Memory effects on energy loss and diffusion of heavy quarks in the quark-gluon plasma

Marco Ruggieri,[1, *] Pooja,[2] Jai Prakash,[2] and Santosh K. Das[2]

[1]*School of Nuclear Science and Technology, Lanzhou University,*
222 South Tianshui Road, Lanzhou 730000, China
[2]*School of Physical Sciences, Indian Institute of Technology Goa, Ponda-403401, Goa, India*

We study the dynamics of heavy quarks in a thermalized quark-gluon plasma with a time-correlated thermal noise, η. In this case it is said that η has memory. We use an integro-differential Langevin equation in which the memory enters via the thermal noise and the dissipative force. We assume that the time correlations of the noise decay exponentially on a time scale, τ, that we treat as a free parameter. We compute the effects of $\tau \neq 0$ on the thermalization time of the heavy quarks, on their momentum broadening and on the nuclear modification factor. We find that overall memory slows down the momentum evolution of heavy quarks: in fact, transverse momentum broadening and the formation of R_{AA} are slowed down by memory and the thermalization time of the heavy quarks become larger. The potential impact on other observables is discussed briefly.

I. INTRODUCTION

A hot and dense phase of nuclear matter, the quark-gluon plasma (QGP), is expected to form in the ultra-relativistic heavy-ion collisions at Relativistic Heavy-ion Collider (RHIC) and the Large Hadron Collider (LHC) energies. Probing and characterizing the bulk properties of QGP is a field of high contemporary interest. Heavy quarks (HQs) [1–12] such as charm and beauty are considered as good probes of the system produced in high-energy nuclear collisions. In fact, they are produced in the very early stage due to the hard partonic scatterings on a time scale $\tau = O(1/m)$ where m is the rest mass of the quark. Due to their large mass and low abundance, they can propagate in the QGP bringing almost no disturbance to it. Consequently, they act as good probes that can experience the whole evolution of the system created in the collisions, from the very early stage up to the hadronization stage.

The standard approach to study the HQ dynamics in the QGP is following their position and momentum evolution by means of the Langevin equations [13–26] (see also [27]) as well as relativistic kinetic theory [9–12, 21, 28–31]. In the approaches based on the Langevin equation, the thermal noise, η, is usually treated as standard Wiener process, thus without correlations in time. In this work we relax this approximation and analyze the case in which η is time-correlated; this class of stochastic processes is called a process with memory.

The prototype of Langevin equation that we consider in this work is

$$\frac{dp}{dt} = -\int_0^t \gamma(t-s)p(s)ds + \eta(t), \qquad (1)$$

where p is the momentum of the particle, η is the stochastic term that models the thermal noise, while the integral term on the right hand side is the dissipative force. In previous studies the latter is replaced by $-\gamma p$ where γ is the drag coefficient: this replacement follows from the Fluctuation-Dissipation Theorem (FDT) when η has no time correlations.

Our goal is to analyze the motion of heavy quarks in a quark-gluon plasma, when the correlations of the thermal noise do not decay instantaneously: instead, we assume that these correlations decay over a specific time scale that we call the memory time, τ. Hydrodynamic fluctuations [32, 33], diffusion in the evolving Glasma [34–43], diffusion of electric charge [44], dilepton yields [45] and the electric conductivity of the quark-gluon plasma [46] are some of the physical problems where memory can play a role; for these τ lies in between 0.1 and 10 fm/c. In this study we treat τ as a free parameter and study its effect on a few physical quantities, namely the momentum broadening of heavy quarks and on the nuclear modification factor, R_{AA}. For the sake of simplicity we consider the interaction of heavy quarks with a thermalized quark-gluon plasma at a fixed, constant temperature T; the diffusion coefficients that we use in the calculations are that obtained by perturbative QCD (pQCD) for high T, and by a quasiparticle model (QPM) for low T, while the dissipative kernel is related to the thermal noise by the Fluctuation-Dissipation Theorem (FDT).

We anticipate the main result, namely that memory delays the dynamics of the heavy quarks in the QGP: we show this by studying the thermalization time, the momentum broadening and the time evolution of R_{AA}. The latter in particular can be potentially of interest for the phenomenology of heavy quarks in the QGP, due to the fact that the slower evolution of R_{AA} would require the use of larger diffusion coefficients in order to reproduce the experimental data and this in turn would require stronger interactions of the heavy quarks with the bulk,

potentially leading to a larger v_2.

The plan of the article is the following: in section II we present the calculations for the equilibration time and the momentum broadening for the Brownian motion with memory in the nonrelativistic limit; in section III we discuss the numerical implementation of the Langevin equation with an integral kernel, while in section IV we present our results. Finally, in section V we draw our conclusions.

II. NONRELATIVISTIC LIMIT

For the sake of illustration, we consider here a simple one-dimensional motion of a heavy particle with mass m in the nonrelativistic limit. Most of the calculations presented here have been obtained firstly in [47], where a gaussian correlator has been considered. Here we consider an exponential correlator instead, therefore we skip many details that have been given in [47], and limit ourselves to write explicit results that stand for the exponential correlator.

The Langevin equation for momentum p is

$$\frac{dp}{dt} = -\int_0^t \gamma(t-s)p(s)ds + \eta(t), \qquad (2)$$

where η is the stochastic term that models the noise, while the integral term on the right hand side is the dissipative force.

The formal solution of Eq. (2) can be obtained by means of Laplace transforms, namely

$$p(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{p_0 + \Xi(s)}{s + \Gamma(s)}e^{st}ds, \qquad (3)$$

where $p_0 = p$ at $t = 0$, and Γ and Ξ denote the Laplace transforms of the dissipative kernel and of the noise respectively; the integral is understood on a Bromwich contour that leaves all the singularities of the integrand on its left side.

We assume that η is a gaussian random variable with correlators given by

$$\langle \eta(t_1)\eta(t_2)\rangle = 2\mathcal{D}f(t_1 - t_2). \qquad (4)$$

Moreover, we assume that η represents the thermal noise in a thermalized bath at temperature T. From the FDT we have then

$$\gamma(t,s) = \frac{1}{mT}\langle \eta(t)\eta(s)\rangle = 2\gamma f(t-s), \qquad (5)$$

with

$$\gamma = \frac{\mathcal{D}}{mT}. \qquad (6)$$

In order to analyze the thermalization time and the momentum broadening of the heavy particle we need to evaluate the following averages:

$$\langle p(t)\rangle = p_0\mathcal{G}, \qquad (7)$$
$$\sigma_p \equiv \langle (p(t) - \langle p\rangle)^2\rangle = \mathcal{J}, \qquad (8)$$

where we have put

$$\mathcal{G} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{1}{s + \Gamma(s)}e^{st}ds, \qquad (9)$$

that describes momentum randomization due to the propagation of the particle in the bath, and

$$\mathcal{J} = \mathcal{L}^{-1}\left[\frac{\langle\Xi(s_1)\Xi(s_2)\rangle}{(s_1 + \Gamma(s_1))(s_2 + \Gamma(s_2))}\right](t,t), \qquad (10)$$

where $\mathcal{L}^{-1}[h](t_1,t_2)$ is the two-dimensional inverse Laplace transform of $h(s_1,s_2)$ that depends on (t_1,t_2). In the above equation we have put

$$\langle\Xi(s_1)\Xi(s_2)\rangle = \int_0^\infty dt_1\int_0^\infty dt_2\langle\xi(t_1)\xi(t_2)\rangle e^{-s_1t_1 + s_2t_2}. \qquad (11)$$

In this work we consider an exponential correlator in Eqs. (4) and (5), namely

$$f(t) = \frac{1}{2\tau}e^{-|t|/\tau}, \qquad (12)$$

which allows to solve the problem analytically. In the following we call τ the memory time, since it sets the time scale over which time correlations of the noise decay. Note that $\lim_{\tau\to 0} f(t) = \delta(t)$ in the distributional sense, therefore it is possible to interpolate between the local and the nonlocal kernel by changing the value of τ. The Laplace transform of the dissipation kernel is

$$\Gamma(s) = \frac{\gamma}{(\tau s + 1)}. \qquad (13)$$

Moreover, a quick calculation shows that

$$\langle\Xi(s_1)\Xi(s_2)\rangle = \frac{\mathcal{D}(2 + \tau s_1 + \tau s_2)}{(s_1 + s_2)(1 + \tau s_1)(1 + \tau s_2)}. \qquad (14)$$

A. Thermalization

Firstly, we examine the thermalization of the heavy particle, which consists in the loss of information about the initial condition and in the equilibration of its kinetic energy with the bath, $\langle E\rangle = T/2$ as required by the Equipartition Theorem. From Eqs. (7), (9) and (13), considering that the zeroes of $s + \Gamma(s)$ are the solutions of the equation $s + \tau s^2 + \gamma = 0$, we get by a straightforward application of the residues theorem

$$\langle p(t)\rangle = p_0\frac{e^{-\frac{t(1+A)}{2\tau}}(-1+A)}{2A} + p_0\frac{e^{-\frac{t(1-A)}{2\tau}}(1+A)}{2A}, \qquad (15)$$

with $A \equiv \sqrt{1 - 4\gamma\tau}$.

Before moving to the thermalization time we pause on the early, pre-thermalization behavior of $\langle p(t)\rangle$, in order to emphasize the differences between the motion with and without memory. In order to simplify the discussion we
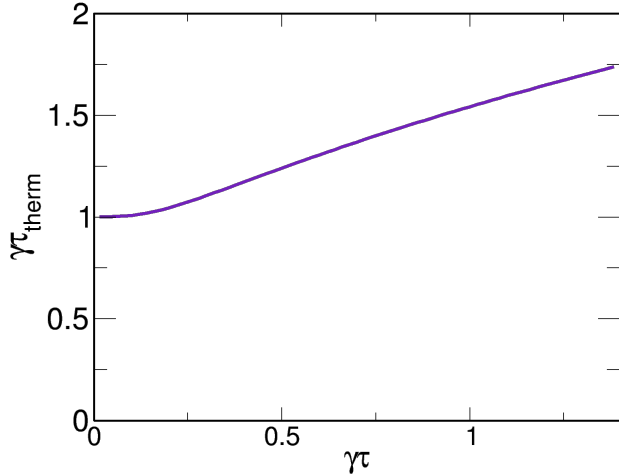
FIG. 1. Thermalization time versus memory time, both measured in units of $1/\gamma$.

assume that $\tau \ll 1/\gamma$ where $1/\gamma$ represents the thermalization time for processes without memory. For $t \ll \tau$ we get

$$\langle p(t) \rangle = p_0 \left(1 - \frac{\gamma t^2}{2\tau} \right), \quad t \ll \tau_{\text{mem}}, \qquad (16)$$

while for $\tau \ll t \ll 1/\gamma$ we get

$$\langle p(t) \rangle = p_0 \left(1 - \gamma t \right), \qquad (17)$$

The memory changes the pre-thermalization evolution of $\langle p(t) \rangle$ from linear to quadratic; in particular, this implies that the thermalization of the particle in a bath with memory is slower than the one in a bath without memory with the same drag coefficient, γ. This result is confirmed by the calculation of the thermalization time that makes use of the full result (15).

We define the thermalization time, τ_{therm}, such that $\langle p(\tau_{\text{therm}}) \rangle = p_0/e$ with $\langle p(t) \rangle$ given by Eq. (15). The results of this calculation are shown in Fig. 1 where we plot τ_{therm} as a function of the memory time, both measured in units of $1/\gamma$. Thermalization time increases with τ in agreement with the discussion above; the quantitative effect of memory is negligible for $\gamma\tau \ll 1$, but becomes substantial $\approx 20\%$ already for $\gamma\tau \approx 0.5$.

The qualitative behavior of the thermalization time can be understood in simple terms. In fact, thermalization implies the loss of the correlation with the initial condition: the heavy particle equilibrates with the bath regardless of its initial momentum distribution. If the thermal noise of the bath has memory, the momentum evolution from time t to $t + \Delta t$ does not delete totally the information of $p(t)$ because of the correlations of the noise. Therefore, it is natural to expect that the loss of the information about the initalization requires more time.

B. Momentum broadening

Next we turn to the momentum broadening, σ_p in Eq. (8). The starting point is the computation of $\mathcal{J}$ in Eq. (10) with correlator given by Eq. (14). We get

$$\sigma_p = \frac{\mathcal{D}}{2\gamma(4\gamma\tau - 1)} e^{-\frac{t(1+A)}{\tau}}$$
$$\times \left[1 - 2\gamma\tau - 4\gamma\tau e^{\frac{tA}{\tau}} - A - 2A^2 e^{\frac{t(1+A)}{\tau}} \right.$$
$$\left. + (1 - 2\gamma\tau + A) e^{\frac{2tA}{\tau}} \right], \qquad (18)$$

where A has been defined right after Eq. (15). Although Eq. (18) is exact, it is quite cumbersome, therefore it is convenient to analyze a few limit situations in which the result (18) is manageable; after that, we will present the full result (18) computed numerically. As in the previous subsection, for the sake of illustration we assume that $\tau \ll 1/\gamma$: this assumption will be removed in the full numerical calculation, see Fig. 2. From the results presented in the previous subsection in this regime we have $\tau_{\text{therm}} \approx 1/\gamma$, see Fig. 1.

The dissipative force becomes relevant on the time scale $t \approx \tau_{\text{therm}} \approx 1/\gamma$, thus for $t\gamma \ll 1$ we can ignore it and put $\gamma = 0$ in Eq. (18). We get

$$\sigma_p = \mathcal{D}(2t - 2\tau + 2\tau e^{-t/\tau}), \quad t \ll 1/\gamma. \qquad (19)$$

In the early stage $t \ll \tau$ the above result gives

$$\sigma_p \approx \frac{\mathcal{D}t^2}{\tau}, \quad t \ll \tau \ll 1/\gamma, \qquad (20)$$

$$\sigma_p \approx 2\mathcal{D}t, \quad \tau \ll t \ll 1/\gamma. \qquad (21)$$

In particular, the result (21) agrees with that we would find for the Brownian motion without memory. We notice that the effect of a finite memory time is to slow down the momentum broadening of the heavy particle, changing the early time evolution from linear to quadratic.

For late times $t \gg 1/\gamma$, $t \gg \tau$, the memory with the exponential kernel has no effect on the equilibration value of momentum broadening: in fact, in the asymptotic limit $\gamma t \gg 1$ we get from Eq. (18)

$$\sigma_p \asymp \frac{\mathcal{D}}{\gamma}, \qquad (22)$$

in agreement with the standard result for the Brownian motion.

III. LANGEVIN EQUATION WITH MEMORY: NUMERICAL IMPLEMENTATION

In this work, we solve the Langevin equation for the heavy quarks in a bath with a colored noise; the latter

has correlations at different times. In order to generate this noise we introduce an ancillary stocastic process, $h(t)$, that evolves simultaneously to (and independently of) the heavy quarks, built up in such a way its correlator at different times does not vanish. In this section we firstly define the ancillary process and specify its correlations at different times; then, we formulate the Langevin equation where the heavy quark is coupled to $h(t)$ and discuss the numerical discretization scheme adopted in the calculations.

A. Ancillary stochastic process

Let us consider the stochastic process a that satisfies the Langevin equation

$$\frac{da}{dt} = -\alpha a + \alpha\xi, \tag{23}$$

where ξ is a gaussian white noise,

$$\langle \xi \rangle = 0, \tag{24}$$

$$\langle \xi(t)\xi(t') \rangle = \frac{1}{\alpha}\delta(t - t'); \tag{25}$$

a is assumed to be dimensionless, so the $1/\alpha$ in (25) is put to balance the time dimension carried by the $\delta-$function because ξ is dimensionless too. The formal solution of Eq. (23) is given by

$$a(t) = a_0 e^{-\alpha t} + e^{-\alpha t}\int_0^t dt_1 \; \alpha\xi(t_1)e^{\alpha t_1}, \tag{26}$$

where $a_0 = a(t=0)$. Clearly we have

$$\langle a(t) \rangle = a_0 e^{-\alpha t}. \tag{27}$$

We define the fluctuating field

$$h(t) \equiv a(t) - \langle a(t) \rangle. \tag{28}$$

This satisfies Eq. (23) with $h(t=0) = 0$, that we rewrite for the sake of future reference:

$$\frac{dh}{dt} = -\alpha h + \alpha\xi. \tag{29}$$

We baptize h as the ancillary process, because we use it as an additional stochastic process to generate the colored noise for the Langevin equation of the heavy quarks, see below.

From the very definition of h it is easy to see that $\langle h(t) \rangle = 0$. Instead the correlator of h at different times is not a $\delta-$function: it can be obtained easily from Eq. (26), namely

$$\langle h(t)h(t') \rangle = \frac{e^{-\alpha|t-t'|} - e^{-\alpha(t+t')}}{2}. \tag{30}$$

Therefore, h is a process with memory and it can be used in any Langevin equation. From Eq. (30) it is obvious that asymptotically

$$\langle h(t)h(t') \rangle \approx \frac{e^{-\alpha|t-t'|}}{2}, \tag{31}$$

namely correlations are washed out on the time scale $1/\alpha \equiv \tau$. Note that for $\alpha \to +\infty$ we get, from Eq. (30),

$$\alpha\langle h(t)h(t') \rangle \approx \delta(t - t'), \tag{32}$$

that is the process h becomes a standard white noise in the limit $\tau \to 0$, as expected.

Before going ahead, it is useful to comment on Eq. (30): we note that the correlator is not a function of $t-t'$ but of t and t' separately. It is convenient to fix t' and analyze the correlator for $t \geq t'$. The addendum $A \equiv e^{-\alpha(t+t')}$ lowers the value of the correlator; on the other hand, A is suppressed when $t' = O(1/\alpha)$. That is, correlations of h develop substantially on a time scale $t \approx \tau = 1/\alpha$ necessary to suppress A. After this transient regime, time correlations are approximately given by Eq. (31) which have also the property to be invariant under time translations. Therefore, the process (29) describes a noise that needs a time $\approx \tau$ to develop memory: after the system enters in this regime, the correlations of h at different times are approximately invariant under time translation, and decay on a time scale $\approx \tau$.

Numerically Eq. (29) can be discretized in the usual manner by the replacements

$$\delta(t - t') \to \frac{\delta_{t,t'}}{\Delta t}, \tag{33}$$

$$\xi(t) = \zeta(t)\sqrt{\frac{1}{\alpha\Delta t}}, \tag{34}$$

where Δt corresponds to the discrete time step implemented in the numerical calculation. With these we have

$$\Delta h = -\alpha h\Delta t + \sqrt{\alpha}\zeta(t)\sqrt{\Delta t}. \tag{35}$$

$\zeta(t)$ will be implemented as a white noise with variance equal to one.

B. Application to the Langevin equation

Next we turn to the solution of the Langevin equation, (2), for heavy quarks in the relativistic limit. We assume the FDT in the relativistic limit, namely

$$\gamma(t,s) = \frac{1}{ET}\langle \eta(t)\eta(s) \rangle, \tag{36}$$

with $E = \sqrt{\boldsymbol{p}^2 + M^2}$. Moreover, the process $\eta(t)$ in (2) is assumed to satisfy $\langle \eta \rangle = 0$ and

$$\langle \eta(t_1)\eta(t_2) \rangle = \frac{2\mathcal{D}}{2\tau}g(t_1 - t_2), \tag{37}$$

where τ is the memory time and g is a dimensionless function that defines the correlation of the noise; for simplicity we assume it satisfies $g(0) = 1$. In the case of a Markov process $\tau \to 0$ and g has to satisfy the condition

$$\frac{1}{2\tau}g(t_1 - t_2) \to \delta(t_1 - t_2). \qquad (38)$$

In this work we generate the noise η by means of the ancillary process h introduced in the previous subsection with $\alpha = 1/\tau$. More precisely we assume that

$$\frac{1}{2\tau}g(t - t') = \frac{1}{\tau}\langle h(t)h(t')\rangle, \qquad (39)$$

see Eq. (30). Therefore, in Eq. (2) we put

$$\eta(t) = \sqrt{\frac{2\mathcal{D}}{\tau}}h(t), \qquad (40)$$

see also Eq. (37). Using Eqs. (32) and (37) we note that in the limit $\tau \to 0$ we get

$$\langle\eta(t)\eta(t')\rangle = 2\mathcal{D}\delta(t - t'), \qquad (41)$$

namely we recover the time correlation of the standard Brownian motion.

We have noted that the ancillary process (29) develops substantial correlations after an initial transient stage that lasts for $t' \approx \tau$; after this transient, the correlator is approximately invariant under time translation and decays exponentially on the time scale τ. We call this stage as the exponential decay regime. In the numerical calculations we start the ancillary process and let it run up to some time t_0, leaving heavy quarks frozen in momentum and coordinate space; then, when the noise is in the exponential decay regime, we start the evolution of the heavy quarks including their interaction with the noise itself. In this regime the correlator takes the form

$$\langle\eta(t)\eta(t')\rangle = 2\mathcal{D}\frac{e^{-|t-t'|/\tau}}{2\tau}. \qquad (42)$$

With the rescaling (40) the Langevin equation (2) reads

$$\frac{dp}{dt} = -\int_{t_0}^{t}\gamma(t,s)p(s)ds + \sqrt{\frac{2\mathcal{D}}{\tau}}h(t), \qquad (43)$$

that we solve for $t > t_0$. In (43) we put

$$\gamma(t,s) = \frac{2\mathcal{D}}{ET}\frac{e^{-|t-s|/\tau}}{2\tau}, \qquad (44)$$

in agreement with the relativistic form of the FDT. The time-discretized version of Eq. (43) is given by

$$\Delta p = -\Delta t\int_{t_0}^{t}\gamma(t,s)p(s)ds + \sqrt{\frac{2\mathcal{D}}{\tau}}h(t)\Delta t. \qquad (45)$$

Note that differently from the Markov process, the stochastic term in (45) does not come with the $\sqrt{\Delta t}$: this

is so because now the noise is not a Wiener process due to the memory. We implement a simple iterative scheme to solve (45), namely

$$p(t) = p(t - \Delta t) - \Delta t\sum_{n=0}^{N}\gamma(t,s_n)p(s_n)\Delta t$$
$$+ \sqrt{\frac{2\mathcal{D}}{\tau}}h(t)\Delta t, \qquad (46)$$

with $s_0 = t_0$, $s_{N-1} = t - \Delta t$ and $s_N = t$. This scheme leads to the self-consistent solution

$$p(t)\left[1 + (\Delta t)^2\gamma(t,t)\right] = p(t - \Delta t)$$
$$-(\Delta t)^2\sum_{n=0}^{N-1}\gamma(t,s_n)p(s_n)$$
$$+ \sqrt{\frac{2\mathcal{D}}{\tau}}h(t)\Delta t. \qquad (47)$$

Equations (35) and (45) allow us to implement the momentum evolution of the heavy quark in a bath with a colored noise. h in Eq. (45) is given by the solution of the ancillary Langevin equation (35). This means that at each time step one has to solve (35) [with the initial condition $h(t = 0) = 0$] and (45) simultaneously. This procedure is different from the one adopted in the literature in the case of a Markov process, in which the white noise would be extracted randomly from a Gaussian distribution at each time step.

For the numerical implementation of the 3-dimensional Langevin equation, we are solving equations (35) and (45) simultaneously for the 3-components (h_x,h_y, h_z, p_x, p_y, p_z) to study HQ momentum evolution coupled with the coordinate evolution

$$dr_i = \frac{p_i}{E}dt \qquad (48)$$

where dr_i is the shift of the coordinate in each time step dt. E and p are the energy and momentum of the heavy quark respectively.

The HQ transport coefficients are computed as follows. At high T we model the thermalized bath by a QGP made of massless quarks and gluons, and use the pQCD kinetic coefficients for the processes $c\ell \to c\ell$, where ℓ denotes either a massless gluon or a quark in the bath. The diffusion coefficients in this case are well known and can be found in the literature, see for example [49, 50]. The squared invariant scattering amplitudes are the Combridge ones, that includes s, u and t channel and their interferences terms. The infrared divergence associated with the t-channel diagrams is screened by the Debye mass, $m_D = g(T)T$. For more details see earlier works [28, 51]. On the other hand, at low T we model the bath with a gas of quarks and gluons quasiparticles, using the so-called quasi-particle models (QPMs); in these models the quasiparticle masses are tuned in order to reproduce Lattice QCD thermodynamics [52, 53].The QPMs account effectively for the non-perturbative effects for
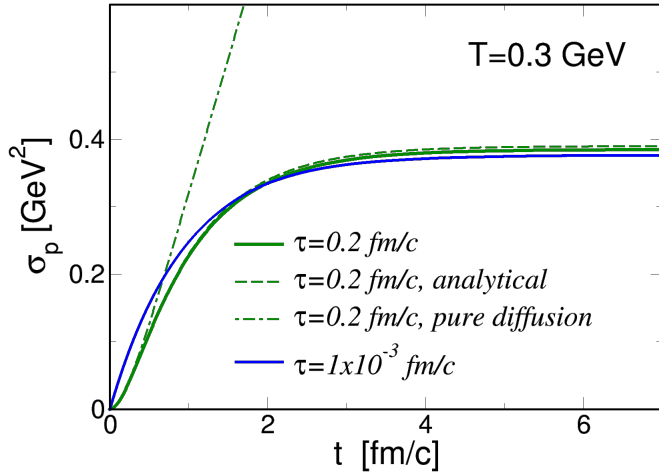
FIG. 2. σ_p versus time for a one-dimensional motion of charm quarks in the nonrelativistic limit, see Eq. (6). σ_p is defined in Eq. (8). We have considered two values of the memory time, τ. Analytical results correspond to Eq. (18).

FIG. 3. σ_p for charm quarks, defined in Eq. (49), versus time, for three values of the temperature. The solid lines correspond to $\tau = 0.2$ fm/c while the dashed, dot-dashed and dot-dot-dashed stand for $\tau = 1$ fm/c.

T close to the quark-hadron transition temperature, T_c. The main feature of the QPM is that the effective coupling is significantly stronger than the one of pQCD near T_c, which enhances the HQ-bulk inteactions. We have evaluated the diffusion coefficient within the QPM starting from the effective coupling with massive quarks and gluons. For details we refer to earlier works [28, 51].

IV. RESULTS

A. Non-relativistic check

In order to check the discretization scheme of the Langevin equation with correlated noise (45), we plot σ_p versus time in Fig. 2; the definition of σ_p is given in Eq. (8), and the Langevin equation has been solved for a one-dimensional motion of charm quarks. We have solved the equation in the nonrelativistic limit as explained in section II, that corresponds to replace the kinetic energy with the mass of the quark in the Fluctuation-Dissipation theorem, see Eq. (6); moreover, we have used a constant diffusion coefficient $\mathcal{D} = 0.2$ GeV^2/fm for illustrative purposes only. Here T referes to the bath temeprature. The green and orange solid lines correspond to $\tau = 0.2$ and $\tau = 0.01$ fm/c.

For comparison, in the same figure we show by dashed lines the analytical result (18), that is valid in the non-relativistic limit: the agreement between the two results is excellent, showing that our numerical scheme works properly. We notice that the memory slows down the
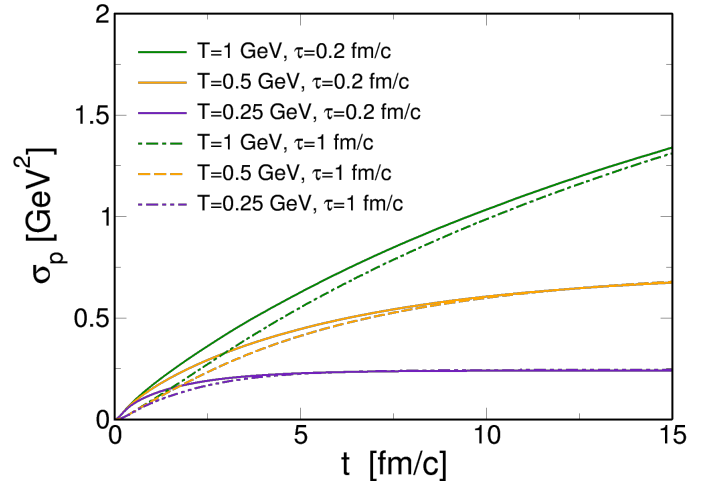
evolution of σ_p as anticipated in section II. Moreover, for $\tau = 0.2$ fm/c we notice that the diffusive motion is characterized by the initial nonlinear increase of σ_p that turns into a linear regime before the drag force becomes relevant and leads to the thermalization of the heavy quark. In the case of the smaller τ the charm quark enters the linear regime immediately, then equilibrates with the medium.

B. Transverse momentum broadening

In this subsection we analyze momentum broadening of heavy quarks in a hot medium; analogously to the previous section we define

$$\sigma_p = \langle (p_T - \langle p_T \rangle)^2 \rangle, \qquad (49)$$

where $p_T = \sqrt{p_x^2 + p_y^2}$ is the transverse momentum.

In Fig. 3 we plot σ_p versus time for three temperatures and two values of τ, namely $\tau = 0.2$ fm/c (solid lines) and $\tau = 1$ fm/c (broken lines). Calculations correspond to the cham quark. For the three cases considered it is clear that increasing the memory time results in the slowing down of momentum broadening; the effect is more visible at large temperature, where the drag force is less effective. At small temperature it is still possible to measure some difference between the results with the two memory times in the early evolution, then for time $t \geq 5$ fm/c the results with and without memory coincide.

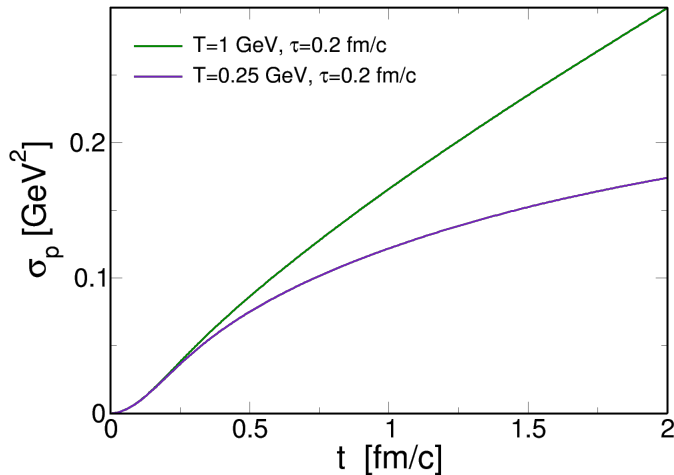In Fig. 4 we plot a selection of the results shown in Fig. 3 for $\tau = 0.2$ fm/c, zooming on the early time evo-

FIG. 4. σ_p for charm quarks, defined in Eq. (49), versus time and for $\tau = 0.2$ fm/c.

FIG. 5. R_{AA} at $t = 1$ fm/c and $t = 3$ fm/c, for $\tau = 0$ and $\tau = 1$ fm/c , at $T = 1$ GeV obtained within pQCD.

lution of σ_p. We notice the nonlinear increase of σ_p, induced by memory in agreement with the discussion of section II, followed by a linear enhancement. In this regime the charm quarks experience an almost diffusive motion, in the sense that the energy loss due to the drag force is still negligible. Qualitatively the different regimes appear also for the small temperature case in the figure: however, in this case the drag force is stronger so the linear regime lasts for a shorter fraction of the evolution, then σ_p bends and eventually saturates, signaling the equilibration of the charm with the medium.

C. Nuclear modification factor

In this subsection we analyze the modification factor R_{AA}, defined as

$$R_{\mathrm{AA}}(p_T) = \frac{(dN/d^2 p_T)_{\mathrm{t}}}{(dN/d^2 p_T)_{\mathrm{FONLL}}}, \qquad (50)$$

where $(dN/d^2 p_T)_{\mathrm{t}}$ denotes the spectrum of charm quarks at time t and $(dN/d^2 p_T)_{\mathrm{FONLL}}$ denotes the spectrum at the initialization time, To this end, at the formation time we assume the prompt spectrum obtained within Fixed Order + Next-to-Leading Log (FONLL) QCD that reproduces the D-mesons spectra in pp collisions after fragmentation [54, 55].

$$\left. \frac{dN}{d^2 p_T} \right|_{\mathrm{prompt}} = \frac{x_0}{(1 + x_3 p_T{}^{x_1})^{x_2}}; \qquad (51)$$

the parameters that we use in the calculations are $x_0 = 20.2837$, $x_1 = 1.95061$, $x_2 = 3.13695$ and $x_3 = 0.0751663$
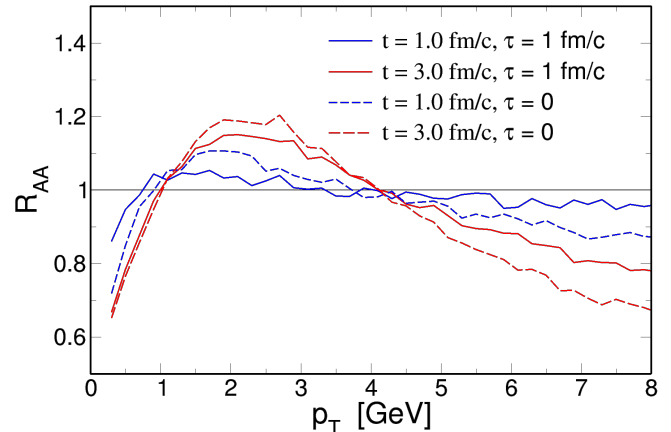
for charm quarks; the slope of the spectrum has been calibrated to a collision at $\sqrt{s} = 5.02$ TeV. $R_{\mathrm{AA}}(p_T) \neq 1$ implies that charm quarks experience interactions with the gluon medium, causing a change in their spectrum. Our motivation is to highlights the impact of memory on $R_{\mathrm{AA}}(p_T)$.

In Fig. 5 we plot R_{AA} versus p_T , for two values of the memory time τ. The shape of R_{AA} for $T = 1$ GeV and up to $p_T \approx 5$ GeV is in agreement with the one advertised in [34, 35, 40]: this is the result of the diffusion-dominated propagation of the heavy quarks in the hot medium at high temeprature, that effectively diffuses low momentum charm quarks to higher momentum states. This explains why R_{AA} remains smaller than one up to $p_T \approx 2$ GeV. For larger p_T the energy loss is important and particles migrate to lower momentum states. At low temperature the drag force is dominant in the whole range of p_T, therefore the general tendency is that particles move to lower p_T states so R_{AA} stays greater than one up to $p_T \approx 2$ GeV.

The effect of $\tau \neq 0$ is to slow down the formation of R_{AA}. We notice a sizable impact of memory on the R_{AA}. To make this poing clearer, in Fig. 6 we show R_{AA} for $\tau = 0$ and $\tau = 1$ fm/c at $T = 0.25$ GeV obtained within QPM. We notice the slower evolution of R_{AA} when $\tau \neq 0$.

In Fig. 7 we plot R_{AA} at $t = 3$ fm/c, for $\tau = 0$ and $\tau = 1$ fm/c, at $T = 0.25$ GeV. Solid lines correspond to the results of the QPM model, while dashed lines stand for the pQCD calculations. We notice that R_{AA} obtained within the QPM model differs considerably from the results of pQCD, which is in agreement with expectations because the cross sections within the former are enhanced with respect to those of the latter, implying that diffusion and drag coefficients are larger.
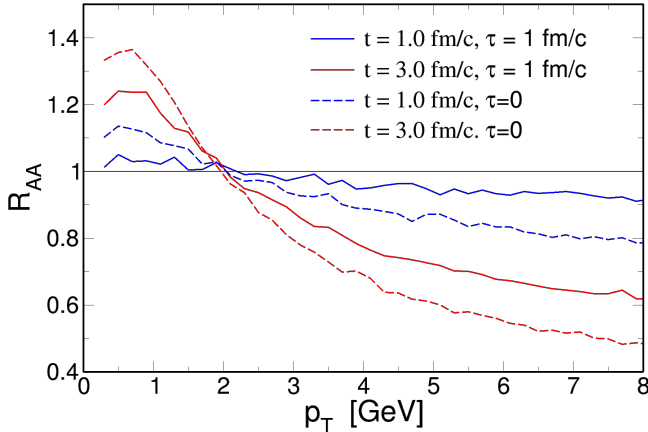
The results in Fig. 7 show that memory slows down the

FIG. 6. R_{AA} at $t = 1$ and $t = 3$ fm/c, for $\tau = 0$ and $\tau = 1$ fm/c, at $T = 0.25$ GeV obtained within QPM.

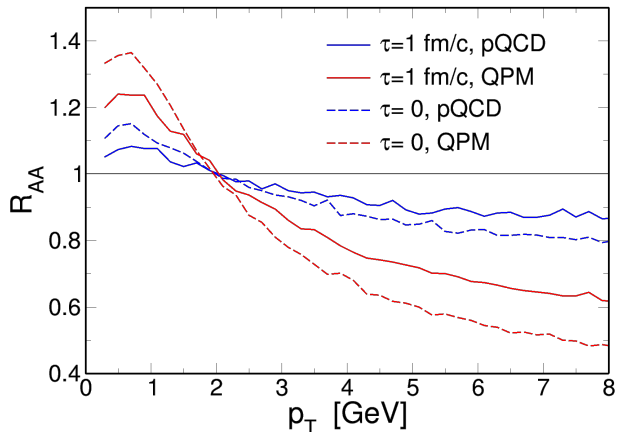FIG. 7. R_{AA} at $t = 3$ fm/c, for $\tau = 0$ and $\tau = 1$ fm/c, at $T = 0.25$ GeV. Red lines correspond to the results of the QPM model, while blue lines stand for the pQCD calculations.

dynamics of the heavy quarks in the whole p_T range. In fact, R_{AA} at large p_T in the process with memory stays above that without memory, meaning that the large p_T particles have lost less energy during the evolution. In other words, memory slows down the energy loss. Ergo, also thermalization in presence of a time-correlated noise is retarded. Similarly, the diffusion to low p_T is slower when memory is present, since R_{AA} in this case remains lower than the one without memory. These effects are more evident in the case of the QPM while they are smaller (but still present) in the pQCD case.

The results discussed in this section allow us to discuss

a potential impact that memory might have on more sophisticated phenomenological calculations. As a matter of fact, our results can be summarized by stating that memory slows down the heavy quark dynamics. For example, the formation of R_{AA} of heavy quarks is slower if the thermal fluctuations of the bath are time correlated. Therefore, if the diffusion coefficient is tuned in order to reproduce the experimental R_{AA}, then a larger coefficient is needed when the bath has memory. The larger diffusion coefficient will then enhance the elliptic flow, v_2. The memory effect has the potential to alter the heavy quark R_{AA}-v_2 dynamics and can improve the simulations description of heavy quark R_{AA}-v_2 [51]. Memory effect may affect other observables like the heavy quark directed flow, particle correlations and so on. Within our present study it is impossible to predict quantitatively this change due to our simplifying assumptions; these simplifications are partly justified by the fact that it is the first time, to our knowledge, that the effects of time correlated fluctuations on heavy quark observables are computed. A more detailed, quantitative study will be the subject of future investigations.

V. CONCLUSIONS

We have studied the effects of a time-correlated thermal noise of a thermalized quark-gluon plasma on the energy loss and the diffusion of heavy quarks. In this case the time correlation of the thermal noise does not decay instantaneously, as instead it is assumed for the standard Brownian motion. We have considered a simple situation in which the time correlations of the noise decay exponentially on the time scale τ, called the memory time, and treated τ as a free parameter. We have considered an integro-differential Langevin equation for the heavy quark momentum, taking into account memory effects both in the thermal noise and in the dissipative force.

We have computed several quantities that characterize the dynamics of heavy quarks in the bath, namely the thermalization time and the transverse momentum broadening. Then we turned to the nuclear modification factor, R_{AA}, that we have computed using kinetic coefficients from pQCD and quasiparticle models. Our results suggest that memory delays the dynamics of the heavy quarks in the QGP: indeed, memory slows down momentum broadening as well as the formation of R_{AA}, retards the energy loss and thus increases the thermalization time.

Our work could be of interest for the phenomenology of heavy quarks in the QGP: in fact, the slower evolution of R_{AA} would require the use of larger diffusion coefficients in phenomenological calculations, for reproducing the experimental R_{AA} and this in turn would require stronger interactions of the heavy quarks with the bulk, potentially leading to a larger v_2. The investigation of this point requires a refined study with realistic initial

conditions, including the initial geometry of the fireball, and will be matter for future studies. In addition to this problem, it is of a certain interest to analyze processes with a long tail memory, in which the time correlations of the noise do not decay exponentially but as power laws. These processes can be generated via Langevin equations with fractional derivatives [56, 57]; this problem will also be the subject of future studies.

[1] F. Prino and R. Rapp, J. Phys. G **43**, no. 9, 093002 (2016)

[2] A. Andronic *et al.*, Eur. Phys. J. C **76**, no. 3, 107 (2016)

[3] R. Rapp, P. B. Gossiaux, A. Andronic, R. Averbeck, S. Masciocchi, A. Beraudo, E. Bratkovskaya, P. Braun-Munzinger, S. Cao and A. Dainese, *et al.* Nucl. Phys. A **979**, 21-86 (2018)

[4] G. Aarts *et al.*, Eur. Phys. J. A **53**, no. 5, 93 (2017)

[5] S. Cao, G. Coci, S. K. Das, W. Ke, S. Y. F. Liu, S. Plumari, T. Song, Y. Xu, J. Aichelin and S. Bass, *et al.* Phys. Rev. C **99**, no.5, 054907 (2019)

[6] X. Dong and V. Greco, Prog. Part. Nucl. Phys. **104**, 97-141 (2019)

[7] Y. Xu, S. A. Bass, P. Moreau, T. Song, M. Nahrgang, E. Bratkovskaya, P. Gossiaux, J. Aichelin, S. Cao and V. Greco, *et al.* Phys. Rev. C **99**, no.1, 014902 (2019)

[8] M. Golam Mustafa, D. Pal and D. Kumar Srivastava, Phys. Rev. C **57**, 889-898 (1998) [erratum: Phys. Rev. C **57**, 3499-3499 (1998)]

[9] J. Uphoff, O. Fochler, Z. Xu and C. Greiner, Phys. Rev. C **84**, 024908 (2011)

[10] T. Song, H. Berrehrah, D. Cabrera, J. M. Torres-Rincon, L. Tolos, W. Cassing and E. Bratkovskaya, Phys. Rev. C **92**, no. 1, 014910 (2015)

[11] S. Cao, T. Luo, G. Y. Qin and X. N. Wang, Phys. Rev. C **94**, no.1, 014909 (2016)

[12] S. Plumari, V. Minissale, S. K. Das, G. Coci and V. Greco, Eur. Phys. J. C **78**, no.4, 348 (2018)

[13] H. van Hees, V. Greco and R. Rapp, Phys. Rev. C **73**, 034913 (2006)

[14] H. Van Hees, M. Mannarelli, V. Greco and R. Rapp, Phys. Rev. Lett. **100**,192301 (2008)

[15] P. B. Gossiaux and J. Aichelin, Phys. Rev. C **78**, 014904 (2008)

[16] S. K. Das, J. e. Alam and P. Mohanty, Phys. Rev. C **82**, 014908 (2010)

[17] W. M. Alberico, A. Beraudo, A. De Pace, A. Molinari, M. Monteno, M. Nardi and F. Prino, Eur. Phys. J. C **71**, 1666 (2011)

[18] T. Lang, H. van Hees, J. Steinheimer, G. Inghirami and M. Bleicher, Phys. Rev. C **93**, no. 1, 014901 (2016)

[19] M. He, R. J. Fries and R. Rapp, Phys. Rev. Lett. **110**, no.11, 112301 (2013)

[20] M. He, H. van Hees, P. B. Gossiaux, R. J. Fries and R. Rapp, Phys. Rev. E **88**, 032138 (2013)

[21] S. K. Das, F. Scardina, S. Plumari and V. Greco, Phys. Rev. C **90**, 044901 (2014)

[22] S. Cao, G. Y. Qin and S. A. Bass, Phys. Rev. C **92**, no. 2, 024907 (2015)

[23] S. K. Das, S. Plumari, S. Chatterjee, J. Alam, F. Scardina and V. Greco, Phys. Lett. B **768**, 260 (2017)

[24] Y. Xu, J. E. Bernhard, S. A. Bass, M. Nahrgang and S. Cao, Phys. Rev. C **97**, no. 1, 014907 (2018)

[25] R. Katz, C. A. G. Prado, J. Noronha-Hostler, J. Noronha and A. A. P. Suaide, Phys. Rev. C **102**, no.2, 024906 (2020)

[26] S. Li, W. Xiong and R. Wan, Eur. Phys. J. C **80**, no.12, 1113 (2020)

[27] J. Schmidt, A. Meistrenko, H. van Hees, Z. Xu and C. Greiner, Phys. Rev. E **91**, no.3, 032125 (2015)

[28] F. Scardina, S. K. Das, V. Minissale, S. Plumari and V. Greco, Phys. Rev. C **96**, no. 4, 044905 (2017)

[29] J. Uphoff, O. Fochler, Z. Xu and C. Greiner, Phys. Lett. B **717**, 430-435 (2012) doi:10.1016/j.physletb.2012.09.069 [arXiv:1205.4945 [hep-ph]].

[30] T. Song, H. Berrehrah, D. Cabrera, W. Cassing and E. Bratkovskaya, Phys. Rev. C **93**, no.3, 034906 (2016) doi:10.1103/PhysRevC.93.034906 [arXiv:1512.00891 [nucl-th]].

[31] M. Nahrgang, J. Aichelin, S. Bass, P. B. Gossiaux and K. Werner, Phys. Rev. C **91**, no.1, 014904 (2015) doi:10.1103/PhysRevC.91.014904 [arXiv:1410.5396 [hep-ph]].

[32] J. I. Kapusta, B. Muller and M. Stephanov, Phys. Rev. C **85**, 054906 (2012)

[33] K. Murase and T. Hirano, Nucl. Phys. A **956**, 276-279 (2016)

[34] J. H. Liu, S. Plumari, S. K. Das, V. Greco and M. Ruggieri, Phys. Rev. C **102**, no.4, 044902 (2020)

[35] J. H. Liu, S. K. Das, V. Greco and M. Ruggieri, Phys. Rev. D **103**, no.3, 034029 (2021)

[36] P. Khowal, S. K. Das, L. Oliva and M. Ruggieri, Eur. Phys. J. Plus **137**, no.3, 307 (2022)

[37] S. Mrowczynski, Eur. Phys. J. A **54**, no. 3, 43 (2018)

[38] A. Ipp, D. I. Müller and D. Schuh, Phys. Lett. B **810**, 135810 (2020)

[39] M. E. Carrington, A. Czajka and S. Mrowczynski, Nucl. Phys. A **1001**, 121914 (2020)

[40] M. Ruggieri and S. K. Das, Phys. Rev. D **98**, no.9, 094024 (2018)

[41] K. Boguslavski, A. Kurkela, T. Lappi and J. Peuron, JHEP **09**, 077 (2020)

[42] A. Ipp, D. I. Müller and D. Schuh, Phys. Rev. D **102**, no.7, 074001 (2020)

[43] Y. Sun, G. Coci, S. K. Das, S. Plumari, M. Ruggieri and V. Greco, Phys. Lett. B **798**, 134933 (2019)

[44] J. I. Kapusta and C. Plumberg, Phys. Rev. C **97**, no.1,

014906 (2018) [erratum: Phys. Rev. C **102**, no.1, 019901 (2020)]

[45] B. Schenke and C. Greiner, Phys. Rev. Lett. **98**, 022301 (2007)

[46] J. Hammelmann, J. M. Torres-Rincon, J. B. Rose, M. Greif and H. Elfner, Phys. Rev. D **99**, no.7, 076015 (2019)

[47] M. Ruggieri, M. Frasca and S. K. Das, Chin. Phys. C **43** (2019) no.9, 094105

[48] M. Ruggieri, L. Oliva, G. X. Peng and V. Greco, Phys. Rev. D **97**, no. 7, 076004 (2018)

[49] B. L. Combridge, Nucl. Phys. B **151**, 429 (1979)

[50] B. Svetitsky, Phys. Rev. D **37**, 2484 (1988).

[51] S. K. Das, F. Scardina, S. Plumari and V. Greco, Phys. Lett. B **747**, 260 (2015)

[52] S. Plumari, W. M. Alberico, V. Greco and C. Ratti, Phys. Rev. D **84**, 094004 (2011)

[53] H. Berrehrah, E. Bratkovskaya, W. Cassing, P. B. Gossi-aux, J. Aichelin and M. Bleicher, Phys. Rev. C **89**, no.5, 054901 (2014)

[54] M. Cacciari, M. Greco and P. Nason, JHEP **9805** (1998) 007 M. Cacciari, S. Frixione and P. Nason, JHEP **0103** (2001) 006

[55] M. Cacciari, S. Frixione, N. Houdeau, M. L. Mangano, P. Nason and G. Ridolfi, JHEP **1210** (2012) 137

[56] Abdon Atangana, Chapter 5 - Fractional Operators and Their Applications, Editor(s): Abdon Atangana, Fractional Operators with Constant and Variable Order with Application to Geo-Hydrology, Academic Press, 2018, Pages 79-112, ISBN 9780128096703.

[57] Abdon Atangana, Chapter 6 - Regularity of a General Parabolic Equation With Fractional Differentiation, Editor(s): Abdon Atangana, Fractional Operators with Constant and Variable Order with Application to Geo-Hydrology, Academic Press, 2018, Pages 113-169, ISBN 9780128096703.