappearance of all the UV-sensitive terms. This is for instance the case of KK theories with Scherk-Schwarz SUSY breaking considered above.

2.5 Compact dimensions and EFTs

The result $\rho^{1l} \sim m_{\rm KK}^4$ (more generally $\rho_d \sim m_{\rm KK}^d$ in d spacetime non-compact dimensions) is sometimes used to argue for a possible general breakdown of EFT methods. On our side, referring to the original $(4+n)D$ theory (with n compact dimensions), we have shown in the previous sections that the EFT approach is perfectly suited to theories with compact extra dimensions, and found for ρ^{1l} the radically different result (2.25). An interesting point of view on these issues has been recently given in [92], where the result $\rho^{1l} \sim m_{\rm KK}^4$ is taken for granted but it is argued that it cannot be used as a signal of general departure from the EFT approach.

Their argument goes as follows. Consider a tower of states with mass spectrum $m_n = f_n\, \mu_{tow}$. Cutting the sum at $n = N$ means that we include in the theory KK modes up to the N-th one, and exclude the states from the $(N+1)$-th up to infinity. In general, integrating out a (finite) set of fields to define a low energy EFT for the lighter ones is a consistent operation only if there is a large mass hierarchy between the fields included and those excluded. When a KK tower is cut, the hierarchy between the heaviest state included and the lightest one excluded is given by f_N/f_{N+1}. Being this ratio $\mathcal{O}(1)$ (except for the case $N = 0$, that defines the 4D EFT), the authors conclude that no EFT estimate with a finite number of KK states can ever be done, and that the $(4+n)D$ theory must necessarily contain the infinite tower.

The observation that an infinite tower of massive states cannot be divided into heavy and light fields to define an EFT for the latter ones is certainly true and interesting in its own right. This line of reasoning, however, might not apply to higher-dimensional theories with compact extra dimensions. In fact, sticking for concreteness to a higher-dimensional 5D theory, we should keep in mind that the KK modes are momentum eigenstates of the original 5D fields (and not an infinitely numerable set of massive 4D fields), and that the 5D theory from which the 4D theory derives is an EFT itself. As for any EFT, this means that the 5D momentum $\hat{p} \equiv (p, n/R)$ in the loops has to be cut at the scale where the theory loses its validity. The fifth component n/R of $\hat{p}$ cannot be disentangled from the other four components, $p \equiv (p_1, p_2, p_3, p_4)$, so that the cut in the KK states results from a physically necessary requirement. No large hierarchy between included and excluded momentum modes is ever needed.

In this respect, we stress that the Wilsonian point of view developed in section 1.4 is perfectly suited for theories with compact extra dimensions. Starting from the 5D action $\mathcal{S}_\Lambda^{(5)}$, and integrating out the modes in the range $[k, \Lambda]$, one obtains the action $\mathcal{S}_k^{(5)}$ at the lower scale k. Due to the discreteness of $p_5 = n/R$, the contribution from the related eigenmodes comes in a stepwise fashion. For $k < 1/R$, no such eigenmodes appear any longer, and the RG evolution becomes effectively of 4D type. It is *only in this sense* that the 4D theory emerges from the 5D one, and no 4D theory with an infinite tower can ever

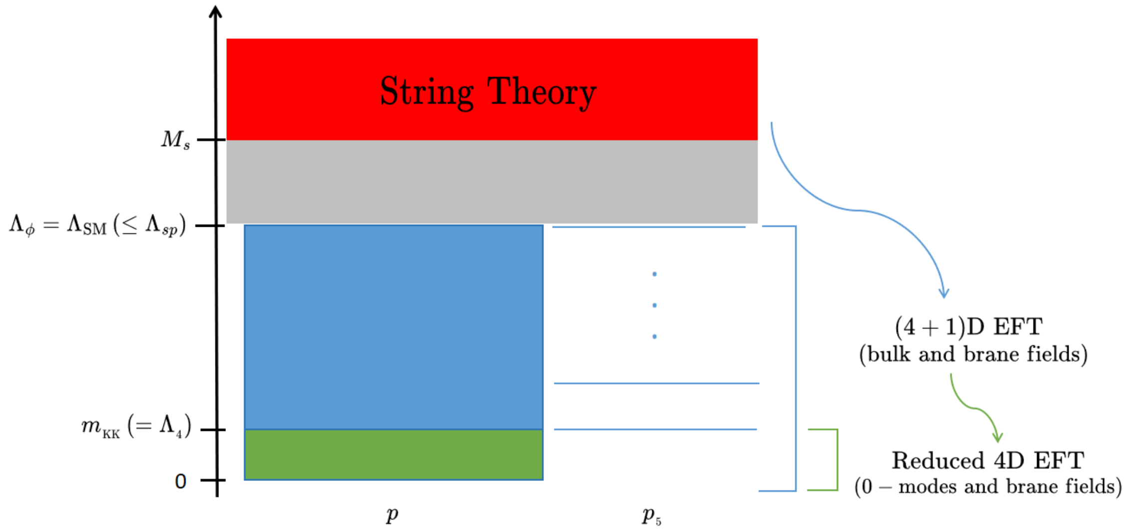

Figure 2.1: Pictorial representation of the energy scales and theories involved in the Dark Dimension scenario. The grey region indicates a possible UV-completion of the $(4+1)$D theory between its rescaled physical cutoff Λ_ϕ (see text above Eq. (2.22)) and the string scale M_s. A particular case could be that Λ_ϕ directly coincides with M_s. Below Λ_ϕ (that coincides with the Standard Model physical cutoff $\Lambda_{\rm SM}$, see Appendix B) the $(4+1)$D theory of the DD scenario (blue region), that contains bulk fields and SM (with possibly other) brane fields, takes place. Below $m_{\rm KK}$, the theory is well described by a 4D EFT (green region) containing the 0-modes of the bulk fields together with brane fields. The components p and p_5 of the $(4+1)$D momentum $\hat{p} \equiv (p, p_5)$, with $p \equiv (p_1, p_2, p_3, p_4)$ and p_5 the fifth component, are reported. The fully colored blue and green rectangles indicate that p takes continuous values (non-compact dimensions), while the blue lines indicate that p_5 takes discrete values starting at $m_{\rm KK}$ (compact dimension).

give an accurate description of the original 5D theory.

This considerations certainly applies to the $(4+1)$D model that implements the DD scenario. To be phenomenologically viable, this model has to include the SM on a 3-brane. An important consequence is that the physical cutoff Λ of the $(4+1)$D theory is an upper bound for the SM cutoff $\Lambda_{\rm SM}$ (the derivation of Eq. (2.35) below is in Appendix B)

$$\Lambda_{\rm SM} \leq \Lambda\,. \tag{2.35}$$

Let us also note that $m_{\rm KK}$ is the cutoff of the "reduced 4D theory", which besides brane fields contains the zero modes of bulk fields (i.e. fields that propagate also along the 5th compact dimension). Moreover, it is worth to observe that, although both the Standard Model and this "reduced theory" are 4-dimensional EFTs, they are certainly not the same theory, and their cutoffs ($\Lambda_{\rm SM}$ and $m_{\rm KK}$ respectively) should not be confused. In Fig. 2.1 , we provide a comprehensive picture of the global framework in which the DD scenario is framed, described in detail in the caption below.

2.6 Summary and conclusions

In this chapter, we carefully analyzed the calculation of the one-loop contribution to the vacuum energy in theories with compact extra dimensions, also in connection with the recent dark dimension proposal [10], according to which the tiny measured value of the cosmological constant might signal the presence of a single compact extra dimension of mesoscopic size (order μm or so). Moving from swampland arguments, this proposal is based on the idea that the cosmological constant fixes the scale m_{KK} of a KK tower (of order the neutrino scale), and relies on the (widely believed) result $\rho_{\text{EFT}} \sim m_{\text{KK}}^4$ for the vacuum energy in the underlying higher-dimensional EFT with one compact extra dimension.

According to [10], both the relation $\rho_{\text{swamp}} \sim m_{\text{KK}}^4$ (that comes from swampland conjectures formulated in an asymptotic corner of the quantum gravity landscape) and the corresponding result for the vacuum energy ρ_{EFT} coming from the EFT limit are at the basis of the dark dimension scenario. Leaving aside problems that might arise in the string theory framework itself, in this chapter we have shown that, due to the presence of previously missed UV-sensitive terms in ρ_{EFT} (proportional to the q charges of bulk fields), the matching between ρ_{swamp} and the corresponding ρ_{EFT} is a delicate issue. In order for this matching to be realized, a physical mechanism that implements the suppression of the strongly UV-sensitive terms in the vacuum energy is needed. In this respect, we stress that in this chapter we mainly focused on the contribution to the vacuum energy from bulk fields. Naturally, there is also a contribution from the Standard Model, that, as already said, lives on a 3-brane. Since the SM physical cutoff Λ_{SM} is at least of the TeV order, this contribution cannot be ignored.

We expect that, for both bulk and brane contributions, the aforementioned mechanism (if any) should be provided by the piling up of quantum fluctuations in the UV $\to$ IR renormalization group (RG) flow that takes into account the whole $(4+1)$D theory, including the SM in the 3-brane. In such a framework, the UV-complete theory (string theory/quantum gravity) should provide the boundary (i.e. UV) value of the vacuum energy (more generally the boundary values of all the physical parameters), so that RG equations should dictate how the measured vacuum energy is reached in the IR (see [11] for examples where this mechanism is realized in $d = 3$ and $d = 4$ scalar theories).

In the next chapter, devoted to a presentation of [14], the problem of the UV-sensitivity of the vacuum energy is studied in a different framework. We consider pure quantum gravity (in four non-compact dimensions) in the Einstein-Hilbert truncation, and calculate the one-loop effective action $\Gamma^{1l}_{\text{grav}}$ paying attention to two aspects often overlooked in previous literature: the use of the correct (diffeomorphism invariant) path integral measure, and a proper introduction of the UV physical cutoff Λ of the theory. Contrary to what is found from usual calculations (typically performed resorting to the heat-kernel formalism introduced in section 1.2), we show that the radiative correction to the vacuum energy (read from $\Gamma^{1l}_{\text{grav}}$) is only logarithmically sensitive to Λ. We also show that this still holds true in the presence of matter fields.

Chapter 3

Path integral measure and the cosmological constant

In the present chapter, we consider (Euclidean) quantum gravity in the Einstein-Hilbert truncation, and re-examine the calculation of the one-loop effective action $\Gamma^{1l}_{\rm grav}$. As said in the Introduction, this calculation is usually performed within the heat-kernel formalism [17, 44] (see also section 1.2), and gives rise to quartically and quadratically UV-sensitive contributions to the vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{\rm cc}}{8\pi G}$, with $\Lambda_{\rm cc}$ and G the cosmological and the Newton constant, respectively. We show that the appearance of these "divergences" is due to the use of a non-diffeomorphism invariant measure in the path integral that defines the effective action, and to an improper identification of the UV physical cutoff Λ. We find that, if the diffeomorphism invariant Fradkin-Vilkovisky measure is used, and the UV physical cutoff Λ is properly introduced, $\rho_{\rm vac}$ presents only a (mild) logarithmic sensitivity to Λ. We also consider a free scalar field and a free Dirac field minimally coupled to gravity, and show that the same holds true even in the presence of matter.

An important point to stress is that these results are obtained without resorting neither to regularization schemes, as dimensional or zeta-function regularization, that operate the cancellation of power-like divergences by construction, nor to any ad hoc physical cancellation mechanism, such as a supersymmetric embedding of the theory.

The results of the present chapter have been published in [14].

3.1 One-loop effective action. Pure gravity

Let us consider the (Euclidean) gravitational action in the Einstein-Hilbert truncation[1]

$$S_{\rm grav} = \frac{1}{16\pi G} \int {\rm d}^4x \, \sqrt{g} \, (-R + 2\Lambda_{\rm cc}) \, . \tag{3.1}$$

To calculate the one-loop effective action $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$, we resort to the geometrical approach pioneered by Vilkovisky [55] and DeWitt [57] (introduced in section 1.3) that allows to obtain a gauge invariant result even off-shell. We follow the strategy put forward

[1] The Einstein-Hilbert truncation is well justified since in the cosmological framework we only need to consider manifolds with typical length scale l much larger than the Planck length, $l \gg M_P^{-1}$, that in turn implies $\Lambda_{\rm cc} \ll M_P^2$.

in [58,104], paying particular attention to the measure in the path integral that defines $\Gamma^{1l}_{\rm grav}$. Considering the background field method [105,106], we write the metric $g_{\mu\nu}$ as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the background and $h_{\mu\nu}$ the fluctuation. As shown in [58], when $\bar{g}_{\mu\nu}$ has spherical symmetry the one-loop Vilkovisky-DeWitt effective action coincides with the standard one calculated with gauge-fixing

$$S_{\rm gf} = \frac{1}{32\pi G\xi}\int {\rm d}^4x\,\sqrt{\bar{g}}\left[\nabla_\mu\left(h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu\, h^\sigma_\sigma\right)\right]^2 \tag{3.2}$$

taking the limit $\xi \to 0$ at the end of the calculation. We then consider a spherical background $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$, where a is the radius of the sphere

$$g_{\mu\nu} = g^{(a)}_{\mu\nu} + h_{\mu\nu}\,. \tag{3.3}$$

For this background, the classical action (3.1) is $(\int {\rm d}^4x\,\sqrt{g^{(a)}} = \frac{8\pi^2}{3}a^4\,,\; R(g^{(a)}) = \frac{12}{a^2})$

$$S^{(a)}_{\rm grav} \equiv S_{\rm grav}[g^{(a)}_{\mu\nu}] = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2\,, \tag{3.4}$$

and the ghost action $S_{\rm ghost}$ corresponding to $S_{\rm gf}$ in (3.2) is

$$S_{\rm ghost} = \frac{1}{32\pi G}\int {\rm d}^4x\,\sqrt{g^{(a)}}\,g^{(a)\,\mu\nu}\,v^*_\mu\left(-\nabla_\rho\nabla^\rho - \frac{3}{a^2}\right)v_\nu\,. \tag{3.5}$$

After the calculation of $\delta S^{1l}_{\rm grav}$, we will identify the one-loop corrections $\delta\left(\frac{\Lambda_{\rm cc}}{G}\right)$ and $\delta\left(\frac{1}{G}\right)$ to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ with the coefficients of a^4 and a^2, respectively [104].

According to the strategy outlined above, the one-loop correction $\delta S^{1l}_{\rm grav}$ to $S^{(a)}_{\rm grav}$ is obtained from

$$e^{-\delta S^{1l}_{\rm grav}} = \lim_{\xi\to 0}\int \left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]e^{-\delta S^{(2)}}\,, \tag{3.6}$$

where

$$\delta S^{(2)} \equiv S_2 + S_{\rm gf} + S_{\rm ghost}\,, \tag{3.7}$$

with S_2 the quadratic term in the expansion of $S_{\rm grav}[g^{(a)}_{\mu\nu} + h_{\mu\nu}]$ around $g^{(a)}_{\mu\nu}$ (below $h \equiv g^{(a)}_{\mu\nu}h^{\mu\nu}$, $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g^{(a)}_{\mu\nu}h$; indexes are raised with $g^{(a)\,\mu\nu}$, and covariant derivatives are in terms of $g^{(a)}_{\mu\nu}$)

$$S_2 \equiv \frac{1}{32\pi G}\int {\rm d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\tilde{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2\Lambda_{\rm cc} + \frac{8}{a^2}\right)h_{\mu\nu} + \frac{h^2}{a^2} - \nabla^\rho\tilde{h}_{\rho\mu}\nabla^\sigma\tilde{h}^\mu_\sigma\right]\,. \tag{3.8}$$

The measure $\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]$ is the one proposed by Fradkin and Vilkovisky in [18,19],

$$\begin{aligned}\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \Big[&g^{(a)\,00}(x)\left(g^{(a)}(x)\right)^{-1}\\ &\times\Big(\prod_{\alpha\leq\beta}{\rm d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho {\rm d}v^*_\rho(x)\Big)\Big(\prod_\sigma {\rm d}v_\sigma(x)\Big)\Big]\,,\end{aligned} \tag{3.9}$$

where they also showed that the terms $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$, coming from the integration over conjugate momenta[2] of the fields, ensure the invariance of the measure (3.9) under general coordinate transformations. We will see that these terms play a crucial role in the calculation of $\delta S^{1l}_{\rm grav}$.

Concerning (3.9), it is worth to note that in the literature other measures have been proposed. A notable example is in [107], where Fujikawa derived his measure requiring the BRST invariance of the configuration space path integral[3]. This "Fujikawa measure" is sometimes claimed to be invariant under general coordinate transformations (see [107, 108] and the more recent work [36]). According to Fradkin and Vilkovisky's proof [18], however, this cannot be true. What is behind this discrepancy is a delicate point that arises in gravitational theories, related to the necessity of a time ordering parameter and a discretization of spacetime (lattice) in the construction of the path integral (see section 1.1.1). As thoroughly discussed in [18, 37, 41, 109], both of them change under general coordinate transformations $x \to x'$, and this is responsible for the appearance of non-trivial transformation factors from the path integral. Unfortunately, these factors seem to be missed in Fujikawa's analysis [107], leading to the belief that this measure is invariant. When they are correctly taken into account, however, we see that the Fujikawa measure is not invariant, and that, instead, the local terms $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$ in the FV measure (3.9) guarantee their cancellation, eventually ensuring the diffeomorphism invariance of the path integral. These issues concerning the transformation properties of the path integral measure under diffeomorphisms have been thoroughly analyzed in [37], that is the subject of chapter 6.

The diffeomorphism invariance of the measure $\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]$ in (3.9), together with the considerations made at the beginning of this section on the Vilkovisky-DeWitt effective action, ensures the diffeomorphism invariance of $\delta S^{1l}_{\rm grav}$ in (3.6). For our purposes, it is convenient to calculate $\delta S^{1l}_{\rm grav}$ considering coordinate systems (as for instance the four angles that parametrize the sphere) where the background metric $g^{(a)}_{\mu\nu}$ can be written as

$$g^{(a)}_{\mu\nu} = a^2\,\widetilde{g}_{\mu\nu}\,, \tag{3.10}$$

being the elements of $\widetilde{g}_{\mu\nu}$ dimensionless and a-independent. The factor $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$ that appears in the measure (3.9) can then be written as

$$g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1} = a^{-10}\,\widetilde{g}^{00}(x)\,\left(\widetilde{g}(x)\right)^{-1}. \tag{3.11}$$

From the above equation we see that the a-dependence of the non-trivial terms in the measure is all contained in the factor a^{-10}. As we will see below, it is convenient to define the dimensionless field

$$\widehat{h}_{\mu\nu} \equiv (32\pi G)^{-1/2}\,a^{-1}h_{\mu\nu}\,, \tag{3.12}$$

[2] The alert reader might notice that the original expression in [18] is $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-\frac{3}{2}}$. The difference is due to the fact that here we take both v and v^* as world vectors, while in [18] a different choice is made. The extra factor $\sqrt{g^{(a)}}$ is the Jacobian that relates these two equivalent functional integration variables [109].

[3] There have been attempts to derive this measure from the phase space path integral [108].

so that $S_2 + S_{\rm gf}$ can be written as

$$S_2 + S_{\rm gf} = \int {\rm d}^4x \, \sqrt{\widetilde{g}} \, \Big[\frac{1}{2} \overline{h}^{\mu\nu} \left(-\nabla_\rho \nabla^\rho - 2a^2 \Lambda_{\rm cc} + 8 \right) \widehat{h}_{\mu\nu} + \widehat{h}^2 - \left(1 - \frac{1}{\xi} \right) \left(\nabla^\rho \overline{h}_{\rho\mu} \right) \left(\nabla^\sigma \overline{h}^{\mu}_{\sigma} \right) \Big] , \tag{3.13}$$

where $\widehat{h} \equiv \widetilde{g}_{\mu\nu} \, \widehat{h}^{\mu\nu}$, $\overline{h}_{\mu\nu} \equiv \widehat{h}_{\mu\nu} - \frac{1}{2} \, \widetilde{g}_{\mu\nu} \, \widehat{h}$, indexes are raised with $\widetilde{g}^{\mu\nu}$ ($\nabla^\rho \equiv \widetilde{g}^{\rho\sigma} \, \nabla_\sigma$), and covariant derivatives are defined in terms of $\widetilde{g}_{\mu\nu}$. Note also that in (3.13) only dimensionless operators, namely $-\nabla_\rho \nabla^\rho$ and $-\nabla^\rho$, appear. Moreover, introducing (3.10) in (3.5), and defining the dimensionless field

$$\widehat{v}_\mu \equiv (32 \pi G)^{-\frac{1}{2}} \, v_\mu \, , \tag{3.14}$$

$S_{\rm ghost}$ can be written as (again indexes are raised with $\widetilde{g}^{\mu\nu}$ and covariant derivatives are in terms of $\widetilde{g}_{\mu\nu}$)

$$S_{\rm ghost} = \int {\rm d}^4x \, \sqrt{\widetilde{g}} \;\; \widetilde{g}^{\mu\nu} \, \widehat{v}^*_\mu \, (- \nabla_\rho \nabla^\rho - 3) \, \widehat{v}_\nu \, . \tag{3.15}$$

Finally, we observe that

$$\prod_{\alpha \leq \beta} {\rm d}h_{\alpha\beta}(x) = (32 \pi G)^5 \, a^{10} \prod_{\alpha \leq \beta} {\rm d}\widehat{h}_{\alpha\beta}(x) \; . \tag{3.16}$$

Using (3.11), (3.14) and (3.16), the measure $\left[\mathcal{D}u(h) \mathcal{D}v^*_\rho \, \mathcal{D}v_\sigma \right]$ in (3.9) can be written as

$$\left[\mathcal{D}u(h) \mathcal{D}v^*_\rho \, \mathcal{D}v_\sigma \right] = \mathcal{A} \prod_x \Big[\Big(\prod_{\alpha \leq \beta} {\rm d}\widehat{h}_{\alpha\beta}(x) \Big) \Big(\prod_\rho {\rm d}\widehat{v}^*_\rho(x) \Big) \Big(\prod_\sigma {\rm d}\widehat{v}_\sigma(x) \Big) \Big] , \tag{3.17}$$

where a-independent terms such as $\prod_x \widetilde{g}^{00}(x) \; (\widetilde{g}(x))^{-1}$ are included in $\mathcal{A}$.

Since equations (3.13) and (3.15) contain the *dimensionless* operators $-\nabla_\rho \nabla^\rho$ and $-\nabla_\rho$, to calculate the path integral in (3.6) we consider the bases for symmetric tensors and for vectors constructed with the eigenfunctions of the dimensionless Laplace-Beltrami operator $-\widetilde{\Box}^{(s)}$ defined as (see Appendix C for details)

$$-\widetilde{\Box}^{(s)} \equiv -a^2 \, \Box^{(s)}_a \, , \tag{3.18}$$

where $-\Box^{(s)}_a$ are the spin-s Laplace-Beltrami operators for a sphere of radius a, with $s = 0, 1, 2$. The dimensionless eigenvalues $\lambda^{(s)}_n$ of $-\widetilde{\Box}^{(s)}$ and the corresponding degeneracies $D^{(s)}_n$ are

$$\lambda^{(s)}_n = n^2 + 3n - s \qquad ; \qquad D^{(s)}_n = \frac{2s+1}{3} \left(n + \frac{3}{2} \right)^3 - \frac{(2s+1)^3}{12} \left(n + \frac{3}{2} \right) , \tag{3.19}$$

where $n = s, s+1, \dots$. Expanding $\widehat{h}_{\mu\nu}$, $\widehat{v}^*_\rho$ and $\widehat{v}_\sigma$ in terms of the bases mentioned above, $\delta S^{1l}_{\rm grav}$ turns out to be

$$\delta S^{1l}_{\rm grav} = - \frac{1}{2} \log \frac{\det_1[-\widetilde{\Box}^{(1)} - 3] \det_2[-\widetilde{\Box}^{(0)} - 6]}{\det_0[-\widetilde{\Box}^{(2)} - 2a^2 \Lambda_{\rm cc} + 8] \det_2[-\widetilde{\Box}^{(0)} - 2a^2 \Lambda_{\rm cc}]} + \frac{1}{2} \log \big(2a^2 \Lambda_{\rm cc} \big) + \mathcal{B} \, , \tag{3.20}$$

where $\mathcal{B}$ is an a-independent term, and the index i in $\det_i$ indicates that the product of eigenvalues starts from $\lambda^{(s)}_{s+i}$. The calculations that lead to (3.20) are presented in Appendix C, and closely follow [104].

Eq. (3.20) is an important outcome of our calculation, and we pause for a moment to comment on it. The term $\frac{1}{2}\log(2a^2\Lambda_{cc})$ comes from the integration over one of the modes of $\widehat{h}_{\mu\nu}$ (see Appendix C), $\mathcal{B}$ (as already said) is an a-independent term. Both terms are irrelevant for our scopes. The only relevant term in (3.20) is the first one. Its peculiarity is that it contains only *dimensionless determinants.* This is a fundamental result of our analysis and is due to the fact that we have appropriately considered the correct measure (3.9). In particular, we stress the importance of the term $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$ whose presence is sometimes overlooked. As Eq. (3.11) shows, in fact, it provides the factor a^{-10} that compensates the factor a^{10} coming from $\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)$ in Eq. (3.16). It is also worth to stress that, when written in terms of $\widehat{h}_{\mu\nu}$ and $\widehat{v}_\mu$, $\delta S^{(2)}$ contains only dimensionless quantum fluctuation operators, see (3.13) and (3.15). In this respect, we note that in typical calculations of $\delta S^{1l}_{\rm grav}$ the arguments in $\det_i$ are dimensionful, and an arbitrary scale μ (supposed to be harmless) is introduced to make them dimensionless. In our result (3.20), the determinants turned out to be *automatically* dimensionless[4], and no such arbitrary scale is ever needed.

We now move to the calculation of the right hand side of (3.20) (keeping as explained above only the first term). We will follow two different strategies: (i) we calculate the determinants performing the product of a finite number N of eigenvalues; (ii) we repeat the calculation using proper-time regularization. Anticipating on the results, from both calculations we will see that the quartically and quadratically divergent contributions to the vacuum energy are absent. We will also explain why in the literature these terms are usually found.

Let us begin with the calculation of the determinants in (3.20) as products of the eigenvalues of the fluctuation operators. We have

$$\delta S^{1l}_{\rm grav} = \frac{1}{2}\sum_{n=2}^{N-2}\Big[D^{(2)}_n \log\left(\lambda^{(2)}_n - 2a^2\Lambda_{cc} + 8\right) + D^{(0)}_n \log\left(\lambda^{(0)}_n - 2a^2\Lambda_{cc}\right) - D^{(1)}_n \log\left(\lambda^{(1)}_n - 3\right) - D^{(0)}_n \log\left(\lambda^{(0)}_n - 6\right)\Big] + \frac{1}{2}\log\big(2a^2\Lambda_{cc}\big) + \mathcal{B}\,, \quad (3.21)$$

where the UV cutoff is introduced in terms of a gauge invariant[5] numerical cut N ($N \gg 1$) on the number of eigenvalues. The choice $N-2$ (rather than N) in the upper limit of the sum is just a matter of convenience and simplifies the expression of $\delta S^{1l}_{\rm grav}$.

Before going on with the calculation, it is worth to stress that a similar numerical cut was introduced in [110–113]. In particular, in [111] the same pure gravity case considered in the present section is studied, while [110] and [112] are devoted to the case of a scalar theory in a gravitational background, a setup that will be considered in section 4 and in chapter 5, where we will treat the case of a free and of an interacting scalar theory, respectively. There is however a crucial difference between the calculation performed in [111] and our calculation. As said above (see comments below (3.9)), the measure to

[4]Note that the radius a and the cosmological constant Λ_{cc} appear only in the dimensionless combination $a^2\Lambda_{cc}$.

[5]The gauge invariance is guaranteed by the fact that N is a cut on the eigenvalues $\lambda^{(s)}_n$.

be used to calculate $\delta S^{1l}_{\rm grav}$ in (3.6) is the Fradkin - Vilkovisky measure in (3.9). The latter contains the non-trivial metric-dependent terms $g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$, that result from the integration over the conjugate momenta in the original Hamiltonian formulation of the theory, and ensure the diffeomorphism invariance of $\delta S^{1l}_{\rm grav}$. In [111], a different measure is used, namely the non-invariant Fujikawa measure [107], and then a non-invariant result is obtained. This is why the conclusions of this work ought to be reconsidered. Similar considerations[6] apply to [110, 112, 113].

Let us go back to the explicit calculation of $\delta S^{1l}_{\rm grav}$. Inserting (3.19) in the right hand side of (3.21) and using the identity $\log\left(x/y\right) = -\int_0^{+\infty} du \left[(x+u)^{-1} - (y+u)^{-1}\right]$, the sum can be put in closed form, though the expression is quite involved. For our purposes, it is sufficient to consider the expansion for $N \gg 1$, that gives

$$\begin{aligned}\delta S^{1l}_{\rm grav} =& -\left(\Lambda_{\rm cc}^2 \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8\log N^2\right) a^2 \\ &+ \frac{N^4}{24}\left(-1 + 2\log N^2\right) + \frac{N^2}{36}\left(203 - 75\log N^2\right) - \frac{779}{90}\log N^2 + \mathcal{B} \\ &+ \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(N^{-2}\right)\,, \end{aligned} \tag{3.22}$$

where $\mathcal{F}(a^2\Lambda_{\rm cc})$ contains only UV-finite terms (no dependence on N), and its explicit expression is given in Appendix D. As already said (see comments below (3.5)), the one-loop corrections $\delta\left(\frac{\Lambda_{\rm cc}}{G}\right)$ and $\delta\left(\frac{1}{G}\right)$ to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ are to be identified with the coefficients of a^4 and a^2 in (3.22) respectively. The cosmological and Newton constant at one-loop turn then out to be

$$\Lambda^{1l}_{\rm cc} = \frac{\Lambda_{\rm cc}\left(1 - \frac{3G\Lambda_{\rm cc}}{\pi}\log N^2\right)}{1 + \frac{G\Lambda_{\rm cc}}{2\pi}\left(N^2 - 8\log N^2\right)} \tag{3.23}$$

$$G^{1l} = \frac{G}{1 + \frac{G\Lambda_{\rm cc}}{2\pi}\left(N^2 - 8\log N^2\right)}\,. \tag{3.24}$$

Since $\Lambda^{1l}_{\rm cc}$ and G^{1l} are the renormalized values of the cosmological and Newton constant, we must have $\Lambda^{1l}_{\rm cc} > 0$ and $G^{1l} > 0$. A simple inspection of (3.23) and (3.24) shows that only positive values of the bare parameters $\Lambda_{\rm cc}$ and G are then admitted.

We now discuss the relation between the numerical cut N and the UV physical cutoff Λ (further comments on this relation are in the next chapter). From (3.1), the classical (de Sitter) solution

$$a_{\rm dS} = \sqrt{\frac{3}{\Lambda_{\rm cc}}} \tag{3.25}$$

is obtained and, since $a_{\rm dS}$ is the (tree-level approximation to the) size of the universe, the connection between N and Λ (that might be for instance M_P) is given by

$$\Lambda \equiv \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}\,. \tag{3.26}$$

Since the physical cutoff Λ is a fixed scale, from (3.26) we see that for increasing values of $a_{\rm dS}$ (i.e. decreasing values of $\Lambda_{\rm cc}$) N grows linearly with $a_{\rm dS}$. This is clearly as expected:

[6] Similarly to [111], in [110, 112, 113] the non-invariant Fujikawa measure is used.

to keep the minimal distance resolution ($\sim 1/\Lambda$) unchanged for increasing values of $a_{\rm dS}$, a finer angular resolution is needed. This requires the inclusion of a larger number of eigenmodes in the decomposition of the fluctuation fields. In other words, for a given value of the physical cutoff Λ the number N of eigenvalues to be included in the sum (3.21) is fixed by the highest eigenvalue $\sim N^2/a_{\rm dS}^2$ of the Laplace-Beltrami operator for the sphere of radius $a_{\rm dS}$.

In this respect, it is worth to recall that for a sphere of generic radius a the dimensionful eigenvalues $\widehat{\lambda}_n^{(s)}$ of the Laplace-Beltrami operator $-\Box_a^{(s)}$ are $\widehat{\lambda}_n^{(s)} \equiv \frac{\lambda_n^{(s)}}{a^2} \sim \frac{n^2}{a^2}$ (see (3.19) where $\lambda_n^{(s)}$ are the eigenvalues of the dimensionless operators $-\widetilde{\Box}^{(s)}$). It might seem natural at first to introduce the physical cutoff Λ through the requirement $\widehat{\lambda}_n^{(s)} \leq \Lambda^2$, rather than through (3.26). However, since the eigenvalues $\widehat{\lambda}_n^{(s)}$ of $-\Box_a^{(s)}$ (that *is not* the dimensionless operator $-\widetilde{\Box}^{(s)}$ that appears in the determinants of Eq. (3.20)) are proportional to a^{-2}, such a requirement would introduce in N a spurious dependence on a, i.e. a spurious dependence on the classical background metric $g_{\mu\nu}^{(a)}$. In other words, the identification $\Lambda^2 \sim \widehat{\lambda}_{\rm max}^{(s)} \sim N^2/a^2$ would give rise to spurious additional terms $\sim a^4$ and $\sim a^2$ in $\delta S_{\rm grav}^{1l}$. Consequently, the results for the one-loop corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ would be altered. In particular, as we will show below, this artificially generates the quartically and quadratically divergent contributions to the vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{\rm cc}}{8\pi G}$ usually found in the literature.

Finally, inserting (3.26) in (3.22), for $\delta S_{\rm grav}^{1l}$ we have

$$\begin{aligned}\delta S_{\rm grav}^{1l} = &- \left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) a^2 \\ &+ \frac{3\Lambda^4}{8\Lambda_{\rm cc}^2}\left(-1 + 2\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) + \frac{\Lambda^2}{12\Lambda_{\rm cc}}\left(203 - 75\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) - \frac{779}{90}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}} + \mathcal{B} \\ &+ \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(\Lambda^{-2}\right) . \end{aligned} \tag{3.27}$$

Eq. (3.27) (and equivalently (3.22)) is one of the most important findings of the present chapter. We will comment on the consequences of this result in the next section. Before doing that, however, it is useful to proceed with the evaluation of $\delta S_{\rm grav}^{1l}$ following the other strategy mentioned above, namely proper-time regularization. We will then conveniently discuss both results together.

Since the operators $(-\widetilde{\Box}^{(s)} - \alpha)$ in (3.20) (with $\alpha = 3,\ 6,\ 2a^2\Lambda_{\rm cc} - 8,\ 2a^2\Lambda_{\rm cc}$) are dimensionless, to regularize the determinants $\det_i(-\widetilde{\Box}^{(s)} - \alpha)$ we introduce the dimensionless proper-time τ, with numerical lower integration bound $1/N^2$ ($N \gg 1$)

$$\det{}_i(-\widetilde{\Box}^{(s)} - \alpha) = e^{-\int_{1/N^2}^{+\infty} \frac{d\tau}{\tau}\, {\rm K}_i^{(s)}(\tau)} . \tag{3.28}$$

The kernel ${\rm K}_i^{(s)}(\tau)$ is ($\lambda_n^{(s)}$ and $D_n^{(s)}$ are the eigenvalues and the degeneracies in (3.19))

$${\rm K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)} . \tag{3.29}$$

To calculate the determinants, we insert (3.29) in (3.28), perform the integration over τ, and then sum over n with the help of the EML formula. For the reader's convenience, we report here this formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} \mathrm{d}x\, f(x) + \frac{f(n_f)+f(n_i)}{2}$$
$$+ \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}\,, \tag{3.30}$$

where p is an integer, B_m are Bernoulli numbers and the rest R_{2p} is given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right)$$
$$= \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} \mathrm{d}x\, f^{(2p)}(x) B_{2p}(x-[x])\,, \tag{3.31}$$

with $B_n(x)$ the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument.

Expanding for $N \gg 1$, we finally get

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log N^2\right) a^4 + \Lambda_{\text{cc}}\left(-N^2 + 8\log N^2\right) a^2$$
$$- \frac{N^4}{12} + \frac{17}{3}N^2 - \frac{1859}{90}\log N^2 + \mathcal{B}$$
$$+ \frac{1}{2}\log\left(2a^2\Lambda_{\text{cc}}\right) + \mathcal{G}(a^2\Lambda_{\text{cc}}) + \mathcal{O}\left(N^{-2}\right)\,, \tag{3.32}$$

where $\mathcal{G}(a^2\Lambda_{\text{cc}})$ contains only UV-finite terms (no dependence on N), and is similar to the term $\mathcal{F}(a^2\Lambda_{\text{cc}})$ in (3.22). As for the previous calculation, the connection between the numerical cut N and the physical cutoff Λ is given by the relation (see (3.26) and comments below)

$$\Lambda \equiv \frac{N}{a_{\text{dS}}} = N\sqrt{\frac{\Lambda_{\text{cc}}}{3}}\,, \tag{3.33}$$

so that (3.32) is written as

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda_{\text{cc}}^2 \log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^4 + \left(-3\Lambda^2 + 8\Lambda_{\text{cc}}\log\frac{3\Lambda^2}{\Lambda_{\text{cc}}}\right) a^2$$
$$- \frac{3\Lambda^4}{4\Lambda_{\text{cc}}^2} + \frac{17\Lambda^2}{\Lambda_{\text{cc}}} - \frac{1859}{90}\log\frac{3\Lambda^2}{\Lambda_{\text{cc}}} + \mathcal{B}$$
$$+ \frac{1}{2}\log\left(2a^2\Lambda_{\text{cc}}\right) + \mathcal{G}(a^2\Lambda_{\text{cc}}) + \mathcal{O}\left(\Lambda^{-2}\right)\,. \tag{3.34}$$

In summary, for $\delta S^{1l}_{\text{grav}}$ we have found (3.34) (or equivalently (3.32)) using the proper-time method, and correspondingly (3.27) (or (3.22)) resorting to the direct product of eigenvalues.

In the next section, we will focus on the most important consequences of Eqs. (3.27) and (3.34), considering the corrections to $\frac{\Lambda_{\text{cc}}}{G}$ and $\frac{1}{G}$ that result from these equations. We will also explain the reason why in previous literature quartically and quadratically divergent contributions to the vacuum energy are typically found.

3.2 One-loop corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$

As already said, the coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$ provide the corrections to $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$, respectively. In the previous section, we calculated $\delta S^{1l}_{\rm grav}$ resorting first to the direct product of eigenvalues, that led to (3.27), and then to the proper-time method, that resulted in (3.34). Though they were derived using two different methods, from both (3.27) and (3.34) we find that up to one-loop order

$$\frac{\Lambda^{1l}_{\rm cc}}{G^{1l}} = \frac{\Lambda_{\rm cc}}{G}\left(1 - \frac{3G\Lambda_{\rm cc}}{\pi}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right) \tag{3.35}$$

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\Lambda^2 - 8\Lambda_{\rm cc}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\right)\right] . \tag{3.36}$$

We now comment on (3.35) and (3.36), starting from the latter. First of all we observe that, since $\Lambda \sim M_P$, taking for the Newton constant the "natural value" $G \sim M_P^{-2}$, from (3.36) we see that the dressing of G does not spoil this natural relation,

$$G^{1l} \sim G \sim \frac{1}{M_P^2} . \tag{3.37}$$

In other words, there is no naturalness problem in connection with the renormalization of the Newton constant.

Moving to (3.35), we immediately see that it contains a surprising result: quantum fluctuations dress the vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{\rm cc}}{8\pi G}$ *only* with logarithmic corrections[7]. The quantum correction to $\rho_{\rm vac}$ goes like $\log M_P$ (again we take $\Lambda \sim M_P$) rather than M_P^4, the latter being the typically acknowledged UV-sensitivity. This strong power-like dependence on the UV scale requires the bare value $\frac{\Lambda_{\rm cc}}{8\pi G}$ of the vacuum energy to be $\sim M_P^4$, with a coefficient that must be enormously fine-tuned for it to cancel (quite exactly) the one-loop generated M_P^4 correction. On the contrary, with Eq. (3.35) the bare cosmological constant $\Lambda_{\rm cc}$ does not need to be $\sim M_P^2$, but might well be $\Lambda_{\rm cc} \ll M_P^2$. Under this latter condition, from (3.35) and (3.37) we have

$$\Lambda^{1l}_{\rm cc} \sim \Lambda_{\rm cc} . \tag{3.38}$$

As a consequence, in pure gravity there is *no naturalness problem* for the cosmological constant. Note also that (3.38) implies that bare and dressed de Sitter radius practically coincide.

It is important at this point to understand why, when the calculation is performed with techniques (the heat-kernel formalism introduced in section 1.2) similar to those we used in the previous section, quartically and quadratically divergent contributions to the vacuum energy are typically found (dimensional regularization is obviously excluded from these considerations since the method itself is constructed ad hoc not to display them). To investigate on this point, we consider (3.32) (equivalently we could consider (3.22)), and for the sake of the present discussion we temporarily realize the connection between N and Λ through the relation

$$\Lambda = \frac{N}{a} , \tag{3.39}$$

[7] As we show in the next section, this holds true also when matter is included.

rather than via $\Lambda = N/a_{\rm dS}$ (Eq. (3.33)). As explained above, Eq. (3.39) would correspond to the (improper) identification of the dimensionful cutoff Λ with the maximal eigenvalue $\widehat{\lambda}^{(s)}_{\rm max}$ of the dimensionful Laplacian $-\Box^{(s)}_a$ (see comments below (3.26) and above (3.33)). Inserting (3.39) in (3.32), for $\delta S^{1l}_{\rm grav}$ we would obtain (we neglect the terms starting from $\mathcal{B}$ as they are inessential for the present discussion)

$$\begin{aligned}\delta S^{1l}_{\rm grav} = &- \left[\Lambda^2_{\rm cc} \log\left(a^2\Lambda^2\right)\right] a^4 + \Lambda_{\rm cc}\left[-\Lambda^2 a^2 + 8\log\left(a^2\Lambda^2\right)\right] a^2 \\ &- \frac{\Lambda^4}{12} a^4 + \frac{17}{3}\Lambda^2 a^2 - \frac{1859}{90}\log\left(\Lambda^2 a^2\right) \,,\end{aligned} \tag{3.40}$$

which is trivially rewritten as

$$\begin{aligned}\delta S^{1l}_{\rm grav} = &- \left[\frac{\Lambda^4}{12} + \Lambda_{\rm cc}\Lambda^2 + \Lambda^2_{\rm cc}\log\left(a^2\Lambda^2\right)\right] a^4 + \left[\frac{17}{3}\Lambda^2 + 8\Lambda_{\rm cc}\log\left(a^2\Lambda^2\right)\right] a^2 \\ &- \frac{1859}{90}\log\left(a^2\Lambda^2\right) \,.\end{aligned} \tag{3.41}$$

Eq. (3.41) is nothing but the well-known result found in the literature when the calculation is performed resorting to heat-kernel techniques. Quartically and quadratically divergent corrections to $\Lambda_{\rm cc}/G$ are found.

The comparison between Eq. (3.40) (that we have written only for the sake of the present discussion using Eq. (3.39) for the physical cutoff Λ) and the original result (3.32) for $\delta S^{1l}_{\rm grav}$ allows to understand how these spurious divergences are generated. This point is crucial to our analysis, and it is worth to examine these terms one after the other. Taking $-\Lambda_{\rm cc} N^2 a^2$ of (3.32), and replacing N^2 according to (3.39), the quadratically divergent term $-\Lambda_{\rm cc}\Lambda^2 a^4$ of (3.41) arises. This is an example of how, when the UV cutoff Λ is improperly identified through (3.39), a term that is originally proportional to a^2 (artificially) becomes an a^4 term. Consequently, a term that originally renormalizes $1/G$ becomes a quadratic divergence that renormalizes $\Lambda_{\rm cc}/G$. Similarly, $-\frac{N^4}{12}$ in the second line of (3.32) becomes $-\frac{\Lambda^4}{12}a^4$, thus (artificially) giving rise to the (in)famous quartically divergent contribution to the vacuum energy $\Lambda_{\rm cc}/8\pi G$. Finally, $\frac{17}{3}N^2$ in the second line of (3.32) becomes $\frac{17}{3}\Lambda^2 a^2$, i.e. a quadratically divergent contribution to $1/G$. Concerning this latter term, a simple inspection of (3.34) and (3.41) shows that (3.39) generates in the one-loop correction to $1/G$ a quadratic divergence that is opposite in sign with respect to the original one (Eq. (3.34)).

The importance of the above results and considerations can hardly be underestimated. What we have just seen is that implementing the cut in the fluctuation determinants taking as physical cutoff the maximal eigenvalues $\widehat{\lambda}^{(s)}_{\rm max}$ (see (3.39)) introduces in $\delta S^{1l}_{\rm grav}$ a spurious dependence on the background metric $g^{(a)}_{\mu\nu}$. As stressed above, the connection between the numerical cut N and the physical cutoff Λ must rather be realised through (3.33), i.e. through the de Sitter radius $a_{\rm dS}$. In section 4.3 of the next chapter, we will show that usual heat-kernel calculations, performed using the non-diffeomorphism invariant Fujikawa measure, automatically implement the improper identification (3.39) of the UV physical cutoff (see also section 5.3 of chapter 5, where the same is shown in the case of a scalar theory on a non-trivial gravitational background), and, according to what we showed above, this is at the origin of the appearance of quartically and quadratically UV-sensitive terms in $\Lambda_{\rm cc}/8\pi G$.

Before ending this section, we comment on the terms in the second and third line of Eq. (3.32). Starting with the third line, we observe that $\frac{1}{2}\log(2a^2\Lambda_{\rm cc})$ and $\mathcal{G}(a^2\Lambda_{\rm cc})$ are $\mathcal{O}(1)$, negligible contributions to $\delta S^{1l}_{\rm grav}$ (the same holds true for the similar terms $\frac{1}{2}\log(2a^2\Lambda_{\rm cc})$ and $\mathcal{F}(a^2\Lambda_{\rm cc})$ in the third line of (3.22)). Concerning the terms in the second line, in our analysis we interpret them as constants, though the high symmetry of the spherical background prevents from a clear distinction between constants and R^2 terms. A less symmetric background should help in clarifying this issue.

In the present and in the previous section, we considered pure gravity, ignoring the matter term $S_{\rm mat}$ in the action. A full treatment of the quantum matter-gravity system is beyond the scopes of the present thesis, and is left for future studies [114]. Still, we can have an indication of the role that matter plays on the renormalization of $\frac{\Lambda_{\rm cc}}{G}$ and $\frac{1}{G}$ considering the simpler cases of free scalar and free fermion fields on a gravitational background. The next section is devoted to this analysis.

3.3 Matter contribution

3.3.1 Free scalar field

Let us consider the free theory[8] of a real scalar field ϕ of mass m_ϕ defined on the spherical gravitational background $g^{(a)}_{\mu\nu}$,

$$S^{(\phi)}_{\rm mat} = \int {\rm d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\,g^{(a)\,\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}\,m_\phi^2\,\phi^2\right]\,. \tag{3.42}$$

Adding $S^{(\phi)}_{\rm mat}$ to the classical gravitational action $S^{(a)}_{\rm grav}$ in (3.4), the total action is

$$S = S^{(a)}_{\rm grav} + S^{(\phi)}_{\rm mat} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \int {\rm d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\,g^{(a)\,\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{1}{2}\,m_\phi^2\,\phi^2\right]\,. \tag{3.43}$$

Integrating out the field ϕ, for the effective gravitational action we get

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \delta S_{\rm grav}\,, \tag{3.44}$$

with $\delta S_{\rm grav}$ given by

$$e^{-\delta S_{\rm grav}} = \int\prod_x\left[\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}{\rm d}\phi(x)\right]e^{-\int {\rm d}^4x\,\sqrt{g^{(a)}}\left[-\frac{1}{2}\phi\,\Box^{(0)}_a\,\phi + \frac{1}{2}m_\phi^2\,\phi^2\right]}\,. \tag{3.45}$$

As for pure gravity, the factors $\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}$ in the measure come from the integration over the conjugate momenta in the original path integral (hamiltonian formalism) [18,109]. Similarly to that case, their presence ensures the diffeomorphism invariance of $\delta S_{\rm grav}$ (this is shown in chapter 6 with the help of an explicit calculation). We then

[8]The interacting theory is considered in chapter 5, where we show that the issues related to the path integral measure and the introduction of the UV physical cutoff have crucial consequences also on the renormalization of the mass of scalar particles.

calculate $\delta S_{\rm grav}$ in a coordinate system where the relation (3.10) holds, from which we have

$$\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}} = a\left(\widetilde{g}^{00}(x)\right)^{\frac{1}{2}}\left(\widetilde{g}(x)\right)^{\frac{1}{4}}\,, \tag{3.46}$$

Eq. (3.45) can be rewritten in terms of dimensionless quantities as

$$e^{-\delta S_{\rm grav}} = \Big[\prod_x \left(\widetilde{g}^{00}(x)\right)^{\frac{1}{2}}\Big]\int\Big[\prod_x \left(\widetilde{g}(x)\right)^{\frac{1}{4}}\,{\rm d}\widehat{\phi}(x)\Big]\,e^{-\int {\rm d}^4x\,\sqrt{\widetilde{g}}\,[-\frac{1}{2}\widehat{\phi}\,\widetilde{\Box}^{(0)}\,\widehat{\phi}+\frac{1}{2}a^2\,m_\phi^2\,\widehat{\phi}^2]}\,, \tag{3.47}$$

where $-\widetilde{\Box}^{(0)}$ is the dimensionless spin-0 Laplacian (see (3.18)), and $\widehat{\phi}$ the dimensionless field defined as $\widehat{\phi}\equiv a\phi$.

Performing the Gaussian integrations we get[9]

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \frac{1}{2}\,{\rm Tr}\log\big(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2\big) + \mathcal{C}_\phi\,, \tag{3.48}$$

where ($\delta^{(4)}(0)$ below is due to the replacement $\sum_x \to \int {\rm d}^4x$)

$$\mathcal{C}_\phi \equiv -\frac{1}{2}\log\Big(\prod_x \widetilde{g}^{00}(x)\Big) = -\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4x\,\log\big(\widetilde{g}^{00}(x)\big) \tag{3.49}$$

comes from the exponentiation of the measure term $\prod_x(\widetilde{g}^{00}(x))^{1/2}$ (see (3.47)). The presence of the non invariant term $\mathcal{C}_\phi$ might lead one to suspect that the above result for $S^{\rm eff}_{\rm grav}$ is not invariant, but this is not the case. In fact, as thoroughly discussed in [18, 20, 46], and more recently in [37] (whose results will be presented in chapter 6), subtleties arise in the calculation of $\log\big(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2\big)$. More specifically, one should carefully take into account the distributional nature of the Green functions of the operator $(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2)$. When this is done, from the calculation of ${\rm Tr}\log\big(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2\big)$ the non-trivial term $\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4x\,\log(\widetilde{g}^{00}(x))$ arises, that cancels the term $\mathcal{C}_\phi$ in (3.48). For this reason, the latter does not appear in the forthcoming expressions. All the other terms from the "Tr log" are diffeomorphism invariant [20].

To calculate the determinant in (3.48), we use the proper-time method (see (3.28)). Being the operator $(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2)$ dimensionless, its determinant is written in terms of a dimensionless proper-time τ with numerical lower integration bound $1/N^2$ ($N \gg 1$)

$$\det(-\widetilde{\Box}^{(0)} + a^2 m_\phi^2) = e^{-\int_{1/N^2}^{+\infty}\frac{{\rm d}\tau}{\tau}\,{\rm K}^{(0)}(\tau)}\,, \tag{3.50}$$

where the kernel ${\rm K}^{(0)}(\tau)$ is ($\lambda_n^{(0)}$ and $D_n^{(0)}$ are the eigenvalues and degeneracies reported in (3.19))

$${\rm K}^{(0)}(\tau) = \sum_{n=0}^{+\infty} D_n^{(0)}\,e^{-\tau\left(\lambda_n^{(0)}+a^2 m_\phi^2\right)}\,. \tag{3.51}$$

Inserting (3.51) in (3.50), performing the integration over τ, summing over n with the help of the EML formula (3.30), and finally expanding for $N \gg 1$, we get

$$\begin{aligned} S^{\rm eff}_{\rm grav} = &\frac{\pi}{3}\left(\frac{\Lambda_{\rm cc}}{G} - \frac{m_\phi^4}{8\pi}\log N^2\right)a^4 - 2\pi\left[\frac{1}{G} - \frac{m_\phi^2}{24\pi}\left(N^2 + 2\log N^2\right)\right]a^2 \\ &- \frac{N^4}{24} - \frac{N^2}{6} - \frac{29}{180}\log N^2 + {\rm finite}\,. \end{aligned} \tag{3.52}$$

---

[9] Note that, had we missed in the measure (3.45) the factors $\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}$, the a-dependence of the fluctuation operator in (3.48) would have been altered.

The similarity of (3.52) with (3.32) (and also with (3.22)) is evident. The numerical cut N is related to the physical cutoff Λ ($\sim M_P$) through the relation (3.33), and as for the pure gravity case the vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{\rm cc}}{8\pi G}$ receives only a (mild) logarithmically divergent correction,

$$\delta\left(\frac{\Lambda_{\rm cc}}{8\pi G}\right) = -\frac{m_\phi^4}{64\pi^2}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\,. \tag{3.53}$$

Now, had we proceeded with the (incorrect) identification of Λ through the relation $\Lambda = \frac{N}{a}$ (see (3.26) and (3.39) and comments below them), spurious quartically and quadratically divergent terms would have appeared in the quantum correction to $\Lambda_{\rm cc}/G$ (see (3.41) and comments below). For instance, with such an identification the term $-\frac{N^4}{24}$ would become $-\frac{\Lambda^4}{24}\,a^4$, that is a (spurious) quartically divergent correction to $\Lambda_{\rm cc}/G$; the term $\frac{N^2}{12}\,m^2\,a^2$ (that originally renormalizes $1/G$, see (3.52)) would become $\frac{\Lambda^2}{12}\,m^2\,a^4$, thus generating an "artificial" quadratically divergent correction to $\Lambda_{\rm cc}/G$.

Eq. (3.53) is instructive also for another reason. If on the one hand it tells us that in the quantum correction to the vacuum energy quartic and quadratic divergences do not appear, at the same time it shows that to fully solve the cosmological constant problem a physical mechanism that allows to get rid of the $\sim m_\phi^4$ contribution to $\Lambda_{\rm cc}/G$ has to be found. If $m_\phi \sim \mathcal{O}(\text{EW scale})$, and again we take $\Lambda \sim M_P$, Eq. (3.53) shows that the cosmological constant problem is reduced from a ~ 120 orders of magnitude problem to a ~ 50 orders of magnitude one. In chapter 5 (see in particular section 5.2), we comment on how this problem of the large masses should be handled following the approach put forward in [11].

In the next section, we continue our investigation on the matter contribution to $\Lambda_{\rm cc}/8\pi G$ adding to $S^{(a)}_{\rm grav}$ the action $S^{(\psi)}_{\rm mat}$ of a free Dirac field on a spherical gravitational background.

3.3.2 Free fermion field

Let us consider the free theory of a massive Dirac field ψ defined on the spherical gravitational background $g^{(a)}_{\mu\nu}$. The action is ($\nabla\!\!\!/$ is the Dirac operator on the sphere, $\nabla\!\!\!/ = \gamma^a e^\mu_a(\partial_\mu + \frac{1}{2}\sigma^{bc}\omega_{bc\mu})$, where $\boldsymbol{e}_a \equiv e^\mu_a\partial_\mu$ and $\omega_{abc} \equiv e^\mu_a\,\omega_{bc\mu}$ are the vielbein and the spin connection on the sphere respectively)

$$S^{(\psi)}_{\rm mat} = \int \mathrm{d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\left(\overline{\psi}(\nabla\!\!\!/\psi) - \overline{(\nabla\!\!\!/\psi)}\,\psi\right) + m_\psi\,\overline{\psi}\psi\right]. \tag{3.54}$$

Adding $S^{(\psi)}_{\rm mat}$ to $S^{(a)}_{\rm grav}$, the total action is

$$S = S^{(a)}_{\rm grav} + S^{(\psi)}_{\rm mat} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \int \mathrm{d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\left(\overline{\psi}(\nabla\!\!\!/\psi) - \overline{(\nabla\!\!\!/\psi)}\,\psi\right) + m_\psi\,\overline{\psi}\psi\right]. \tag{3.55}$$

Integrating over $\overline{\psi}$ and ψ, the correction $\delta S_{\rm grav}$ to the classical Einstein-Hilbert action is given by [18, 109]

$$e^{-\delta S_{\rm grav}} = \int \prod_x \left[\left(g^{(a)}(x)\,g^{(a)\,00}(x)\right)^{-2}\left(\prod_{s=1}^{4}\mathrm{d}\overline{\psi}_s(x)\right)\left(\prod_{s'=1}^{4}\mathrm{d}\psi_{s'}(x)\right)\right]$$
$$\times\, e^{-\int \mathrm{d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\left(\overline{\psi}(\nabla\!\!\!/\psi) - \overline{(\nabla\!\!\!/\psi)}\,\psi\right) + m_\psi\,\overline{\psi}\psi\right]}\,. \tag{3.56}$$

Now, proceeding in the same way as done for the pure gravity and the scalar field cases, observing that from (3.10)

$$\left(g^{(a)}(x)\, g^{(a)\,00}\right)^{-2} = a^{-12}\left(\widetilde{g}(x)\,\widetilde{g}^{00}(x)\right)^{2}\,, \tag{3.57}$$

it is immediate to see that (3.56) can be written as[10]

$$e^{-\delta S_{\rm grav}} = \Big[\prod_x (\,\widetilde{g}^{00}(x))^{-2}\Big]\int\Big[\prod_x (\widetilde{g}(x))^{-2}\Big(\prod_{s=1}^{4}\mathrm{d}\widehat{\overline{\psi}}_s(x)\Big)\Big(\prod_{s'=1}^{4}\mathrm{d}\widehat{\psi}_{s'}(x)\Big)\Big]$$
$$\times\, e^{-\int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\left[\frac{1}{2}\left(\widehat{\overline{\psi}}(\widetilde{\nabla\!\!\!/}\widehat{\psi})-(\overline{\widetilde{\nabla\!\!\!/}\widehat{\psi}})\,\widehat{\psi}\right)+a\,m_\psi\,\widehat{\overline{\psi}}\widehat{\psi}\right]}\,, \tag{3.58}$$

where $\widetilde{\nabla\!\!\!/}$ is the dimensionless Dirac operator $\widetilde{\nabla\!\!\!/}\equiv a\nabla\!\!\!/$, and $\widehat{\psi}$ the dimensionless field defined as $\widehat{\psi}\equiv a^{\frac{3}{2}}\psi$.

Performing the Gaussian integrations we get

$$S^{\rm eff}_{\rm grav} \equiv S^{(a)}_{\rm grav} + \delta S_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 - \mathrm{Tr}\log\left(\widetilde{\nabla\!\!\!/} + a\,m_\psi\right) + \mathcal{C}_\psi\,, \tag{3.59}$$

where

$$\mathcal{C}_\psi \equiv 2\log\Big(\prod_x \widetilde{g}^{00}(x)\Big) = 2\,\delta^{(4)}(0)\int \mathrm{d}^4x\,\log\big(\,\widetilde{g}^{00}(x)\big)\,, \tag{3.60}$$

for which similar considerations hold as those developed for the term $\mathcal{C}_\phi$ in (3.49), and for this reason $\mathcal{C}_\psi$ does not appear in the forthcoming expressions. Since

$$\mathrm{Tr}\log\left(\widetilde{\nabla\!\!\!/} + a\,m_\psi\right) = \frac{1}{2}\,\mathrm{Tr}\log\left(\widetilde{\nabla\!\!\!/}^{\,2} + a^2 m_\psi^2\right)\,, \tag{3.61}$$

introducing the dimensionless proper-time τ with numerical cut $N \gg 1$ (see (3.50)), for $S^{\rm eff}_{\rm grav}$ we get

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \frac{1}{2}\int_{1/N^2}^{+\infty}\frac{\mathrm{d}\tau}{\tau}K^{(1/2)}(\tau)\,, \tag{3.62}$$

where

$$\mathrm{K}^{(1/2)}(\tau) = \sum_{n=1/2}^{+\infty} D_n^{(1/2)}\, e^{-\tau\left(\lambda_n^{(1/2)}+a^2m_\psi^2\right)} \tag{3.63}$$

is the kernel of the operator $\nabla\!\!\!/^{\,2} + a^2 m_\psi^2$. The eigenvalues $\lambda_n^{(1/2)}$ and the corresponding degeneracies $D_n^{(1/2)}$ of $\nabla\!\!\!/^{\,2}$ are [115]

$$\lambda_n^{(1/2)} = \left(n+\frac{3}{2}\right)^2 \quad ; \quad D_n^{(1/2)} = \frac{(2n+1)(2n+3)(2n+5)}{12} \quad , \quad \text{with}\ \ n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \tag{3.64}$$

---

[10]Note that $\overline{\psi}$ and ψ are Grassmann variables, and the change of integration variables $(\overline{\psi},\psi)\to(\widehat{\overline{\psi}},\widehat{\psi})$ involves the inverse Jacobian, that is a factor a^{12} in the measure. The latter is canceled by the factor a^{-12} in (3.57).

Inserting (3.63) in (3.62), performing the integration over τ, summing over n with the help of the EML formula (3.30), and finally expanding for $N \gg 1$, we get

$$S^{\rm eff}_{\rm grav} = \frac{\pi}{3}\left(\frac{\Lambda_{\rm cc}}{G} + \frac{m_\psi^4}{4\pi}\log N^2\right)a^4 - 2\pi\left[\frac{1}{G} + \frac{m_\psi^2}{12\pi}\left(N^2 - \log N^2\right)\right]a^2$$
$$+ \frac{N^4}{12} - \frac{N^2}{6} + \frac{11}{360}\log N^2 + \text{finite}\,. \tag{3.65}$$

The similarity of (3.65) with (3.52) and (3.32) is evident. Again, the numerical cut N is related to the physical cutoff Λ ($\sim M_P$) through the relation (3.33), and as for the previous cases the vacuum energy $\rho_{\rm vac} = \frac{\Lambda_{\rm cc}}{8\pi G}$ receives only a logarithmically divergent correction,

$$\delta\left(\frac{\Lambda_{\rm cc}}{8\pi G}\right) = \frac{m_\psi^4}{32\pi^2}\log\frac{3\Lambda^2}{\Lambda_{\rm cc}}\,. \tag{3.66}$$

As in sections 3.2 and 3.3.1, we observe that had we proceeded with the (incorrect) identification of Λ through the relation $\Lambda = N/a$, spurious quartically and quadratically divergent terms would have appeared in the correction to $\frac{\Lambda_{\rm cc}}{G}$ (see (3.41), (3.53) and comments below these equations). Comparing (3.66) (fermions) with (3.53) (bosons) we see that both contributions to the vacuum energy are proportional to the forth power of the respective masses, although with opposite signs. Naturally, the total correction depends on the particle content of the full theory. Moreover, as already stressed at the end of section 3.3.1 for the scalar case, considering the physical cutoff of order the Planck scale, and the masses of order the Fermi scale, the absence of quartic and quadratic divergences in both (3.66) and (3.53) alleviates the cosmological constant problem of about 70 orders of magnitude.

It is important to stress that the fact that the vacuum energy has only a logarithmic sensitivity to the UV physical cutoff (see (3.34), (3.53) and (3.66)) comes from a careful treatment of the path integral that defines the effective action (in a curved spacetime background), and no supersymmetric embedding of the theory (supergravity) is needed to get rid of power-like divergences. These results represent a first step towards a more complete analysis of the cosmological constant problem.

3.4 Summary and conclusions

Considering pure gravity within the (Euclidean) Einstein-Hilbert truncation, we performed the calculation of the one-loop effective action $\Gamma^{1l}_{\rm grav}$ paying particular attention to important aspects that in the past were either missed or mistreated. In previous literature, this calculation was usually done within the heat-kernel formalism and gave rise to quartically and quadratically UV-sensitive contributions to the vacuum energy $\rho_{\rm vac}$. As a consequence, if the UV physical cutoff Λ is $\sim M_P$, a discrepancy of more than 120 orders of magnitude between the theoretical prediction and the measured value of $\rho_{\rm vac}$ is found. In this respect, it is worth to stress that up to now two main approaches have been adopted to dispose of these quartically and quadratically divergent terms, one formal, the other physical. The formal one consists in performing the calculation resorting to regularization schemes, such as dimensional regularization, that cancel power-like divergences by construction. Obviously, the application of these methods cannot be regarded as a physical way to solve the original problem: they simply implement a technical cancellation. On the more physical side, where the presence of these divergences is acknowledged, their cancellation is realized resorting to a supersymmetric embedding of the theory (supergravity).

The calculations of the present chapter show that, if the correct (diffeomorphism invariant) measure is used in the path integral that defines the effective gravitational action, and the physical cutoff Λ is properly introduced, $\rho_{\rm vac}$ presents only a (mild) logarithmic sensitivity to Λ. In other words, when the measure and the UV cutoff are treated carefully, the quartically and quadratically UV-sensitive contributions to $\rho_{\rm vac}$ are *automatically* absent. Our results are neither formal, nor resort to any ad hoc physical cancellation mechanism. They show that in $\rho_{\rm vac}$ quartic and quadratic divergences simply do not appear. Moreover, considering the free theory of a scalar and a Dirac field on a spherical gravitational background, we have shown that this holds true even in the presence of matter fields.

In the next chapter, the analysis will be extended considering the renormalization group flow (see section 1.4 for an introduction to the RG approach) of the pure gravitational action found in [23, 24]. Taking for the running action the Einstein-Hilbert truncation, we derive the RG equations for the Newton and cosmological constant. We show that, when this is done using the diffeomorphism invariant FV path integral measure and introducing properly the running scale k, the beta functions turn out to be significantly different from those of previous literature. In particular, we see that they do not possess the non-trivial UV-attractive fixed point of the asymptotic safety scenario [29,30].

Chapter 4

Path integral measure and RG equations for gravity

The Einstein-Hilbert (EH) theory of gravity is perturbatively non-renormalizable. Two scenarios are then possible [25]: either it is an effective field theory valid up to a maximal energy scale above which its UV completion (maybe string theory) takes over, or it is non-perturbatively renormalizable through the existence of a (non-trivial) UV-attractive fixed point of its RG flow. This latter possibility was dubbed asymptotic safety scenario [26].

As said in the Introduction, first studies on the possible realization of the AS scenario for gravity in $d = 4$ dimensions date back to the late nineties [29–31], and were conducted within the effective average action Γ_k formalism introduced in [33]. In [30], considering for Γ_k the Einstein-Hilbert truncation, a non-trivial UV-attractive fixed point with positive Newton and cosmological constant was found (in addition to the Gaussian fixed point). The existence of this fixed point was later confirmed resorting to the proper-time formalism [34].

In the present chapter, we consider the RG evolution of the pure gravitational action S_k. Taking for the latter the Einstein-Hilbert truncation, we derive RG equations for the running Newton and cosmological constant, G_k and Λ_k, respectively. We show that, if the diffeomorphism invariant FV measure is used in the path integral that defines S_k, and the running scale k is introduced according to the analysis of the previous chapter, the RG equations for G_k and Λ_k turn out to be substantially different from those of [29, 30] and [34]. In particular, they do not possess the non-trivial UV-attractive fixed point of the AS scenario. We also show why (and how) the latter is artificially generated by the way the RG flow is implemented in [29, 30, 34].

The analysis of the present chapter has been published in [23, 24].

4.1 RG equations for the cosmological and Newton constant

With the help of the results and techniques presented in the previous chapter for the calculation of the one-loop effective action $\Gamma^{1l}_{\rm grav}$, in this section we derive the RG equation for the pure gravitational action, considering for the latter the Einstein-Hilbert truncation (3.1). As before, we write the metric $g_{\mu\nu}$ as the sum of a spherical background $g^{(a)}_{\mu\nu}$

plus the fluctuation $h_{\mu\nu}$ (see (3.3)). Our RG strategy is as follows (see section 1.4 for an introduction to the Wilsonian RG). The (bare) action S_N at the "UV scale" N, that in terms of the dimensionful physical cutoff Λ ($= N/a_{\rm dS}$, see (3.26)) can be indicated as S_Λ, contains modes of $\widehat{h}_{\mu\nu}$ (see (3.12)) up to the N-th ones. The action S_L at the "lower scale" L ($< N$) is obtained integrating out the modes within the range $[L, N]$. The action $S_{L-\delta L}$ at the scale $L - \delta L$ is obtained from S_L integrating out the modes in the "infinitesimal shell" $[L - \delta L, L]$ ($\frac{\delta L}{L} \ll 1$)

$$S_{L-\delta L}[\, g^{(a)}_{\mu\nu}] = S_L[\, g^{(a)}_{\mu\nu}] + \delta S_L \,, \tag{4.1}$$

where δS_L is given by the first line of Eq. (3.20) with $\Lambda_{\rm cc}$ replaced by Λ_L and the fluctuation determinants restricted to the subspace of modes in the shell[1] $[L - \delta L, L]$.

Following steps similar to those presented in section 3.1, we now derive δS_L resorting to two different methods: (i) direct sum over the eigenvalues of the fluctuation operators; (ii) proper-time. For S_L we take the Einstein-Hilbert truncation ansatz

$$S_L = \frac{1}{16\pi G_L} \int \mathrm{d}^4x \, \sqrt{g}\, \left(-R + 2\Lambda_L\right) \underset{g_{\mu\nu} = g^{(a)}_{\mu\nu}}{=} \frac{\pi \Lambda_L}{3G_L} a^4 - \frac{2\pi}{G_L} a^2 \,, \tag{4.2}$$

whose minimum is $a_L^{\rm dS} = \sqrt{3/\Lambda_L}$. Note that, according to the notation introduced in the above equation, the bare action (3.1) is obtained for $L = N$ with

$$\Lambda_N \equiv \Lambda_{\rm cc} \qquad \text{and} \qquad G_N \equiv G \,. \tag{4.3}$$

From the equation (4.1) for S_L, we will finally derive the RG equations for the running cosmological and Newton constant, Λ_L and G_L respectively. As we will see, both methods (i) and (ii) give rise to the same equations.

4.1.1 Method (i): product of eigenvalues

In this case, δS_L is obtained taking the product of eigenvalues of the fluctuation operators in the shell $[L - \delta L, L]$. We have

$$S_{L-\delta L}[\, g^{(a)}_{\mu\nu}] = S_L[\, g^{(a)}_{\mu\nu}] + \delta S_L = S_L[\, g^{(a)}_{\mu\nu}] + \sum_{n \in\, [L-\delta L, L]} f_L(n), \tag{4.4}$$

with $f_L(n)$ given by (see Eq. (3.20) with $\Lambda_{\rm cc}$ replaced by Λ_L and Eq. (3.19))

$$\begin{aligned} f_L(n) = {} & D_n^{(2)} \log\left(\lambda_n^{(2)} - 2a^2\Lambda_L + 8\right) + D_n^{(0)} \log\left(\lambda_n^{(0)} - 2a^2\Lambda_L\right) \\ & - \, D_n^{(1)} \log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)} \log\left(\lambda_n^{(0)} - 6\right) \,. \end{aligned} \tag{4.5}$$

In differential form, Eq. (4.4) is written as[2]

$$\frac{\partial S_L}{\partial L} = - \left(\frac{\partial}{\partial L} \sum_{n=2}^{L-2} f_L(n) \right)_{\Lambda_L, G_L} \,, \tag{4.6}$$

[1]For each shell $[L - \delta L, L]$, the contribution of the gauge-fixing and ghost terms is taken into account considering in $S_{\rm gf}$ and $S_{\rm ghost}$ only the modes within such a shell.

[2]Note that the right hand side of (4.6) is nothing but the derivative with respect to L of the one-loop contribution δS_L^{1l} calculated with numerical cut L (see (3.22)).

where the subscripts Λ_L and G_L indicate that the derivative with respect to L is performed keeping Λ_L and G_L fixed. Note that the minimal value for L is $L_{\min} = 4$. Eq. (4.6) describes the evolution of S_L with the running L. The right hand side of (4.6) can be evaluated using the identity $\log(x/y) = -\int_0^{+\infty} \mathrm{d}z \left[(x+z)^{-1} - (y+z)^{-1}\right]$. Expanding the result for[3] $L \gg 1$, we get

$$L\,\frac{\partial S_L}{\partial L} = 2\Lambda_L^2\, a^4 + 2\Lambda_L \left(L^2 - 8\right) a^2 - \frac{L^2(2L^2-25)}{6}\log L^2 - \frac{64L^2}{9} + \frac{779}{45} + \mathcal{O}\left(\frac{1}{L^2}\right). \quad (4.7)$$

Inserting in (4.7) the Einstein-Hilbert truncation (4.2) for S_L, from the identification of the coefficients of a^4 and a^2 of the first member with those of the second member we get the RG equations

$$L\,\frac{\mathrm{d}}{\mathrm{d}L}\frac{\Lambda_L}{G_L} = \frac{6}{\pi}\Lambda_L^2 \quad (4.8)$$

$$L\,\frac{\mathrm{d}}{\mathrm{d}L}\frac{1}{G_L} = -\,\frac{\Lambda_L}{\pi}\left(L^2 - 8\right), \quad (4.9)$$

that are easily translated into the equivalent equations

$$L\,\frac{\mathrm{d}\Lambda_L}{\mathrm{d}L} = \frac{G_L\Lambda_L^2}{\pi}\left(L^2 - 2\right) \quad (4.10)$$

$$L\,\frac{\mathrm{d}G_L}{\mathrm{d}L} = \frac{G_L^2\Lambda_L}{\pi}\left(L^2 - 8\right). \quad (4.11)$$

Before proceeding with the study of these equations, and search for their solution, it is convenient (the reason will be clear in the next section) to move first to the derivation of the RG equations resorting to the proper-time method.

4.1.2 Method (ii): proper-time

Let us move now to the proper-time method. In this case, δS_L is given by (3.20) with Λ_{cc} replaced by Λ_L and the derminants $\det_i$ calculated using (3.28) and (3.29) with integration over the dimensionless proper-time τ restricted to the interval $\left[1/L^2\,,\, 1/(L-\delta L)^2\right]$. Therefore, δS_L is the combination of terms of the kind ($\alpha = 3,\ 6,\ 2a^2\Lambda_L - 8,\ 2a^2\Lambda_L$, see (3.28) and (3.29))

$$\frac{1}{2}\log\det_i(-\widetilde{\Box}^{(s)} - \alpha) = -\frac{1}{2}\int_{1/L^2}^{1/(L-\delta L)^2} \frac{\mathrm{d}\tau}{\tau}\,\mathrm{K}_i^{(s)}(\tau) \qquad \left(\mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}\right). \quad (4.12)$$

In differential form, the RG equation for the running action S_L is

$$\frac{\partial S_L}{\partial L} = -\left(\frac{\partial\left(\delta S_L^{1l}\right)}{\partial L}\right)_{\Lambda_L,\, G_L} \quad (4.13)$$

where δS_L^{1l} is given by (3.32) with N replaced by L and Λ_{cc} by Λ_L. Performing the derivative in the right hand side of (4.13) (Λ_L and G_L fixed as in (4.6)), we finally get

$$L\,\frac{\partial S_L}{\partial L} = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2 - 8\right) a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45} + \mathcal{O}\left(\frac{1}{L^2}\right). \quad (4.14)$$

[3] Since $L_{\min} = 4$, $L \gg 1$ is realised practically in the whole range of L.

Inserting the Einstein-Hilbert truncation (4.2) in the left hand side of (4.14), and identifying the coefficients of a^4 and a^2 of the first and second member, we obtain the RG equations for Λ_L/G_L and $1/G_L$. Remarkably, they turn out to be the same as those obtained resorting to the direct product of eigenvalues (previous section). Therefore, independently of the method used for their derivation, the RG equations for Λ_L and G_L turn out to be those in Eqs. (4.10) and (4.11).

Before ending this section, it is worth to stress that the proper-time formalism was already used to derive RG equations for the cosmological and Newton constant [34], but, as we will see in section 4.1.4, the RG equations of [34] are substantially different from those found in the present chapter. We will comment on this difference in section 4.1.4 and more in detail in section 4.3.

In the next section, we look for the solution to Eqs. (4.10) and (4.11), and we will see that under a well controlled approximation they can be solved analytically. We will also solve them numerically, and compare the analytic and numerical results.

4.1.3 Solution of the RG equations

Let us consider the RG equations (4.10) and (4.11) for Λ_L and G_L. For $L \gg 1$ (remember that $L_{\text{min}} = 4$, so that this condition is verified for all but only few modes near L_{min}) they can be approximated as

$$L\,\frac{\mathrm{d}\Lambda_L}{\mathrm{d}L} = \frac{G_L \Lambda_L^2}{\pi} L^2 \tag{4.15}$$

$$L\,\frac{\mathrm{d}G_L}{\mathrm{d}L} = \frac{G_L^2 \Lambda_L}{\pi} L^2\,. \tag{4.16}$$

The interest of this approximation is that Eqs. (4.15) and (4.16) can be solved analytically. For the UV boundary conditions $\Lambda_N = \Lambda_{\text{cc}}$ and $G_N = G$ the solution is

$$\Lambda_L = \frac{\Lambda_{\text{cc}}}{\sqrt{1 + \frac{G\,\Lambda_{\text{cc}}}{\pi}(N^2 - L^2)}} \tag{4.17}$$

$$G_L = \frac{G}{\sqrt{1 + \frac{G\,\Lambda_{\text{cc}}}{\pi}(N^2 - L^2)}}\,. \tag{4.18}$$

According to (4.17) and (4.18), the sign of both Λ_L and G_L is fixed and given by the sign of Λ_{cc} and G respectively. Moreover, from a simple inspection of (4.10) and (4.11) we see that this is not related to the approximation considered here, but holds true in general. On the contrary, while (4.17) and (4.18) predict that the vacuum energy $\rho_L = \frac{\Lambda_L}{8\pi G_L}$ is constant (no running with L), from the numerical solution of (4.10) and (4.11) we see that instead this is due to the approximation. Nevertheless, the running of ρ_L is so slow that it is very well approximated by the constant behaviour given by (4.17) and (4.18).

In view of the above considerations, and since the measured values of the cosmological and Newton constant are both positive, from now on (unless explicitly stated) we restrict ourselves to consider positive UV boundaries $\Lambda_{\text{cc}} > 0$ and $G > 0$. For completeness, in Appendix E we will also consider (and speculate on) the unphysical cases where one or both of them are negative. With the help of (3.26), we now replace N in (4.17) and (4.18)

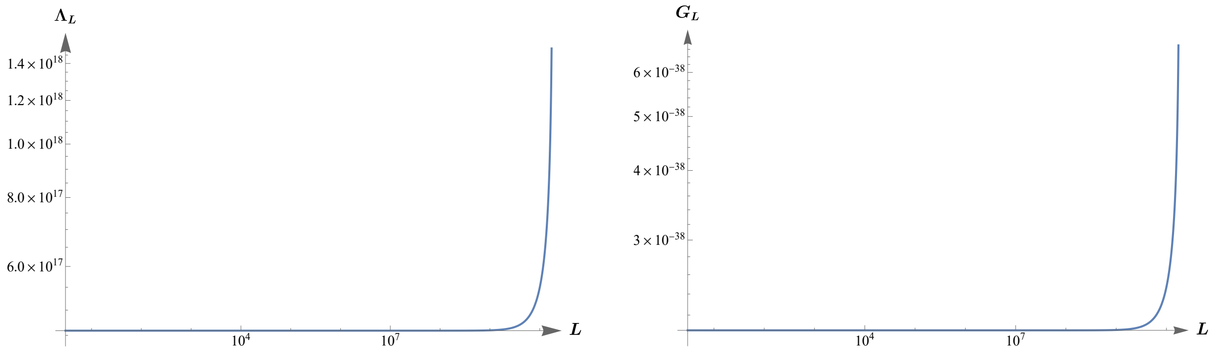

Figure 4.1: *Left panel*: Log-log plot in the range $L_{\min} = 4 \leq L \leq 10^{10}$ of the cosmological constant flow (4.19), for $\Lambda_{\rm cc} = 10^{-20} M_P^2$, $G = 10 M_P^{-2}$ and $\Lambda = M_P$. *Right panel*: Log-log plot in the same range of L of the Newton constant flow (4.20), for the same values of Λ, $\Lambda_{\rm cc}$ and G. In both panels, GeV units are used.

with the UV physical cutoff Λ, and get

$$\Lambda_L = \frac{\Lambda_{\rm cc}}{\sqrt{1 + \frac{G}{\pi}(3\Lambda^2 - \Lambda_{\rm cc} L^2)}} \tag{4.19}$$

$$G_L = \frac{G}{\sqrt{1 + \frac{G}{\pi}(3\Lambda^2 - \Lambda_{\rm cc} L^2)}} \,. \tag{4.20}$$

According to (4.19) and (4.20), as long as L is not much lower than N, the cosmological and Newton constant decrease for decreasing L (the rapidity of the descent depends on the value of $G\Lambda_{\rm cc}$) and then practically freeze to the renormalized values

$$\Lambda_{\rm IR} \sim \frac{\Lambda_{\rm cc}}{\sqrt{1 + \frac{3G\Lambda^2}{\pi}}} \tag{4.21}$$

$$G_{\rm IR} \sim \frac{G}{\sqrt{1 + \frac{3G\Lambda^2}{\pi}}} \,. \tag{4.22}$$

This behaviour is seen in Fig. 4.1, where we show a log-log plot of the flow (4.19) (left panel) and (4.20) (right panel) for $\Lambda = M_P$, $\Lambda_{\rm cc} = 10^{-20} M_P^2$ and $G = 10\, M_P^{-2}$. Solving numerically equations (4.10) and (4.11) for different UV boundary values of Λ_L and G_L, we see that (4.19) and (4.20) provide an excellent approximation to the exact solution.

Eqs. (4.21) and (4.22) show an important outcome of our analysis. Since $\Lambda \sim M_P$, for natural values of the Newton constant, i.e. $G \sim M_P^{-2}$, from (4.21) and (4.22) we have that $\Lambda_{\rm IR} \sim \Lambda_{\rm cc}$ and $G_{\rm IR} \sim G$. This means that our RG equations (4.10) and (4.11) give only a mild dressing of the cosmological and Newton constants. In other words, quantum fluctuations do not modify significantly the UV values $\Lambda_{\rm cc}$ and G: *no naturalness problem* arises in pure gravity.

In the next section, we connect the "numerical scale" L to the physical running scale k and write the RG equations (4.10) and (4.11) in terms of k. In this new framework, we will partially repeat the study of the present section and add further comments.

4.1.4 From L to the physical running scale k

According to (3.26) (see comments below this equation), the relation between the UV numerical cut N and the physical cutoff Λ is given by $\Lambda = N/a_{\rm dS} = N\sqrt{\Lambda_N/3}$. Therefore, the relation between the numerical "running scale" L and the physical running scale k is

$$k = \frac{L}{a_L^{\rm dS}}\,, \tag{4.23}$$

where $a_L^{\rm dS}$ is the de Sitter radius that minimizes the action $S_L[g^{(a)}_{\mu\nu}]$ in (4.2)

$$a_L^{\rm dS} = \sqrt{\frac{3}{\Lambda_L}}\,. \tag{4.24}$$

As seen in the previous section, since we are considering positive UV boundaries[4] $\Lambda_{\rm cc} > 0$ and $G > 0$, the running cosmological constant Λ_L decreases monotonically for decreasing L and remains positive in the whole range $L_{\rm min} \le L \le N$ ($L_{\rm min} = 4$ is the minimal value of L, see (4.6)). Eq. (4.23) establishes then a one-to-one correspondence between L and k, with $k \in [k_{\rm IR}, \Lambda]$ where

$$k_{\rm IR} = \sqrt{\frac{16\Lambda_4}{3}}\,. \tag{4.25}$$

In this respect, let us also observe that, independently of (4.23), the smallest scale k is the inverse of the universe radius $R_{\rm U}$ (see [116]), that means $k_{\rm min} \sim 1/R_{\rm U} \equiv 1/a^{\rm dS}_{L_{\rm min}=4}$. This latter relation is satisfied by (4.23) (see (4.25)). One might think to replace $a_L^{\rm dS}$ in (4.23) with the off-shell radius a, i.e. to relate L and k through $k = L/a$ (see the discussion in section 3.2). As we will show in section 4.3 (see also section 5.3), in fact, this identification of the running scale k is realized by usual implementations of the RG flow within the effective average action and/or the proper-time formalism. If this were done, we would obtain $k_{\rm min} \sim 1/a$. However, since $k_{\rm min}$ is a fixed scale ($k_{\rm min} \sim R_{\rm U}^{-1}$), this dependence on the off-shell radius a appears to be unphysical.

With the help of (4.23) and (4.24), Eqs.(4.10) and (4.11) can be written in terms of the running scale k as ($\Lambda_k \equiv \Lambda_L$ and $G_k \equiv G_L$)

$$k\frac{{\rm d}\Lambda_k}{{\rm d}k} = \frac{3G_k}{\pi}\,\frac{\Lambda_k\left(k^2 - \frac{2}{3}\Lambda_k\right)}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)} \tag{4.26}$$

$$k\frac{{\rm d}G_k}{{\rm d}k} = \frac{3G_k^2}{\pi}\,\frac{k^2 - \frac{8}{3}\Lambda_k}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)}\,. \tag{4.27}$$

Before going on with our analysis, it is worth to remind what we found in the previous sections: the RG equations for the cosmological and Newton constant are (4.10) and (4.11) independently of the method used for their derivation (direct product of eigenvalues, section 4.1.1; proper-time method, section 4.1.2). Obviously, the same holds true for (4.26) and (4.27). Sticking to the proper-time method, we recall that this formalism was already used in [34] to derive RG equations for Λ_k and G_k, but these equations are substantially different from our Eqs. (4.26) and (4.27). In section 4.3, we will show what is at the origin

[4]As already said, in Appendix E we will also consider (and speculate on) the unphysical cases where this condition is released.

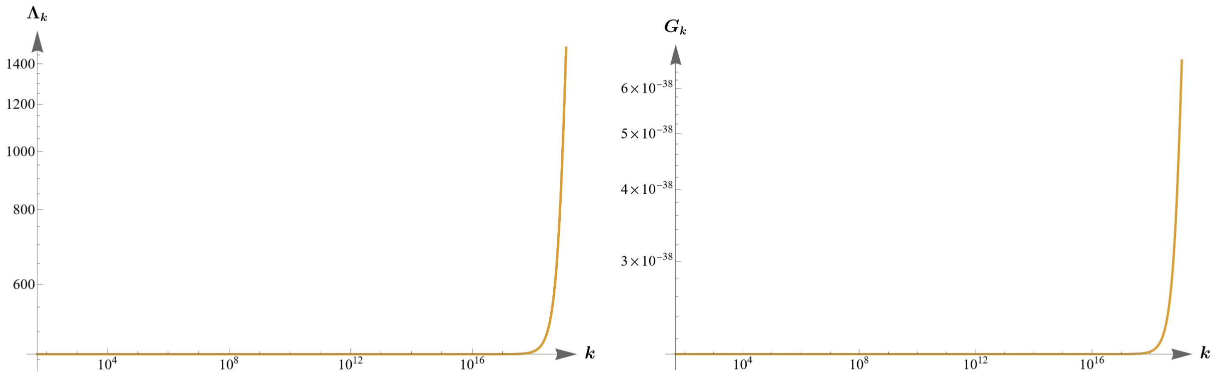

Figure 4.2: *Left panel*: Log-log plot of the cosmological constant flow (GeV units) in the range $k_{\rm IR} \simeq 50\,G$eV $\leq k \leq M_P$, for $\Lambda_{\rm cc} = 10^{-35}M_P^2$, $G = 10M_P^{-2}$ and $\Lambda = M_P$. The approximated flow (4.30) is plotted together with the corresponding (exact) flow from the numerical solution of (4.26)-(4.27). The two curves practically coincide. Around the scale $k \sim 10^{18}\,G$eV the flow creates an elbow that marks the transition from the k^2-running to a quick freezing to the value $\Lambda_{\rm IR} \simeq 459(G\text{eV})^2$. *Right panel*: Log-log plot of the Newton constant flow (GeV units) in the same range of k, and for the same Λ, $\Lambda_{\rm cc}$ and G. The approximated flow (4.31) is plotted together with the corresponding (exact) flow from the numerical solution of (4.26)-(4.27). As for Λ_k, the two curves practically coincides, and around $k \sim 10^{18}\,G$eV the RG flow again forms an elbow that marks the transition from the k^2-running to the frozen value $G_{\rm IR} \simeq 2.07 \cdot 10^{-38}(G\text{eV})^{-2}$.

of this difference, and explain why (4.26) and (4.27) should be the correct RG equations for Λ_k and G_k.

The system (4.26)-(4.27) can be solved numerically, and the solutions are obviously the same as those found in the previous section for the system (4.10)-(4.11), although given in terms of k, Λ_k and G_k. Since $\Lambda_k/k^2 = 3/L^2 \ll 1$ (see also footnote 1), we can expand (4.26) and (4.27) around $\Lambda_k/k^2 \sim 0$. Retaining only the first non-trivial order in both equations we get

$$k\frac{\mathrm{d}\Lambda_k}{\mathrm{d}k} = \frac{3G_k}{\pi}\frac{k^2\Lambda_k}{1+\frac{3G_k}{2\pi}k^2} \tag{4.28}$$

$$k\frac{\mathrm{d}G_k}{\mathrm{d}k} = \frac{3G_k^2}{\pi}\frac{k^2}{1+\frac{3G_k}{2\pi}k^2}\,. \tag{4.29}$$

This system can be solved analytically. Taking at $k = \Lambda$ the UV boundary values $\Lambda_{\rm cc}$ and G for Λ_k and G_k respectively, we get

$$\Lambda_k = \frac{k^2\Lambda_{\rm cc}}{2\left(\Lambda^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right] \tag{4.30}$$

$$G_k = \frac{k^2 G}{2\left(\Lambda^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right]\,. \tag{4.31}$$

In the left panel of Fig. 4.2, the analytic solution Λ_k in (4.30) is plotted (log-log plot) together with the corresponding numerical solution of the system (4.26)-(4.27), with $\Lambda = M_P$, $\Lambda_{\rm cc} = 10^{-35}M_P^2$ and $G = 10M_P^{-2}$. The two curves are practically indistinguishable in the whole range $k_{\rm IR} \leq k \leq M_P$. A similar plot for G_k is shown in the right panel. We verified that (4.30) and (4.31) very well approximate the numerical solutions to the full equations (4.26) and (4.27) for different boundary values $\Lambda_{\rm cc}$ and G. From Fig. 4.2 and from (4.30) and (4.31), we see that Λ_k and G_k run quadratically with the scale k until the latter reaches the "transition scale"

$$k_{\rm tr} \sim \left(\frac{\Lambda^2 + \frac{\pi}{3G}}{G} \right)^{1/4} . \tag{4.32}$$

Around this scale, both Λ_k and G_k show an "elbow" that marks the transition from the quadratic running to a constant behaviour. This is seen in Fig. 4.2, where the elbow is in the region around the scale $k \sim 10^{18}\,G$eV. Below $k_{\rm tr}$ the running cosmological and Newton constant rapidly freeze to their IR values

$$\Lambda_{\rm IR} = \frac{\Lambda_{\rm cc}}{\sqrt{1 + \frac{3G\Lambda^2}{\pi}}} \qquad\qquad G_{\rm IR} = \frac{G}{\sqrt{1 + \frac{3G\Lambda^2}{\pi}}} \tag{4.33}$$

already found in the previous section (Eqs. (4.21) and (4.22)). We observe that for $G \gtrsim \Lambda^{-2} \sim M_P^{-2}$ the elbow appears at the very early stages of the UV running, i.e. $k_{\rm tr} \sim M_P$ (see Fig. 4.2), which means that the running freezes very early. In other words, the quadratic evolution in k is restricted to a very narrow window in the UV region.

In the next section we continue the study of the RG flow (4.26)-(4.27), and perform the fixed points analysis.

4.2 Fixed points and RG flow

We introduce (as usual) the "RG time" $t \equiv \log(k/k_0)$ ($k_0 \leq \Lambda$ arbitrary scale) and the dimensionless running cosmological and Newton constant

$$\lambda(t) \equiv \Lambda_k / k^2 \qquad , \qquad g(t) \equiv k^2 G_k \,. \tag{4.34}$$

In terms of λ, g, the RG equations (4.26) and (4.27) can be written as

$$\frac{{\rm d}\lambda}{{\rm d}t} = -2\lambda + \frac{2g\lambda\,(3-2\lambda)}{2\pi + g\,(3-2\lambda)} \equiv \beta_\lambda(\lambda, g) \tag{4.35}$$

$$\frac{{\rm d}g}{{\rm d}t} = \quad 2g + \frac{2g^2\,(3-8\lambda)}{2\pi + g\,(3-2\lambda)} \equiv \beta_g(\lambda, g) \,. \tag{4.36}$$

The fixed points (λ_i, g_i) are the solutions of $\beta_\lambda(\lambda, g) = 0$ and $\beta_g(\lambda, g) = 0$ (see section 1.4.1). We find

$$(\lambda_1, g_1) = (0, 0) \tag{4.37}$$

$$(\lambda_2, g_2) = \left(0, -\frac{\pi}{3}\right) . \tag{4.38}$$

In section 4.1.3, we showed that only positive UV boundary values of the cosmological and Newton constant, $\Lambda_{\rm cc} > 0$ and $G > 0$, are physically relevant (see comments below (4.17) and (4.18)). We also showed that Λ_k ($= \Lambda_L$) and G_k ($= G_L$) do not change sign all along their flow, and obviously the same holds true for λ and g. Therefore, the point

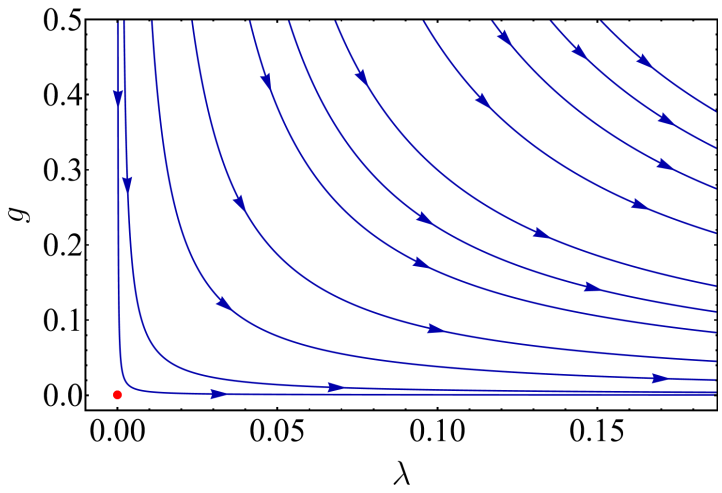

Figure 4.3: RG flow from the numerical solution of (4.35) and (4.36) in the physical quadrant $(\lambda > 0, g > 0)$. The red dot is the Gaussian fixed point $(\lambda_1, g_1) = (0, 0)$. The arrows point towards the IR.

(λ_2, g_2) has to be excluded from the present analysis[5], and we are left with the Gaussian fixed point (λ_1, g_1) only. As anticipated, in contrast with the so-called asymptotic safety scenario [29,30,34], our analysis does not show any sign of a non-trivial fixed point $\lambda_{\text{FP}} > 0$, $g_{\text{FP}} > 0$. Comments on the absence of such a fixed point are postponed to section 4.3, where we will also explain what is at the origin of its appearance in [29, 30, 34].

From the analysis of the stability matrix

$$M(\lambda, g) = \begin{pmatrix} \frac{\partial \beta_\lambda}{\partial \lambda} & \frac{\partial \beta_\lambda}{\partial g} \\ \frac{\partial \beta_g}{\partial \lambda} & \frac{\partial \beta_g}{\partial g} \end{pmatrix} \tag{4.39}$$

we find that (λ_1, g_1) has a UV-repulsive eigendirection and a UV-attractive one, the axes $\lambda = 0$ and $g = 0$ respectively.

As already said, the only physically relevant region is the quadrant $(\lambda > 0\,, g > 0)$, and we now move to a more complete study of the RG flow (4.35)-(4.36) in this quadrant. Solving numerically these latter equations for different boundary conditions, we find the RG trajectories presented in Fig. 4.3. The red dot is the Gaussian fixed point. The arrows point towards the IR and all the trajectories end at the minimal IR value of λ allowed by (4.25), namely $\lambda_{\text{IR}} = \Lambda_4/k_{\text{IR}}^2 = 3/16$. As already seen from the stability analysis, the $\lambda = 0$ and $g = 0$ axes are the corresponding UV-repulsive and UV-attractive eigendirections respectively.

In the previous section, we have seen that (4.28) and (4.29) provide an excellent approximation to the RG equations (4.26) and (4.27). For this reason, it is worth to write the former equations also in terms of the dimensionless couplings λ and g. We get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -2\lambda + \frac{6\lambda g}{2\pi + 3g} \tag{4.40}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \quad 2g + \frac{6g^2}{2\pi + 3g}\,. \tag{4.41}$$

[5]For completeness, in Appendix E we will also consider the remaining (unphysical) cases.

Eqs. (4.40) and (4.41) can also be obtained directly from (4.35) and (4.36) expanding the right hand side of both equations for $\lambda \ll 1$ up to the first non-trivial order. For boundary conditions $\lambda_0 > 0$ and $g_0 > 0$ at $t = 0$, they admit the analytic solutions

$$\lambda = \frac{\lambda_0}{2\left(1+\frac{\pi}{3g_0}\right)} \left[1 + \sqrt{1 + \frac{4\pi\, e^{-4t}}{3g_0}\left(1+\frac{\pi}{3g_0}\right)}\,\right] \tag{4.42}$$

$$g = \frac{g_0\, e^{4t}}{2\left(1+\frac{\pi}{3g_0}\right)} \left[1 + \sqrt{1 + \frac{4\pi\, e^{-4t}}{3g_0}\left(1+\frac{\pi}{3g_0}\right)}\,\right], \tag{4.43}$$

that are nothing but Eqs. (4.30) and (4.31) written for λ and g. The solutions (4.42) and (4.43) very well approximate the numerical solution to the original equations (4.35) and (4.36) (plotted in Fig. 4.3) in the whole range considered for t.

In view of the profound difference between the results of the present chapter and the asymptotic safety scenario [29–31, 34], that in the last decades has gained a certain popularity and has been considered for applications that range from black holes physics to inflation, it is worth to investigate on the origin of such a difference. This is the subject of the next section.

4.3 Comparison with existing literature

The analysis that we performed in the previous sections did not show any sign of the UV-attractive fixed point of the asymptotic safety scenario found in [30] (where the effective average action formalism was used) and later confirmed in [34] (resorting to the proper-time formalism). Pushing further our analysis, we now investigate on the reasons at the origin of the appearance of such a fixed point.

To this end, we begin by considering our RG equation (4.14) for the running action S_L, that for the reader's convenience we report below

$$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2-8\right)a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}\,. \tag{4.44}$$

As already explained, the relation between the numerical "running scale" L and the physical running scale k is given by $k = L/a_L^{\mathrm{dS}} = L\sqrt{\Lambda_L/3}$ (see (4.23) and (4.24) and comments therein). For the purposes of the present analysis, we temporarily introduce the running scale k in a different manner, namely through the relation

$$k = \frac{L}{a}\,, \tag{4.45}$$

i.e. we replace a_L^{dS} in (4.23) with the generic "off-shell" radius a of the background metric $g_{\mu\nu}^{(a)}$.

Obviously, this different identification of k profoundly alters the a-dependence (powers of a) in the right hand side of (4.44). In this respect, we recall that the running of $\frac{\Lambda_{cc}}{8\pi G}$ and $\frac{1}{G}$ is determined once we insert the Einstein-Hilbert truncation (4.2) in the left hand side of (4.44) and identify the coefficients of a^4 and a^2 of the first member with the corresponding ones of the second member (see comments above (4.8) and (4.9)). Inserting (4.45) in (4.44) we then obtain the incorrect RG equation

$$k\frac{\partial}{\partial k}S_k = \left[\frac{k^4}{3} + 2\Lambda_k\left(k^2+\Lambda_k\right)\right]a^4 - \left(\frac{34k^2}{3} + 16\Lambda_k\right)a^2 + \frac{1859}{45}\,, \tag{4.46}$$

which gives rise to the (incorrect) equations ($\Lambda_L \equiv \Lambda_k$, $G_L \equiv G_k$)

$$k\frac{\mathrm{d}}{\mathrm{d}k}\frac{\Lambda_k}{G_k} = \frac{k^4 + 6\Lambda_k\left(k^2+\Lambda_k\right)}{\pi} \tag{4.47}$$

$$k\frac{\mathrm{d}}{\mathrm{d}k}\frac{1}{G_k} = \frac{17k^2+24\Lambda_k}{3\pi}\,, \tag{4.48}$$

that are easily translated in

$$k\frac{\mathrm{d}\Lambda_k}{\mathrm{d}k} = \frac{G_k}{3\pi}\left(3k^4 + k^2\Lambda_k - 6\Lambda_k^2\right) \tag{4.49}$$

$$k\frac{\mathrm{d}G_k}{\mathrm{d}k} = -\frac{G_k^2}{3\pi}\left(17k^2+24\Lambda_k\right)\,. \tag{4.50}$$

These equations have to be compared with our correct RG equations (4.26) and (4.27). The two systems are profoundly different, and lead to significantly different results. Introducing the dimensionless running λ and g as in the previous section (see (4.34) and comments above), (4.49) and (4.50) are written as

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -2\lambda + \frac{g}{3\pi}\left(3+\lambda-6\lambda^2\right) \equiv \beta_\lambda(\lambda,g) \tag{4.51}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = 2g - \frac{g^2}{3\pi}\left(17+24\lambda\right) \equiv \beta_g(\lambda,g)\,. \tag{4.52}$$

If we now search for the fixed points of this (incorrect) system, besides the Gaussian fixed point $(\lambda_1, g_1) = (0,0)$ we find the two other fixed points:

$$(\lambda_{\rm A}, g_{\rm A}) = \left(\frac{\sqrt{154}-8}{30}, \frac{2\pi}{23}\left(53-4\sqrt{154}\right)\right) \simeq (0.147, 0.918) \tag{4.53}$$

$$(\lambda_{\rm B}, g_{\rm B}) = \left(-\frac{8+\sqrt{154}}{30}, \frac{2\pi}{23}\left(53+4\sqrt{154}\right)\right) \simeq (-0.680, 28.039)\,. \tag{4.54}$$

Let us perform now the stability analysis for these fixed points[6]. For the Gaussian fixed point, the stability matrix $M(\lambda, g)$ of Eq. (4.39) has a positive ($\theta_1 = 2$) and a negative ($\theta_2 = -2$) eigenvalue, that correspond to a UV-repulsive ($g = 4\pi\lambda$) and a UV-attractive ($g = 0$) eigendirection. Similarly, $M(\lambda_{\rm B}, g_{\rm B})$ has a positive ($\theta_1 \simeq 28.453$) and a negative ($\theta_2 \simeq -5.190$) eigenvalue, corresponding to a UV-repulsive ($g \simeq -65.741\,\lambda$) and a UV-attractive ($g \simeq 627.551\,\lambda$) eigendirection. Finally, for the fixed point $(\lambda_{\rm A}, g_{\rm A})$ we find that the eigenvalues θ_1 and θ_2 of the stability matrix $M(\lambda_{\rm A}, g_{\rm A})$ are $\theta_{1,2} \simeq -2.037 \pm 0.828\,i$.

[6] For all the fixed points, in the following we could write the exact (more involved) eigenvalues and eigendirections of the corresponding stability matrix (Eq. (4.39)), but this is not necessary as it does not provide any useful additional information. We report all the numbers approximated to the third digit.

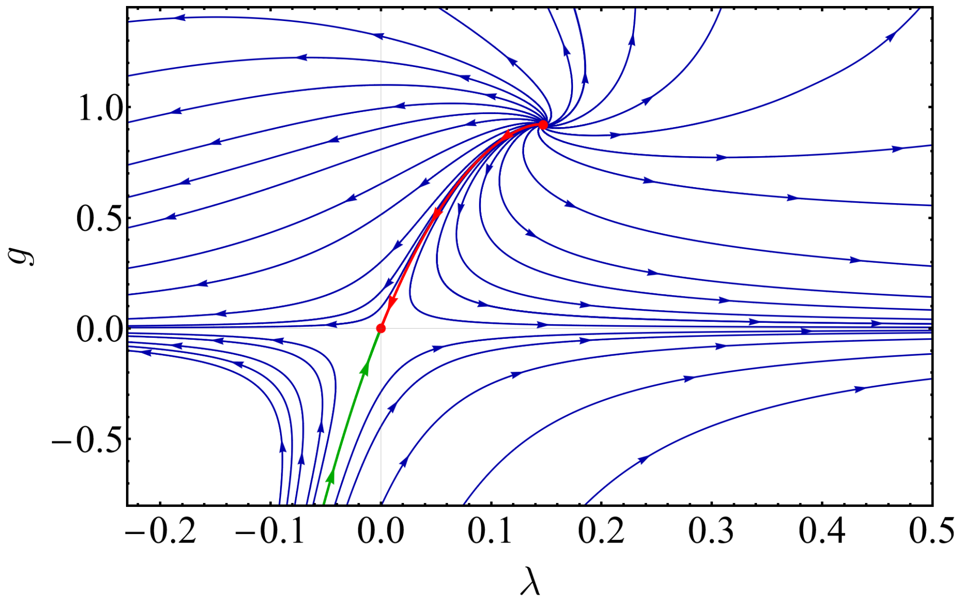

Figure 4.4: Flow from the numerical solution of the (incorrect) RG equations (4.51) and (4.52). The red dots are the Gaussian fixed point $(\lambda_1, g_1) = (0, 0)$ and the non-trivial UV-attractive fixed point $(\lambda_A, g_A) \simeq (0.147, 0.918)$. In the half-plane $g > 0$, the trajectories approach (λ_A, g_A) in the UV with a spiralling behaviour. This is the typical flow of the asymptotic safety scenario to be compared with Fig. 12 of [30]. The red and green lines are the separatrix curves; the red one connects (λ_A, g_A) to (λ_1, g_1). Arrows point towards the IR.

This means that in the half plane $g > 0$ the RG flow is UV-attracted towards this fixed point (negative real part) and has a spiralling behaviour in its vicinity (imaginary eigenvalues with opposite imaginary parts). Therefore, the point (λ_A, g_A) is nothing but the fixed point of the asymptotic safety scenario [29, 30, 34], and the analysis above shows that its appearance is due to the improper identification of k through the relation (4.45). In the following we will see that this identification is unfortunately what is implemented in the RG equations for Λ_k and G_k derived in [29, 30, 34].

Before doing that, we must stress another important result contained in Eqs. (4.51) and (4.52). These equations, in fact, not only admit the non-trivial UV-attractive fixed point (λ_A, g_A), but also provide an RG flow very similar to that of the asymptotic safety scenario. This can be immediately seen comparing our Fig. 4.4, where we plot the RG trajectories obtained solving numerically the system (4.51)-(4.52) for different boundary conditions, with Fig. 12 of [30]. For the purposes of this comparison, in Fig. 4.4 we consider the same range of values for the λ and g axes of Fig. 12 of [30]. In Fig. 4.5, we show the same RG flow, but to better appreciate the impact on this flow of the fixed point (λ_B, g_B) we consider a wider range[7] for the λ and g axes. In this respect, we note that a third fixed point as (λ_B, g_B) does not appear in the plot presented in Fig. 12 of [30]. It should be stressed, however, that this plot was obtained with a specific choice of the regulator R_k that appears in the RG equation for the effective average action (see (4.55) below). Actually, it can be seen that different choices of R_k can lead to the appearance of additional non-trivial fixed points other than the one of the asymptotic safety scenario. In particular,

[7]Clearly, as the fixed point (λ_B, g_B) is located along the g axis much above (λ_A, g_A), in Fig. 4.5 the portion of the RG flow shown in Fig. 4.4 appears to be flattened along this axis.

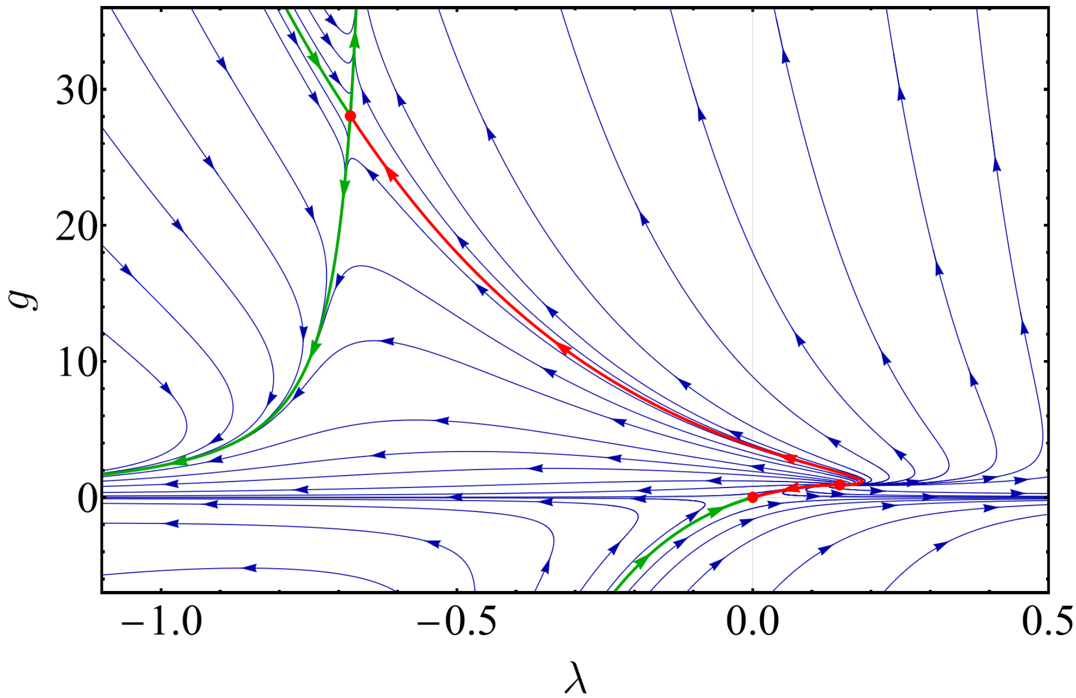

Figure 4.5: Same flow as in Fig. 4.4, but with a wider range for the λ and g axes (to evidentiate the presence and the impact on the flow of the fixed point $(\lambda_{\rm B}, g_{\rm B})$). The red dots are the fixed points $(\lambda_1, g_1) = (0, 0)$, $(\lambda_{\rm A}, g_{\rm A}) = (0.147, 0.918)$ and $(\lambda_{\rm B}, g_{\rm B}) = (-0.680, 28.039)$. The red and green lines are the separatrix curves; the red ones connect the UV-attractive fixed point $(\lambda_{\rm A}, g_{\rm A})$ to the other fixed points.

for some of these choices (actually for infinitely many choices) a non-trivial fixed point with the same characteristics of $(\lambda_{\rm B}, g_{\rm B})$ is generated. Similar considerations apply also to the RG equations derived with the proper-time method, where again the results show a dependence on the cutoff function $f_k(s)$ used in this framework (see (4.61) below).

To show now that within the effective average action formalism [29, 30], as well as with the proper-time method [34], the identification of the running scale k in the derivation of the RG equations for Λ_k and G_k is realized through the improper relation (4.45), we now reconsider the derivation of these equations with both these methods.

Effective average action - Let us begin by considering the effective average action formalism. In this framework, the flow of λ and g is derived from the RG equation for the effective average action Γ_k. To see how this implementation of the RG transformation automatically incorporates the identification of k through (4.45), we now consider some relevant steps that lead to the RG equations of [29, 30]. The effective average action Γ_k obeys the RG equation (as before, t is the "RG time" $t = \log \frac{k}{k_0}$)

$$\begin{aligned}\partial_t \Gamma_k[g, \bar{g}] =& \frac{1}{2}\,\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g, \bar{g}] + R_k^{\rm grav}[\bar{g}]\right)^{-1} \partial_t R_k^{\rm grav}[\bar{g}]\right] \\ &- \mathrm{Tr}\left[\left(-\mathcal{M}[g, \bar{g}] + R_k^{\rm gh}[\bar{g}]\right)^{-1} \partial_t R_k^{\rm gh}[\bar{g}]\right], \qquad (4.55)\end{aligned}$$

where $\kappa \equiv (32\pi G)^{-\frac{1}{2}}$, with G bare Newton constant, $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$, with $\bar{g}_{\mu\nu}$ a fixed gravitational background and $\bar{h}_{\mu\nu}$ the classical field, and $\mathcal{M}[g, \bar{g}]$ is the classical kinetic term of the ghosts

$$\mathcal{M}[g,\bar{g}]^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_{\lambda}\left(g_{\rho\nu}D_{\sigma} + g_{\sigma\nu}D_{\rho}\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_{\lambda}g_{\sigma\nu}D_{\rho}\,, \tag{4.56}$$

with D_μ and $\bar{D}_\mu$ covariant derivatives that involve the Christoffel symbols for $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ respectively. The background metric $\bar{g}_{\mu\nu}$ is eventually taken to be a sphere. $\Gamma_k^{(2)}[g,\bar{g}]$ is the Hessian of $\Gamma_k[g,\bar{g}]$ with respect to $g_{\mu\nu}$ at fixed $\bar{g}_{\mu\nu}$. $R_k^{\rm grav}[\bar{g}]$ and $R_k^{\rm gh}[\bar{g}]$ are regulators that appear in the definition of Γ_k (for the gravitational and ghost contribution respectively), both having the shape[8]

$$R_k[\bar{g}] \sim\; k^2 R^{(0)}(-\Box/k^2)\ , \quad \text{with} \quad \Box \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu\,, \tag{4.57}$$

where $R^{(0)}(x)$ is a dimensionless function that interpolates between $R^{(0)}(0) = 1$ and $\lim_{x\to\infty} R^{(0)}(x) = 0$. In the effective average action method [33], the cutoff functions R_k implement the "Wilsonian" shell by shell elimination of modes, since they ensure that the (functional) traces in the right hand side of (4.55) are effectively (i.e. in a smooth rather than a sharp sense) restricted to the eigenmodes of the Laplace-Beltrami operator $-\Box$ whose corresponding eigenvalues p^2 lie "around" k^2: $p^2 \sim k^2$. In the present case ($\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$), this means that the running scale k is identified with the eigenvalues of the Laplace-Beltrami operator $-\Box$ on the background sphere of radius a, i.e. through the relation (4.45).

The above considerations, the results found with the incorrect equations (4.51) and (4.52), and the results of section 4.2 put into question the existence of the UV-attractive fixed point of the asymptotic safety scenario and indicate that its appearance is due to the improper identification of k through (4.45). Below we show that in the proper-time formalism [34] the same identification of k is realized, and the same conclusions can be drawn also in this case.

Proper-time method - Let us move now to the proper-time method. Indicating with Ω a typical fluctuation operator, the one-loop correction to the classical action in the proper-time representation is written as sum of contributions of the kind

$$\mathrm{Tr}\log\Omega = -\,\mathrm{Tr}\int_0^{+\infty}\frac{\mathrm{d}s}{s}e^{-s\,\Omega}, \tag{4.58}$$

where s, the so-called proper-time, is a parameter with dimension $(\text{mass})^{-2}$ and the UV divergence due to the lower bound of integration is regulated through the replacement $0 \to 1/\Lambda^2$. According to [34], the Wilsonian RG strategy (shell by shell elimination of modes) is implemented in this framework considering the one-loop correction to the classical action, introducing an IR regulator k that "suppresses contributions from large proper-times $s \gtrsim k^{-2}$", i.e. making the replacement

$$\int_{1/\Lambda^2}^{+\infty}\mathrm{d}s \quad\longrightarrow\quad \int_{1/\Lambda^2}^{1/k^2}\mathrm{d}s\ , \tag{4.59}$$

taking the derivative with respect to k, and finally realizing the RG improvement of the one-loop result. All these steps lead to the RG equation for the gravitational action. A

[8]We do not write here factors that are irrelevant to the present discussion. See [29,30] for details.

technical remark. In [34], after presenting the hard cutoff implementation of (4.59), in the actual calculation the authors implement equivalent smooth cutoffs introducing in the proper-time integral of (4.58) functions $f_k(s)$ that smoothly interpolate between $f_k(s) \approx 0$ for $s \gg k^{-2}$ and $f_k(s) \approx 1$ for $s \ll k^{-2}$ (similarly for the UV). Accordingly, they write the one-loop result for the gravitational action as

$$\widehat{S}_k[g,\bar{g}] \equiv \widehat{S}\,[\bar{h};g] + \Gamma_1[g,\bar{g}]_{\text{reg}} \equiv \widehat{S}\,[\bar{h};g] - \frac{1}{2}\,\text{Tr}\int_0^{+\infty} \frac{\text{d}s}{s} f_k(s)\,\left[e^{-s\,\widehat{S}^{(2)}} - 2\,e^{-s\,S^{(2)}_{\text{ghost}}}\right], \quad (4.60)$$

where $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ are as in (4.55), $\widehat{S}\,[\bar{h};g] \equiv S[\bar{g}+\bar{h}] + S_{\text{gf}}[\bar{h};\bar{g}]$ (where S is the classical action and S_{gf} the gauge-fixing term whose corresponding ghost action is S_{ghost}), $\widehat{S}^{(2)}$ is the matrix of the second functional derivatives of $\widehat{S}\,[\bar{h};g]$ with respect to $\bar{h}_{\mu\nu}$, and likewise for $S^{(2)}_{\text{ghost}}$. Clearly, the hard cutoff of (4.59) is implemented in (4.60) choosing $f_k(s) = \theta\,(s - 1/\Lambda^2)\,\theta\,(1/k^2 - s)$. Finally, the RG equation for the running action $\widehat{S}_k[g,\bar{g}]$ is obtained taking the derivative of both members of (4.60) with respect to the "RG time" $t = \log\frac{k}{k_0}$ and eventually replacing in the right hand side $\widehat{S}$ with $\widehat{S}_k$

$$\partial_t\,\widehat{S}_k[g,\bar{g}] = -\frac{1}{2}\,\text{Tr}\int_0^{+\infty} \frac{\text{d}s}{s}\,\partial_t f_k(s)\,\left[e^{-s\,\widehat{S}^{(2)}_k} - 2\,e^{-s\,S^{(2)}_{\text{ghost}}}\right]. \quad (4.61)$$

To see that the RG equation (4.61) implements the identification (4.45) for the running scale k, we now make the following observations. Taking the background metric $\bar{g}_{\mu\nu}$ to be the metric $g^{(a)}_{\mu\nu}$ of a sphere of radius a, and considering for $\widehat{S}_k$ the Einstein-Hilbert truncation, we see that $\widehat{S}^{(2)}_k$ contains dimensionful Laplace-Beltrami operators $-\Box$ for the sphere of radius a (and different spins $0,1,2$) whose eigenvalues $\widehat{\lambda}_n$ go like $\widehat{\lambda}_n \sim \frac{n^2}{a^2}$. Moreover, in the right hand side of (4.61) the term $\partial_t f_k(s)$ effectively selects the eigenmodes of $-\Box$ whose corresponding eigenvalues lie in a narrow range ("infinitesimal shell") around k^2, i.e. $\widehat{\lambda}_n \sim k^2$. Therefore, as it is the case for the effective average action formalism, in the RG equation (4.61) the running scale k is identified through the relation (4.45), and the same conclusions on the UV-attractive fixed point of the asymptotic safety scenario hold true.

In summary, we have shown that both the effective average action formalism and the proper-time RG implement the improper identification of the running scale k through the relation (4.45), and that the appearance of the non-trivial UV-attractive fixed point of the asymptotic safety scenario is due to this identification.

4.4 Conclusions

In this chapter, we considered the Einstein-Hilbert truncation for the running action in Euclidean quantum gravity and, taking a spherical gravitational background, we derived the renormalization group equations for the running cosmological and Newton constant. We have shown that, as for the one-loop calculation of the previous chapter, there are two crucial aspects to which attention must be paid in order to derive the RG equations. One concerns the measure in the path integral that defines the running action. This measure contains terms coming from the integration over the conjugate momenta in the original Hamiltonian formulation of the theory that are often neglected or mistreated. The other aspect concerns the identification of the physical running scale k. If the latter is not properly introduced, the RG flow is substantially altered.

We have shown that in usual implementations of the RG transformation, that typically resort to the effective average action and/or to the proper-time formalism, the running scale k is improperly introduced. This results in altered RG equations for the cosmological and Newton constant, that lead to the generation of the UV-attractive fixed point of the asymptotic safety scenario. Moreover, we have shown that in the physically relevant quadrant $(\lambda > 0, g > 0)$ only the Gaussian fixed point exists, with a UV-attractive and a UV-repulsive eigendirection.

In the next chapter, that is based on [35], we consider the calculation of the one-loop effective action for an interacting scalar theory on a non-trivial gravitational background. We show that the issues concerning the path integral measure and the introduction of the UV physical cutoff, which were at the heart of the analysis performed in the two previous chapters, have a crucial impact also on the calculation of the radiative correction δm^2 to the mass m^2 of scalar particles. More specifically, we will see that, contrary to usual results, when these two aspects are properly taken into account no quadratic divergences are present in δm^2.

Chapter 5

Gravity and the Higgs boson mass

The Standard Model (SM) of particle physics is one of the greatest achievements of modern theoretical and experimental physics, a synthesis started in the late sixthies/early seventhies of the last century with the emergence of the electroweak and strong interaction theory. The SM, that has received several experimental confirmations over the years, among which the discovery of the Higgs boson at LHC [117, 118], provides the basic ground to describe and understand a great variety of phenomena. Still, it is not a fully fledged theory. It is an effective field theory valid up to a maximal energy scale, the UV physical cutoff Λ (say GUT scale $M_{\rm GUT}$, Planck scale M_P, string scale M_s).

Typical approaches to go beyond the SM consider supersymmetric extensions and/or models where the Higgs boson appears as a composite particle. In both cases, the UV completion is a field theory whose validity extends to energy regimes higher than those typically explored in SM physics. One of the motivations for supersymmetry is to realize the cancellation (thanks to the presence of superpartners) of quadratic divergent contributions to the Higgs mass. In fact, performing the calculation of the propagator and/or the effective action both in flat and curved spacetime, the bare Higgs boson mass $m^2(\Lambda)$ receives contributions δm^2 proportional to Λ^2, due to unsuppressed quantum fluctuations (no symmetry protection). In this respect, it is important to stress that, if regularization schemes as dimensional or zeta function regularization are used, this quadratic UV-sensitivity of the mass is not seen. However, these regularizations operate by construction a cancellation of power-like divergences and cannot be regarded as a physical mechanism responsible for the suppression of the aforementioned strong UV-sensitivity [22].

As stressed above, the SM has to be considered as an EFT valid up to the UV scale Λ. The fact that $\delta m^2 \sim \Lambda^2$ means that $m^2(\Lambda)$ must be $\mathcal{O}(\Lambda^2)$ too, with a coefficient that has to be finely tuned for the Higgs boson mass $m_{\rm H}^2 = m^2(\Lambda) + \delta m^2$ to coincide with the measured value $\sim (125\,{\rm GeV})^2$. Since $\Lambda \gg m_{\rm H}$, the bare mass $m^2(\Lambda)$ has to be taken *unnaturally* large[1] with respect to $m_{\rm H}^2$ [127–131]. This is one aspect of the so-called naturalness problem for the Higgs boson mass, to which we will refer in the following as "physical cutoff problem" (PCP).

Another aspect of this long-standing issue arises when the SM is viewed as embedded in a higher energy theory where the Higgs field $H(x)$ is coupled to fields of large masses. Let us consider for instance a supersymmetric extension of the SM, where SUSY is broken by a large stop mass $\widetilde{m}_t \gg m_{\rm H}$. The Higgs mass receives the correction (y_t is the top

[1]For recent discussions on naturalness, renormalization and the flow of the mass see [11, 22, 119–126].

Yukawa coupling, μ the renormalization/subtraction scale)

$$\delta m^2 \sim y_t\, \widetilde{m}_t^2 \ln \frac{\widetilde{m}_t^2}{\mu^2}\,, \tag{5.1}$$

and again we have to cope with a quadratic radiative correction to the Higgs boson mass. In the following, we refer to this other aspect of the naturalness problem as "large masses problem" (LMP).

Building on the analysis of [14, 18, 19, 23, 24, 37, 109], that was the subject of the two previous chapters, we now revisit the PCP. We find that, when the SM is considered on a smooth gravitational background (with curvature much smaller than the inverse Planck length squared), no problem of quadratic sensitivity to the physical cutoff arises. It has to be emphasized that this is not due to the use of a regularization scheme that automatically cancels out quadratically UV-sensitive terms. As we will see, it comes from a correct treatment of the path integral measure, and a proper introduction of the UV physical cutoff Λ, two aspects often overlooked in the literature. Usually, the calculation is performed resorting to the heat-kernel formalism [17], and gives rise (as in flat spacetime) to a quadratically sensitive radiative correction $\delta m^2 \sim \Lambda^2$. On the contrary, we will show that, when the two aforementioned points are taken into account, quantum fluctuations provide only a mild logarithmic correction $\delta m^2 \sim \log\Lambda$ to the mass of the Higgs boson.

Concerning the second aspect of the naturalness problem, the LMP, we stress that terms of the kind (5.1) in the radiative correction to the Higgs boson mass arise even when the path integral measure and the UV physical cutoff are properly treated. We will discuss how this problem might be handled along the lines of [11].

For the purposes of our analysis, it is not necessary to take into account the full Higgs sector of the SM; it is sufficient to consider the case of a single component scalar field on a non-trivial gravitational background. We then calculate and analyse the one-loop effective action Γ^{1l} for this simpler theory.

The analysis of the present chapter has been performed in [35].

5.1 One-loop effective action

Let us add to the purely gravitational Einstein-Hilbert (Euclidean) action the contribution of a single component real scalar field ϕ non-minimally coupled to gravity

$$S[g_{\mu\nu},\phi] = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\,(-R+2\Lambda_{\rm cc}) + \int \mathrm{d}^4x\,\sqrt{g}\left[\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{\xi}{2}R\phi^2 + V(\phi)\right]. \tag{5.2}$$

Taking for $g_{\mu\nu}$ the metric $g^{(a)}_{\mu\nu}$ of a sphere of radius a, the action (5.2) becomes ($\int \mathrm{d}^4x\,\sqrt{g^{(a)}} = \frac{8\pi^2}{3}a^4\,,\ R(g^{(a)}) = \frac{12}{a^2}$)

$$S^{(a)}[\phi] = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2 + \int \mathrm{d}^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\,g^{(a)\,\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + \frac{\xi}{2}\,\frac{12}{a^2}\,\phi^2 + V(\phi)\right]. \tag{5.3}$$

Different powers of a correspond to different gravitational terms. We will use this feature to identify the gravitational operators in the one-loop correction δS^{1l} to $S^{(a)}$. To calculate δS^{1l}, we resort to the background field method [105, 106] and write $\phi = \Phi + \eta$, where

Φ is a constant background and η the fluctuation. Expanding $S^{(a)}[\phi]$ around Φ up to quadratic terms in η, we have (see section 1.1.2)

$$e^{-\delta S^{1l}} = \int \big[\mathcal{D}u(\eta)\big]\, e^{-S_2}\,, \tag{5.4}$$

where

$$S_2 \equiv \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g^{(a)}}\;\eta\Big[-\Box_a + \frac{12\,\xi}{a^2} + V''(\Phi)\Big]\eta\,, \tag{5.5}$$

with $-\Box_a$ the spin-0 Laplace-Beltrami operator for a sphere of radius a and $V''(\Phi)$ the second derivative of the potential with respect to Φ. The measure $\big[\mathcal{D}u(\eta)\big]$ is given by

$$\big[\mathcal{D}u(\eta)\big] = \prod_x \Big[\big(g^{(a)\,00}(x)\big)^{\frac{1}{2}}\,\big(g^{(a)}(x)\big)^{\frac{1}{4}}\,\mathrm{d}\eta(x)\Big]\,, \tag{5.6}$$

where the factors $\big(g^{(a)\,00}(x)\big)^{\frac{1}{2}}\,\big(g^{(a)}(x)\big)^{\frac{1}{4}}$ arise from the integration over the conjugate momenta of $\phi(x)$ in the original (Hamiltonian) formulation of the theory [14, 18, 19, 23, 24, 37, 109] (see also chapters 3, 4 and 6). As already stressed for the case of pure gravity (see comments below (3.9)), despite the presence of $g^{(a)\,00}$ factors, this measure is diffeomorphism invariant. As we will show in the next chapter (based on [37]), this invariance emerges from a delicate balance between the different elements involved in the definition of the path integral. Among them, the necessity of introducing a time ordering parameter and a discretization (lattice) of spacetime. Under a general coordinate transformation, the time ordering parameter and the lattice both transform, and this induces the appearance of non-trivial terms. The $g^{(a)\,00}$ factors in (5.6) ensure the cancellation of these non-trivial terms, and ultimately guarantee the diffeomorphism invariance[2] of the measure. From the invariance of both $\big[\mathcal{D}u(\eta)\big]$ in (5.6) and S_2 in (5.5), we have that δS^{1l} in (5.4) is also invariant.

For our purposes, as we have done in chapter 3 for the pure gravity case, it is convenient to calculate δS^{1l} considering coordinate systems (as for instance the four angles that parametrize the sphere) where the metric $g^{(a)}_{\mu\nu}$ can be written as

$$g^{(a)}_{\mu\nu} = a^2\,\widetilde{g}_{\mu\nu}\,, \tag{5.7}$$

where the elements of $\widetilde{g}_{\mu\nu}$ are dimensionless and a-independent. Using (5.7), the factor $\big(g^{(a)\,00}(x)\big)^{\frac{1}{2}}\,\big(g^{(a)}(x)\big)^{\frac{1}{4}}$ in (5.6) is written as

$$\big(g^{(a)\,00}(x)\big)^{\frac{1}{2}}\,\big(g^{(a)}(x)\big)^{\frac{1}{4}} = a\,\big(\widetilde{g}^{00}(x)\big)^{\frac{1}{2}}\,\big(\widetilde{g}(x)\big)^{\frac{1}{4}}\,. \tag{5.8}$$

As we will see, the effect of these terms in the measure (5.6) is conveniently taken into account if we define the dimensionless field

$$\widehat{\eta} \equiv a\eta\,. \tag{5.9}$$

Inserting (5.7) and (5.9) in (5.5), for S_2 we have

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\;\widehat{\eta}\Big[-\widetilde{\Box} + 12\,\xi + a^2\,V''(\Phi)\Big]\widehat{\eta}\,, \tag{5.10}$$

[2] In [36] a different conclusion is reached. As it is shown in next chapter, the reason for such a difference is that the authors of [36] miss the non-trivial terms mentioned above.

where $-\widetilde{\Box}$ is the dimensionless spin-0 Laplace-Beltrami operator defined as

$$-\widetilde{\Box} \equiv -a^2 \, \Box_a \,. \tag{5.11}$$

We now observe that, since

$$\mathrm{d}\eta(x) = a^{-1} \, \mathrm{d}\widehat{\eta}(x) \;, \tag{5.12}$$

after insertion of (5.8) and (5.12) in (5.6), the measure $\big[\mathcal{D}u(\eta)\big]$ becomes

$$\big[\mathcal{D}u(\eta)\big] = \Big[\prod_x \big(\widetilde{g}^{00}(x)\big)^{\frac{1}{2}} \Big] \Big[\prod_x \big(\widetilde{g}(x)\big)^{\frac{1}{4}} \, \mathrm{d}\widehat{\eta}(x) \Big] . \tag{5.13}$$

The factor a^{-1} in (5.12) has been exactly compensated by the factor a in (5.8), eventually resulting in the a-independent path integral measure above.

Inserting (5.10) and (5.13) in (5.4) we obtain

$$e^{-\delta S^{1l}} = \Big[\prod_x \big(\widetilde{g}^{00}(x)\big)^{\frac{1}{2}} \Big] \int \Big[\prod_x \big(\widetilde{g}(x)\big)^{\frac{1}{4}} \, \mathrm{d}\widehat{\eta}(x) \Big] \, e^{-\frac{1}{2}\int \mathrm{d}^4x \sqrt{\widetilde{g}}\, \widehat{\eta}\left[-\widetilde{\Box}+12\,\xi+a^2\,V''(\Phi)\right]\widehat{\eta}} . \tag{5.14}$$

Finally, performing the Gaussian integrations we get[3]

$$\Gamma^{1l} = S^{(a)}[\Phi] + \frac{1}{2} \,\mathrm{Tr} \log \Big(-\widetilde{\Box} + 12\xi + a^2 V''(\Phi) \Big) + \mathcal{C} \,, \tag{5.15}$$

where ($\delta^{(4)}(0)$ below is due to the replacement $\sum_x \to \int \mathrm{d}^4x$)

$$\mathcal{C} \equiv -\frac{1}{2} \log \Big(\prod_x \widetilde{g}^{00}(x) \Big) = -\frac{\delta^{(4)}(0)}{2} \int \mathrm{d}^4x \log \big(\widetilde{g}^{00}(x) \big) \tag{5.16}$$

comes from the exponentiation of the measure term $\prod_x (\widetilde{g}^{00}(x))^{1/2}$ (see (5.14)). The presence of the non invariant term $\mathcal{C}$ might lead one to suspect that the above result for Γ^{1l} is not invariant. This is not the case. In fact, as thoroughly discussed in [18, 20, 46], and as it will be discussed also in the next chapter (see in particular section 6.4), subtleties arise in the calculation of $\log\big(-\widetilde{\Box}+12\xi+a^2V''(\Phi)\big)$: one has to carefully take into account the distributional nature of the Green's function of the operator $(-\widetilde{\Box} + 12\xi + a^2V''(\Phi))$ [20]. When this is done, from the calculation of $\mathrm{Tr}\log\big(-\widetilde{\Box} + 12\xi + a^2V''(\Phi)\big)$ the non-trivial term $\frac{\delta^{(4)}(0)}{2} \int \mathrm{d}^4x \log(\widetilde{g}^{00}(x))$ arises, that eventually cancels $\mathcal{C}$ in (5.15). This is why in the forthcoming expressions $\mathcal{C}$ does not appear. All the other terms coming from the calculation of "Tr log" are diffeomorphism invariant [20].

Let us calculate now the right hand side of (5.15). As for the pure gravity case, for the regularization of the trace, we will follow two different strategies: (i) we consider the sum over a finite number N of eigenvalues; (ii) we repeat the calculation using proper-time regularization. Taking for the potential the truncation $V(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$, from both calculations we will see that the usually acknowledged quadratic divergence in the one-loop correction to m^2 is *absent*. We will also see that neither quartic nor quadratic

[3] Had we missed in the measure (5.6) the factors $\big(g^{(a)\,00}(x)\big)^{1/2}\,\big(g^{(a)}(x)\big)^{1/4}$, the a-dependence of the fluctuation operator in (6.49) would have been altered.

divergences appear in the one-loop contribution to the vacuum energy[4]. Moreover, we will show that the appearance of these power-like divergences in the literature is due to the fact that usual calculations implement the improper introduction (3.39) of the UV physical cutoff.

Let us begin with the calculation of the trace in (5.15) considering the sum over the eigenvalues of the fluctuation operator. To this end, we recall that the eigenvalues λ_n of $-\widetilde{\Box}$ and the corresponding degeneracies D_n are

$$\lambda_n = n^2 + 3n \qquad ; \qquad D_n = \frac{1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{1}{12}\left(n+\frac{3}{2}\right) . \tag{5.17}$$

The regularization is implemented taking a finite number N ($\gg 1$) of eigenvalues λ_n ($n = 0, 1, \ldots, N$). From (5.15) we have (the choice of $N-2$ rather than N as upper limit is for convenience and simplifies the expression of δS^{1l})

$$\delta S^{1l} = \frac{1}{2}\sum_{n=0}^{N-2}\Big[D_n \log\left(\lambda_n + 12\xi + a^2 V''(\Phi)\right)\Big] . \tag{5.18}$$

The numerical UV cutoff N in (5.18) implements a gauge invariant regularization[5]. The connection between N and the UV physical cutoff Λ [14, 23, 24] will be considered in the next section.

Inserting (5.17) in the right hand side of (5.18), and considering the identity $\log(x/y) = -\int_0^{+\infty} \mathrm{d}u \left[(x+u)^{-1} - (y+u)^{-1}\right]$, the sum can be performed and put in closed form. For our purposes, it is sufficient to consider the expansion for $N \gg 1$. We get

$$\begin{aligned}\delta S^{1l} = {}& \frac{8\pi^2}{3}a^4\Big[-\frac{(V''(\Phi))^2}{64\pi^2}\log N^2 + \frac{12}{a^2}\frac{V''(\Phi)}{384\pi^2}\left(N^2 + 2(1-6\xi)\log N^2\right)\Big] \\ & + \frac{N^4}{48}\left(-1 + 2\log N^2\right) - \frac{N^2}{72}\left(13 - 72\xi + 3\log N^2\right) + \left(2\xi(1-3\xi) - \frac{29}{180}\right)\log N^2 \\ & + \mathcal{H}(a^2 V''(\Phi)) + \mathcal{O}\left(N^{-2}\right) ,\end{aligned} \tag{5.19}$$

where $\mathcal{H}(a^2 V''(\Phi))$ contains only UV-finite (N-independent) terms. Its expression is given in Appendix F.

Up to now, we have not considered any specific form of $V(\Phi)$. Let us take the self-interacting potential $V(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$. For the one-loop effective action $\Gamma^{1l} = S^{(a)}[\Phi] + \delta S^{1l}$ we have (below the inessential terms in the third line of (5.19) are omitted)

$$\begin{aligned}\Gamma^{1l} = {}& \frac{8\pi^2}{3}a^4\Big\{-\frac{1}{16\pi G}\Big[1 - \frac{G\,m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\Big]\frac{12}{a^2} + \frac{\Lambda_{cc}}{8\pi G}\Big[1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log N^2\Big] \\ & + \frac{\xi}{2}\Big[1 + \frac{\lambda}{384\pi^2\,\xi}\Big(N^2 + 2(1-6\xi)\log N^2\Big)\Big]\frac{12}{a^2}\Phi^2 \\ & + \frac{m^2}{2}\Big[1 - \frac{\lambda}{32\pi^2}\log N^2\Big]\Phi^2 + \frac{\lambda}{4!}\Big[1 - \frac{3\lambda}{32\pi^2}\log N^2\Big]\Phi^4\Big\} \\ & + \frac{N^4}{48}\left(-1 + 2\log N^2\right) - \frac{N^2}{72}\left(13 - 72\xi + 3\log N^2\right) + \left(2\xi(1-3\xi) - \frac{29}{180}\right)\log N^2 .\end{aligned} \tag{5.20}$$

[4]The same result was found in section 3.3.1, where the case of a free scalar field was considered.
[5]The gauge invariance is guaranteed by the fact that N is a cut on the eigenvalues λ_n.

Eq. (5.20) is the central result of the present chapter, and we will comment on its consequences in the next section. Before doing that, we find it useful to proceed with the evaluation of δS^{1l} following the second strategy mentioned above, namely proper-time regularization. We will then conveniently discuss both results together.

Since the operator $(-\widetilde{\Box}+12\xi+a^2V''(\Phi))$ in (5.15) is dimensionless, as done in chapter 3 (see section 3.1) for the case of pure quantum gravity, to regularize its determinant we introduce the dimensionless proper-time τ, with numerical lower integration bound $1/N^2$ ($N \gg 1$)

$$\det(-\widetilde{\Box}+12\xi+a^2V''(\Phi)) = e^{-\int_{1/N^2}^{+\infty}\frac{\mathrm{d}\tau}{\tau}\,\mathrm{K}(\tau)}\,. \tag{5.21}$$

The kernel $\mathrm{K}(\tau)$ is (λ_n and D_n are the eigenvalues and degeneracies reported in (5.17))

$$\mathrm{K}(\tau) = \sum_{n=0}^{+\infty} D_n\, e^{-\tau\left(\lambda_n+12\xi+a^2V''(\Phi)\right)}\,. \tag{5.22}$$

To calculate the determinant, we now insert (5.22) in (5.21), perform the integration over τ, and then sum over n with the help of the EML formula (3.30).

Expanding the resulting expression of δS^{1l} for $N \gg 1$, we finally get

$$\begin{aligned}\delta S^{1l} = {}& \frac{8\pi^2}{3}a^4\Big[-\frac{(V''(\Phi))^2}{64\pi^2}\log N^2 + \frac{12}{a^2}\frac{V''(\Phi)}{384\pi^2}\left(N^2+2\,(1-6\xi)\log N^2\right)\Big]\\ &-\frac{N^4}{24}-\frac{1-6\xi}{6}\,N^2+\left(2\xi\,(1-3\xi)-\frac{29}{180}\right)\log N^2\\ &+\mathcal{Z}(a^2\,V''(\Phi))+\mathcal{O}\left(N^{-2}\right)\,,\end{aligned} \tag{5.23}$$

where $\mathcal{Z}(a^2\,V''(\Phi))$ contains only UV-finite terms (no dependence on N) and is similar to the term $\mathcal{H}(a^2\,V''(\Phi))$ in (5.19).

Considering as before the potential $V(\Phi)=\frac{m^2}{2}\Phi^2+\frac{\lambda}{4!}\Phi^4$, the one-loop effective action Γ^{1l} becomes (below the inessential terms in the third line of (5.23) are omitted)

$$\begin{aligned}\Gamma^{1l} = {}& \frac{8\pi^2}{3}a^4\Big\{-\frac{1}{16\pi G}\Big[1-\frac{G\,m^2}{24\pi}\left(N^2+2(1-6\xi)\log N^2\right)\Big]\frac{12}{a^2}+\frac{\Lambda_{\rm cc}}{8\pi G}\Big[1-\frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log N^2\Big]\\ &+\frac{\xi}{2}\Big[1+\frac{\lambda}{384\pi^2\,\xi}\Big(N^2+2\,(1-6\xi)\log N^2\Big)\Big]\frac{12}{a^2}\Phi^2\\ &+\frac{m^2}{2}\Big[1-\frac{\lambda}{32\pi^2}\log N^2\Big]\,\Phi^2+\frac{\lambda}{4!}\Big[1-\frac{3\lambda}{32\pi^2}\log N^2\Big]\Phi^4\Big\}\\ &-\frac{N^4}{24}-\frac{1-6\xi}{6}\,N^2+\left(2\xi\,(1-3\xi)-\frac{29}{180}\right)\log N^2\,.\end{aligned} \tag{5.24}$$

Apart from irrelevant a and Φ independent terms (fourth line of both equations), the two results (5.20) and (5.24) for Γ^{1l} coincide. Therefore, in the following we can equivalently consider either of these two expressions, and for concreteness we will refer to (5.24).

In the next section, we derive the one-loop corrected expressions of the parameters $1/G$, $\Lambda_{\rm cc}/G$, m^2, λ and ξ, focusing in particular on the main concern of the present chapter, namely the correction δm^2 to m^2. As anticipated in the introduction to this chapter, we will see that no quadratic divergence appears in δm^2: the mass receives only a mild logarithmic correction.

5.2 One-loop corrected parameters

Let us consider the constant background $\phi = \Phi$ and the potential $V(\Phi) = \frac{m^2}{2}\Phi^2 + \frac{\lambda}{4!}\Phi^4$. Comparing Γ^{1l} in (5.24) with the classical (bare) action $S^{(a)}$ in (5.3), we see that Γ^{1l} depends on a and Φ in the same way as $S^{(a)}[\Phi]$. We can then easily read the radiative corrections to $1/G$, $\Lambda_{\rm cc}/G$, m^2, λ and ξ, and for the one-loop corrected parameters $1/G^{1l}$, $\Lambda_{\rm cc}^{1l}/G^{1l}$, ... we find

$$\frac{1}{G^{1l}} = \frac{1}{G}\Big[1 - \frac{G\,m^2}{24\pi}\left(N^2 + 2(1-6\xi)\log N^2\right)\Big] \tag{5.25}$$

$$\frac{\Lambda_{\rm cc}^{1l}}{G^{1l}} = \frac{\Lambda_{\rm cc}}{G}\Big[1 - \frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log N^2\Big] \tag{5.26}$$

$$m_{1l}^2 = m^2\Big[1 - \frac{\lambda}{32\pi^2}\log N^2\Big] \tag{5.27}$$

$$\lambda^{1l} = \lambda\Big[1 - \frac{3\lambda}{32\pi^2}\log N^2\Big] \tag{5.28}$$

$$\xi^{1l} = \xi\Big[1 + \frac{\lambda}{384\pi^2\,\xi}\left(N^2 + 2\,(1-6\xi)\log N^2\right)\Big]\,. \tag{5.29}$$

Since $\Lambda_{\rm cc}^{1l}$ and G^{1l} are the renormalized values of the cosmological and Newton constant, they have to be positive. From (5.25) and (5.26), we see that for this to hold only positive values of the bare $\Lambda_{\rm cc}$ and G should be taken.

Let us consider now the relation between the numerical cut N and the UV physical cutoff Λ to which we referred in the previous section. As discussed in [14, 23, 24], and also in chapters 3 and 4, the connection between N and Λ is given by

$$\Lambda = \frac{N}{a_{\rm m}}\,, \tag{5.30}$$

where $a_{\rm m}$ is the radius that minimizes [6] the action $S^{(a)}[\Phi]$ in (5.3). Inserting (5.30) in (5.24), the effective action Γ^{1l} is written in terms of Λ

$$\begin{aligned}\Gamma^{1l} = \frac{8\pi^2}{3}a^4\Big\{ &-\frac{1}{16\pi G}\Big[1 - \frac{G\,m^2}{24\pi}\Big(a_{\rm m}^2\Lambda^2 + 2(1-6\xi)\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\Big]\frac{12}{a^2} + \frac{\Lambda_{\rm cc}}{8\pi G}\Big[1 - \frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log\big(a_{\rm m}^2\Lambda^2\big)\Big]\\ &+ \frac{\xi}{2}\Big[1 + \frac{\lambda}{384\pi^2\,\xi}\Big(a_{\rm m}^2\Lambda^2 + 2\,(1-6\xi)\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\Big]\frac{12}{a^2}\Phi^2\\ &+ \frac{m^2}{2}\Big[1 - \frac{\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big]\,\Phi^2 + \frac{\lambda}{4!}\Big[1 - \frac{3\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big]\Phi^4\Big\}\\ &- \frac{1}{24}\,a_{\rm m}^4\Lambda^4 - \frac{1-6\xi}{6}\,a_{\rm m}^2\Lambda^2 + \Big(2\xi\,(1-3\xi) - \frac{29}{180}\Big)\log\big(a_{\rm m}^2\Lambda^2\big)\,.\end{aligned} \tag{5.31}$$

[6] The minimum $a_{\rm m}$ of $S^{(a)}[\Phi]$ is obtained solving the classical equations of motion for a and Φ, and depends on the parameters in $S^{(a)}[\Phi]$ and on the minimum $\Phi_{\rm m}$. For $m^2 > 0$ and $\xi > 0$, $a_{\rm m}$ is the de Sitter solution $a_{\rm m} = \sqrt{3/\Lambda_{\rm cc}}$ (with $\Phi_{\rm m} = 0$).

Similarly, inserting (5.30) in (5.25)-(5.29), we have

$$\frac{1}{G^{1l}} = \frac{1}{G}\Big[1 - \frac{G\,m^2}{24\pi}\Big(a_{\rm m}^2\Lambda^2 + 2(1-6\xi)\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\Big] \tag{5.32}$$

$$\frac{\Lambda_{\rm cc}^{1l}}{G^{1l}} = \frac{\Lambda_{\rm cc}}{G}\Big[1 - \frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log\big(a_{\rm m}^2\Lambda^2\big)\Big] \tag{5.33}$$

$$m_{1l}^2 = m^2\Big[1 - \frac{\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big] \tag{5.34}$$

$$\lambda^{1l} = \lambda\Big[1 - \frac{3\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big] \tag{5.35}$$

$$\xi^{1l} = \xi\Big[1 + \frac{\lambda}{384\pi^2\,\xi}\Big(a_{\rm m}^2\Lambda^2 + 2\,(1-6\xi)\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\Big]\,. \tag{5.36}$$

Few comments are in order. From (5.35) we see that the quartic self-coupling λ receives only a logarithmic correction. This is the usual result. On the contrary, it is immediately apparent that the result for the mass m_{1l}^2 in (5.34) is significantly different from the usual one: *no quadratic divergence* appears in the one-loop correction δm^2. Actually, δm^2 goes like $\log\Lambda$ rather than Λ^2. In this respect, we stress that the usual result $\delta m^2 \sim \Lambda^2$ enforces a Λ^2 dependence in the bare mass $m^2(\Lambda)$, with a coefficient that must be finely tuned for it to cancel (almost exactly) this quadratically sensitive contribution. Differently from that, Eq. (5.34) shows that $m_{1l}^2 \sim m^2$, so that the bare mass $m^2(\Lambda)$ may well be $m^2(\Lambda) \ll \Lambda^2$.

This result implies that, when the diffeomorphism invariant path integral measure (5.6) is used and the UV physical cutoff is introduced as in (5.30), the PCP (physical cutoff problem) aspect of the naturalness problem for scalar particles does not arise. Further comments on this point are below, where we also discuss the other aspect of the naturalness problem, namely the LMP (large masses problem), that arises when the Higgs boson is coupled to particles of large mass M.

Let us move to the one-loop corrected non-minimal coupling ξ^{1l}, Eq. (5.36). We see that, in addition to a logarithmic correction (the one usually found), ξ receives a quadratically divergent contribution. The comparison of our results with the usual ones shows that the UV behaviours of m^2 and ξ are inverted: m^2 is only logarithmically sensitive to the physical cutoff Λ, while ξ carries a quadratic sensitivity. In the next section, we will further investigate on these points and show why in usual calculations, that are performed within the heat-kernel formalism, the quadratic divergence appears in m^2 rather than in ξ. In this respect, it is worth to make the following remark. While the Higgs boson mass has to be confronted with the measured value $m_{\rm H}^2 \sim (125\,{\rm GeV})^2$, and a quadratic sensitivity to the UV physical cutoff Λ gives rise to a severe naturalness problem, much less is known on the value of ξ, and a correction $\delta\xi \sim \Lambda^2$ does not appear to be worrisome.

We now comment on the two aspects of the naturalness problem mentioned above, starting from the PCP. As already said, the SM is an effective field theory, i.e. a theory valid up to a certain physical scale Λ, the built-in physical cutoff. We have shown that the expected quadratic sensitivity of m^2 to Λ is absent. In this respect, we stress that two main approaches have been typically adopted in the literature to dispose of the quadratically divergent contributions to m^2, one formal, the other physical. The formal one consists in performing the calculation resorting to regularization schemes (such as dimensional regularization) where power-like divergences are absent by construction[7]. By

[7] For calculations with dimensional regularization see [132] (see also [133], [134]).

no means can these methods be regarded as a "solution" to the original problem [22]. On the physical side, there have been several attempts to obtain a finite Higgs boson mass (finite Higgs effective potential). Typically, quadratic divergences are cancelled considering a supersymmetric embedding of the theory [135–137], or models where the Higgs boson is regarded as a composite particle [138–142]. Other attempts, that in the past gained a certain popularity, consider models with compact extra dimensions [8, 9, 69, 70], though it has been recently suggested that UV-sensitive terms were missed in these works that would undermine their conclusions [12, 13, 71] (see section 1.5 and chapter 2). The calculations of the present chapter are not based on formal methods, nor resort to any physical cancellation mechanism. We have seen that the quadratic sensitivity of m^2 to Λ is simply absent. In the next section, we will show that the appearance in usual results of radiative corrections to m^2 proportional to Λ^2 is due to an improper implementation of the calculations.

We move now to the other aspect of the naturalness problem, the LMP (large masses problem). Assuming that the SM is embedded in a higher energy theory (SUSY, GUT, ...) that contains fields of heavy mass M coupled to the Higgs field, the Higgs boson mass m^2 receives large contributions proportional to M^2. Clearly, the solution of the LMP requires the existence of a physical mechanism that disposes of these contributions, and ultimately gives $m_{\mathrm{H}}^2 \sim (125\,\mathrm{GeV})^2$.

This question should be framed and understood within the Wilsonian paradigm [66] (see section 1.4 for an introduction to the Wilsonian RG approach). Let us consider the ultimate theory, namely the Theory of Everything (TOE). The renormalization group flow that emanates from the TOE connects theories, T_1, T_2, ..., whose range of validity is restricted to lower and lower energy regimes. Schematically, we can represent this RG flow as: TOE $\to T_1 \to T_2 \to \ldots$. The SM is part of this chain of effective theories, say SM $\equiv T_n$, and emerges at a certain energy scale[8] Λ_{SM}. The theory T_{n-1} is then the higher energy theory considered above (SUSY, GUT, ...) that embeds the SM, and is typically referred to as its "UV completion" valid above Λ_{SM}. Consider now the running of the Higgs boson mass $m^2(\mu)$ within the SM. The boundary $m^2(\Lambda_{\mathrm{SM}})$ at the scale Λ_{SM} is provided by the theory T_{n-1}. Such a boundary value is in turn inherited from the higher energy theory T_{n-2}, and ultimately from the TOE. In this framework, $m^2(\Lambda_{\mathrm{SM}})$ is the *precise* boundary of the Wilsonian RG flow that drives $m^2(\mu)$ to the measured value $m_{\mathrm{H}}^2 \sim (125\,\mathrm{GeV})^2$ at the Fermi scale. This scenario was dubbed "physical tuning" in [11].

Let us comment now on (5.33). Similarly to what we have found for m^2, the radiative correction to the vacuum energy $\frac{\Lambda_{\mathrm{cc}}}{8\pi G}$ goes like $\log \Lambda$: it does not contain the quartic and quadratic divergences usually found in the literature[9]. At the same time, we observe that this logarithmic contribution is proportional to m^4. Therefore, even disregarding any embedding of the SM in a higher energy theory, we see that, due to the very low value of the vacuum energy, a physical mechanism is needed that disposes of these contributions proportional to the fourth power of the masses of SM particles. Moreover, if as before we suppose that the SM is embedded in a higher energy theory (the T_{n-1} theory considered above) with fields of heavy mass M, contributions to the vacuum energy proportional to M^4 would also be present, and this makes the problem even more severe. As for the

[8]Different UV completions of the SM have different values of Λ_{SM}, that can range from few TeV all the way up to the Planck scale.

[9]The same result was found in section 3.3.1 for the case of a free scalar theory.

naturalness problem LMP related to the Higgs boson mass (see above), the small value of the vacuum energy (cosmological constant) should be understood as the result of a "physical tuning" of the boundary value of the vacuum energy at the UV physical scale $\Lambda_{\rm SM}$ dictated by the RG flow that emanates from the TOE. A similar RG scenario was proposed in [13] (see sections 2.5 and 2.6), when discussing the physical viability of the dark dimension proposal [10].

Finally, from (5.32) we see that the inverse Newton constant $1/G$ receives a quadratically UV-sensitive contribution (note that if the physical cutoff Λ coincides with the Planck scale M_P, for the inverse Newton constant we have $1/G \sim M_P^2$).

What is left at this point is to understand why, contrary to the results of the present chapter, usual calculations of the one-loop effective action Γ^{1l}, typically performed within the heat-kernel formalism, give rise to power-like divergent contributions[10] to m^2 and $\Lambda_{\rm cc}/8\pi G$. The next section is devoted to this investigation.

5.3 Comparison with previous literature

The aim of the present section is to understand why usual results show the appearance of power-like divergences both in the mass of a scalar particle and in the vacuum energy, while our calculations show only a mild logarithmic sensitivity of these quantities to the UV scale Λ. We will proceed along the lines of sections 3.2 and 4.3.

Let us consider the expression (5.24) of Γ^{1l}. For the sake of the present discussion, we temporarily realize the connection between the numerical cut N and the UV physical cutoff Λ via the relation (a is the radius of the off-shell background)

$$\Lambda = \frac{N}{a}\,, \tag{5.37}$$

rather than through $\Lambda = N/a_{\rm m}$ given in (5.30). This means that, to establish the relation between the UV numerical cut N and the physical cutoff Λ, we are temporarily using the off-shell radius a in place of the classical solution $a_{\rm m}$. It is important to recall here that, as stressed above, the different powers of the off-shell radius a identify the different gravitational terms in the (effective) Lagrangian. We used that to determine the radiative corrections to $1/G$, $\Lambda_{\rm cc}/G$, m^2, λ and ξ. Inserting (5.37) (rather than (5.30)) in (5.24), for Γ^{1l} we obtain the "would-be" result

$$\begin{aligned}
\Gamma^{1l} = \frac{8\pi^2}{3}a^4\Big\{ &-\frac{1}{16\pi G}\Big[1-\frac{G\,m^2}{24\pi}\left(a^2\Lambda^2+2(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\Big]\frac{12}{a^2}+\frac{\Lambda_{\rm cc}}{8\pi G}\Big[1-\frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log\left(a^2\Lambda^2\right)\Big]\\
&+\frac{\xi}{2}\Big[1+\frac{\lambda}{384\pi^2\,\xi}\Big(a^2\Lambda^2+2\,(1-6\xi)\log\left(a^2\Lambda^2\right)\Big)\Big]\frac{12}{a^2}\Phi^2\\
&+\frac{m^2}{2}\Big[1-\frac{\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\Big]\Phi^2+\frac{\lambda}{4!}\Big[1-\frac{3\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\Big]\Phi^4\Big\}\\
&-\frac{a^4\Lambda^4}{24}-\frac{1-6\xi}{6}\,a^2\Lambda^2+\Big(2\xi\,(1-3\xi)-\frac{29}{180}\Big)\log\left(a^2\Lambda^2\right)\,,
\end{aligned} \tag{5.38}$$

[10]Regularization schemes as dimensional or zeta-function regularization, which automatically implement the cancellation of power-like divergences, are excluded from these considerations.

which is trivially rewritten as

$$
\begin{aligned}
\Gamma^{1l} = \frac{8\pi^2}{3}a^4 \Big\{ & -\frac{1}{16\pi G}\Big[1+\frac{1-6\xi}{12\pi}G\Lambda^2 - \frac{G\,m^2}{24\pi}\left(2(1-6\xi)\log\left(a^2\Lambda^2\right)\right)\Big]\frac{12}{a^2} + \frac{\Lambda_{\rm cc}}{8\pi G}\Big[1-\frac{G}{8\pi\Lambda_{\rm cc}}\Lambda^4 \\
& + \frac{m^2 G}{4\pi\Lambda_{\rm cc}}\Lambda^2 - \frac{G\,m^4}{8\pi\Lambda_{\rm cc}}\log\left(a^2\Lambda^2\right)\Big] + \frac{\xi}{2}\Big[1+\frac{\lambda}{384\pi^2\,\xi}\Big(2\,(1-6\xi)\log\left(a^2\Lambda^2\right)\Big)\Big]\frac{12}{a^2}\Phi^2 \\
& + \frac{m^2}{2}\Big[1+\frac{\lambda\,\Lambda^2}{32\pi^2\,m^2} - \frac{\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\Big]\Phi^2 + \frac{\lambda}{4!}\Big[1-\frac{3\lambda}{32\pi^2}\log\left(a^2\Lambda^2\right)\Big]\Phi^4\Big\} \\
& + \Big(2\xi\,(1-3\xi) - \frac{29}{180}\Big)\log\left(a^2\Lambda^2\right)\,. \qquad (5.39)
\end{aligned}
$$

Interestingly, Eq. (5.39) reproduces the well-known result found in the literature when the calculation is performed within the heat-kernel formalism. We immediately note the presence of the (in)famous quadratically divergent correction to m^2, and of the equally (in)famous quartically and quadratically divergent contributions to the vacuum energy $\Lambda_{\rm cc}/8\pi G$.

Comparing (5.39) with the original result (5.24) for Γ^{1l}, we understand how these "spurious divergences" are generated. This point is crucial to our analysis, and it is worth to examine the different terms in detail. Let us begin with the term $\frac{1}{2}\left(\frac{\lambda N^2}{384\pi^2}\right)\frac{12}{a^2}\Phi^2$ of (5.24), that provides a correction to the non-minimal coupling ξ. Now, if we replace N^2 according to (5.37), the quadratically divergent term $\frac{1}{2}\left(\frac{\lambda\Lambda^2}{32\pi^2}\right)\Phi^2$ of (5.39) arises. This is how, due to the improper identification (5.37) of the UV physical cutoff Λ, a term that originally renormalizes the non-minimal coupling ξ is artificially turned into the well-known quadratically divergent contribution to the mass m^2. Similarly, using again (5.37), the term $-\frac{N^4}{24}$ in the fourth line of (5.24) artificially becomes $-\frac{\Lambda^4}{24}a^4$, that is the well-known quartically divergent contribution to the vacuum energy $\Lambda_{\rm cc}/8\pi G$. Moreover, the term $\left(\frac{m^2N^2}{384\pi^2}\right)\frac{12}{a^2}$, that in the original expression (5.24) renormalizes $1/G$, becomes $\frac{m^2\Lambda^2}{32\pi^2}$, that is the usual quadratically divergent contribution to the vacuum energy. Finally, always with the replacement (5.37), the term $-\frac{1-6\xi}{6}N^2$ in the fourth line of (5.24) becomes $\frac{8\pi^2}{3}a^4\left(-\frac{1-6\xi}{192\pi^2}\Lambda^2\frac{12}{a^2}\right)$, thus giving a quadratically divergent contribution to $1/G$.

The importance of the above results can hardly be underestimated. We observe that the identification of the UV physical cutoff Λ through (5.37) introduces in Γ^{1l} a spurious dependence on the off-shell radius a, i.e. on the background metric $g^{(a)}_{\mu\nu}$. We now show that such an improper identification is implicitly implemented in usual calculations. To this end, we begin by writing the "usual expression" of the one-loop effective action for a generic gravitational background $g_{\mu\nu}$ and a constant background scalar field Φ

$$
\Gamma^{1l} = S[g_{\mu\nu},\Phi] + \frac{1}{2}\,\mathrm{Tr}\log\left(-\Box + \xi R(g) + V''(\Phi)\right)\,, \qquad (5.40)
$$

where $S[g_{\mu\nu},\Phi]$ is the action in (5.2), and $-\Box$ the dimensionful Laplace-Beltrami operator for the metric $g_{\mu\nu}$. The trace in (5.40) is usually calculated with the proper-time method[11]

$$\mathrm{Tr}\log\left(-\Box+\xi R(g)+V''(\Phi)\right)=-\,\mathrm{Tr}\int_{1/\Lambda^2}^{+\infty}\frac{\mathrm{d}s}{s}e^{-s\,(-\Box+\xi R(g)+V''(\Phi))}\equiv-\int_{1/\Lambda^2}^{+\infty}\frac{\mathrm{d}s}{s}\mathrm{K}(s), \quad (5.41)$$

where s, the so-called proper-time, is a parameter with dimension $(\text{mass})^{-2}$ and the UV divergences are regulated through the replacement $0\to 1/\Lambda^2$ in the lower bound of integration. The kernel $\mathrm{K}(s)$ is calculated resorting to the heat-kernel expansion [17], and the UV divergences are due to the first terms of this expansion (see section 1.2).

To see that (5.41) implements the identification (5.37) for the UV physical cutoff Λ, we now specify to the spherical background $g_{\mu\nu}=g^{(a)}_{\mu\nu}$, for which the kernel takes the form

$$\mathrm{K}(s)=\sum_{n=0}^{+\infty}D_n\,e^{-\tau\left(\widehat{\lambda}_n+\xi\frac{12}{a^2}+V''(\Phi)\right)}\,, \quad (5.42)$$

with $\widehat{\lambda}_n$ the *a-dependent* eigenvalues of $-\Box_a$ (Laplace-Beltrami operator for a sphere of radius a, see (5.5)) and D_n the corresponding degeneracies

$$\widehat{\lambda}_n=\frac{n^2+3n}{a^2}\qquad;\qquad D_n=\frac{1}{3}\left(n+\frac{3}{2}\right)^3-\frac{1}{12}\left(n+\frac{3}{2}\right). \quad (5.43)$$

The cut $1/\Lambda^2$ in the proper-time integral (5.41) effectively restricts (in a smooth way) the sum in (5.42) to the N eigenvalues such that $\widehat{\lambda}_n\lesssim\Lambda^2$. The highest eigenvalue is then $\widehat{\lambda}_N\sim\frac{N^2}{a^2}\sim\Lambda^2$. This shows that the usual implementation (5.41) of the proper-time method automatically identifies the UV physical cutoff Λ through the relation (5.37).

It is worth to stress at this point that, as shown in section 5.1, when the diffeomorphism invariant Fradkin-Vilkovisky measure [18, 37] is used, the fluctuation operator is the one in (5.15). The latter is *dimensionless* and contains the dimensionless Laplace-Beltrami operator $-\widetilde{\Box}$. As a consequence, the determinant in (5.21) is written in terms of the *dimensionless* proper-time τ, and is regularized through the numerical lower integration bound $1/N^2$ ($N\gg 1$), that in turn is related to the physical cutoff Λ as in (5.30) (see [14, 23, 24] for further details). To repeat ourselves, the important outcome of this calculation is that no quadratic divergences are found in the quantum correction to m^2.

Before ending this section, we would like to speculate on the way flat spacetime calculations should be reanalyzed in light of the results of the present chapter. Given the evidence for a small positive vacuum energy, that might indicate a positive small curvature of space, we find it reasonable to argue that flat spacetime should be seen only as a limit, and that physical quantities typically calculated in the Minkowski QFT framework should rather be obtained from calculations performed on a non-trivial gravitational background $g_{\mu\nu}$ (with $g_{\mu\nu}$ a smooth background with characteristic length $l\gg l_{_P}\,(=M_P^{-1})$) through the limiting procedure $g_{\mu\nu}\to\delta_{\mu\nu}$. This limit is quite delicate. In fact, we have seen that the results obtained starting directly with $g_{\mu\nu}=\delta_{\mu\nu}$ (flat background) *do not*

[11] As already stressed, the "usual expression" (5.40) misses an important term, coming from the Fradkin-Vilkovisky path integral measure, that contains the time-time component of the inverse metric, see (5.15) and (5.16). We have also seen that the usual implementation (5.41) of the proper-time method in turn misses a term that is opposite in sign to the previous one. These two terms cancel, and this eventually ensures the diffeomorphism invariance of the one-loop effective action (see comments below (5.16)).

coincide with those obtained performing the calculations with a non-trivial metric $g_{\mu\nu}$, and taking only after the limit $g_{\mu\nu} \to \delta_{\mu\nu}$. We have shown that, when the calculations are performed in this latter way, power-like divergences in the mass m^2 of the scalar field and in the vacuum energy $\Lambda_{cc}/8\pi G$ are automatically absent. There is no need to resort to any "physical cancellation", as the one obtained for instance with a supersymmetric embedding of the theory, nor to resort to any "technical cancellation", as the one realized with dimensional regularization.

We finally observe that from our result (5.31) for Γ^{1l}, the effective potential $V^{1l} \equiv \Gamma^{1l}/\big(8\pi^2 a^4/3\big)$ ($\frac{8\pi^2}{3}a^4$ is the volume) in the flat limit $a \to \infty$ is

$$V^{1l}_{a\to\infty} = \frac{\Lambda_{cc}}{8\pi G}\Big(1 - \frac{G\,m^4}{8\pi\Lambda_{cc}}\log\big(a_{\rm m}^2\Lambda^2\big)\Big) + \frac{m^2}{2}\Big(1 - \frac{\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\,\Phi^2 + \frac{\lambda}{4!}\Big(1 - \frac{3\lambda}{32\pi^2}\log\big(a_{\rm m}^2\Lambda^2\big)\Big)\,\Phi^4\,. \tag{5.44}$$

This is nothing but the usual Φ^4 one-loop effective potential, where power-like divergences are *automatically* absent (no cancellation).

5.4 Conclusions

In the present chapter we considered a scalar theory on a non-trivial gravitational background (we used a spherical background) and calculated the one-loop effective action Γ^{1l}. Our first concern was to use a path integral measure that ensures the diffeomorphism invariance of the effective action. According to [18, 37] (see also the next chapter, where the results of [37] will be presented), such a measure is the one proposed by Fradkin and Vilkovisky. Moreover, we carefully introduced the UV physical cutoff Λ of the theory according to the analysis of [14, 23, 24] (see chapters 3 and 4).

From Γ^{1l}, we derived the radiative correction to the mass m^2 of the scalar particle. We showed that, differently from typical (well-know) results, where m^2 receives contributions quadratic in Λ, quantum fluctuations provide only a (mild) logarithmic contribution to m^2. This addresses one aspect of the Higgs naturalness problem, namely the sensitivity of the mass to the UV physical cutoff. We dubbed this aspect PCP (*physical cutoff problem*). This result arises from the use of the diffeomorphism invariant Fradkin - Vilkovisky measure together with the proper identification of the physical cutoff Λ of the theory. It is not obtained resorting to a "physical" cancellation (as it would be the case for a supersymmetric embedding of the theory), nor from regularization schemes where the cancellation of power-like divergences is automatically implemented (as it is the case, for instance, of dimensional regularization).

In addition, we showed that usual calculations, that are performed within the heat-kernel formalism, actually implement an improper identification of the UV physical cutoff. This results in a distorted dependence of Γ^{1l} on the gravitational background. We also showed that this is why in usual calculations the well-known power-like divergences are generated in the mass of the scalar particle and in the vacuum energy.

There is another facet to the Higgs naturalness problem, that we dubbed LMP (*large masses problem*). It arises when the SM is viewed as embedded in a higher energy theory that contains particles of large mass M coupled to the Higgs boson. The Higgs mass receives contributions proportional to M^2, and a physical mechanism that disposes of these large contributions, ultimately providing $m_{\rm H}^2 \sim (125\,{\rm GeV})^2$, is needed. This question should be framed within the Wilsonian paradigm. As stressed in section 5.2, in fact, the ultimate theory is the TOE, and the RG flow that emanates from the TOE connects effective theories whose validity extends to lower and lower energy regimes. The SM is part of this chain of theories, and $\Lambda_{\rm SM}$ is the physical scale where the SM takes over. Concerning the running of the Higgs boson mass $m^2(\mu)$, the effective theory that "completes" the SM above $\Lambda_{\rm SM}$ provides the *precise* boundary $m^2(\Lambda_{\rm SM})$ such that the Wilsonian RG flow drives $m^2(\mu)$ to the measured value $m_{\rm H}^2 \sim (125\,{\rm GeV})^2$ at the Fermi scale. This *precise* boundary $m^2(\Lambda_{\rm SM})$ is in turn inherited from the "previous" higher energy theory, and ultimately from the TOE. This is the "physical tuning" scenario introduced in [11].

The results of the present chapter (absence of power-like divergences in the mass of scalar particles and in the vacuum energy on a gravitational background), together with the evidence for a non-zero positive vacuum energy that might indicate a positive small curvature of space, led us to the following speculation. Flat spacetime should be regarded only as a limit, and physical quantities typically calculated in the Minkowski QFT framework should rather be obtained from calculations performed on a non-trivial smooth gravitational background $g_{\mu\nu}$ through the limiting procedure $g_{\mu\nu} \to \delta_{\mu\nu}$. The results obtained starting directly with $g_{\mu\nu} = \delta_{\mu\nu}$ (flat background) *do not coincide* with those obtained with a non-trivial metric $g_{\mu\nu}$ and taking only after the limit $g_{\mu\nu} \to \delta_{\mu\nu}$. We have shown that, when the calculations are performed according to this limiting procedure, power-like divergences in the mass m^2 of the scalar field and in the vacuum energy $\Lambda_{\rm cc}/8\pi G$ are automatically absent. There is no need for a supersymmetric embedding of the theory, nor to resort to any "technical cancellation".

A crucial element at the basis of the results presented in the last three chapters is the use of the Fradkin-Vilkovisky path integral measure [18]. As already stressed more than once throughout this thesis, despite the presence in it of non-covariant factors of the time-time component $g^{(a)\,00}$ of the inverse background metric, this measure is diffeomorphism invariant. Though this was proved explicitly by Fradkin and Vilkovisky for the case of pure gravity in [18], in the literature some authors have made claims to the contrary (see for instance [36]). To bring clarity on this question, part of my activity has been dedicated to a thorough investigation on the transformation properties of the path integral measure under diffeomorphisms. The results of this analysis have been published in [37], and will be presented in the next chapter. With the help of a detailed calculation, we show that, despite the presence of the non-covariant factors of $g^{(a)\,00}$, the FV measure is diffeomorphism invariant. We will see that the invariance arises from a delicate balance of all the elements involved in the definition of the path integral (see section 1.1.1 for the detailed construction of the path integral).

Chapter 6

Diffeomorphism invariance of the effective gravitational action

The results of chapters 3, 4 and 5 were derived using the Fradkin-Vilkovisky path integral measure [18, 19, 143]. In the literature, two different measures are mainly considered, the one proposed by Fradkin and Vilkovisky and the one by Fujikawa [107]. It is sometimes claimed that, since the FV measure contains non-covariant factors of the time-time component $g^{00}(x)$ of the inverse metric, it is not diffeomorphism invariant, and that (on the contrary) the invariant measure is the one proposed by Fujikawa. This point of view has been recently taken up in [36].

The subject of the present chapter is the analysis we performed in [37]. We thoroughly investigate on the transformation properties of the effective gravitational action, focusing in particular on the path integral measure. We show that, despite the presence of g^{00} factors, the FV measure is diffeomorphism invariant. With the help of detailed calculations, we see that this result is intimately related to the *very definition* of the path integral, that involves the introduction of a time ordering parameter and of a discretization of spacetime (lattice). When a general coordinate transformation is performed, both the time ordering parameter and the lattice transform, giving rise to non-trivial terms. Thanks to the g^{00} factors, these non-trivial terms are canceled, ultimately ensuring the diffeomorphism invariance of the FV measure.

By the same token, we also show that the result by which the Fujikawa measure appears to be diffeomorphism invariant is due to a *formal* treatment of the path integral, resorting to which the aforementioned points are overlooked. As a consequence, the non-trivial terms mentioned above, that appear when the path integral undergoes a general coordinate transformation, are missed. Once these terms are taken into account, it turns out that the FV measure is diffeomorphism invariant, while the Fujikawa measure is not.

6.1 Diffeomorphisms and path integral

Let us consider the gravitational action $S_{\rm g}[g]$ (think for instance of the Einstein-Hilbert action (3.1)) in the presence of a scalar field with action $S_{\rm m}[\phi, g]$. The total action is (from now on we indicate with g the metric $g_{\mu\nu}$ when it appears as the argument of a function/functional, its determinant otherwise; the signature $(-,+,+,+)$ is used)

$$S[\phi,g]=S_{\rm g}[g]+S_{\rm m}[\phi,g]=S_{\rm g}[g]+\int {\rm d}^4x\,\mathcal{L}_{\rm m}(\phi(x),\partial_\mu\phi(x),g(x)) \tag{6.1}$$

where $\mathcal{L}_{\rm m}(\phi(x),\partial_\mu\phi(x),g(x))$ is the matter Lagrangian density

$$\mathcal{L}_{\rm m}(\phi(x),\partial_\mu\phi(x),g(x))=-\frac{1}{2}\sqrt{-g(x)}\left(g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x)+m^2\phi^2(x)\right)\,. \tag{6.2}$$

The effective gravitational action[1] $\Gamma[g]$ is given by

$$e^{i\Gamma[g]}=\int e^{iS[\phi(x),\,g(x)]}\prod_x\Big[M(g(x))\,{\rm d}\phi(x)\Big]\,, \tag{6.3}$$

where $M(g(x))$ is a non-trivial term in the configuration space measure $\prod_x\big[M(g(x))\,{\rm d}\phi(x)\big]$. In the literature, two different expressions for $M(g(x))$ are mainly considered. For the FV measure [18],

$$M_{\rm FV}(g(x))=(-g^{00}(x))^{1/2}(-g(x))^{1/4}\,, \tag{6.4}$$

while for the Fujikawa measure [107] (μ is an arbitrary mass scale)

$$M_{\rm Fuji}(g(x))=\mu\,(-g(x))^{1/4}\,. \tag{6.5}$$

Within the canonical formalism, the FV measure is obtained from the phase space path integral measure $\prod_x\big[{\rm d}\pi(x)\,{\rm d}\phi(x)\big]$ (Liouville), while the Fujikawa measure [107] from $\prod_x\big[(g^{00}(x))^{-1/2}\,{\rm d}\pi(x)\,{\rm d}\phi(x)\big]$ [108]. After integration over the conjugate momenta, the FV configuration space measure (6.4) contains $[g^{00}(x)]^{1/2}$ factors, while the Fujikawa measure (6.5) does not.

A fundamental aspect to be stressed is the following. In (6.3), $\prod_x$ indicates that in the definition of the path integral a discretization of spacetime is introduced (see section 1.1.1), which in turn implies that $\prod_x\big[M(g(x))\,{\rm d}\phi(x)\big]$ in (6.3) can (obviously) be written as $\big[\prod_x M(g(x))\big]\big[\prod_x {\rm d}\phi(x)\big]$. This latter (trivial) observation will be useful in the following. Let us stress a crucial point first raised in [41], and later deeply investigated in [18]. In any formulation of quantum field theory, and in particular in the formulation of gauge theories, the construction of the basic transition amplitudes $\langle\phi'',t''|\phi',t'\rangle$ (definition of the S matrix) requires the identification of a parameter in terms of which a time ordering is introduced. Differently from the case of Yang-Mills theories, where the gauge transformation only affects the form of the fields, in gravity a general coordinate transformation acts on the coordinates, and thus on the argument of the fields. As we will see, this has a non-trivial impact on the time ordering. In the path integral formulation, the time ordering parameter can be introduced as follows. Given a coordinate system x^μ, if the points of the lattice are obtained from the intersection between hypersurfaces $x^0=$ const and curves $x^i=$ const, the time ordering parameter is identified[2] with x^0 [18,41]. The notation $\prod_x$ in Eq. (6.3) indicates the product over all the points Q_i of the lattice $\mathcal{E}_1$ defined in this way. Note that the construction of the lattice and the identification of the ordering parameter can be done only once the coordinate system has been specified.

[1]For general considerations on the effective gravitational action see for instance [132,144,145]

[2]The quantization can also be realized considering a more general space-like hypersurface $\tau(x)=$ const, in which case the role of ordering parameter is played by τ.

Let us call Σ the reference frame with coordinates x and $\hat{\Sigma}$ another frame with coordinates $\hat{x}$. In $\hat{\Sigma}$, the action (6.1) is written as (we use the notation $\hat{\partial}_\mu \equiv \frac{\partial}{\partial \hat{x}^\mu}$)

$$\hat{S}[\hat{\phi}, \hat{g}] = \hat{S}_{\rm g}[\hat{g}] + \hat{S}_{\rm m}[\hat{\phi}, \hat{g}] = \hat{S}_{\rm g}[\hat{g}] + \int {\rm d}^4\hat{x}\, \mathcal{L}_{\rm m}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))\,, \tag{6.6}$$

and for the effective action $\hat{\Gamma}[\hat{g}]$ we have

$$e^{i\hat{\Gamma}[\hat{g}]} = \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \prod_{\hat{x}} \Big[M(\hat{g}(\hat{x}))\, {\rm d}\hat{\phi}(\hat{x}) \Big]\,. \tag{6.7}$$

Similarly to what we said for (6.3), the notation $\prod_{\hat{x}}$ in (6.7) indicates the product over the points P_i of the lattice $\mathcal{E}_2$ obtained from the intersection of hypersurfaces $\hat{x}^0 = \text{const}$ with curves $\hat{x}^i = \text{const}$. In this case, the time ordering parameter is $\hat{x}^0$. We will see that, due to this change in the time ordering parameter, the path integral measure transforms non-trivially under diffeomorphisms[3]. This important aspect is missed in [36]: the measure $\prod_x \big[M(g(x))\, {\rm d}\phi(x) \big]$ should not be treated formally, and attention should be paid to the discretization and to the time ordering underlying this expression.

Let us investigate now on the diffeomorphism invariance of the effective action, performing the transition from $\hat{\Sigma}$ to Σ. Our aim is to see for which choice of $M(g(x))$, either the FV one in (6.4) or the Fujikawa one in (6.5), we have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$. Since the classical action is diffeomorphism invariant, i.e. $\hat{S}[\hat{\phi}, \hat{g}] = S[\phi, g]$, from (6.3) and (6.7) we see that $\hat{\Gamma}[\hat{g}] = \Gamma[g]$ if

$$\prod_{\hat{x}} \Big[M(\hat{g}(\hat{x}))\, {\rm d}\hat{\phi}(\hat{x}) \Big] = \prod_x \Big[M(g(x))\, {\rm d}\phi(x) \Big]\,. \tag{6.8}$$

We stress again that in $\hat{\Sigma}$ and Σ two different lattices and time ordering parameters are considered; this is encoded in the two product symbols $\prod_{\hat{x}}$ of (6.7) and $\prod_x$ of (6.3). To realize the bridge between $\hat{\Gamma}[\hat{g}]$ (defined in $\hat{\Sigma}$) and $\Gamma[g]$ (defined in Σ), we now proceed in two steps.

In the first step, while remaining in the frame $\hat{\Sigma}$, we move from the lattice $\mathcal{E}_2$ of points P_i (and time ordering parameter $\hat{x}^0$) to the lattice $\mathcal{E}_1$ of points Q_i (and time ordering parameter x^0). This is realized going from $\prod_{\hat{x}} \big[M(\hat{g}(\hat{x}))\, {\rm d}\hat{\phi}(\hat{x}) \big]$ to $\prod_x \big[M(\hat{g}(\hat{x}(x)))\, {\rm d}\hat{\phi}(\hat{x}(x)) \big]$ and performing in the action $\hat{S}[\hat{\phi}, \hat{g}]$ the change of integration variables from $\hat{x}$ to x through the relations $\hat{x} = \hat{x}(x)$, $\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}) = \hat{g}_{\mu\nu}(\hat{x}(x))$. We get (see comments below (6.10) for the factors A and B)

$$\begin{aligned} e^{i\hat{\Gamma}[\hat{g}]} &= \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \Big[\prod_{\hat{x}} M(\hat{g}(\hat{x})) \Big] \Big[\prod_{\hat{x}} {\rm d}\hat{\phi}(\hat{x}) \Big] \\ &= \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]} \Big[B \prod_x M(\hat{g}(\hat{x}(x))) \Big] \Big[A \prod_x {\rm d}\hat{\phi}(\hat{x}(x)) \Big]\,, \end{aligned} \tag{6.9}$$

[3]With an abuse of notation, throughout this chapter we will use the same symbol $\hat{x}$ to indicate both the action of the diffeomorphism on x and its corresponding image, i.e. $\hat{x} = \hat{x}(x)$.

where $\hat{S}$ in the right-hand side is written as

$$\begin{aligned}\hat{S}[\hat{\phi},\hat{g}] &= \hat{S}_{\rm g}[\hat{g}] + \hat{S}_{\rm m}[\hat{\phi},\hat{g}] \\ &= \hat{S}_{\rm g}[\hat{g}] - \frac{1}{2}\int {\rm d}^4x\, J\sqrt{-\hat{g}(\hat{x}(x))}\left(\hat{g}^{\mu\nu}(\hat{x}(x))\frac{\partial x^\rho}{\partial\hat{x}^\mu}\frac{\partial x^\sigma}{\partial\hat{x}^\nu}\partial_\rho\hat{\phi}(\hat{x}(x))\partial_\sigma\hat{\phi}(\hat{x}(x)) + m^2\hat{\phi}^2(\hat{x}(x))\right) \\ &\equiv \hat{S}_{\rm g}[\hat{g}] + \int {\rm d}^4x\, \widetilde{\mathcal{L}}_{\rm m}(\hat{\phi}(\hat{x}(x)),\partial_\mu\hat{\phi}(\hat{x}(x)),\hat{g}(\hat{x}(x)))\,, \end{aligned} \tag{6.10}$$

with $J \equiv \left|\det\frac{\partial\hat{x}}{\partial x}\right|$ the Jacobian. Note that $\hat{S}[\hat{\phi},\hat{g}]$ is the action for the fields $\hat{\phi}$ and $\hat{g}_{\mu\nu}$, that is the action in the frame $\hat{\Sigma}$. We stress that up to now only a change of integration variables has been performed (not a change of reference frame). Moreover, $\widetilde{\mathcal{L}}_{\rm m}$ in the third line of (6.10) is the matter Lagrangian density[4], still in the frame $\hat{\Sigma}$, written in terms of the variables x.

Let us comment now on the factors A and B that appear in the right hand side of (6.9). One might naïvely expect that the step performed above ($\prod_{\hat{x}} \to \prod_x$) consists of a trivial reshuffling of (spacetime) points, in which case one would have $A = B = 1$. This is what is implicitly implemented in [36]. However, we have seen that in going from the left to the right hand side of (6.9), a non-trivial change of lattice and of ordering parameter occurs (see above). This is what ultimately leads to the appearance of non-trivial factors $A \neq 1$ and $B \neq 1$ in (6.9). The latter are calculated in the next section.

In the second step, we move from $\hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}(x))$ to $\phi(x)$ and $g_{\mu\nu}(x)$. This will eventually lead to the relation between $\hat{\Gamma}[\hat{g}]$ and $\Gamma[g]$. From (6.9), we get (below the invariance of the classical action, $\hat{S}[\hat{\phi},\hat{g}] = S[\phi,g]$, is used)

$$\begin{aligned} e^{i\hat{\Gamma}[\hat{g}]} &= \int e^{i\,\hat{S}[\hat{\phi},\hat{g}]}\Big[B\prod_x M(\hat{g}(\hat{x}(x)))\Big]\Big[A\prod_x {\rm d}\hat{\phi}(\hat{x}(x))\Big] \\ &= \int e^{i\,S[\phi,g]}\Big[B\,C\prod_x M(g(x))\Big]\Big[A\,E\prod_x {\rm d}\phi(x)\Big]\,. \end{aligned} \tag{6.11}$$

In the above equation, the factor C arises when we express $M(\hat{g}(\hat{x}(x)))$ in terms of $g_{\mu\nu}(x)$, while E is the Jacobian of the transformation $\hat{\phi}(\hat{x}(x)) \to \phi(x)$. As for A and B, the calculation of C and E is performed in the next section.

From (6.3) and (6.11), we see that for the effective action to be invariant under general coordinate transformations, i.e. to have $\hat{\Gamma}[\hat{g}] = \Gamma[g]$, it is necessary that $A\,B\,C\,E = 1$. In the coming sections 6.2 and 6.3, we will see that while this is the case for the FV measure (see $M_{\rm FV}$ in (6.4)), it fails to be true for the Fujikawa measure (see $M_{\rm Fuji}$ in (6.5)). Let us proceed now to the calculation of A, B, C and E.

6.2 Transformation factors

In the present section, we calculate the transformation factors A, B, C and E that appear in (6.11) following [18], where similar calculations have been performed for the pure gravity

[4]It is easy to see that $\widetilde{\mathcal{L}}_{\rm m}$ gives rise to the same equations of motion and the same energy-momentum tensor for the field $\hat{\phi}$ as $\mathcal{L}_{\rm m}$ in (6.6). The only difference is that they are written in terms of the variables x using the relations $\hat{x} = \hat{x}(x)$, $\hat{\phi}(\hat{x}) = \hat{\phi}(\hat{x}(x))$ and $\hat{g}_{\mu\nu}(\hat{x}) = \hat{g}_{\mu\nu}(\hat{x}(x))$.

case. In particular, we will find that the factors A and B, that are related to the transition $\prod_{\hat{x}} \to \prod_x$ involved in the transformation of the effective action under diffeomorphisms, turn out to be non-trivial ($\neq 1$). As anticipated, this shows that $\prod_{\hat{x}} \to \prod_x$ is not a trivial reshuffling of points, but rather a delicate step to be performed carefully while establishing how the effective action transforms under diffeomorphisms. Overlooking these terms, one is led to think that the FV measure is not diffeomorphism invariant, due to the presence of the g^{00} factors (see (6.4)). In fact, the opposite is true. As it will be seen in section 6.3, the invariance of the effective action emerges from a balance between these non-trivial terms and those coming from the g^{00} factors in the FV measure. Let us proceed now with the calculation.

The factor E - We begin with the factor E. Since ϕ is a scalar field ($\hat{\phi}(\hat{x}(x)) = \phi(x)$) and E is the Jacobian of the transformation $\hat{\phi}(\hat{x}(x)) \to \phi(x)$, clearly $E = 1$.

The factor A - Let us move now to the factor A. As mentioned in the previous section, the FV measure $\prod_x \left[M_{\rm FV}(g(x))\, {\rm d}\phi(x)\right]$ (see (6.4)) is obtained within the canonical formalism from the phase space path integral measure $\prod_x \left[{\rm d}\pi(x)\, {\rm d}\phi(x)\right]$, while the Fujikawa measure $\prod_x \left[M_{\rm Fuji}(g(x))\, {\rm d}\phi(x)\right]$ (see (6.5)) from $\prod_x \left[(g^{00}(x))^{-1/2}\, {\rm d}\pi(x)\, {\rm d}\phi(x)\right]$. To calculate A, we consider both cases at once writing (below $f(g) = 1$ for the FV measure and $f(g) = (g^{00}(x))^{-1/2}$ for the Fujikawa one)

$$e^{i\Gamma_{\mathcal{E}_1}[g]} = e^{iS_{\rm g}[g]} \int e^{i\int {\rm d}^4x [\partial_0\phi(x)\pi(x) - \mathcal{H}(\pi(x),\, \phi(x))]} \Big[\prod_x f(g(x))\ {\rm d}\pi(x)\, {\rm d}\phi(x)\Big] \tag{6.12}$$

where $\pi(x)$ is the momentum conjugate to the field $\phi(x)$

$$\pi(x) = \frac{\partial \mathcal{L}_{\rm m}(\phi(x), \partial_\mu\phi(x), g(x))}{\partial(\partial_0\phi(x))} = -(-g(x))^{1/2}\, g^{0\mu}(x)\, \partial_\mu\phi(x)\,, \tag{6.13}$$

$\mathcal{L}_{\rm m}(\phi(x), \partial_\mu\phi(x), g(x))$ the Lagrangian density in the frame Σ given in (6.1), and $\mathcal{H}(\pi(x),\, \phi(x))$ is the Hamiltonian density (here we use the shorthand notation $\mathcal{H}(\pi, \phi)$ for $\mathcal{H}(\pi, \phi, \partial_i\phi, g)$)

$$\mathcal{H}(\pi, \phi) = \pi(x)\, \partial_0\phi(x) - \mathcal{L}_{\rm m}(\phi(x), \partial_\mu\phi(x), g(x)) = \frac{1}{2}(-g^{00})^{-1}(-g)^{-1/2}\pi^2 + (-g^{00})^{-1} g^{0i}\, \pi\, \partial_i\phi$$
$$+ \frac{1}{2}(-g^{00})^{-1}(-g)^{1/2} g^{0i} g^{0j} \partial_i\phi\, \partial_j\phi + \frac{1}{2}(-g)^{1/2} g^{ij} \partial_i\phi\, \partial_j\phi + \frac{1}{2}(-g)^{1/2}\, m^2\phi^2\,. \tag{6.14}$$

As stressed in the previous section (see comments below (6.5)), $\prod_x$ in (6.12) means that the discretization is realized considering the points Q_i of the lattice $\mathcal{E}_1$ and that the time ordering parameter is x^0 (this is further stressed by the subscript $\mathcal{E}_1$ in $\Gamma_{\mathcal{E}_1}[g]$). In this respect, we stress that $\pi(x)$ is correctly defined by (6.13) since in this case x^0 is the time ordering parameter. As said above (see footnote 2), we could also quantize the theory considering a more general family of space-like hypersurfaces $\tau(x) = {\rm const}$, in which case the time ordering parameter would be τ, and the conjugate momenta would be obtained differentiating the Lagrangian with respect to $\partial_\tau\phi$.

For the effective action in $\hat{\Sigma}$ we have (now the lattice $\mathcal{E}_2$ of points P_i is used)

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_{\rm g}[\hat{g}]} \int e^{i\int {\rm d}^4\hat{x} \left[\hat{\partial}_0\hat{\phi}(\hat{x})\hat{\pi}(\hat{x}) - \mathcal{H}(\hat{\pi}(\hat{x}),\, \hat{\phi}(\hat{x}))\right]} \Big[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\ {\rm d}\hat{\pi}(\hat{x})\, {\rm d}\hat{\phi}(\hat{x})\Big] \tag{6.15}$$

where $\hat{\pi}(\hat{x})$ is the momentum conjugate to the field $\hat{\phi}(\hat{x})$

$$\hat{\pi}(\hat{x}) = \frac{\partial \mathcal{L}_{\rm m}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))}{\partial(\hat{\partial}_0 \hat{\phi}(\hat{x}))} = -(-\hat{g}(\hat{x}))^{1/2}\, \hat{g}^{0\mu}(\hat{x})\, \hat{\partial}_\mu \hat{\phi}(\hat{x})\,, \tag{6.16}$$

$\mathcal{L}_{\rm m}(\hat{\phi}(\hat{x}), \hat{\partial}_\mu \hat{\phi}(\hat{x}), \hat{g}(\hat{x}))$ the Lagrangian density in the frame $\hat{\Sigma}$ (see (6.6)), and $\mathcal{H}(\hat{\pi}(\hat{x}),\, \hat{\phi}(\hat{x}))$ the corresponding Hamiltonian density (see (6.14)). The symbol $\prod_{\hat{x}}$ indicates that the discretization is realized considering the points P_i of the lattice $\mathcal{E}_2$. The time ordering parameter is now $\hat{x}^0$, and accordingly the conjugate momentum $\hat{\pi}(\hat{x})$ is given by (6.16).

As explained in the previous section, the bridge between $\hat{\Gamma}[\hat{g}]$ and $\Gamma[g]$ is realized in two steps. In the first one (from which the factor A emerges), while remaining in the frame $\hat{\Sigma}$ we write $\hat{\Gamma}[\hat{g}]$ switching from the lattice $\mathcal{E}_2$ to the lattice $\mathcal{E}_1$ (that in turn implies switching from the time ordering parameter $\hat{x}^0$ to x^0). Considering then $\mathcal{E}_1$, for $\hat{\Gamma}[\hat{g}]$ we have

$$\begin{aligned} e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} &= e^{i\hat{S}_{\rm g}[\hat{g}]} \\ \times \int & e^{i\int {\rm d}^4x \left[\partial_0\hat{\phi}(\hat{x}(x))\widetilde{\pi}(\hat{x}(x)) - \widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)),\, \hat{\phi}(\hat{x}(x)))\right]} \Big[F \prod_x f(\hat{g}(\hat{x}(x)))\Big] \Big[\prod_x {\rm d}\widetilde{\pi}(\hat{x}(x))\, {\rm d}\hat{\phi}(\hat{x}(x))\Big]\,. \end{aligned} \tag{6.17}$$

In the above equation, F is the factor that arises when $\prod_{\hat{x}} f(\hat{g}(\hat{x}))$ is written in terms of $\prod_x f(\hat{g}(\hat{x}(x)))$, i.e. when we switch from $\mathcal{E}_2$ to $\mathcal{E}_1$: $\prod_{\hat{x}} f(\hat{g}(\hat{x})) = F \prod_x f(\hat{g}(\hat{x}(x)))$. We will shortly see that we do not need to calculate explicitly the transformation factor F. Moreover, since in (6.17) the time ordering parameter is x^0, the momentum $\widetilde{\pi}(\hat{x}(x))$ conjugate to $\hat{\phi}(\hat{x}(x))$ is given by (see comments below (6.10) and (6.13))

$$\widetilde{\pi}(\hat{x}(x)) = \frac{\partial \widetilde{\mathcal{L}}_{\rm m}(\hat{\phi}(\hat{x}(x)), \partial_\mu\hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))}{\partial(\partial_0\hat{\phi}(\hat{x}(x)))} = -(-\hat{g}(\hat{x}(x)))^{1/2}\, G^{0\mu}(\hat{x}(x))\, \partial_\mu\hat{\phi}(\hat{x}(x))\,, \tag{6.18}$$

and the Hamiltonian density to be used is

$$\begin{aligned} &\widetilde{\mathcal{H}}(\widetilde{\pi}(\hat{x}(x)), \hat{\phi}(\hat{x}(x))) = \widetilde{\pi}(\hat{x}(x))\, \partial_0\hat{\phi}(\hat{x}(x)) - \widetilde{\mathcal{L}}_{\rm m}(\hat{\phi}(\hat{x}(x)), \partial_\mu\hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x))) \\ &= \frac{1}{2}(-G^{00})^{-1}(-\hat{g})^{-1/2}\,\widetilde{\pi}^{\,2} + (-G^{00})^{-1}G^{0i}\,\widetilde{\pi}\,\partial_i\hat{\phi} \\ &+ \frac{1}{2}(-G^{00})^{-1}(-\hat{g})^{1/2}\, G^{0i}\, G^{0j}\partial_i\hat{\phi}\,\partial_j\hat{\phi} + \frac{1}{2}(-\hat{g})^{1/2}G^{ij}\partial_i\hat{\phi}\,\partial_j\hat{\phi} + \frac{1}{2}J\,(-\hat{g})^{1/2}\, m^2\hat{\phi}^2\,, \end{aligned} \tag{6.19}$$

where $\widetilde{\mathcal{L}}_{\rm m}(\hat{\phi}(\hat{x}(x)), \partial_\mu\hat{\phi}(\hat{x}(x)), \hat{g}(\hat{x}(x)))$ is the Lagrangian density in (6.10) and we define ($J \equiv \left|\det \frac{\partial \hat{x}}{\partial x}\right|$ is the Jacobian related to the change of variables $\hat{x} \to x$)

$$G^{\mu\nu}(\hat{x}(x)) \equiv J\hat{g}^{\rho\sigma}(\hat{x}(x))\frac{\partial x^\mu}{\partial \hat{x}^\rho}\frac{\partial x^\nu}{\partial \hat{x}^\sigma}\,. \tag{6.20}$$

Let us perform now in (6.15) the functional integration over $\hat{\pi}(\hat{x})$. We get

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_{\rm g}[\hat{g}]}\Big[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\Big] \int e^{i\hat{S}_{\rm m}[\hat{\phi},\,\hat{g}]}\Big[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\Big]\prod_{\hat{x}} {\rm d}\hat{\phi}(\hat{x}) \tag{6.21}$$

where we define

$$W(\hat{g}(\hat{x})) \equiv (-\hat{g}(\hat{x}))^{1/4}(-\hat{g}^{00}(\hat{x}))^{1/2}\,. \tag{6.22}$$

Similarly, the integration over $\widetilde{\pi}(\hat{x}(x))$ in (6.17) gives

$$e^{i\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]} = e^{i\hat{S}_{\rm g}[\hat{g}]}\, F\Big[\prod_x f(\hat{g}(\hat{x}(x)))\Big] \int e^{i\hat{S}_{\rm m}[\hat{\phi},\,\hat{g}]}\Big[\prod_x Y(\hat{g}(\hat{x}(x)))\Big]\prod_x {\rm d}\hat{\phi}(\hat{x}(x))\ , \tag{6.23}$$

where we define

$$Y(\hat{g}(\hat{x}(x))) \equiv J^{1/2}(-\hat{g}(\hat{x}(x)))^{1/4}\big[-\hat{g}^{\mu\nu}(\hat{x}(x))\,\hat{\partial}_\mu x^0\,\hat{\partial}_\nu x^0\big]^{1/2}\,. \tag{6.24}$$

Moreover, writing $\prod_{\hat{x}} {\rm d}\hat{\phi}(\hat{x})$ in terms of $\prod_x {\rm d}\hat{\phi}(\hat{x}(x))$ (see Eq. (6.9)), Eq. (6.21) can be written as

$$e^{i\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]} = e^{i\hat{S}_{\rm g}[\hat{g}]}\Big[\prod_{\hat{x}} f(\hat{g}(\hat{x}))\Big] \int e^{i\hat{S}_{\rm m}[\hat{\phi},\,\hat{g}]}\Big[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\Big]\Big[A\prod_x {\rm d}\hat{\phi}(\hat{x}(x))\Big]\,. \tag{6.25}$$

Since $\hat{\Gamma}_{\mathcal{E}_2}[\hat{g}]$ and $\hat{\Gamma}_{\mathcal{E}_1}[\hat{g}]$ are the same effective action in $\hat{\Sigma}$, simply written using the two lattices $\mathcal{E}_2$ and $\mathcal{E}_1$, respectively, from (6.23) and (6.25) we finally get

$$A = \frac{\prod_x Y(\hat{g}(\hat{x}(x)))}{\prod_{\hat{x}} W(\hat{g}(\hat{x}))}\,. \tag{6.26}$$

We now write A in a convenient form. Let us consider an infinitesimal coordinate transformation (in the following we only need to keep terms up to first order in ε)

$$\hat{x}^\mu = x^\mu + \varepsilon^\mu(x)\,, \tag{6.27}$$

and switch from $\prod_x$ to $\prod_{\hat{x}}$ in the numerator of (6.26). As we will see below (Eq. (6.31)), this allows to factor out a term that cancels the denominator. Observing that[5] (as usual, $\delta^{(4)}(0)$ comes from $\sum_x \to \int {\rm d}^4x$)

$$\prod_x J^{1/2} = \exp\Big(\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4x\log(1+\partial_\mu\varepsilon^\mu)\Big) = \exp\Big(\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4x\,\partial_\mu\varepsilon^\mu\Big) = 1\,, \tag{6.28}$$

where we have used

$$\frac{\partial\hat{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\varepsilon^\mu \quad ; \quad {\rm d}^4\hat{x} = {\rm d}^4x\,J \equiv {\rm d}^4x\,\Big|\det\frac{\partial\hat{x}}{\partial x}\Big| = {\rm d}^4x\,(1+\partial_\mu\varepsilon^\mu(x))\,, \tag{6.29}$$

and calculating the third factor in the right hand side of (6.24) ($\hat{\partial}_\mu x^0 = \delta^0_\mu - \partial_\mu\varepsilon^0$),

$$-\hat{g}^{\mu\nu}(\hat{x}(x))\,\hat{\partial}_\mu x^0\,\hat{\partial}_\nu x^0 = -\hat{g}^{00} + 2\,\hat{g}^{0\mu}\,\partial_\mu\varepsilon^0 = -\hat{g}^{00}\Big(1 - 2\,\frac{\hat{g}^{0\mu}}{\hat{g}^{00}}\,\partial_\mu\varepsilon^0\Big)\,, \tag{6.30}$$

from (6.22), (6.24), (6.28), (6.30) we get

$$\prod_x Y(\hat{g}(\hat{x}(x))) = \exp\Big(\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4\hat{x}\,(1-\partial_\mu\varepsilon^\mu)\log\Big[(-\hat{g}(\hat{x}))^{1/2}\big(-\hat{g}^{00}(\hat{x})\big)\Big(1-2\,\frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})}\,\partial_\mu\varepsilon^0\Big)\Big]\Big)$$
$$= \Big[\prod_{\hat{x}} W(\hat{g}(\hat{x}))\Big]\,\exp\Big(-\frac{\delta^{(4)}(0)}{2}\int {\rm d}^4\hat{x}\,\Big[\partial_\mu\varepsilon^\mu\log\Big((-\hat{g}(\hat{x}))^{1/2}\big(-\hat{g}^{00}(\hat{x})\big)\Big) + 2\,\frac{\hat{g}^{0\mu}(\hat{x})}{\hat{g}^{00}(\hat{x})}\,\partial_\mu\varepsilon^0\Big]\Big)\,. \tag{6.31}$$

[5]Note that the exponent in the third term of (6.28) has the form of a surface integral. It is then equal to zero in the case of boundary-less manifolds and/or asymptotically flat (smooth) gravitational fields, in which latter case gauge transformations such that $\varepsilon \to 0$ at infinity are considered to preserve the asymptotics.

Finally, inserting (6.31) in (6.26) we have

$$A = \exp\biggl(-\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x \left[\partial_\mu\varepsilon^\mu \log\Bigl((-g(x))^{1/2}\bigl(-g^{00}(x)\bigr)\Bigr) + 2\,\frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu\varepsilon^0\right]\biggr). \tag{6.32}$$

The factor B - Let us calculate now B. We begin by observing that $M(\hat{g}(\hat{x}))$ in (6.9) is nothing but the product $f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))$ in (6.21). We then have (below we make use of (6.29))

$$\begin{aligned}\prod_{\hat{x}} M(\hat{g}(\hat{x})) &= \prod_{\hat{x}}\left[f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x}))\right] = \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4\hat{x}\log(f(\hat{g}(\hat{x}))W(\hat{g}(\hat{x})))\biggr)\\ &= \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu\log(f(g(x))W(g(x)))\biggr)\prod_x\left[f(\hat{g}(\hat{x}(x)))W(\hat{g}(\hat{x}(x)))\right]\\ &= \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu\log(M(g(x)))\biggr)\prod_x M(\hat{g}(\hat{x}(x)))\,.\end{aligned} \tag{6.33}$$

Recalling that $\prod_{\hat{x}} M(\hat{g}(\hat{x})) = B\prod_x M(\hat{g}(\hat{x}(x)))$ (see (6.9)), from (6.33) we have

$$B = \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu\log(M(g(x)))\biggr). \tag{6.34}$$

The factor C - Finally, we calculate the factor C, that is defined by $\prod_x M(\hat{g}(\hat{x}(x))) = C\prod_x M(g(x))$ (see (6.11)). At first order in ε, we have

$$\begin{aligned}M(\hat{g}^{\mu\nu}(\hat{x}(x))) &= M\bigl(g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma\varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma\varepsilon^\mu(x)\bigr) =\\ &= M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right),\end{aligned} \tag{6.35}$$

where we used

$$\hat{g}^{\mu\nu}(\hat{x}) = \frac{\partial\hat{x}^\mu(x)}{\partial x^\rho}\frac{\partial\hat{x}^\nu(x)}{\partial x^\sigma}\,g^{\rho\sigma}(x) = g^{\mu\nu}(x) + g^{\mu\sigma}(x)\partial_\sigma\varepsilon^\nu(x) + g^{\nu\sigma}(x)\partial_\sigma\varepsilon^\mu(x)\,. \tag{6.36}$$

We then get

$$\begin{aligned}\prod_x M(\hat{g}(\hat{x}(x))) &= \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\log\Bigl[M(g) + \frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\Bigr]\biggr)\\ &= \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\,\frac{1}{M(g)}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\biggr)\prod_x M(g(x))\,,\end{aligned} \tag{6.37}$$

from which

$$C = \exp\biggl(\delta^{(4)}(0)\int \mathrm{d}^4x\,[M(g)]^{-1}\frac{\partial M(g)}{\partial g^{\alpha\beta}(x)}\left(g^{\alpha\sigma}(x)\partial_\sigma\varepsilon^\beta(x) + g^{\beta\sigma}(x)\partial_\sigma\varepsilon^\alpha(x)\right)\biggr). \tag{6.38}$$

Having calculated the factors A, B, C and E, we can now see how the FV and the Fujikawa measure transform under diffeomorphisms. This is the subject of the next section.

6.3 Fradkin-Vilkovisky versus Fujikawa measure

As shown in section 6.1 (see (6.11) and comments below), for the effective action to be diffeomorphism invariant, it must be $A\,B\,C\,E = 1$. Having found in the previous section the general expressions for A, B, C and E, we can now calculate these terms for both $M(g(x)) = M_{\rm FV}(g(x))$ and $M(g(x)) = M_{\rm Fuji}(g(x))$. From a simple inspection of (6.32), (6.34) and (6.38) we see that, while B and C depend on the specific form of $M(g(x))$, A does not. Moreover, we have already seen at the beginning of section 6.2 that $E = 1$. We have then to calculate B and C for $M_{\rm FV}(g(x))$ and $M_{\rm Fuji}(g(x))$, respectively.

Fradkin-Vilkovisky measure - Inserting (6.4) in (6.34) and (6.38) we get

$$B_{\rm FV} = \exp\!\left(\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu \log[(-g(x))^{1/2}(-g^{00}(x))]\right) \tag{6.39}$$

$$C_{\rm FV} = \exp\!\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,\frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu\varepsilon^0\right), \tag{6.40}$$

from which (see also (6.32))

$$B_{\rm FV}\,C_{\rm FV} = A^{-1}\,. \tag{6.41}$$

Eq. (6.41), together with the result $E = 1$, shows that for the FV measure

$$A\,B_{\rm FV}\,C_{\rm FV}\,E = 1\,, \tag{6.42}$$

which means that in this case (see (6.11)): $\hat{\Gamma}[\hat{g}] = \Gamma[g]$. As anticipated, the effective action calculated using the FV measure *is* diffeomorphism invariant.

Fujikawa measure - Inserting (6.5) in (6.34) and (6.38) we get

$$B_{\rm Fuji} = \exp\!\left(\delta^{(4)}(0)\int \mathrm{d}^4x\,\partial_\mu\varepsilon^\mu \log\big((-g(x))^{1/4}\mu\big)\right) \tag{6.43}$$

$$C_{\rm Fuji} = 1\,. \tag{6.44}$$

From (6.32), (6.43) and (6.44) (together with $E = 1$) we obtain

$$A\,B_{\rm Fuji}\,C_{\rm Fuji}\,E = \exp\!\left(\delta^{(4)}(0)\int \mathrm{d}^4x\left[\partial_\mu\varepsilon^\mu \log\!\left(\frac{\mu}{(-g^{00})^{1/2}}\right) - \frac{g^{0\mu}(x)}{g^{00}(x)}\,\partial_\mu\varepsilon^0\right]\right), \tag{6.45}$$

which means that in this case (see (6.11)): $\hat{\Gamma}[\hat{g}] \neq \Gamma[g]$. As anticipated, and contrary to what is claimed in [36], the effective action calculated using the Fujikawa measure *is not* diffeomorphism invariant.

6.4 Comparison with existing literature

The main result of the analysis presented in the previous sections is that the FV measure [18] is diffeomorphism invariant, while the Fujikawa measure [107] is not. The naïve argument according to which the FV measure is claimed to be non-invariant is that it contains factors of the time-time component g^{00} of the inverse metric. Actually, we have shown that the opposite is true. As thoroughly discussed in the previous sections, the

g^{00} factors in the FV measure guarantee the diffeomorphism invariance of the effective action. In fact, their presence is imposed by the necessity of having a time ordering parameter to define S matrix elements. By the same token, we have shown that the Fujikawa measure is not diffeomorphism invariant. Moreover, our calculations allow to understand why in the recent work [36] the opposite conclusion is reached, namely that the Fujikawa measure rather than the FV one is diffeomorphism invariant. The detailed analysis on the transformation of the effective action under diffeomorphisms developed in the previous sections shows that some of the terms involved in this transformation are missed in that paper.

Since the question of the diffeomorphism invariance of the effective action, sometimes source of discussions and controversies, is central to the present thesis, it is worth to make a more detailed comparison between the results of this chapter and those of [36]. We will see that the calculations and claims of that paper are flawed.

The authors of [36] consider the scalar theory defined by (6.1) and (6.2) above (see their section II). A simple inspection of their Eqs. (2.14) and (2.19) shows that they overlook the transition between the two lattices involved in the coordinate transformation (together with the related change of time ordering parameter). This is tantamount to assume that this transition is nothing but a trivial reshuffling of points (see (6.9) and comments below). Actually, the authors of [36] consider *only* the factors related to the transformation $(-\hat{g})^{1/4}\hat{\phi} \to (-g)^{1/4}\phi$. According to the notation that we introduced in the previous sections, this amounts to consider *only* the factors C and E in (6.11), while missing both the non-trivial factors A and B. It is precisely because they miss these latter terms that the authors of [36] are led to conclude that the Fujikawa measure is diffeomorphism invariant. The same issues arise when the authors extend their considerations to pure quantum gravity (see their section VI). Also in this case, they consider the Fujikawa measure [107], and attempt to demonstrate that the latter is BRST invariant. Once again, they miss non-trivial transformation factors in the transition $\hat{x} \to x$, as it is immediately seen in their Eqs. (6.3)-(6.10).

Actually, all the arguments and conclusions in [36] are flawed, since they are based on the alleged diffeomorphism invariance of the Fujikawa measure. For instance, they claim that the path integral measure to be used in phase space is [108] (see section III of [36], in particular the comments below Eqs. (3.9) and (3.27))

$$\prod_x \left[(g^{00}(x))^{-1/2} \, \mathrm{d}\pi(x) \, \mathrm{d}\phi(x) \right] . \tag{6.46}$$

Starting with (6.46), in fact, after integration over the conjugate momenta π the Fujikawa measure in configuration space is obtained. As shown in the previous sections, however, the diffeomorphism invariant measure in configuration space is the Fradkin-Vilkovisky one. The latter is obtained after integration over π starting from the phase space path integral measure

$$\prod_x \left[\mathrm{d}\pi(x) \, \mathrm{d}\phi(x) \right] , \tag{6.47}$$

that is nothing but the Liouville measure. This is what one would expect from general considerations based on the path integral construction of the basic transition amplitude $\langle \phi'', t'' | \phi', t' \rangle$ (see section 1.1.1). For R^2 higher-derivative theories of gravity, a similar derivation of the configuration space measure from the Liouville one was later done by Buchbinder and Lyakhovich [146].

To support their claim on the diffeomorphism invariance of the Fujikawa measure, the authors of [36] (see their section IV) consider the contribution δS to the effective gravitational action from a free scalar field, obtained using this measure (μ is an arbitrary mass scale in the Fujikawa measure, see (6.5))

$$\delta S[g] = \frac{1}{2}\,\mathrm{Tr}\log\left(\frac{-\Box + m^2}{\mu^2}\right), \tag{6.48}$$

where $-\Box$ is the spin-0 Laplace-Beltrami operator for the metric $g_{\mu\nu}$. They evaluate the trace resorting to two different methods: (i) sum over the eigenvalues of $-\Box + m^2$; (ii) proper-time formalism, and get for δS the results (4.7) and (4.15) of [36], respectively. Their calculations, however, overlook a quite delicate point (see below). This oversight leads them in both cases to a result for δS expressed in terms of diffeomorphism invariant quantities, so they conclude that the Fujikawa measure is diffeomorphism invariant. The delicate point is that, in the case of gravitational theories, subtleties arise in the calculation of $\log(-\Box + m^2)$ (that appears in their Eqs. (4.4) and (4.13)), and one should carefully take into account the distributional nature of the Green's function of $(-\Box+m^2)$ [18,20,46] (see also the discussion below (5.16) in section 5.1). When this is done, a non-trivial term contributing to the trace in (6.48) is found. This term is missed in [36]. Taking x^0 as time ordering parameter (see comments below Eqs. (6.5) and (6.7)), this latter term turns out to be $\delta^{(4)}(0)\int \mathrm{d}^4x \log(g^{00})$ [20], which is not diffeomorphism invariant. As a consequence, the quantum correction δS calculated with the Fujikawa measure *is not* diffeomorphism invariant.

Let us consider now the result for $\delta S[g]$ obtained when the FV measure rather than the Fujikawa one is used, which is what we calculated in section 3.3.1 of the present thesis. The result is similar to (6.48) (obtained with the Fujikawa measure), but in addition to the "Tr log" it contains an extra term (see second line of (6.49)), whose crucial importance will be soon clear. The calculation is performed taking the metric $g^{(a)}_{\mu\nu}$ of a sphere[6] of radius a, and gives (see (3.48) and (3.49))

$$\begin{aligned}\delta S[g^{(a)}] &= \frac{1}{2}\log\left[\det\left(-\widetilde{\Box} + a^2m^2\right)\right] - \frac{1}{2}\log\left(\prod_x \widetilde{g}^{00}(x)\right)\\ &= \frac{1}{2}\,\mathrm{Tr}\log\left(-\widetilde{\Box} + a^2m^2\right) - \frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x \log\left(\widetilde{g}^{00}(x)\right),\end{aligned} \tag{6.49}$$

where $-\widetilde{\Box}$ is the Laplace-Beltrami operator for the metric $\widetilde{g}_{\mu\nu}$ of a unit sphere, and $\widetilde{g}^{00}$ is the time-time component of the inverse of $\widetilde{g}_{\mu\nu}$. To avoid misunderstandings, we stress again that, despite the presence in (6.49) of $\widetilde{g}_{\mu\nu}$ and $-\widetilde{\Box}$, $\delta S[g^{(a)}]$ is calculated for a sphere of *generic* radius a. As clearly explained in chapters 3 and 4 (to which we refer for details), these dimensionless quantities ($-\widetilde{\Box}$, $\widetilde{g}_{\mu\nu}$ and a^2m^2) appear thanks to the presence of the FV measure. As stressed several times throughout this thesis, and contrary to what is incorrectly claimed in [149], in [14, 23] we do not take the radius a as reference scale. This is immediately evident if for instance we note that in (6.49) the combination a^2m^2 appears.

[6]This useful choice is often considered in the literature, see for instance [111,112,147,148].

Some crucial points need to be stressed. First of all, we observe that the term $\frac{1}{2}\log\big(\prod_x \widetilde{g}^{00}(x)\big)$ in (6.49) comes from the exponentiation of the measure term $\prod_x(\widetilde{g}^{00}(x))^{1/2}$ (see (6.3), with $M(g)$ given in (6.4)). In this respect, we recall (see comments below (6.5)) that the discretization involved in the definition of the path integral, encoded in $\prod_x$, allows the trivial splitting $\prod_x \big[(g^{00}(x))^{1/2}(g(x))^{1/4}\,\mathrm{d}\phi(x)\big] = \big[\prod_x(g^{00}(x))^{1/2}\big]\big[\prod_x(g(x))^{1/4}\,\mathrm{d}\phi(x)\big]$, which is what we operate to get (6.49). Moreover, as it should be clear from the comments below (6.48), the calculation of $\frac{1}{2}\,\mathrm{Tr}\log(-\widetilde{\Box}+a^2m^2)$ in (6.49) gives rise to the term $\frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x\log\big(\widetilde{g}^{00}(x)\big)$ [18, 20]. This term *cancels* the last one in the second line[7] of (6.49). All the other terms that come from the "Tr log" turn out to be diffeomorphism invariant. Therefore, the presence of the $g^{00}(x)$ factors in the measure not only does not spoil the diffeomorphism invariance of the effective action (as one would naïvely expect) but it is rather necessary[8] to *compensate* non-invariant terms that arise from the calculation of the "Tr log". Similar considerations apply to the pure gravity case [18] considered in section 3.1.

We have just seen that δS is given by (6.49), and that the presence of $g^{00}(x)$ factors in the configuration space measure is crucial to have a diffeomorphism invariant result. In [36], the more general measure $\prod_x\big[(g(x))^{1/4}\,\Omega_g(x)\,\mathrm{d}\phi(x)\big]$ is also considered, and the authors write (see their section V)

$$\delta S[g] = \frac{1}{2}\,\mathrm{Tr}\log[\Omega_g^{-2}(-\Box+m^2)]\,. \tag{6.50}$$

Based on their idea that the diffeomorphism invariance of δS is ensured if the fluctuation operator $\Omega_g^{-2}(-\Box+m^2)$ is covariant, they claim that the only possibility for Ω_g is to be a scalar. The analysis of the present chapter (see in particular the three previous paragraphs) shows that, contrary to this claim, the only possibility to have a diffeomorphism invariant effective action is to take $\Omega_g = (g^{00})^{1/2}$.

In fact, as explained in detail above, the diffeomorphism invariance of the effective action is not naïvely related to the fact that only covariant functions and functionals are present in the path integral. The invariance emerges from a delicate balance between *all* the elements that enter the definition of the path integral. We have seen that the delicate point concerns the product $\prod_x$ of all these elements, more specifically the time ordering and the discretization (lattice) involved in the definition of the path integral and encoded in this product.

Going back to (6.50), for a scalar Ω_g the authors of [36] manage to write $\Omega_g^{-2}(-\Box+m^2)$ in the form of a minimal Laplace-type operator, i.e. an operator of the kind $(-\Box+m^2)$ as the one in (6.48) (see Eqs. (A5) and (A6) of [36]). They then calculate the "Tr log" resorting to proper-time techniques, but fail to treat the $\log(-\Box)$ with the due care, thus missing relevant terms (see the thorough discussion below (6.48)) [20]. As a consequence, the proper-time RG equation (5.6) of [36], that the authors derive differentiating with respect to the scale Λ their (incomplete) regularized action S_Λ, cannot be trusted, and the considerations and conclusions they draw starting from this equation are flawed.

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[7] See [41] for earlier discussions on the appearance and cancellation of $\delta(0)$ divergences.

[8] In [14], the term $\frac{1}{2}\log\big(\prod_x \widetilde{g}^{00}(x)\big) = \frac{\delta^{(4)}(0)}{2}\int \mathrm{d}^4x\log\big(\widetilde{g}^{00}(x)\big)$ is indicated with $\mathcal{C}$ (see Eq. (48) of [14]). There, we do not refer to the cancellation discussed in this work since the terms involved are a-independent, and have no impact in the calculation of the quantum corrections to the Newton and cosmological constant (the objective of [14]).

The authors then move to consider the pure gravity case (in the EH truncation; see section VII of [36]), and develop considerations that mirror (as they say) those made for the contribution to the effective gravitational action $\Gamma[g]$ from the scalar field. Even in this pure gravity case, in fact, the contribution δS_{grav} to Γ from the graviton is calculated considering traces similar to (6.50). They claim that if one uses the Fuijikawa measure [107] a diffeomorphism invariant RG flow is obtained that (in four dimensions) contains terms of the kind (see their Eqs. (7.1) and (7.2))

$$\sim \Lambda^4 \int \mathrm{d}^4 x \, \sqrt{g} \tag{6.51}$$

and

$$\sim \Lambda^2 \int \mathrm{d}^4 x \, \sqrt{g} \, R \, . \tag{6.52}$$

Moreover, they observe that these terms, that depend quartically and quadratically on the running scale Λ, are at the origin of the UV-attractive fixed point of the asymptotic safety scenario. We will comment on the existence/absence of this fixed point at the end of the present section.

In [36], it is also claimed that the use of the FV measure in the path integral amounts to make in (6.51) and (6.52) (that are obtained using the Fujikawa measure) the replacement $\Lambda^2 \to N^2 g^{00}$, where N is a dimensionless (running) cutoff. They then get

$$\Lambda^4 \int \mathrm{d}^4 x \, \sqrt{g} \quad \to \quad N^4 \int \mathrm{d}^4 x \, (g^{00})^2 \sqrt{g} \tag{6.53}$$

and

$$\Lambda^2 \int \mathrm{d}^4 x \, \sqrt{g} \, R \quad \to \quad N^2 \int \mathrm{d}^4 x \, \sqrt{g} \, g^{00} \, R \, . \tag{6.54}$$

Accordingly, they claim that the FV measure leads to the presence of non-invariant operators in the running action $S_\Lambda[g]$. These claims are incorrect. In fact, from the thorough discussion above, it is clear that these claims are due to the loss of important terms in their calculation of $S_\Lambda[g]$. We have shown that when the FV measure is used, not only non-invariant terms of the kind (6.53) and (6.54) do not appear, but also that the presence of g^{00} factors in this measure is necessary to guarantee the diffeomorphism invariance of $S_\Lambda[g]$ (see (6.49) and comments therein). More in detail, we have seen that the g^{00} factors in the FV measure turn out to cancel terms that arise when the different $\log\left(-\Box^{(s)}\right)$ in δS_{grav} are correctly calculated (here $-\Box^{(s)}$ are the Laplace-Beltrami operators for the different spins, $s = 0, 1, 2$), and only diffeomorphism invariant terms are left in the final result. Among them are the terms[9] $N^4/3$ and $-34N^2/3$ in Eq. (4.14) of chapter 4. Contrary to what is incorrectly claimed in [36], they are diffeomorphism invariant and do not come from (6.53) and (6.54) evaluated in a coordinate system where $g^{00} = R$.

Let us finally go back to the question of the possible existence of the UV-attractive fixed point of the AS scenario. It is true that this fixed point would exist if terms of the kind (6.51) and (6.52) appeared in the RG equation for the running action. However, thanks to a careful treatment of the path integral measure and a proper introduction of the

[9] In [23], the calculations are performed with a spherical background. Moreover, the dimensionless (running) cutoff N is indicated with L. For the sake of clarity, here we uniform our notation to [36] and use N.

physical running scale, in chapter 4 ([23, 24]) we showed that a term like (6.51) is absent, and that (more generally) the beta functions for the Newton and the cosmological constant are significantly different from those of the AS literature [30, 34]. They do not possess the non-trivial UV-attractive fixed point of the AS scenario. Finally, concerning the beta functions of the AS literature, it is worth to add here that a gauge and parametrization dependence of these RG functions has been repeatedly reported. See for instance the very recent work [150], where it is shown that the existence of the UV attractive fixed point of the AS scenario depends on the value of the gauge fixing parameter α. In a certain range of values of α, this fixed point even disappears. A clear indication that the UV behaviour of the theory depends on the gauge fixing parameter was previously given in [151], where it was shown that, depending on the value of α, the UV behaviour of the Newton constant in fact changes from screening to antiscreening.

6.5 Conclusions

In the present chapter we have thoroughly considered the controversial issue concerning the diffeomorphism invariance of the path integral measure in quantum gravity. It is usually thought that, due to the presence of non-covariant g^{00} factors, the FV measure cannot be diffeomorphism invariant.

With the help of the detailed analysis developed in sections 6.1, 6.2 and 6.3, we have shown that, contrary to this naïve expectation, the g^{00} factors that appear in the FV measure are necessary to ensure the diffeomorphism invariance of the effective action $\Gamma[g]$. Actually, we have seen that the invariance emerges from a delicate balance between *all* the elements involved in the definition of the path integral. In particular, we have shown that a crucial point is related to the necessity of introducing a time ordering parameter and a discretization (lattice) of spacetime. Differently from other gauge theories, where the gauge transformation does not affect the spacetime coordinates, when considering a general coordinate transformation $x \to \hat{x}$, two different lattices and time ordering parameters are involved in the two coordinate systems. As a consequence, non-trivial factors appear when the path integral undergoes a general coordinate transformation. The g^{00} factors of the FV measure are necessary to *exactly* compensate for these non-trivial terms, and this is what ensures the diffeomorphism invariance of the effective action.

Another important point concerns the derivation of the configuration space measure from the phase space one. The FV measure $\prod_x \left[(g^{00}(x))^{1/2}(g(x))^{1/4}\, \mathrm{d}\phi(x)\right]$ is obtained after integration over the conjugate momenta π if the phase space measure is the Liouville one, namely $\prod_x \left[\mathrm{d}\pi(x)\, \mathrm{d}\phi(x)\right]$. The latter is what one would naturally expect from considerations based on the path integral construction of the basic transition amplitude $\langle \phi'', t'' | \phi', t' \rangle$ (as seen in section 1.1.1). On the contrary, the Fujikawa measure is obtained in configuration space if $\prod_x \left[(g^{00}(x))^{-1/2}\, \mathrm{d}\pi(x)\, \mathrm{d}\phi(x)\right]$ is assumed to be the phase space measure. This rather bizarre form of the phase space measure was first proposed in [108], though the arguments presented in that paper are far from being well-grounded. This proposal was recently taken up by the authors of [36]. Based on their idea that the diffeomorphism invariant measure in configuration space is the Fujikawa one, they derive this bizarre phase space measure from the request that, after integration over the conjugate fields π, the resulting configuration space measure is the Fujikawa one. The detailed analysis presented in this chapter, where we have shown that it is rather the FV measure to be invariant, indicates that such an artificial distortion of the natural Liouville measure in phase space has to be rejected (as one also expects on the basis of simpler arguments).

In conclusion, we have shown that the question of the measure to be used in the calculation of the effective gravitational action $\Gamma[g]$ presents delicate aspects that can be missed if one resorts to *formal* calculations. Paying attention to the construction of the path integral involved in the very definition of $\Gamma[g]$, it turns out that the measure proposed by Fradkin and Vilkovisky ensures the diffeomorphism invariance of $\Gamma[g]$, while the measure proposed by Fujikawa does not.

Conclusions and outlook

The present thesis was mainly devoted to an investigation on the UV-sensitivity of the vacuum energy $\rho_{\rm vac}$ in quantum gravity. From usual calculations, typically performed within the heat-kernel formalism, the radiative corrections to $\rho_{\rm vac}$ turn out to be quartically and quadratically sensitive to the UV physical cutoff Λ of the theory. When confronted with the observed value of the vacuum energy, this unveils a strong naturalness problem, the strongest facet to the long-standing cosmological constant problem.

Among the different attempts in the literature to dispose of the aforementioned power-like "divergences", models with compact extra dimensions received particular attention. In chapter 2, however, we showed that the automatically finite (no fine-tuning) result $\rho_{\rm EFT} \sim m_{\rm KK}^4$ ($m_{\rm KK}$ is the scale of a KK tower), typically obtained in these models for the vacuum energy, is due to an improper implementation of the EFT logic. In fact, we have seen that usual calculations mistreat the asymptotics of the five-dimensional loop momentum, and this results in an *artificial* washout of UV-sensitive terms. Performing a proper EFT calculation, we found that previously missed UV-sensitive contributions are present in $\rho_{\rm EFT}$. The latter do not cancel even in a SUSY theory since they are proportional to the boundary charges of superpartners, that have to be different to trigger the SUSY breaking.

These results have a significant impact on the recent dark dimension proposal [10], according to which our universe might possess a compact extra dimension of micrometer size dictated by the measured value of the vacuum energy. This DD scenario is based on swampland conjectures in string theory/quantum gravity and phenomenological bounds that lead to the relation $\rho_{\rm swamp} \sim m_{\rm KK}^4$ between the vacuum energy $\rho_{\rm swamp}$ and the size of the extra dimension $m_{\rm KK}^{-1}$, and also on the corresponding result $\rho_{\rm EFT}$ from the EFT limit. The presence in the latter of previously missed UV-sensitive terms renders the matching between $\rho_{\rm swamp}$ and $\rho_{\rm EFT}$ a non-trivial issue, and requires to find (if any) a physical mechanism responsible for the suppression of these large contributions. We commented on the possibility that such a mechanism is provided by the piling up of quantum fluctuations operated by the UV $\to$ IR renormalization group flow for the whole $(4+1)$D theory that includes the SM in a 3-brane. In such a framework, the RG flow would connect the boundary (i.e. UV) value of the vacuum energy, which is dictated by the UV-complete theory (string theory/quantum gravity), to its measured value in the IR. Further studies on the implementation of the RG approach in models with compact extra dimensions would then be of help to investigate on the possibility that the DD scenario might be (or not) a physical reality.

The issue of the strong UV-sensitivity in the vacuum energy was then studied in a different framework. In chapter 3, we considered pure gravity in four dimensions within the Einstein-Hilbert truncation, and re-examined the calculation of the one-loop (Euclidean) effective action $\Gamma^{1l}_{\rm grav}$. We showed that the appearance of quartically and quadratically UV-sensitive contributions to the vacuum energy $\rho_{\rm vac}$ in usual calculations (typically performed within the heat-kernel formalism) is due to the fact that two important aspects are typically overlooked in the derivation of $\Gamma^{1l}_{\rm grav}$, one concerning the path integral measure, the other the introduction of the UV physical cutoff Λ of the theory. We also showed that, when the diffeomorphism invariant Fradkin-Vilkovisky measure is used, and the UV cutoff Λ properly introduced, $\rho_{\rm vac}$ turns out to be only logarithmically sensitive to Λ. In this respect, it is important to stress that this result was obtained without resorting to a supersymmetric embedding of the theory, nor to regularization techniques, as dimensional regularization, where power-like divergences are absent by construction.

Considering a free scalar field and a free Dirac field on a non-trivial gravitational background, we found that the above result hols true even in the presence of matter fields. The vacuum energy receives only contributions that are proportional to the logarithm of the UV scale Λ, with coefficient given by the fourth power of the masses of matter fields (see below for further comments on these contributions).

In chapter 4, we extended the analysis considering the RG flow of the gravitational action. Taking for the latter the Einstein-Hilbert truncation, and specifying to a spherical gravitational background, we derived the renormalization group equations for the running cosmological and Newton constant. We showed that, as for the one-loop calculation performed in chapter 3, a proper treatment of the path integral measure and of the running scale k is crucial in the derivation of these equations. If again the Fradkin-Vilkovisky measure is used and the running scale k properly introduced, the RG flow turns out to be profoundly different from that of previous literature. In particular, we found no sign of the non-trivial UV-attractive fixed point of the asymptotic safety scenario that characterizes, instead, the typical RG flows derived within the effective average action and the (usual) proper-time formalism. We have shown that in the physically relevant quadrant (positive cosmological and Newton constant) only the Gaussian fixed point exists, with a UV-attractive and a UV-repulsive eigendirection.

In chapter 5, we considered the theory of an interacting scalar field on a non-trivial gravitational background, and showed that the issues concerning the path integral measure and the introduction of the UV physical cutoff Λ have a crucial impact also on another long-standing problem in theoretical physics, the Higgs naturalness problem. In particular, in this chapter we mainly addressed the issue related to the strong (quadratic) sensitivity of the mass m^2 to the UV cutoff Λ, which is one of the two aspects of the Higgs naturalness problem that we dubbed PCP (*physical cutoff problem*).

Specifying to a spherical gravitational background, we calculated the one-loop (Euclidean) effective action Γ^{1l}, from which we derived the radiative correction δm^2 to the mass m^2 of the scalar particle. Contrary to usual results, typically performed within the heat-kernel formalism, we showed that, if the diffeomorphism invariant Fradkin-Vilkovisky measure is used and the UV cutoff Λ properly introduced, δm^2 does not contain contributions proportional to Λ^2, but it presents only a mild logarithmic sensitivity to Λ. Again, this result is not obtained resorting to a "physical" cancellation (as it would be the case for a supersymmetric embedding of the theory), nor to regularization schemes where the cancellation of power-like divergences is automatically implemented (as it is the case, for instance, of dimensional regularization).

The other facet to the Higgs naturalness problem is what we dubbed LMP (*large* *masses problem*), and arises in beyond the SM models where the Higgs boson is coupled to particles of large mass M. The Higgs mass receives contributions proportional to M^2, and a physical mechanism that disposes of these large contributions, ultimately providing $m_{\rm H}^2 \sim (125\,\text{GeV})^2$, is needed. Similarly to the global RG picture discussed in connection with the dark dimension scenario, we commented on the fact that this question should be framed within the Wilsonian paradigm. The SM is part of the chain of effective theories that are connected by the RG flow that emanates from the ultimate theory, the TOE. Let us indicate with $\Lambda_{\rm SM}$ the scale at which the SM takes over. Considering the flow of the Higgs mass $m^2(\mu)$, its boundary value $m^2(\Lambda_{\rm SM})$ at $\Lambda_{\rm SM}$ is provided by the higher energy theory that embeds the SM above this scale, and it is the *precise* boundary such that the Wilsonian RG flow drives $m^2(\mu)$ to the measured value $m_{\rm H}^2 \sim (125\,\text{GeV})^2$ at the Fermi scale. In turn, such a *precise* value for $m^2(\Lambda_{\rm SM})$ is inherited from the previous higher energy theory, and ultimately from the TOE. This RG scenario was dubbed "physical tuning" in [11].

The same considerations hold for the contributions to the vacuum energy proportional to the fourth power of vevs/masses mentioned above. Even if we suppose that only SM particles exist, these contributions are enormous compared to the very low value of the vacuum energy (clearly, the presence of heavier particles would increase such a discrepancy, giving rise to an even more severe naturalness problem). A physical mechanism is then required that disposes of these large contributions. As for the LMP related to the Higgs boson mass, we proposed a picture where the small value of the vacuum energy (cosmological constant) emerges in the IR as the result of a "physical tuning" of its UV boundary at the physical scale $\Lambda_{\rm SM}$ dictated by the RG flow that emanates from the TOE.

It is worth to point out here that the calculations of chapters 3, 4 and 5 have been performed within the Euclidean signature. Certainly, these issues deserve further investigation in a Lorentzian setting.

The final chapter of the present thesis, chapter 6, was devoted to a thorough analysis on the transformation properties of the path integral measure under general coordinate transformations, also in light of some recent claims in the literature according to which the Fradkin-Vilkovisky measure used in the present thesis would not be invariant. The reason typically advocated is that it contains non-covariant factors of the time-time component g^{00} of the inverse metric.

With the help of a detailed calculation, we have shown that, contrary to this naïve expectation, the g^{00} factors that appear in the FV measure are crucial to ensure its diffeomorphism invariance. This result comes from a careful treatment of all the elements involved in the definition of the path integral. Among them, crucial is the necessity of introducing a time ordering parameter and a discretization (lattice) of spacetime. In fact, two different lattices and time ordering parameters are involved in the two coordinate systems related by a general coordinate transformation $x \to \hat{x}$, and this is at the origin of the appearance of non-trivial transformation terms from the path integral. The g^{00} factors in the FV measure ensure the cancellation of these non-trivial terms, and ultimately guarantee the diffeomorphism invariance of the effective action.

Appendix A

Species scale cutoff

In this appendix we perform the calculation of the vacuum energy ρ^{1l} using the species scale cutoff $\Lambda_{\rm sp}$. In a 4D theory with N particle states, $\Lambda_{\rm sp} = M_P/\sqrt{N}$. In a 5D theory with one compact dimension the identification of $\Lambda_{\rm sp}$ is done counting the number of KK states that respect the condition $m_n^2 \leq \Lambda_{\rm sp}^2$. The inequality is saturated when

$$m_\phi^2 + \left(\frac{n+q}{R_\phi}\right)^2 = \Lambda_{\rm sp}^2 \to n_\pm = \left[-q \pm R_\phi\sqrt{\Lambda_{\rm sp}^2 - m_\phi^2}\,\right], \tag{A.1}$$

where $n_\pm$ reduce to $n_{\rm max}$ and $n_{\rm min}$ of (2.32) in the text when $m_\phi = 0$, and the brackets [...] indicate "integer part" (that in the following we neglect for simplicity). The number of states between n_+ and n_- is

$$N = n_+ + |n_-| + 1 = 2R_\phi\sqrt{\Lambda_{\rm sp} - m_\phi^2} + 1 \tag{A.2}$$

and the species scale is then obtained as

$$\Lambda_{\rm sp} = \frac{a}{3} + \frac{X^{1/3}}{3\cdot 2^{1/3}} - \frac{2^{1/3}\,(3b - a^2)}{3\cdot X^{1/3}} \tag{A.3}$$

with

$$X = 3\sqrt{3}\sqrt{4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2} + 2a^3 - 9ab + 27c \tag{A.4}$$

and

$$a = m_\phi^2 + \frac{1}{4R_\phi^2}; \quad b = \frac{M_P^2}{2R_\phi^2}; \quad c = \frac{M_P^4}{4R_\phi^2}. \tag{A.5}$$

Expanding for $m_\phi, R_\phi^{-1} \ll M_P$, we get

$$\Lambda_{\rm sp}^2 = \frac{M_P^{4/3}}{(2R_\phi)^{2/3}} - \frac{M_P^{2/3}}{3\,(2R_\phi^4)^{1/3}} + \frac{m_\phi^2 + \frac{1}{4R_\phi^2}}{3} + \mathcal{O}(M_P^{-2/3}). \tag{A.6}$$

The first term of this expansion is the one typically referred to in the literature, where only a rough estimate of $\Lambda_{\rm sp}$ is reported (see for instance [100]).

The contribution of a bosonic (or fermionic, adding an overall minus sign) tower to the vacuum energy is

$$\rho^{1l} \sim \sum_{n=n_-}^{n_+} \int^{(\Lambda_{\rm sp})} \frac{{\rm d}^4 p}{(2\pi)^4} \log \frac{p^2 + \frac{(n+q)^2}{R_\phi^2} + m_\phi^2}{\mu^2}, \tag{A.7}$$

where the upper case $(\Lambda_{\rm sp})$ in the integral means that the modulus of the four-dimensional momentum is cut at $\Lambda_{\rm sp}$. Performing the integration over p we find

$$\begin{aligned}\rho^{1l} = \frac{1}{64\pi^2} \sum_{n=n_-}^{n_+} \Big\{ &- \Lambda_{\rm sp}^4 + 2\Lambda_{\rm sp}^2 \Big(m_\phi^2 + \frac{(n+q)^2}{R_\phi^2} \Big) + 2 \Big(m_\phi^2 + \frac{(n+q)^2}{R_\phi^2} \Big)^2 \log \frac{m_\phi^2 + \frac{(n+q)^2}{R_\phi^2}}{\Lambda_{\rm sp}^2 + m_\phi^2 + \frac{(n+q)^2}{R_\phi^2}} \\ &+ 2\Lambda_{\rm sp}^4 \log \frac{\Lambda_{\rm sp}^2 + m_\phi^2 + \frac{(n+q)^2}{R_\phi^2}}{\mu^2} \Big\} \equiv \sum_{n=n_-}^{n_+} G(n)\,. \end{aligned} \tag{A.8}$$

As in the text (see (3.30), (2.24) and the text therein, where B_i and $B_i(x)$ are defined), the sum can be calculated by means of the EML formula,

$$\rho^{1l} = \int_{n_-}^{n_+} {\rm d}x\ G(x) + \frac{G(n_+) + G(n_-)}{2} + \sum_{j=1}^{r} \frac{B_{2j}}{(2j)!} \Big(G^{(2j-1)}(n_+) - G^{(2j-1)}(n_-) \Big) + R_{2r} \tag{A.9}$$

where

$$\begin{aligned} R_{2r} &= \sum_{k=r+1}^{\infty} \frac{B_{2j}}{(2j)!} \left(G^{(2j-1)}(n_+) - G^{(2j-1)}(n_-) \right) \\ &= \frac{(-1)^{2r+1}}{(2r)!} \int_{n_-}^{n_+} {\rm d}x\ G^{(2r)}(x) B_{2r}(x - [x]). \end{aligned} \tag{A.10}$$

In the physically meaningful limit $m_\phi, R_\phi^{-1} \ll \Lambda_{\rm sp}$, the result for the vacuum energy is

$$\begin{aligned} \rho^{1l} = &\frac{20 \log\left(\frac{4M_P^2}{5\mu^3 R_\phi}\right) + 12\pi - 57}{2^{-1/3} \cdot 3840\pi^2 R_\phi^{2/3}} M_P^{10/3} + \frac{-4\log\left(\frac{4M_P^2}{\mu^3 R_\phi}\right) - 6\pi + 27}{2^{-2/3} \cdot 2304\pi^2 R_\phi^{4/3}} M_P^{8/3} \\ &+ \frac{12\pi - 35}{4608\pi^2 R_\phi^2} M_P^2 + \frac{\left(4\, m_\phi^2 R_\phi^2 + 1\right) \log\left(\frac{M_P^2}{2\mu^3 R_\phi}\right) - 3(5-4\pi) m_\phi^2 R_\phi^2}{1152\pi^2 R_\phi^2} M_P^2 \\ &+ \frac{-20\log\left(\frac{M_P^2}{\mu^3 R_\phi}\right) - 120\pi + 309 + 104\log 2}{2^{-1/3} \cdot 124416\pi^2 R_\phi^{8/3}} M_P^{4/3} + \frac{3(19-8\pi) - 4\log\left(\frac{4M_P^2}{\mu^3 R_\phi}\right)}{2^{-1/3} \cdot 3456\pi^2 R_\phi^{8/3}} (m_\phi R_\phi)^2 M_P^{4/3} \\ &+ \frac{525\pi + 367\log 2 - 1953 + 35\log\left(\frac{M_P^2}{\mu^3 R_\phi}\right)}{2^{-2/3} 1866240\pi^2 R_\phi^{10/3}} M_P^{2/3} + \frac{9\log\left(\frac{M_P^2}{\mu^3 R_\phi}\right) + 135\pi - 432 + 99\log 2}{2^{-2/3} 46656\pi^2 R_\phi^{10/3}} m_\phi^2 R_\phi^2 M_P^{2/3} \\ &+ \frac{2\log\left(\frac{M_P^2}{\mu^3 R_\phi}\right) + 3\pi - 30 - 14\log 2}{2^{-2/3} 1728\pi^2 R_\phi^{10/3}} m_\phi^4 R_\phi^4 M_P^{2/3} + \frac{61 - 18\pi + 40(17-6\pi) m_\phi^2 R_\phi^2 + 80(33-9\pi) m_\phi^4 R_\phi^4}{138240\pi^2 R_\phi^4} \\ &+ \frac{m_\phi^5 R_\phi}{60\pi} + R_4 + \mathcal{O}(M_P^{-2/3})\,, \end{aligned} \tag{A.11}$$

with R_4 given in (2.26), (1.120).

Few comments are in order. Limiting ourselves to the leading order relation $\Lambda_{\rm sp} \sim R_\phi^{-1/3} M_P^{2/3}$ (see (A.6)), we observe that the powers $M_P^{10/3}$, M_P^2 and $M_P^{2/3}$ correspond to the powers Λ^5, Λ^3 and Λ respectively in terms of a generic cutoff Λ. However, comparing the coefficients of M_P^2 and $M_P^{2/3}$ in (A.11) with the corresponding coefficients of Λ^3 and Λ in (2.25), we note that they have a different structure. Moreover, powers of M_P other than those mentioned above (that do not find any correspondence in (2.25)) are also present. These differences have a twofold origin: they are due both to the fact that the species scale cut is cylindrical in 5D momentum space (see [71] for a thorough discussion on the difference between the implementation of a cylindrical and a spherical cutoff on the 5D momentum) and to the fact that, as per (A.6), the relation $\Lambda_{\rm sp} \sim R_\phi^{-1/3} M_P^{2/3}$ holds only at the leading (large M_P) order.

An even more important difference between the results (2.25) and (A.11) is that the latter does not contain any UV-sensitive term proportional to the boundary charge q. Actually (A.11) comes from a physically illegitimate operation. In fact, rather than a cut on $p_5^2 = e^{-2\beta\phi} n^2/R^2$, $\Lambda_{\rm sp}$ implements a cut on the "KK masses" $m_n^2 = m_\phi^2 + (n+q)^2/R_\phi^2$. As discussed in the text, the (unphysical) introduction of the combination $n+q$ in the cutoff is at the origin of the artificial washout of the q-dependent UV-sensitive terms. This makes the result (A.11), and more generally the introduction of the species scale cutoff in higher-dimensional theories with compact extra dimensions, unreliable. The species scale cut only arises as a result of a too literal interpretation of the 5D theory in terms of a 4D theory with towers of massive 4D fields. These warnings do not apply to the case of a bona fide 4D theory with a large number N of fields coupled to gravity, where $\Lambda_{\rm sp}$ truly is the quantum gravity physical cutoff.

It is also worth to point out that (A.11) provides an example of a hard cutoff calculation where no q-dependent UV-sensitive terms are generated. In previous literature, the opinion was widely expressed that the use of a hard cutoff was at the origin of UV-sensitive terms, that were considered as spurious [89, 90, 152]. The above result shows that the presence of these terms is rather due to a correct treatment of the asymptotics of the loop momenta.

Appendix B

Standard Model cutoff

In this Appendix we derive the relation between the cutoff Λ of a $(4+1)$D theory and the 4D cutoff $\Lambda_{\rm SM}$ of the Standard Model localized on a 3-brane. Let us consider the $(4+1)$D theory (with compact space dimension in the shape of a circle of radius R) defined by

$$\mathcal{S} = \mathcal{S}_{\rm grav} + \mathcal{S}_{\rm mat}, \tag{B.1}$$

where $\mathcal{S}_{\rm grav}$ is given by (2.6) in the text, and as an example for $\mathcal{S}_{\rm mat}$ we take the action (2.13) of a $(4+1)$D complex scalar field $\hat{\Phi}(x,z)$ that obeys the boundary condition $\hat{\Phi}(x, z+2\pi R) = \hat{\Phi}(x,z)$. Parametrizing the $(4+1)$D metric $\hat{g}_{MN}$ as in (2.7), and considering $g_{\mu\nu}(x)$, $A_\mu(x)$ and $\phi(x)$ that depend only on x, the integration over z gives the 4D gravitational action (2.8) in the text.

Considering the Fourier decomposition of $\hat{\Phi}(x,z)$, for the 4D matter action (2.13) we have

$$\mathcal{S}^{(4)}_{\rm mat} = \int {\rm d}^4x\,\sqrt{-g}\,\sum_n \left[|D\varphi_n|^2 - \left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} m^2 + e^{\sqrt{6}\frac{\phi}{M_P}} \frac{n^2}{R^2} \right) |\varphi_n|^2 \right], \tag{B.2}$$

where $D_\mu \equiv \partial_\mu - i\,(n/R)\,A_\mu$, and $\varphi_n(x)$ are the KK modes of $\hat{\Phi}(x,z)$.

Taking a constant background for the radion (that for notational simplicity we continue to indicate with ϕ) and the trivial background for A_μ, the metric (2.7) becomes

$$\hat{g}^0_{MN} = \begin{pmatrix} e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}}\eta_{\mu\nu} & 0 \\ 0 & -e^{-2\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \end{pmatrix}. \tag{B.3}$$

From (B.2) we define the ϕ-dependent radius $R_\phi \equiv R\,e^{-\sqrt{\frac{3}{2}}\frac{\phi}{M_P}}$. With such a definition, we immediately see that, when computing radiative corrections, the $(4+1)$D momentum $\hat{p} \equiv (p, n/R)$ is cut as (see also comments below (2.21) in the text)

$$\hat{p}^2 = e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \left(p^2 + \frac{n^2}{R_\phi^2} \right) \leq \Lambda^2. \tag{B.4}$$

This latter equation is conveniently rewritten as

$$p^2 + \frac{n^2}{R_\phi^2} \leq \Lambda_\phi^2, \tag{B.5}$$

where we defined $\Lambda_\phi \equiv \Lambda\, e^{\frac{1}{\sqrt{6}}\frac{\phi}{M_P}}$. In terms of the dimensionless ϕ of (2.7) and (2.8), and before using $\alpha = 1/\sqrt{12}$, it is $\Lambda_\phi = e^{\alpha\phi}\Lambda = m_{\rm KK}^{1/3} R^{1/3}\Lambda$.

As p^2 in (B.5) is nothing but the modulus of the four-momentum on the brane, this equation tells us that Λ_ϕ is the cutoff $\Lambda_{\rm SM}$ of the SM (or more generally of the BSM model that lives on the 3-brane, where fields have $n = 0$). Therefore:

$$\Lambda_{\rm SM} = \Lambda_\phi = \Lambda\, e^{\frac{1}{\sqrt{6}}\frac{\phi}{M_P}}. \tag{B.6}$$

Finally, as the DD scenario is realized for negative values of ϕ, from (B.6) we see that $\Lambda_{\rm SM} \leq \Lambda$, i.e. the SM cutoff is lower than the cutoff of the $(4+1)$-dimensional EFT that implements the DD scenario.

Before closing this Appendix, let us note that here we considered a spherical cutoff. Naturally, we can make a different choice, taking for instance a cylindrical cutoff [71]

$$p^2 \leq \Lambda_\phi^2 \qquad \text{and} \qquad \frac{n^2}{R_\phi^2} \leq \Lambda_\phi^2.$$

This choice, that is closer to what is typically done when using the species scale $\Lambda_{\rm sp}$ as the cutoff [100], does not change the above considerations.

Appendix C

Mode expansion

In this Appendix we derive Eq. (3.20) in the text, closely following the strategy put forward in [104]. Let us indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the dimensionless Laplace-Beltrami operator $-\widetilde{\Box}^{(s)}$ normalized as (i is the degeneracy index)

$$\int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\;h_n^{\mu\nu(i)}(x)h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\;\xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{\widetilde{g}}\;\phi_n^{(i)}(x)\,\phi_m^{(j)}(x) = \delta^{ij}\delta_{nm}\,, \tag{C.1}$$

and the corresponding eigenvalues with $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively (see (3.19) in the text). The modes $\{h_n^{\mu\nu},\,v_n^{\mu\nu},\,w_n^{\mu\nu},\,z_n^{\mu\nu}\}$, where (from now on the degeneracy indexes are omitted)

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)}-3\right)\right]^{-\frac{1}{2}}\nabla^{(\mu}\xi_n^{\nu)}\,, \quad n=2,\dots\,,\\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)}-3\right)\right]^{-\frac{1}{2}}\left(\nabla^\mu\nabla^\nu-\frac{1}{4}\widetilde{g}^{\mu\nu}\widetilde{\Box}\right)\phi_n\,, \quad n=2,\dots\,,\\
z_n^{\mu\nu} &= \frac{1}{2}\widetilde{g}^{\mu\nu}\phi_n\,, \quad n=0,1,2,\dots\,,
\end{aligned} \tag{C.2}$$

form the orthonormal basis for symmetric tensors. Defining now the longitudinal vector modes

$$l_n^\mu = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}}\,\nabla^\mu\phi_n\,, \quad n=1,2,\dots\,, \tag{C.3}$$

we observe that, together with the transverse modes ξ_n^μ, they form the orthonormal basis for vectors.

Expanding the dimensionless graviton field $\widehat{h}^{\mu\nu}$ (see Eq. (3.12) in the text) as

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu}\,, \tag{C.4}$$

the ghost field $\widehat{v}^\mu$ (Eq. (3.14) in the text) as

$$\widehat{v}^\mu = \sum_{n=1}^{\infty} g_n\,\xi_n^\mu + \sum_{n=1}^{\infty} f_n\,l_n^\mu\,, \tag{C.5}$$

and finally inserting (C.4) and (C.5) in (3.13) and (3.15) respectively, we get

$$
\begin{aligned}
2\left(S_2+S_{\rm gf}\right) = & \sum_{n=2}^{\infty} a_n^2 \left[\lambda_n^{(2)} - 2a^2\Lambda_{\rm cc} + 8\right] + \sum_{n=2}^{\infty} b_n^2 \left[\xi^{-1}\left(\lambda_n^{(1)}-3\right) - 2a^2\Lambda_{\rm cc} + 6\right] \\
& + \sum_{n=2}^{\infty} c_n^2 \left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)} - 6\right) - \frac{\lambda_n^{(0)}}{2} - 2a^2\Lambda_{\rm cc} + 6\right] \\
& + \sum_{n=0}^{\infty} e_n^2 \left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)} + 2a^2\Lambda_{\rm cc}\right] + \sum_{n=2}^{\infty} 2e_n c_n(\xi^{-1}-1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)}-3\right)\right]^{\frac{1}{2}}
\end{aligned}
\tag{C.6}
$$

and

$$
S_{\rm ghost} = \sum_{n=1}^{\infty} g_n^* \, g_n \left(\lambda_n^{(1)} - 3\right) + \sum_{n=1}^{\infty} f_n^* \, f_n \left(\lambda_n^{(0)} - 6\right) \, . \tag{C.7}
$$

The functional measure (3.17) in the text can then be written as [104]

$$
\left[\mathcal{D}u(h)\mathcal{D}v_\rho^* \, \mathcal{D}v_\sigma\right] \sim \prod_{n=2}^{\infty} {\rm d}a_n \prod_{n=2}^{\infty} {\rm d}b_n \prod_{n=2}^{\infty} {\rm d}c_n \prod_{n=0}^{\infty} {\rm d}e_n \prod_{n=2}^{\infty} {\rm d}g_n^* \prod_{n=2}^{\infty} {\rm d}g_n \prod_{n=1}^{\infty} {\rm d}f_n^* \prod_{n=1}^{\infty} {\rm d}f_n \, . \tag{C.8}
$$

Clearly there is no gaussian integration along the flat directions g_1^* and g_1 (zero modes) of $S_{\rm ghost}$, and the related ghost fields are proportional to the Killing vectors ξ_1^μ (see [104] for details). Finally, introducing (C.8) in (3.6) and performing the gaussian integrations, Eq. (3.20) in the text is obtained. Note that the term $\frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right)$ in (3.20) comes from the integration over e_0.

Appendix D

Calculation of $\delta S^{1l}_{\text{grav}}$: product of eigenvalues

In this Appendix we derive Eq. (3.22) in the text. To this end, we report here Eq. (3.21) that was obtained considering the direct product of eigenvalues

$$\delta S^{1l}_{\text{grav}} = \frac{1}{2}\sum_{n=2}^{N-2}\Big[D_n^{(2)}\log\left(\lambda_n^{(2)} - 2a^2\Lambda_{\text{cc}} + 8\right) + D_n^{(0)}\log\left(\lambda_n^{(0)} - 2a^2\Lambda_{\text{cc}}\right) - D_n^{(1)}\log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)}\log\left(\lambda_n^{(0)} - 6\right)\Big] + \frac{1}{2}\log\big(2a^2\Lambda_{\text{cc}}\big) + \mathcal{B}\,. \tag{D.1}$$

Using the identity

$$\log\left(x/y\right) = -\int_0^{+\infty} \mathrm{d}u\,\left[(x+u)^{-1} - (y+u)^{-1}\right]\,, \tag{D.2}$$

Eq. (D.1) can be written as

$$\delta S^{1l}_{\text{grav}} = -\frac{1}{2}\int_0^{+\infty}\mathrm{d}u\sum_{n=2}^{N-2}\Big[\frac{D_n^{(2)}}{u+\lambda_n^{(2)} - 2a^2\Lambda_{\text{cc}} + 8} + \frac{D_n^{(0)}}{u+\lambda_n^{(0)} - 2a^2\Lambda_{\text{cc}}} - \frac{D_n^{(1)}}{u+\lambda_n^{(1)} - 3} - \frac{D_n^{(0)}}{u+\lambda_n^{(0)} - 6} - \frac{D_n^{(2)} - D_n^{(1)}}{u+1}\Big] + \frac{1}{2}\log\big(2a^2\Lambda_{\text{cc}}\big) + \mathcal{B}\,. \tag{D.3}$$

Performing now the sum over n, the integration over u, and finally expanding the result for $N \gg 1$, we obtain Eq. (3.22) in the text for $\delta S^{1l}_{\text{grav}}$, with $\mathcal{F}(a^2\Lambda_{\text{cc}})$ given by

$$\begin{aligned}
\mathcal{F}(a^2\Lambda_{\text{cc}}) &= 9\,a^2\Lambda_{\text{cc}} - \frac{a^2\Lambda_{\text{cc}}\sqrt{8a^2\Lambda_{\text{cc}}+9}}{6}\log\Gamma\left(\frac{7-\sqrt{8a^2\Lambda_{\text{cc}}+9}}{2}\right) - 5\,a^2\Lambda_{\text{cc}}\,\psi^{(-2)}\left(\frac{7+\sqrt{8a^2\Lambda_{\text{cc}}-15}}{2}\right) \\
&-5\,a^2\Lambda_{\text{cc}}\,\psi^{(-2)}\left(\frac{7-\sqrt{8a^2\Lambda_{\text{cc}}-15}}{2}\right) - a^2\Lambda_{\text{cc}}\,\psi^{(-2)}\left(\frac{7+\sqrt{8a^2\Lambda_{\text{cc}}+9}}{2}\right) \\
&-a^2\Lambda_{\text{cc}}\,\psi^{(-2)}\left(\frac{7-\sqrt{8a^2\Lambda_{\text{cc}}+9}}{2}\right) + \frac{a^2\Lambda_{\text{cc}}\,\sqrt{8a^2\Lambda_{\text{cc}}+9}}{6}\log\Gamma\left(\frac{7+\sqrt{8a^2\Lambda_{\text{cc}}+9}}{2}\right) - 5\log(120) \\
&+\frac{49\log(A)}{3} - 2\sqrt{\frac{11}{3}}\log\Gamma\left(\frac{7+\sqrt{33}}{2}\right) - \frac{5\left(a^2\Lambda_{\text{cc}}-5\right)\sqrt{8a^2\Lambda_{\text{cc}}-15}}{6}\log\Gamma\left(\frac{7-\sqrt{8a^2\Lambda_{\text{cc}}-15}}{2}\right) \\
&-\frac{\sqrt{8a^2\Lambda_{\text{cc}}+9}}{6}\log\Gamma\left(\frac{7-\sqrt{8a^2\Lambda_{\text{cc}}+9}}{2}\right) + 3\left(\psi^{(-4)}(1) + \psi^{(-4)}(6)\right) + \psi^{(-4)}\left(\frac{7-\sqrt{33}}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\psi^{(-4)}\left(\frac{7+\sqrt{33}}{2}\right)-5\left(\psi^{(-4)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)+\psi^{(-4)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)\right)\\
&-\psi^{(-4)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)-\psi^{(-4)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)+\frac{15}{2}\left(\psi^{(-3)}(1)-\psi^{(-3)}(6)\right)\\
&-\frac{\sqrt{33}}{2}\,\psi^{(-3)}\left(\frac{7+\sqrt{33}}{2}\right)-\frac{5\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\,\psi^{(-3)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)\\
&-\frac{\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\,\psi^{(-3)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)+\frac{33}{4}\left(\psi^{(-2)}(1)+\psi^{(-2)}(6)\right)\\
&+\frac{49}{12}\left(\psi^{(-2)}\left(\frac{7-\sqrt{33}}{2}\right)+\psi^{(-2)}\left(\frac{7+\sqrt{33}}{2}\right)\right)\\
&+\frac{175}{12}\left(\psi^{(-2)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)+\psi^{(-2)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)\right)\\
&-\frac{13}{12}\left(\psi^{(-2)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)+\psi^{(-2)}\left(\frac{7-\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)\right)+\frac{\sqrt{33}}{2}\,\psi^{(-3)}\left(\frac{7-\sqrt{33}}{2}\right)\\
&+2\sqrt{\frac{11}{3}}\,\log\Gamma\left(\frac{7-\sqrt{33}}{2}\right)+\frac{5\left(a^2\Lambda_{\rm cc}-5\right)\sqrt{8a^2\Lambda_{\rm cc}-15}}{6}\,\log\Gamma\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)\\
&+\frac{5\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\,\psi^{(-3)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}-15}}{2}\right)+\frac{\sqrt{8a^2\Lambda_{\rm cc}+9}}{6}\,\log\Gamma\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)\\
&+\frac{\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\,\psi^{(-3)}\left(\frac{7+\sqrt{8a^2\Lambda_{\rm cc}+9}}{2}\right)+\frac{7\zeta(3)}{4\pi^2}-\frac{2\,\zeta'(-3)}{3}-\frac{20801}{1080}\,, \qquad \text{(D.4)}
\end{aligned}
$$

where A is the Glaisher's constant ($A \simeq 1.282427$), $\zeta(z)$ is the Riemann zeta function ($\zeta(3) \simeq 1.20206$ and $\zeta'(-3) \simeq 0.00538$), and $\psi^{(-n)}(z)$ (with n positive integer) are the polygamma functions of negative order defined as [153]

$$
\psi^{(-n)}(z) = \frac{1}{(n-2)!}\int_0^z \mathrm{d}t\,(z-t)^{n-2}\log\,\Gamma(t) \qquad \text{for } \mathrm{Re}(z)>0\,. \qquad \text{(D.5)}
$$

Appendix E

RG equations for Λ_k and G_k: generic boundary values

As said in the text (chapter 4), although only positive UV boundary values for the running cosmological and Newton constant are physically relevant, $\Lambda_{\rm cc} > 0$ and $G > 0$, for completeness in this Appendix we consider (and speculate on) the case of generic boundary values. Let us begin with $\Lambda_{\rm cc} > 0$ and $G < 0$. As for the case $\Lambda_{\rm cc} > 0$ and $G > 0$, the relation between the "running scale" L and the physical running scale k is given by (4.23), and the RG equations for Λ_k and G_k are (4.26) and (4.27). Considering the corresponding equations (4.35) and (4.36) for the dimensionless couplings λ and g, reported below for the reader's convenience

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -2\lambda + \frac{2g\lambda\,(3-2\lambda)}{2\pi + g\,(3-2\lambda)} \equiv \beta_\lambda(\lambda, g) \tag{E.1}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \quad 2g + \frac{2g^2\,(3-8\lambda)}{2\pi + g\,(3-2\lambda)} \equiv \beta_g(\lambda, g)\,, \tag{E.2}$$

we find the non-trivial fixed point

$$(\lambda_2, g_2) = \left(0, -\frac{\pi}{3}\right) \tag{E.3}$$

in the $(\lambda > 0, g < 0)$ quadrant. In the main text this fixed point was already found (see Eq. (4.38)), but it was discarded as we were interested only in the physical quadrant $(\lambda > 0, g > 0)$. Performing the stability analysis, we find that the matrix $M(\lambda_2, g_2)$ (see Eq. (4.39)) has two negative degenerate eigenvalues $(\theta_{1,2} = -4)$. To study the behaviour of the RG flow in the neighbourhood of (λ_2, g_2) we have to linearize (4.35) and (4.36) around this point. We find (λ_2, g_2) to be a UV-attractive fixed point.

Let us consider now the case of negative UV boundary values for the cosmological constant, $\Lambda_{\rm cc} < 0$. As stressed in section 4.1.3, the running cosmological constant Λ_L cannot change sign along its flow, so that in this case it is $\Lambda_L < 0$ in the whole range $4 \leq L \leq N$. We introduce the running physical scale k as[1] (see (4.23))

$$k = L\sqrt{-\frac{\Lambda_L}{3}} = L\sqrt{\frac{|\Lambda_L|}{3}}\,. \tag{E.4}$$

[1]Note that k varies in the range $[k_{\rm IR}, M_P]$. From (E.4) we see that $k_{\rm IR} = \sqrt{\frac{16|\Lambda_4|}{3}}$.

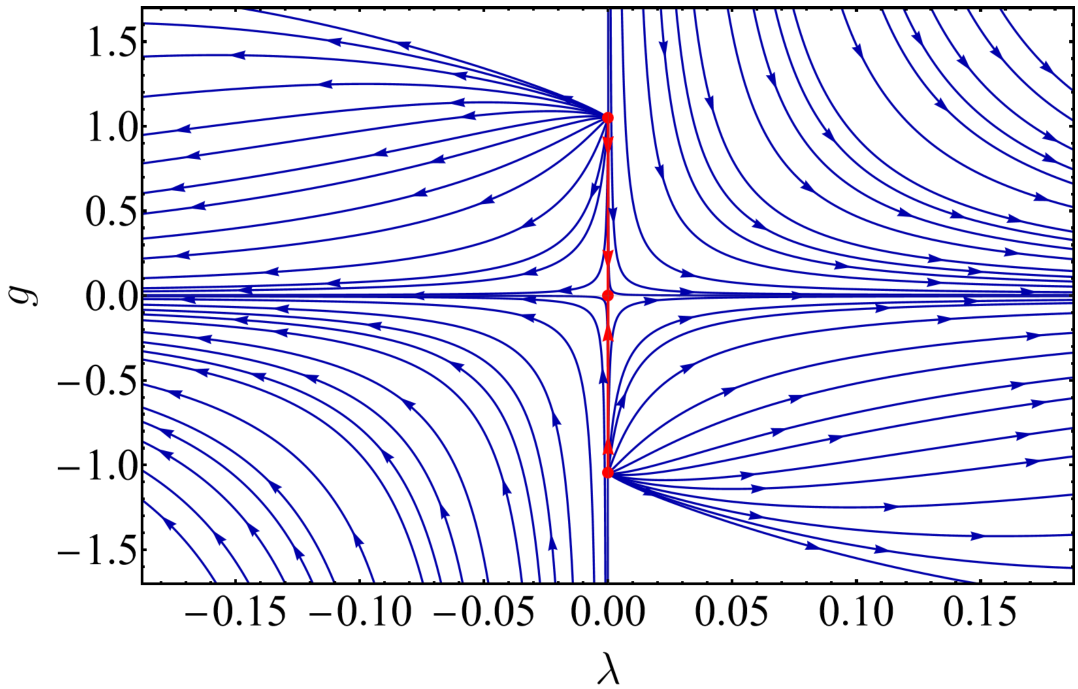

Figure E.1: RG flow in the whole plane (λ, g) from the numerical solution of (E.1)-(E.2) for $\Lambda_{cc} > 0$ and $G \gtrless 0$, and of (E.7)-(E.8) for $\Lambda_{cc} < 0$ and $G \gtrless 0$. The red dots are the three fixed points $(\lambda_1, g_1) = (0, 0)$, $(\lambda_2, g_2) = (0, -\pi/3)$ and $(\lambda_3, g_3) = (0, \pi/3)$. The trajectories in the second quadrant $(\lambda < 0, g > 0)$ are UV-attracted by the fixed point (λ_3, g_3), those in the fourth quadrant $(\lambda > 0, g < 0)$ by the fixed point (λ_2, g_2). The arrows point towards the IR, and the red lines connect the two non-trivial fixed points (λ_2, g_2) and (λ_3, g_3) to the Gaussian one. The trajectories in the second and fourth quadrant are symmetric with respect to $(0, 0)$. The same holds true for the trajectories in the first and third quadrant. The flow in the first (physical) quadrant is the one plotted in Fig. 4.3.

Inserting (E.4) in (4.10) and (4.11) we finally get the RG equations ($\Lambda_k \equiv \Lambda_{\scriptscriptstyle L}$ and $G_k \equiv G_{\scriptscriptstyle L}$)

$$k\frac{\mathrm{d}\Lambda_k}{\mathrm{d}k} = -\frac{3G_k}{\pi}\frac{\Lambda_k\left(k^2 + \frac{2}{3}\Lambda_k\right)}{1 - \frac{3G_k}{2\pi}\left(k^2 + \frac{2}{3}\Lambda_k\right)} \tag{E.5}$$

$$k\frac{\mathrm{d}G_k}{\mathrm{d}k} = -\frac{3G_k^2}{\pi}\frac{k^2 + \frac{8}{3}\Lambda_k}{1 - \frac{3G_k}{2\pi}\left(k^2 + \frac{2}{3}\Lambda_k\right)}\,, \tag{E.6}$$

that introducing the "RG time" t and the dimensionless running cosmological and Newton constant $\lambda(t)$ and $g(t)$ as in section 4.2 (see (4.34)) can be written as

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -2\lambda - \frac{2g\lambda\,(3 + 2\lambda)}{2\pi - g\,(3 + 2\lambda)} \equiv \beta_\lambda(\lambda, g) \tag{E.7}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \quad 2g - \frac{2g^2\,(3 + 8\lambda)}{2\pi - g\,(3 + 2\lambda)} \equiv \beta_g(\lambda, g)\,. \tag{E.8}$$

Beyond the Gaussian fixed point, we also find

$$(\lambda_3, g_3) = \left(0, \frac{\pi}{3}\right)\,, \tag{E.9}$$

that from the stability analysis turns out to be a UV attractive fixed point as (λ_2, g_2).

Having now at our disposal the system (E.1)-(E.2) for $\Lambda_{cc} > 0$ and $G \gtrless 0$, and the system (E.7)-(E.8) for $\Lambda_{cc} < 0$ and $G \gtrless 0$, we can now study the RG flow in the whole (λ, g) plane. Solving numerically these equations, we find the RG trajectories presented in Fig. E.1. The red dots are the three fixed points (4.37), (E.3) and (E.9), namely the Gaussian fixed point (λ_1, g_1) and the two non-trivial ones (λ_2, g_2) and (λ_3, g_3). The red lines connect the two non-trivial fixed points to the Gaussian one. Arrows point towards the IR. All the trajectories (blue lines) in the half-plane $\lambda > 0$ end at the minimal IR value of λ allowed by (4.25), namely $\lambda_{\text{IR}} = \Lambda_4/k_{\text{IR}}^2 = 3/16$. The trajectories in the half-plane $\lambda < 0$ end at the minimal IR value of λ allowed by (E.4), i.e. $\lambda_{\text{IR}} = -|\Lambda_4|/k_{\text{IR}}^2 = -3/16$. The plot in Fig. E.1 shows what already found with the stability analysis: (i) the $\lambda = 0$ and $g = 0$ axes are the UV-repulsive and UV-attractive eigendirections for the Gaussian fixed point; (ii) the RG trajectories are UV-attracted by (λ_2, g_2) in the quadrant $(\lambda > 0, g < 0)$ and by (λ_3, g_3) in the quadrant $(\lambda < 0, g > 0)$.

Appendix F

The function $\mathcal{H}$

In this Appendix we report the function $\mathcal{H}(a^2V''(\Phi))$ that appears in Eq. (5.19). As already stressed in the text, its specific form is not of interest for the analysis presented in chapter 5, where we are focused on the UV-sensitivity of the one-loop effective action Γ^{1l}. For completeness, however, we write its expression below

$$\begin{aligned}
\mathcal{H}(a^2V'') = &\frac{1}{12}\Big(- \sqrt{9-4a^2V''-48\xi}\Big(a^2V''+12\xi-2\Big)\log\Gamma\Big(\frac{3+\sqrt{9-4a^2V''-48\xi}}{2}\Big) \\
&+ \Big(a^2V''+12\xi-2\Big)\sqrt{9-4a^2V''-48\xi}\,\log\Gamma\Big(\frac{3-\sqrt{9-4a^2V''-48\xi}}{2}\Big) \\
&- 12\Big[\psi^{(-4)}\Big(\frac{3+\sqrt{9-4a^2V''-48\xi}}{2}\Big)+\psi^{(-4)}\Big(\frac{3-\sqrt{9-4a^2V''-48\xi}}{2}\Big)\Big] \\
&+ \Big(6a^2V''+72\xi-13\Big)\Big[\psi^{(-2)}\Big(\frac{3+\sqrt{9-4a^2V''-48\xi}}{2}\Big)+\psi^{(-2)}\Big(\frac{3-\sqrt{9-4a^2V''-48\xi}}{2}\Big)\Big] \\
&+ 6\sqrt{9-4a^2V''-48\xi}\Big[\psi^{(-3)}\Big(\frac{3+\sqrt{9-4a^2V''-48\xi}}{2}\Big)-\psi^{(-3)}\Big(\frac{3-\sqrt{9-4a^2V''-48\xi}}{2}\Big)\Big] \\
&- 4\zeta'(-3)\Big) - \frac{a^2V''}{12} + \frac{13\log(A)}{6} - \xi + \frac{3\zeta(3)}{8\pi^2} + \frac{319}{2160} + \frac{1}{2}\log(2\pi)\,. \qquad \text{(F.1)}
\end{aligned}$$

In the above equation, A is the Glaisher's constant ($A \simeq 1.282427$), $\zeta(z)$ is the Riemann zeta function ($\zeta(3) \simeq 1.20206$ and $\zeta'(-3) \simeq 0.00538$), and $\psi^{(-n)}(z)$ (with n positive integer) are the polygamma functions of negative order defined as [153]

$$\psi^{(-n)}(z) = \frac{1}{(n-2)!}\int_0^z \mathrm{d}t\,(z-t)^{n-2}\log\Gamma(t) \qquad \text{for } \mathrm{Re}(z) > 0\,. \qquad \text{(F.2)}$$

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