University of Catania
Department of Physics and Astronomy

Ph.D. Thesis in Complex Systems

FITNESS AND FLAVOR IN *d*-DIMENSIONAL COMPLEX NETWORKS

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Deep, persistent problems are never solved by accident; they are solved only by people who are obsessed with them and set out to solve them directly.

Lee Smolin

A good scientist is someone who works hard enough to make every possible mistake before coming to the right answer.

R. Feynmann

It's better to be connected with your neighbour than with the Pope, at least if you need salt.

Constantino Tsallis

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Introduction

The world seems to be relational. Many, probably the majority of, natural and artificial systems show emergent behavior, i.e. properties that cannot be explained within a reductionist framework, rather they appear *because of the relations* between the elements of the system. All phenomena seem to show relational properties whether you look at a very small spatial/energy scale (fundamental particles, many-body systems, molecules, proteins. etc..) or at a very large one (planetary systems, systems of stars, galaxies, orbiting black holes).

Modern theories of quantum gravity indicate that spacetime itself is relational [1–14], if that will reveal true, it would mean that *relationalism* is ubiquitous: even at the deepest level the cosmos is filled with relations. Therefore it does make sense to describe Nature via relational mathematical tools: by construction Graph Theory is such a tool. Indeed in section §1.1 I briefly intruduce basic graph-theoretical definitions and metrics to give to the reader a first approach to the tools we are using today to describe a very huge spectrum of phenomena.

Many systems show some *complex* characteristics: what we call complex systems (CS). Someone says it is easier to tell what is not a complex system than what it is, however everyone agrees upon some of those characteristics. Among the others, CS are made of a large number of elements that in themselves can be simple, the interactions are non-linear, CS are far from equilibrium, show emerging behavior (and when you take only part of the original system seldom you get similar behavior, think about taking half the human body, you pick the half!). During the last decades the field has grown so much to include brand new disciplines, Socio-physics and Econophysics among the others; no time but now could be more exciting for a complex scientist.

When a system is relational and complex you talk about a complex network (CN). CN theory [15] has become a field in its own and, funny enough, it seems to growth non-linearly with hundreds of articles popping up every month. In section §1.2 I shortly describe some model of CN somehow in historical order (but not giving a complete historical picture) and discuss the utility of this new formalism.

As the field has grown, some generalizations of graphs and CN appeared; hypergaphs is the name (preferred by mathematicians). Two possible generalizations of CN are *Multilayer Networks* and *Simplicial Complexes* (names preferred by physicists, even though they are a subclass of hypergraphs). In section §1.3 I will only scratch the surface of the latter giving to the reader the essential informations needed for the work that follows in chapter 3. For the former see [16–18].

Of particular importance, and representative of many natural systems, are the so-called asymptotically-scale free networks, or simply scale-free networks. In all those

models a major role is played by the *attachment rule*, that is the way in which each element (node) of the system (network) gets new connections (links). The attachment probability for a node to win over the others could be proportional to its degree, i.e. how many connections it already has (full description will be depicted in the following chapters). If the system is geographically constrained, as for ecological systems, power grids, public transportation, social face-to-face interactions, there will be (usually an inverse) proportionality to the geographical distance between the nodes. In this case the importance of the distance between the nodes can be regulated by the introduction of a parameter (α_A in what follows). Furthermore, if each node has any ability (or inability) to attract new nodes, a ***fitness*** parameter can be introduced for every node. The possible values that the fitness parameter can take, i.e., the fitness parameter distribution, and the importance of the distances between nodes, open the doors for different models to emerge. Among these the Barabási-Albert model [21] where there is no dependence on the distances and all the nodes have the same ability to attract new ones (fitness parameter equal one for all nodes). A possible extension of this model is the well-known Bianconi-Barabási one [22] where now the fitness parameter is (introduced and chosen) uniformly between zero and one.

As I will explain better later, it turned out that these are particular cases of more general models [39, 40] where the dependence on the distance of the growing mechanism and therefore the role of dimensionality of the system is introduced. The nodes are placed in a specific geographical position (in dimension 1,2,3 and 4) based on an isotropic distribution; then the topology of the network is dictated by the degree, the fitness and distances of the nodes. In one of the works presented in this thesis, and as the result of a research conducted together with prof. Constantino Tsallis at Centro Brasileiro de Pesquisas Fisica (CBPF) (and prof. Andrea Rapisarda) [19], we generalized the fitness distribution obtaining a new landscape of models of which the above cited are particular cases. We recover them by tuning a new introduced parameter to particular values as illustrated in chapter §2.

The cited Bianconi-Barabási model is also a classical example of the phenomenon in which the dynamics of a classical network is mathematically described by quantum statistics (QS). QS have been shown to emerge spontaneously in the description of growing network models with fitness of the nodes. The implications of this mapping are profound. In particular the mapping of the Bianconi-Barabási model with a Bose gas is able to predict a topological phase transition [56, 65] in the network in which the dynamics of the networks is not stationary anymore but instead it is dominated by the sequence of nodes with *high fitness* that arrive in the network and eventually become *super-hubs*. Similarly the mapping of a so-called growing Cayley tree with fitness of nodes to the Fermi-Dirac distribution [58] leads to the analytical description of Invasion Percolation on these structures.

Recently these classical results of network theory have been related to the properties of growing simplicial complexes [66–70]. A simplicial complex [71–75] is a generalized network structure that allows the description of many-body interactions between a set of nodes (see §1.3 for more on that). In particular simplicial complexes are not only formed by nodes and links, like in the networks, but they are also formed by triangles, tetrahedra and so on. Given that a simplicial complex is built by geometrical

blocks, simplicial complexes are natural structures to study network geometry. As such simplicial complexes have been widely used in quantum gravity to describe the discrete (or discretised) structure of space-time [76–79]. In the last five years simplicial complexes are becoming increasingly popular to describe complex systems as well including collaboration networks, social networks, financial networks, nano-structures, and brain networks [74, 75, 80–85].

In that respect *Network Geometry with Flavor* (NGF) [66–70] is a non-equilibrium model of growing simplicial complexes with fitness that has been proposed to study emergent network geometry. In fact the NGFs evolve thanks to purely combinatorial rules that make no use of any embedding space, but when the same length is attributed to each link of the simplicial complex they are able to generate structures with an emergent hyperbolic geometry [68].
The ***flavor*** (s in what follows) of the NGF is a parameter that can change the topological nature of the simplicial complexes and their evolution. For different values of the flavor (see chapter §3 and references therein) you can get manifolds or networks that grow by uniform attachment or still networks evolving according to a generalized preferential attachment rule.

Interestingly NGFs have a *stochastic* topology that is described by quantum statistics [66, 67]. In particular, as explained later, for $s = -1$ the $(d-1)$-dimensional faces are described by the Fermi-Dirac statistics. Moreover the lower dimensional faces are described by either the Boltzmann or the Bose-Einstein statistics. For instance in a NGF with flavor $s = -1$ and dimension $d = 3$ the statistical properties of the triangles, links and nodes of the simplicial complexes are described by the Fermi-Dirac, the Boltzmann and the Bose-Einstein statistics respectively.

In the work presented in this thesis, as the result of my period of research at Queen Mary University (a winner of an Erasmus scholarship), in collaboration with prof. Ginestra Bianconi (and prof. Andrea Rapisarda) [20], we extended the study of this model to Network Geometry with *Fractional Flavor*. In principle, for reasons that will become clearer in what follows, for these networks we might expect to find that $(d-1)$-faces are described by fractional statistics [86–88]. Contrary to this naive expectation we found that also in this case $(d-1)$-dimensional faces are described by the Fermi-Dirac statistics and that instead the main difference with the NGF with integer flavor is that we do not find any face described by the Boltzmann statistics. This result sheds light on the effect that dimensionality and flavor have on the emergence of quantum statistics in NGFs.

Chapter 1

Basic concepts

1.1 What is a graph?

Let start with some useful basic definitions so to prepare the terrain for what comes next in this work. For more on the mathematical description of graphs see [26–35]. As we said graph theory is the natural tool to describe relational systems, let's start then with the following

Definition. A **graph** G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$ and a relation R that associates with each edge two vertices called its endpoints.

One can draw the graph on paper by placing each vertex at a point and represanting each edge by a curve joining the locations of its endpoints. This is a *representation* of the graph G.

Definition. A **loop** is an edge whose endpoints are equal. **Multiple edges** are edges having the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges.

A simple graph can be specified by its vertex set and edge set. To refer to a specific edge e with endpoints i,j or its (whatever) property p, one can simply write e_{ij} and p_{ij} respectively. When i and j are the endpoints of an edge, they are *adjacent* and are *neighbors*.

Definition. A graph is **finite** if its vertex set and edge set are finite.

In this dissertation we will consider, if not otherwise specified, finite simple graphs.
In graph theory and network science can be useful to know if an element is connected to all the others (or a subset of them) in that case the following definitions hold

Definition. A **clique** in a graph is a set of pairwise adjacent vertices.

Definition. A **complete graph** is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted k_n.

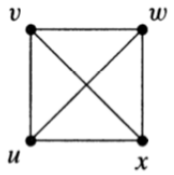


Figure 1.1: Example of a clique; it can be a subgraph (see definition below) of a graph G

While a *complete graph* is defined as a graph whose vertices are pairwise adjacent, a *clique* is a set of pairwise adjacent vertices *in* a graph.

Definition. A **path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consequtively along the circle.

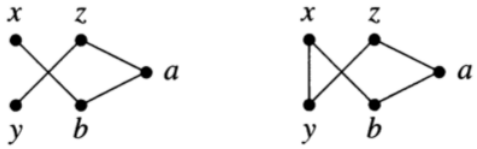


Figure 1.2: A path on the left, a cycle on the right: dropping an edge from a cycle produces a path

Definition. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We write $H \subseteq G$ and say that G *contains* H.

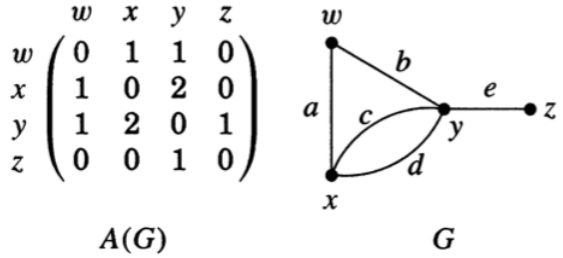
Definition. A graph is **connected** if each pair of vertices in G belongs to a path; otherwise, G is **disconnected**.

Instead of listing the vertices and edges a more useful representation of a graph is obtained via its adjacency matrix defined as follows (if not otherwise specified, in the rest of the chapter we refer to graphs without loops).

Definition. Given a graph G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_n\}$. The **adjacency matrix** of G, $A(G)$, is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges in G with endpoints $\{v_i, v_j\}$

Definition. If vertex v is an endpoints of edge e then v and e are **incident**. The **degree** of vertex v is the number of incident edges.

Every adjacency matrix is **simmetric** ($a_{i,j} = a_{j,i}$). An adjacency matrix of a simple graph G has entries 0 and 1, with 0s on the diagonal. The degree of v is the sum of the



entries in the row for v in $A(G)$.
The ith row and column in $A(G)$ correspond to the ith vertex of the graph. This means that the adjacency matrix names the vertices by the order of the row. Nevertheless many properties of interest do not depend on the names. The structural properties of G and H will be the same if we can rename the vertices of G using the vertices in H so that G will become H. It could be useful to keep in mind the following

Definition. An **isomorphism** from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is **isomorphic to** H" and we write $G \cong H$.

Definition. A **walk** is a list v_0, e_1, v_1, ..., e_k, v_k of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A u, v-**walk** has first vertex u and last vertex v, its endpoints. A u, v-**path** is a path whose vertices of degree 1 (its endpoints) are u and v, the others are **internal vertices**.

Proposition 1. If G is a simple graph in which every vertex has degree at least k, then G contains a path of length at least k. If $k \geq 2$, then G also contains a cycle of length at least $k + 1$.

When studing a relational system one can be intersested in knowing if from an element of it any other element can be reached (to pass information, energy etc..)

Definition. A graph is **connected** if it has a u,v-**path** whenever $u, v \in G$, otherwise is **disconnected**.

This means that a graph is connected if from each vertex there is a walk to one particular vertex.

Definition. A **cut-edge** or **cut-vertex** of a graph is an edge or vertex whose deletion increases the number of components. We write $G - e$ or or $G - M$ for the subgraph of G obtained by deleting an edge e or set of edges M. We write $G - v$ or or $G - S$ for the subgraph of G obtained by deleting a vertex v or set of vertices S.

Definition. The **union** of graphs $G_1, \ldots, G_k$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

Proposition 2. Every graph with a nonloop edge has at least two vertices that are not cut-vertices.

As we will see, the degrees of the vertices of a graph are fundamental parameters. Let's give again the definition with more notations.

Definition. The **degree** of vertex v in a graph G, written $d_G(v)$ or (stealing the notation from Networks Science) k, is the number of edges incident to v. The maximum degree is $\Delta(G)$, the minimum degree is $\delta(G)$, and G is **regular** if $\Delta(G) = \delta(G)$. It is k-**regular** if the common degree is k.

Proposition 3. If G is a graph, then

$$\sum_{v \in V(G)} d_G(v) = 2e(G) \tag{1.1}$$

Definition. The **degree sequence** or **degree distribution** of a graph is the list of vertex degrees.

Definition. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A tree is connected acyclic graph. A **spanning subgraph** of G is a subgraph with vertex set V(G). A **spanning tree** is a spanning subgraph that is a tree.

Theorem 1. For an n-vertex graph G, the following are equivalent (and characterize the trees with n vertices).

A) G is connected and has no cycles.

B) G is connected and has $n - 1$ edges.

C) G has $n - 1$ edges and no cycles.

D) For $u, v \in V(G)$, G has exactly one u, v-path.

In the next section we will introduce different kind of network models and one of the way to differentiate them is related to their *diameter*

Definition. If G has a u, v-path, then the **distance** from u to v, written $d_G(u, v)$, is the least length of a u, v-path. If G has no such path, then $d_G(u, v) = \infty$. The **diameter** (diam G) is $max_{u,v \in V(G)} d(u, v)$.

An entire book would not be enough to describe all the beauty and power of graph theory; we focused on some very basic definitions in order to give to the reader a wide self-consistent picture of the main work treated in this thesis.
When sociologists first and physicists later, put their hands on graph theory, the subject turned into a new field with a wider range of applications. Let's have a look at models useful to those applications.

1.2 What can you do with that? - Complex Networks

Physiscists prefer to talk about *nodes* (and set of nodes N) instead of vertices (and set of vertices V), and *links* instead of edges. Usually the notation is, let's say i and j for the nodes and l_{ij} for the link connecting them. A part a change of notation, networks theory borrows definitions and theorems from graph theory.
One of the key characteristics in the study of networks is the *degree distribution* $P(k)$ (or p_k) i.e. the fraction of nodes with degree k. One can calculate the nth-moment of $P(k)$

$$\langle k^n \rangle = \sum_k k^n P(k) \tag{1.2}$$

that gives useful informatiions on the degree distribution. For instance the first moment $\langle k \rangle$ gives the mean of the degrees of the network, the 2nd moment gives the variance and so on.
It could happen that the probability of a node of degree k of being connected to a node of degree k' depends on k, we say the graphs are *correlated* and we write $P(k|k')$; the two following relations hold

$$\sum_{k'} P(k|k') = 1 \tag{1.3}$$

$$kP(k'|k)P(k) = k'P(k|k')P(k') \tag{1.4}$$

Using the adjacency matrix defined in the previous section one can define the *the first-neighbors mean degree* of a node i, $k_{nm,i} = \frac{1}{k_i}\sum_{j=1}^{N} a_{ij}k_j$and in terms of the conditional probability above the first-neighbors mean degree of a node of degree k is defined as

$$k_{nm}(k) = \sum_{k'} k'P(k'|k). \tag{1.5}$$

Assortative graphs are graphs whose nodes tend to attach to others that are similar usually in terms of degree ($k_{nm}(k)$ increasing function of k); *dissortative* graphs are graphs whose nodes tend to attach to others that are not similar; high degree nodes tend to attach to low degree ones ($k_{nm}(k)$ decreasing function of k).
Another very useful coefficient is the so-called local clustering coefficient, for a node i

$$c_i = \frac{\sum_{j,m;jm} a_{ij}a_{jm}a_{mi}}{k_1(k_i - 1)} \tag{1.6}$$

from which the (total) clustering coefficient

$$C = \langle c \rangle = \frac{1}{N}\sum_i c_i \tag{1.7}$$

which is just the probability of a pair of neighbors to be interconnected.
What do real networks look like and how can we describe them? Let's start with some models?

1.2.1 RANDOM GRAPHS

The simplest type of networks are the random graphs introduced by Erdös and Rényi in 1959. One can construct a random graph by starting with N nodes and connecting pairs of them at random. In principle you can add up to $\frac{N(N-1)}{2}$ links (obtaining a complete graph). There are a large number of ways to distribute the links, therefore a particular random graph would be an element of a statistical ensamble. You can build a random graph by starting with N isolated nodes and connecting pair of them with probability $0 < p < 1$. The probability that a particular realization of a random graph has exactly L links is

$$P_L = \binom{\frac{N(N-1)}{2}}{L} p^L (1-p)^{N(N-1)/2-L} \tag{1.8}$$

Taking p equal for each pair gives rise to the Erdös-Rényi version of random networks. Changing it gives rise to different random models. Given the critical probability $p_c = \frac{1}{N}$, it can be shown

- if $p < p_c$, almost certaintly the graph doesn't contain subgraphs of size bigger than $\mathcal{O}(\ln N)$ and no component has more than one cycle;
- if $p = p_c$ almost certaintly the biggest component is of size $\mathcal{O}(N^{2/3})$;
- if $p > p_c$ the graph has about $\mathcal{O}(N)$ components with $\mathcal{O}(N)$ cycles and no other component has more than $\mathcal{O}(\ln N)$ nodes and more than one cycle.

The expected number of links in a random graph is

$$\langle L \rangle = p \frac{N(N-1)}{2} \tag{1.9}$$

from which one can obatin the average degree of a random graph

$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1) \tag{1.10}$$

where $N-1$ is the maximum number of links a node can have (in a graph of size N). The number of links in a random graph is therefore determined only by two numbers, N and p. Increasing the latter makes the graph denser, that is the average number of links increases from $\langle L \rangle = 0$ to $L_{\max}$ while the average degree of a node increases from $\langle k \rangle = 0$ to $\langle k \rangle = N-1$. In random networks some nodes gain numerous links, while others only a few, this is captured by the degree distribution

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k} \tag{1.11}$$

that is a binomial distribution.
In most real networks it happens that $\langle k \rangle << N$, in this case we say that they are *sparse*. In this limit the degree distribution can be approximated by the Poisson distribution

$$p_k = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}. \tag{1.12}$$

Random network models predict that all nodes have comparable number of links; this is not in agreement with real networks, where the so-called hubs exist (elements with a very high number of links).
As we saw before, in the $p < \frac{1}{N}$ case for $\langle k \rangle = 0$ the network consists of N isolated nodes. As we increase $\langle k \rangle$ we add $N\langle k \rangle = pN(N-1)/2$ links to the network. Therefore we observe numerous tiny components of comparable sizes and no giant components. A giant component appear as $p > \frac{1}{N}$ (with a phase transition at $p = \frac{1}{N}$). Near the critical point ($\langle k \rangle = 1$) the size of the giant component is

$$N_G \approx (p - p_c)N \tag{1.13}$$

it means it contains a finite fraction of nodes. As you move from the critical point, a larger fraction of nodes will be included. As a consequence numerous isolated components (trees) coexist with the giant component (as soon as $\langle k \rangle = 1$). When $\langle k \rangle > \ln N$, $N_G \approx N$ the giant component absorbs all nodes and the network becomes connected with no isolated nodes.
Summarizing, in random network models as you vary $\langle k \rangle$ the isolated nodes and small components (for small valus of $\langle k \rangle$) collapse into a giant component (for high values of $\langle k \rangle$) through a phase transition.
These two characteristics of random networks (that a giant component emerge for $\langle k \rangle > 1$ and it contains all nodes for $\langle k \rangle > \ln N$) are seldom satisfied by real networks: they exceed the $\langle k \rangle = 1$ treshold and are fragmented into multiple components.
Given a node, what is the density of links between its neighborhood? It can be easily shown that the local clustering coefficient of a random network is

$$c_i = \frac{\langle k \rangle}{N} \tag{1.14}$$

i.e., the larger the network, the smaller the clustering coefficient; moreover for each node it is independent of the node's degree. The average clustering coefficient $\langle C \rangle$ also follows eq. 1.14.
For real networks, if one plots $\langle C \rangle / \langle k \rangle$, it is independent of N, not decreasing as N^{-1} thus eq. 1.14 is violated. Also the average clustering coefficient does depend on the node degree $C = C(k)$ contrary to what the random model predicts.

1.2.2 SMALL-WORLD NETWORKS

Whether a network is random or not it can show small world properties. The small world phenomenon or *six degrees of separation*, implies that if you choose any two elements of a network you will find a short path (of at most six elements) between them.
In a random network with average degree $\langle k \rangle$ a node has on average, up to a distance d, a number of nodes

$$N(d) \approx 1 + \langle k \rangle + \langle k \rangle^2 + \cdots + \langle k \rangle^d = \frac{\langle k \rangle^{d+1} - 1}{\langle k \rangle - 1} \tag{1.15}$$

but because it cannot be bigger than the total number of nodes there must be a maximum distance $d_{\max}$ (the diameter of the network) that can be found imposing

$$N(d_{\max}) \approx N \tag{1.16}$$

and assuming that $\langle k \rangle >> 1$ we find

$$d_{\max} \approx \frac{\ln N}{\ln \langle k \rangle}. \tag{1.17}$$

Since $\ln N << N$, distances in a random network are smaller than the size of the network. Usually this is the definition of the "small world phenomenon": the diameter depends logaritmically on the size of the system. For random network models this property is due to the fact that the number of nodes at distance d from a node increases exponentially with d.
Real networks are not random but still show small world properties. They also have a higher clustering coefficient with respect to the one predicted by random models. Watts and Strogatz [42] proposed an extension of random networks to account for high $\langle C \rangle$ and the small world property.
You start from a reguar lattice, actually a ring of nodes connected to their immediate and next neighbors, which has high clustering coefficient but no small-world property. Some links are re-wired with probability p to a node choosen at random. For a certain range of values of the rewiring parameters the long-range links decrease the distances between the nodes. This gives rise to a low average path length mantaining the high clustering coefficient. At $p = 1$ all links have been rewired and the network becomes a random one.
The resulting degree distrubution is Poisson-like, thus nodes with high degree are absent and also it predicts a k-independent clustering coefficient

$$C(p) \approx \frac{3(m-1)}{2(2m-1)}(1-p)^3 \tag{1.18}$$

with $m = k_{\min}$ the minimum degree.
In order to get model closer to real networks we must start from the correct degree distribution and then look for small-world and high clustering properties.

1.2.3 SCALE-FREE NETWORKS

Empirical observations show that many real networks present a degree distribution quite different from the Poissson-like, they show a degree distribution well approximated with

$$p_k \approx k^{-\gamma} \tag{1.19}$$

this is called a *power law distribution*, while the exponent its *degree exponent*. We call a *scale-free network* a network whose degree distribution follows a power law. A more precise formula for the degree distribution is

$$p(k) = (\gamma - 1) k_{\min}^{(\gamma-1)} k^{-\gamma} \tag{1.20}$$

The main difference between a random and a scale-free network comes in the high-k region of the degree distribution p_k. A scale-free network has a large number of small-degree nodes (absent in a random network) much more high-degree nodes, or hubs, and

less nodes with degree $k \approx \langle k \rangle$ compared to a random network.
For scale-free networks one can easily compute the maximum (and the minimum) degree, $k_{\max}$ ($k_{\min}$), of the degree distribution p_k and estimate the expected size of the largest hub

$$k_{\max} = k_{\min} + \frac{\ln N}{\lambda} \tag{1.21}$$

using eq.1.20 one gets

$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}} \tag{1.22}$$

this means that the larger the network, the larger the biggest hub's degree. In very large scale-free network there are orders of magnitude diferences in size between the smallest node and the biggest one. While in random networks most nodes have comparable degrees and there are no hubs, in scale-free networks hubs are common and their size grows polynomially with the network size (in random networks the size of the largest node grows logaritmically with N).
Consider now the n^{th} moment of the degree distribution

$$\langle k^n \rangle = \sum_{k_{\min}}^{\infty} k^n p_k \approx \int_{k_{\min}}^{\infty} k^n p(k) dk \tag{1.23}$$

that is

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1} \tag{1.24}$$

since the degree of the largest hub, $k_{\max}$, increases with the system size, we take the limit $k_{\max} \to \infty$ for very large networks.
If $n - \gamma + 1 \leq 0$ all the moments (up to $\gamma - 1$) are finite. If $n - \gamma + 1 > 0$ all moments larger than $\gamma - 1$ diverge. For many networks the exponent γ is between 2 and 3 thus, in those cases, in the $N \to \infty$ limit only the first moment is finite, higher moments go to inifinity. The divergence of the second moment, $\langle k^2 \rangle$ and hence the standard deviation, indicates that the fluctuations around the average can be arbitrarily large. From this the name *scale-free*: there is no internal scale, nodes with very different degrees coexist in the same network. A large number of real networks have been found to be scale-free, from biological to social networks, but not all networks display the scale-free property; many important networks, from the power grid to networks appearing in material science, among the others, are not scale-free. Moreover, many degree distributions observed in real networks deviate from a pure power law and often this can be attributed to the processes that give rise to a particular network. In general, the scale-free property is absent in systems that limit the number of links a node can have.
What about small-world properties in scale-free networks? It can be shown that the average distance $\langle d \rangle$ for scale-free networks is

$$\langle d \rangle \approx \begin{cases} cost. & \text{for } \gamma = 2 \\ \ln \ln N & \text{for } 2 < \gamma < 3 \\ \frac{\ln N}{\ln \ln N} & \text{for } \gamma = 3 \\ \ln N & \text{for} \gamma > 3 \end{cases} \tag{1.25}$$

In the case $\gamma = 2$ all nodes connect to the same central hub and the average path length does not depend on N. In the case $2 < \gamma < 3$ the many hubs reduce the path length creating short distances between nodes of small degree. At the critical point $\gamma = 3$ (the second moment of the degree distribution no longer diverges), due to the correction $\ln \ln N$ distances are shorter compared to a random network of similar size. For $\gamma > 3$ small-world properties hold as in random models, hubs exist but are not enough to significantly impact the distance between the nodes. We can conclude that the larger and more numerous the hubs are, the more they shrink the distances between nodes.
Hubs is the main difference between random and scale-free networks. In order to understand why scale-free property is so common in real networks is good to try to model them and figure out the mechanism that allows the emergence of such property.

1.2.4 Barabási-Albert model

In random network models the number of nodes N is fixed, while in real networks it grows since new nodes are continually added. Morevover, in real networks (contrary to random models) nodes tend to link to more connected nodes. This alone can make a huge difference in the shape of the degree distribution of a network. The first model introduced to take into account the growth and the *preferential attachment* was the so-called Barabási-Albert model [21]. One starts with m_0 nodes, with arbitrarily chosen links between them. At each time step a new node is added with m links connected to m nodes already present in the network. The probability $\Pi(k)$ that a link of the new node connects to a node i depends on the degree k_i

$$\Pi(k) = \frac{k_i}{\sum_i k_i} \tag{1.26}$$

while each node is free to connect to any other node a high-degree node has higher probability to attract the node. At time t you get a network of size $N = t + m_0$ and $m_0 + mt$ links.
It can be shown that the degree of a node i at time t is

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \tag{1.27}$$

hence all nodes follow the same law (with the same dynamical exponent β); each new node has more nodes to link to than previous nodes and the earlier the node i arrived, the higher its degree $k_i(t)$: older nodes have an advantage and eventually turn into hubs. The degree distribution of the Barabási-Albert model is

$$p_k = \frac{2m(m+1)}{k(k+1)(k+2)} \tag{1.28}$$

that for large k reduces to $p_k \approx k^3$ giving a power-law with exponent degree $y = 3$. Notice that γ is independent of m in agreement with observations of many real networks. Also the degree distribuiton p_k is indipendent of the time t and the size of the system

N, indicating that the model generates a stationary scale-free network and explaining why networks with different evolution, size and age develop similar (power-law) degree distributions.
It is worth to notice that if the probability that a new node links to a node with degree k_i is independent of k_i (new nodes choose randomly the nodes to link to) the degree increases logarithmically with time

$$k_i(t) = m \ln \left(e \frac{m_0 + t - 1}{m_0 + t_i - 1} \right) \tag{1.29}$$

therefore slower than the power law and as a consequence the degree distribution is an exponential

$$p(k) = \frac{e}{m} \exp \left(-\frac{k}{m} \right) \tag{1.30}$$

which decays faster than a power law not supporting hubs.
The absence of preferential attachment seems to lead to a growing network with an exponential degree distribution. This indicates that preferential attachment is necessary for the emergence of scale-free property, at least for *ordinary* complex networks. Indeed in chapter §3 I will show that this does not necessarily apply to simplicial complexes.

To conclude the section let's look at the small-world properties of this model. It can be shown that, for $m > 1$ and large N the average distance scales as

$$\langle d \rangle \approx \frac{\ln N}{\ln \ln N} \tag{1.31}$$

thus distances in Barabási-Albert networks are smaller than those observed in random networks. At same time, the clustering coefficient of the Barabási-Albert model turns out to be

$$\langle C \rangle \approx \frac{(\ln N)^2}{N} \tag{1.32}$$

larger than the clustering coefficient $(1/N)$ of random networks. This makes Barabási-Albert model more clustered than a random graph.

The Barabási-Albert model is a very basic one which, while being useful to explain the emergence of the scale-free property, does not fully capture details and characteristics of real networks.

1.2.5 Bianconi-Barabási model

In reality the growth rate of a node in real networks does not depend only on its age. Nodes have qualities or abilities that influence the rate at which they aquire links. The ability of a node to attract nodes is called its fitness. In order to take it into account, Barabási-Albert model can be generalized introducing a fitness parameter η. The resulting model is called fitness model or Bianconi-Barabási model. The network is built as follows: At each timestep a new node j with m links and fitness η_j is added

to the network. A node's fitness does not change over time; it is chosen from a fitness distribution $p(\eta)$. The probability that a link of a new node connects to node i is

$$\Pi(k) = \frac{\eta_i k_i}{\sum_j \eta_j k_j} \tag{1.33}$$

This implies that between two nodes with same degree, the one with higher fitness is selected with a higher probability. This way even a node with initially few links can aquire more links if it has larger fitness with respect to other nodes. Assuming that the time evolution of k_i follows a power law

$$k(t, t_i, \eta_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)} \tag{1.34}$$

(since a node with higher fitness will increase its degree faster) it can be shown that the degree distribution is

$$p_k \approx C \int d\eta \frac{p(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{C}{\eta}+1} \tag{1.35}$$

which when all fintesses are equal to $\eta = 1$, thus $p(\eta) = \delta(\eta - 1)$, yealds to $C = 2$. With $\beta = 1/2$ you get $p_k \approx k^{-3}$ i.e. the Barabási-Albert model.
If η is uniformly distributed in the $[0, 1]$ interval, nodes can have different fitness picked at random in that interval, the dynamic exponent become $\beta(\eta_i) = \eta_i / C'$ and consequently the degree distribution becomes

$$p_k \approx \int_0^1 d\eta \frac{C'}{\eta} \frac{1}{k^{1+C'/\eta}} \approx \frac{k^{-(1+C')}}{\ln k} \tag{1.36}$$

where $C' = 1.255$ (estimated numerically). Hence the degree distribution follows a power law with exponent $\gamma = 2.255$ with a logarithmic correction $1/\ln k$.
This model accounts for the ability of nodes to aquire links at different rates. The node's growth rate is determined by its fitness η, choosen uniformly from the distribution $p(\eta)$. For most fitness distributions the degree of each node is ultimately detemined by its fitness. The fittest node becomes the largest hub, and at any moment the degree distribution follows a power law, indicating that the generated network has a scale-free topology. The largest hub is closely followed by slightly smaller hubs, with almost the same number of links as the fittest node. The model with uniform fitness distribution is in this scale-free phase. Changing the fitness distribution $p(\eta)$ can change the topology of the network and/or its degree distribution, as we will see in later sections.

1.2.6 Geographically-constrained networks: The role of dimensionality

Many real networks have physical background, meaning that they are located somewhere in the physical world: their elements are geographically constrained. We say those networks are geographically-constrained networks. Clearly the dimensionality d of the constraint can make a huge difference for the network topology and eventually for its degree distribution. Usually, a high-dimensional system have more degrees of freedom than a low-dimensional one. Distances in physical world can play a major role. Closer

nodes can have higher probability to link to each other due to their proximity. Therefore the dimensionality and the actual physical distances must be taken into account when we study physical networks. The Bianconi-Barabási model can be generalized [40] to include them and still have a preferential attachment mechanism (for simplicity, each new node will bring only one link to connect to another node, thus $m = 1$). The model is built as follow: you fix a reference frame and place a node at the origin. At each time step a new node is added and located at a certain euclidean distance $r \geq 1$ from the center of mass ($r = |x|$ in one dimension, $r = \sqrt{x^2 + y^2}$, in two dimensions, $r = \sqrt{x^2 + y^2 + z^2}$ in three dimensions and so on).
The distance is picked from the d-dimensional isotropic distribution

$$p(r) = \frac{1}{r^{d+\alpha_G}} \tag{1.37}$$

where $\alpha_G > 0$ ("G" stands for *growth*) is introduced to make the distribution $p(r)$ normalizable. The newly arrived node will then be connected to one of the pre-existing nodes of the network through the preferential attachment rule given by

$$\Pi(k) = \frac{\eta_i k_i r_{ij}^{-\alpha_A}}{\sum_j \eta_j k_j r_{ij}^{-\alpha_A}} \tag{1.38}$$

where α_A controls the importance of the distance and the subindex A stands for *attachment*. In this model three factors can influence the probability of the nodes to receive the new connections: the nodes degree, the Euclidean distance of these nodes to the new arrived node, and the fitness of these nodes. Notice that the importance of the distance is less pronounced when α_A is close to zero and completely disappears for $\alpha_A = 0$. In this limit, we recover the Bianconi-Barabási model. When $\alpha_A > 0$ no exact solution is known but numerical simulations strongly suggest that the degree distribution is well fitted by a q-exponential

$$p_k = p_0 e_q^{-k/\kappa} \tag{1.39}$$

with $q > 1$ and $\kappa > 1$. For $\alpha_A \to \infty$ ($q \to 1$) the BG limit is reached and p_k tends to the standard exponential function. In this limit, independently of the system dimension, the network present typical connectivity between sites, characteristic of random networks. In fact fact this represents a phase transition (from scale-free to random networks) associated with the α_A parameter.
As we further generalized this model [19], in the next chapter, I will explain in details more properties of it that are in common with our model.

1.3 More than graphs: Simplicial complexes

Graphs/networks can be (and indeed have been) extended in what is the very general concept of *hypergraphs*. In chapter 3, I describe a model, and its behavior, of a particular class of hypergraphs: growing simplicial complexes. Let me give here a few basic definitions; for further reading on hypergraphs see [36–38]

A simplicial complex describes the many body interactions between a set of N nodes. In particular a simplicial complex is formed by simplices glued along their faces. A δ-dimensional simplex is a set of $\delta+1$ nodes. Therefore a 0-dimensional simplex is a node, a 1-dimensional simplex is a link, a 2-dimensional simplex is a triangle, and so on. A δ'-dimensional *face* α' of a δ-dimensional simplex α, is a simplex formed by a subset of $\delta'+1$ nodes of α, i.e. $\alpha' \subset \alpha$. A d-dimensional *simplicial complex* $\mathcal{K}$ is formed by a set of simplices of dimensions $0 \leq \delta \leq d$ (including at least a d-dimensional simplex) that obey the following two conditions:

(a) if a simplex α belongs to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ then also all its faces $\alpha' \subset \alpha$ belong to the simplicial complex, i.e. $\alpha' \in \mathcal{K}$;

(b) if two simplices α and α' belong to the simplicial complex, i.e. $\alpha \in \mathcal{K}$ and $\alpha' \in \mathcal{K}$, then either their intersection is the null set $\alpha \cap \alpha' = \emptyset$ or their intersection belongs to the simplicial complex, i.e. $\alpha \cap \alpha' \in \mathcal{K}$.

A d-dimensional simplicial complex is called *pure* if it is only formed by d-dimensional simplices and their faces. From a simplicial complex it is always possible to extract a network called the 1-*skeleton* by considering only the nodes and links of the simplicial complex.
We will focus on pure d-dimensional simplicial complexes $\mathcal{K}$. In the following we will indicate with $\mathcal{Q}_\delta(N)$ the set of all possible δ simplices in a simplicial complex of N nodes and with $\mathcal{S}_{d,\delta}$ the set of all the δ-dimensional faces of the pure d-dimensional simplicial complex $\mathcal{K}$.
The topology of a pure d-dimensional simplicial complex $\mathcal{K}$ is fully specified by the adjacency tensor $\mathbf{a}$ of elements a_α with $\alpha \in \mathcal{Q}_d(N)$ given by

$$a_\alpha = \begin{cases} 1 & \text{if } \alpha \in \mathcal{K}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.40}$$

The *generalized degree* $k_{d,\delta}(\alpha)$ [67, 73] of the δ-face α is defined as the number of d-dimensional simplices incident to it. Using the adjacency tensor we can evaluate $k_{d,\delta}$ of a δ-face α as

$$k_{d,\delta}(\alpha) = \sum_{\alpha' \in Q_d(N) | \alpha' \supset \alpha} a_{\alpha'}. \tag{1.41}$$

Therefore, in $d=2$, the generalized degree $k_{2,1}(\alpha)$ is the number of triangles incident to a link α while the generalized degree $k_{2,0}(\alpha)$ indicates the number of triangles incident to a node α. Similarly in a pure $d=3$ dimensional simplicial complex, the generalized degrees $k_{3,2}$, $k_{3,1}$ and $k_{3,0}$ indicate the number of tetrahedra incident respectively to a triangular face, a link or a node. The generalized degrees of faces are not independent of the generalized degree of the simplices to which they belong [73]. In fact the generalized degree of a $\delta-$face α is related to the generalized degree of the δ'-dimensional faces incident to it, with $\delta' > \delta$, by the simple combinatorial relation

$$k_{d,\delta}(\alpha) = \frac{1}{\begin{pmatrix} d-\delta \\ \delta'-\delta \end{pmatrix}} \sum_{\alpha' \in \mathcal{Q}_d(N) | \alpha' \supset \alpha} k_{d,\delta'}(\alpha'). \tag{1.42}$$

Moreover, since every d-dimensional simplex belongs to $\begin{pmatrix} d+1 \\ \delta+1 \end{pmatrix}$ δ-dimensional faces, in a simplicial complex with M d-dimensional simplices we have

$$\sum_{\alpha \in \mathcal{S}_{d,\delta}} k_{d,\delta}(\alpha) = \begin{pmatrix} d+1 \\ \delta+1 \end{pmatrix} M. \tag{1.43}$$

The $(d-1)$-dimensional faces of a pure d-dimensional simplicial complex deserve some special attention. In particular to each $(d-1)$-dimensional face α we associate an *incidence number* n_α given by the number of incident d-dimensional simplices minus one, i.e.

$$n_\alpha = k_{d,d-1}(\alpha) - 1. \tag{1.44}$$

Interestingly a simplicial complex can define a discrete d-dimensional manifold only if $n_\alpha \in \{0, 1\}$, i.e. a discrete d-dimensional manifold must have all its $(d-1)$-dimensional faces incident at most to two d-dimensional simplices. Therefore if $n_\alpha > 1$ at least for one face $\alpha \in S_{d,d-1}$ then the simplicial complex is not a discrete manifold.

Chapter 2

Case 1: Asymptotically-scale-free geographical networks

As a representative example of the role of fitness in complex networks I present the work (submitted to JSTAT at the time of writing) in which we generalized the fitness distribution by introducing a new parameter that allows us to regulate it. This model turned out to be a very general case able to include many well-known ones, as explained in the following sections.

2.1 How to build the network

2.1.1 The parameters

We build the network by successively including one node at a time. We start with the first node placed at the origin, then we add a second node, a third one and so on up to N. Each node is located at a certain Euclidean distance $r \geq 1$ from the center of mass calculated among all the preexisting nodes, and it is picked from a d-dimensional *isotropic* distribution

$$p(r) \propto \frac{1}{r^{d+\alpha_G}} \tag{2.1}$$

where $d = 1, 2, 3$ and $\alpha_G > 0$ is chosen to make the distribution $p(r)$ normalizable. Here G stands for *geographical* to distinguish it from the other parameter introduced here below. As it was shown in [39], α_G does not relevantly affect the growth of the network. Therefore we shall typically fix it to $\alpha_G = 2$.
At each time step the degree k of each node is updated (after the connections are created as explained here below). Also a fitness parameter $\eta \in [0, 1]$ is attached to the new arrived node. The main novelty of the present model is the probability distribution of η (see Fig. 2.1).

$$P(\eta) = \begin{cases} (1+\rho)\eta^{\rho}, & \text{for } \rho > 0 \\ 1, & \text{for } \rho = 0 \\ (1-\rho)(1-\eta)^{-\rho}, & \text{for } \rho < 0 \end{cases} \tag{2.2}$$

where the prefactors $(1+\rho)$ and $(1-\rho)$ come from the normalization. The new introduced parameter $\rho \in (-\infty, \infty)$ regulates the fitness parameter distribution. Tuning it to particular values allows us to recover various well-known models and a variety of new possibilities that we shall discuss along the thesis.

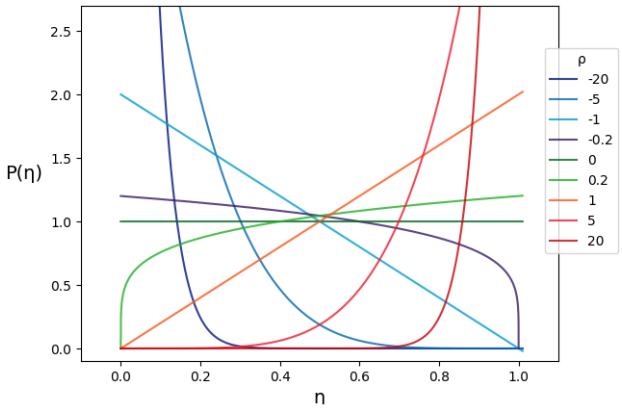


Figure 2.1: Fitness distributions $P(\eta)$ for typical values of $\rho \in (-\infty, \infty)$.

2.1.2 Preferential attachment rule

When a new node j is added it will be attached to one of the preexisting nodes i following the preferential attachment rule

$$\Pi_i = \frac{k_i \eta_i r_{ij}^{-\alpha_A}}{\Sigma_i k_i \eta_i r_{ij}^{-\alpha_A}} \quad (\alpha_A \geq 0) \tag{2.3}$$

where r_{ij} is the geographical distance between node i and node j and α_A is the parameter that regulates the importance of distances in the attachment rule (A stands for *attachment*). Clearly, in the $\alpha_A = 0$ limit distances play no role and the connectivity is dictated only by how many connections a node already has (k) and its ability to get new ones (η). Basically the topology associated with our model is influenced by a couple of parameters (α_A, ρ), in addition to the dimensionality d of the system. Despite its simplicity, the present model is able to reproduce a landscape of models, some well-known plus a variety of previously unexplored ones (see the following section for the details).

2.2 Role of dimensionality and of stochastic fitness

2.2.1 Degree distribution

The first natural thing to check is the digree distribution. It is found of the form $P(k) \propto e_q^{-k/\kappa}$ (see Fig.2.2) where the q-exponential

$$e_q^z \equiv [1 + (1-q)z]^{1/(1-q)} \tag{2.4}$$

optimizes the nonadditive entropy

$$S_q = k\frac{1-\Sigma_i p_i^q}{q-1} \tag{2.5}$$

($q \in \mathbb{R}; S_1 = -k\Sigma_i p_i \log p_i$). Since this q-exponential distribution optimizes the nonadditive entropy S_q, this model constitutes but a particular system within nonextensive statistical mechanics (see [44] for a review), where k plays the role of energy and κ plays the role of temperature. Later we show how (q, κ) depend on $(\alpha_A/d,\ \rho)$. As expected we found that the generated networks are asymptotically scale-free.

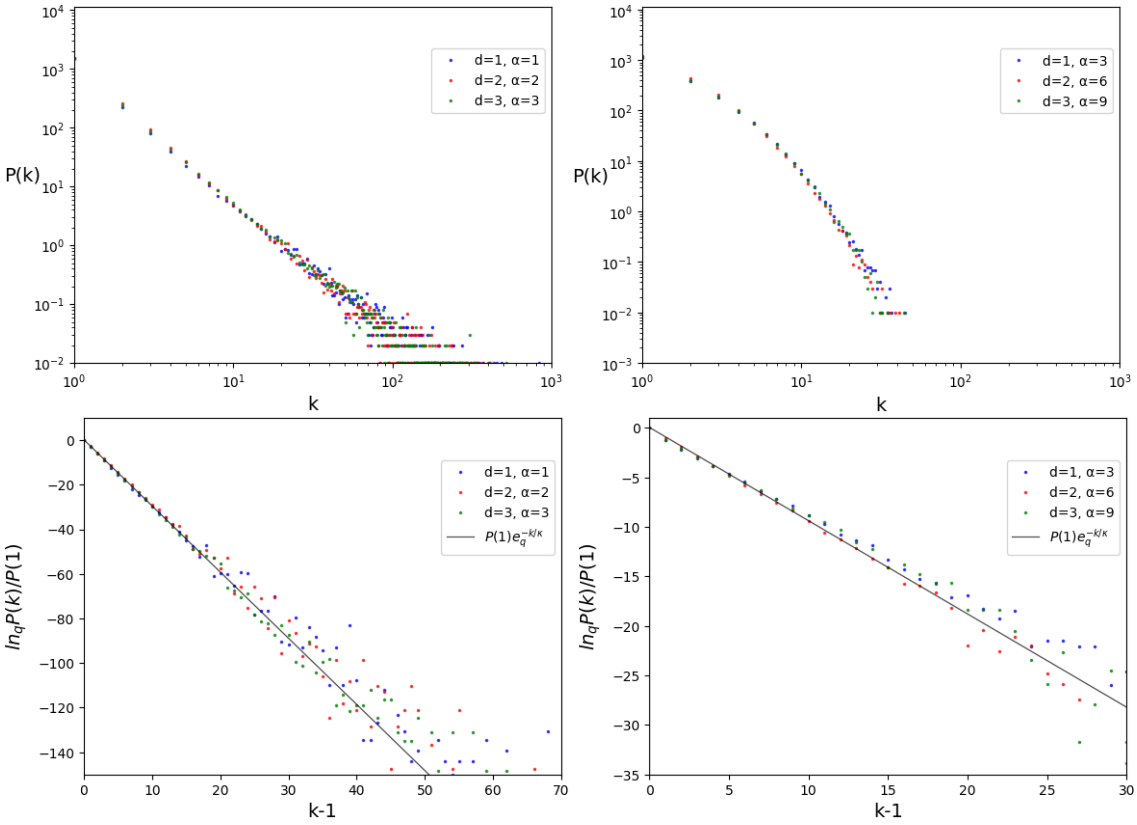


Figure 2.2: Tipical examples of $P(k)$ for $\rho = -100$ (as a good approximation for $\rho \to -\infty$)

As said before, the model strictly depends on the couple of parameters (α_A, ρ) plus the dimensionality of the system d. In particular for $(\alpha_A, \rho) = (0, 0)$ and $(\alpha_A, \rho) = (0, \infty)$ this model recovers respectively the Bianconi-Barabási and the Barabási-Albert ones since for $\rho = 0$ from eq. (2.2) the fitness does not enter anymore in the model while for $\rho \to \infty$ you get a Dirac-delta function. The region where $\rho \in [-\infty, 0)$ is brand new and has never been explored before. We built networks with ρ varying in the interval $[-\infty, \infty]$; as espected and shown in fig. 2.3 (for extreme cases) the parameter ρ doesn't affect much the topology of the network while α_A does.

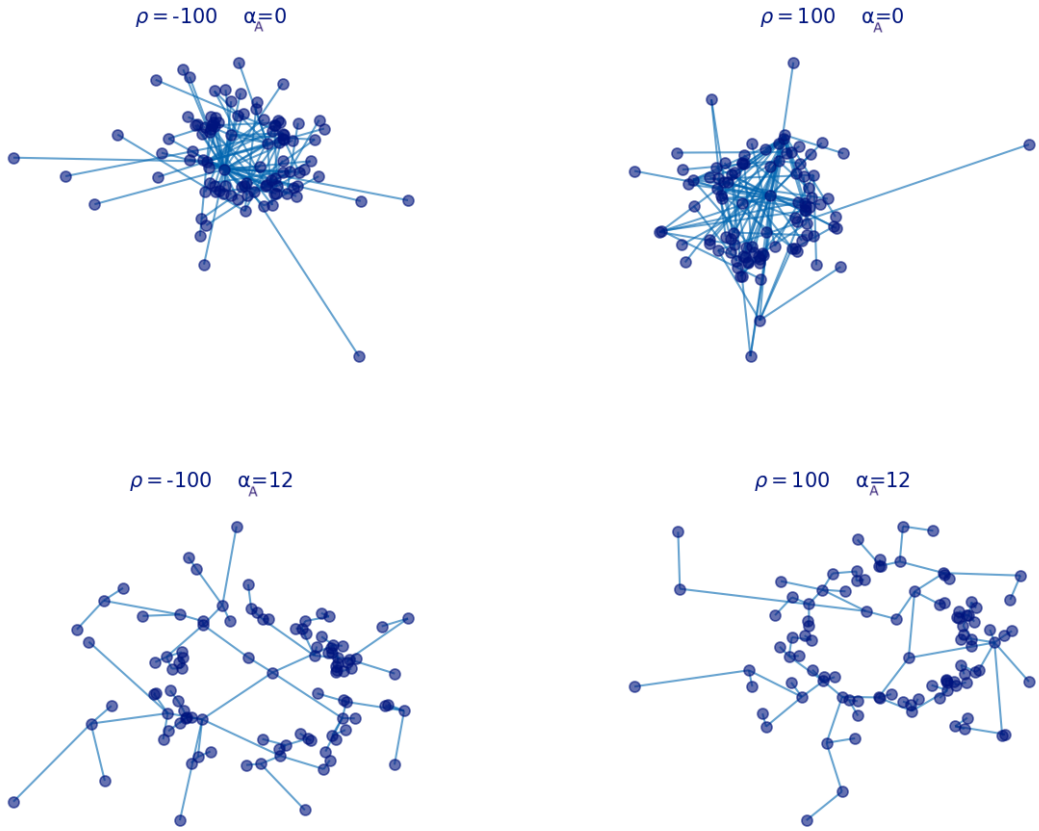


Figure 2.3: $d = 2$ stochastic realizations with $N = 100$.

Notice in the same figure that the spatial disposition of the nodes is not influenced neither from ρ nor from α_A as for they enter after the positions are choosen.

2.2.2 Universal behavior

As anticipated (q, κ), coming from the q-exponential distribution (2.4), depend on $(\alpha_A/d, \rho)$. Consider first the case q versus ρ see fig.2.5. Is interesting to notice that q varies from a maximum constant value for the part of the spectrum where $\rho \to -\infty$ to a minimum constant value for the part of the spectrum where $\rho \to +\infty$. In the region near $\rho = 0$ a more drastic change happens for the values of q (even though all values of q stay in a small interval). In particular $\rho = 0$ constitutes an infletion point for q, $\forall\, \alpha_A/d$. Observe the collapse of the curves when $\alpha_A/d = 1$ (the upper set of points) or $\alpha_A/d = 2$ (the middle set of points) showing the main dependence of q on the ratio α_A/d more then α_A and d taken alone. The value of q for $0 \leq \alpha_A/d \leq 1$ numerically approaches $3/2$, $7/5$ and $4/3$ for ρ approaching $-\infty$, 0 and ∞ respectively. Intriguingly enough, these three values of q respectively correspond to the divergences of the moments $\langle k \rangle$,

$\langle k^{3/2} \rangle$ and $\langle k^2 \rangle$ of a q-exponential distribution.

$$\begin{cases} q = \frac{3}{2} & \text{for } \rho \to -\infty & < k > \to \infty \\ q = \frac{7}{5} & \text{for } \rho = 0 & < k^{\frac{3}{2}} > \to \infty \\ q = \frac{4}{3} & \text{for } \rho \to \infty & < k^2 > \to \infty \end{cases} \tag{2.6}$$

Similar but opposite behavior for κ as a function of ρ. It goes from a minimum value, when $\rho \to -\infty$ to a maximum value when $\rho \to +\infty$, in the region near $\rho = 0$, that constitutes again an inflection point, κ values change more rapidly. In any case all points distribute in a narrow interval. Notice also the same collapse of the curves as for the case of q showing the strict dependence of κ on the ratio α_A/d more then α_A and d taken alone. The inflexion point emerges for q and κ as a function of ρ for $\rho = 0$, $\forall\, \alpha_A/d$. The opposite behavior of q and ρ shows up in the relation we found between the two (see the lowest graph in fig.2.5). All the (q, κ) data closely lie within the straight line

$$q = 1.54 - 0.29\,\kappa \quad \forall\, (\alpha_A/d, \rho) \tag{2.7}$$

A collapse of the data could be observed also plotting q and κ as functions of α_A/d see fig. 2.4.

We analized different cases, changing the ρ parameter, and we found that the behavior is very similar for the entire spectrum of ρ. Here we show the case for $\rho \to -\infty$, that is an interesting previously unexplored case. In the top two plots you can see the dependnce of q and κ, for diffrent dimensions, on the values of α_A. They have similar behavior but differentiate for different dimensions d. In the low two plots it is shown the collapse of the curves when one considers q and κ as functions of α_A/d. All the curves collapse in a single universal one. As said, this is true for the whole spectrum of ρ, showing that our model, with the new introduced parameter is able to reproduce many well-known models and that the universal dependence of q and κ on α_A/d is valid for $\rho \in [-\infty, \infty]$, therefore it is a universal behavior for a pletora of models.

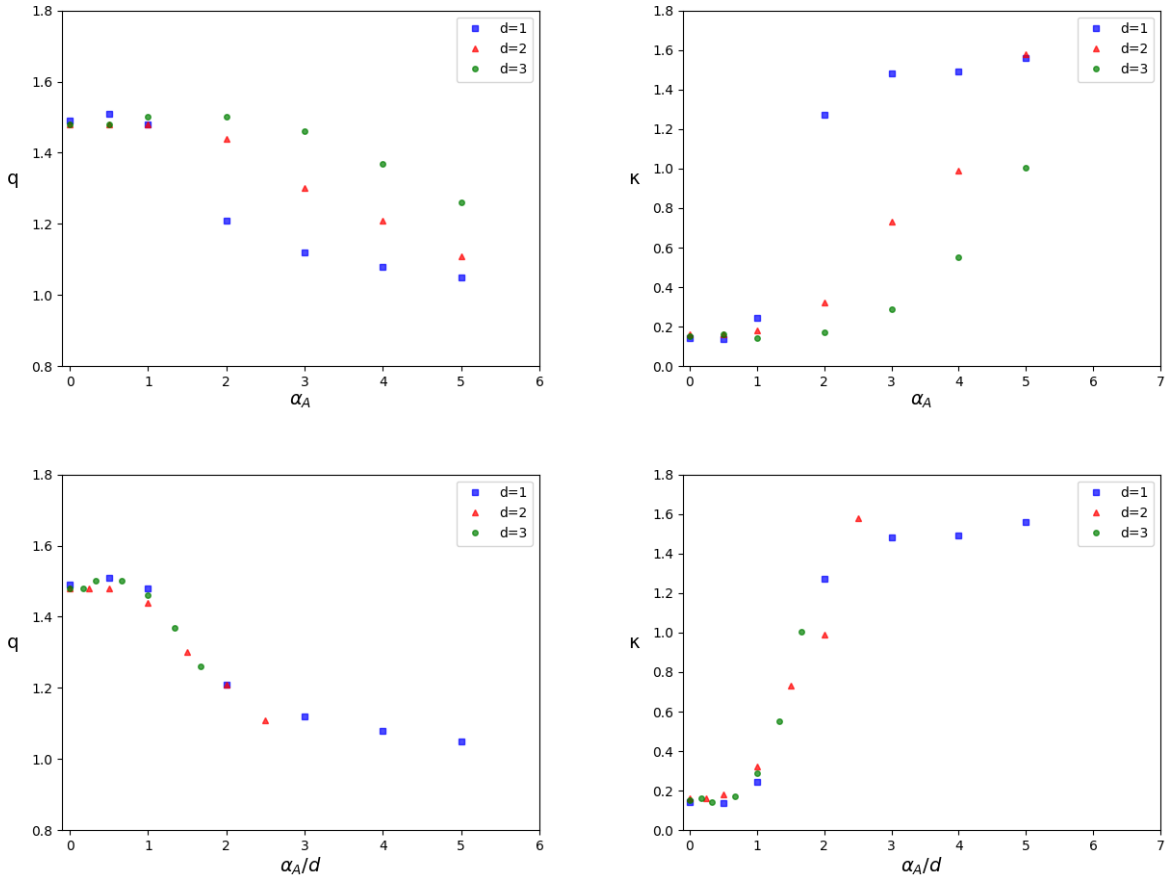


Figure 2.4: q and κ as funtions of α_A and α_A/d for $\rho = -100$.

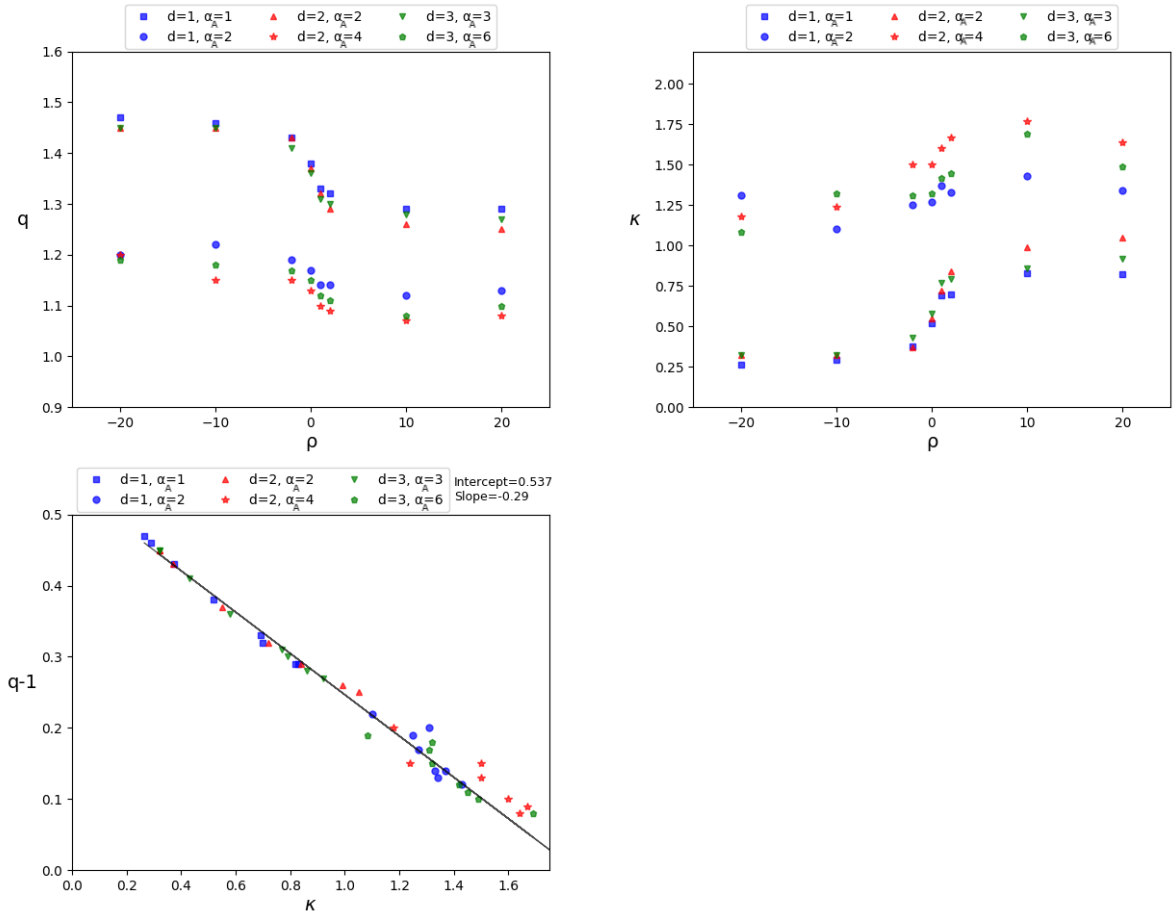


Figure 2.5: q and κ as functions of ρ for typical values of α_A/d.

Chapter 3

Case 2: Network Geometry with Fractional Flavor

3.1 Network Geometry with Flavor

3.1.1 Energy and fitness of the simplices

In the Network Geometry with Flavor each simplex $\alpha \in \mathcal{K}$ is associated to an energy ϵ_α that does not change in time. The energy of a face describes its intrinsic and heterogeneous properties and has an important effect on the simplicial complex evolution.

The energy ϵ_i of node i is drawn randomly from a given distribution $g(\epsilon)$. To every δ-face $\alpha \in \mathcal{S}_{d,\delta}$ with $0 < \delta \leq d$ we associate an *energy* ϵ_α given by the sum of the energy of the nodes that belong to the face α,

$$\epsilon_\alpha = \sum_{i \in \alpha} \epsilon_i. \tag{3.1}$$

Therefore, the energy of a link is given by the sum of energies of the two nodes that belong to it, the energy of a triangular face is given by the sum of the energy of the three nodes belonging to it and so on. The energy $\epsilon_{(i,j)}$ of the generic link $\alpha = (i, j)$ belonging to any given triangle of the NGF formed by the nodes i, j and r satisfy the triangular inequality

$$|\epsilon_{(i,r)} - \epsilon_{(j,r)}| \leq \epsilon_{(i,j)} \leq \epsilon_{(i,r)} + \epsilon_{(j,r)}. \tag{3.2}$$

This result remains valid for any permutation of the order of the nodes i, j and r belonging to the triangle. Therefore it is possible to consider the energy of the link as a possible candidate for the length of the link.

Finally to each simplex $\alpha \in \mathcal{K}$ we associate a *fitness* η_α given by

$$\eta_\alpha = e^{-\beta \epsilon_\alpha}, \tag{3.3}$$

where $\beta \leq 0$ is an external parameter of the model called *inverse temperature*. If $\beta = 0$ we have that $\eta_\alpha = 1$ for every simplex $\alpha \in \mathcal{K}$, therefore every simplex has the same fitness independently of their differences in energy. On the contrary when β is large small differences in energy lead to large differences in the fitness of different simplices.

3.1.2 Evolution of the Network Geometry with Flavor

Network Geometry with Flavor (NGF) is a growing model generating pure d-dimensional simplicial complexes. The stochastic evolution of NGF is determined by a parameter s called the *flavor* and by the fitness of the simplices of the simplicial complex. The evolution of the NGF obeys a simple iterative algorithm.
Initially at time $t = 1$ the simplicial complex is formed by a single d-dimensional simplex. At each time $t > 1$ we glue a d-dimensional simplex to a $(d-1)$-face α chosen with probability

$$\Pi_{d,d-1}(\alpha) = \frac{\eta_\alpha(1 + sn_\alpha)}{Z^{[s]}}, \tag{3.4}$$

where $Z^{[s]}$ is called the *partition function* of the NGF and is given by

$$Z^{[s]}(t) = \sum_{\alpha' \in S_{d,d-1}} \eta_{\alpha'}(1 + sn_{\alpha'}). \tag{3.5}$$

3.1.3 Possible values of the flavor and their topological implications

The Network Geometry with Flavor describes a growing simplicial complex that depends on the value of the flavor s. Let us consider the attachment probability $\Pi_{d,d-1}(\alpha)$ for $\beta = 0$ and the integer flavors $s \in \{-1, 0, 1\}$. In this case we have that the attachment probability satisfies

$$\Pi_{d,d-1}(\alpha) \propto (1 + sn_\alpha) = \begin{cases} 1 - n_\alpha & \text{for} \quad s = -1, \\ 1 & \text{for} \quad s = 0, \\ k_{d,d-1}(\alpha) & \text{for} \quad s = 1. \end{cases} \tag{3.6}$$

Therefore the flavor $s = -1$ enforces the generation of a manifold. In fact we have $\Pi_{d,d-1}(\alpha) > 0$ if $n_\alpha = 0$ and $\Pi_{d,d-1}(\alpha) = 0$ if $n_\alpha = 1$. Therefore $n_\alpha \in \{0, 1\}$ for every $(d-1)$-dimensional face α of the simplicial complex. However in both cases $s = 0$ and $s = 1$ the incidence number can take any integer value $n_\alpha \geq 0$. The flavor $s = 0$ corresponds to a uniform attachment of d-dimensional simplices of $(d-1)$-dimensional faces, while $s = 1$ corresponds to a higher dimensional preferential attachment of d-dimensional simplices of $(d-1)$-dimensional faces.

The NGF with integer flavor reduces to several known models for different values of the parameters s, d and β. For $d = 1, s = 1, \beta = 0$ the NGF reduces to the Barabási-Albert [89] model while for $d = 1, s = 1, \beta > 0$ it reduces to the Bianconi-Barabási model [56, 57]. For $d = 2, s = 0, \beta = 0$ it reduces to the model proposed in Ref. [90] Finally for $d = 3, s = -1, \beta = 0$ it reduces to a random Apollonian network [91].

Values of the flavor s different from the values $\{-1, 0, 1\}$ are also allowed as long as they lead to a suitable probability $\Pi_{d,d-1}(\alpha) \in [0, 1]$ for every face $\alpha \in \mathcal{K}$. Therefore positive values of the flavor $s \in \mathbb{R}^+$ are always allowed. In this case via a rescaling of the attachment probability it is easy to show that NGFs display a stochastic topology with statistical properties equivalent to NGF with flavor $s = 1$.
For negative values of the flavor s the requirement of observing a well defined attachment probability $\Pi_{d,d-1}(\alpha) \in [0, 1]$ implies instead some restriction on the possible values of s.

Table 3.1: Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta = 0$. For $d \geq 2\delta + 2 - s$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Bimodal	Exponential	Power-law
$\delta = d - 2$	Exponential	Power-law	Power-law
$\delta \leq d - 3$	Power-law	Power-law	Power-law

Table 3.2: The average $\langle k_{d,\delta} - 1 | \epsilon \rangle$ of the generalized degrees $k_{d,\delta} - 1$ of δ-faces with energy ϵ in a d-dimensional NGF of flavor s follows either the Fermi-Dirac, the Boltzmann or the Bose-Einstein statistics depending on the values of the dimensions d and δ.

flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$\delta = d - 2$	Boltzmann	Bose-Einstein	Bose-Einstein
$\delta \leq d - 3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

In particular if $s < 0$, then s should be of the form

$$s = -\frac{1}{m}, \tag{3.7}$$

with $m \in \mathbb{N}$. For such values of the flavor s the incidence number of any $(d-1)$-dimensional face α can only take $m + 1$ values, i.e.

$$n_\alpha \in \{0, 1, 2, \ldots, m\}. \tag{3.8}$$

Therefore as long as $m > 1$ NGF with $s = -1/m$ are not anymore manifolds, but they have a generalized degree of the $(d-1)$-dimensional faces bounded by $m + 1$, i.e.

$$k_{d,d-1}(\alpha) \in \{0, 1, 2, \ldots, m + 1\}. \tag{3.9}$$

This case, that we will call NGF with Fractional Flavor, is therefore expected to display statistical properties that are not equivalent to the ones observed for any of the integer flavors $s \in \{-1, 0, 1\}$.

3.2 Network Geometry with Integer Flavor

The distribution $P_{d,\delta}(k)$ of generalized degrees $k_{d,\delta} = k$ of δ-dimensional faces on the d-dimensional NGF has been derived for integer flavors $s \in \{-1, 0, 1\}$ in Ref. [67]. It has been found that for $\beta = 0$ the generalized degree distribution $P_{d,\delta}(k)$ can follow a bimodal, exponential or power-law distribution (see Table 3.1) depending on the dimension δ and the flavor s of the NGF.
For $\beta > 0$ emergent quantum statistics describe the statistical properties of NGFs as long as the NGF has a stationary generalized degree distribution, i.e for sufficiently low value of $\beta \leq \beta_c$. Specifically it has been found in Ref. [66, 67] that the average of the generalized degree minus one, $k_{d,\delta} - 1$, over δ-dimensional faces of energy ϵ can follow the Fermi-Dirac, the Boltzmann or the Bose-Einstein distribution depending

Table 3.3: Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta = 0$. Only for $d - 2\delta \geq 2 + \frac{3}{m}$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

flavor	$s = -1/m$
$\delta = d - 1$	Bounded $k \leq m + 1$
$\delta \leq d - 2$	Power-law

on the dimension δ and the flavor s of the NGF (see Table 3.2). For instance in a NGF with $s = -1$ and $d = 3$ the average of the generalized degree distribution minus one performed over faces of energy ϵ follows the Fermi-Dirac, the Boltzmann or the Bose-Einstein distribution for triangular faces, links and nodes respectively.

In the next section we will show how these statistical properties change for NGF with Fractional Flavor.

3.3 Network Geometry with Fractional Flavor and $\beta = 0$

3.3.1 Main results

In this section we will evaluate the generalized degree distribution $P_{d,\delta}(k)$ of NGF with Fractional flavor $s = -1/m$ and $m > 1$ for $\beta = 0$. In particular we will show that differently from the cases $s = -1$ and $s = 0$ the generalized degree distributions are never exponential. In fact for $s = -1/m$ and $m > 1$ we obtain that the $(d-1)$-faces have a generalized degree distribution with bounded support with $k \leq m + 1$ and the δ-dimensional faces with $0 \leq \delta < d - 1$ have a generalized degree distribution which is power-law (see Table 3.3). In order to proof these results, in the following paragraph we first derive the generalized attachment probability. Subsequently we derive the generalized degree distribution first using the mean-field approximation and finally using the master equation approach providing exact asymptotic results.

3.3.2 Attachment probability

For fractional flavor $s = -\frac{1}{m}$ the attachment probability for $\beta = 0$, given by Eq. (3.4) can also be expressed as

$$\Pi_{d,d-1}(\alpha) = \frac{m - n_\alpha}{\tilde{Z}} = \frac{m + 1 - k_{d,d-1}(\alpha)}{\tilde{Z}}, \tag{3.10}$$

where $\tilde{Z}$ is given by

$$\tilde{Z} = \sum_{\alpha \in \mathcal{S}_{d,d-1}} (m - n_\alpha) = \sum_{\alpha \in \mathcal{S}_{d,d-1}} (m + 1 - k_{d,d-1}(\alpha)).$$

Therefore the normalization constant $\tilde{Z}$ counts each $(d-1)$-dimensional face α with a degeneracy $m - n_\alpha \in \{0, 1, 2, \ldots, m\}$.
Since at time $t = 1$ we have $d + 1$ $(d-1)$-dimensional faces with degeneracy m we have that $\tilde{Z}(t = 1) = m(d + 1)$. Moreover at each time we add d new $(d-1)$-dimensional

faces with degeneracy m and we reduce the degeneracy of the $(d-1)$-faces α to which we add the new d-dimensional simplex by 1. Therefore at each time $\tilde{Z}$ increases by $md-1$. It follows that the normalization constant $\tilde{Z}$ is given by

$$\tilde{Z} \;\; = \;\; = (md-1)t + 1 + m \simeq (md-1)t, \tag{3.11}$$

where the last expression is valid for $t \gg 1$. The probability $\Pi_{d,\delta}(\alpha)$ that a new d-dimensional simplex is attached to a $\delta \leq d-2$ dimensional face α is given by

$$\Pi_{d,\delta}(\alpha) = \sum_{\alpha' \in S_{d,d-1} | \alpha' \supset \alpha} \frac{m - n_{\alpha'}}{\tilde{Z}}. \tag{3.12}$$

In order to calculate the numerator of this expression we make the following considerations. If we assume that the face α has incidence number $n_\alpha = 0$, then the numerator of Eq. (3.12) is given by $d-\delta$. In fact every δ-dimensional face α with $\delta < d-1$ and generalized degree $k_{d,\delta}(\alpha) = 1$ is incident to

$$\binom{d-\delta}{d-\delta-1} = d-\delta \tag{3.13}$$

$(d-1)$-dimensional faces with degeneracy m. This follows from the fact that its incident $(d-1)$-dimensional faces must contain $d-\delta-1$ nodes that do not belong to the face α. These nodes should chosen among the $(d-\delta)$ nodes of the single d-dimensional simplex that contains the face α and are external to face α. Following a similar argument it is easy to check that at each time we add a d-dimensional simplex to the δ-dimensional face the number of $(d-1)$-dimensional faces with degeneracy m increases by

$$\binom{d-\delta-1}{d-\delta-2} = d-\delta-1 \tag{3.14}$$

and additionally we reduce the degeneracy of the $(d-1)$-dimensional face to which we attach the new d-dimensional simplex by one. Therefore for $t \gg 1$ we have

$$\Pi_{d,\delta}(\alpha) \simeq \begin{cases} \frac{m+1-k_{d,\delta}(\alpha)}{(md-1)t} & \text{for} \quad \delta = d-1, \\ \\ \frac{[m(d-\delta-1)-1]k_{d,\delta}(\alpha)+m+1}{(md-1)t} & \text{for} \quad \delta \leq d-2. \end{cases} \tag{3.15}$$

Finally given the above expression we can express the probability $\tilde{\Pi}_{d,\delta}(k)$ that a new d-dimensional simplex is attached to a δ-dimensional face α with generalized degree $k_{d,\delta}(\alpha) = k$ as

$$\tilde{\Pi}_{d,\delta}(k) \simeq \begin{cases} \frac{m+1-k}{(md-1)t} & \text{for} \quad \delta = d-1, \\ \\ \frac{[m(d-\delta-1)-1]k+m+1}{(md-1)t} & \text{for} \quad \delta \leq d-2. \end{cases} \tag{3.16}$$

3.3.3 Mean-field approach

In order to find an approximated generalized degree distribution we can consider the popular mean field approach [92]. In this case we assume that the generalized degrees can be approximated by their average over different network realizations and that they evolve by a deterministic equation

$$\frac{dk_{d,\delta}(\alpha)}{dt} = \Pi_{d,\delta}(\alpha). \tag{3.17}$$

Let us consider separately the case in which $\delta = d-1$ and the case $\delta \leq d-2$. The mean-field equation for the generalized degree of $\delta = d-1$ dimensional faces is given by

$$\frac{dk_{d,d-1}}{dt} = \frac{m+1-k_{d,d-1}}{(md-1)t}, \tag{3.18}$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. It follows that in the mean-field approximation the generalized degree of a $(d-1)$-face added at time t_α evolves in time as

$$k_{d,d-1}(t) = m+1-m\left(\frac{t_\alpha}{t}\right)^{1/(md-1)}. \tag{3.19}$$

If follows that, as expected, the generalized degrees of the $(d-1)$-dimensional faces are bounded and asymptotically in time saturate to the the value $k_{d,d-1} = m+1$. In order to derive the generalized degree distribution in the mean-field approximation we calculate the probability $\mathcal{P}(k_{d,\delta} > k)$. This is given by

$$\begin{aligned}\mathcal{P}(k_{d,d-1} > k) &= \mathcal{P}\left(t_\alpha < \left(\frac{m+1-k}{m}\right)^{md-1} t\right) \\ &= \left(\frac{m+1-k}{m}\right)^{md-1}.\end{aligned} \tag{3.20}$$

Therefore in the mean-field approximation the generalized degree distribution is given by

$$\tilde{P}_{d,d-1}(k) = -\frac{\mathcal{P}(k_{d,d-1} > k)}{dk} = \frac{md-1}{m}\left(\frac{m+1-k}{m}\right)^{md-2}, \tag{3.21}$$

valid for $1 \leq k \leq m$. Let us now consider the mean-field equation for the $\delta \leq d-2$ dimensional faces. This equation reads

$$\frac{dk_{d,d-1}}{dt} = \frac{[m(d-\delta-1)-1]k_{d,\delta}+m+1}{(md-1)t} \tag{3.22}$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. Therefore in the mean field approximation the generalized degree of a δ-face added at time t_α grows in time as

$$k_{d,d-1}(t) = (1+1)\left(\frac{t}{t_\alpha}\right)^{\frac{m(d-\delta-1)-1}{md-1}} - a \tag{3.23}$$

where for convenience we have defined the constant a as

$$a = \frac{m+1}{m(d-\delta-1)-1}. \tag{3.24}$$

Given this definition we can express the generalized degree distribution of $\delta \leq d-2$ faces in the mean field approximation. In fact we have that the cumulative generalized degree distribution is given by

$$\begin{aligned} \mathcal{P}(k_{d,\delta} > k) &= \mathcal{P}\left(t_\alpha < \left(\frac{1+a}{k+a}\right)^{\frac{md-1}{m(d-\delta-1)-1}} t\right) \\ &= \left(\frac{1+a}{k+a}\right)^{\frac{md-1}{m(d-\delta-1)-1}}. \end{aligned} \tag{3.25}$$

By differentiating this expression we find the generalized degree distribution of $\delta \leq d-2$ faces in the mean field approximation is given by

$$\begin{aligned} \tilde{P}_{d,\delta}(k) &= -\frac{\mathcal{P}(k_{d,\delta} > k)}{dk} \\ &= \frac{md-1}{m(d-\delta)} \left(\frac{1+a}{k+a}\right)^{\frac{md-1}{m(d-\delta-1)-1}+1}. \end{aligned} \tag{3.26}$$

Therefore we find in this approximation that generalized degrees have a bounded distribution for $\delta \leq d-1$ faces and a power-law distribution for $\delta \leq d-2$ faces.

3.3.4 Master equation approach

The mean-field approach gives only approximate results for the generalized degree distribution. In order to get the exact asymptotic results we need to consider the master equation approach [92]. The master equation describes the evolution of the average number $N^t_{d,\delta}(k)$ of δ-dimensional faces that at time t have generalized degree k in a d-dimensional NGF with flavor $s = -\frac{1}{m}$. We notice that at each time we add

$$m_{d,\delta} = \binom{d}{\delta}$$

number of δ-faces of generalized degree $k_{d,\delta} = 1$ and the average number of δ-faces of generalized degree $k_{d,\delta} = k$ increases by

$$\Pi_{d,\delta}(k-1) N^t_{d,\delta}(k-1)$$

if $k > 1$ and decreases by

$$\Pi_{d,\delta}(k) N^t_{d,\delta}(k).$$

Therefore the master equation reads

$$\begin{aligned} N^{t+1}_{d,\delta}(k) &= N^t_{d,\delta}(k) + \Pi_{d,\delta}(k-1) N^t_{d,\delta}(k-1)(1-\delta_{k,1}) \\ &\quad -\Pi_{d,\delta}(k) N^t_{d,\delta}(k) + m_{d,\delta}\delta_{k,1}. \end{aligned} \tag{3.27}$$

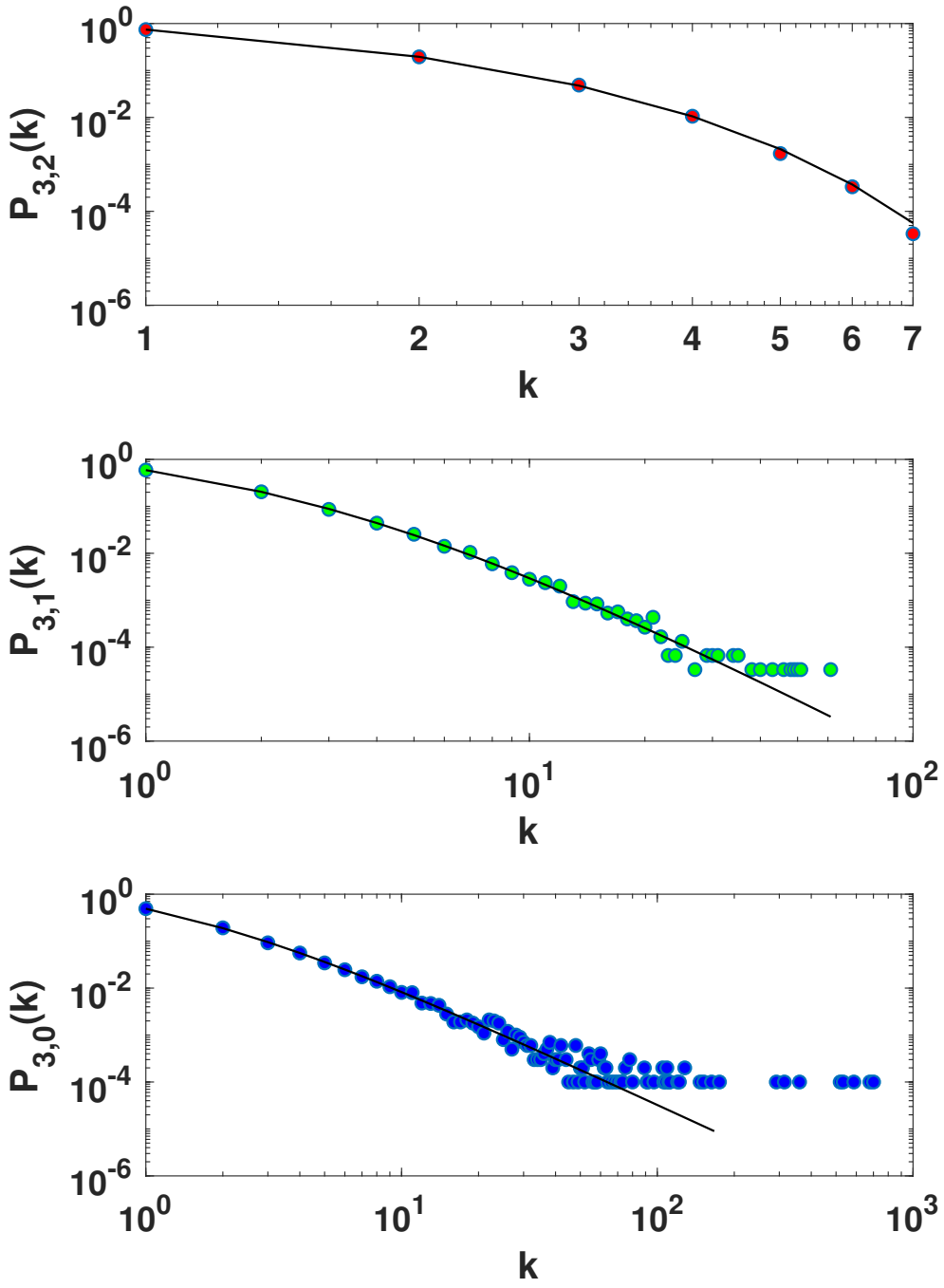


Figure 3.1: Generalized degree distribution $P_{d,\delta}(k)$ of nodes ($\delta = 0$), links ($\delta = 1$) and triangles ($\delta = 2$) of a NGF with $N = 5000$ nodes, flavor $s = -1/6$, dimension $d = 3$ and inverse temperature $\beta = 0$. The symbols indicate the results of simulations, the solid lines indicate the theoretical predictions obtained using the master equation approach.

For large network sizes when $t \gg 1$ the average number of δ-dimensional faces is given by

$$N^t_{d,\delta}(k) \simeq m_{d,\delta} t P_{d,\delta}(k) \tag{3.28}$$

where $P_{d,\delta}(k)$ is the generalized degree distribution of the δ-dimensional faces. Inserting this asymptotic expression in the Eq. (3.27) we can derive the generalized degree distribution as explained in the following by distinguishing between the case in which $\delta = d-1$ and the case in which $\delta \leq d-2$.

For the $\delta = d-1$ dimensional faces, by using the expression

$$\Pi_{d,d-1}(k) = \frac{m+1-k}{(md-1)t}, \tag{3.29}$$

and the asymptotic scaling of $N^t_{d,\delta}(k)$ given by Eq. (3.28) the master equation can be re-written in terms of the generalized degree distribution obtaining

$$\begin{aligned} P_{d,d-1}(k) &= \frac{m+2-k}{(md-1)} P_{d,d-1}(k-1)(1-\delta_{k,1}) + \\ &\quad -\frac{m+1-k}{(md-1)} P_{d,d-1}(k) + \delta_{k,1}. \end{aligned} \tag{3.30}$$

obtaining

$$P_{d,d-1}(k) = \frac{md-1}{m(d+1)-1} \frac{\Gamma(m+1)}{\Gamma(md+m-1)} \frac{\Gamma(md+m-k)}{\Gamma(m-k+2)},$$

valid for $1 \leq k \leq m$.

For the $\delta \leq d-2$ dimensional faces by using the expression

$$\Pi_{d,\delta}(k) = \frac{[m(d-\delta-1)-1]k + m + 1}{(md-1)t}, \tag{3.31}$$

and the asymptotic scaling of $N^t_{d,\delta}(k)$ given by Eq. (3.28) the master equation can be re-written in terms of the generalized degree distribution obtaining

$$\begin{aligned} P_{d,\delta}(k) &= \frac{[m(d-\delta-1)-1]k + m + 1}{md-1} P_{d,\delta}(k-1)(1-\delta_{k,1}) \\ &\quad -\frac{[m(d-\delta-1)-1]k + m + 1}{md-1} P_{d,\delta}(k) + \delta_{k,1}. \end{aligned}$$

This latter recursive equation has explicit solution

$$\begin{aligned} P_{d,\delta}(k) &= \frac{md-1}{m(2d-\delta)-1} \frac{\Gamma\left(2 + \frac{m(d+1)}{m(d-\delta-1)-1}\right)}{\Gamma\left(1 + \frac{m+1}{m(d-\delta-1)-1}\right)} \\ &\quad \times \frac{\Gamma\left(k + \frac{m+1}{m(d-\delta-1)-1}\right)}{\Gamma\left(k+1 + \frac{m(d+1)}{m(d-\delta-1)-1}\right)}. \end{aligned} \tag{3.32}$$

This distribution for large $k \gg 1$ decays as a power-law

$$P_{d,\delta}(k) \simeq k^{-\gamma_{d,\delta}} \tag{3.33}$$

with

$$\gamma_{d.\delta} = 1 + \frac{md-1}{m(d-\delta-1)-1}. \tag{3.34}$$

These distributions are therefore scale-free, i.e. $\gamma \leq 3$ if and only if

$$d - 2\delta \geq 2 + \frac{3}{m}. \tag{3.35}$$

These theoretical predictions show that the generalized degree distribution is indeed bounded for $\delta = d-1$ dimensional faces and power-law for $\delta \leq d-2$ dimensions. As expected these results perfectly match the simulation results providing exact asymptotic expression for the generalized degree distribution $P_{d,\delta}(k)$ for NGF with fractional flavor $s = -\frac{1}{m}$ (see Figure 3.1).

3.4 Network Geometry with Fractional Flavor and $\beta > 0$

3.4.1 Main results

Quantum statistics have been shown to characterize the statistical properties of NGF with integer flavor $s \in \{-1, 0, 1\}$. In particular d-dimensional NGFs with flavor $s = -1$ have an average degree of δ faces with energy ϵ described by the Fermi-Dirac (for $\delta = d-1$), the Boltzmann (for $\delta = d-2$) and the Bose-Einstein distribution (for $\delta \leq d-3$). On the contrary on NGF with flavor $s = 0$ the average degree of δ faces with energy ϵ can be only described by the Boltzmann and the Bose-Einstein distribution. Finally in NGF with flavor $s = 1$ all the faces, independently of their dimension δ, have an average degree described only by the Bose-Einstein distribution. Interestingly if we consider integer flavors $s \in \{-1, 0, 1\}$ the Fermi-Dirac distribution emerges as the natural distribution characterizing the statistical properties of $\delta = d-1$ faces only if the flavor is given by $s = -1$, which corresponds to the case in which the incidence number n_α fo the $\delta = d-1$ faces can only take the values $n_\alpha \in \{0, 1\}$. This suggests a relation between the emergence of the Fermi-Dirac statistics and the constraint imposed by the flavor $s = -1$ on the possible values of the incidence number. It is therefore interesting to investigate the properties of NGF with factional flavor $s = -\frac{1}{m}$ in which the only allowed values of the incidence number are $n_\alpha \in \{0, 1, \ldots, m\}$. In principle one could expect that in this case the statistical properties of the generalized degree of the NGF would be described by generalized quantum statistics such as the Gentile statistics [86] or the anyons statistics [87, 88]. To our surprise instead the result of our calculations has revealed that in NGF with fractional flavor $s = -\frac{1}{m}$ and $m > 1$ does not display any fractional statistics (see Table 3.4). If we compare the results obtained for to the NGFs with fractional flavor $s = -\frac{1}{m}$ and $m > 1$ to the results obtained for NGF with $s = -1$ we observe that

Table 3.4: The average $\langle k_{d,\delta} - 1|\epsilon\rangle$ of the generalized degrees $k_{d,\delta}$ of δ-faces with energy ϵ minus one in a d-dimensional NGF of flavor $s = -\frac{1}{m}$ follows either the Fermi-Dirac or the Bose-Einstein statistics depending on the values of the dimensions d and δ.

	s=-1/m
$\delta = d-1$	Fermi-Dirac
$\delta \leq d-2$	Bose-Einstein

- the average degree of $(d-1)$-dimensional faces with energy ϵ still remain described by the Fermi-Dirac statistics also if the incidence number of $\delta = d-1$ faces can take values $n_\alpha \in \{0, 1, \ldots, m\}$ with $m > 1$;
- the average degree of $\delta = (d-2)$ dimensional faces with energy ϵ is already characterized by the Bose-Einstein statistics and not by the Boltzmann statistics;
- the average degree of $\delta < d-2$-dimensional faces with energy ϵ is characterized by the Bose-Einstein statistics like in the case $s = -1$.

In particular these results imply that when we consider the fractional flavor $s = -\frac{1}{m}$ and $m > 1$ we have that already for $d = 2$ dimensional NGF we can observe the coexistence of faces with statistical properties described respectively by the Fermi-Dirac and Bose-Einstein distribution while for observing the co-existence of these two statistics in NGF with flavor $s = -1$ we should have dimension $d \geq 3$.

3.4.2 Attachment probability and chemical potentials

When $\beta > 0$ the generalized degree distribution of the NGF can be solved by extending the self-consistent approach proposed for solving the Bianconi-Barabási model [56, 57] which constitutes the NGF model for $s = 1$ and $d = 1$. In this approach it is assumed that the statistical properties of the NGF reach a steady state and that it is possible to define suitable parameters $\mu_{d,\delta}$ called *chemical potentials*. In particular the chemical potential $\mu_{d,d-1}$ of the $\delta = d-1$ faces is defined as

$$e^{\beta\mu_{d,d-1}} = \lim_{t\to\infty} \frac{t}{mZ^{[s]}}, \tag{3.36}$$

while the chemical potential $\mu_{d,\delta}$ of the $\delta < d-1$ faces is defined as

$$e^{\beta\mu_{d,\delta}} = \lim_{t\to\infty} \left\langle \frac{\sum_{\alpha'\in S_{d,d-1}|\alpha\subset\alpha'} e^{-\beta(\epsilon_{\alpha'}-\epsilon_\alpha)}(1+sn_{\alpha'})t}{Z^{[s]}(a+k_{d,\delta}(\alpha))} \right\rangle,$$

where here the average is done over δ-dimensional faces $\alpha \in S_{d,\delta}$. In both cases it is assumed that if the network evolution reaches a stationary state, then the chemical potential is self-averaging, i.e. it does not depend on the specific network realization of the NGF over which the limit $t \to \infty$ is performed. As long as the chemical potentials are well determined and self-averaging quantities the attachment probabilities can be expressed in terms of the chemical potentials ,and it can be easily shown that the

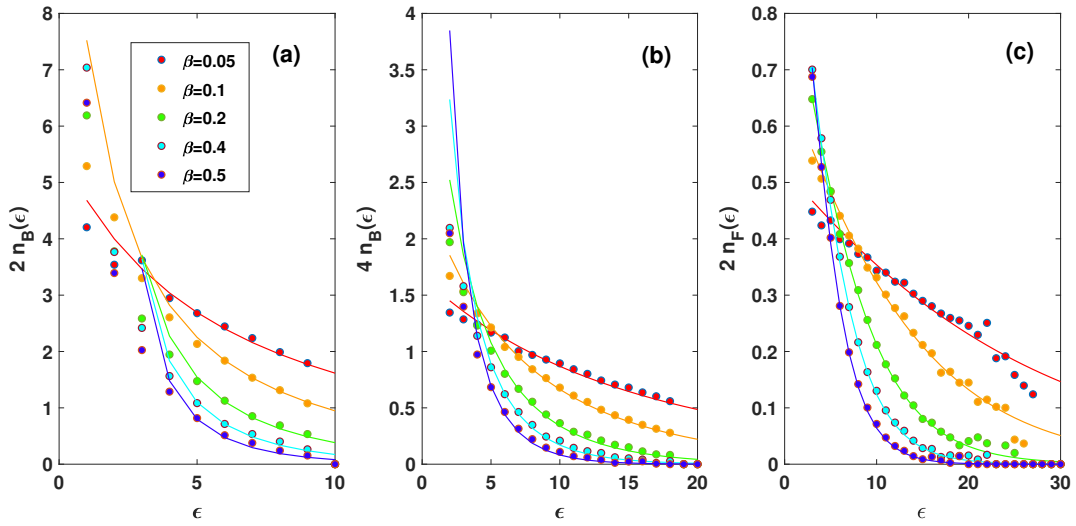


Figure 3.2: Average generalized degree minus one, over faces of energy ϵ and dimension $\delta = 0$ (panel a) $\delta = 1$ (panel b) or $\delta = 2$ (panel c) for $d = 3$ dimensional NGF with fractional flavor $s = -1/2$ are plotted for different values of β (symbols) and compared to the theoretical expectations (Fermi-Dirac and Bose-Einstein statistics). The simulations are performed for NGF with $N = 3000$ nodes. The data are averaged over 50 NGF realizations.

probability $\Pi_{d,\delta}(\alpha)$ that a new d-dimensional simplex is attached to a new δ-dimensional face α is given by

$$\Pi_{d,\delta}(\alpha) \simeq \begin{cases} e^{-\beta(\epsilon_\alpha - \mu_{d,d-1})} \frac{m+1-k_{d,\delta}(\alpha)}{t} & \text{for} \quad \delta = d-1, \\ e^{-\beta(\epsilon_\alpha - \mu_{d,d-1})} \frac{k_{d,\delta}(\alpha)+a}{t} & \text{for} \quad \delta \leq d-2, \end{cases} \tag{3.37}$$

where a is given by Eq. (3.24). From this it follows that the probability $\tilde{\Pi}_{d,\delta}(k)$ that a new d-dimensional simplex is attached to a δ-face with generalized degree $k_{d,\delta}(\alpha) = k$ is given by

$$\tilde{\Pi}_{d,\delta}(k) \simeq \begin{cases} e^{-\beta(\epsilon_\alpha - \mu_{d,d-1})} \frac{m+1-k}{t} & \text{for} \quad \delta \leq d-1, \\ e^{-\beta(\epsilon_\alpha - \mu_{d,d-1})} \frac{k+a}{t} & \text{for} \quad \delta \leq d-2. \end{cases} \tag{3.38}$$

3.4.3 Mean-field approach

Let us consider first the results that can be obtained within the mean field approximation. As mentioned before for the case $\beta = 0$ in the mean-field approximation we neglect the fluctuations and we consider a deterministic evolution of the generalized degree that is assumed to be equal to the average generalized degree over different NGF realizations. Let us consider separately the case in which the cases $\delta = d-1$ and $\delta \leq d-2$. By assuming that the chemical potential $\mu_{d,d-1}$ is well defined, and using Eq. (3.38) for the attachment probability, the mean-field equation (Eq. (3.17)) for generalized degree

$k_{d,d-1}(\alpha) = k_{d,d-1}$ of the generic $(d-1)$-face α with energy $\epsilon_\alpha = \epsilon$ can be written explicitly as

$$\frac{dk_{d,d-1}}{dt} = \frac{e^{-\beta(\epsilon-\mu_{d,d-1})}(m+1-k_{d,d-1})}{t}, \tag{3.39}$$

with initial condition $k_{d,d-1}(t_\alpha) = 1$. This equation has solution

$$k_{d,d-1}(t|\epsilon) = m + 1 - m\left(\frac{t_\alpha}{t}\right)^{e^{-\beta(\epsilon-\mu_{d,d-1})}}, \tag{3.40}$$

which like in the case $\beta = 0$ clearly implies that the generalized degree of the $(d-1)$-dimensional faces is bounded. Interestingly in the mean-field approximation we can evaluate the average of the generalized degrees minus one over faces with energy ϵ getting

$$\langle k_{d,d-1} - 1|\epsilon\rangle = \frac{m}{e^{\beta(\epsilon_\alpha-\mu_{d,d-1})}+1} = mn_F(\epsilon). \tag{3.41}$$

Therefore this quantity is proportional to the Fermi-Dirac distribution $n_F(\epsilon)$ with chemical potential $\mu_{d,d-1}$. Interestingly, as we will show in the next paragraph this result is exact, in fact it is a result that concerns the average of the generalized degrees and therefore is not affected by the mean-field approximation. However the generalized degree distribution of $d-1$ faces that can be derived from the mean-field approach is instead an approximation. By proceeding similarly to the case $\beta = 0$ we obtain that in the mean field approximation the probability $\tilde{P}_{d,d-1}(k|\epsilon)$ that a $(d-1)$-dimensional face with energy ϵ has generalized degree $k_{d,d-1}(\alpha) = k$ is given by

$$\tilde{P}_{d,d-1}(k|\epsilon) = \frac{1}{m}e^{\beta(\epsilon-\mu_{d,\delta})}\left(\frac{m+1-k}{m}\right)^{e^{\beta(\epsilon-\mu_{d,\delta})}-1}. \tag{3.42}$$

We can proceed similarly for the $\delta \leq d-2$ dimensional faces. In particular in this case the mean-field equations read

$$\frac{dk_{d,\delta}}{dt} = \frac{e^{-\beta(\epsilon-\mu_{d,\delta})}(k_{d,\delta}+a)}{t}, \tag{3.43}$$

with initial condition $k_{d,\delta}(t_\alpha) = 1$. Here we have assumed that the chemical potential $\mu_{d,\delta}$ is well defined and we have used Eq. (3.38) for the attachment probability $\Pi_{d,\delta}$. The above mean-field equations have the solution

$$k_{d,\delta}(t|\epsilon) \;=\; (1+a)\left(\frac{t}{t_\alpha}\right)^{e^{-\beta(\epsilon_\alpha-\mu_{d,\delta})}} - a. \tag{3.44}$$

By using this expression it is possible to calculate the average of the generalized degree minus one over faces of energy ϵ finding

$$\langle k_{d,\delta} - 1|\epsilon\rangle = A_\delta \frac{1}{e^{\beta(\epsilon_\alpha-\mu_{d,\delta})}-1} = A_\delta n_B(\epsilon), \tag{3.45}$$

where

$$A_\delta = 1 + a = \frac{m(d-\delta)}{m(d-\delta-1)-1}. \tag{3.46}$$

Therefore we find that $\delta \leq d-2$ dimensional faces of energy ϵ have statistical properties described by the Bose-Einstein distribution $n_B(\epsilon)$ with chemical potential $\mu_{d,\delta}$. However in the mean-field approximation the derived generalized degree distribution is not exact but approximated. Proceeding as in the previous case we find that in the mean field approximation the probability $\tilde{P}_{d,d-1}(k|\epsilon)$ that a $(d-1)$-dimensional face with energy ϵ has generalized degree $k_{d,d-1}(\alpha) = k$ is given by

$$\tilde{P}_{d,\delta}(k|\epsilon) = e^{\beta(\epsilon-\mu_{d,\delta})}(1+a)\left(\frac{1+a}{k+a}\right)^{e^{\beta(\epsilon-\mu_{d,\delta})}+1} \tag{3.47}$$

3.4.4 Master equation approach

For $\beta > 0$ we can find the exact asymptotic result for the generalized degree distribution of faces of given energy ϵ. The master equation from which we start is written for the number $N^t_{d,\delta}(k|\epsilon)$ of δ-dimensional faces with energy ϵ and reads

$$\begin{aligned} N^{t+1}_{d,\delta}(k|\epsilon) &= N^t_{d,\delta}(k|\epsilon) + \Pi_{d,\delta}(k-1)N^t_{d,\delta}(k-1|\epsilon)(1-\delta_{k,1}) \\ &\quad -\Pi_{d,\delta}(k)N^t_{d,\delta}(k|\epsilon) + \rho_{d,\delta}(\epsilon)\delta_{k,1}. \end{aligned}$$

where $\Pi_{d,\delta}(k-1)$ is given by Eq. (3.38) and where $\rho_{d,\delta}(\epsilon)$ indicates the density of new faces with energy ϵ that we add at time t. For large network sizes when $t \gg 1$ the average number of δ-dimensional faces with energy ϵ is given by

$$N^t_{d,\delta}(k|\epsilon) \simeq \rho_{d,\delta}(\epsilon) t P_{d,\delta}(k|\epsilon). \tag{3.48}$$

Inserting this asymptotic expression we get the exact asymptotic result for the generalized degree distribution $P_{d,\delta}(k|\epsilon)$ of δ-dimensional faces with energy ϵ. Specifically in the case $\delta = d-1$ we obtain the bounded distribution

$$\begin{aligned} P_{d,d-1}(k|\epsilon) &= \frac{e^{\beta(\epsilon-\mu_{d,d-1})}}{(e^{\beta(\epsilon-\mu_{d,d-1})}+m)}\frac{\Gamma(m+1)}{\Gamma\left(m+e^{\beta(\epsilon-\mu_{d,d-1})}\right)} \\ &\quad \times\frac{\Gamma\left(m-k+1+e^{\beta(\epsilon-\mu_{d,d-1})}\right)}{\Gamma(m-k+2)}, \end{aligned} \tag{3.49}$$

for $1 \leq k \leq m+1$. For $\delta \leq d-2$ we obtain instead the power-law distribution

$$\begin{aligned} P_{d,\delta}(k|\epsilon) &= \frac{e^{\beta(\epsilon-\mu_{d,\delta})}[m(d-\delta-1)-1]}{e^{\beta(\epsilon-\mu_{d,\delta})}[m(d-\delta-1)-1]+m(d-\delta)} \\ &\quad \times\frac{\Gamma\left(2+a+e^{\beta(\epsilon-\mu_{d,\delta})}\right)}{\Gamma(1+a)} \\ &\quad \times\frac{\Gamma(k+a)}{\Gamma\left(k+1+a+e^{\beta(\epsilon-\mu_{d,\delta})}\right)}. \end{aligned} \tag{3.50}$$

Therefore for $k \gg 1$ the generalized degree distribution of δ-dimensional faces with energy ϵ decays as a power-law with an energy dependent power-law exponent $\gamma(\epsilon)$, i.e.

$$P_{d,\delta}(k|\epsilon) \simeq k^{-\gamma(\epsilon)} \tag{3.51}$$

with

$$\gamma(\epsilon) = 1 + e^{\beta(\epsilon-\mu_{d,\delta})}. \tag{3.52}$$

Having the exact asymptotic results of the generalised degree distribution $P_{d,\delta}(k|\epsilon)$ valid as long as the chemical potentials $\mu_{d,\delta}$ are well defined, we can perform the average over all δ-faces with energy ϵ, i.e.

$$\langle k_{d,\delta} - 1|\epsilon\rangle = \sum_k (k-1) P_{k,\delta}(k|\epsilon). \tag{3.53}$$

In this result, we obtain total agreement with the mean-field results, i.e.

$$\langle k_{d,d-1} - 1|\epsilon\rangle = m n_F(\epsilon) \qquad \text{for} \quad \delta = d-1, \tag{3.54}$$

$$\langle k_{d,\delta} - 1|\epsilon\rangle = A_\delta n_B(\epsilon) \qquad \text{for} \quad \delta \leq d-2. \tag{3.55}$$

Therefore the generalized degree minus one averaged over faces of energy ϵ is proportional to the Fermi-Dirac distribution for $\delta = d-1$ while it is proportional to the Bose-Einstein distribution for faces of dimension $\delta \leq d-2$. In Figure 3.2 we compare the simulation results with the theoretical predictions showing very good agreement as long as the inverse temperature β is sufficiently low. For higher values of β the system does not reach a stationary state and the description of this phase transition is beyond the scope of this work.

3.5 Spectral properties of the NGF with Fractional Flavor

The spectral dimension [93] determines the properties of a diffusion process defined on the 1-skeleton of the NGF, i.e. the network constructed by starting from the simplicial complex by considering exclusively its nodes and links. Given the Laplacian matrix $\mathbf{L}$ with elements defined as

$$L_{ij} = K_i \delta_{ij} - a_{ij} \tag{3.56}$$

where $\mathbf{a}$ is the adjacency matrix of the network, K_i indicates the degree of the generic node i, and the density of eigenvalues $\rho(\lambda)$ for $\lambda \ll 1$ obeys the power-law scaling

$$\rho(\lambda) \simeq \lambda^{d_S/2-1} \tag{3.57}$$

we say that the network has *spectral dimension* d_S. Note that in this case the cumulative density of eigenvalues $\rho_c(\lambda)$ obeys the scaling

$$\rho_c(\lambda) \simeq \lambda^{d_S/2} \tag{3.58}$$

for $\lambda \ll 1$. The NGFs with integer flavor $s \in \{-1, 0, 1\}$ have been shown to display a finite spectral dimension [70, 94, 95]. Therefore it is interesting to investigate here how the spectral dimension changes for NGF with fractional flavor. By calculating numerically the spectrum of large NGF we found that for $\beta = 0$ the spectral dimension d_S of the NGF with flavor $s = -\frac{1}{m}$ is an increasing function of m. Therefore it achieves its smallest value for $s = -1$ and increases as m increases (see Figure 3.3a). Moreover the spectral properties of the NGF changes also with β. In particular numerical results indicate that the spectral dimension d_S decreases as the inverse temperature β increases (see Figure 3.3b).

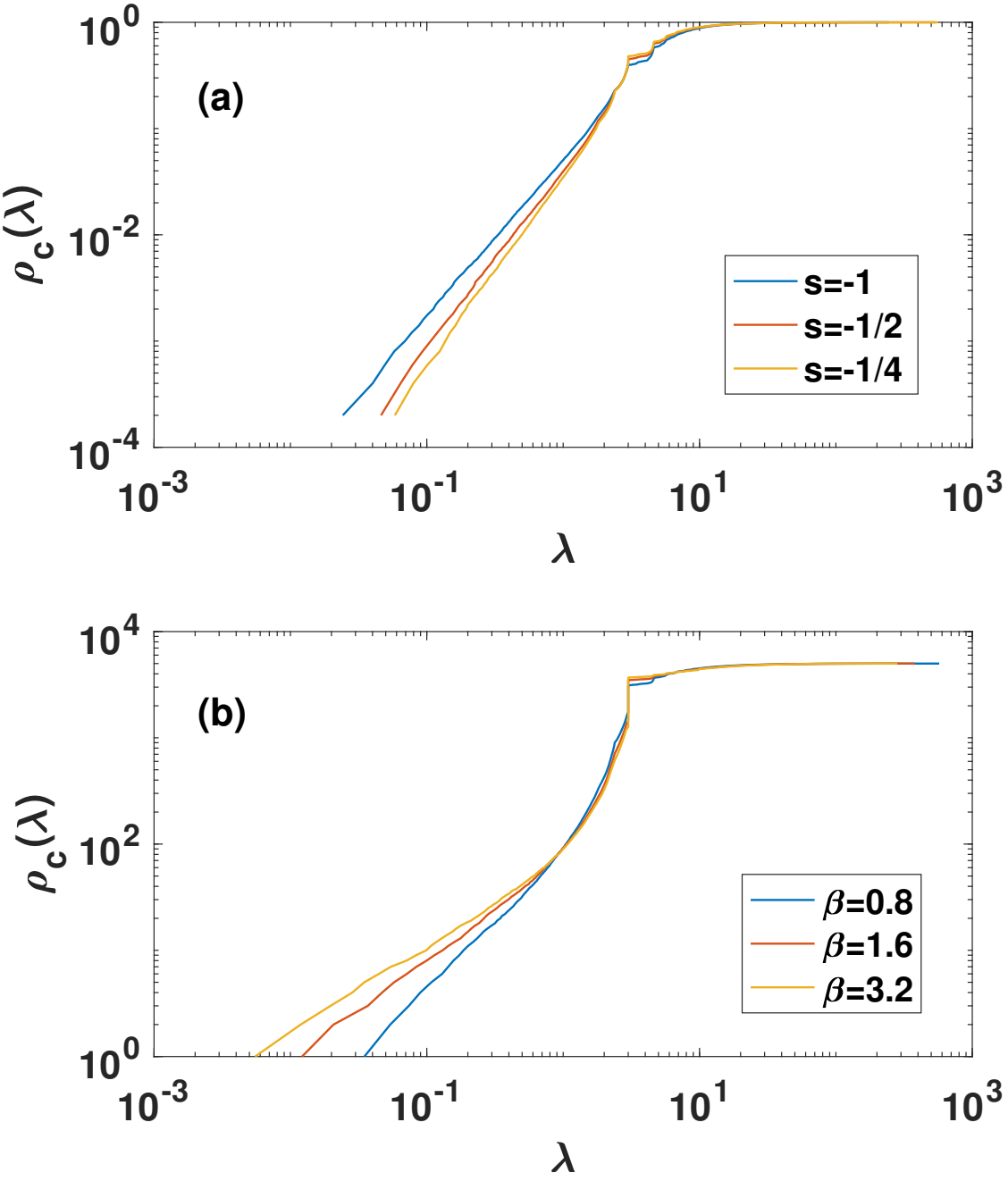


Figure 3.3: Cumulative distribution $\rho_c(\lambda)$ of the eigenvalues of the Laplacian for NGFs of dimension $d = 3$ formed by $N = 5000$ nodes. In panel (a) we show the cumulative distribution $\rho_c(\lambda)$ as a function of s for $\beta = 0$. In panel (b) we show the cumulative distribution $\rho_c(\lambda)$ as a function of β for $s = -1/2$. Every spectrum is averaged over 10 realizations of the NGFs.

Conclusion

As we have seen complex networks are ubiquitous and the tendency (very often in science) is either to generalize models able to reproduce the previously known ones, fitting the data (with an equal or better precision) or to extend the range of applicabilty and study the behavior of the new case. With the works presented here we did exactly that.

In chapter 2 we have presented a network model with a wider spectrum for the fitness of the nodes. We did this by introducing a new probability distribution for the fitness, eq. (2.2): we considered the parameter ρ that allows us to fix several different distributions. In particular, fixing $\rho = \infty$ (and $\alpha_A = 0$) allows to recover the Barabási-Albert model, while fixing $\rho = 0$ (and $\alpha_A = 0$) it is possible to recover the Bianconi-Barabási one. The degree distribution is numerically shown to be $p(k) \propto e_q^{-k/\kappa}$. We have also shown that q and κ depend on ρ and α_a/d. In the first case, as functions of $\rho \in [-\infty, \infty]$, all the values lie in a narrow decreasing (increasing) interval respectively. On the other hand, $\rho = 0$ turned out to be an inflection point for both the parameters, $\forall\, \alpha_A/d$.
It was also shown that q and κ are dependent on the ratio α_A/d more then α_A and d taken alone. Interestingly the value of q for $0 \leq \alpha_A/d \leq 1$ numerically approaches $3/2$, $7/5$ and $4/3$ for ρ approaching $-\infty$, 0 and ∞ respectively. These values of q respectively correspond to the divergences of the moments $\langle k \rangle$, $\langle k^{3/2} \rangle$ and $\langle k^2 \rangle$ of a q-exponential distribution.
We have also found, see fig. 2.5, an unespected universal relation between q and κ since all the data closely lie within the straight line $q = 1.54 - 0.29\,\kappa$, $\forall\, (\alpha_A/d, \rho)$ (eq. 2.7). Moreover the new introduced ρ parameter, as shown in fig. 2.3, does not affect much the topology of the network, as α_A does. This actually is good news for our model to be a generalization for asymptotically-scale-free networks.

In chapter 3 we have extended the model Network Geometry with Flavor to fractional negative values of the flavor $s = -\frac{1}{m}$. This choice of parameters enforces the condition that each $(d-1)$-dimensional face of the pure d-dimensional simplicial complexes generated by the model is incident at most to $m+1$ d-dimensional simplices. For the limiting case $m = 1$ this model generates discrete manifolds where $(d-1)$-dimensional faces have incidence numbers $n_\alpha \in \{0, 1\}$. For $m > 1$ instead the simplicial complexes generated by the model are not anymore manifolds and have incidence number $n_\alpha \in \{0, 1, 2, \ldots, m\}$. In previous studies it has been shown that NGF displays emergent quantum statistics. In particular the generalized degrees of δ faces for energy ϵ in the NGFs with $s = -1$ can be simply related to the Fermi-Dirac, Boltzmann and Bose-Einstein statistics depending on the dimensionality δ. This result implies that for a NGF in $d = 3$ with $s = -1$ the triangles, the links, and the nodes of given

energy ϵ have an average generalized degree minus one given by the Fermi-Dirac, the Boltzmann and the Bose-Einstein statistics respectively. Here we show that when we consider NGF with flavor $s = -\frac{1}{m}$ we still observe different statistics as a function of the dimensionality of the faces but the only two types of statistics emerging are the Fermi-Dirac and Bose-Einstein statistics as long as the NGF evolution reaches a steady state. This implies that for $d = 2$ we observe links and nodes of energy ϵ whose average generalized degree follows the Fermi-Dirac and Bose-Einstein distribution respectively. Therefore already in $d = 2$ we observe the co-existence of the two quantum statistics determining properties of faces of different dimension.

As a main conclusion, whether you consider generilzed complex network models or extended definitions of a network (the simplicial complexes) the geometry is shaped by a couple of parameters; the fitness, togheter with the flavor in the second case, plays a major role.

Chapter 4

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