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Bounds for invariants of numerical semigroups and Wilf's conjecture

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Abstract

Given coprime positive integers $g_1 < \ldots < g_e$, the Frobenius number $F = F(g_1, \ldots, g_e)$ is the largest integer not representable as a linear combination of $g_1, \ldots, g_e$ with non-negative integer coefficients. Let n denote the number of all representable non-negative integers less than F; Wilf conjectured that $F + 1 \leq en$. We provide bounds for g_1 and for the type of the numerical semigroup $S = \langle g_1, \ldots, g_e \rangle$ in function of e and n, and use these bounds to prove that $F + 1 \leq qen$, where $q = \left\lceil \frac{F+1}{g_1} \right\rceil$, and $F + 1 \leq en^2$. Finally, we give an alternative, simpler proof for the Wilf conjecture if the numerical semigroup $S = \langle g_1, \ldots, g_e \rangle$ is almost-symmetric.

Keywords Wilf conjecture · Numerical semigroups · Multiplicity · Embedding dimension · Type · Almost symmetric numerical semigroup

Mathematics Subject Classification 05A99 · 11B75 · 20M14

1 Introduction

The classical money-changing problem consists of finding what sums of money can be changed, using e different denominations of coins $2 \leq g_1 < \ldots < g_e$. Assuming, without loss of generality, that $\gcd(g_1, \ldots, g_e) = 1$, it is well-known that only a finite number of sums cannot be changed, and there exists a maximum integer $F = F(g_1, \ldots, g_e)$ which cannot be represented as a linear combination of the *generators* $g_1, \ldots, g_e$, with coefficients in the set of natural numbers $\mathbb{N}$.

Determining this maximum integer F, called the Frobenius number, is the subject of the Diophantine Frobenius Problem. This challenging problem has been extensively studied

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over the past decades, and presents applications in several areas of mathematics, including Commutative Algebra, Combinatorics and Coding Theory (see [9] for a monograph on this problem). Nonetheless, as of today there is an exact solution only for the special case $e = 2$, where Sylvester showed that $F = g_1 g_2 - g_1 - g_2$. In the general case, it is known that no polynomial formula for F in function of $g_1, \ldots, g_e$ can exist (cf. [3]), and presently the literature is mostly focused on finding algorithms and bounds for F.

In 1978, H.S. Wilf proposed an upper bound for the Frobenius number F (cf. [11]), namely

$$F + 1 \leq en, \tag{1}$$

where n is the number of solutions of the money-changing problem for $g_1, \ldots, g_e$ less than F (actually, Wilf's original question was a lower bound for e, but we choose this equivalent and simpler formulation of his conjecture).

This problem, now known as the Wilf Conjecture, has been considered by several authors; however, only special cases have been solved. For instance, it is known that the Conjecture is true in the following cases: $e \leq 3$ (cf. [7]), $|\mathbb{N}\backslash S| \leq 65$ (where S denotes the numerical semigroup generated by $g_1, \ldots, g_e$; cf. [2]), $e \geq \frac{g_1}{3}$ (cf. [6]), when g_1 is large enough and its prime factors are not smaller than $\left\lceil \frac{g_1}{e} \right\rceil$ (cf. [8]), if $F + 1 \leq 3g_1$ (cf. [5]). Most notably, the last case, due to Eliahou, coupled with a previous result by Zhai (cf. [12]), infers that the Wilf Conjecture is, in a sense, asymptotically true; the survey [4] describes the state of the research on the Wilf Conjecture.

Despite this vibrant literature, the general case is still very elusive, and in fact, no bound for F in function of e and n, that holds true for all numerical semigroups, is known. In this work, we provide such a bound, by virtue of a bound for the smallest generator g_1 in function of e and n.

Theorem 1 *Let $g_1 < \ldots < g_e$ be coprime positive integers larger than 1, let F be the Frobenius number, n be the number of integers less than F which are representable as a linear combination with coefficients in $\mathbb{N}$ of $g_1, \ldots, g_e$, and $q = \left\lceil \dfrac{F+1}{g_1} \right\rceil$. Then*

(1) $F + 1 \leq qen$;
(2) $F + 1 \leq en^2$.

Then, we provide a bound for the *type* of the numerical semigroup $S = \langle g_1, \ldots, g_e \rangle$ in function of e and n, and use this bound to give an alternative proof of the Wilf Conjecture when the numerical semigroup S is almost-symmetric.

2 Main result

Let $\mathbb{Z}$ denote the set of integers, and $\mathbb{N}$ the set of non-negative integers. Given $e \geq 2$ and $g_1, \ldots, g_e \in \mathbb{N}$ such that $\gcd(g_1, \ldots, g_e) = 1$, it is well-known that the set

$$S = \langle g_1, \ldots, g_e \rangle = \{a_1 g_1 + \ldots + a_e g_e \mid a_i \in \mathbb{N}\}$$

is a submonoid of $(\mathbb{N}, +)$ such that the set $\mathbb{N}\backslash S$ is finite; a monoid S satisfying this property is called a *numerical semigroup* (see [10] for a detailed monograph on this algebraic structure). With the notation $S = \langle g_1, \ldots, g_e \rangle$ we will assume that $\{g_1, \ldots, g_e\}$ is a minimal generating system (which is unique for any numerical semigroup) for S, and we will thus say that e

is the *embedding dimension* of S. We also denote by F the *Frobenius number* of S, that is, $F = \max \mathbb{Z}\backslash S$. Denote by $N(S) = S \cap [0, F]$ the set of elements of S less than F (called *small elements*), and let $n = |N(S)|$.

Given an element $m \in S$, define the *Apéry set* of S with respect to s as

$$Ap(S, m) = \{\omega \in S \mid \omega - m \notin S\}.$$

Clearly $Ap(S, m)$ consists of the smallest elements of S in every residual class modulo m, therefore $0 \in Ap(S, m)$, $\max Ap(S, m) = F + m$ and $|Ap(S, m)| = m$.

Our first result is a bound for the smallest generator g_1 (often called the *multiplicity*) of S, in function of e and n. The main result is a direct corollary of this bound.

Theorem 2 *Let $2 \le g_1 < \ldots < g_e$ be coprime positive integers, and let $S = \langle g_1, \ldots, g_e\rangle$. Then*

$$g_1 \le (e-1)n + 1.$$

Proof Define the map

$$\varphi : Ap(S, g_1) \setminus \{0\} \to \mathcal{P}(N(S) \times \{g_2, \ldots, g_e\}), \quad \varphi(\omega) = \{(\omega - g_i, g_i) \mid \omega - g_i \in S\}.$$

This map is well defined since, if $\omega \in Ap(S, g_1)\backslash\{0\}$, then $\omega \le F + g_1$, therefore $\omega - g_i \in S$ implies $\omega - g_i \in N(S)$. Moreover, for every $\omega \in Ap(S, g_1)\backslash\{0\}$, there exists a generator g_i such that $\omega - g_i \in S$, and therefore $\varphi(\omega) \ne \emptyset$. Finally, for $\omega_1, \omega_2 \in Ap(S, g_1) \setminus \{0\}$, if $(s, g_i) \in \varphi(\omega_1) \cap \varphi(\omega_2)$ then $s = \omega_1 - g_i = \omega_2 - g_i$ and thus $\omega_1 = \omega_2$: hence the sets $\varphi(\omega)$ are pairwise disjoint. Therefore the collection $\{\varphi(\omega)\}_{\omega \in Ap(S, g_1)\backslash\{0\}}$ is a partition of a subset of $N(S) \times \{g_2, \ldots, g_e\}$, and thus we conclude that

$$g_1 - 1 = |Ap(S, g_1) \setminus \{0\}| \le \sum_{\omega \in Ap(S, g_1)\backslash\{0\}} |\varphi(\omega)| \le |N(S) \times \{g_2, \ldots, g_e\}| = n(e-1).$$

□

Proof of Theorem 1 By Theorem 2, we know that $g_1 \le (e-1)n + 1$, thus multiplying by q and remembering that $n \ge 1$, we obtain

$$F + 1 \le g_1 q \le q(e-1)n + q = qen - qn + q \le qen.$$

Finally, since by definition of q we have $\{0, g_1, 2g_1, \ldots, (q-1)g_1\} \subseteq S \cap [0, F] = N(S)$, we have $q \le n$, therefore

$$F + 1 \le qen \le en^2.$$

□

For a numerical semigroup $S = \langle g_1, \ldots, g_e\rangle$, define the set of *pseudo-Frobenius numbers* of S as the set

$$PF(S) = \{\omega \notin S \mid \omega + s \in S \text{ for every } s \in S \setminus \{0\}\}.$$

The cardinality of $PF(S)$ is called the *type* of S, denoted by t. Since for every $\omega \in PF(S)$ and $m \in S\backslash\{0\}$, $\omega + m \in Ap(S, m)\backslash\{0\}$, we have $t \le g_1 - 1$. Our next result is a bound for t in function of e and n.

Theorem 3 *Let* $2 \leq g_1 < \ldots < g_e$ *be positive coprime integers, let* $S = \langle g_1, \ldots, g_e \rangle$*, and define* $q = \left\lceil \frac{F+1}{g_1} \right\rceil \geq 1$*. Then*

$$t \leq (e-2)[n-q+1]+2 \leq (e-2)n+2.$$

Proof Assume that there are two elements $f_1, f_2 \in PF(S)$ such that $f_1 = \lambda_1 g_2 - g_1$ and $f_2 = \lambda_2 g_2 - g_1$, with $\lambda_1, \lambda_2 \in \mathbb{N}$ and $\lambda_1 > \lambda_2$; then $s = f_1 - f_2 = (\lambda_1 - \lambda_2) g_2 \in S$, yielding $f_2 + s = f_1 \in S$, a contradiction. Then there is at most one element of the form $\lambda g_2 - g_1$ in the set $PF(S)$; let f_2 be such an element (if it exists), and let $PF'(S) = PF(S) \setminus \{F, f_2\}$ (if f_2 does not exist, then take $PF'(S) = PF(S)$).

Define the function $\varphi : PF'(S) \to \mathcal{P}(N(S) \times \{g_3, \ldots, g_e\})$ as $\varphi(f) = \{(s, g_i) \mid s = f + g_1 - g_i \in S\}$. This function is well-defined since $(s, g_i) \in \varphi(f)$ is such that $s = f + g_1 - g_i < f \leq F$, $\varphi(f) \neq \emptyset$ (because, being $f \neq f_2$, $f + g_1$ cannot be of the form $K g_2$, for some integer K), and clearly $\varphi(f) \cap \varphi(f') = \emptyset$ if $f \neq f'$, since $(s, g_i) \in \varphi(f) \cap \varphi(f')$ would imply $f = s + g_i - g_1 = f'$. Therefore the collection $\{\varphi(f)\}_{f \in PF'(S)}$ is a partition of a subset of $N(S) \times \{g_3, \ldots, g_e\}$. Moreover, our choice of q means that for $i = 1, \ldots, q-1$, $i g_1 \in N(S)$, but if $(i g_1, g_i) \in \varphi(f)$ for some f and g_i, then $f = i g_1 + g_i - g_1 \in S$, which is impossible. Therefore for every $i = 1, \ldots, q-1$ and $j = 3, \ldots, e$, $(i g_1, g_j)$ cannot belong to any set $\varphi(f)$. Combining these facts, and remembering that $q \geq 1$, we obtain

$$t - 2 \leq |PF'(S)| \leq \sum_{f \in PF'(S)} |\varphi(f)| = |\bigcup_{f \in PF'(S)} \varphi(f)| \leq (e-2)[n-q+1] \leq (e-2)n.$$

□

Let $S = \langle g_1, \ldots, g_e \rangle$ be a numerical semigroup. We say that S is *almost-symmetric* if, for every $x \notin S$, either $F - x \in S$ or $\{x, F - x\} \subseteq PF(S)$. Partitioning the interval $[0, F]$ in couples $\{x, F - x\}$, it is simple to see that, for an almost-symmetric numerical semigroup, $2n + t = F + 2$. Then Theorem 3 can be used to provide an alternative proof of Wilf's Conjecture for almost-symmetric numerical semigroups (see [1] for the original proof).

Corollary 4 *Almost symmetric numerical semigroups satisfy Wilf's conjecture.*

Proof Let S be an almost symmetric numerical semigroup, $S \neq \{0, g_1, \to\}$ (in this case it is immediate to check that the Wilf Conjecture still holds). Then in Theorem 3 we have $q \geq 2$, and thus

$$t \leq [e-2][n-1]+2 = [e-2]n - e + 4.$$

By definition of almost symmetric numerical semigroup we have $2n + t = F + 2$, hence assuming that $e \geq 4$ (we recall that Wilf's Conjecture holds in case $e \leq 3$) we have

$$F + 1 \leq en - e + 3 < en.$$

□

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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