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EQUILIBRIA IN EPIDEMIOLOGY AND IN FINANCE

PhD Thesis

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Contents

Chapter 1

Introduction

Mathematics is often applied to model real phenomena. One of the main approaches that mathematicians use to describe real situations are mathematical models. Indeed, a mathematical model translates mathematical equations and formulations into concrete achievements concerning the world around us. It describes a real situation using mathematical concepts and languages. A model may help us to explain what we are describing and to make predictions about the future. For these reasons, mathematical models are used in many fields: natural sciences, (such as physics, biology, earth science, meteorology, computer science), engineering disciplines, (such as artificial intelligence), and in the social sciences (such as economics, psychology, sociology and political science).

The models can take many forms, including but not limited to dynamical systems, statistical models, differential equations, as well as variational inequality formulations. The choice of the mathematical instruments we use depends on the model that we want to describe and on the approach that is more suitable for those situations.

In this thesis we focus our attention on two applications of mathematical modeling to epidemiology and finance. In particular, we want to model two real situations: the spread of anorexia and bulimia among society and a financial network. The choice to model these two phenomena is that we want to model them using two different techniques from a mathematical point of view. In both studied problems, we look for the equilibrium solution of the model and we want to predict the future. In addition, we model just these phenomena because the exponential growth of eating disorders among young people in Western countries in the recent decades leads mathematicians to find parameters that could control these social phenomena; moreover, the recent economics crisis leads to the research of models that describe the financial markets as real as possible.

We are interested in formulating the model that describes both phenomena and in finding equilibrium points. The equilibria play an important role: in the first case they describe the presence or not of anorexic and bulimic; and in the second case they perform the optimal quantity of sector assets, liabilities, and instrument prices for the financial network. When we obtain such a result, we use two different approaches. In order to study the spread of anorexia and bulimia, we use a nonlinear dynamical system; on the other hand, when we study the financial network, we formulate a variational inequality problem.

The model proposed to describe the spread of anorexia and bulimia takes into account, among other things, the effects of peers' influence, media and education. We prove the existence of three possible equilibria, that without media influences are disease-free, bulimic-endemic, and endemic. Neglecting media and education effects we investigate the stability of such equilibria, and we prove that under the influence of media, only one of such equilibria persists and becomes a global attractor. Which of the three equilibria becomes global attractor depends on the other parameters of the model.

To study the financial network we present a financial economy in the case when the financial volumes depend on time and on the expected solution, in order to take into account the influence of the expected equilibrium distribution for assets and liabilities on the investments on all financial instruments. We derive the quasi-variational formulation which characterizes the equilibrium of the dynamical financial model. The main result is a general existence theorem for quasi-variational inequalities under general assumptions, which is also applied to the financial model. We also study some numerical examples.

1.1 Dynamical Systems

The theory of dynamical systems has been defined, in the book of Scheinerman [126], *the mathematics of the time*; indeed the term dynamical refers to processes that produce changes that evolve over time. A dynamical system is a function with an attitude: it makes the same thing over and over again. For this reason dynamical systems are useful to model many different kinds of phenomena. The difficulty is that virtually anything that evolves over time can be thought of as a dynamical system.
An important role in dynamical system is played by the time. Moreover, the variable t, used to measure time, can be thought as a real number, then we say that the time varies continuously, or as a natural number, i.e. $t = 0, 1, 2, ...$, and then we will say that the time varies discretely, taking multiple values of a given unit of time as an hour, a day, a year, depending on the time scale of the system we are describing. In the first case we study dynamical systems in continuous time, in the second case we study them in discrete time. Which of the two representations is more suitable to describe a real system depends on the situation that is being analyzed.
Once we have created a model, we would like to use it to make predictions, finding its solutions. In fact, determining the state for all future times requires to solve the system or to integrate the system. Once the system can be solved, given an initial point, it is possible to determine all its future positions, a collection of points known as a *trajectory* or *orbit*. The behavior of trajectories as a function of a parameter may be what is needed for applications, as varying the parameter, the dynamical system could change its behavior, so it gives a meaningful reply for the phenomena that we are studying. Unfortunately, it is also too common that the dynamical system, which we are interested in, does not yield an analytic solution. The problem is that in many situations a dynamical system depends on many parameters that are not often known precisely, so it is very difficult to find its analytical solution. For example, a dynamical system modeling global weather might have millions of variables accounting for temperature, pressure, wind speed, and so on at points all

around the world.
Before the advent of computers, finding an orbit required sophisticated mathematical techniques and could be possible only for a small class of dynamical systems. Numerical methods implemented on electronic computing machines have simplified the task of determining the orbits of a dynamical system. So, now, we are able to find at least a numerical solution (for further details see the book of Perko [116] and the book of Hirsch and Smale [70]).
The equilibrium points play an important role in the study of a model.
A fixed point of a function or transformation is a point that is mapped to itself by the function or transformation. If we regard the evolution of a dynamical system as a series of transformations, then there may or may not be a point which remains fixed under each transformation.The study of the stability or less of equilibrium points of a system involves an important aspect in the applications.

From a formal point of view, to know a dynamical system means to find a function that, once assigned the initial state vector at a certain instant, allows uniquely to determine the system state at each following instant. In reality, it is not easy at all to find the function, but we try to know it through the formulation of motion equations. In the case of continuous dynamical systems, the laws of motion are expressed by differential equations, which describe how the speed of change of each state variable, expressed by the first derivative with respect to time, depends on itself and other variables, so it is very difficult to find the solution, expecially for models described by PDEs (partial differential equations).
We can apply similar considerations to discrete dynamical systems. In this case, the motion law is described by difference equations.
A classic example of a dynamical system can be found in problems of mechanics, where the system state at any given moment is determined by all the positions and velocities of the bodies that constitute it, but there are many applications of the dynamical systems in many settings: physics, metereology, engineering, economy, biology, ecology, epidemiology.
In this thesis we study an important application of dynamical systems to

epidemiology and, in particular, the spread of anorexia and bulimia. The model has been formulated and studied by Ciarcià, Falsaperla, Giacobbe and Mulone [29].
We are interested in the study of this phenomenon. The model that we will present, has a significant interest from the mathematical point of view; its applications have an important social interest and it is an example of a mathematical model, studied as an epidemiological model.
Our model is inspired by an article of Gonzalez et al. [59], in which a general model is suggested for anorexia and bulimia considered as epidemics. In that article the authors restrict their attention to the spread of bulimia dividing the infected individuals in two classes and analyzing the existence of endemic equilibria. That article concludes the analysis by fixing the parameters according to previous medical literature, and numerically investigating the evolution of simple and advanced bulimic depending on the net infective force. We extend this investigation to a model that describes both infective classes: anorexia and bulimia but considering only one group of individuals for each class. Our model includes several alternative routes of infection/recovery: peer pressure, media effect, education.
The difficulty to study this system is due to the presence of many parameters that change the behavior of the model. For example, the parameter that describes media influence, has a strong influence on the evolution of the system, in fact the influence of media causes the disease-free equilibrium to disappear. In chapter 3 we will prove the existence of three possible equilibria, that without media influences are disease-free, bulimic-endemic, and endemic. Neglecting media and education effects we investigate the stability of such equilibria, and we prove that under the influence of media, only one of such equilibria persists and becomes a global attractor. Which of the three equilibria becomes global attractor depends on the other parameters.

1.2 Epidemiology

Mathematical epidemiology has a long history, going back to the smallpox model of Daniel Bernoulli in 1760.

The first contributions to modern mathematical epidemiology are due to P.D. En'ko between 1873 and 1894, and the foundations of the entire approach to epidemiology based on compartmental models were laid by public health physicians such as Sir R.A. Ross, W.H. Hamer, A.G. McKendrick and W.O. Kermack between 1900 and 1935 and there has been steady progress since that time. Sir Ronald Ross, who received the Nobel Prize in medicine for his work on malaria (1902), founded the field of mathematical epidemiology. After Ross formulated a mathematical model that predicted that malaria outbreaks could be avoided if the mosquito population could be reduced below a critical threshold level, now called *reproduction number*, field trials supported his conclusions and led to sometimes brilliant successes in malaria control.

Mathematical epidemiology seems to have grown exponentially starting in the middle of the 20th century so that a great variety of models have been formulated, mathematically analyzed and applied to infectious diseases.

In the recent years, models have been formulated to control the 2002-2003 epidemics of SARS (Severe Acute Respiratory Syndrome) by Anderson [1], Riley [123] and Wang [152], the H1N1 influenza of 2009 by Bajardi [4], Matrajt and Longini [92], Shim and Galvani [132], Tizzoni et al. [146], the outbreaks of Ebola in Congo and Uganda by Chowell et al. [27], Lekone and Finkenstädt [79] and Pandey et al. [113], and to predict negative habits and social behaviors, such as the spread of heroin studied by Mulone and Straughan [96], the spread of alcoholism among people studied by Mulone and Straughan [97] and Walters, Straughan and Kendal [150] (see also [81, 98, 151] for other epidemiological models).

1.3 Networks

Network analysis is usually associated with transportation problems, electrical power transmission, telecommunications, etc. However, its methods apply not only to physical networks, where the nodes and the links have tangible embodiments, but also to a much wider class of problems where these concepts need have no physical counterparts.

The study of networks and their applications has had a long tradition in engineering, operations research/management science, and in computer science. More recently, the fields of finance and economics have come to be rich and fascinating sources of network-based problems and applications. Interest from such disciplines has been supported, in part, by the greater availability of powerful network-based methodologies and tools that allow for enhancedmodeling as well as computation of their solutions. The role of networks in finance and economics has gained new prominence for a variety of reasons, including: the emergence of network industries from transportation and logistics to telecommunications; the recognition of the interdependence among many network systems, such as telecommunications with finance and telecommunications with transportation in the form of electronic commerce; new relationships between economic decision-makers in terms of cooperation and competition which are yielding new supply chains as well as financial networks; the realization of the importance of networks and the pricing of their usage, and interest surrounding networks and their evolution over space and time.

Financial applications covered include: portfolio optimization with transaction costs, integrated pension and corporate planning, evolutionary financial networks, international finance and electronic transactions as well as hedging instruments for transportation networks (see [102] for more details).

In this thesis we study a financial network where sectors invest their amount of money in financial instruments as assets and liabilities, assuming that the investments depend on time and on the expected solution. The link with previous studies of static networks can be made in a natural way: a static configuration represents a snapshot at a fixed moment of time of an evolving real phenomenon. Therefore, studying the static case can be considered only a first approach to the understanding of the reality, which is useful and also essential for further developments. Moreover, another important motivation for considering time-dependent phenomena has been pointed out by Beckman and Wallace [13] where they claim that the time dependent formulation of network equilibrium problems allows one to explore the dynamics of markets (or traffic flows, financial investments, Walrasian prices, ...) adjustment processes in which a delay on time response is operating. It is worth remarking that the introduction of time-dependent models allows us to take into account the delay effect.

1.4 Variational Inequalities

The evolutionary variational inequalities (EVI) were introduced originally by Lions and Stampacchia [83] and by Brezis [23] to solve problems arising principally from mechanics. They provided also a theory for the existence and uniqueness of the solution of such problems.
On the other hand, Steinbach [138] studied an obstacle problem with a memory term by means of a variational inequality. In particular, under a suitable assumption on the time-dependent conductivity, he established existence and uniqueness results. In this paper, we are interested in studying an evolutionary variational inequality in the form proposed by Daniele, Maugeri, and Oettli [41], [42]. They modeled and studied the traffic network problem with feasible path flows which have to satisfy time-dependent capacity constraints and demands. They proved that the equilibrium conditions (in the form of generalized Wardrop [153] conditions) can be expressed by means of an EVI, for which existence theorems and computational procedures were given. The algorithm proposed was based on the subgradient method. In addition, EVI for spatial price equilibrium problems (see Daniele and Maugeri [37] and Daniele [32], [35]) and for financial equilibria (see Daniele [33]) have been derived. The same framework has been used also by Scrimali in [127], who studied a

special convex set K which depends on the solution of the evolutionary variational inequality and gives rise to an evolutionary quasi-variational inequality. See also the recent work of Bliemer and Bovy [17] in multi-class traffic networks. For an overview of dynamic traffic problems, see Ran and Boyce [124]. For additional background on variational inequalities and quasi-variational inequalities, see Baiocchi and Capelo [3].
In Gwinner [61], the author presents a survey of several classes of time-dependent variational inequalities. Moreover, he deals with projected dynamical systems in a Hilbert space framework. Raciti [121], [122] applied these ideas to the dynamic traffic network problem. Both Gwinner and Raciti used known results in Aubin and Cellina [2] for establishing the existence of infinite-dimensional PDS (Projected Dynamical System), see for more details [30].
Later the evolutionary models where the set of constraints depends on the equilibrium solution have been studied, so the variational inequality (VI) becomes a quasi-variational inequality (QVI). This generalization was introduced by Bensoussan et al. [15] in the context of impulse control problems. Such problems were studied by many authors [3], [25], [95]. Many applications of these mathematical tools are known, for instance, we may refer to Bensoussan [14] and Harker [64], who recognized the connection between generalized Nash games and quasi-variational inequalities, Pang and Fukushima [114] applied this result in order to formulate the noncooperative multi-leader-follower game in terms of generalized Nash games, Bliemer and Bovy [17] discussed a quasi-variational inequality formulation of the dynamic traffic assignment problem. Applications to some economic and financial models can be found in [128], [129].

1.5 Financial Networks

General multitiered financial network problems with intermediation were introduced by Nagurney and Ke [105] and extended by Nagurney and Ke [106] to include electronic transactions. Specifically, Nagurney and Ke considered decision-markers with fixed sources of funds, financial intermediaries, as well as consumers, who were associated with different tiers of the financial network (see also [103]). The decision-makers within one tier of the financial network were allowed to compete with one another in a noncooperative manner. However, decision-makers belonging to different tiers needed to cooperate in order to complete the financial transactions. The authors assumed that the decision-makers with sources of funds (and located at the top tier of the network) and the financial intermediaries (at the middle tier) optimized their own objective functions, which consisted of both net revenue maximization and risk minimization. The consumers, in turn, sought to obtain the financial products such that the price of the financial products charged by the intermediaries or the decision-makers with sources of funds (in the case of direct electronic transactions) plus the respective transaction costs was not greater than the price that consumers were willing to pay for the financial product. The authors assumed that the demand function at each demand market was known, and then formulated the governing equilibrium conditions as a variational inequality. Nagurney and Ke also provided qualitative analysis as well as an algorithm for computing the equilibrium financial flow and price pattern.

We note that financial systems were first conceptualized as networks in 1758 by Quesnay [120], were the circular flow of funds in an economy was considered as a network. Thore [142], in turn, introduced networks and utilized linear programming for the study of systems of linked portfolios (see also Charnes and Cooper [26]). Thore [143] then extended the basic network model to handle holdings of financial reserves in the case of uncertainity (see also Ferguson and Dantzig [54] and Dantzig and Madansky [44]). Storoy, Thore and Boyer [139] developed a network model of the interconnection of capital markets and applied decomposition theory of mathematical programming on the computation of equilibrium. Thore [144] presented network models of linked portfolios with financial intermediation and made the use of decentralization/decomposition theory in the computation. However, the state-of-the-art of that time was not sufficiently developed to allow for the formulation and computation of solutions to general financial network problems with intermediation, which may include competitive behavior in the sense of Nash [110], [111], asymmetric functions, etc. Moreover, financial electronic transactions did not even exist in that era. The book by Nagurney and Siokos [107] provides an overview of a variety of financial network optimization and equilibrium models to that date.

1.6 Equilibria and optimal solutions

In both studied problems, we look for the equilibrium solution of the model.
In the case of epidemiology the equilibrium point represents the persistence or not of the disease that we are studying, as the coordinates of the equilibrium point stand for the number of people that are susceptible to get sick, that are infective and that are recovered.
In the case of the financial model, the equilibrium point, found as the solution of a suitable time-dependent variational inequality, represents the optimal investment of a sector in financial assets and liabilities.
Specifically, the evolution of time allows to settle the development of the financial market, predicting, also, economic crisis through the use of mathematical instruments as the *evaluation index*.
The techniques used to the study the two models are different; in the first problem we use the theory of dynamical systems, while the variational formulation is used to study the second problem. It is possible applying the theory of dynamical systems to the case of economic models assuming that the interest rate depends on time.
We are interested in finding optimization results also in epidemiology when we are looking for the threshold parameters with the study of stability of the equilibria. On the other hand, the variational theory has its foundations in the problems of mechanics as we can see in the works by Lions and Stampacchia [83] and Brezis [23] who solve problems of mechanics with the variational formulation, or in the work by Steinbach [138] who studied an obstacle problem with a memory term by means of

a variational inequality.

The plan of the thesis is as follows. The thesis is settled in five chapters. The model of anorexia and bulimia and the model of finance are presented in the chapters 3 and 5, respectively. For the sake of completeness, we recall some properties and results of dynamical systems and their applications to epidemiology in chapter 2. In chapter 4 we present the financial network and the mathematical formulations.

Chapter 2

Dynamical Systems and application to epidemiology

In this chapter we introduce continuous nonlinear systems and them properties [70, 116, 126], because they are important for the epidemiological models as we can see in the next chapter. We deal with fixed points and their stability and we present two methods for assessing stability: linearization and Lyapunov functions.

Moreover we deal with epidemiology and we present some examples of mathematical models applied to epidemics. We also pay attention to the *basic reproduction number* and its computation.

2.1 Nonlinear systems

The nonlinear system of differential equations is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.1}$$

where $\mathbf{f} : E \to \mathbb{R}^n$ and E is an open subset of $\mathbb{R}^n$. We show that under certain conditions on the function $\mathbf{f}$, the nonlinear system (3.2) has a unique solution through each point $x_0 \in E$ defined on a maximal interval

of existence $(\alpha, \beta) \subset \mathbb{R}$.
In general, it is not possible to solve the nonlinear system (3.2); however, the Hartman-Grobman theorem lets us know that topologically the local behavior of the nonlinear system (3.2) near an equilibrium point x_0 where $\mathbf{f}(x_0) = 0$ is typically determined by the behavior of the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ near the origin when the matrix $\mathbf{A} = D\mathbf{f}(x_0)$, the derivate of $\mathbf{f}$ at x_0.
We establish the fundamental existence-uniqueness theorem for a nonlinear autonomous system of ordinary differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.2}$$

under the hypothesis that $\mathbf{f} \in C^1(E)$ where E is an open subset of $\mathbb{R}^n$.

Definition 2.1.1 *Suppose that $\mathbf{f} \in C(E)$ where E is an open subset of $\mathbb{R}^n$. Then $x(t)$ is a solution of the differential equation (2.2) on an interval I if $x(t)$ is differentiable on I and if for all $t \in I$, $x(t) \in E$ and*

$$\dot{x}(t) = \mathbf{f}(x(t)) .$$

And given $x_0 \in E$, $x(t)$ is a solution of the initial value problem

$$\dot{x} = \mathbf{f}(x)$$
$$x(t_0) = x_0$$

on an interval I if $t_0 \in I$, $x(t_0) = x_0$ and $x(t)$ is a solution of the differential equation (2.2) on the interval I.

Theorem 2.1.1 (The fundamental existence-uniqueness theorem)

Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem

$$\dot{x} = \mathbf{f}(x)$$
$$x(0) = x_0$$

has a unique solution $x(t)$ on the interval $[-a, a]$.

Remark 2.1.1 *If we consider the initial value problem*

$$\dot{x} = \mathbf{f}(x)$$
$$x(t_0) = x_0$$

it has a unique solution on some interval $[t_0 - a, t_0 + a]$.

2.2 Stability

Definition 2.2.1 *Let E be an open subset of $\mathbb{R}^n$ and let $\mathbf{f} \in C^1(E)$. For $x_0 \in E$, let $\Phi(t, x_0)$ be the solution of the initial value problem defined on its maximal interval of existence of the solution $I(\mathbf{x}_0)$. Then for $t \in I(x_0)$, the set of mappings $\{\Phi_t(\mathbf{x}_0)\}_{t \in I(\mathbf{x}_0)}$ defined by*

$$\Phi_t(\mathbf{x}_0) = \Phi(t, \mathbf{x}_0)$$

*is called the **flow** of the differential equation $\dot{x} = f(x)$.*

Definition 2.2.2 *Let $S \subset E$ is invariant with respect to the flow $\{\Phi_t\}$ if*

$$\Phi_t(S) \subset S \qquad \forall t \in \mathbb{R}$$

and in particular

- *S is positive invariant if $\Phi_t(S) \subset S \qquad \forall\, t \geq 0$,*
- *S is negative invariant if $\Phi_t(S) \subset S \qquad \forall\, t \leq 0$.*

Remark 2.2.1
It is possible to rescale the time in any C^1-system so that for all $\mathbf{x}_0 \in E$ the maximal interval of existence is $I(\mathbf{x}_0) =]-\infty, +\infty[$.

We now that $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is an equilibrium point of the system (2.2) if satisfies $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. In general, we can have *stable* or *unstable* equilibrium points as it follows.

Definition 2.2.3 (stable fixed point) *An equilibrium point* $\tilde{\mathbf{x}} \in E$ *of a continuous dynamical system is said* ***stable*** *if*

$$\forall \epsilon > 0,\ \exists\ \delta_\epsilon > 0 : |\mathbf{x}_0 - \tilde{\mathbf{x}}| < \delta_\epsilon \Rightarrow |\mathbf{x}(t) - \tilde{\mathbf{x}}| < \epsilon \quad \forall t \geq 0,$$

where $\mathbf{x}(t)$ *is the solution of the system.*

Definition 2.2.4 (asymptotically stable fixed point) *A fixed point* $\tilde{\mathbf{x}} \in E$ *of a continuous dynamical system is said* ***asymptotically stable*** *if is table and furthermore*

$$\exists\ \delta > 0 : |\mathbf{x}_0 - \tilde{\mathbf{x}}| < \delta \Rightarrow \lim_{t\to+\infty} \mathbf{x}(t) = \tilde{\mathbf{x}}.$$

Definition 2.2.5 (unstable fixed point) *A fixed point* $\tilde{\mathbf{x}} \in E$ *of a continuous dynamical system is said* ***unstable*** *if is not stable.*

Figure (2.1) illustrates these possibilities.

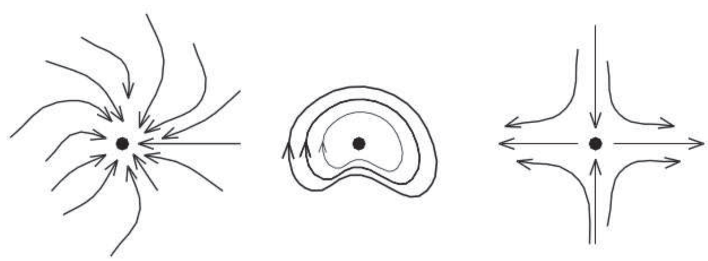

Figure 2.1: The fixed point on the left is asymptotically stable: all trajectories which begin near $\tilde{\mathbf{x}}$ remain near, and converge to, $\tilde{\mathbf{x}}$. The fixed point in the center is stable: trajectories which begin near $\tilde{\mathbf{x}}$ stay nearby but never converge to $\tilde{\mathbf{x}}$. Finally, the fixed point on the right is unstable: there are trajectories which start near $\tilde{\mathbf{x}}$ and move far away from $\tilde{\mathbf{x}}$.

2.3 Linearization

Definition 2.3.1 *The function* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is* ***differentiable*** *at* $x_0 \in \mathbb{R}^n$ *if there is a linear transformation* $D\mathbf{f}(x_0) \in L(\mathbb{R}^n)$ *that satisfies*

$$\lim_{|h|\to 0} \frac{|\mathbf{f}(x_0 + h) - \mathbf{f}(x_0) - D\mathbf{f}(x_0)h|}{|h|} = 0$$

The linear transformation $D\mathbf{f}(x_0)$ *is called the* ***derivate of*** $\mathbf{f}$ *at* x_0*.*

The following theorem gives us a method for computing the derivate in coordinates.

Theorem 2.3.1
If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable at* x_0*, then the partial derivatives* $\dfrac{\partial \mathbf{f}_i}{\partial x_j}$*,* $i, j = 1, \ldots, n$*, all exist at* x_0 *and for all* $x \in \mathbb{R}^n$*,*

$$D\mathbf{f}(x_0)x = \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial x_j}(x_0)x_j\,.$$

Thus, if $\mathbf{f}$ is a differentiable function, the derivate $D\mathbf{f}$ is given by the $n \times n$ **Jacobian matrix**

$$D\mathbf{f} = \left[\frac{\partial \mathbf{f}_i}{\partial x_j}\right].$$

Definition 2.3.2 *Suppose that* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable on* E*. Then* $\mathbf{f} \in C^1(E)$ *if the derivate* $D\mathbf{f} : E \to \mathbb{R}^n$ *is continuous on* E*.*

The next theorem gives a simple test for deciding whether or not a function $\mathbf{f} : E \to \mathbb{R}^n$ belongs to $C^1(E)$.

Theorem 2.3.2
Suppose that E *is an open subset of* $\mathbb{R}^n$ *and that* $\mathbf{f} : E \to \mathbb{R}^n$*. Then* $\mathbf{f} \in C^1(E)$ *if the partial derivatives* $\dfrac{\partial \mathbf{f}_i}{\partial x_j}$*,* $i, j = 1, \ldots, n$*, exist and are continuous on* E*.*

Definition 2.3.3 *Let $\tilde{\mathbf{x}} \in \mathbf{R}^n$ is an* ***hyperbolic*** *equilibrium point if all the eigenvalues of the matrix $D\mathbf{f}(\tilde{\mathbf{x}})$ have the real parts nonzero; instead the equilibrium point is said* ***nonhyperbolic****.*

We suppose that the equilibrium point $\tilde{\mathbf{x}}$ is hyperbolic.
To linearize $\mathbf{f}(\mathbf{x})$ near the hyperbolic fixed point $\tilde{\mathbf{x}}$ applying the formula of Taylor arrested at the first derivative, so we obtain a good approximation.
In the one-dimensional case we obtain

$$\mathbf{f}(x) \approx \mathbf{f}(\tilde{x}) + \mathbf{f}'(\tilde{x})(x - \tilde{x}).$$

In the multidimensional case we have

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\tilde{\mathbf{x}}) + D\big(\mathbf{f}(\tilde{\mathbf{x}})\big)(\mathbf{x} - \tilde{\mathbf{x}}).$$

where

$$D\big(\mathbf{f}(\mathbf{x})\big) = \begin{bmatrix} \frac{\partial \mathbf{f}_1(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{f}_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial \mathbf{f}_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial \mathbf{f}_2(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{f}_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial \mathbf{f}_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{f}_n(\mathbf{x})}{\partial x_1} & \frac{\partial \mathbf{f}_n(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial \mathbf{f}_n(\mathbf{x})}{\partial x_n} \end{bmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \mathbf{f}_1(\mathbf{x}) \\ \mathbf{f}_2(\mathbf{x}) \\ \vdots \\ \mathbf{f}_n(\mathbf{x}) \end{bmatrix}.$$

In particular, if $\tilde{\mathbf{x}} = \mathbf{0}$, then

$$\mathbf{f}(\mathbf{x}) \approx D\big(\mathbf{f}(\mathbf{0})\big)\mathbf{x}$$

as, from the definition of fixed points, $\mathbf{f}(\tilde{\mathbf{x}}) = \mathbf{f}(\mathbf{0}) = \mathbf{0}$.
So in this case a good approximation of a non linear dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

is the following linear dynamical system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

with $\mathbf{A} = D\big(\mathbf{f}(\mathbf{0})\big)$.

Definition 2.3.4 *Let* $\mathbf{A} = D\big(\mathbf{f}(\tilde{\mathbf{x}})\big)$ *the Jacobian matrix of* $\mathbf{f}$ *calculate at the fixed point* $\tilde{\mathbf{x}}$ *The fixed point* $\tilde{\mathbf{x}}$ *is said:*

- ***sink*** *if all the eigenvalues of* $\mathbf{A}$ *have negative real part.*
- ***source*** *if all the eigenvalues of* $\mathbf{A}$ *have positive real part.*
- ***saddle*** *if is hyperbolic and exists at least one eigenvalue of* $\mathbf{A}$ *with positive real part and at least one with negative real part.*

Remark 2.3.1 *In particular, in the epidemiological models the stability of the equilibrium disease-free depends on a parameter* R_0*, the basic reproduction number, that has an important epidemiological meaning, as we shall see in section (2.9).*
When $R_0 < 1$ *we predict that the infection will be not spread, while if* $R_0 > 1$*, the disease will spread. Moreover, the behaviour of* R_0 *is strictly connected with the real part of the eigenvalues of the Jacobian matrix, indeed the condition* $R_0 < 1$ *is equivalent to the condition that all eigenvalues of the Jacobian matrix has negative real part and the condition* $R_0 > 1$ *is equivalent to the condition that at least one eigenvalues of the Jacobian matrix has positive real part. So, the disease-free equilibrium point is locally stable if and only if* $R_0 < 1$*.*

2.4 Topological equivalence

Definition 2.4.1 *Let* X *be a metric space and let* A *and* B *be a subsets of* X*. A **homeomorphism** of* A *onto* B *is a continuous map of* A *onto* B*,* $H : A \to B$*, such that* $H^{-1} : B \to A$ *is continuous. The sets* A *and* B *are called **topologically equivalent** if there is a homeomorphism of* A *onto* B*.*

The Hartman-Grobman theorem is an important result in the local qualitative theory of ordinary differential equations. The theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{2.3}$$

has the same qualitative structure as the linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{2.4}$$

with $\mathbf{A} = D\mathbf{f}(x_0)$.

Definition 2.4.2 *Two autonomous systems of differential equations such as* (2.3) *and* (2.4) *are said to be **topologically equivalent** in a neighborhood of the origin if there is a homeomorphism H mapping an open set U containing the origin onto an open set V containing the origin which maps trajectories of* (2.3) *in U onto trajectories of* (2.4) *in V and preserves their orientation by time in the sense that if a trajectory is directed from x_1 to x_2 in U, then its image is directed from $H(x_1)$ to $H(x_2)$ in V. If the homeomorphism H preserves the parameterization by time, then the systems* (2.3) *and* (2.4) *are said to be **topologically conjugate** in a neighborhood of the origin.*

Theorem 2.4.1 (The Hartman-Grobman theorem)
Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $\mathbf{f} \in C^1(E)$, and let Φ_t be the flow of the nonlinear system (2.3). *Suppose that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and that the matrix $\mathbf{A} = D\mathbf{f}(\mathbf{0})$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \circ \Phi_t(x_0) = e^{\mathbf{A}t} H(x_0)$$

i.e. H maps trajectories of (2.3) *near the origin onto trajectories of* (2.4) *near the origin and preserves the parameterization by time.*

The consequence of all this is that every equilibrium points of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that are sinks they are asymptotically stable; instead every equilibrium points that are sources they are unstable.
When you can apply the Hartman-Grobman theorem is said that is valid the *test of linearization.*
Hence, the behavior of the nonlinear and continuous dynamical system is:

- if $\forall Re(\lambda) < 0 \;\Rightarrow\;$ the fixed point $\tilde{\mathbf{x}}$ is stable
- if $\exists\lambda :\; Re(\lambda) > 0 \;\Rightarrow\;$ the fixed point $\tilde{\mathbf{x}}$ is unstable
- if $\forall Re(\lambda) \leq 0$, $\exists\lambda :\; Re(\lambda) = 0 \;\Rightarrow\;$ the test of linearization *fails*

where $\tilde{\mathbf{x}}$ is a fixed point and λ is an eigenvalues of the matrix $\mathbf{A} = D\mathbf{f}(\tilde{\mathbf{x}})$.

2.5 Lyapunov function

The question if an equilibrium point is stable, asymptotically stable or unstable is a delicate problem. The following method, due to Lyapunov function is very useful in answering this question [84].

Definition 2.5.1 *If $\mathbf{f} \in C^1(E)$, $V \in C^1(E)$ and Φ_t is the flow of the differential equation $\dot{x} = \mathbf{f}(x)$, then for $x \in E$ the derivate of the function $V(x)$ along the solution $\Phi_t(x)$*

$$\dot{V}(x) = \frac{\mathrm{d}}{\mathrm{d}t} V(\Phi_t(x))|_{t=0} = DV(x)\mathbf{f}(x).$$

Theorem 2.5.1
Let E be an open subset of $\mathbb{R}^n$ containing x_0. Suppose that $\mathbf{f} \in C^1(E)$ and that $\mathbf{f}(x_0) = 0$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then

(a) *if $\dot{V}(x) \leq 0$ for all $x \in E \;\Rightarrow\; \tilde{x}$ is* ***stable****,*

(b) *if $\dot{V}(x) < 0$ for all $x \in E - \{\tilde{x}\} \;\Rightarrow\; \tilde{x}$ is* ***asymptotically stable****,*

(c) *if $\dot{V}(x) > 0$ for all $x \in E - \{\tilde{x}\} \;\Rightarrow\; \tilde{x}$ is* ***unstable****,*

Remark 2.5.1
If $\dot{V}(x) = 0$ for all $x \in E$ then the trajectories of $\dot{x} = \mathbf{f}(x)$, lie on the surface in $\mathbb{R}^n$ defined by

$$V(x) = c$$

where c is a constant.

Definition 2.5.2 *Each function $V : E \to \mathbb{R}$ that satisfies* (a) *and* (b) *is called* ***Lyapunov function****. Moreover, if satisfies* (c) *is called* ***strictly Lyapunov function****.*

The Lyapunov theorem allows as to draw conclusions on the stability of the fixed points without knowing explicitly the solutions of the system. However, the main problem is to determine the Lyapunov function and there isn't a precise law to find a good Lyapunov function.

Theorem 2.5.2 (Krasovskii - La Salle principle)
Let $\tilde{x} \in E$ an equilibrium point of $\dot{x} = \mathbf{f}(x)$. Let $V : E \to \mathbb{R}$ a Lyapunov function for $\tilde{x}$. Let $\overline{X}$ a neighborhood of $\tilde{x}$ closed on E positive invariant. Suppose that doesn't exist an orbit $x(t)$, solution of $\dot{x} = \mathbf{f}x$, defined for all $t \in \mathbb{R}$, on $\overline{X} - \{\tilde{x}\}$ in which $V = c$ (c is a constant), then $\tilde{x}$ is asymptotically stable and $\overline{X} \subset B(\tilde{x})$, where $B(\tilde{x})$ is a basin of attraction for $\tilde{x}$.

2.6 Attractors

Definition 2.6.1 *Let $A \subset \mathbb{R}^n$. A neighborhood of A is an open set U that contains B.*

Definition 2.6.2 *Is said that $x(t) \to B$ for $t \to \infty$ if the distance $\mathrm{d}(x(t), B) \to 0$ for $t \to \infty$.*

Definition 2.6.3 *Let B an open subset of $\mathbb{R}^n$. A closed invariant subset $A \subset B$, is said an* ***attracting set*** *of the system $\dot{x} = \mathbf{f}x$ if exists a neighborhood U of A that*

$$\forall x \in U, \quad \Phi_t(x) \in U \quad \forall t \geq 0 \quad \textit{and} \quad \Phi_t(x) \to A \quad \textit{for} \quad t \to \infty$$

An ***attractor*** *of the system is an attracting set which contains a dense orbit.*

2.7 Dynamics of infectious diseases

Terminology, notations and assumptions, that are given in this section, are based on the papers of *Hethcote* [67], [68] and the book of *Murray* [99].
An epidemic is an occurence of a disease in excess of normal expectancy, while a disease is called endemic if it is habitually present; however, communicable disease models of all types are often referred to as epidemic models and the study of disease occurence is called epidemiology.
The population or community under consideration is divided into disjoint compartments which change with time t:

- the **susceptible** class, S, consists of those individuals who can incur the disease but are not yet infective (the number of individuals in this class is denoted with $S(t)$).
- the **infective** class, I, consists of those who are transmitting the disease to others (the number of individuals in this class is denoted with $I(t)$).
- the **removed** class, R, consists of those who are removed from the susceptible-infective interaction by recovery with immunity, isolation or death (the number of individuals in this class is denoted with $R(t)$).

Sometimes, there would be another two compartments that are often omitted because they are not crucial for the susceptible-infective interaction:

- the **passively immune** class, M, that contains infants with passive immunity as their mother has been infected (the number of individuals in this class is denoted with $M(t)$).
- the **exposed** class, E, that contains susceptibles that became infected but not yet infectious during the latent period (the number of individuals in this class is denoted with $E(t)$).

The choice of which compartments to include in a model depends on the characteristics of the particular disease being modeled and the purpose of the model. Acronyms for epidemiology models are often based on the flow patterns between the compartments such as MSEIR, SEIR, SIS, SIR and so on. For example, in the MSEIR model, shown in Figure 2.2, passively immune newborns first become susceptible, then exposed in the latent period, then infectious and then removed with permanent immunity.

If recovery does not give immunity, then the model is called an SIS

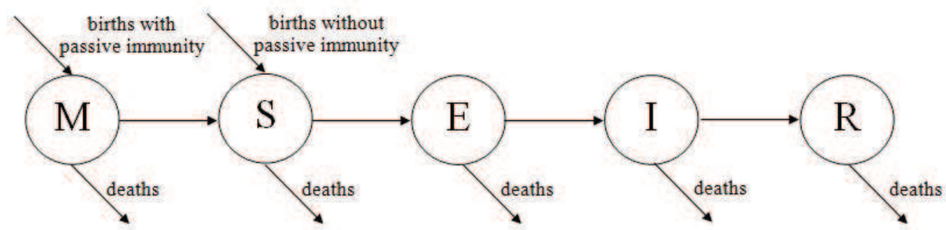

Figure 2.2: The general transfer diagram for the MSEIR model.

model, since individuals move from the susceptible class to the infective class and then back to the susceptible class upon recovery. If individuals recover with immunity, then the model is an SIR model. If individuals do not recover, then the model is an SI model. In general, SIR models are appropriate for viral agent diseases such as measles, mumps and smallpox; while SIS models are appropriate for some bacterial agent diseases such as meningitis, plague and veneral diseases.

2.8 Formulating epidemiology models

In the previous section we have said that we consider *compartmental models* as the population under studying is divided into disjoint compartments. The rates of transfer between compartments are expressed

mathematically as derivatives with respect to time of the sizes of the compartments, so as a result our models are formulated as *differential equations*. We note that all parameters in the differential equations are nonnegative, and only nonnegative solution are considered, since negative solutions have no epidemiological significance.
We assume that the population considered has constant size N which is sufficiently large so that the sizes of each class can be considered as continuous variables instead of discrete variables. If the model include vital dynamics, then it is assumed that births and deaths occur at equal rates and that all newborns are susceptible. Moreover, we suppose that the population is uniform and homogeneously mixing.
The horizontal incidence shown in Figure 2.2 is the infection rate of susceptible individuals through their contacts with infectives.
If $S(t)$ is the number of susceptibles at time t, $I(t)$ is the number of infectives and N is the total population size, then $s(t) = \frac{S(t)}{N}$ and $i(t) = \frac{I(t)}{N}$ are the susceptible and infectiuos fractions, respectively.
If β is the average number of adquate contacts (i.e. contacts sufficient for transmission) of a person per unit time, then $\beta \frac{I}{N} = \beta i$ is the average number of contacts with infectives per unit time of one susceptible, and $\beta \left(\frac{I}{N}\right) S = \beta N i s$ is the number of new cases per unit time due to the $S = Ns$ susceptibles.
This form of the horizontal incidence is called the *standard incidence*.

2.9 The *Basic reproduction number* R_0

A basic concept in epidemiology is existence of a threshold quantity. This threshold, for many epidemiology models, is the *basic reproduction number* R_0, which is defined as the expected number of individuals infected by a single infected individual, during his or her entire infectious period, in a population which is entirely susceptible. From this definition, it is immediately clear that when $R_0 < 1$, each infected individual produces, on average, less than one new infected individual, and we therefore predict that the infection will be cleared from the population, while an infection can get started in a fully susceptible population if and only if $R_0 > 1$, as in this case the pathogen is able to invade the susceptible population. This threshold behaviour is the most important and useful aspect of the R_0 concept. In an endemic infection, we can determine which control measures, and at what magnitude, would be most effective in reducing R_0 below one, providing important guidance for public health initiatives.

R_0 is often found through the study and computation of the eigenvalues of the Jacobian matrix at the disease-free or infectious-free equilibrium point.

Moreover, to compute R_0 you can follow a method, called *next generation operator approach*, introduced by *Diekmann et al.* [48, 49] (a number of salient examples of this method are in [24, 66, 148]). Now, we explain this method.

We consider a heterogeneous population whose individuals are distinguishable by age, behavior, spatial position and/or stage of disease, but can be grouped into n homogeneous compartments.

A compartment is called a *disease compartment* if the individuals therein are infected. Note that this use of the term *disease* is broader than the clinical definition and includes asymptomatic stages of infection as well as symptomatic.

Suppose there are n disease compartment and m non disease compartments, and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ the subpopulations in each of these compartments. Further, denote by $\mathcal{F}_i$ the rate secondary infections increase the ith disease compartment and by $\mathcal{V}_i$ the rate disease progression, death and recovery decrease the ith compartment. Moreover, let $\mathcal{V}_i = \mathcal{V}_i^- - \mathcal{V}_i^+$, where $\mathcal{V}_i^+$ is the rate of transfer of individuals into compartment i by all other means and $\mathcal{V}_i^-$ is the rate of transfer of individuals out of the ith compartment.Observe that $\mathcal{F}_i$ should include only infections that are newly arising, but does not include terms which describe the transfer of infectious individuals from one infected compartment to another. From this definition, note that the difference $\mathcal{F}_i - \mathcal{V}_i$ gives the rate of change of individuals in the compartments.

The compartmental model can then be written in the following form:

$$x_i' = \mathcal{F}_i(x, y) - \mathcal{V}_i(x, y), \qquad i = 1, \ldots, n, \tag{2.5a}$$
$$y_j' = g_j(x, y), \qquad j = 1, \ldots, m, \tag{2.5b}$$

where $'$ denotes differentiation with respect to time.
Note that the decomposition of the dynamics into $\mathcal{F}$ and $\mathcal{V}$ and the designation of compartments as infected or uninfected may not be unique; different decompositions correspond to different epidemiological interpretations of the model. The definitions of $\mathcal{F}$ and $\mathcal{V}$ used here differ slightly from those in [22] and [148].

The derivation of the basic reproduction number is based on the linearization of the ODE model about a disease-free equilibrium.
In the next section there will be made assumptions to ensure the existence of this equilibrium and to ensure the model is well posed. Assume that $\mathcal{F}_i$ and $\mathcal{V}_i$ meet the conditions outlined by *Castillo-Chavez et al.* [24] and [148] *Diekmann et al.* (1990) [48]:

(A1) Assume $\mathcal{F}_i(0, y) = 0$ and $\mathcal{V}_i(0, y) = 0 \quad \forall\ y \geq 0$ and for $i = 1, \ldots, n$. All new infections are secondary infections arising from infected hosts; there is no immigration of individuals into the disease compartments.

(A2) Assume $\mathcal{F}_i(x, y) \geq 0 \quad \forall\, y \geq 0,\ x \geq 0$ and $i = 1, \ldots, n$.
The function $\mathcal{F}$ represents new infections and cannot be negative.

(A3) Assume $\mathcal{V}_i(x, y) \leq 0$ whenever $x_i = 0$, $i = 1, \ldots, n$. Each component, $\mathcal{V}_i$, represents a net outflow from compartment i and must be negative (inflow only) whenever the compartment is empty.

(A4) Assume $\sum_{i=1}^{n} \mathcal{V}_i(x, y) \geq 0 \quad \forall x > 0,\ y \geq 0$.
This sum represents the total outflow from all infected compartments. Terms in the model leading to *increases* in $\sum_{i=1}^{n} x_i$ are assumed to represent secondary infections and therefore belong in $\mathcal{F}$.

(A5) Assume the disease-free system $y^{'} = g(0, y)$ has a unique equilibrium that is asymptotically stable. That is, all solution with initial conditions of the form $(0, y)$ approach a point $(0, y_0)$ as $t \to \infty$. We refer to this point as the *disease-free equilibrium*.

Note that the assumption (A1) ensures that the disease-free set, which consists of all points of the form $(0, y)$, is invariant. That is, any solution with no infected individuals at some point in time will be free of infection for all time. This in turn ensures that the disease-free equilibrium is also an equilibrium of the full system.

Suppose a single infected person is introduced into a population originally free of disease. The initial ability of the disease to spread through the population is determined by an examination of the linearization of (2.5a) about the disease-free equilibrium $(0, y_0)$.
Using assumption (A1), it can be shown that

$$\frac{\partial \mathcal{F}_i}{\partial y_j}(0, y_0) = \frac{\partial \mathcal{V}_i}{\partial y_j}(0, y_0) = 0$$

for every pair (i, j). This implies that the linearized equations for the disease compartments, x, are decoupled from the remaining equations and can be written as

$$x^{'} = (F - V)x \tag{2.6}$$

where F and V are the $n \times n$ matrices with entries

$$F = \frac{\partial \mathcal{F}_i}{\partial x_j}(0, y_0) \quad \text{e} \quad V = \frac{\partial \mathcal{V}_i}{\partial x_j}(0, y_0).$$

that are Jacobian matrices.
Note that using assumption (A5), linear stability of the system (2.5) is completely determined by the linear stability of $(F - V)$ in (2.6). They define R_0 as the spectral radius of the *next generation operator*, that is the matrix FV^{-1}.
To interpret the entries of FV^{-1} and develop a meaningful definition

of R_0, consider the fate of an infected individual introduced into compartment k of a disease free population. The $(_j, _k)$ entry of V^{-1} is the average length of time this individual spends in compartment j during its lifetime, assuming that the population remains near the disease-free equilibrium and barring reinfection. The $(_i, _j)$ entry of F is the rate at which infected individuals in compartment j produce new infections in compartment i. Hence, the $(_i, _k)$ entry of the product FV^{-1} is the expected number of new infections in compartment i produced by the infected individual originally introduced into compartment k.
Following [48], we call FV^{-1} the *next generation matrix* for the model and set:

$$R_0 = \rho(FV^{-1}) \tag{2.7}$$

where $\rho(FV^{-1})$ denotes the spectral radius of a matrix FV^{-1}.
As we shall see, the next generation matrix, $K = FV^{-1}$, is nonnegative and therefore has a nonnegative eigenvalue, $R_0 = \rho(FV^{-1})$, such that there are no other eigenvalues of K with modulus greater than R_0 and there is a *nonnegative eigenvector* w associated with R_0. Thus, R_0 and the associated eigenvector w suitably define a *typical* infective and the basic reproduction number can be rigorously defined as the spectral radius of the next generation matrix, K. The spectral radius of a matrix K, denoted $\rho(K)$, is the maximum of the moduli of the eigenvalues of K.
If K is irreducible, then R_0 is a simple eigenvalue of K. However, if K is reducible, which is often the case for diseases with multiple strains, then K may have several positive real eigenvectors corresponding to reproduction numbers for each competing strain of the disease. If the reproduction numbers, $R_0 = \rho(FV^{-1})$, computed in the next examples are consistent with differential equation model, then it should follow that the disease-free equilbrium is stable if $R_0 < 1$ and unstable if $R_0 > 1$. This is shown through a series of lemmas.
If each entry of a matrix T is nonnegative we write $T \geq 0$ and refer to T as a *nonnegative matrix*. A matrix of the form $A = sI - B$, with $B \geq 0$, is said to have the Z sign pattern. These are matrices whose offdiagonal

entries are negative or zero. If in addition, $s \geq \rho(B)$, then A is called an M-matrix. Note that in this section, I denotes an identity matrix, not a population of infectious individuals.
The following theorems are results from [16].

Proposition 2.9.1
If A has the Z sign pattern, then $A^{-1} \geq 0$ if and only if A is a nonsingular M-matrix.

From assumptions (A1) and (A2) it follows that each entry of F is nonnegative. From assumptions (A1) and (A3) it follows that the offdiagonal entries of V are negative or zero. Thus V has the Z sign pattern. Assumption (A4) with assumption (A1) ensures that the column sums of V are positive or zero, which, together with the Z sign pattern, implies that V is a (possibly singular) M-matrix. In what follows, it is assumed that V is nonsingular. In this case, $V^{-1} > 0$, by Lemma 2.9.1. Hence, $K = FV^{-1}$ is also nonnegative.

Proposition 2.9.2
If F is nonnegative and V is a nonsingular M-matrix, then $R_0 = \rho(FV^{-1})$ < 1 if and only if all eigenvalues of $(F - V)$ have negative real parts.

Theorem 2.9.1
Consider the disease transmission model given by (2.5). The disease-free equilibrium point of (2.5) is locally asymptotically stable if $R_0 < 1$, but unstable if $R_0 > 1$.

2.10 Examples of epidemiological models

Using the notation and the assumptions in section 2.8, we present some classic epidemiological models that describe the spread of a disease.
A particular case of the MSEIR model, in Figure 2.2, is the SIR model, in which the passively immune class M and the exposed class E are omitted. We distinguish two cases the model without and with vital dynamics (births and deaths). The SIR models without vital dynamics

(the *epidemic* models)might be appropriate for describing an epidemic outbreak during a short time period, whereas the SIR model with vital dinamics (the *endemic* models) would be appropriate over a longer time period.
However, these simple, classic SIR models have obvious limitations. They unrealistically assume that the population is uniform and homogeneously mixing, whereas it is known that mixing depends on many factors including age. Moreover, different geographic and social-economic groups have different contact rates. Despite their limitations, the classic SIR models can be used to obtain some estimates and comparisons.

2.11 The epidemic SIR model

McKendrick, like Sir Ronald Ross, was a physician commissioned by the English Army to India. McKendrick became involved in the study of epidemic diseases using mathematical models through the direct encouragement of Ross. His simple epidemic model was published in a joint paper with Kermack (*Kermack and McKendrick* (1927) [76]). It involved the study of the transmission dynamics of a communicable disease that provide permanent immunity after recovery. Their mathematical work led to the first widely recognized threshold theorem in epidemiology. Kermack and Mckendrick's model is an SIR (Susceptible-Infected-Recovered) model without vital (births and deaths) dynamics.
Using the notation in sections 2.8 and 2.7, the classic epidemic model is given by the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = -\beta S \dfrac{I}{N} & S(0) = S_0 \geq 0 \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta S \dfrac{I}{N} - \gamma I & I(0) = I_0 \geq 0 \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma I & R(0) = R_0 \geq 0 \end{cases} \tag{2.8}$$

where $S(t)$, $I(t)$ and $R(t)$ are the numbers in these classes, such that $S(t) + I(t) + R(t) = N$.

This model uses the standard incidence and has recovery at rate γI, corresponding to an exponential waiting time $e^{-\gamma t}$; since the time period is short, this model has no vital dynamics.
Dividing the equations in (2.8) by the constant total population size N yields the normalized system

$$\begin{cases} \frac{\mathrm{d}s}{\mathrm{d}t} = -\beta si \\ \frac{\mathrm{d}i}{\mathrm{d}t} = \beta si - \gamma i \\ \frac{\mathrm{d}r}{\mathrm{d}t} = \gamma i \end{cases}$$

If we consider $r(t) = 1 - s(t) - i(t)$, where $s(t)$, $i(t)$ and $r(t)$ are the fractions in the classes, we obtain:

$$\begin{cases} \frac{\mathrm{d}s}{\mathrm{d}t} = -\beta is & s(0) = s_0 \geq 0 \\ \frac{\mathrm{d}i}{\mathrm{d}t} = \beta is - \gamma i & i(0) = i_0 \geq 0 \end{cases} . \tag{2.9}$$

The triangle T in the si phase plane given by

$$T = \{(s,i) | \quad s \geq 0, \quad i \geq 0, \quad s + i \leq 1\} \tag{2.10}$$

is positively invariant and unique solutions exist in T for all positive time, so that the model is mathematically and epidemiologically well posed. To compute R_0 we note that the disease compartment is only the infected class I, while the nondisease compartments are the classes S and R.
Moreover, the unique *disease-free* equilibrium is $(1, 0, 0)$. So we pose:

$$\mathcal{F} = \beta si \qquad \mathcal{V} = \gamma i$$

$\mathcal{F}$ and $\mathcal{V}$ satisfy assumptions from (A1) to (A5), so we compute:

$$F(1, 0, 0) = \beta \qquad V(1, 0, 0) = \gamma$$

Hence, immediately, we obtain R_0 from the definition (2.7):

$$R_0 = \frac{\beta}{\gamma} .$$

Note that β is the average number of susceptibles infected by one infectious individual per unit time and $\frac{1}{\gamma}$ is the mean length of infectious period; therefore $R_0 = \frac{\beta}{\gamma}$ gives the number of secondary infectious cases produced by an infectious individual who has been introduced into a population of susceptibles during the individual's period of infectiousness.

2.12 The endemic SIR model

The classic endemic models the SIR model with vital dynamics, proposed by Kermack and McKendrick; where N, the total population, is constant, and γ is the per capita natural death rate.
The equation of the model became:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu N - \mu S - \dfrac{\beta I S}{N} & S(0) = S_0 \geq 0 \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \dfrac{\beta I S}{N} - \gamma I - \mu I & I(0) = I_0 \geq 0 \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R & R(0) = R_0 \geq 0 \end{cases} \tag{2.11}$$

with $S(t) + I(t) + R(t) = N$.
This SIR model is almost the same as the SIR epidemic model (2.8) above, except that it has an inflow of newborns into the susceptible class at rate μN and deaths in the classes at rates μS, μI and μR. The deaths balance the births, so that the population size N is constant.
Dividing the equations in (2.11) by the constant total population size N yields

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = -\beta i s + \mu - \mu s & s(0) = s_0 \geq 0 \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = \beta i s - (\gamma + \mu) i & i(0) = i_0 \geq 0 \end{cases} \tag{2.12}$$

with $r(t) = 1 - s(t) - i(t)$. The triangle T in the si phase plan given by (2.10) is positively invariant and the model is well posed.
For this model the threshold quantity is given by $R_0 = \frac{\beta}{\gamma+\mu}$, which is the contact rate β times the average death-adjusted infectious period $\frac{1}{\gamma+\mu}$.

The normalized system is:

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = \mu - \beta s i - \mu s \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = \beta s i - (\gamma + \mu) i \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \gamma i - \mu r \end{cases}$$

with:

$$1 = s + i + r$$

The disease and nondisease compartments are the same of the previous model.
Hence we pose:

$$\mathcal{F} = \beta s i \qquad \mathcal{V} = (\gamma + \mu) i$$

$\mathcal{F}$ e $\mathcal{V}$ satisfy assumptions from (A1) to (A5), so we compute:

$$F(1, 0, 0) = \beta \qquad V(1, 0, 0) = \gamma + \mu\,.$$

From definition (2.7) the basic reproduction number is:

$$R_0 = \frac{\beta}{\gamma + \mu}\,.$$

Note that β is the average number of susceptibles infected by one infectious individual per unit time and $\frac{1}{\gamma+\mu}$ is the mean length of infectious period; therefore $R_0 = \frac{\beta}{\gamma+\mu}$ gives the number of secondary infectious cases produced by an infectious individual who has been introduced into a population of susceptibles during the individual's period of infectiousness.
The stability for $R_0 > 1$ can be also proved using a Lyapunov function. If we pose

$$s = s_e\,(1 + u)$$
$$i = i_e\,(1 + v)$$

with $(s_e, i_e) = \left(\frac{1}{R_0}, \frac{\mu(R_0-1)}{\beta}\right)$.
Notice that u and v are the perturbations.
Then the system (2.12) becomes

$$\begin{cases} \dot{u} = -\beta i_e u(1+v) - \beta i_e v - \mu u \\ \dot{v} = (\gamma+\mu)(1+v)u \,. \end{cases}$$

Hence the triangle (2.10) becomes

$$T^* = \{(u,v)| \quad u \geq -1; \quad v \geq -1; \quad s_e + u_e \leq 1 - s_e - i_e\} \,.$$

We can introduce a Lyapunov function

$$L = \frac{u^2}{2} + R_0 i_e [v - \ln(1+v)]$$

and it will be

$$\dot{L} = -\beta i_e u^2(1+v) - \mu u^2 \leq 0 \,.$$

Hence, thanks to the Lasalle theorem 2.5.2 the equilibrium point $(0,0)$ in the uv phase plane is globally asymptotically stable in T^*.

2.13 The SEIR model

Another epidemiological model is the SEIR model, that describes, above all, childhood disease as measles.
The population is divided into four compartments: susceptible individuals, S, exposed and latently infected, E, infectious individuals, I, and recovered individuals with immunity, R.
New infections in compartment E arise by contacts between susceptible and infected individuals in compartments S and I at a rate βSI. Individuals progress from compartment E to I at a rate k and develop immunity at a rate γ. In addition, natural mortality claims individuals at a rate μ. For simplicity, the model assumes a constant recruitment, Π, of susceptible individuals.

The model is:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Pi - \mu S - \beta S \dfrac{I}{N} \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = \beta S \dfrac{I}{N} - (k+\mu)E \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = kE - (\gamma+\mu)I \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma I - \mu R \end{cases}$$

with:

$$N = S + E + I + R\,.$$

The normalized system is:

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = \dfrac{\Pi}{N} - \beta si - \mu s \\ \dfrac{\mathrm{d}e}{\mathrm{d}t} = \beta si - (k+\mu)e \\ \dfrac{\mathrm{d}i}{\mathrm{d}t} = ke - (\gamma+\mu)i \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \gamma i - \mu r \end{cases}$$

with:

$$1 = s + e + i + r\,.$$

The system has a unique disease-free equilibrium $(\,s_0,\,0,\,0,\,0)$ with $s_0 = \frac{\Pi}{\mu N}$. Taking the infected compartments to be E and I gives

$$\mathcal{F} = \begin{pmatrix} \beta si \\ 0 \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} (k+\mu)e \\ -ke + (\gamma+\mu)i \end{pmatrix}$$

$$F(\,s_0,\,0,\,0,\,0) = \begin{pmatrix} 0 & \beta s_0 \\ 0 & 0 \end{pmatrix} \qquad V(\,s_0,\,0,\,0,\,0) = \begin{pmatrix} k+\mu & 0 \\ -k & \gamma+\mu \end{pmatrix}$$

and the next generation matrix is

$$K = FV^{-1} = \begin{pmatrix} \dfrac{k\beta s_0}{(k+\mu)(\gamma+\mu)} & \dfrac{\beta s_0}{\gamma+\mu} \\ & \\ 0 & 0 \end{pmatrix} .$$

Notice that the $({}_1, {}_2)$ entry of K is the expected number of secondary infections produced in compartment E by an individual initially in compartment I over the course of its infection. To interpret this term, recall that βs_0 is the rate of infection for our single infected individual in a population of s_0 susceptible individuals, and $\frac{1}{\gamma+\mu}$ is the expected duration of the infectious period. The ratio $\frac{k}{\mu+k}$ is the fraction of individuals that progress from E to I. Hence, the $({}_1, {}_1)$ entry of K is the expected number of secondary infections produced in compartment E by an infected individual originally in compartment E.
From definition (2.7):

$$R_0 = \frac{k\beta s_0}{(k+\mu)(\gamma+\mu)} .$$

Chapter 3

Anorexia and bulimia: a mathematical model in the presence of media's influence

In this chapter we propose a mathematical model to study the dynamics of anorexic and bulimic populations presented in [29]. The model proposed takes into account, among other things, the effects of peers' influence, media influence, and education.

In section 3.2 we describe and formalize the model, that we denote $SABR$ because the compartments are: susceptible, anorexic, bulimic, and recovered. After a proof of positive invariance of the admissible region and of the existence of three equilibria in section 3.3, we consider at first, in section 3.4, the case in which the influence of media and education are neglected. In such case we analyze: existence and spectral stability of the equilibria, global stability of the disease-free equilibrium with a Lyapunov function, basic reproduction number and its sensitivity with respect to the parameters. In section 3.5 we finally discuss, partly analytically and partly numerically, the effect of education and media. We numerically prove the existence and the global stability of a unique endemic equilibrium.

3.1 Eating disorders

The prevalence of eating disorders has increased over the last 50 years and they have, recently, had a major impact on the physical and mental health of young women. Anorexia and bulimia are related to eating disorders. Both of these disorders revolve around the fear of obesity or obsessive desire to remain thin, and the biological necessity of consuming food.

In the United States, where statistics are generally complete and easy to access, 8 million people (90% of which are women, for this reason studies on eating disorders frequently look at women) suffer from eating disorders. Anorexia is suffered by 0.5% of women, 2 to 3% of women suffer of bulimia [156]. Statistics reveal that the situation is really alarming: in some EU countries 0.93% of woman older than 18 suffer from anorexia. In particular, in Italy eating disorders involve 3.3% of woman and man older than 18 (see [119]). To these numbers, however, we should add another 8% of individuals who don't show all the features which are essential for the diagnosis of anorexia or bulimia, but have sub-clinical forms of the diseases.

These disorders are very serious: anorexia nervosa is the third most common chronic illness in the United States [52]. In Australia, eating disorders are the seventh major cause of mental disorders, and treatment for anorexia nervosa represents the second highest cost to the private hospital field [91].

Although eating disorders are prevalent in western countries, recent studies have shown that the incidence of anorexia has risen sharply in Asian countries such as China and westernization is one of the causes of the development of eating disorders in Chinese population [74].

3.1.1 Harmful psychological influences and prevention

In [19] the authors investigate the meaning of body image and the role it plays during the adolescence. *Body image* is the internal representation of one's own outer appearance, which reflects physical and perceptual dimensions. In the age-range 10–15, 20% to 50% of girls in the United States say that they feel too fat [77] and 20% to 40% of girls feel overweight [134]. An important study has shown that 40% of adolescent girls believed that they were overweight, even though most of these girls fell in the normal weight range [112].

The family acts as a primary socialization agent by transmitting certain messages to adolescents, often differently according to gender [51]. Peers also are important in shaping body image and eating patterns. Girls who compare their appearance with that of their female peers have a greater risk of body dissatisfaction [82, 115, 125, 155].

Media's effect on adolescent girls is strikingly strong [60, 94]. Studies from the United States, Britain, and New Zealand offer evidence that increased media use, especially the number of hours per day spent watching television, is associated with greater BMI (Body Mass Index) and greater risk of obesity among children and adolescents [63, 90].

Media propose also an unrealistic ideal to be thin. In particular, investigators have explored the hypothesis that an increasingly thin standard of female beauty has led to increases in weight and shape anxiety, dieting, and disordered eating in girls and women. Investigators from a range of disciplines (e.g., anthropology, communications, history, philosophy, and psychology) have used a variety of methods to examine the relationship between media and how girls and women regard their bodies [11, 12, 71, 154]. Important works are those of the anthropologist Ann Becker [11, 12]. In her studies of Fijian girls' self perception during the three-year period in which western media were introduced to Fiji, Becker observed that dieting and disordered eating appeared in adolescent girls for the first time ever in Fijian culture. The influence of Western media in Fiji is particularly significant given that the thin ideal of beauty directly contradicts traditional Fijian norms. In another Australian qualitative study, girls associated the media's portrayal of the thin ideal with pressure to be thin [154].

Messages about body weight and appearance are now common also in the Internet. Although there are many sites that convey positive health messages to young people, several web sites contain health-related information that can be harmful, portraying disordered eating in a positive light. A very deep and interesting analysis of pro-eating disorder web sites can be found in [20] where they describe different kind of messages to which users are exposed. These sites characterize anorexia and bulimia as a lifestyle choice, not a clinical disease [19, 21, 56].

Frequent magazine readers, usually adolescent girls, also are more likely to engage in anorexic and bulimic behaviors, such as taking appetite control or weight-loss pills. Research suggests that several factors contribute to harmful attitudes and behaviours, but exposure and desire to resemble media ideals are significant factors that must be taken into consideration [38, 57].

For many patients suffering from severe anorexia nervosa hospitalization does not lead to full remission, since typically residual psychopathological features persist after weight-recovery [78]. In fact some Individuals achieve complete recovery while others are ravaged by a chronic disorder, and some die from it. Predicting course and outcome of anorexia nervosa is complicated by the intrinsic complexity of the disorder [53, 117].

Eating disorders research has moved toward attempted prevention. To prevent eating disorders, one needs to first understand what causes them and then to institute programs in order to mitigate those causes or to teach individuals how to deal with them. There are many educational campaigns to prevent eating disorders promoted by schools, colleges, social institutions and so on [131]. According to the National Institute of Mental Health it is important to increase the awareness that eating disorders are a public health problem and that prevention efforts are warranted [5, 136, 140], especially prevention at school [145]. Furthermore researches revealed significant reductions in disordered eating patterns and disturbed attitudes about eating and body shape, as well as significant increases in healthy eating patterns after a prevention program also in a high risk school setting [80, 135].

3.2 The SABR Model

The time evolution of anorexia and bulimia has been also analyzed in the context of epidemiological models (see for example [49, 67, 99]). In this work we focus on the spread of anorexia and bulimia nervosa and we investigate a mathematical model in which anorexia and bulimia depend not only on peer pressure (related to parameters β_1, β_2) but also on the influence of media (related to parameters m_1, m_2). This last factor has a strong influence on the evolution of the system. As it will become apparent from the dynamical equations (3.1), the influence of media causes the disease-free equilibrium to disappear. Recovery from these pathological conditions can be obtained through pharmacological therapy with antidepressants and with cognitive-behavioral therapy, which fosters the development of healty body images in order to prevent re-sensitization (i.e. to become susceptible once again). We model the effects of treatment using the parameters γ_1, γ_2 and of (re)sensitization using the parameter ν [69].

We also study the positive effect of a parameter related to education, that we call ξ. In this model we consider only the possibility that anorexic individual can become bulimic because the rate of bulimic individuals that become anorexic can be disregarded in a first approximation. The main scope of this research is to consider the positive effects of education and the negative effects of some media, to compute their influence on the reproduction numbers and the equilibria, and finally to investigate strategies to mitigate the effects of the disorders on population acting on such parameters.

Our model is inspired by an article of Gonzalez et al. [59], in which a general model is suggested for anorexia and bulimia considered as epidemics. In that article the authors restrict their attention to the spread of bulimia dividing the infected individuals in two classes and analyzing the existence of endemic equilibria. That article concludes the analysis fixing the parameters according to previous medical literature, and numerically investigating the evolution of simple and advanced bulimic depending on the net infective force. We extend this investigation to a model that describes both infective classes: anorexia and bulimia but considering only one group of individuals for each class. Our model includes several alternative routes of infection/recovery: peer pressure, media effect, education. In particular we divide the population into four classes: susceptible class, S, in which individuals are at risk of becoming anorexic or bulimic; anorexia class, A, in which an individual has the symptoms of anorexia; bulimia class, B, in which an individual has the symptoms of bulimia; and recovered/educated class, R, in which individuals have been taught healthy eating behaviors and body images. We are able to perform the investigation in a rigorous mathematical setting almost up to the general case, and we resort to a numerical investigation only at the very end, to prove with certainty the existence of a unique endemic equilibrium.

Let S, A, B, R denote respectively the number of susceptible individuals, the number of anorexics, the number of bulimics, and the number of recovered/educated individuals. The at-risk population S, can develop either anorexia A, or bulimia B because of contact with peers or the influence of media. Once anorexic, an individual may become bulimic. An anorexic or bulimic can recover from this condition, and move to the recovered class R. Once recovered, an individual may become again susceptible.

According to the Introduction, we consider also the case in which a susceptible becomes not sensitive to negative peer pressures and media influences thanks to an education campaign.

The model is described by Figure 3.1 assuming that the parameters appearing near the arrows are multiplied by the class from which the arrows go out, as proposed by Hethcote in [68]. This model is more appropriate to describe a homogeneous population (for instance young women in the age range 12-25), because such part of the population is primarily at risk. In fact females are more susceptible than males, and they enter the susceptible group as they enroll in Junior High School and begin frequenting other adolescents, while they leave the group by finding a job or creating a family of their own.

The parameters of the model, all non-negative constants, are:

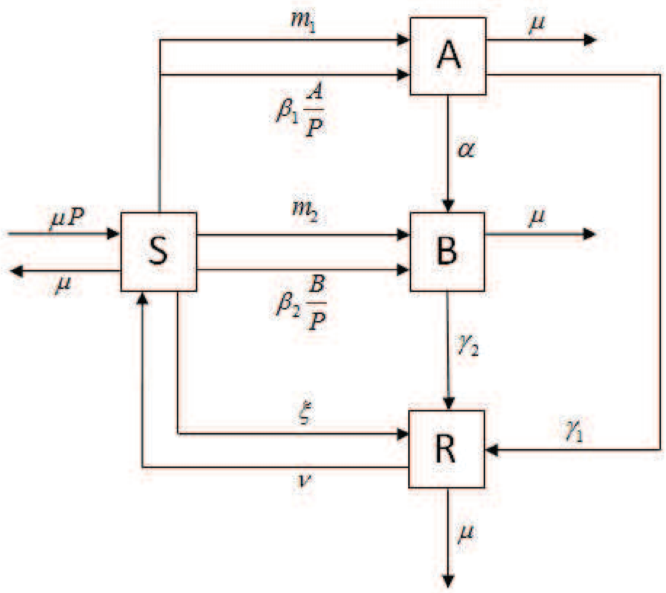

Figure 3.1: The SABR model that describes the spread of eating disorders in a community of susceptible (S), anorexic (A), bulimic (B) and recovered people (R). Arrows indicate the direction of movement into or out of a group.

- m_1: rate of individuals becoming anorexics due to media influences per unit time;
- m_2: rate of individuals becoming bulimics due to media influences per unit time;
- β_1: anorexics' peer-pressure contact rate per unit time;
- β_2: bulimics' peer-pressure contact rate per unit time;
- α: rate of anorexics that become bulimics per unit time;
- γ_1: rate of anorexics that recover for medicine or due to social campaigns per unit time;
- γ_2: rate of bulimics that recover for medicine or due to social campaigns per unit time;
- ξ: education rate per unit time;
- μ: entry and exit rates of the general population per unit time;
- ν: sensitization rate.

Observe that the class of recovered population R contains the individuals that have healthy body images and contains individuals that have never had eating disorders but are immune through education or because of a strong personality, and those who have had them but have been treated. The (re)sensitization rate ν we use in our model is an average of the sensitization rates of the two families.

We assume that the rate at which anorexia and bulimia spread depends on how often susceptible individuals meet people with eating disorders and how successful those encounters are in transmitting eating disorder habits, and how persuasive media are. These social factors are embedded in the recruitment rate as we noted above. The number of individuals who develop eating disorders depends on the relative sizes of the healthy and ill population. The probabilities of meeting an anorexic or a bulimic individual is proportional to the fraction of the two groups

A/P and B/P in the total population $P = S+A+B+R$, and such contacts can drive some individuals to the corresponding eating disorder. This fact (perhaps surprising given the visible devastation suffered by many anorexic) is well-documented in the literature [18, 31, 55, 118]. In the model we consider, the possibility that a bulimic becomes anorexic is disregarded because, according to the American Psychiatric Association, half of anorexic patients do develop bulimia, while only a few bulimic patients develop anorexia. This fact is in accordance with the introduction of [75] and with the mathematical model of [59].

To model the system we use the mathematical formulations of section 2.1

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} S \\ A \\ B \\ R \end{pmatrix}; \qquad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix};$$

where

$$\begin{aligned}
f_1 &= \mu P - (\mu + \xi + m_1 + m_2)S - (\beta_1 A + \beta_2 B)\,\frac{S}{P} + \nu R,\\
f_2 &= m_1 S + \beta_1 A \frac{S}{P} - (\mu + \alpha + \gamma_1)\,A,\\
f_3 &= m_2 S + \beta_2 B \frac{S}{P} + \alpha A - (\mu + \gamma_2)B,\\
f_4 &= \xi S + \gamma_1 A + \gamma_2 B - (\mu + \nu)R.
\end{aligned}$$

So, this model of spread of anorexia and bulimia can be cast mathematically as a set of the following four nonlinear ordinary differential equations that describe the changes in the populations S, A, B and R

over time

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \mu P - (\mu + \xi + m_1 + m_2)S - (\beta_1 A + \beta_2 B)\,\dfrac{S}{P} + \nu R \\ \dfrac{\mathrm{d}A}{\mathrm{d}t} = m_1 S + \beta_1 A \dfrac{S}{P} - (\mu + \alpha + \gamma_1)\,A \\ \dfrac{\mathrm{d}B}{\mathrm{d}t} = m_2 S + \beta_2 B \dfrac{S}{P} + \alpha A - (\mu + \gamma_2) B \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \xi S + \gamma_1 A + \gamma_2 B - (\mu + \nu) R\,. \end{cases} \tag{3.1}$$

As discussed above, we assume that the population under study is part of a larger population at demographic equilibrium, so that we can take it to be constant. It is hence natural to normalize the quantities by introducing the new variables to be constant, with equal entry and exit rates μ,

$$S = s \cdot P \quad A = a \cdot P \quad B = b \cdot P \quad R = r \cdot P\,,$$

obtaining the normalized model

$$\begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = \mu - (\mu + \xi + m_1 + m_2)\,s - (\beta_1\,a + \beta_2\,b)\,s + \nu\,r \\ \dfrac{\mathrm{d}a}{\mathrm{d}t} = m_1\,s + \beta_1\,a\,s - (\mu + \alpha + \gamma_1)\;a \\ \dfrac{\mathrm{d}b}{\mathrm{d}t} = m_2\,s + \beta_2\,b\,s + \alpha\,a - (\mu + \gamma_2)\,b \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \xi\,s + \gamma_1\,a + \gamma_2\,b - (\mu + \nu)\,r \end{cases} \tag{3.2}$$

subject to the constraint

$$1 = s + a + b + r. \tag{3.3}$$

We hence can use the integral of motion (4.1) and reduce the normalized system (3.2) to a system consisting of the three differential equations

$$\begin{cases} \dfrac{\mathrm{d}a}{\mathrm{d}t} = -\beta_1\, a^2 - \beta_1\, a\, b - \beta_1\, a\, r + (\beta_1 - m_1 - \mu - \alpha - \gamma_1)\, a - m_1\, b - m_1\, r + m_1 \\ \dfrac{\mathrm{d}b}{\mathrm{d}t} = -\beta_2\, b^2 - \beta_2\, a\, b - \beta_2\, b\, r + (\alpha - m_2)\, a + (\beta_2 - m_2 - \mu - \gamma_2)\, b - m_2\, r + m_2 \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = (\gamma_1 - \xi)\, a + (\gamma_2 - \xi)\, b - (\xi + \mu + \nu)\, r + \xi\,. \end{cases} \tag{3.4}$$

We observe that in this general case the *disease-free* equilibrium, i.e. the case in which the whole population belongs to the susceptibles or the removed, does not exist. In fact, if all the parameters are positive constants, to the choice $a = b = 0$ correspond a positive time derivative of the solutions $a(t), b(t)$.

The analysis of stability of equilibria for this system is complicated. In the following sections we consider some particular cases where this investigation becomes possible, and we use the results to draw conclusions for the general case. We start by considering m_1, m_2 and ξ equal to zero, we then consider the case $\xi > 0$ with $m_1, m_2 = 0$ and finally we investigate the effect of media ($m_1, m_2 > 0$) on the equilibria.

3.3 General properties of the model

3.3.1 Positive invariance of the unit tetrahedron

Representing percentages of a population, the three quantities a, b, r must be positive and have sum less than 1, i.e. must belong to the tetrahedron

$$\mathcal{T} = \{(a, b, r) \,|\, a + b + r \leq 1,\ a\ ,\ b\ , r\ > 0\}.$$

In this subsection we analyze the positive invariance of such tetrahedron, i.e. we show that any solutions starting inside that region can never leave it.

Positive invariance is equivalent to the fact that the vector field X whose associated O.D.E. is the system of equations (3.4) is always entering the faces of the tetrahedron, that is, its scalar product with the inner normal vector of the boundary of $\mathcal{T}$ is always positive. The tetrahedron is composed of 4 faces:

- the face of $\mathcal{T}$ lying in the b, r-plane has inner normal $(1, 0, 0)$, and its scalar product with X is $m_1(1-b-r)$. This function is positive on that face, in which precisely $b + r < 1$ (and $b, r > 0$);

- the face of $\mathcal{T}$ lying in the a, r-plane has inner normal vector $(0, 1, 0)$, and its scalar product with X is $a\,\alpha - a\,m_2 - m_2\,r + m_2$. Letting $h = \alpha/m_2$ the equation becomes $(1-h)a+r < 1$, and this equation is satisfied in a set that includes the face of $\mathcal{T}$;

- the face of $\mathcal{T}$ lying in the a, b-plane has inner normal vector $(0, 0, 1)$, and its scalar product with X is $a\,(\gamma_1-\xi)+b\,(\gamma_2-\xi)+\xi$. Denoting $h = \gamma_1/\xi$ and $k = \gamma_2\xi$ the equation becomes $a\,(1-h)+b\,(1-k) < 1$, and this equation is satisfied in a set that includes the face of $\mathcal{T}$;

- the face of $\mathcal{T}$ inside the first octant has equations $a+b+r = 1$ and inner normal $n = (-1, -1, -1)$. The scalar product $n \cdot X$ equals $\mu + \nu\,r$, that is positive on the face.

This proves the positive invariance of $\mathcal{T}$. We will prove that this system always *has three equilibria*, but only some of them belong to $\mathcal{T}$, and hence have meaning in this model. In the sequel we will say that an equilibrium *exists* or that it is *admissible* if it belongs to $\mathcal{T}$. The main goal of our treatment is to determine the existence and stability of admissible equilibria.

3.3.2 The equilibria

In this section we prove that the system always admits three equilibrium solutions. To be meaningful, an equilibrium must have coordinates a, b, r which are positive and such that $a + b + r \leq 1$.

From the first and third components of the vector field at the equilibrium one has that, posing $\chi = \mu + \nu$,

$$1 - r(a,b) = \frac{\beta_1 a^2 + \beta_1 ab + (m_1 + \mu + \alpha + \gamma_1)a + m_1 b}{\beta_1 a + m_1} \quad (3.5)$$

$$r(a,b) = \frac{a\,\gamma_1 + b\,\gamma_2 + \xi\,(1 - a - b)}{\chi + \xi} \quad (3.6)$$

(in our model we assume $\chi + \xi \neq 0$.) These two equations imply respectively that $r(a,b) < 1$ when a, b are positive and $r(a,b) > 0$ when a, b have a sum less than 1, hence the equilibrium point $(a,\ b,\ r(a,b))$ is always admissible whenever $a, b > 0$ and $a + b < 1$.

Substituting (3.6) in the first two components of the vector field and equating to zero, one obtains the two equations

$$(a\ \beta_1 + m_1\ )\,(c_1 + a\,(\gamma_1 + \chi) + b\ \ (\gamma_2 + \chi)) = c_2 \quad (3.7)$$

$$\left(-b\ \beta_2 - m_2 - \frac{\alpha(\chi + \xi)}{\gamma_1 + \chi}\right)(c_3 + a\ \ (\gamma_1 + \chi) + b\,(\gamma_2 + \chi)) = c_4 \quad (3.8)$$

with

$$c_1 = \frac{(\alpha + \gamma_1 + \mu)(\chi + \xi)}{\beta_1} - \chi, \qquad c_2 = m_1 \frac{(\alpha + \gamma_1 + \mu)(\chi + \xi)}{\beta_1},$$

$$c_3 = \frac{\alpha\,(\gamma_2 + \chi)\,(\chi + \xi)}{\beta_2\,(\gamma_1 + \chi)} + \frac{(\chi + \xi)\,(\gamma_2 + \mu)}{\beta_2} - \chi$$

and

$$c_4 = \frac{\chi + \xi}{\beta_2}\left(m_2\,(\gamma_2 + \mu) - \frac{\alpha^2\,(\gamma_2 + \chi)\,(\chi + \xi)}{(\gamma_1 + \chi)^{\,2}} - \frac{\alpha\,(\mu\xi + \chi\,(-\beta_2 + \gamma_2 + \mu - m_2) + \gamma_2\,(\xi - m_2))}{\gamma_1 + \chi}\right)$$

It is clear that, unless $\beta_1 = 0$ or $\beta_2 = 0$, such equations are those of two hyperbolas, and we are in the case in which two asymptotes are parallel, hence the two hyperbola intersect only in three points (see the

Appendix). When either β_1 or β_2 are zero, then some of the algebraic passages need a little more care, and one can see that one of the two hyperbolas degenerates to a line, and the intersection then consists of two points. If both $\beta_1 = \beta_2 = 0$ then the intersection is a unique point. We note that in our model we assume always $\beta_1 > 0$, $\beta_2 > 0$, so the system presents three equilibria.

The expression of the equilibria in the generic case is too cumbersome to write explicitly, and it is difficult to discuss existence and stability. We begin our investigation with the case in which the effect of media and education are absent (i.e. $m_1 = m_2 = \xi = 0$). We then discuss partly analytically and partly numerically what happens when these parameters move away from zero.

3.4 The simplified case: $m_1 = m_2 = \xi = 0$

Disregarding media and education coefficients m_1, m_2, ξ system (3.4) becomes

$$\begin{cases} \dfrac{\mathrm{d}a}{\mathrm{d}t} = -\beta_1\, a^2 - \beta_1\, a\, b - \beta_1\, a\, r + (\beta_1 - \mu - \alpha - \gamma_1)\, a \\ \dfrac{\mathrm{d}b}{\mathrm{d}t} = -\beta_2\, b^2 - \beta_2\, a\, b - \beta_2\, b\, r + \alpha\, a + (\beta_2 - \mu - \gamma_2)\, b \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \gamma_1\, a + \gamma_2\, b - \chi\, r. \end{cases} \tag{3.9}$$

We see immediately that the *disease-free* equilibrium $E_0 = (0, 0, 0)$ is a solution of (3.9).

To discuss the local stability of E_0, we consider the Jacobian matrix J_0 associated to system (3.9) in E_0. A simple computation gives

$$J_0 = \begin{bmatrix} \beta_1 \dfrac{R_a - 1}{R_a} & 0 & 0 \\ \alpha & \beta_2 \dfrac{R_b - 1}{R_b} & 0 \\ \gamma_1 & \gamma_2 & -\chi \end{bmatrix},$$

where we introduced the quantities

$$R_a = \frac{\beta_1}{\mu + \alpha + \gamma_1}, \qquad R_b = \frac{\beta_2}{\mu + \gamma_2}.$$

The eigenvalues of matrix J_0, since it is lower triangular, are given by the diagonal elements, and then local stability of E_0 is ensured by the conditions $R_a < 1$, $R_b < 1$. The quantity R_0 defined by

$$R_0 = \max\{R_a, R_b\}$$

guarantees that the disease free equilibrium is linearly stable for $R_0 < 1$, instead if $R_0 > 1$ the pathological behaviors will spread in the susceptible population. So R_0 is the *basic reproduction number*. We compared this result with the computation of R_0 based on the *the next generation operator approach* introduced by Diekmann et al. [49, 48] (a number of salient examples of this method are in [24, 66, 148]). We use the unreduced system (3.2), for which the disease-free equilibrium has $s = 1$ and $a = b = r = 0$. The disease-compartments are in this case a and b, so that, using the notations of the references above, we have

$$\mathcal{F} = \begin{pmatrix} \beta_1 a\, s \\ \\ \beta_2 b\, s \end{pmatrix} \qquad \mathcal{V} = \begin{pmatrix} (\mu + \alpha + \gamma_1) a \\ \\ -\alpha a + (\mu + \gamma_2) b \end{pmatrix}.$$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ on the *disease-free* equilibrium are

$$F(1, 0, 0, 0) = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix} \qquad V(1, 0, 0, 0) = \begin{pmatrix} \mu + \alpha + \gamma_1 & 0 \\ -\alpha & \mu + \gamma_2 \end{pmatrix},$$

and so the *next generation operator* is

$$FV^{-1} = \begin{pmatrix} R_a & 0 \\ \\ \frac{\alpha}{\beta_1} R_a R_b & R_b \end{pmatrix}$$

whose spectral radius is

$$R_0 = \max\{R_a, R_b\}$$

which coincides with our previous result. A simple analysis shows that R_0 is most sensitive to changes in the value of β_1 or β_2.

3.4.1 Global stability of E_0

We want to prove global stability of the equilibrium E_0 using the theory of Lyapunov functions [84]. Let us consider the two-parametric family of functions

$$V(a, b, r) = a + h\,b + k\,r$$

with h, k strictly positive reals. Note that $V(E_0) = 0$ and $V > 0$ for any $(a, b, r) \neq E_0$ in the positive octant. To prove global stability of the E_0 equilibrium we compute the orbital derivative of V, which is

$$\begin{aligned}\dot{V} = &- \beta_1(a + b + r)a + (\beta_1 - \mu - \alpha - \gamma_1)a \, - h\,\beta_2(b + a + r)b + h\,\alpha\,a + \\ &h\,(\beta_2 - \mu - \gamma_2)\,b \, + k(\gamma_1 a + \gamma_2 b) - k\,\chi\,r.\end{aligned}$$

From a linear stability analysis we know that E_0 is locally stable if

$$R_a < 1, \quad R_b < 1 \qquad \text{i.e.} \qquad \beta_1 - \mu - \alpha - \gamma_1 < 0, \quad \beta_2 - \mu - \gamma_2 < 0, \tag{3.10}$$

but these conditions do not guarantee in general $\dot{V} < 0$. We investigate then the effect of different values of h, k. By choosing $h = k = 1$, one obtains that

$$\dot{V} < (\beta_1 - \mu)\,a \, + (\beta_2 - \mu)\,b,$$

and global stability follows when β_1, $\beta_2 < \mu$. A stricter condition can be obtained observing that

$$\begin{aligned}\dot{V} &< (\beta_1 - \mu - \alpha - \gamma_1)a \, + h\,\alpha\,a + h\,(\beta_2 - \mu - \gamma_2)\,b \, + k(\gamma_1 a + \gamma_2 b) = \\ &= (\beta_1 - \mu - \alpha(1 - h) - \gamma_1(1 - k))\,a + h\left(\beta_2 - \mu - \left(1 - \frac{k}{h}\right)\gamma_2\right)b,\end{aligned}$$

and $\dot{V}$ is globally negative for

$$\beta_1 - \mu - \alpha(1 - h) - \gamma_1(1 - k) < 0, \qquad \beta_2 - \mu - \left(1 - \frac{k}{h}\right)\gamma_2 < 0.$$

Such conditions can be made arbitrarily close to the linear instability conditions (3.10) if we choose $0 < k \ll h \ll 1$. We have hence proved that when the equilibrium E_0 is spectrally stable, then it also is globally stable in the positive octant.

3.4.2 The endemic equilibria

In this section we calculate the endemic equilibria of the model and we introduce the quantities

$$\lambda_1 = \beta_1 \frac{R_a - 1}{R_a} \qquad \lambda_2 = \beta_2 \frac{R_b - 1}{R_b}$$

to simplify our computation, which imply immediately

$$R_a > 1 \Leftrightarrow \lambda_1 > 0 \qquad R_b > 1 \Leftrightarrow \lambda_2 > 0$$

So, we consider again system (3.9), and find the further two equilibria

- $E_1 = \left(0,\ \frac{\chi\lambda_2}{\beta_2(\chi+\gamma_2)},\ \frac{\gamma_2\lambda_2}{\beta_2(\chi+\gamma_2)}\right)$ the anorexia-free endemic equilibrium, where only bulimia is endemic, that we call for the sake of shortness *bulimic-endemic* equilibrium;
- $E_2 = \left(\chi\frac{\lambda_1\,\rho_1}{\beta_1\,\rho_2},\ \chi\frac{\alpha\lambda_1}{\rho_2},\ \gamma_1\frac{\lambda_1\,\rho_1}{\beta_1\,\rho_2} + \gamma_2\frac{\alpha\lambda_1}{\rho_2}\right)$, the *endemic* equilibrium;

where

$$\rho_1 = \beta_1\beta_2\frac{R_a - R_b}{R_a R_b} = \lambda_1\beta_2 - \lambda_2\beta_1$$
$$\rho_2 = \rho_1(\chi+\gamma_1) + \alpha\beta_1(\chi+\gamma_2).$$

Remark 3.4.1 *When the coordinates of an equilibrium are positive, then their sum is less than or equal to one. In particular the sums of the coordinates of E_1, E_2 are λ_2/β_2 and λ_1/β_1 respectively.*

Let us now investigate existence and stability of E_1 and E_2 relative to the stability conditions of E_0.

The expression of E_1 shows that this equilibrium exists only when $R_b > 1$ (i.e. $\lambda_2 > 0$) and hence when E_0 becomes unstable. The stability of E_1 can be determined by evaluating the Jacobian matrix J_1 in E_1

$$J_1 = \begin{bmatrix} \dfrac{\rho_1}{\beta_2} & 0 & 0 \\ \alpha - \dfrac{\chi\lambda_2}{\chi+\gamma_2} & -\dfrac{\chi\lambda_2}{\chi+\gamma_2} & -\dfrac{\chi\lambda_2}{\chi+\gamma_2} \\ \gamma_1 & \gamma_2 & -\chi \end{bmatrix} .$$

One eigenvalue is ρ_1/β_2 and sum and product of the other two eigenvalues are respectively

$$-\chi - \frac{\chi\lambda_2}{\chi+\gamma_2}, \qquad \chi\lambda_2 \, .$$

Since E_1 exists for $\lambda_2 > 0$, these eigenvalues are negative and stability of E_1 depends entirely on the sign of ρ_1. Then E_1 is spectrally stable only if $\rho_1 < 0$ and then $R_a < R_b$.

The expression of E_2 shows that this equilibrium exists if $\rho_1 > 0$ (i.e. $R_a > R_b$) and $\lambda_1/\rho_2 > 0$. Since $\rho_1 > 0$ implies $\rho_2 > 0$, these conditions turn out to be $R_a > R_b$ and $R_a > 1$.

The Jacobian matrix J_2 associated to the equilibrium E_2 is

$$\begin{pmatrix} -\dfrac{\chi\lambda_1\rho_1}{\rho_2} & -\dfrac{\chi\lambda_1\rho_1}{\rho_2} & -\dfrac{\chi\lambda_1\rho_1}{\rho_2} \\ \alpha - \dfrac{\alpha\chi\beta_2\lambda_1}{\rho_2} & -\dfrac{\rho_1}{\beta_1} - \dfrac{\alpha\chi\beta_2\lambda_1}{\rho_2} & -\dfrac{\alpha\chi\beta_2\lambda_1}{\rho_2} \\ \gamma_1 & \gamma_2 & -\chi \end{pmatrix} .$$

The characteristic polynomial of J_2 is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

where

$$a_2 = -\mathrm{tr}(A) = \frac{\chi\,\lambda_1\rho_1}{\rho_2} + \frac{\rho_1}{\beta_1} + \frac{\alpha\,\chi\,\beta_2\lambda_1}{\rho_2} + \chi$$
$$a_1 = \frac{\chi\,\left(\lambda_1\beta_1\gamma_1\rho_1 + \lambda_1\beta_1\gamma_2\alpha\,\beta_2 + \rho_1\rho_2 + \alpha\,\chi\,\beta_2\lambda_1\beta_1 + \chi\,\lambda_1\rho_1\beta_1 + \lambda_1\rho_1\alpha\,\beta_1 + \lambda_1{\rho_1}^2\right)}{\beta_1\rho_2}$$
$$a_0 = -det(A) = \frac{\chi\,\lambda_1\rho_1}{\beta_1}\,.$$

We notice that these coefficients are all positive. To prove the local stability of E_2 by the Routh-Hurwitz criterion we should also prove that

$$a_2\,a_1 - a_0 \tag{3.11}$$

is positive. This condition is difficult to be proved analytically because of the many parameters. However numerical sampling of expression (3.11) in the space of parameters and a numerical minimization of (3.11) show that when E_2 exists it is also locally stable. We can summarize these results in Table 3.1.

	E_0	E_1	E_2
$R_a < 1, R_b < 1$	stable	does not exist	does not exist
$R_a > 1;\ R_b < 1$	unstable	does not exist	stable(1)
$R_a < 1;\ R_b > 1$	unstable	stable	does not exist
$R_a > 1;\ R_b > 1,\ R_a < R_b$	unstable	stable	does not exist
$R_a > 1;\ R_b > 1,\ R_a > R_b$	unstable	unstable	stable(1)

Table 3.1: the scheme of equilibrium points and their stability for $m_1 = m_2 = \xi = 0$. (1) The stability of E_2 is proved only numerically.

It is instructive to analyze the existence of the equilibrium E_1 in the plane β_1, β_2 when the other parameters are fixed. The geometry of such region is always qualitatively the same: the half-plane $R_b > 1$ (see Figure 3.2).

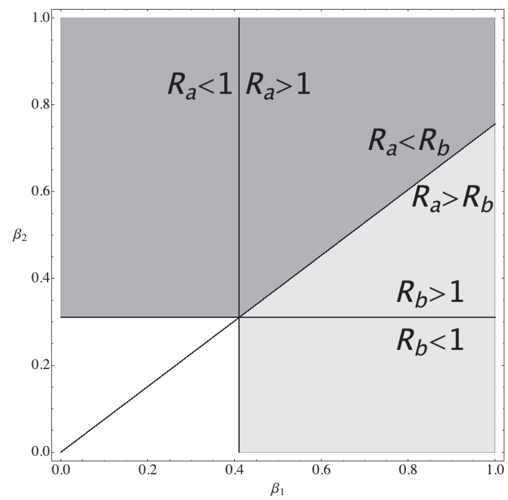

Figure 3.2: The equilibria E_1 and E_2 are stable for choices of parameters in the dark and light shaded regions respectively. The regions are plotted in β_1, β_2-space, and they are qualitatively the same for every choice of the other parameters $\alpha, \gamma_1, \gamma_2, \mu, \nu$. In this particular case we have chosen $\alpha = 0.3, \gamma_1 = 0.1, \gamma_2 = 0.3, \mu = 0.01, \nu = 0.1$. The disease-free equilibrium E_0 is stable only when the parameters belong to the unshaded region, that is when $R_a < 1$ and $R_b < 1$.

3.5 Case with influences of education and media

3.5.1 Case with $\xi > 0$ and $m_1 = m_2 = 0$

Let us now consider the effect of positive values of the education coefficient ξ. System (3.4) becomes

$$\begin{cases} \dfrac{\mathrm{d}a}{\mathrm{d}t} = -\beta_1\, a^2 - \beta_1\, a\, b - \beta_1\, a\, r + (\beta_1 - \mu - \alpha - \gamma_1)\, a \\ \dfrac{\mathrm{d}b}{\mathrm{d}t} = -\beta_2\, b^2 - \beta_2\, a\, b - \beta_2\, b\, r + \alpha\, a + (\beta_2 - \mu - \gamma_2)\, b \\ \dfrac{\mathrm{d}r}{\mathrm{d}t} = \gamma_1\, a + \gamma_2\, b - \chi\, r + \xi\, (1 - a - b - r). \end{cases} \tag{3.12}$$

In this case the disease-free equilibrium is

$$E_0^{'} = \left(0,\, 0,\, \frac{\xi}{\xi + \chi}\right),$$

which is always admissible. Moreover, we note that in this state the number of susceptibles is $s = \chi/(\xi + \chi)$.
To discuss the stability of E_0', we proceed as in section 4.4.
The Jacobian matrix J_0' associated to system (3.12) at E_0' is

$$J_0' = \begin{bmatrix} \lambda_1' & 0 & 0 \\ \alpha & \lambda_2' & 0 \\ \gamma_1 - \xi & \gamma_2 - \xi & -\xi - \chi \end{bmatrix}$$

with

$$\lambda_1' = \beta_1 \frac{R_a' - 1}{R_a}\,, \quad \lambda_2' = \beta_2 \frac{R_b' - 1}{R_b}\,.$$

In these expressions we introduce the new reproduction numbers

$$R_a' = R_a \frac{\chi}{\xi + \chi}\,, \quad R_b' = R_b \frac{\chi}{\xi + \chi}\,,$$

which include the fraction $\chi/(\xi + \chi)$ of the population susceptible to eating disorders in the disease-free state when an education campaign is considered. The eigenvalues of J_0' are λ_1', λ_2' and $-(\xi+\chi)$, so the stability is guaranteed by the conditions $R_a' < 1, R_b' < 1$.
We checked these results with the *next generation operator approach*, obtaining

$$FV^{-1} = \begin{bmatrix} R_a' & 0 \\ \dfrac{\alpha}{\beta_1} R_a' R_b & R_b' \end{bmatrix}.$$

So, $R_0' = \max(R_a', R_b')$ is the *control* reproductive number.
It is straightforward that λ_1', λ_2', R_a', R_b' are strictly decreasing functions of ξ. It follows that, as expected, ξ has a *stabilizing effect* on the disease-free equilibrium.
Also in this case we have a *bulimic-endemic* equilibrium and an *endemic* equilibrium, but their explicit expressions and the study of their stability are mathematically cumbersome so we will not report them here.

3.5.2 General case

In this section we describe what happens when the parameters m_1, m_2 become positive. It is possible to show, partly analytically and partly numerically that in this general case *there is always only one endemic equilibrium in the unit tetrahedron.*

An increase in m_1, m_2 will increase the percentage of anorexic and bulimic, but which of the three possible equilibria E_0', E_1', E_2' described in the case without the influence of media will become the endemic equilibrium depends on the other parameters.

When $m_1 = 0$ and $m_2 > 0$ we denote by E_0'', E_1'', E_2'' the prolongation of the equilibria E_0', E_1', E_2'. It can be proved that E_0'', E_1'' are bulimic-endemic but remain anorexic-free. Their analytic expression is

$$\left(0, -\frac{1}{2\beta_2(\gamma_2+\chi)}\left(G \pm \sqrt{G^2 + 4\beta_2 m_2 \chi (\gamma_2 + \chi)}\right), *\right) \tag{3.13}$$

where $G = (\gamma_2 + \mu)\xi + (\gamma_2 + \chi)m_2 - \lambda_2\chi$ and r is not explicitly written. Which of the two expressions (with plus or minus) is the prolongation of

E_0' and which of E_1' is possible to say only once the parameters are fixed, and depends on the sign of G. From this expression one can analytically prove that

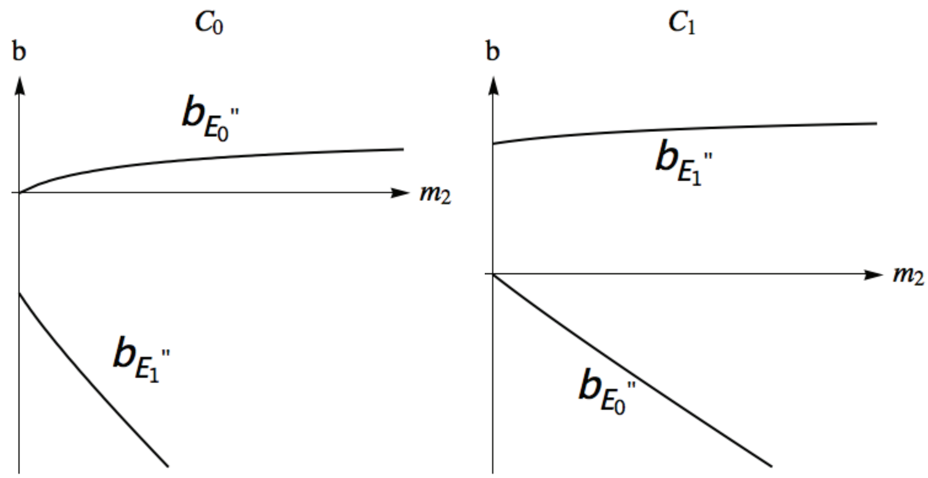

Figure 3.3: The evolution of the b-component of the equilibria E_0'', E_1'' as m_2 becomes positive while $m_1 = 0$. The left panel corresponds to a choice of parameters for which E_1' does not exist when $m_2 = 0$ (that is $\lambda_2 < 0$), and shows that the disease-free equilibrium becomes bulimic-endemic. The right panel corresponds to a choice of parameters for which E_1' does exist when $m_2 = 0$ (that is $\lambda_2 > 0$) and shows that the disease-free equilibrium exits from the admissible region while E_1'' remains admissible.

Proposition 3.5.1 *As soon as m_2 is increased from zero then only one of the two anorexic-free equilibria (the disease-free E_0' and the bulimic-endemic E_1') will be in the unit tetrahedron, while the other will move out of the unit tetrahedron $\mathcal{T}$. So there are two possible cases:*

- C_0 *the bulimic-endemic equilibrium E_1' does not belong to the tetrahedron when $m_2 = 0$, then also when $m_2 > 0$ its prolongation E_1'' does not belong to the tetrahedron while E_0'' becomes bulimic-endemic.*

C_1 *The bulimic-endemic equilibrium E_1' does belong to the tetrahedron when $m_2 = 0$, then also when $m_2 > 0$ the equilibrium E_1'' does belong to the tetrahedron and is anorexic-free and bulimic-endemic, while E_0'' moves out of the unit tetrahedron.*

From now on we call this bulimic-endemic and anorexic-free equilibrium E_{01}'' (the subscript $_{01}$ indicates that the prolongation of either E_0' or E_1' play the role of such equilibrium, and we cannot know a-priori which of the two will be). The two possibile events described in the Proposition above are summarized in Figure 3.3.

As we discussed above, when $m_1 = 0$, there still exist two anorexic-free equilibria (i.e. with $a = 0$) that we denote E_0'', E_1''. The value of their b-component is written in formula (3.13).

Regardless the sign of G, one of the two components becomes negative while the other becomes bulimic-endemic (i.e. with b positive) and anorexic-free (i.e. with $a = 0$). Which of the two depends on the sign of G. There are hence two possibilities described in the result above. The plots of the two possible scenarios is depicted, for two different choices of all the parameters except m_2 (and with m_1 fixed to zero) in Figure 3.3 left for the case C_0 and in Figure 3.3 right for the case C_1.

When $m_2 > 0$ and also m_1 is increased from zero, the coordinates of the equilibria do not have simple analytical expression. They are the roots of a cubic polynomial in a whose coefficients depend on the parameters and hence can be obtained using Cardano's formula. Not only the investigation of their positivity is extremely difficult, but it is also complicate to decide which of the three expressions tend to E_0'', E_1'', E_2'' respectively when m_1 tends to zero. We outline the evolution of the equilibria as m_1 grows away from zero by resorting to the numerical analysis plotted in Figure 3.4. Also in this case there are two possibilities

Remark 3.5.1 *If $m_1 \neq 0, m_2 \neq 0$ we denote by E_0''', E_1''', E_2''' the three equilibria that tend respectively to E_0'', E_1'', E_2'' as m_1 tends to zero. Only one of such solutions lies in the interior of the unit tetrahedron $\mathcal{T}$, giving a system with precisely one endemic equilibrium. There are two possibilities*

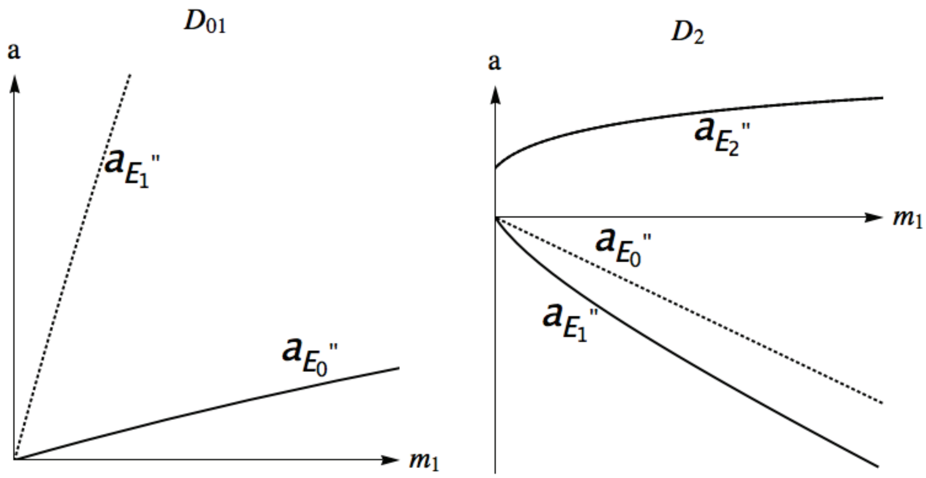

Figure 3.4: The evolution of the a-component of the equilibria E_0''', E_1''', E_2''' as m_1 becomes positive. The solid lines represent equilibria whose b-component is positive, the dotted lines to equilibria whose b-component is negative. The left panel is associated to a choice of parameters for which E_2'' does not exist, the right panel to a choice for which E_2'' does exist when m_1 is set to zero.

D_{01} *The endemic equilibrium E_2'' does not belong to the tetrahedron when $m_1 = 0$, then also when $m_1 > 0$ the equilibrium E_2''' does not belong to the tetrahedron while E_{01}'' becomes endemic (i.e. either E_0''' or E_1''' moves in the interior of the unit tetrahedron).*

D_2 *The endemic equilibrium E_2'' does belong to the the unit tetrahedron when $m_1 = 0$, then also when $m_1 > 0$ the equilibrium E_2''' belongs to the tetrahedron and remains endemic. In this case E_0''' and E_1''' move out of the unit tetrahedron (one of them already did not belong to such tetrahedron already when $m_1 = 0$).*

In this case, the two possible scenarios can be proven only numerically. In the left panel of Figure 3.4 we plot the case D_{01} under the hypothesis that C_0 is verfied. In the a-axis there are only solutions with

$a = 0$. Such solutions are the two E_0'' and E_1''. Increasing m_1 the two a-components of those solutions become positive, but one of them has negative b-component, while the other has positive b-component and is hence admissible, i.e. belongs to the unit tetrahedron.

In the right panel of Figure 3.4 we plot the case D_2. In the a-axis there are two solutions with $a = 0$ (E_0'' and E_1'') and one with positive a (the endemic equilibrium E_2''). Increasing m_1 the two a-components of E_0'', E_1'' become negative, and these solutions are hence non-admissible. One of them also has negative b-component, while the other has positive b-component. On the other hand, the equilibrium E_2''' has a-component which increases with m_1, and remains an endemic solutions with higher percentage of anorexics when m_1 becomes larger.

3.5.3 Numerical illustration

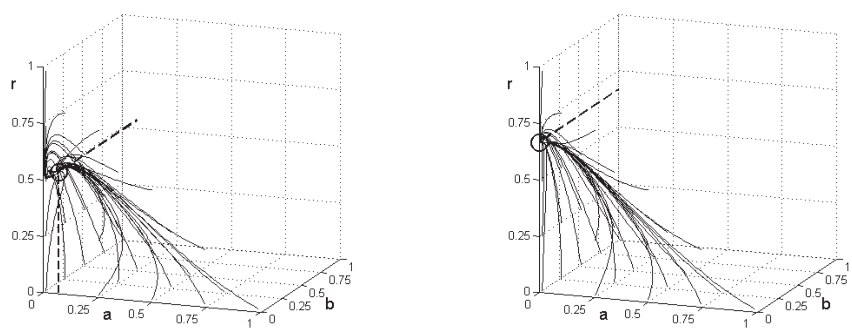

Figure 3.5: Orbits of system (3.12) starting from a regular grid of points inside the unit tetrahedron and converging to the equilibrium point indicated by a small circle. Dashed lines show the projection of the equilibrium point on the coordinate planes. In the left panel we show a global endemic equilibrium E_2', with $\xi = 0.05$. In the right panel we change only ξ to 0.1 obtaining a global disease-free equilibrium.

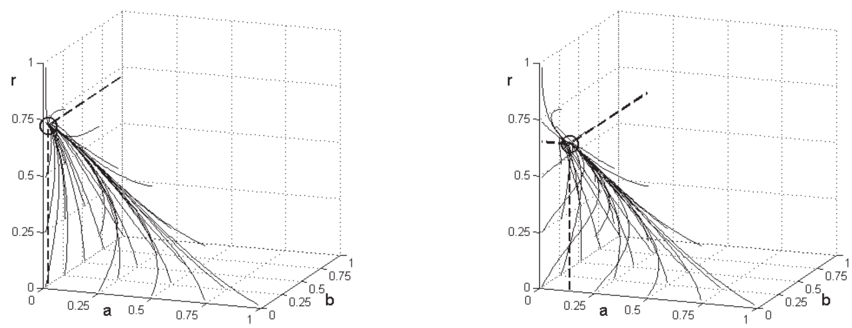

Figure 3.6: Orbits of system (3.4) starting from a regular grid of points inside the unit tetrahedron with a different choice of parameters. In the left panel we show the effect of media pressure on bulimic population ($m_2 = 0.05$, $\xi = 0.1$). In the right panel we show the effect of media on anorexic population ($m_1 = 0.05$, $\xi = 0.1$).

In this section we examine numerically the competing effect of the education factor ξ and the media influence on the onset of anorexia m_1 and bulimia m_2. We consider the initial set of parameters $\beta_1 = 0.4$, $\beta_2 = 0.3$, $m_1 = m_2 = 0$, $\alpha = 0.05$, $\gamma_1 = 0.05$, $\gamma_2 = 0.2$, $\mu = 0.05$, $\nu = \xi = 0$. With this choice of parameters we have $R_a = 2.67 > R_b = 1.2 > 1$, and, as expected, the only endemic equilibrium $E_2 = (0.16, 0.06, 0.40)$ (see Table 3.1).

Introducing the education factor $\xi = 0.05$ the new reproductive numbers are $R'_a = 1.3, R'_b = 0.6$. The endemic equilibrium is $E'_2 = (0.07, 0.02, 0.54)$, which, by numerical evidence, is still globally stable (see Figure 3.5-left). What we see is that both the anorexic and bulimic population have noticeably shrunk but they are still present.

A further increase of ξ to $\xi = 0.1$ has the effect of making $R'_a = 0.89$ and $R'_b = 0.4$ both less than 1. The numerically globally stable equilibrium is in this case the disease free state $E'_0 = (0, 0, 0.67)$ (see Figure 3.5-right). The effect is exactly what we expected from a strong educational campaign.

We introduce now the negative influences of media on bulimia, by setting $m_2 = 0.05$. Note that we have still a strong educational effect from $\xi = 0.1$, but we know that a state with no bulimics can no longer be an equilibrium, since now $\mathrm{d}b/\mathrm{d}t \neq 0$ even for $b = 0$. The new bulimic-endemic equilibrium $E_1'' = (0, 0.06, 0.71)$ is shown in Figure 3.6-left

As expected, a worse effect derives from a promotion of anorexic behaviour, modeled in our numerical computation by $m_1 = 0.05, m_2 = 0$. The new equilibrium is $(0.13, 0.03, 0.65)$. Even in this case there is numerical evidence that such state is globally stable (see Figure 3.6-right).

Remark 3.5.2 *What are the reasonable values of some of the coefficients is a complicated question. Any conjecture should be tested with the help of the medical community. The values of the rates β_1, β_2, and their related unit of time is highly sensitive to the particular environment in which the recruited population live. For instance such rates can be profoundly higher when dealing with a high-risk environment such as a ballet school [135]. We have not found explicit values of these parameters in literature.*

Chapter 4

Financial Models and Mathematical Formulations

This chapter offers a comprehensive analysis of dynamic networks and evolutionary variational inequalities applied to a financial network and an introduction to the quasi-variational inequalities.

4.1 The Financial Model

In 1992 Nagurney, Dong and Hughes [104] were the first to develop a multi-sector, multi-instrument financial equilibrium problem using the theory of variational inequalities and recognized the network structure underlying the problem. That contribution was subsequently extended by Nagurney [100] to include more general utility functions and by Nagurney and Siokos [107], [108] who formulated a dynamic financial equilibrium model and analyzed it qualitatively using the theory of projected dynamical systems. Many other dynamic financial models, along with their variational inequality formulations at the equilibrium state, can be found in the book by Nagurney and Siokos [107].
We notice that Daniele [36] proposed an alternative approach to the one described above, indeed the dynamics are now modeled not using the theory of projected dynamical systems [109], but studied by means of

evolutionary variational inequalities and these are infinite - rather than finite-dimensional. In addition, the variance-covariance matrices (see also [88], [89]) which allow us for risk minimization are now time-varying as well as the financial volumes held by the sectors.
Infinite-dimensional variational inequalities have been used for many purposes in finance by Jaillet, Lamberton and Lapeyre [73] for the pricing of American options, and by Tourin and Zariphopoulou [147] for single-agent investment modelling and computation. Stochastic variational inequalities, in turn, have been used by McLean [93] for the non-linear portfolio choice problem and by Gurkan, Ozge and Robinson [62] for the pricing of American options. For additional background on financial problems and variational inequalities see [101].
The papers by Daniele and Maugeri [37] and by Daniele, Maugeri and Oettli [41], [42] discuss other time-dependent applications using the approach revealed in [36] for the first time for financial equilibrium problems.
We start by introducing a first general evolutionary model for the formulation and analysis of multi-sector, multi-instrument financial equilibrium problems, proposed in [33], which will be improved later on. The functional setting is the Lebesgue space $L^2([0,T],\mathbb{R}^p)$. The time dependence of the model in the $L^2([0,T])$ space allows the model to follow the financial behavior, even in the presence of a possibly very irregular evolution, whereas the equilibrium conditions are required to hold almost everywhere (see [37], [41], [42] for analogous problems). The variance-covariance matrices associated with the sectors' risk perceptions will be required to have $L^\infty([0,T])$-entries.
Analitically, consider a financial economy consisting of m sectors, with a typical sector denoted by i, and of n instruments, with a typical financial instrument denoted by j, in the time horizon $[0,T]$.
Examples of sectors include domestic businesses, banks and other financial institutions, as well as state and local governments. Examples of financial instruments, in turn, are mortgages, mutual funds, savings deposits, money markets funds, etc.
Let $s_i(t)$ denote the total financial volume held by sector i at time t,

which is considered to depend on time $t \in [0, T]$. As in the presence of uncertainty and of risky perspectives, the volume s_i held by each sector cannot be considered stable and may decrease or increase depending on unfavorable or favorable economic conditions. As a consequence, the amounts of the assets and of the liabilities of the sectors will depend on time.
For this reason, at time t, denote the amount of instrument j held as an asset in sector i's portfolio by $x_{ij}(t)$ and the amount of instrument j held as a liability in sector i's portfolio by $y_{ij}(t)$. The assets in sector i's portfolio are grouped into the column vector $x_i(t) = [x_{i1}(t), x_{i2}(t), ..., x_{ij}(t), ..., x_{in}(t)]^T$ and the liabilities in sector i's portfolio are grouped into the column vector $y_i(t) = [y_{i1}(t), y_{i2}(t), ..., y_{ij}(t), ..., y_{in}(t)]^T$. Moreover, group the sector asset vectors into the matrix

$$x(t) \in L^2([0, T], \mathbb{R}^{nm})$$

i.e.

$$x(t) = \begin{bmatrix} x_1(t) \\ \dots \\ x_i(t) \\ \dots \\ x_m(t) \end{bmatrix} = \begin{bmatrix} x_{11}(t) & \dots & x_{1j}(t) & \dots x_{1n}(t) \\ \dots & & & \\ x_{i1}(t) & \dots & x_{ij}(t) & \dots \quad x_{in}(t) \\ \dots & & & \\ x_{m1}(t) & \dots & x_{mj}(t) & \dots \quad x_{mn}(t) \end{bmatrix}$$

and the sector liability vectors into the matrix

$$y(t) \in L^2([0, T], \mathbb{R}^{nm})$$

i.e.

$$y(t) = \begin{bmatrix} y_1(t) \\ \dots \\ y_i(t) \\ \dots \\ y_m(t) \end{bmatrix} = \begin{bmatrix} y_{11}(t) & \dots & y_{1j}(t) & \dots y_{1n}(t) \\ \dots & & & \\ y_{i1}(t) & \dots & y_{ij}(t) & \dots \quad y_{in}(t) \\ \dots & & & \\ y_{m1}(t) & \dots & y_{mj}(t) & \dots \quad y_{mn}(t) \end{bmatrix}.$$

In order to determine for each sector i the optimal composition of instruments held as assets and as liabilities, first we consider the influence due to the risk-aversion. Following the concept that assessment of risk is based on a variance-covariance matrix denoting the sector's assesment of the standard deviation of prices for each instrument, we use as a measure of this aversion the $2n \times 2n$ variance-covariance matrix.

$$Q^i(t) \;=\; \left[\begin{array}{c} Q^i_{11}(t) Q^i_{12}(t) \\ Q^i_{21}(t) Q^i_{22}(t) \end{array}\right]$$

associated with sector i's assets and liabilities, which, in general, will evolve in time as well and which we assume to be symmetric and positive definite and with $L^\infty([0,T])$ entries. Further, denote by $[Q^i_{\alpha\beta}(t)]_j$ the j-th column of $[Q^i_{\alpha\beta}(t)]$ where $\alpha = 1,2$ and $\beta = 1,2$. Then the aversion to the risk at time $t \in [0,T]$ is given by:

$$\left[\begin{array}{c} x_i(t) \\ y_i(t) \end{array}\right]^T Q_i(t) \left[\begin{array}{c} x_i(t) \\ y_i(t) \end{array}\right] .$$

The second component that we have to consider in the process of optimization of each sector in the financial economy is the desire to maximize the value of its asset holdings and to minimize the value of its liabilities. These objectives are related to the prices of each instrument, which, in turn, depend on time and appear as variables in our problem. We denote the price of instrument j at time t by $r_j(t)$ and group the instrument prices into the vector $r(t) = [r_1(t), r_2(t), ..., r_i(t), ..., r_n(t)]^T$.
Assuming as the functional setting the Lebesgue space $L^2([0,T], \mathbb{R}^p)$, the set of feasible assets and liabilities becomes:

$$P_i \;=\; \Big\{[x_i(t), y_i(t)]^T \in L^2([0,T], \mathbb{R}^{2n}) :$$

$$\sum_{j=1}^n x_{ij}(t) = s_i(t),\; \sum_{j=1}^n y_{ij}(t) = s_i(t) \text{ a.e. in } [0,T],$$

$$x_{ij}(t) \geq 0,\; y_{ij}(t) \geq 0, \text{ a.e. in}[0,T]\Big\} .$$

In Figure 1 we depict the network structure associated with the above feasible set and the financial economy out of equilibrium. The set of feasible assets and liabilities associated with each sector corresponds to budget constraints.
We now can give the following definition of an equilibrium of the financial model.

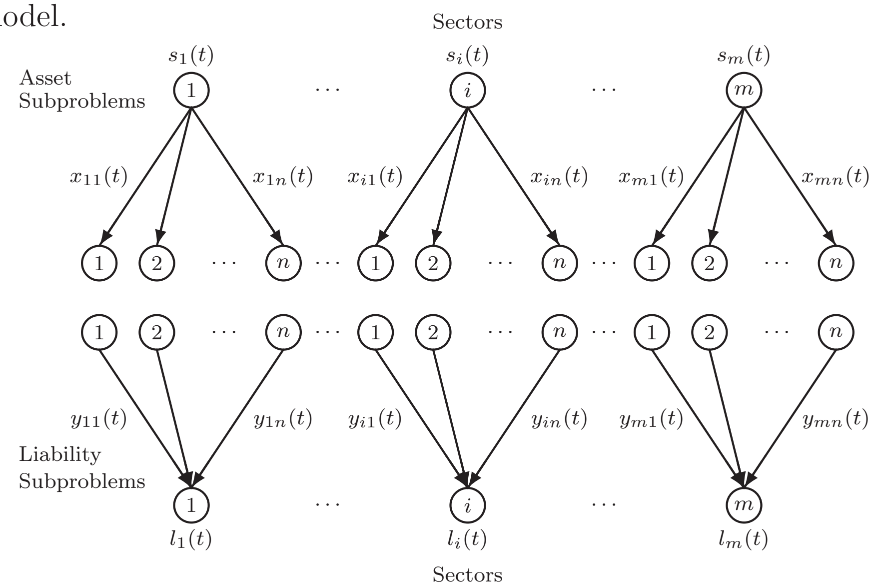

Figure 4.1: Network structure of the sectors' optimization problems a.e. in $[0, T]$.

Definition 4.1.1 *A vector of sector assets, liabilities, and instrument prices* $(x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^m P_i \times L^2([0,T], \mathbb{R}^n_+)$ *is an equilibrium of the evolutionary financial model if and only if it satisfies simultaneously the system of inequalities*

$$2[Q_{11}^i(t)]_j^T x_i^*(t) \,+\, 2[Q_{21}^i(t)]_j^T y_i^*(t) \,-\, r_j^*(t) \,-\, \mu_i^{(1)}(t) \geq 0, \qquad (4.1)$$

and

$$2[Q_{12}^i(t)]_j^T x_i^*(t) \,+\, 2[Q_{22}^i(t)]_j^T y_i^*(t) \,+\, r_j^*(t) \,-\, \mu_i^{(2)}(t) \geq 0, \qquad (4.2)$$

and equalities

$$x_{ij}^*(t)\left[2[Q_{11}^i(t)]_j^T x_i^*(t) + 2[Q_{21}^i(t)]_j^T y_i^*(t) - r_j^*(t) - \mu_i^{(1)}(t)\right] = 0, \quad (4.3)$$

$$y_{ij}^*(t)\left[2[Q_{12}^i(t)]_j^T x_i^*(t) + 2[Q_{22}^i(t)]_j^T y_i^*(t) + r_j^*(t) - \mu_i^{(2)}(t)\right] = 0, \quad (4.4)$$

where $\mu_i^1(t)$*,* $\mu_i^2(t) \in L^2([0,T])$ *are Lagrangean functionals, for all sectors* $i : i = 1, 2, ..., m$*, and for all instruments* $j : j = 1, 2, ..., n$*, and the system*

$$\begin{cases} \displaystyle\sum_{i=1}^m (x_{ij}^*(t) - y_{ij}^*(t)) \geq 0, & \text{a.e. in } [0,T] \\ \displaystyle\sum_{i=1}^m (x_{ij}^*(t) - y_{ij}^*(t)) r_j^*(t) = 0, & r^*(t) \in L^2([0,T], \mathbb{R}_+^n)\,. \end{cases} \quad (4.5)$$

If we consider a group of conditions (4.1) – (4.4) for a fixed $r(t)$, then we realize that they are necessary and sufficient conditions to ensure that $(x^*(t), y^*(t))$ is the minimum of the problem:

$$\min_{P_i} \int_0^T \left\{ \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} - r(t) \times [x_i(t) - y_i(t)] \right\} dt, \quad (4.6)$$

$$\forall \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \in \prod_{1=1}^m P_i$$

Equilibrium conditions (4.1) – (4.5) are characterized by the following variational inequality.

Theorem 4.1.1 (Variational Inequality Formulation)
A vector of sector assets, liabilities and instrument prices

$$(x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^m P_i \times L^2([0,T], \mathbb{R}_+^n)$$

is an evolutionary financial equilibrium if and only if it satisfies the following variational inequality:

$$\text{Find } (x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^{m} P_i \times L^2([0,T], \mathbb{R}^n_+)$$

$$\int_0^T \left\{ \sum_{i=1}^{m} \left[2[Q_{11}^i(t)]^T x_i^*(t) + 2[Q_{21}^i(t)]^T y_i^*(t) - r^*(t) \right] \times [x_i(t) - x_i^*(t)] \right.$$
$$+ \sum_{i=1}^{m} \left[2[Q_{12}^i(t)]^T x_i^*(t) \; + \; 2[Q_{22}^i(t)]^T y_i^*(t) \; + \; r^*(t) \right] \times [y_i(t) - y_i^*(t)] \quad (4.7)$$
$$\left. + \sum_{i=1}^{m} (x_i^*(t) - y_i^*(t)) \times [r(t) - r^*(t)] \right\} dt \geq 0,$$
$$\forall (x(t), y(t), r(t)) \in \prod_{i=1}^{m} P_i \times L^2([0,T], \mathbb{R}^n_+).$$

The proof of the variational inequality formulation of the governing equilibrium conditions is obtained in the following way. In a first step we prove that conditions (4.1) – (4.4), for a fixed $r(t)$, are necessary and sufficient to ensure that $(x_i^*(t), y_i^*(t))$ is the maximum of the problem

$$\max_{P_i} \int_0^T \left\{ - \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + r(t) \times [x_i(t) - y_i(t)] \right\} dt, \quad (4.8)$$
$$\forall \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} \in \prod_{1=1}^{m} P_i$$

which is analogous to problem (4.6).

Problem (4.8) means that each sector maximizes his utility. Since the feasible set P_i is a bounded, convex, and closed subset of the Hilbert space, then it is also weakly compact, hence such a maximum exists (see [72], Lemma 2.11, p. 15). Then we may prove that problem (4.8) is equivalent to a first variational inequality (4.9). In a second step we prove a variational formulation of the equilibrium condition related to the instrument prices (4.5). Therefore, the following Theorems hold true.

Theorem 4.1.2 $(x_i^*(t), y_i^*(t))$ *is a solution to (4.8) if and only if it is a solution to the variational inequality*

$$\int_0^T \left\{ \sum_{i=1}^m \left[2[Q_{11}^i(t)]^T x_i^*(t) + 2[Q_{21}^i(t)]^T y_i^*(t) - r^*(t) \right] \times [x_i(t) - x_i^*(t)] \right.$$

$$+ \sum_{i=1}^m \left[2[Q_{12}^i(t)]^T x_i^*(t) \; + \; 2[Q_{22}^i(t)]^T y_i^*(t) \; + \; r^*(t) \right] \times [y_i(t) - y_i^*(t)] \quad (4.9)$$

$$\forall (x_i(t), y_i(t)) \in P_i,$$

for a given $r^*(t) \in L^2([0,T], \mathbb{R}^n_+)$.

We may state (see [40] for the proof) the equivalence between problem (4.8) or problem (4.9) and the equilibrium conditions (4.1) – (4.4).

Theorem 4.1.3 $(x_i^*(t), y_i^*(t))$ *is a solution to (4.8) or (4.9) if and only if it satisfies, a.e. in* $[0,T]$, *conditions (4.1) – (4.4), where* $\mu_i^{(1)}(t)$, $\mu_i^{(2)}(t)$ $\in L^2([0,T])$ *are Lagrangean functions.*

We can show now the following characterization of the equilibrium condition related to the instrument prices (see [40] for the proof).

Theorem 4.1.4 *Condition (4.5) is equivalent to the problem*
Find $r^*(t) \in \mathcal{R}$ *such that*

$$\int_0^T \sum_{i=1}^m [x_{ij}^*(t) - y_{ij}^*(t)] \; \times \; [r_j(t) - r_j^*(t)] \, dt \geq 0, \; \forall r(t) \; \in \; L^2([0,T], \mathbb{R}^n_+). \quad (4.10)$$

From Theorems 4.1.2, 4.1.3, 4.1.4, it immediately follows that if $(x^*(t), y^*(t), r^*(t)) \in \prod_{i=1}^m P_i \times L^2([0,T], \mathbb{R}^n_+)$ is a financial equilibrium, then it satisfies variational inequalities (4.9), (4.10) and hence variational inequality (4.7) and viceversa. Thus Theorem 4.1.1 is completely proved.

4.2 Quasi-variational inequalities

A generalization of the variational inequality problem is the quasi-variational inequality problem, introduced by Bensoussan et al. [15] in the context of impulse control problems. Such problems were studied by many authors [3], [25], [95].
Many applications of these mathematical tools are known, for instance, we may refer to Bensoussan [14] and Harker [64], who recognized the connection between generalized Nash games and quasi-variational inequalities, Pang and Fukushima [114] applied this result in order to formulate the noncooperative multi-leader-follower game in terms of generalized Nash games, Bliemer and Bovy [17] discussed a quasi-variational inequality formulation of the dynamic traffic assignment problem. Applications to some economic and financial models can be found in [128], [129].
Shortly, we recall that a model for a traffic network with fixed demand due to Smith [133], leads to the following problem: to find a vector $H \in \mathbb{R}^m_+$ such that

$$H \in K : \quad C(H)(F - H) \geq 0, \quad \forall F \in K, \tag{4.11}$$

with

$$K := \left\{ F \in \mathbb{R}^m_+ : \phi F = \rho \right\}.$$

Here, m is the number of paths connecting all the l O/D pairs, $C(\cdot) : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ is the path cost function, $\rho \in \mathbb{R}^l_+$ is the fixed demand and ϕ is a $l \times m$ incidence matrix whose elements are:

$$\phi_{jr} = \begin{cases} 1 & if R_r \in \mathcal{R}_j \\ 0 & if R_r \notin \mathcal{R}_j \end{cases}$$

with $r = 1, 2, ..., m, \quad j = 1, 2, ..., l$, and where R_r is the r-th path and $\mathcal{R}_j$ is the set of those paths connecting the O/D pair j.
A solution H of the variational inequality (4.11) is an equilibrium pattern flow in the sense of J. G. Wardrop (1952), that is:

$$\forall\ O/D \text{ pair j}, \qquad \forall R_r, R_s \in \mathcal{R}_j, \quad if \quad C_r(H) > C_s(H) \Rightarrow H_r = 0\,.$$

Hence, we have equilibrium costs C_j for every O/D pair j, obtained considering those paths on which the equilibrium flows are greater than zero. So, we can require that the demand ρ depends on this equilibrium costs or, more directly, on the equilibrium pattern flow H.
Thus, if we put, for each $H \in \mathbb{R}^m_+$

$$K(H) := \left\{ F \in \mathbb{R}^m_+ \, : \, \phi F = \rho(H) \right\} ,$$

the Variational Inequality (4.11) becomes the following Quasi-Variational Inequality (QVI): to find

$$H \in K(H) : \qquad C(H)(F - H) > 0, \qquad \forall F \in K(H). \tag{4.12}$$

In order to find a numerical solution you can use the direct method proposed in [85] for Variational Inequalities, also generalized for Q.V.I. in [46].
In the following, some theorems for the existence of solutions to finite dimensional quasi-variational inequalities are recalled.

Theorem 4.2.1 ([65], [127]) *Let C, K be continuous functions and $\forall\, H \in B$, B a Banach space, let $K(H)$ be a nonempty, closed and convex subset of $\mathbb{R}^m_+$. Then problem (4.12) admits a solution.*

Theorem 4.2.2 ([47], [127]) *Let K be a continuous function, $\forall\, H \in B$ let $K(H)$ be a nonempty, closed and convex subset of B and let C satisfy the condition*

$$\{H \in B : C(H)F \leq 0\} \text{ is closed } \forall\, F \in B - B.$$

Then problem (4.12) admits a solution.

Theorem 4.2.3 ([45], [127]) *Let $C : B \to 2^{\mathbb{R}^m_+}$ be a multifunction (possibly discontinuous) such that:*

$$\forall\, F \in B - B \text{ the set } G_F = \left\{ H \in B \, : \, \inf_{z \in C(H)} zF \leq 0 \right\} \text{ is closed.}$$

Then, under the same assumptions of Theorem 4.2.1 on $K(H)$, there exist $H \in K(H) \cap B$ and $z \in C(H)$ such that $z(F - H) \geq 0 \; \forall F \in K(H) \cap B$.

The following result is due to Nguyen Xuan Tan [141] and concerns infinite dimensional quasi-variational inequalities:

Theorem 4.2.4 ([127], [141]) *Let X be a topological linear locally convex Hausdorff space, $C \subset X$ a convex compact nonempty subset. Let $P : C \to 2^{X^*}$ be an u.s.c. (upper semi-continuous) multivalued mapping with $P(x)$, $x \in C$, convex, compact, nonempty and let $E : C \to 2^C$ be a closed l.s.c. (lower semi-continuous) multivalued mapping with $E(x)$, $x \in C$ convex, compact, nonempty and let $\phi : C \to \mathbb{R}$ be a proper convex, lower semi-continuous function. Then, there exists $x^* \in C$ such that:*

(i) $x^* \in E(x^*)$

(ii) *there exists $y^* \in P(x^*)$ for which*

$$\langle x - x^*, y^* \rangle + \varphi(x) - \varphi(x^*) \geq 0, \qquad \forall\, x \in E(x^*)\,.$$

Chapter 5

The Financial Model with volumes depending on the expected solution

In this chapter we present a multi-sector, multi-instrument financial equilibrium problem, using the variational inequality theory presented in [28]. The model is assumed evolving in time and the equilibrium conditions are considered in dynamic sense. Moreover the amount of investment as liabilities and as assets is assumed depending on the expected solutions, namely we require that the set of feasible solutions is flexible and adaptive and this objective is achieved just assuming that the equality constraints depend on the variational solution. This leads to a quasi-variational formulation. We prove an existence theorem for quasi-variational inequalities under general and reasonable assumptions, namely assumptions really satisfied in concrete situations. Indeed we shall prove a general existence theorem (see Theorem 5.2.2) which, roughly speaking, under some kind of monotonicity and the Fan-hemicontinuity of the operator, along with natural growth conditions, ensures the existence of solutions for a general variational inequality. The chapter is also enriched by the study of numerical examples on financial networks with adaptive constraint sets.

The chapter is structured as follows. In section 5.1 we present the original model and a brief history of the improvements in the framework. In Section 5.2 we study the model with data depending on the expected solution, giving the equilibrium definition and obtaining the quasi-variational formulation. In Section 5.3 we prove the existence result. In Section 5.4 we recall some concepts on infinite-dimensional duality and introduce the evaluation index. Finally, in Section 5.4 we present some numerical examples.

5.1 Introduction

In the paper [33] P. Daniele presents, for the first time in literature, a model of time-dependent financial flows in the case of quadratic utility functions

$$U_i(t, x_i(t), y_i(t)) = -\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix} + r(t) \times [x_i(t) - y_i(t)]$$

where $Q^i(t) = \begin{bmatrix} Q^i_{11}(t) Q^i_{12}(t) \\ Q^i_{21}(t) Q^i_{22}(t) \end{bmatrix}$ is a $2n \times 2n$ variance-covariance matrix.
Then in [34] the evolutionary financial model has been generalized choosing as a utility function a general function:

$$U_i(t, x_i(t), y_i(t)) = u_i(t, x_i(t), y_i(t)) + r(t)(x_i(t) - y_i(t))$$

where $u_i(t, x_i(t), y_i(t))$ is a concave and differentiable function. The assumption of concavity on $u_i(t, x_i(t), y_i(t))$ is essential in order to obtain a characterization of the evolutionary financial equilibrium and the existence of the financial equilibrium.
This model has been generalized in [40] to allow for the incorporation of policy interventions in the form of taxes and price control. From the policy intervention aspect, denote the ceiling price and the floor price associated with instrument j respectively by $\overline{r}_j(t)$ and $\underline{r}_j(t)$. Denote the given tax rate levied on sector i's net yeld on financial instrument

j as $\tau_{ij}(t)$. Assume the tax rates lie in the interval $[0, 1)$ and belong to $L^\infty(0, T)$. Therefore, the government in this model has the flexibility of levying a distinct tax rate across both sectors and instruments and the possibility of adjusting the tax rate following the evolution of the system. Then if, at time t, $x_{ij}(t)$ denotes the amount of instrument j held as an asset in sector i's portfolio and $y_{ij}(t)$ denotes the amount of instrument j held as a liability in sector i's portfolio, the equilibrium condition for the price $r_j(t)$ of instrument j is the following:

$$\sum_{i=1}^{m}(1-\tau_{ij})(x_{ij}(t)-y_{ij}(t)) = \begin{cases} \leq 0 & \text{if} \quad r_j(t) = \overline{r}_j(t) \\ = 0 & \text{if} \quad \underline{r}_j(t) < r_j(t) < \overline{r}_j(t) \\ \geq 0 & \text{if} \quad r_j(t) = \underline{r}_j(t) \end{cases} .$$

In other words, if there is a real supply excess of an instrument in the economy, then its price must be the floor. If the price of an instrument is greater than the floor price, but not at the ceiling, then the market of that instrument must clear; analogously if there is an effective excess demand for an instrument in the economy, then the price must be at the ceilng. Subsequently, in [6] the authors present the first evolutionary model with different prices for assets and liabilities. Moreover, they choose a general utility function and include the expenses for the management of the financial instrument $h_j(t)$ that is a nonnegative function defined into $[0, T]$ and belonging to $L^\infty([o, T])$; so the utility function became

$$U_i(t, x_i(t), y_i(t)) = u_i(t, x_i(t), y_i(t)) + r(t)(x_i(t) - (1 + h_j(t))y_i(t)) .$$

Moreover, they introduce, for the first time in literature, the portion of financial transactions per unit employed to cover the expenses of the financial institutions including possible dividends and manager bonus, F_j.
In [7, 8, 39] the authors consider as the utility function $U_i(t, x_i(t), y_i(t))$, for each sector i, the following function

$$U_i(t, x_i(t), y_i(t)) =$$
$$u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 + h_j(t))y_{ij}(t)] ,$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent and $r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 + h_j(t))y_{ij}(t)]$ represents the value of the difference between the asset holdings and the value of liabilities. Here, $\tau_{ij}(t)$ still denotes denotes the given tax rate levied on sector i's net yield on financial instrument j and lie in the interval $[0, 1)$ and belongs to $L^\infty([0, T], \mathbb{R})$, as in [10]. Moreover, a simple but useful indicator of the economy, introduced in this works, is the *Evaluation Index* $E(t)$, that we will define in section 5.4 and use in the numerical examples.
Finally, in [10] the authors assume that the total amount of investment as liabilities and as assets depends on the expected solutions and the measure of the financial risk they use is of Markovitz type.
In our thesis we take inspiration from [10], that is a financial economy in the case when the financial volumes depend on time and on the expected solution, in order to take into account the influence of the expected equilibrium distribution for assets and liabilities on the investments on all financial instruments. But unlike in [10], where the measure of the risk is of a Markowitz type, we consider as a measure of the financial risk a general function.

5.2 The model

The model, that we consider, evolves in time and the equilibrium conditions are considered in a dynamic sense. Moreover, the amount of investment as liabilities and as assets is assumed depending on the expected solutions, namely we require that the set of feasible solutions is flexible and adaptive and this objective is achieved just assuming that the equality constraints depend on the variational solution. This leads to a quasi-variational formulation. For a quasi-variational approach to other economic problems see also [50] and [149].
We consider a financial economy consisting of m sectors (such as domestic businesses, banks, and other financial institutions), with a typical sector denoted by i, and of n instruments (such as mortgages, mutual

funds, savings deposits, money market funds), with a typical financial instrument denoted by j, in the horizon $[0, T]$.
Let $s_i(t)$ denote the total financial volume held by sector i at time t as assets, and let $l_i(t)$ denote the total financial volume held by sector i at time t as liabilities and assume they depend on time in order to describe the unstable behavior of the economy. We denote, at time t, the amount of instrument j held as an asset in sector i's portfolio by $x_{ij}(t)$ and the amount of instrument j held as a liability in sector i's portfolio by $y_{ij}(t)$. The assets in sector i's portfolio are grouped into the column vector $x_i(t) = [x_{i1}(t), x_{i2}(t), ..., x_{ij}(t), ..., x_{in}(t)]^T$ and the liabilities in sector i's portfolio are grouped into the column vector $y_i(t) = [y_{i1}(t), y_{i2}(t), ..., y_{ij}(t), ..., y_{in}(t)]^T$. Moreover, group the sector asset vectors into the matrix

$$x(t) \in L^2([0, T], \mathbb{R}^{nm})$$

and the sector liability vectors into the matrix

$$y(t) \in L^2([0, T], \mathbb{R}^{nm}).$$

We denote the price of instrument j held as an asset at time t by $r_j(t)$ and the price of instrument j held as liability at time t by $(1 + h_j(t))r_i(t)$, where h is a nonnegative function defined into $[0, T]$ and belonging $L^\infty([0, T])$. We introduce the term $h_j(t)$ because the prices of liabilities are generally greater than or equal to the prices of assets so that we can describe in a more consistent fashion the behavior of the markets for which the liabilities are more expensive than the assets. Under the assumption of perfect competition, each sector will behave as if it has no influence on the instrument prices or on the behavior of the other sectors, but the equilibrium prices depend on the total amount of investments and liabilities of each sector. The total financial volumes s_i and l_i depend on time t and on the expected solution, namely by $\int_0^T w^*(s)ds$, so s_i is given by $s_i\left(t, \int_0^T w^*(s)ds\right)$ and l_i is given by $l_i\left(t, \int_0^T w^*(s)ds\right)$.
In such a way we are taking into account the influence, by means of the

average value, of the expected equilibrium distribution for assets and liabilities on the investments on all financial instruments. In the literature this kind of constraints is called *elastic* or *adaptive constraints*. In order to make the model more consistent with reality, we introduce the government intervention through a taxation on the profits, so we denote the given tax rate levied on sector i's net revenue on financial instrument j by τ_{ij}, with $\tau_{ij} \in [0, 1)$ a.e.

We adhere to the existing formulation of the financial model determining for each sector i the optimal composition of instruments held as assets and as liabilities, namely also in our paper we consider the following utility function:

$$U_i(t, x_i(t), y_i(t), r(t))$$

$$= u_i(t, x_i(t), y_i(t)) + \sum_{j=1}^{n} r_j(t)(1 - \tau_{ij}(t))[x_{ij}(t) - (1 + h_j(t))y_{ij}(t)],$$

where the term $-u_i(t, x_i(t), y_i(t))$ represents a measure of the risk of the financial agent and $r_j(t)(1 - \tau_{ij}(t))[x_i(t) - (1 + h_j(t))y_i(t)]$ represents the value of the difference between the asset holdings and the value of liabilities. An example of utility function is obtained by using variance-covariance matrices denoting the sectors assessment of the standard deviation of prices for each instrument (see [88] and [89]).

First, we make the following assumptions *Hypotheses 1*, which will be denoted by
Hp. 1:

- *The sector's utility function $U_i(t, x_i(t), y_i(t))$ is defined on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, is measurable in t and is continuous with respect to x_i and y_i.*

- $\dfrac{\partial u_i}{\partial x_{ij}}$ *and* $\dfrac{\partial u_i}{\partial y_{ij}}$ *exist and that they are measurable in t and continuous with respect to x_i and y_i.*

- *$\forall i = 1, \ldots, m$, $\forall j = 1, \ldots, n$, and a.e. in $[0, T]$ the following growth conditions hold true:*

$$|u_i(t, x, y)| \leq \alpha_i(t)\|x\|\|y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{5.1}$$

and

$$\left|\frac{\partial u_i(t, x, y)}{\partial x_{ij}}\right| \leq \beta_{ij}(t)\|y\|, \quad \left|\frac{\partial u_i(t, x, y)}{\partial y_{ij}}\right| \leq \gamma_{ij}(t)\|x\|, \tag{5.2}$$

where α_i, β_{ij}, γ_{ij} are non-negative functions of $L^\infty([0, T])$.

- *The function $u_i(t, x, y)$ is concave.*

- $-\dfrac{\partial u_i\left(t, x_i(t), y_i(t)\right)}{\partial x_{ij}}$ *and* $-\dfrac{\partial u_i\left(t, x_i(t), y_i(t)\right)}{\partial y_{ij}}$ *are strictly monotone functions.*

Now we present in detail the model. The prices are unknown variables and they are determined by a demand-supply law, namely for $j = 1, \ldots, n$ and a.e. in $[0, T]$

$$\sum_{i=1}^{m}(1 - \tau_{ij}(t))\left[x^*_{ij}(t) - (1 + h_j(t))y^*_{ij}(t)\right] + F_j(t)$$
$$\begin{cases} \geq 0 \text{ if } r^*_j(t) = \underline{r}_j(t) \\ = 0 \text{ if } \underline{r}_j(t) < r^*_j(t) < \overline{r}_j(t) \\ \leq 0 \text{ if } r^*_j(t) = \overline{r}_j(t), \end{cases} \tag{5.3}$$

where $w^* = (x^*, y^*, r^*)$ is the equilibrium solution for the investments as assets and as liabilities and for the prices and $F_j(t) \in L^2([0, T])$ is the quantity of financial transactions per unit employed to cover the expenses of the financial institutions including dividends and manager bonus.

For technical reasons, we shall choose as the functional setting

$$L^2([0, T], \mathbb{R}^p) = \left\{x : [0, T] \to \mathbb{R}^p | \int_0^T \|x(t)\|^2_{\mathbb{R}^p}\, dt < +\infty\right\}$$

where $\left(\int_0^T \|x(t)\|^2_{\mathbb{R}^p}\, dt\right)^{\frac{1}{2}} = \|x\|_{L^2([0,T],\mathbb{R}^p)}$. To denote the norm in the Hilbert space $L^2([0,T],\mathbb{R}^p)$ we shall use the symbol $\|x\|$ when there is no possibility of confusion. As it is well known, the dual space of $L^2([0,T],\mathbb{R}^p)$ is still $L^2([0,T],\mathbb{R}^p)$ and we define the canonical bilinear form in $L^2([0,T],\mathbb{R}^p) \times L^2([0,T],\mathbb{R}^p)$ as:

$$\ll G, x \gg = \int_0^T \langle G(t), x(t)\rangle\, dt,\ G,\ x \in L^2([0,T],\mathbb{R}^p).$$

Where $\langle G(t), x(t)\rangle$ denotes the scalar product in $\mathbb{R}^p$.

In order to define the constraint set, let us introduce the set

$$E = \big\{ w = (x(t), y(t), r(t)) \in L^2([0,T],\mathbb{R}^{2mn+n}) : x(t) \geq 0,$$

$$y(t) \geq 0,\ \underline{r}(t) \leq r(t) \leq \overline{r}(t) \text{ a.e. in } [0,T] \big\},$$

with $\underline{r}(t) \leq \overline{r}(t) \in L^2([0,T],\mathbb{R}^n)$, $0 \leq \underline{r}(t) \leq \overline{r}(t)$ a.e. in $[0,T]$. It is easy to verify that E is a convex, bounded and closed subset of $L^2([0,T],\mathbb{R}^{2mn+n})$.
If $\mathbb{K} : E \to 2^E$ is the set-valued map defined as

$$\begin{aligned} \mathbb{K}(w^*) \ = \ & \Bigg\{ w = (x(t), y(t), r(t)) \in E : \sum_{j=1}^n x_{ij}(t) = s_i\left(t, \int_0^T w^*(s)\, ds\right), \\ & \sum_{j=1}^n y_{ij}(t) = l_i\left(t, \int_0^T w^*(s)\, ds\right) \text{ a.e. in } [0,T]\ i = 1,\ldots,m \Bigg\}, \end{aligned} \quad (5.4)$$

then $\mathbb{K}(w^*)$ is the feasible set for every $w^* \in E$.
Now, we can give different but equivalent equilibrium conditions, each of which is useful to illustrate particular features of the equilibrium.

Definition 5.2.1 *A vector of sector assets, liabilities and instrument prices $w^* \in \mathbb{K}(w^*)$ is an equilibrium of the dynamic financial model if* *and only if* $\forall i = 1, \dots, m$, $\forall j = 1, \dots, n$, *and a.e. in* $[0, T]$, *it satisfies the system of inequalities*

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \mu_i^{(1)*}(t) \geq 0, \tag{5.5}$$

$$-\frac{\partial u_i(t, x^*, y^*)}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t)) r_j^*(t) - \mu_i^{(2)*}(t) \geq 0, \tag{5.6}$$

and equalities

$$x_{ij}^*(t) \Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) - \mu_i^{(1)*}(t) \Big] = 0, \tag{5.7}$$

$$y_{ij}^*(t) \Big[-\frac{\partial u_i(t, x^*, y^*)}{\partial x_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t)) r_j^*(t) - \mu_i^{(2)*}(t) \Big] = 0, \tag{5.8}$$

where $\mu_i^{(1)*}(t)$, $\mu_i^{(2)*}(t) \in L^2([0, T])$ *are the Lagrange functions associated to the constraints*

$$\sum_{j=1}^{n} x_{ij}(t) = s_i \left(t, \int_0^T w^*(s)\, ds \right) \text{ and } \sum_{j=1}^{n} y_{ij}(t) = l_i \left(t, \int_0^T w^*(s)\, ds \right)$$

respectively, and verifies conditions (5.3) *a.e. in* $[0, T]$.

For additional details on Definition 5.2.1, see, for instance, [6] and [7]. Also in this formulation we are dealing with Lagrange multipliers which are unknown a priori, but this has no influence because, as we shall see by means of Theorem 5.2.1, Definition 5.2.1 is equivalent to a variational inequality in which $\mu_i^{(1)^*}(t)$ and $\mu_i^{(2)^*}(t)$ do not appear.

Indeed, under assumptions *Hp. 1*, such an equilibrium is characterized by the following variational formulation.

Theorem 5.2.1 *A vector* $w^* \in \mathbb{K}(w^*)$ *is a dynamic financial equilibrium if and only if it satisfies the following variational inequality:*

$$\begin{aligned}
&\textit{Find } w^* \in \mathbb{K}(w^*):\\
&\sum_{i=1}^{m}\int_0^T \Bigg\{ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t)) r_j^*(t) \Big] \\
&\qquad\qquad \cdot [x_{ij}(t) - x_{ij}^*(t)] \\
&+ \sum_{j=1}^{n} \Big[-\frac{\partial u_i(t, x_i^*(t), y_i^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t)) r_j^*(t)(1+h_j(t)) \Big] \\
&\qquad\qquad \cdot [y_{ij}(t) - y_{ij}^*(t)] \Bigg\} dt \\
&+ \sum_{j=1}^{n}\int_0^T \sum_{i=1}^{m} \big\{ (1-\tau_{ij}(t)) \big[x_{ij}^*(t) - (1+h_j(t)) y_{ij}^*(t) \big] + F_j(t) \big\} \\
&\qquad\qquad \cdot \big[r_j(t) - r_j^*(t) \big] dt \geq 0, \quad \forall w \in \mathbb{K}(w^*). \qquad (5.9)
\end{aligned}$$

For the proof see [6], [8], [9].

The aim of this paper is to prove an existence result for (5.9). We shall get such an existence result by proving a general theorem which is interesting in itself.

Let $F : [0, T] \times \mathbb{R}^{2mn} \to \mathbb{R}^{2mn}$ be such that the following condition is satisfied:

$$\begin{gathered}
F \text{ is measurable in } t \ \forall w \in \mathbb{R}^{2mn}, \text{ continuous in } w \text{ a.e. in } [0,T] \\
\text{and there exists } \overline{\delta} \in L^2([0,T]) \text{ such that} \qquad (F) \\
\|F(t,w)\| \leq \overline{\delta}(t) + \|w\| \ \text{ a.e. in } [0,T],\ w \in \mathbb{R}^{2mn}.
\end{gathered}$$

First of all, we recall the definitions of a strongly monotone and a Fan-hemicontinuous mapping.

Definition 5.2.2 *Let $F : [0, T] \times E \to L^2([0, T], \mathbb{R}^{2mn})$. We say that F is strongly monotone in x and y and monotone in r if there exists $\nu > 0$:*

$$\langle F(t, w_1(t)) - F(t, w_2(t)), w_1(t) - w_2(t)\rangle \geq \nu(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2)$$
$$\forall w_1(t) = (x_1(t), y_1(t), r_1(t)),\ w_2(t) = (x_2(t), y_2(t), r_2(t)) \in E.$$

Definition 5.2.3 *Let* $F : [0, T] \times E \to L^2([0, T], \mathbb{R}^{2mn})$*. We say that* F *is Fan-hemicontinuous if* $\forall v \in E$ *the function*

$$w \to \ll F(t, w), w - v \gg$$

is weakly lower semicontinuous on E*.*

Further, we make the following assumptions (α):

- *the functions* s *and* l *are Carathéodory functions, which means they are measurable in* t *and continuous with respect to the second variable;*
- *there exist* $\delta_1(t) \in L^2([0, T])$ *and* $c_1 \in \mathbb{R}$ *such that:*

$$\|s(t, x)\| \le \delta_1(t) + c_1, \ \forall x \in \mathbb{R}^{mn};$$

- *there exist* $\delta_2(t) \in L^2([0, T])$ *and* $c_2 \in \mathbb{R}$ *such that:*

$$\|l(t, y)\| \le \delta_2(t) + c_2, \ \forall y \in \mathbb{R}^{mn}.$$

Now let us consider the following variational inequality:

$$\text{Find } w^* \in \mathbb{K}(w^*) : \ \langle\langle F(t, w^*), w - w^* \rangle\rangle \ge 0 \ \ \forall w \in \mathbb{K}(w^*). \tag{5.10}$$

We can prove the following existence theorem.

Theorem 5.2.2 *Let* $F : [0, T] \times E \to L^2([0, T], \mathbb{R}^{2m+n})$ *be a bounded, strongly monotone in* x *and* y*, monotone in* r*, Fan-hemicontinuous mapping and satisfying conditions (F) and* (α)*. Then variational inequality (5.10) admits a solution.*

Now, let us remark that variational inequality (5.9) can be rewritten by using the operator

$$A(t, w) : [0, T] \times \mathbb{R}^{2mn} \to \mathbb{R}^{2mn}$$

defined as follows:

$$\begin{aligned} A(t,w) &= A(t,x(t),y(t),r(t)) \\ &= \left(\left[-\frac{\partial u_i\,(t,x_i(t),y_i(t))}{\partial x_{ij}} - r_j(t)\,(1-\tau_{ij}(t)) \right]_{ij}, \right. \\ &\quad \left[-\frac{\partial u_i\,(t,x_i(t),y_i(t))}{\partial y_{ij}} + r_j(t)\,(1-\tau_{ij}(t))\,(1+h_j(t)) \right]_{ij}, \\ &\quad \left. \left[\sum_{i=1}^{m} (1-\tau_{ij}(t))\,[x_{ij}(t) - (1+h_j(t))\,y_{ij}(t)] + F_j(t) \right]_j \right). \end{aligned}$$

Then variational inequality (5.9) becomes:

$$\langle\langle A(t,w^*), w-w^* \rangle\rangle \geq 0 \quad \forall w \in \mathbb{K}(w^*). \tag{5.11}$$

From Theorem 5.2.2 we deduce an existence theorem for variational inequality (5.11), namely we shall prove the following theorem.

Theorem 5.2.3 *Under assumptions Hp. 1 and (α_1)-(α_3) variational inequality (5.11) admits a solution.*

Remark 5.2.1 *IWe assume that $\forall w \in E$, then $w \in \mathbb{K}(w)$. Hence, setting*

$$\overline{\mathbb{K}}(w) = \mathbb{K}(w) - w,$$

we get $0 \in \overline{\mathbb{K}}(w)$. As a consequence, replacing $\mathbb{K}(w)$ with $\overline{\mathbb{K}}(w)$, the assumption $0 \in \overline{\mathbb{K}}(w)$ is always satisfied. In the sequel we shall use the notation $\mathbb{K}(w)$.

5.3 Proof of Theorems

In order to prove Theorem 5.2.2, let us recall that the following existence result holds true (see [130], Theorem 3.2):

Theorem 5.3.1 *Let $F(t, w)$ satisfy all assumptions in Theorem 5.2.2 provided that the monotonicity on r is replaced by the strong monotonicity on r. Then, variational inequality (5.10) admits a solution.*

We remark that in this theorem it is required that the operator $F(t, w)$ is strongly monotone in x, y and r. Hence, the aim of Theorem 5.2.2 is to show that, for the existence of solutions, it is enough to assume the operator F only monotone in r.
To prove Theorem 5.2.2, let us consider the following variational inequality, for each fixed $n \in \mathbb{N}$:

$$\text{Find } w_n^* \in \mathbb{K}(w_n^*) : \langle \overline{F}(t, w_n^*), w - w_n^* \rangle = \langle F(t, w_n^*), w - w_n^* \rangle$$
$$+ \frac{1}{n} \int_0^T r_{nj}^*(t) \, (r_j(t) - r_{nj}^*(t)) \, dt \, \geq 0, \quad \forall w \in \mathbb{K}(w_n^*), \tag{5.12}$$

where

$$\mathbb{K}(w_n^*) = \Bigg\{ (x_n(t), y_n(t), r_n(t)) \in L^2 \left([0, T], \mathbb{R}^{2mn} \right) :$$
$$\sum_{j=1}^n x_{nij}(t) = s_i \left(t, \int_0^T w_n^*(s) \, ds \right), \; \sum_{j=1}^n y_{nij}(t) = l_i \left(t, \int_0^T w_n^*(s) \, ds \right)$$
$$\text{a.e. in } [0, T], x_n(t) \geq 0, \; y_n(t) \geq 0, \; \underline{r}(t) \leq r_n(t) \leq \overline{r}(t) \text{ a.e. in } [0, T] \Bigg\}.$$

Variational inequality (5.12) is a perturbation of the previous variational inequality (5.10). We are following the same procedure presented by Stampacchia in [137] (see Sect. 3).

For the operator $\overline{F}$, as it is easy to see, all the assumptions of Theorem 5.3.1 are fulfilled. Then, there exists a unique solution $w_n^* \in \mathbb{K}(w_n^*)$ to (5.12) $\forall n \in \mathbb{N}$. The set of solutions $\{w_n^*\}$ turns out to be bounded in L^2 by the definition of $\mathbb{K}(w_n^*)$. Then, we can extract from $\{w_n^*\}$ a subsequence, denoted, for the sake of simplicity, by $\{w_n^*\}$, weakly convergent to some $\widetilde{w} \in L^2$ (see, for instance, [23], Theor. III, 16):

$$w_n^* \rightharpoonup \widetilde{w}.$$

We shall show that
$$\widetilde{w} \in \mathbb{K}(\widetilde{w}).$$
Taking into account that $w_n^* \in \mathbb{K}(w_n^*)$, we have
$$\sum_{j=1}^{n} x_{nij}^*(t) = s_i\left(t, \int_0^T w_n^*(s)\,ds\right) \text{ and } x_{nij}^* \rightharpoonup \widetilde{x}_{ij}(t),$$
hence
$$\sum_{j=1}^{n} x_{nij}^*(t) \rightharpoonup \sum_{j=1}^{n} \widetilde{x}_{ij}(t).$$
Now, for every test function $\varphi \in L^2([0,T])$, we get:
$$\lim_n \int_0^T \varphi \sum_{j=1}^{n} x_{nij}^*(t) = \lim_n \int_0^T \varphi\, s_i\left(t, \int_0^T w_n^*(s)ds\right) dt = \int_0^T \varphi \sum_{j=1}^{n} \widetilde{x}_{ij}(t).$$
For assumption (α_1) and taking into account that $\lim_n \int_0^T x_n^*(s)\,ds = \int_0^T \tilde{x}(s)\,ds$, we get
$$\lim_n s_i\left(t, \int_0^T w_n^*(s)ds\right) = s_i\left(t, \int_0^T \widetilde{w}(s)ds\right)$$
and
$$\lim_n \int_0^T \varphi\, s_i\left(t, \int_0^T w_n^*(s)ds\right) dt = \int_0^T \varphi\, s_i\left(t, \int_0^T \widetilde{w}(s)ds\right) dt.$$
As a consequence for every φ we have:
$$\int_0^T \varphi \left[s_i\left(t, \int_0^T \widetilde{w}(s)ds\right) - \sum_{j=1}^{n} \widetilde{x}_{ij}(t)\right] dt = 0$$
and therefore
$$s_i\left(t, \int_0^T \widetilde{w}(s)\,ds\right) = \sum_{j=1}^{n} \widetilde{x}_{ij}(t).$$

An analogous procedure can be applied to obtain:

$$l_i\left(t, \int_0^T \widetilde{w}(s)\,ds\right) = \sum_{j=1}^n \widetilde{y}_{ij}(t).$$

It remains to prove that $x_i(t) \geq 0$ a.e. in $[0,T]$. For the sake of generality and forseeing future extensions of the model, we shall prove

$$\underline{x}_i(t) \leq x_i(t) \leq \overline{x}_i(t) \text{ a.e. in } [0,T].$$

We shall just prove that $\underline{x}_i(t) \leq \overline{x}_i(t)$. We know that $\underline{x}_i(t) \leq \widetilde{x}^*_{ni}(t)$. Then, for every nonnegative function $\varphi \in L^2([0,T])$, in virtue of the weak convergence, we have:

$$0 \leq \lim_n \int_0^T \varphi \underbrace{(x^*_{ni}(t) - \underline{x}_i(t))}_{\geq 0}\,dt = \int_0^T \varphi(\widetilde{x}_i(t) - \underline{x}_i(t))dt. \qquad (5.13)$$

Now, let us assume ad absurdum there exists a subset $E \subset [0,T]$ with positive measure such that $\widetilde{x}_i(t) - \underline{x}_i(t) < 0$ in E. Then one chooses:

$$\overline{\varphi} = \begin{cases} 0 & \text{in } [0,T] \setminus E \\ \varphi > 0 & \text{in } E. \end{cases}$$

Hence, (5.13) becomes:

$$\int_0^T \varphi\,(\widetilde{x}_i(t) - \underline{x}_i(t)) \;=\; \int_E \varphi\,(\widetilde{x}_i(t) - \underline{x}_i(t)) < 0$$

which is an absurdity. An analogous procedure can be applied to show the other constraints and then obtain $\widetilde{w} \in \mathbb{K}(\widetilde{w})$.
Let us prove now that $\widetilde{w}$ satisfies variational inequality:

$$\langle\langle F(t,\widetilde{w}), w - \widetilde{w}\rangle\rangle \geq 0 \quad \forall w \in \mathbb{K}(\widetilde{w}). \qquad (5.14)$$

Since $F(t, \widetilde{w})$ is monotone, according to Minty's Lemma (see [137], Lemma 2.2), (5.14) is equivalent to prove that $\widetilde{w}$ is a solution to the following Minty variational inequality:

$$\langle\langle F(t,w), w-\widetilde{w}\rangle\rangle \geq 0 \quad \forall w \in \mathbb{K}(\widetilde{w}). \tag{5.15}$$

Let us observe that if (5.12) is satisfied, then, according to Minty's Lemma again, w_n^* is also a solution to:

$$\langle\langle \overline{F}(t,w), w-w_n^*\rangle\rangle = \langle\langle F(t,w), w-w_n^*\rangle\rangle + \frac{1}{n}\int_0^T r_j(t)\,(r_j(t)-r_{nj}^*(t))\,dt \geq 0 \forall w \in \mathbb{K}(w_n^*). \tag{5.16}$$

In order to prove (5.14), let us recall that in [130] (see Lemma 3.1) the author has proved that the set $\mathbb{K}(w)$ is Mosco convergent; in particular we have that:

$$\forall w \in \mathbb{K}(\widetilde{w})\ \exists \{w_n\} \subseteq \mathbb{K}(w_n) : w_n \to w \text{ in } L^2(\mathbb{R}^{2mn}).$$

Since (5.16) is satisfied in $\mathbb{K}(w_n^*)$, we choose as $w \in \mathbb{K}(w_n^*)$ in (5.16) the element w_n such that $w_n \to w$.
Then (5.16) becomes:

$$\langle\langle F(t,w_n), w_n-w_n^*\rangle\rangle + \frac{1}{n}\int_0^T r_{nj}(t)(r_{nj}(t)-r_{nj}^*(t))\,dt \geq 0. \tag{5.17}$$

We want to prove that (5.17) converges to (5.15). To this end we shall work separately on the two terms of (5.17). The first term is:

$$\begin{aligned}&\langle\langle F(t,w_n), w_n-w_n^*\rangle\rangle = \langle\langle F(t,w_n), w_n-\widetilde{w}+\widetilde{w}-w_n^*\rangle\rangle = \\ &= \langle\langle F(t,w_n), w_n-\widetilde{w}\rangle\rangle + \langle\langle F(t,w_n), \widetilde{w}-w_n^*\rangle\rangle.\end{aligned}$$

Let us remark that:

- $w_n \to w$, hence $\lim_n \langle\langle F(t,w_n), w_n-\widetilde{w}\rangle\rangle = \langle\langle F(t,w), w-\widetilde{w}\rangle\rangle$;

- From assumption (F), F is a Nemitski operator, so $F(t, w_n) \to F(t, w)$ in $L^2(\mathbb{R}^{2m+n})$ when $w_n \to w$ and $\widetilde{w} - w_n^* \rightharpoonup 0$, hence $\lim_n \langle\langle F(t, w_n), \widetilde{w} - w_n^* \rangle\rangle = \langle\langle F(t, w), \widetilde{w} - \widetilde{w} \rangle\rangle = 0$, indeed: $\lim_n \langle\langle F(t, w_n), \widetilde{w} - w_n^* \rangle\rangle = \lim_n \langle\langle F(t, w_n) - F(t, w), \widetilde{w} - w_n^* \rangle\rangle + \lim_n \langle\langle F(t, w), \widetilde{w} - w_n^* \rangle\rangle = 0$.

Therefore:

$$\lim_n \langle\langle F(t, w_n), w_n - w_n^* \rangle\rangle = \langle\langle F(t, w), w - \widetilde{w} \rangle\rangle \quad \forall w \in \mathbb{K}(\widetilde{w})$$

Now we examine the second term: $\lim_n \dfrac{1}{n} \displaystyle\int_0^T r_{nj}(t)(r_{nj}(t) - r_{nj}^*(t))\, dt = 0$, since $\dfrac{1}{n}$ converges to 0, $r_{nj}^* \rightharpoonup r_j(t)$ and so $r_j(t) - r_{nj}^*(t)$ is bounded. As a consequence:

$$\lim_n \left\langle \langle F(t, w_n), w_n - w_n^* \rangle\rangle + \frac{1}{n} \int_0^T r_{nj}(t)(r_{nj}(t) - r_{nj}^*(t))\, dt \right\}$$
$$= \langle F(t, w), w - \widetilde{w} \rangle, \forall w \in \mathbb{K}(\widetilde{w}).$$

Then the theorem is proved. □

Now we are in position to prove Theorem 5.2.3.
We apply Theorem 5.2.2 by choosing as the operator F the new operator A defined as follows:

$$A(t, w) : [0, T] \times \mathbb{R}^{2mn} \to \mathbb{R}^{2mn}$$

defined as follows:

$$\begin{aligned} A(t, w) &= A(t, x(t), y(t), r(t)) \\ &= \Bigg(\left[-\frac{\partial u_i\,(t, x_i(t), y_i(t))}{\partial x_{ij}} - r_j(t)\,(1 - \tau_{ij}(t)) \right]_{ij}, \\ &\quad \left[-\frac{\partial u_i\,(t, x_i(t), y_i(t))}{\partial y_{ij}} + r_j(t)\,(1 - \tau_{ij}(t))\,(1 + h_j(t)) \right]_{ij}, \\ &\quad \left[\sum_{i=1}^m (1 - \tau_{ij}(t))\,[x_{ij}(t) - (1 + h_j(t))\,y_{ij}(t)] + F_j(t) \right]_j \Bigg). \end{aligned}$$

The boundedness of $A(t,w)$, which means:

$$\exists c \in \mathbb{R} : \|A(t,w)\|_{L^2([0,T],\mathbb{R}^{2mn+n})} \leq c,$$

trivially follows from assumption α_2.
Let us prove that $A(t,w)$ is strongly monotone with respect to x and y, namely:

$$\exists \nu > 0 : \langle\langle A(t,w_1) - A(t,w_2), w_1 - w_2 \rangle\rangle \geq \nu \left[\|x^1 - x^2\|^2_{L^2} + \|y^1 - y^2\|^2_{L^2} \right].$$

We have:

$$\begin{aligned}
&\langle\langle A(t,w_1) - A(t,w_2), w_1 - w_2 \rangle\rangle \\
&= \sum_{i=1}^m \int_0^T \Bigg\{ \sum_{j=1}^n \left(-\frac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial x_{ij}^1} + \frac{\partial u_i\left(t, x_i^2(t), y_i^2(t)\right)}{\partial x_{ij}^2} \right) \left(x_{ij}^1(t) - x_{ij}^2(t) \right) \\
&\quad - \sum_{j=1}^n (r_j^1(t) - r_j^2(t))(1 - \tau_{ij}(t))(x_{ij}^1(t) - x_{ij}^2(t)) \\
&\quad + \sum_{j=1}^n \left(-\frac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial y_{ij}^1} + \frac{\partial u_i\left(t, x_i^2(t), y_i^2(t)\right)}{\partial y_{ij}^2} \right) \left(y_{ij}^1(t) - y_{ij}^2(t) \right) \\
&\quad + \sum_{j=1}^n (r_j^1(t) - r_j^2(t))(1 - \tau_{ij}(t))(1 + h_j(t))(y_{ij}^1(t) - y_{ij}^2(t)) \Bigg\} dt \\
&\quad + \sum_{j=1}^n \int_0^T \Bigg\{ \sum_{i=1}^m (1 - \tau_{ij}(t))(x_{ij}^1(t) - x_{ij}^2(t))(r_j^1(t) - r_j^2(t)) \\
&\quad - \sum_{i=1}^m (1 - \tau_{ij}(t))(1 + h_j(t))(y_{ij}^1(t) - y_{ij}^2(t))(r_j^1(t) - r_j^2(t)) \Bigg\} dt \\
&= \sum_{i=1}^m \int_0^T \Bigg\{ \sum_{j=1}^n \left(-\frac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial x_{ij}^1} + \frac{\partial u_i\left(t, x_i^2(t), y_i^2(t)\right)}{\partial x_{ij}^2} \right) \left(x_{ij}^1(t) - x_{ij}^2(t) \right) \\
&\quad + \left(-\frac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial y_{ij}^1} + \frac{\partial u_i\left(t, x_i^2(t), y_i^2(t)\right)}{\partial y_{ij}^2} \right) \left(y_{ij}^1(t) - y_{ij}^2(t) \right) \Bigg\} dt
\end{aligned}$$

$$\geq \nu \left[\|x^1 - x^2\|_{L^2}^2 + \|y^1 - y^2\|_{L^2}^2 \right].$$

The last inequality is easily achieved since, by assumptions, the functions $-\dfrac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial x_{ij}^1}$ and $-\dfrac{\partial u_i\left(t, x_i^1(t), y_i^1(t)\right)}{\partial y_{ij}^1}$ are strictly monotone.
Hence, $A(t,w)$ is strongly monotone in x and y and only monotone in r.

Now, we can prove that $A(t,w)$ is Fan-hemicontinuous, namely:

$$\langle\langle A(t,w), w - \xi\rangle\rangle \text{ is weakly lower semicontinuous,}$$

where $\xi = (\xi^1, \xi^2, \xi^3)$ is fixed. So we need to prove that

$$\liminf_n \langle\langle A(t,w_n), w_n - \xi\rangle\rangle \geq \langle\langle A(t,w), w - \xi\rangle\rangle, \quad \forall \{w_n\} \text{ such that } w_n \rightharpoonup w.$$

We have:

$$\langle\langle A(t,w_n), w_n - \xi\rangle\rangle$$

$$= \int_0^T \sum_{i=1}^m \sum_{j=1}^n \left(-\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial x_{ij}} - r_j^n(t)(1 - \tau_{ij}(t)) \right) \left(x_{ij}^n(t) - \xi_{ij}^1 \right) dt$$

$$+ \int_0^T \sum_{i=1}^m \sum_{j=1}^n \left(-\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial y_{ij}} + r_j^n(t)(1 - \tau_{ij}(t))(1 + h_j(t)) \right)$$

$$\left(y_{ij}^n(t) - \xi_{ij}^2 \right) dt$$

$$+ \sum_{i=1}^m \int_0^T \left\{ \sum_{j=1}^n (1 - \tau_{ij}(t)) \left[x_{ij}^n(t) - (1 + h_j(t)) y_{ij}^n(t) \right] + F_j(t) \right\} (r_j^n(t) - \xi_j^3)\, dt$$

$$= \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial x_{ij}} \left(x_{ij}^n(t) - \xi_{ij}^1 \right) dt$$

$$+ \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial y_{ij}} \left(y_{ij}^n(t) - \xi_{ij}^2 \right) dt$$

$$+ \sum_{i=1}^m \sum_{j=1}^n \int_0^T \Big[- r_j^n(t)(1 - \tau_{ij}(t)) x_{ij}^n(t) + r_j^n(t)(1 - \tau_{ij}(t)) \xi_{ij}^1$$

$$+r_j^n(t)(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^n(t) - r_j^n(1-\tau_{ij}(t))(1+h_j(t))\xi_{ij}^2$$

$$+r_j^n(t)(1-\tau_{ij}(t))x_{ij}^n(t) - x_{ij}^n(t)(1-\tau_{ij}(t))\xi_j^3 - r_j^n(t)(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^n(t)$$

$$\left. +(1-\tau_{ij}(t))(1+h_j(t))y_{ij}^n(t)\xi_j^3 + F_j(t)(r_j^n(t)-\xi_j^3)\right] dt$$

$$= \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial x_{ij}} \left(x_{ij}^n(t) - \xi_{ij}^1\right) \, dt$$

$$+ \int_0^T \sum_{i=1}^m \sum_{j=1}^n -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial y_{ij}} \left(y_{ij}^n(t) - \xi_{ij}^2\right) \, dt$$

$$+ \sum_{i=1}^m \sum_{j=1}^n \int_0^T \left[r_j^n(t)(1-\tau_{ij}(t))\xi_{ij}^1 - r_j^n(1-\tau_{ij}(t))(1+h_j(t))\xi_{ij}^2 \right.$$

$$\left. -x_{ij}^n(t)(1-\tau_{ij}(t))\xi_j^3 + (1-\tau_{ij}(t))(1+h_j(t))y_{ij}^n(t)\xi_j^3 + F_j(t)(r_j^n(t)-\xi_j^3)\right] dt.$$

For the weak convergence, we have:

$$\lim_{n\to\infty} \sum_{i=1}^m \sum_{j=1}^n \int_0^T r_j^n(t)(1-\tau_{ij}(t))\xi_{ij}^1 \, dt = \sum_{i=1}^m \sum_{j=1}^n \int_0^T r_j(t)(1-\tau_{ij}(t))\xi_{ij}^1 \, dt;$$

$$\lim_{n\to\infty} \sum_{i=1}^m \sum_{j=1}^n \int_0^T -r_j^n(1-\tau_{ij}(t))(1+h_j(t))\xi_{ij}^2 \, dt$$

$$= \sum_{i=1}^m \sum_{j=1}^n \int_0^T -r_j(1-\tau_{ij}(t))(1+h_j(t))\xi_{ij}^2 \, dt;$$

$$\lim_{n\to\infty} \sum_{i=1}^m \sum_{j=1}^n \int_0^T -x_{ij}^n(t)(1-\tau_{ij}(t))\xi_j^3 \, dt = \sum_{i=1}^m \sum_{j=1}^n \int_0^T -x_{ij}(t)(1-\tau_{ij}(t))\xi_j^3 \, dt;$$

$$\lim_{n\to\infty} \sum_{i=1}^m \sum_{j=1}^n \int_0^T (1-\tau_{ij}(t))(1+h_j(t))y_{ij}^n(t)\xi_j^3 \, dt$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T (1 - \tau_{ij}(t))(1 + h_j(t)) y_{ij}(t) \xi_j^3 \, dt;$$

$$\lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T F_j(t)(r_j^n(t) - \xi_j^3)\, dt = \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T F_j(t)(r_j(t) - \xi_j^3)\, dt.$$

It remains to prove that

$$\liminf_{n\to\infty} \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial x_{ij}} \left(x_{ij}^n(t) - \xi_{ij}^1\right) \, dt$$

$$\geq \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial u_i\left(t, x_i(t), y_i(t)\right)}{\partial x_{ij}} \left(x_{ij}(t) - \xi_{ij}^1\right) \, dt$$

and the analogous relation for $y_{ij}(t)$. The operators $-\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial x_i}$ are Nemitsky operators (see [58]) and then they are L^2-continuous. Hence, we have

$$\lim_{n\to\infty} \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial u_i\left(t, x_i^n(t), y_i^n(t)\right)}{\partial x_{ij}} \left(x_{ij}^n(t) - \xi_{ij}^1\right) \, dt$$

$$= \int_0^T \sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial u_i\left(t, x_i(t), y_i(t)\right)}{\partial x_{ij}} \left(x_{ij}(t) - \xi_{ij}^1\right) \, dt$$

$$\forall \{x_n\} : x_n \longrightarrow x \text{ in } L^2,$$

namely the strong continuity of the operator $-\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial x_{ij}} \left(x_{ij}(t) - \xi_{ij}^1\right)$.

Then, taking into account that $-\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial x_{ij}}$ and $-\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial y_{ij}}$ are convex and monotone, it is easy to prove that the operator

$$\sum_{i=1}^{m} \sum_{j=1}^{n} -\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial x_{ij}} \left(x_{ij}(t) - \xi_{ij}^1\right)$$

$$+\sum_{i=1}^{m}\sum_{j=1}^{n} -\frac{\partial u_i(t, x_i(t), y_i(t))}{\partial y_{ij}} \left(y_{ij}(t) - \xi_{ij}^2\right)$$

is convex (see [87]). Indeed, if a function is continuous and convex, then it is weakly lower semicontinuous. Hence function $A(t, w)$ is Fan-hemicontinuous. □

5.4 Notes on infinite dimensional duality

In this section we recall the infinite dimensional duality (see [7], [43], [86] and [129]) for the general financial equilibrium problem expressed by variational inequality (5.9). The infinite dimensional duality allows us to define the evaluation index $E(t)$, which will be used in the sequel.

The first step is to introduce the Lagrange functional for this general model. To this end, as usual, let us set

$$\begin{aligned} f(x, y, r) \;=\; & \int_0^T \Bigg\{ \sum_{i=1}^{m}\sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1 - \tau_{ij}(t)) r_j^*(t) \right] \\ & \quad \times [x_{ij}(t) - x_{ij}^*(t)] \\ + & \sum_{i=1}^{m}\sum_{j=1}^{n} \left[-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1 - \tau_{ij}(t))(1 + h_j(t)) r_j^*(t) \right] \\ & \quad \times [y_{ij}(t) - y_{ij}^*(t)] \\ + & \sum_{j=1}^{n} \left[\sum_{i=1}^{m} (1 - \tau_{ij}(t)) \left[x_{ij}^*(t) - (1 + h_j(t)) y_{ij}^*(t)\right] + F_j(t) \right] \\ & \quad \times \left[r_j(t) - r_j^*(t)\right] \Bigg\} dt. \end{aligned}$$

Then the Lagrange functional is

$$\begin{aligned}
&\mathcal{L}(x, y, r, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}) \\
&= f(x, y, r) - \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(1)}(t) x_{ij}(t)\, dt \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} \int_0^T \lambda_{ij}^{(2)} y_{ij}(t)\, dt - \sum_{i=1}^{m} \int_0^T \mu_i^{(1)}(t) \left(\sum_{j=1}^{n} x_{ij}(t) - \hat{s}_i(t) \right) dt \\
&- \sum_{i=1}^{m} \int_0^T \mu_i^{(2)}(t) \left(\sum_{j=1}^{n} y_{ij}(t) - \hat{l}_i(t) \right) dt \\
&+ \sum_{j=1}^{n} \int_0^T \rho_j^{(1)}(t)(\underline{r}_j(t) - r_j(t))\, dt \\
&+ \sum_{j=1}^{n} \int_0^T \rho_j^{(2)}(t)(r_j(t) - \overline{r}_j(t))\, dt, \qquad (5.18)
\end{aligned}$$

where $(x, y, r) \in L^2([0, T], \mathbb{R}^{2mn+n})$, $\lambda^{(1)}, \lambda^{(2)} \in L^2([0, T], \mathbb{R}_+^{mn})$, $\mu^{(1)}, \mu^{(2)} \in L^2([0, T], \mathbb{R}^m)$, $\rho^{(1)}, \rho^{(2)} \in L^2([0, T], \mathbb{R}_+^n)$.

Remember that $\lambda^{(1)}$, $\lambda^{(2)}$, $\rho^{(1)}$, $\rho^{(2)}$ are the Lagrange multipliers associated, a.e. in $[0, T]$, to the sign constraints $x_i(t) \geq 0$, $y_i(t) \geq 0$, $r_j(t) - \underline{r}_j(t) \geq 0$, $\overline{r}_j(t) - r_j(t) \geq 0$, respectively. The functions $\mu^{(1)}(t)$ and $\mu^{(2)}(t)$ are the Lagrange multipliers associated, a.e. in $[0, T]$, to the equality constraints $\sum_{j=1}^{n} x_{ij}(t) = s_i \left(t, \int_0^T w^*(s)\, ds \right) = \hat{s}_i(t)$ and $\sum_{j=1}^{n} y_{ij}(t) = l_i \left(t, \int_0^T w^*(s)\, ds \right) = \hat{l}_i(t)$, respectively.

Now we recall the following theorem (see [7] for the proof).

Theorem 5.4.1 *Let* $w^* = (x^*, y^*, r^*) \in \mathbb{K}(w^*)$ *be a solution to variational inequality* (5.9) *and let us consider the associated Lagrange functional* (5.18). *Then, the strong duality holds and there exist*

$$\lambda^{(1)*},\ \lambda^{(2)*} \in L^2([0, T], \mathbb{R}_+^{mn}),$$

$$\mu^{(1)*},\ \mu^{(2)*} \in L^2([0,T],\mathbb{R}^m),$$
$$\rho^{(1)*},\ \rho^{(2)*} \in L^2([0,T],\mathbb{R}^n_+)$$

such that $(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*},\ \mu^{(1)*},\ \mu^{(2)*},\ \rho^{(1)*}, \rho^{(2)*})$ *is a saddle point of the Lagrange functional, namely*

$$\begin{aligned} & \mathcal{L}(x^*, y^*, r^*, \lambda^{(1)}, \lambda^{(2)}, \mu^{(1)}, \mu^{(2)}, \rho^{(1)}, \rho^{(2)}) \\ \leq\ & \mathcal{L}(x^*, y^*, r^*, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) = 0 \\ \leq\ & \mathcal{L}(x, y, r, \lambda^{(1)*}, \lambda^{(2)*}, \mu^{(1)*}, \mu^{(2)*}, \rho^{(1)*}, \rho^{(2)*}) \end{aligned}$$

$\forall (x,y,r) \in L^2([0,T],\mathbb{R}^{2mn+n})$, $\forall \lambda^{(1)}, \lambda^{(2)} \in L^2([0,T],\mathbb{R}^{mn}_+)$, $\forall \mu^{(1)}, \mu^{(2)} \in L^2([0,T],\mathbb{R}^m)$, $\forall \rho^{(1)}, \rho^{(2)} \in L^2([0,T],\mathbb{R}^n_+)$ *and, a.e. in* $[0,T]$,

$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial x_{ij}} - (1-\tau_{ij}(t))r_j^*(t) - \lambda_{ij}^{(1)*}(t) - \mu_i^{(1)*}(t) = 0,$$
$$\forall i = 1,\dots,m,\ \forall j = 1\dots,n;$$
$$-\frac{\partial u_i(t, x^*(t), y^*(t))}{\partial y_{ij}} + (1-\tau_{ij}(t))(1+h_j(t))r_j^*(t) - \lambda_{ij}^{(2)*}(t) - \mu_i^{(2)*}(t) = 0,$$
$$\forall i = 1,\dots,m,\ \forall j = 1\dots,n;$$

$$\sum_{i=1}^m (1-\tau_{ij}(t))\left[x_{ij}^*(t) - (1+h_j(t))y_{ij}^*(t)\right] + F_j(t) + \rho_j^{(2)*}(t) = \rho_j^{(1)*}(t),$$
$$\forall j = 1,\dots,n; \tag{5.19}$$

$$\lambda_{ij}^{(1)*}(t)x_{ij}^*(t) = 0,\ \lambda_{ij}^{(2)*}(t)y_{ij}^*(t) = 0, \quad \forall i = 1,\dots,m,\ \forall j = 1,\dots,n \tag{5.20}$$

$$\mu_i^{(1)*}(t)\left(\sum_{j=1}^n x_{ij}^*(t) - \hat{s}_i(t)\right) = 0, \quad \mu_i^{(2)*}(t)\left(\sum_{j=1}^n y_{ij}^*(t) - \hat{l}_i(t)\right) = 0,$$
$$\forall i = 1,\dots,m; \tag{5.21}$$

$$\rho_j^{(1)*}(t)(\underline{r}_j(t) - r_j^*(t)) = 0, \rho_j^{(2)*}(t)(r_j^*(t) - \overline{r}_j(t)) = 0,$$
$$\forall j = 1,\dots,n. \tag{5.22}$$

Definition 5.4.1 *We define "Evaluation Index" the value*

$$E(t) = \frac{\sum_{i=1}^{m} \hat{l}_i(t)}{\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t)},$$

where

$$\tilde{s}_i(t) = \frac{\hat{s}_i(t)}{1 + i(t)} \text{ and } \tilde{F}_j(t) = \frac{F_j(t)}{1 + i(t) - \theta(t) - \theta(t)\, i(t)}.$$

We state that, if $E(t)$ is greater than or equal to 1, the evaluation of the financial equilibrium is positive, whereas if $E(t)$ is less than 1, the evaluation of the financial equilibrium is negative.

In fact, we know that

$$E(t) = 1 - \frac{\sum_{j=1}^{n} \rho_j^{(1)*}(t)}{(1-\theta(t))(1+i(t)) \left(\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t) \right)} + \frac{\sum_{j=1}^{n} \rho_j^{(2)*}(t)}{(1-\theta(t))(1+i(t)) \left(\sum_{i=1}^{m} \tilde{s}_i(t) + \sum_{j=1}^{n} \tilde{F}_j(t) \right)}.$$

If $E(t) < 1$, it means that $\sum_{j=1}^{n} \rho_j^{(2)*}(t) < \sum_{j=1}^{n} \rho_j^{(1)*}(t)$ and this implies that the sum of deficits exceeds the sum of surpluses and we find a negative outlook.

On the other hand if $E(t) \geq 1$, then $\sum_{j=1}^{n} \rho_j^{(2)*}(t) \geq \sum_{j=1}^{n} \rho_j^{(1)*}(t)$ and we have already considered the positive effects of this surplus.

The Evaluation Index has been fixed assuming that the taxes $\tau_{ij}(t)$, $i = 1, \dots, m$, $j = 1, \dots, n$ have a common value $\theta(t)$, and the increments $h_j(t)$, $j = 1, \dots, n$, have a common value $i(t)$. However it can also be considered the case when $\tau_{ij}(t)$, $h_j(t)$ are all different, assuming $\theta(t)$ and $i(t)$ as the averages of $\tau_{ij}(t)$ and $h_j(t)$, namely

$$\theta(t) = \frac{\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\tau_{ij}(t)}{mn} \text{ and } i(t) = \frac{\displaystyle\sum_{j=1}^{n} h_j(t)}{n},$$

respectively. Also in this case the Evaluation Index gives reliable information.

5.5 Numerical Examples

5.5.1 Example 1

We consider a special utility function, specifically the quadratic utility function obtained by means of the variance-covariance matrix which denotes the sector's assessment of the standard deviation of prices for each instrument. In detail, we consider an economy with two agents and two financial instruments. The variance-covariance matrices of the two agents are:

$$Q^1 = \begin{bmatrix} 1 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.5 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the time interval $[0, 1]$, the term $u_i(t, x_i(t), y_i(t))$ is given by:

$$u_i(t, x_i(t), y_i(t)) = -\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}.$$

We choose as the feasible set for assets, liabilities and prices:

$$\mathbb{K}(w^*) = \Big\{ w = (x(t), y(t), r(t)) \in L^2([0, 1], \mathbb{R}^{10}_+) :$$

$$x_{11}(t) + x_{12}(t) = \alpha \int_0^1 r_1^*(s)\, ds + \beta,\ x_{21}(t) + x_{22}(t) = \alpha \int_0^1 r_2^*(s)\, ds + \beta,$$

$$y_{11}(t) + y_{12}(t) = \gamma,\ y_{21}(t) + y_{22}(t) = \delta, \text{ a.e. in } [0,1]$$

$$\left.\text{and } 2t \leq r_1(t) \leq 5t \text{ and } t \leq r_2(t) \leq 10t \text{ a.e. in } [0,1]\right\}$$

where α, β and δ are positive parameters to be appropriately fixed. It follows that variational inequality (5.9) becomes the problem:
Find $w^* \in \mathbb{K}(w^*)$:

$$\begin{aligned}
\int_0^1 \Big(&\left[2(x_{11}^*(t) - 0.5 y_{11}^*(t)) - (1-\tau_{11}(t))\, r_1^*(t)\right] (x_{11}(t) - x_{11}^*(t)) + \\
&+ \left[2x_{12}^*(t) - (1-\tau_{12}(t))\, r_2^*(t)\right] (x_{12}(t) - x_{12}^*(t)) + \\
&+ \left[2x_{21}^*(t) - (1-\tau_{21}(t))\, r_1^*(t)\right] (x_{21}(t) - x_{21}^*(t)) + \\
&+ \left[2(x_{22}^*(t) - 0.5 y_{21}^*(t)) - (1-\tau_{22}(t))\, r_2^*(t)\right] (x_{22}(t) - x_{22}^*(t)) + \\
&+ \left[2(y_{11}^*(t) - 0.5 x_{11}^*(t)) + (1-\tau_{11}(t))\, r_1^*(t)(1 + h_1(t))\right] \\
&\quad (y_{11}(t) - y_{11}^*(t)) + \\
&+ \left[2y_{12}^*(t) + (1-\tau_{12}(t))\, r_2^*(t)(1 + h_2(t))\right] (y_{12}(t) - y_{12}^*(t)) + \\
&+ \left[2(y_{21}^*(t) - 0.5 x_{22}^*(t)) + (1-\tau_{21}(t))\, r_1^*(t)(1 + h_1(t))\right] \\
&\quad (y_{21}(t) - y_{21}^*(t)) + \\
&+ \left[2y_{22}^*(t) + (1-\tau_{22}(t))\, r_2^*(t)(1 + h_2(t))\right] (y_{22}(t) - y_{22}^*(t)) + \\
&+ \{(1-\tau_{11}(t))\, [x_{11}^*(t) - (1 + h_1(t))\, y_{11}^*(t)] + \\
&+ (1-\tau_{21}(t))\, [x_{21}^*(t) - (1 + h_1(t))\, y_{21}^*(t)] + F_1(t)\} \,(r_1(t) - r_1^*(t)) + \\
&+ \{(1-\tau_{12}(t))\, [x_{12}^*(t) - (1 + h_2(t))\, y_{12}^*(t)] + \\
&+ (1-\tau_{22}(t))\, [x_{22}^*(t) - (1 + h_2(t))\, y_{22}^*(t)] + F_2(t)\} \\
&\quad (r_2(t) - r_2^*(t)) \Big)\, \mathrm{d}t \geq 0 \qquad \forall w \in \mathbb{K}(w^*).
\end{aligned}$$

From the conservation laws, we get:

$$\begin{aligned}
x_{12}(t) &= \alpha \int_0^T r_1^*(s)\, \mathrm{d}s - x_{11}(t) + \beta, & x_{21}(t) &= \alpha \int_0^T r_2^*(s)\, \mathrm{d}s - x_{22}(t) + \beta \\
y_{12}(t) &= -\, y_{11}(t) + \gamma, & y_{21}(t) &= -\, y_{22}(t) + \delta\,.
\end{aligned}$$

So, we have:

$$\int_0^1 \Bigg(\left[4\, x_{11}^*(t) - y_{11}^*(t) - 2\,\alpha \int_0^T r_1^*(s)\,\mathrm{d}s - 2\beta - (1-\tau_{11}(t))r_1^*(t) + (1-\tau_{12}(t))r_2^*(t)\right]$$

$$(x_{11}(t) - x_{11}^*(t))$$

$$+ \left[4\, x_{22}^*(t) + y_{22}^*(t) - 2\,\alpha \int_0^T r_2^*(s)\,\mathrm{d}s - 2\beta - \delta + (1-\tau_{21}(t))r_1^*(t) - (1-\tau_{22}(t))r_2^*(t)\right]$$

$$(x_{22}(t) - x_{22}^*(t))$$

$$+ [4\, y_{11}^*(t) - x_{11}^*(t) - 2\,\gamma + (1-\tau_{11}(t))\, r_1^*(t)\,(1+h_1(t)) - (1-\tau_{12}(t))\, r_2^*(t)\,(1+h_2(t))]$$

$$(y_{11}(t) - y_{11}^*(t))$$

$$+ [4\, y_{22}^*(t) + x_{22}^*(t) - 2\,\delta - (1-\tau_{21}(t))\, r_1^*(t)\,(1+h_1(t)) + (1-\tau_{22}(t))\, r_2^*(t)\,(1+h_2(t))]$$

$$(y_{22}(t) - y_{22}^*(t)) \tag{5.23}$$

$$+\Bigg\{ (1-\tau_{11}(t))[x_{11}^*(t) - (1+h_1(t))y_{11}^*(t)]$$

$$+(1-\tau_{21}(t)) \left[\alpha \int_0^T r_2^*(s)\,\mathrm{d}s - x_{22}^*(t) + \beta - (1+h_1(t))(-y_{22}^*(t) + \delta)\right] + F_1(t) \Bigg\}$$

$$(r_1(t) - r_1^*(t))$$

$$+\Bigg\{ (1-\tau_{12}(t)) \left[\alpha \int_0^T r_1^*(s)\,\mathrm{d}s - x_{11}^*(t) + \beta - (1+h_2(t))(-y_{11}^*(t) + \gamma)\right]$$

$$+(1-\tau_{22}(t))[x_{22}^*(t) - (1+h_2(t))y_{22}^*(t)] + F_2(t) \Bigg\} (r_2(t) - r_2^*(t))\ \mathrm{d}t \geq 0\,.$$

Applying the direct method (see [?]) and choosing $\tau_{ij} = \frac{1}{4}$ $\forall i, j$ and $h_j = 1$ $\forall j$, we find that the solution to the variational inequality (5.23) is given by solving the system

$$\begin{cases} \Gamma_1 = 4x_{11}^*(t) - y_{11}^*(t) - 2\alpha \int_0^1 r_1^*(s)\,\mathrm{d}s - 2\beta - \frac{3}{4}r_1^* + \frac{3}{4}r_2^* = 0 \\ \Gamma_2 = 4x_{22}^*(t) + y_{22}^*(t) - 2\alpha \int_0^1 r_2^*(s)\,\mathrm{d}s - 2\beta - \delta + \frac{3}{4}r_1^* - \frac{3}{4}r_2^* = 0 \\ \Gamma_3 = 4y_{11}^*(t) - x_{11}^*(t) - 2\gamma + \frac{3}{2}r_1^* - \frac{3}{2}r_2^* = 0 \\ \Gamma_4 = 4y_{22}^*(t) + x_{22}^*(t) - 2\delta - \frac{3}{2}r_1^* + \frac{3}{2}r_2^* = 0 \\ \Gamma_5 = \frac{3}{4}\left[x_{11}^*(t) - 2y_{11}^*(t)\right] + \frac{3}{4}\left[\alpha \int_0^1 r_2^*(s)\,\mathrm{d}s - x_{22}^* + \beta - 2(-y_{22}^* + \delta)\right] \\ \qquad + F_1 > 0 \\ \Gamma_6 = \frac{3}{4}\left[\alpha \int_0^1 r_1^*(s)\,\mathrm{d}s - x_{11}^* + \beta - 2(-y_{11}^* + \gamma)\right] + \frac{3}{4}\left[x_{22}^*(t) - 2y_{22}^*(t)\right] \\ \qquad + F_2 > 0\,. \end{cases}$$

Since $\Gamma_5 > 0$ and $\Gamma_6 > 0$, the direct method ensures that

$$r_1^*(t) = \underline{r_1}(t) = 2t \qquad r_2^*(t) = \underline{r_2}(t) = t.$$

Moreover, since

$$\int_0^1 r_1^*(t)\,\mathrm{d}t = \int_0^1 2t\,\mathrm{d}t = 1 \text{ and } \int_0^1 r_2^*(t)\,\mathrm{d}t = \int_0^1 t\,\mathrm{d}t = \frac{1}{2},$$

the system:

$$\begin{cases} \Gamma_1 = 0 \iff 4x_{11}^*(t) - y_{11}^*(t) = 2\alpha + 2\beta + \frac{3}{4}t \\ \Gamma_2 = 0 \iff 4x_{22}^*(t) + y_{22}^*(t) = \alpha + 2\beta + \delta - \frac{3}{4}t \\ \Gamma_3 = 0 \iff 4y_{11}^*(t) - x_{11}^*(t) = 2\gamma - \frac{3}{2}t \\ \Gamma_4 = 0 \iff 4y_{22}^*(t) + x_{22}^*(t) = 2\delta + \frac{3}{2}t \end{cases}$$

yields:

$$\begin{cases} y^*_{11}(t) = \frac{1}{15}\left(2\alpha + 2\beta + 8\gamma - \frac{21}{4}t\right) \\ y^*_{22}(t) = \frac{1}{15}\left(-\alpha - 2\beta + 7\delta + \frac{27}{4}t\right) \\ x^*_{11}(t) = \frac{8}{15}(\alpha + \beta) + \frac{2}{15}\gamma + \frac{1}{10}t \\ x^*_{22}(t) = \frac{4}{15}(\alpha + 2\beta) + \frac{2}{15}\delta - \frac{3}{10}t. \end{cases}$$

Further, $\Gamma_5 > 0$ and $\Gamma_6 > 0$ mean:

$$F_1 > \frac{7}{10}\gamma + \frac{9}{10}\delta - \frac{11}{40}\alpha - \frac{7}{20}\beta - \frac{3}{2}t$$

$$F_2 > \frac{4}{5}\gamma + \frac{3}{5}\delta - \frac{17}{20}\alpha - \frac{23}{20}\beta + \frac{3}{2}t$$

Now, we verify that the following conditions are fulfilled:

$$0 \le x^*_{11}(t) \le \beta + \alpha \qquad 0 \le y^*_{11}(t) \le \gamma$$

$$0 \le x^*_{22}(t) \le \beta + \frac{1}{2}\alpha \qquad 0 \le y^*_{22}(t) \le \delta$$

We have:

- $x^*_{11}(t) \le \beta + \alpha \iff \alpha + \beta \ge \frac{2}{7}\gamma + \frac{3}{14}$ in $[0, 1]$
- $x^*_{22}(t) \ge 0 \iff \alpha + 2\beta \ge -\frac{1}{2}\delta + \frac{9}{8}$ in $[0, 1]$
- $x^*_{22}(t) \le \beta + \frac{1}{2}\alpha \iff \alpha + 2\beta \ge \frac{4}{7}\delta$ in $[0, 1]$
- $y^*_{11}(t) \ge 0 \iff \alpha + \beta \ge -4\gamma + \frac{21}{8}$ in $[0, 1]$
- $y^*_{11}(t) \le \gamma \iff \alpha + \beta \le \frac{7}{2}\gamma$ in $[0, 1]$

- $y_{22}^*(t) \geq 0 \iff \alpha + 2\beta \leq 7\delta$ in $[0,1]$
- $y_{22}^*(t) \leq \delta \iff \alpha + 2\beta \geq \dfrac{27}{4} - 8\delta$ in $[0,1]$

So:

$$\max\left\{-\frac{1}{2}\delta + \frac{9}{8},\ \frac{4}{7}\delta,\ \frac{27}{4} - 8\delta\right\} \leq \alpha + 2\beta \leq 7\delta \text{ in } [0,1]$$
$$\max\left\{\frac{2}{7}\gamma + \frac{3}{14},\ -4\gamma + \frac{21}{8}\right\} \leq \alpha + \beta \leq \frac{7}{2}\gamma \text{ in } [0,1]\,.$$

Now, we choose $\alpha = 2$, $\beta = 5$, $\delta = 2$, $\gamma = 10$ and, replacing these values in the equilibrium solution, we obtain a.e. in $t \in [0,1]$:

$$\left\{\begin{array}{lcl} x_{11}^*(t) &=& \dfrac{76}{15} + \dfrac{1}{10}t, \\[2ex] x_{22}^*(t) &=& \dfrac{52}{15} - \dfrac{3}{10}t, \\[2ex] y_{11}^*(t) &=& \dfrac{94}{15} - \dfrac{7}{20}t, \\[2ex] y_{22}^*(t) &=& \dfrac{2}{15} + \dfrac{9}{20}t, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} x_{12}^*(t) &=& \dfrac{29}{15} - \dfrac{1}{10}t, \\[2ex] x_{21}^*(t) &=& \dfrac{38}{15} + \dfrac{3}{10}t, \\[2ex] y_{12}^*(t) &=& \dfrac{56}{15} + \dfrac{7}{20}t, \\[2ex] y_{21}^*(t) &=& \dfrac{28}{15} - \dfrac{9}{20}t. \end{array}\right.$$

Finally, conditions $\Gamma_5 > 0$ and $\Gamma_6 > 0$ yield: $F_1 > \dfrac{13}{2}$ in $[0,\ 1]$ and $F_2 > \dfrac{13}{4}$ in $[0,1]$. From formulas (5.19)and (5.22) we know that:

$$\Gamma_5 + \rho_1^{(2)*}(t) = \rho_1^{(1)*}(t) \ \text{ and } \ \Gamma_6 + \rho_2^{(2)*}(t) = \rho_2^{(1)*}(t)$$

and

$$\rho_1^{(1)*}(t)(\underline{r}_1(t) - r_1^*(t)) = 0 \ \text{ and } \ \rho_1^{(2)*}(t)(r_1^*(t) - \overline{r}_1(t)) = 0,$$
$$\rho_2^{(2)*}(t)(\underline{r}_2(t) - r_2^*(t)) = 0 \ \text{ and } \ \rho_2^{(2)*}(t)(r_2^*(t) - \overline{r}_2(t)) = 0.$$

Since $r_1^*(t) = \underline{r}_1(t)$, we obtain $\rho_1^{(1)*}(t) > 0$ and $\rho_1^{(2)*}(t) = 0$; hence:

$$\Gamma_5 = \rho_1^{(1)*}(t) > 0.$$

Analogously, since $r_2^*(t) = \underline{r}_2(t)$, we obtain $\rho_2^{(1)*}(t) > 0$ and $\rho_2^{(2)*}(t) = 0$; hence:

$$\Gamma_6 = \rho_2^{(1)*}(t) > 0.$$

But $\rho_1^{(1)*}(t)$ and $\rho_2^{(1)*}(t)$ are the deficit variables and are positive. So the economy is in a phase of regression. The same conclusion is confirmed by the evaluation index whose value, in this example, is

$$E(t) = \frac{72}{83} < 1.$$

5.5.2 Example 2

With the same data as in Example 1, but with the new feasible set defined as:

$$\mathbb{K}(w^*) = \Big\{ w = (x(t), y(t), r(t)) \in L^2([0,1], \mathbb{R}^{10}_+) :$$

$$x_{11}(t) + x_{12}(t) = \alpha \int_0^1 r_1^*(s)\,ds + \beta,\ \ x_{21}(t) + x_{22}(t) = \alpha \int_0^1 r_2^*(s)\,ds + \beta,$$

$$y_{11}(t) + y_{12}(t) = \gamma,\ \ y_{21}(t) + y_{22}(t) = \delta,\ \text{a.e. in } [0,1]$$

$$\text{and } t \le r_1(t) \le 2t \text{ and } t \le r_2(t) \le 10t \text{ a.e. in } [0,1] \Big\}$$

now we solve the system:

$$\begin{cases} \Gamma_1 = 0 \\ \Gamma_2 = 0 \\ \Gamma_3 = 0 \\ \Gamma_4 = 0 \\ \Gamma_5 < 0 \\ \Gamma_6 > 0 \end{cases}$$

and the direct method ensures that

$$r_1^*(t) = \overline{r}_1(t) = 2t \text{ and } r_2^*(t) = \underline{r}_2(t) = t,$$

which yield: $F_1 < 5$ and $F_2 > \dfrac{13}{4}$. With the same remarks as in Example 1, here we have:

$$\Gamma_5 = \rho_1^{(2)*}(t)$$

and the evaluation index is $E(t) = \dfrac{72}{67} > 1$. So the economy is positive.

5.5.3 Example 3

Let's modify the conservations laws as follows:

$$\mathbb{K}(w^*) = \Bigg\{ w = (x(t), y(t), r(t)) \in L^2([0,1], \mathbb{R}^{10}_+) :$$

$$x_{11}(t) + x_{12}(t) = \alpha \int_0^1 x_1^*(s)\, ds + \beta,\ x_{21}(t) + x_{22}(t) = \alpha \int_0^1 x_2^*(s)\, ds + \beta,$$

$$y_{11}(t) + y_{12}(t) = \gamma,\ y_{21}(t) + y_{22}(t) = \delta, \text{ a.e. in } [0,1]$$

$$\text{and } 2t \le r_1(t) \le 5t \text{ and } t \le r_2(t) \le 10t \text{ a.e. in } [0,1] \Bigg\}$$

Then, variational inequality (5.23) becomes:

$$\int_0^1 \Bigg(\Bigg[4\, x_{11}^*(t) - y_{11}^*(t) - 2\,\alpha \int_0^T x_1^*(s)\, \mathrm{d}s - 2\beta - (1 - \tau_{11}(t)) r_1^*(t) + (1 - \tau_{12}(t)) r_2^*(t) \Bigg]$$

$$(x_{11}(t) - x_{11}^*(t))$$

$$+ \Bigg[4\, x_{22}^*(t) + y_{22}^*(t) - 2\,\alpha \int_0^T x_2^*(s)\, \mathrm{d}s - 2\beta - \delta + (1 - \tau_{21}(t)) r_1^*(t) - (1 - \tau_{22}(t)) r_2^*(t) \Bigg]$$

$$(x_{22}(t) - x_{22}^*(t))$$

$$+\left[4\,y^*_{11}(t)-x^*_{11}(t)-2\,\gamma+(1-\tau_{11}(t))\,r^*_1(t)\,(1+h_1(t))-(1-\tau_{12}(t))\,r^*_2(t)\,(1+h_2(t))\right]$$

$$(y_{11}(t)-y^*_{11}(t))$$

$$+\left[4\,y^*_{22}(t)+x^*_{22}(t)-2\,\delta-(1-\tau_{21}(t))\,r^*_1(t)\,(1+h_1(t))+(1-\tau_{22}(t))\,r^*_2(t)\,(1+h_2(t))\right]$$

$$(y_{22}(t)-y^*_{22}(t))$$

$$+\{(1-\tau_{11}(t))[x^*_{11}(t)-(1+h_1(t))y^*_{11}(t)]$$

$$+(1-\tau_{21}(t))\left[\alpha\int_0^T x^*_2(s)\,\mathrm{d}s-x^*_{22}(t)+\beta-(1+h_1(t))(-y^*_{22}(t)+\delta)\right]+F_1(t)\}$$

$$(r_1(t)-r^*_1(t))$$

$$+\{(1-\tau_{12}(t))\left[\alpha\int_0^T x^*_1(s)\,\mathrm{d}s-x^*_{11}(t)+\beta-(1+h_2(t))(-y^*_{11}(t)+\gamma)\right]$$

$$+(1-\tau_{22}(t))[x^*_{22}(t)-(1+h_2(t))y^*_{22}(t)]+F_2(t)\}\,(r_2(t)-r^*_2(t))\,\mathrm{d}t\geq 0\,.$$

Applying the direct method and choosing $\tau_{ij}=\frac{1}{4}\ \forall i,\,j$ and $h_j=1\ \forall j$, we obtain the solution to variational inequality (5.23) given by the solution to the system:

$$\begin{cases}
\Gamma_1=4x^*_{11}(t)-y^*_{11}(t)-2\alpha\displaystyle\int_0^1 x^*_{11}(s)\mathrm{d}s-2\beta-\frac{3}{4}(r^*_1(t)-r^*_2(t))=0\\
\Gamma_2=4x^*_{22}(t)+y^*_{22}(t)-2\alpha\displaystyle\int_0^1 x^*_{22}(s)\mathrm{d}s-2\beta-\delta+\frac{3}{4}(r^*_1(t)-r^*_2(t))=0\\
\Gamma_3=4y^*_{11}(t)-x^*_{11}(t)-2\gamma+\dfrac{3}{2}(r^*_1(t)-r^*_2(t))=0\\
\Gamma_4=4y^*_{22}(t)+x^*_{22}(t)-2\delta-\dfrac{3}{2}(r^*_1(t)-r^*_2(t))=0\\
\Gamma_5=\dfrac{3}{4}\left[x^*_{11}(t)-2y^*_{11}(t)\right]+\dfrac{3}{4}\left[\alpha\displaystyle\int_0^1 x^*_{22}(s)\mathrm{d}s-x^*_{22}+\beta-2(-y^*_{22}+\delta)\right]\\
\qquad +F_1>0\\
\Gamma_6=\dfrac{3}{4}\left[\alpha\displaystyle\int_0^1 x^*_{11}(s)\mathrm{d}s-x^*_{11}+\beta-2(-y^*_{11}+\gamma)\right]+\dfrac{3}{4}\left[x^*_{22}(t)-2y^*_{22}(t)\right]\\
\qquad +F_2>0\,.
\end{cases}$$

Since $\Gamma_5 > 0$ and $\Gamma_6 > 0$, the direct method guarantees that

$$r_1^*(t) = \underline{r}_1(t) = 2t \ \text{ and } \ r_2^*(t) = \underline{r}_2(t) = t.$$

Moreover, since

$$\int_0^1 r_1^*(t)\,\mathrm{d}t = \int_0^1 2t\,\mathrm{d}t = 1 \text{ and } \int_0^1 r_2^*(t)\,\mathrm{d}t = \int_0^1 t\,\mathrm{d}t = \frac{1}{2},$$

choosing $\beta = 3$ and $r_1^*(t) - r_2^*(t) = t$ the previous system gives:

$$\begin{aligned}
x_{11}^*(t) &= \frac{1}{10}t + \frac{8}{5} + \frac{2}{15}\gamma + \frac{66}{5(15-8\alpha)}\,\alpha + \frac{16}{15(15-8\alpha)}\,\alpha\,\gamma \\
y_{11}^*(t) &= -\frac{7}{20}t + \frac{2}{5} + \frac{8}{15}\,\gamma + \frac{33}{10(15-8\alpha)}\,\alpha + \frac{4}{15(15-8\alpha)}\,\alpha\,\gamma \\
x_{22}^*(t) &= -\frac{3}{10}t + \frac{8}{5} + \frac{2}{15}\,\delta + \frac{58}{5(15-8\alpha)}\,\alpha + \frac{16}{15(15-8\alpha)}\,\alpha\,\delta \\
y_{22}^*(t) &= \frac{9}{20}t - \frac{2}{5} + \frac{7}{15}\delta - \frac{29}{10(15-8\alpha)}\,\alpha - \frac{4}{15(15-8\alpha)}\,\alpha\,\delta\,.
\end{aligned}$$

Now we verify:

$$\begin{aligned}
&0 \le x_{11}^*(t) \le \alpha \int_0^1 x_{11}^*(s)\mathrm{d}s + \beta \\
&0 \le x_{22}^*(t) \le \alpha \int_0^1 x_{22}^*(s)\mathrm{d}s + \beta \\
&0 \le y_{11}^*(t) \le \gamma \\
&0 \le y_{22}^*(t) \le \delta
\end{aligned}$$

The first one equation is verified for $\alpha = 1$ and $\beta = 3$ and $\forall \gamma$; the second one for $\alpha = 1$, $\beta = 3$ and $\forall \delta$; the third one for $\alpha = 1$ and $\gamma \ge 183/90$; the last one for $\alpha = 1$ and $\delta \ge 171/90$.
If we choose $\alpha = 1,\ \beta = 3,\ \gamma = 6,\ \delta = 6$, we obtain a.e. in $[0,1]$ the

following solution:

$$\begin{cases} x_{11}^*(t) &= \dfrac{1}{10}t + \dfrac{182}{35} \\[2ex] x_{22}^*(t) &= -\dfrac{3}{10}t + \dfrac{174}{35} \\[2ex] y_{11}^*(t) &= -\dfrac{7}{20}t + \dfrac{301}{70} \\[2ex] y_{22}^*(t) &= \dfrac{9}{20}t + \dfrac{123}{70} \end{cases} \quad \text{and} \quad \begin{cases} x_{12}^*(t) &= -\dfrac{1}{10}t + \dfrac{61}{20} \\[2ex] x_{21}^*(t) &= \dfrac{3}{10}t + \dfrac{57}{20} \\[2ex] y_{12}^*(t) &= \dfrac{7}{20}t + \dfrac{17}{10} \\[2ex] y_{21}^*(t) &= -\dfrac{9}{20}t + \dfrac{297}{70}\,. \end{cases}$$

Moreover, $\Gamma_5 > 0$ and $\Gamma_6 > 0$ mean $F_1 > \dfrac{591}{112}$ and $F_2 > \dfrac{75}{112}$ in $[0,1]$. Also in this case we have a negative economy as stated by the evaluation index whose value is $E(t) = \dfrac{1008}{1067} < 1$.

Chapter 6

Conclusion

In this thesis we focus our attention on modeling, one related to epidemiology and one related to finance.

In chapter 3 we propose a mathematical model for the dynamics of anorexic and bulimic population. The model proposed takes into account, among other features, the effects of *peers' influence*, *media influence* and *education*. As far as we know, this problem with both kinds of disease compartments has not been yet investigated, and we are not modeling a specific study on a well delimited population. For this reason the values of many parameters are purely indicative.

We begin the analysis ignoring the effects of media pressure and education, and we obtain conditions for global stability of the disease-free equilibrium introducing two reproduction numbers R_a, R_b associated to the anorexic and bulimic population. We show that there exist at most two endemic equilibria: the *purely bulimic* one and the *endemic*, both of which can be stable under certain conditions.

We then consider the influence of an educational campaign. In this last case we notice that the reproduction number rescaled with the coefficient $\chi/(\chi+\xi)$, indicates the fraction of population susceptible to eating disorder in the disease-free state when an education campaign is considered.

We finally study the case in which media influence plays a role. In such a case only one of the equilibria becomes endemic and belongs to the admissible region, while the other two become non-admissible. The equilibrium that becomes admissible is the one which was attracting when $m_1, m_2 = 0$, and as the media parameters are increased it moves into the endemic region with increasing percentages of anorexic and bulimic, remaining an attractor. Naturally the final scenario is radically different whether the equilibrium that becomes the attractor was disease-free, bulimic-endemic, or endemic when $m_1 = m_2 = 0$: the percentages of ill population differ greatly. Despite the effects of mass media, models such as this one serve the practical purpose of deriving reproductive numbers which can predict the possible effects of combating media pressure. In particular, if $R_0' > 1$ then even eliminating mass media influence will not be sufficient to end the disorders.

We conclude observing that, for simplicity, in this paper we have made the assumption that the exit rate is the same for every compartment and it is equal to the entry rate. A more general and biologically more significant model would require different entry and exit rates. The problem could also be enriched by adding multi-group components to capture heterogeneity in the mixing, adding a risk-structure or an age-structure, adding to the incidence functions a saturation term such as, for example, $\beta_1 A/(P + \alpha A + \beta B)$, dividing the recovered in treated and non-susceptible, with different sensitization rates. Most of these refinements can be captured by an averaging assumption on population groups, while others will be made in future works, and some require a mathematical analysis that exceeds our capacities. For this reason we have settled on this model, that could be a good compromise and could be in line with the systems that are being considered nowadays by a mathematical community (of course a numerical investigation with properly chosen parameters can easily be made on more complicate systems).

In chapter 5 we have proved a general existence theorem for quasi-variational inequalities. Moreover, we have applied such a general result to the quasi-variational inequality governing the financial model in the case when the financial volumes depend on time and on the expected solution. Such a dependence is required in order to take care also of the influence of the expected equilibrium distribution for assets and liabilities on the investments on all financial instruments.

We have also recalled the definition of the evaluation index, a very useful tool which allows us to realize if the financial equilibrium is positive or negative.

Finally we have presented three numerical examples and have obtained the equilibrium solutions by using the direct method. The study cases show both regression and positive situations, as proved also by the evaluation indices.

In this thesis we have considered the case in which the assets and liabilities satisfy only nonnegative constraints, i.e. $x_{ij}(t) \geq 0$ and $y_{ij}(t) \geq 0$, but in future works we can examine a more realistic situation when assets and liabilities have to satisfy capacity constraints, i.e. $\underline{x}_{ij}(t) \leq x_{ij}(t) \leq \overline{x}_{ij}(t)$ and $\underline{y}_{ij}(t) \leq y_{ij}(t) \leq \overline{y}_{ij}(t)$. This new formulation of the constraint set seems to be more concrete, since there is always a financial budget to consider in an economy.

The importance of the proved general existence theorem for quasi-variational inequalities is evident because it can be applied to many other problems coming from economy, physics, engineering and so on.

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