Università degli studi di Catania

Dipartimento di Matematica e Informatica

Simona Maria Puglisi

Regularity results for some elliptic and parabolic problems

Ph.D. Thesis in Mathematics

Tutor:
prof.ssa G. Rita Cirmi

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Contents

Introduction

The present thesis deals with the regularity of solutions to nonlinear partial differential equations of elliptic and parabolic type, and it collects some results obtained during the last three years under the supervision of Prof. Ugo Gianazza.

The first part concerns the study of singular porous medium type equations: we start proving Hölder regularity for solutions of variable sign, then we continue with an Harnack type inequality for positive solutions.

The second part treats existence and regularity results for a class of non-coercive elliptic equations with discontinuous coefficients, extending a paper by Prof. Lucio Boccardo.

Elliptic equations

The study of the regularity of elliptic partial differential equations essentially began with the pioneering works by De Giorgi [21] and Moser [50, 51] around 1950-1960. They consider the following linear problem

$$u \in H^1_{loc}(\Omega) : \quad \left(a_{ij}(x)\, u_{x_i}\right)_{x_j} = 0 \qquad \text{weakly in } \Omega, \tag{0.1}$$

where Ω is a domain in $\mathbb{R}^N$ and $\{a_{ij}\}$ is a bounded, measurable matrix, satisfying the ellipticity condition

$$a_{ij}(x)\, \xi_i \cdot \xi_j \geq \lambda |\xi|^2.$$

The celebrated De Giorgi's Theorem states that the local solutions of equation (0.1) are Hölder continuous, while the Moser's iteration technique allows to obtain Harnack inequality for non-negative solutions, which in turn implies Hölder regularity.

Later on, Stampacchia [62, 63] considered the non-coercive Dirichlet problem

$$u \in H^1_0(\Omega) : \quad -\left(a_{ij}(x)\, u_{x_i} + d_j(x) u\right)_{x_j} + b_i(x)\, u_{x_i} + cu = f(x) \qquad \text{weakly in } \Omega,$$

where b_i, d_i, c and f satisfy some suitable summability properties, proving existence of weak solutions and showing regularity results for these solutions depending on the summability of the data f.

De Giorgi's and Moser's approaches can be extended to quasi-linear elliptic equations of p-laplacian type, namely

$$\text{div}\mathcal{A}(x,u,Du) = \mathcal{B}(x,u,Du) \qquad\qquad \text{weakly in } \Omega, \tag{0.2}$$

with $\mathcal{A}$ and $\mathcal{B}$ satisfying the structural conditions

$$\begin{cases} \mathcal{A}(x,z,\xi)\cdot\xi \geq C_0|\xi|^p - \varphi(x) \\ |\mathcal{A}(x,z,\xi)| + |\mathcal{B}(x,z,\xi)| \leq C_1|\xi|^{p-1} + \varphi(x), \end{cases} \tag{0.3}$$

being $0 \leq C_0 \leq C_1$, $\varphi \in L^\infty_{loc}(\Omega)$, $\varphi \geq 0$ and $p > 1$.
Notice that, as soon as $p \neq 2$, the principal part $\mathcal{A}(x,u,Du)$ has a nonlinear dependence with respect to Du and a nonlinear growth with respect to $|Du|$.
These extensions have been proved by Ladyženskaja and Ural'tzeva [47] following the method by De Giorgi, and by Serrin [61] and Trudinger [64] following the one by Moser.

Parabolic equations

In 1954 Hadamard [41] and Pini [55] proved Harnack inequality for solutions to the heat equation

$$u_t - \Delta u = 0. \tag{0.4}$$

Ten years later, this result was generalized by Moser [52] to the linear problem

$$\begin{cases} u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \\ u_t - (a_{ij}(x,t)u_{x_i})_{x_j} = 0 \qquad\qquad \text{in } \Omega_T, \end{cases}$$

where, for $T > 0$, we denote by $\Omega_T = \Omega \times (0,T]$ a cylindrical domain.

Concerning nonlinear parabolic equations, the most famous examples are the p-laplacian

$$u_t - \text{div}\left(|Du|^{p-2}Du\right) = 0, \tag{0.5}$$

where $p > 1$, and the porous medium equation

$$u_t - \Delta\left(u^m\right) = 0, \tag{0.6}$$

in which $m > 0$.
Both the p-laplacian and the porous medium equation reduce to the heat equation (0.4)

when $p = 2$ and $m = 1$ respectively.
Equation (0.5) (resp. (0.6)) shows a different behaviour for $p > 2$ and $1 < p < 2$ (resp. for $m > 1$ and $0 < m < 1$). In particular, the first case is referred to as DEGENERATE and the second one as SINGULAR. For the porous medium equation one also speaks about SLOW (if $m > 1$) and FAST DIFFUSION (if $0 < m < 1$).
Sometimes people refer to porous medium equation only for the case $m > 1$.

In many physical settings the restriction $u \geq 0$ naturally appears; it is mathematically convenient and thus often followed. In the general case when u is not imposed to be non-negative, (0.6) becomes the so-called SIGNED porous medium equation, i.e.

$$u_t - \Delta\left(|u|^{m-1}u\right) = 0. \tag{0.7}$$

Parabolic equations of p-laplacian type

Since in the elliptic case it was possible to pass from the linear framework (0.1) to the quasi-linear one of p-laplacian type (0.2)-(0.3), one could expect that the De Giorgi's and Moser's techniques allow to treat nonlinear parabolic equations of the form

$$u_t - \mathrm{div}\mathcal{A}(x,t,u,Du) = \mathcal{B}(x,t,u,Du) \qquad \text{weakly in } \Omega_T, \tag{0.8}$$

under the p-growth assumptions

$$\begin{cases} \mathcal{A}(x,t,z,\xi)\cdot\xi \geq C_0|\xi|^p - \varphi_0(x) \\ |\mathcal{A}(x,t,z,\xi)| \leq C_1|\xi|^{p-1} + \varphi_1(x) \\ |\mathcal{B}(x,t,z,\xi)| \leq C_2|\xi|^p + \varphi_2(x), \end{cases} \tag{0.9}$$

where C_i are positive constants and φ_i are non-negative functions satisfying suitable summability properties.
Unfortunately, De Giorgi's argument cannot be applied to equation (0.8), while Moser's method can only reach the quadratic case $p = 2$, as shown by Aronson and Serrin [3], and by Trudinger [65].

Analogous results were obtained by Nash [54] and by Kružkov [43, 44, 45] by means of different approaches.

Notice that the prototype of the class of parabolic equations (0.8)-(0.9) is the p-laplacian (0.5).

Many developments were made in the 1980's starting from the innovative paper by DiBenedetto [23], where it has been proved the Hölder regularity of local, weak solutions to p-laplacian type equations (0.8)-(0.9), in the degenerate case $p > 2$.

Another crucial step forward consisted in the proof of Hölder continuity also for the singular case $1 < p < 2$ obtained by Chen and DiBenedetto [17] (see also [26]) at the beginning of the 1990's.

Parabolic equations of porous medium type

The porous medium equation (0.6) is one of the simplest nonlinear evolution equations. It appears in the description of different natural phenomena, and its theory and property depart strongly from those of the heat equation (0.4).

There are a number of physical applications in which this simple model appears in a natural way. Some of the best known are the description of the flow of an isentropic gas through a porous medium, modeled independently by Leibenzon [48] and Muskat [53] around 1930, or the heat radiation in plasmas, developed by Zeldovich and Raizer [70] around 1950. Other applications have been proposed in mathematical biology, spread of viscous fluids, boundary layer theory, and other fields.

In 1952, Barenblatt [4] found a similarity solution for the porous medium equation, which resembles the fundamental solution of the heat equation.
In the 1980's Caffarelli and Friedman [16] proved that non-negative solutions to the Cauchy problem in $\mathbb{R}^N \times (0, +\infty)$ associated with the porous medium equation, for $m > 1$ and positive initial data, are Hölder continuous. In the same period, well-posedness in classes of general data was established by Aronson and Caffarelli in [2] and by Bénilan, Crandall and Pierre [6], while the study of solutions with measure data was started by Brezis and Friedman [14], and continued by Dahlberg and Kenig [20].
Few years later, Herrero and Pierre [42] showed existence of solutions to the Cauchy problem associated to signed porous medium equation with $0 < m < 1$ and with locally integrable initial data. While these results are non-local, a more local point of view was adopted in [15, 22].

For a complete survey on the porous medium equation we refer to the book by Vázquez [67].
The porous medium equation admits a more general formulation too. More precisely the quasi-linear parabolic equation (0.8) is said of POROUS MEDIUM TYPE if the following conditions hold

$$\begin{cases} \mathcal{A}(x,t,z,\xi)\cdot\xi \geq C_0|z|^{m-1}|\xi|^2 - \varphi_0(x) \\ |\mathcal{A}(x,t,z,\xi)| \leq C_1|z|^{m-1}|\xi| + \varphi_1(x) \\ |\mathcal{B}(x,t,z,\xi)| \leq C_2\big(\,|z|^{m-1}|\xi|\,\big)^2 + \varphi_2(x). \end{cases} \tag{0.10}$$

On the relation between the two model equations

Let us notice a further link between p-laplacian and porous medium type equations.
Let u be a local solution to (0.5), for $p > 2$. Then the function $v = |Du|^2$ formally satisfies (see [66, 36])

$$v_t - \left(a_{\ell\, k}\, v^{\frac{p}{2}-1} v_{x_k}\right)_{x_\ell} \leq 0,$$

where

$$a_{\ell\, k} = \delta_{\ell\, k} + (p-2)\frac{u_{x_\ell} u_{x_k}}{|Du|^2},$$

and $\delta_{\ell\, k}$ is the Kronecker's delta.
This is a quasi-linear version of the porous medium equation (0.6) with $m = \dfrac{p}{2}$. Therefore, the study of the local behaviour of solutions to porous medium type equations is useful to understand also the p-laplacian.

Intrinsic rescaling and Harnack inequality

Let $(x_0, t_0) \in \Omega_T$, $\rho > 0$, and $B_\rho(x_0)$ be the ball of radius ρ centered at x_0.
The cylinders associated to the heat equation (0.4) and to the porous medium equation are of the type

$$B_\rho(x_0) \times (t_0 - \rho^2, t_0 + \rho^2),$$

since the structure of these equations is invariant under those space-time variable transformations such that $\dfrac{|x|^2}{t}$ remains constant.
Instead, in the case of the p-laplacian, the space-time variable transformations for which the structure remains unchanged are those that keep constant the ratio $\dfrac{|x|^p}{t}$; therefore the cylinders associated to this class of parabolic equations are

$$B_\rho(x_0) \times (t_0 - \rho^p, t_0 + \rho^p).$$

The evolution of the diffusion process scales differently in space and time, and the different scaling depends on the pointwise value of u itself. This observation lead DiBenedetto and Friedman ([27, 28]) to introduce the INTRINSIC RESCALING, that is, a rescaling determined by the function itself.
To explain this concept, let us think about the prototype equations (0.4), (0.5), (0.6). Given a non-negative solution u, for any constant $\alpha > 0$ let us consider $v = \alpha u$.
In the case of the heat equation we have

$$u_t - \Delta u = 0 \qquad \Longleftrightarrow \qquad v_t - \Delta v = 0,$$

while for the p-laplacian or the porous medium equation, we find respectively

$$
\begin{aligned}
v_t - \operatorname{div}\left(|Dv|^{p-2}Dv\right) = 0 \qquad &\Longleftrightarrow \qquad u_t - \alpha^{p-2}\operatorname{div}\left(|Du|^{p-2}Du\right) = 0,\\
v_t - \Delta\left(v^m\right) = 0 \qquad &\Longleftrightarrow \qquad u_t - \alpha^{m-1}\Delta\left(u^m\right) = 0.
\end{aligned}
$$

As a consequence, while for the heat equation v is again a solution for any choice of the constant α, in the other cases this is no more true, except trivially for $\alpha = 1$. The homogeneity is restored once we further stretch the time variable by a factor and we work with suitable intrinsic cylinders (see (0.12) and (0.14) below).
This has been a main tool to prove local Hölder regularity (see [23, 17, 18]) and Harnack inequalities (see [24, 25, 35, 19] and [30, 31, 32, 33]) for solutions to (0.8) with conditions (0.9) or (0.10) and some assumptions on p or m.
Let us describe more in detail the results concerning Harnack type inequalities starting from the p-laplacian case, letting u be a non-negative, local weak solutions to (0.8)-(0.9). If $p > 2$, then u satisfies the parabolic Harnack inequality in some intrinsic form. Precisely, there exist positive constants γ and c depending only upon data such that

$$
\gamma^{-1} \sup_{x \in B_\rho(x_0)} u(x, t_0 - \theta\rho^p) \le u(x_0, t_0) \le \gamma \inf_{x \in B_\rho(x_0)} u(x, t_0 + \theta\rho^p), \tag{0.11}
$$

where

$$
\theta = \left(\frac{c}{u(x_0, t_0)}\right)^{p-2},
$$

whenever the intrinsic parabolic cylinder

$$
B_{2\rho}(x_0) \times \left(t_0 - \theta(2\rho)^p, t_0 + \theta(2\rho)^p\right) \tag{0.12}
$$

is contained within the domain of definition of the solution.
Consider now the singular case $1 < p < 2$: here there is a critical value of p, namely

$$
p_* = \frac{2N}{N+1},
$$

to be taken into account. If p is supercritical, i.e. $p_* < p < 2$, then (see [32]) u satisfies forward, backward and elliptic Harnack inequalities. More precisely, there exist positive constants $\tilde{\gamma}$ and δ depending only upon the data, such that

$$
\tilde{\gamma}^{-1} \sup_{x \in B_\rho(x_0)} u(x, \tau) \le u(x_0, t_0) \le \tilde{\gamma} \inf_{x \in B_\rho(x_0)} u(x, \tau), \tag{0.13}
$$

for all

$$
t_0 - \delta\left[u(x_0, t_0)\right]^{2-p} \rho^p \le \tau \le t_0 + \delta\left[u(x_0, t_0)\right]^{2-p} \rho^p,
$$

for each cylinder (0.12) contained in the domain of u.

The first Harnack type inequality in the subcritical range $1 < p \leq p_*$ was given by Bonforte, Iagar and Vázquez (see [11]) who proved forward, backward and elliptic Harnack inequalities only for the prototype equation (0.5), by letting the constant $\tilde{\gamma}$ depend on the solution itself through the ratio of suitable integral norms of u and slightly changing the intrinsic geometry.
Finally, a recent paper by Fornaro and Vespri [37] extends the above result [11] to the quasi-linear p-laplacian case, using a comparison principle.

We conclude this section by briefly addressing to the case of the porous medium type equations. Some of the techniques just described for the p-laplacian type equations (for instance, those in [23, 17, 30, 33]) can be adapted; however, the porous medium case shows some peculiarities which need to be treated in different ways.
In particular, in the degenerate case $m > 1$ one can show the same inequality of Harnack type (0.11) corresponding to $p > 2$. In the singular case $m < 1$, there exists again a critical exponent

$$m_* = \frac{N-2}{N},$$

and the inequality (0.13) holds in the subcritical range $m_* < m < 1$. In these results, the constant θ becomes

$$\theta = \left(\frac{c}{u(x_0, t_0)} \right)^{m-1}$$

and, as one can expect, the intrinsic cylinders take the form

$$B_\rho(x_0) \times \big(t_0 - \theta\rho^2, t_0 + \theta\rho^2\big). \tag{0.14}$$

Results of the thesis

The results of this Ph.D. thesis follow basically three directions. In the first part, which regards the study of quasi-linear porous medium type equations, we show local Hölder continuity for signed solutions, and Harnack inequality for positive solutions; in the second part, we obtain existence and regularity results for a certain class of non-coercive elliptic equations. We now describe these results more in detail.

• HÖLDER CONTINUITY. As we already mentioned above, in some physical applications it is natural to consider positive solutions to quasi-linear parabolic equations of the form (0.8), and it is also a very useful simplification from the mathematical point of view. Therefore, most of the papers directly deal with positive solutions.

A first Hölder regularity result for signed solutions were obtained in 1988 by Chen and DiBenedetto [17], who treated the case of singular p-laplacian type equations. Later on, in 1993 Porzio and Vespri [57] considered the case of a degenerate doubly non-linear equation, whose prototype is

$$u_t - \operatorname{div}\big(|u|^{m-1}|Du|^{p-2}Du\big) = 0,$$

for $p \geq 2$ and $m \geq 1$. Notice that this kind of equations admits as a particular case both the degenerate p-laplacian type equations (for $m = 1$ and $p > 2$) and the degenerate porous medium type equations (for $p = 2$ and $m > 1$). As a consequence, it only remained open the case of the singular porous medium type equations. This is precisely our first result, Theorem 2.1. Our proof essentially follows the lines of [17]; an important point of our strategy is to work with a different equation, apparently more complicate but instead easier to handle with, to which we can reduce thanks to a change of variable introduced by Vespri in [68]. This result is contained in the forthcoming paper [59].

• Harnack inequality. Let us now pass to the results about the Harnack inequality. We briefly recall the different steps of the history, that we have already outlined above.

In the case of p-laplacian type equations, a parabolic Harnack type inequality has been proved for the degenerate case $p > 2$ by DiBenedetto in [24] (see also [26]). For the singular case $p < 2$, a backward, forward and elliptic type Harnack inequality has been proved for the supercritical case $p_* < p < 2$ by DiBenedetto, Gianazza and Vespri in [31], and for the subcritical case $1 < p \leq p_*$ first by Bonforte, Iagar and Vazquez in [11] only for the prototype equation, and then by Fornaro and Vespri in [37] for the general case. In particular, in [37] the authors make use of a comparison principle and of a higher integrability result given in [38].

In the case of porous medium type equations, instead, the results for the degenerate case $m > 1$ and for the singular supercritical case $m_* < m < 1$ have been obtained together with their p-laplacian counterpart, while for the subcritical case $0 < m \leq m_*$ the only available result is for the prototype equation, and it has been obtained by Bonforte and Vazquez in [12].

Our second result, Theorem 3.1 provides a Harnack type inequality for the general porous medium type equation in the whole range $0 < m < 1$, thus in particular completing the analysis for the subcritical case. Our construction resembles that of [37], again using a comparison principle (see Section 3.2) and a higher integrability result obtained by Fugazzola in [39]. Our result is the object of the paper [60].

• Results for the elliptic problem. The work of the second part of this thesis (whose results are contained in the recent paper [58]) started after a fruitful discussion with Lucio Boccardo. He has recently dealt in [7] with the elliptic non-coercive problem

$$\begin{cases} -\mathrm{div}\big(M(x)\,Du\big) = -\mathrm{div}\big(u\,E(x)\big) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.15}$$

where M is a bounded, elliptic, measurable matrix, and E and f are measurable functions satisfying suitable summability properties. He has established existence and regularity results, which depend on the summability of $|E|$ and f.
We are able to prove the analogous of the results of [7] for the equation

$$\begin{cases} -\mathrm{div}\,(M(x)\,Du) = -\mathrm{div}\,\big(|u|^{\theta-1}\,u\,E(x)\big) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < \theta < 1$. Notice that our problem is a sort of interpolation between the linear case, corresponding to $\theta = 0$, and (0.15), corresponding to $\theta = 1$.
In fact, also all our results need assumptions on $|E|$ and on f which are weaker than those in [7], and reduce exactly to those of [7] when $\theta \to 1$.

Chapter 1

Notations and functional spaces

Throughout this chapter we consider Ω to be a bounded domain in $\mathbb{R}^N$.

For $\rho > 0$ and $y \in \mathbb{R}^N$, denote by $B_\rho(y)$ the ball of radius ρ centered at y and by $K_\rho(y)$ the cube centered at y with edge 2ρ, i.e.

$$K_\rho(y) = \left\{ x \in \mathbb{R}^N : \max_{1\le i\le N} |x_i - y_i| < \rho \right\}.$$

If y coincides with the origin, we let $B_\rho(0) = B_\rho$ and $K_\rho(0) = K_\rho$.
As usual, given $A \subseteq \Omega$, we will denote by $|A|$ the Lebesgue measure of the set A, by ∂A its boundary, and by $\bar{A}$ its closure.

If $f \in L^q(\Omega)$, for some $1 \le q \le \infty$, we denote by $\|f\|_{q,\Omega}$ the L^q norm of f over Ω; we also write $\|f\|_q$ whenever the specification of the domain Ω is unambiguous from the context.
Let us recall some basic facts we will need in the sequel, for the proofs related to this part we will refer to [13] and [1].

Given $1 \le p \le \infty$ and $f \in L^p(\Omega)$, we say that $f \in W^{1,p}(\Omega)$ if there exist $g_i \in L^p(\Omega)$, for $i = 1, \dots, N$ such that

$$\int_\Omega f \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega g_i \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

For any $f \in W^{1,p}(\Omega)$, we let $\dfrac{\partial f}{\partial x_i} = g_i$ and $Df = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \dots, \dfrac{\partial f}{\partial x_N} \right)$.
Let us remind that $W^{1,p}(\Omega)$ is a Banach space with norm

$$\|f\|_{W^{1,p}} = \|f\|_{L^p} + \sum_{i=1}^N \left\| \frac{\partial f}{\partial x_i} \right\|_{L^p} \qquad \forall f \in W^{1,p}(\Omega).$$

Recall also that the space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$.

Theorem 1.1. (Sobolev)
For $1 \leq p < N$, $W^{1,p}(\Omega) \hookrightarrow L^{p*}(\Omega)$, *with* p^* *given by* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, *where* $\hookrightarrow$ *denotes a continuous injection.*
In particular, there exists a positive constant S *depending only upon* p, N *and* Ω, *such that*

$$\|f\|_{L^{p^*}} \leq S\|Df\|_{L^p} \qquad\qquad \forall f \in W_0^{1,p}(\Omega).$$

Theorem 1.2. (Poincaré)
If $1 \leq p < \infty$, *then there exists a constant* P, *depending upon* p *and* Ω, *such that*

$$\|f\|_{L^p} \leq P\|Df\|_{L^p} \qquad\qquad \forall f \in W_0^{1,p}(\Omega).$$

In particular, this implies that $\|Df\|_{L^p}$ is a norm in $W_0^{1,p}(\Omega)$, which is equivalent to $\|f\|_{W^{1,p}}$.

If $k \in \mathbb{R}$ and $v \in W^{1,p}(E)$, introduce the truncated functions

$$(v-k)_\pm = \max\{\pm(v-k), 0\}.$$

We have the following result, due to Stampacchia [63].

Lemma 1.1. *Let* $v \in W^{1,p}(\Omega)$. *Then* $(v-k)_\pm \in W^{1,p}(\Omega)$, *for all* $k \in \mathbb{R}$; *if we assume also that the trace of* v *on* $\partial\Omega$ *is essentially bounded and*

$$\|v\|_{\infty,\partial\Omega} \leq M \qquad\qquad \textit{for some } M > 0,$$

then, for all $k \geq M$, $(v-k)_\pm \in W_0^{1,p}(\Omega)$.

One can find a simple proof of the previous lemma in [40].

1.1 Some technical lemmas

Let us recall the general Young inequality.

Lemma 1.2. *For* $p, q > 1$ *conjugate exponents* $\left(\textit{i.e. } \frac{1}{p} + \frac{1}{q} = 1\right)$ *and* $\varepsilon > 0$, *one has*

$$ab \leq \frac{\varepsilon^p}{p} a^p + \frac{1}{\varepsilon^q q} b^q \qquad\qquad \forall a, b > 0.$$

The following lemma, proved in [21] by De Giorgi, will be very useful in the sequel.

Lemma 1.3. *Let $v \in W^{1,1}(K_\rho(y))$ and let $k, l \in \mathbb{R}$, with $k < l$. There exists a positive constant c, depending only upon N and p, such that*

$$(l-k)|\{v > l\}| \leq c \, \frac{\rho^{N+1}}{|\{v < k\}|} \int_{\{k<v<l\}} |Dv| \, dx. \tag{1.1}$$

The previous inequality (1.1) is a particular case of a more general Poincaré type inequality (see [26]).

Let us state now a lemma on fast geometric convergence one can find in [21]; for a simpler proof see again [26] and also [46].

Lemma 1.4. *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers satisfying*

$$Y_{n+1} \leq C b^n Y_n^{1+\alpha},$$

being $C, b > 1$ and $\alpha > 0$. If

$$Y_0 \leq C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}},$$

then Y_n converges to 0, as n tends to $+\infty$.

Here we state a measure-theoretical lemma, recently obtained in [29], see also [34].

Lemma 1.5. *Let $v \in W^{1,1}(K_\rho)$ satisfy*

$$\|v\|_{W^{1,1}(K_\rho)} \leq \alpha \, \rho^{N-1}, \qquad |\{v > 1\}| \geq \beta |K_\rho| \tag{1.2}$$

for some $\alpha > 0$ and $\beta \in (0,1)$. Then for every $\delta, \lambda \in (0,1)$, there exist $x_0 \in K_\rho$ and $\varepsilon \in (0,1)$ depending upon $\alpha, \beta, \delta, \lambda, N$ such that

$$\big|\{v > \lambda\} \cap K_{\varepsilon\rho}(x_0)\big| > (1-\delta) \, \big|K_{\varepsilon\rho}(x_0)\big|.$$

Roughly speaking, the previous lemma asserts that if the set where v is bounded away from zero occupies a sizeable portion of K_ρ, then there exists at least one point x_0 and a neighborhood $K_{\varepsilon\rho}(x_0)$ such that v remains large in a large portion of $K_{\varepsilon\rho}(x_0)$. Thus the set where v is positive clusters about at least one point of K_ρ.

1.2 Parabolic spaces

For $T > 0$ denote the cylindrical domain

$$\Omega_T = \Omega \times (0, T],$$

and let $\Gamma = \partial\Omega_T \setminus \bar{\Omega} \times \{T\}$ be the parabolic boundary of Ω_T.

Let us introduce the space

$$L^{q,r}(\Omega_T) := L^r\left(0, T; L^q(\Omega)\right)$$

with $q, r \geq 1$, that is the space of the functions f defined and measurable in Ω_T such that

$$\|f\|_{q,r;\Omega_T} := \left(\int_0^T \left(\int_\Omega |f|^q dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}$$

is finite.
Also $f \in L^{q,r}_{loc}(\Omega_T)$ if, for every compact set $\mathcal{K} \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$, the following integral

$$\int_{t_1}^{t_2} \left(\int_{\mathcal{K}} |f|^q dx\right)^{\frac{r}{q}} dt$$

is finite.
If $q = r$, $L^{q,r}(\Omega_T) = L^q(\Omega_T)$ and $L^{q,r}_{loc}(\Omega_T) = L^q_{loc}(\Omega_T)$. These definitions are extended in the obvious way when either q or r are infinity.
Here we introduce spaces of functions in which typically solutions to parabolic equations in divergence form are found (see [46]).
Let $m, p \geq 1$ and consider the Banach spaces

$$V^{m,p}(\Omega_T) := L^\infty\big(0, T; L^m(\Omega)\big) \cap L^p\big(0, T; W^{1,p}(\Omega)\big)$$
$$V_0^{m,p}(\Omega_T) := L^\infty\big(0, T; L^m(\Omega)\big) \cap L^p\big(0, T; W_0^{1,p}(\Omega)\big)$$

both equipped with the norm

$$\|v\|_{V^{m,p}(\Omega_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{m,\Omega} + \|Dv\|_{p,\Omega_T};$$

when $m = p$, set $V^{p,p}(\Omega_T) = V^p(\Omega_T)$ and $V_0^{p,p}(\Omega_T) = V_0^p(\Omega_T)$.

These spaces are embedded in $L^q(\Omega_T)$, for some $q > p$, in the following way.

Proposition 1.1. *If $v \in V_0^{m,p}(\Omega_T)$, then there exists a positive constant $c = c(N, p, m)$ such that*

$$\iint_{\Omega_T} |v|^q dxdt \leq c^q \left(\iint_{\Omega_T} |Dv|^p dxdt\right) \left(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v|^m dx\right)^{\frac{p}{N}} \tag{1.3}$$

with $q = p\,\dfrac{N+m}{N}$. In particular

$$\|v\|_{q,\Omega_T} \leq c\|v\|_{V^{m,p}(\Omega_T)}.$$

Proposition 1.2. *If $\partial\Omega$ is piecewise smooth, and $v \in V^{m,p}(\Omega_T)$, then there exists a positive constant $c = c(N, p, m, \partial\Omega)$ such that*

$$\|v\|_{q,\Omega_T} \leq c\left(1 + \frac{T}{|\Omega|^{\frac{N(p-m)+mp}{Nm}}}\right)^{\frac{1}{q}} \|v\|_{V^{m,p}(\Omega_T)},$$

with q as in Proposition 1.1.

For a proof of the previous propositions one can see [26].
Notice that, taking $m = p$ in Proposition 1.1 and 1.2, and applying Hölder inequality, one obtains

Corollary 1.1. *Let $p > 1$. If $v \in V_0^p(\Omega_T)$, then there exists a positive constant c depending only upon N and p such that*

$$\|v\|_{p,\Omega_T}^p \leq c\, |\{|v| > 0\}|^{\frac{p}{N+p}}\; \|v\|_{V^p(\Omega_T)}^p.$$

Corollary 1.2. *Let $p > 1$. If $v \in V^p(\Omega_T)$, then there exists a positive constant c depending only upon N, p and $\partial\Omega$ such that*

$$\|v\|_{p,\Omega_T}^p \leq c\left(1 + \frac{T}{|\Omega|^{\frac{p}{N}}}\right)^{\frac{N}{N+p}} |\{|v| > 0\}|^{\frac{p}{N+p}}\; \|v\|_{V^p(\Omega_T)}^p.$$

We can give now a sort of parabolic version of Lemma 1.1.

Lemma 1.6. *Let $v \in V^{m,p}(\Omega_T)$. Then $(v-k)_\pm \in V^{m,p}(\Omega_T)$ for all $k \in \mathbb{R}$. If in addition $\partial\Omega$ is piecewise smooth, the trace of $v(\cdot,t)$ on $\partial\Omega$ is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{\infty,\partial\Omega} \leq M \qquad\qquad \text{for some } M > 0,$$

then $(v-k)_\pm \in V_0^{m,p}(\Omega_T)$ for all $k \geq M$.

Part I

Singular porous medium type equation

Let Ω be an open set in $\mathbb{R}^N$, for $T > 0$ denote the cylindrical domain

$$\Omega_T = \Omega \times (0, T],$$

and let $\Gamma = \partial\Omega_T \setminus \bar{\Omega} \times \{T\}$ be the parabolic boundary of Ω_T.

We consider quasi-linear homogeneous parabolic partial differential equations of the form

$$u_t - \text{div}\mathcal{A}(x, t, u, Du) = 0 \quad \text{weakly in } \Omega_T, \tag{1.4}$$

where $\mathcal{A} : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ is measurable and subject to the structure conditions

$$\begin{cases} \mathcal{A}(x, t, z, \xi) \cdot \xi \geq C_0 m|z|^{m-1}|\xi|^2 \\ |\mathcal{A}(x, t, z, \xi)| \leq C_1 m|z|^{m-1}|\xi| \end{cases} \tag{1.5}$$

for a.e. $(x, t) \in \Omega_T$, for every $z \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, where C_0, C_1 are given positive constants and $m > 0$.

The prototype of this class of parabolic equations is the porous medium equation

$$u_t - \text{div}\left(m|u|^{m-1}Du\right) = 0 \qquad \text{weakly in } \Omega_T. \tag{1.6}$$

The modulus of ellipticity of this class of parabolic equations is $|u|^{m-1}$. Whenever $m > 1$, such a modulus vanishes when u vanishes and for this reason we say that the equation (1.4) is DEGENERATE. Whenever $0 < m < 1$, such a modulus goes to infinity when $u \to 0$ and for this reason we say that the equation (1.4) is SINGULAR.

We are interested only in *local* solutions to *singular* porous medium type equation. Let us give the notion of weak solution for this kind of equations as follows.

A function $u \in C_{loc}(0, T; L^2_{loc}(\Omega))$ with $|u|^m \in L^2_{loc}(0, T; H^1_{loc}(\Omega))$ is a local weak sub (super)-solution to (1.4) if for every compact set $\mathcal{K} \subset \Omega$ and every subinterval $[t_1, t_2] \subset (0, T]$

$$\int_{\mathcal{K}} u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} [-u\varphi_t + \mathcal{A}(x, t, u, Du) \cdot D\varphi] dxdt \leq (\geq)\, 0, \tag{1.7}$$

for all non-negative test functions $\varphi \in W^{1,2}_{loc}(0, T; L^2(\mathcal{K})) \cap L^2_{loc}(0, T; W_0^{1,2}(\mathcal{K}))$.

The parameters $\{N, m, C_0, C_1\}$ are the data of our problem. In this first part, the letter c will be used to denote a constant depending upon the data, which can be quantitavely determined a priori only in terms of the indicated parameters. As usual, the constant c may change from line to line.

Chapter 2

Hölder continuity of solutions of variable sign to singular porous medium type equation

The aim of this chapter is to show that local solutions of variable sign to our problem (1.4)-(1.5), with $0 < m < 1$, are locally Hölder continuous.

Let us introduce the parabolic m-distance from a compact set $\mathcal{K} \subset \Omega_T$ to the parabolic boundary Γ in the following way

$$m\text{-}\operatorname{dist}(\mathcal{K}, \Gamma) = \inf_{(x,t)\in\mathcal{K},\,(y,s)\in\Gamma} \left(\|u\|_{\infty,\Omega_T}^{\frac{1-m}{2}} |x-y| + |t-s|^{\frac{1}{2}} \right).$$

Now, we can state the main result of this chapter.

Theorem 2.1. *Let u be a locally bounded, local, weak solution to* $(1.4)-(1.5)$*. Then u is locally Hölder continuous in Ω_T and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ such that for every compact set $\mathcal{K} \subset \Omega_T$*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \, \|u\|_{\infty,\Omega_T} \left(\frac{\|u\|_{\infty,\Omega_T}^{\frac{1-m}{2}} |x_1 - x_2| + |\, t_1 - t_2|^{\frac{1}{2}}}{m\text{-}\operatorname{dist}(\mathcal{K},\Gamma)} \right)^{\alpha},$$

for every pair of points $(x_1,t_1), (x_2,t_2) \in \mathcal{K}$.

The constant γ depends only upon the data, the norm $\|u\|_{\infty,\mathcal{K}}$ and m-$\operatorname{dist}(\mathcal{K},\Gamma)$, while the constant α depends only upon the data and the norm $\|u\|_{\infty,\mathcal{K}}$.

We will prove this result by applying a technique due to DiBenedetto [26] via an alternative argument. We introduce a suitable change of variable, so that (1.4) is rewritten

as (2.2) below. The Hölder continuity of a solution u to (2.2) will be heuristically a consequence of the following fact: for every $(x_0, t_0) \in \Omega_T$ there exists a family of nested and shrinking cylinders in which the essential oscillation of u goes to zero in a way that can be quantitavely determined in terms of the data.
Since this result is well known for non-negative solutions (see [34]), it will suffice to consider the case in which the infimum of our solution is negative and the supremum is positive.

2.1 Change of variables

In order to justify some of the following calculations, we assume u to be smooth. In no way this is a restrictive assumption: indeed the modulus of continuity of u will play no role in the forthcoming calculations.

Let us consider $n \in \mathbb{N}$ such that

$$n > \frac{1}{m},$$

and define

$$|v|^{n-1}v = u,$$

which is equivalent to

$$v = |u|^{\frac{1}{n}-1}u.$$

Observe that

$$Du = n|v|^{n-1}Dv, \qquad Dv = \frac{1}{n}|u|^{\frac{1}{n}-1}Du.$$

With this substitution equation (1.4) becomes

$$\left(|v|^{n-1}v\right)_t - \mathrm{div}\tilde{\mathcal{A}}(x,t,v,Dv) = 0 \qquad \text{weakly in } \Omega_T,$$

where

$$\tilde{\mathcal{A}}(x,t,v,Dv) = \mathcal{A}(x,t,u,Du)\big|_{u=|v|^{n-1}v}.$$

Now, let us see what the structure conditions become

$$\begin{aligned}\tilde{\mathcal{A}}(x,t,v,Dv)\cdot Dv = \frac{1}{n}|u|^{\frac{1}{n}-1}\mathcal{A}(x,t,u,Du)\cdot Du &\geq \frac{m}{n}\,C_0|u|^{\frac{1}{n}+m-2}|Du|^2\\ &= nmC_0|v|^{1+nm-2n}|v|^{2(n-1)}|Dv|^2 = nmC_0|v|^{nm-1}|Dv|^2;\end{aligned}$$

since the exponent is $nm-1>0$, the equation is now "degenerate".
In the same way

$$\begin{aligned}|\tilde{\mathcal{A}}(x,t,v,Dv)| = |\mathcal{A}(x,t,u,Du)| &\leq mC_1|u|^{m-1}|Du|\\ &= mC_1|v|^{n(m-1)}n|v|^{n-1}|Dv| = nmC_1|v|^{nm-1}|Dv|.\end{aligned}$$

If we denote our variable with u again, then we consider equations of the type

$$\left(|u|^{n-1}u\right)_t - \mathrm{div}\tilde{\mathcal{A}}(x,t,u,Du) = 0,$$

with structure conditions

$$\begin{cases} \tilde{\mathcal{A}}(x,t,z,\xi)\cdot\xi \geq nmC_0|z|^{nm-1}|\xi|^2 \\ |\tilde{\mathcal{A}}(x,t,z,\xi)| \leq nmC_1|z|^{nm-1}|\xi|, \end{cases} \tag{2.1}$$

for a.e. $(x,t)\in\Omega_T$ and for every $z\in\mathbb{R}$, $\xi\in\mathbb{R}^N$. Without loss of generality, we can assume n to be odd; in this case

$$|u|^{n-1}u = u^n,$$

and we can rewrite the equation in the following way

$$\left(u^n\right)_t - \mathrm{div}\tilde{\mathcal{A}}(x,t,u,Du) = 0. \tag{2.2}$$

Hence we have reduced problem (1.4)-(1.5) to (2.2) with structure conditions (2.1).
Let us see now how the notion of weak solution becomes in this new setting.
A function u such that $u^n\in C_{loc}(0,T;L^2_{loc}(\Omega))$ with $|u|^{nm}\in L^2_{loc}(0,T;W^{1,2}_{loc}(\Omega))$ is a local weak sub(super)-solution to (2.2) if for every compact set $\mathcal{K}\subset\Omega$ and every subinterval $[t_1,t_2]\subset(0,T]$

$$\int_{\mathcal{K}} u^n\varphi\,dx\,\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{\mathcal{K}}[-u^n\varphi_t + \tilde{\mathcal{A}}(x,t,u,Du)\cdot D\varphi]dxdt \leq (\geq)\,0$$

for all non-negative test functions $\varphi\in W^{1,2}_{loc}(0,T;L^2(\mathcal{K}))\cap L^2_{loc}(0,T;W^{1,2}_0(\mathcal{K}))$.

2.2 Preliminary estimates

Given $(y,s)\in\Omega_T$ and $\lambda, R>0$, we denote the generic cylinder as

$$(y,s)+Q_R(\lambda) := K_R(y)\times(s-\lambda,s].$$

Let us prove energy estimates we will need later. We start with estimates for super-solutions, then we will state the analogous ones for sub-solutions.

Proposition 2.1. (Energy estimates for super-solutions)
Let u be a local, weak super-solution to (2.2) in Ω_T. There exists a positive constant c,

depending only upon data, such that for every cylinder $(y,s)+Q_R(\lambda)\subset\Omega_T$, every level $k\in\mathbb{R}$ and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(y)$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\lambda<t\le s}\int_{K_R(y)}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta^2(x,t)\,dx\\
&\quad+\iint_{(y,s)+Q_R(\lambda)}|u|^{nm-1}|D[(u-k)_-\zeta]|^2 dxd\tau\\
&\le c\left\{\int_{K_R(y)}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta^2(x,s-\lambda)\,dx\right.\\
&\quad+\iint_{(y,s)+Q_R(\lambda)}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta\,|\zeta_\tau|\,dxd\tau\\
&\quad\left.+\iint_{(y,s)+Q_R(\lambda)}|u|^{nm-1}(u-k)_-^2|D\zeta|^2 dxd\tau\right\}.
\end{aligned}\tag{2.3}$$

Proof. After a translation we may assume that (y,s) coincides with the origin and it suffices to prove (2.3) for the cylinder $Q_R(\lambda)$. In the weak formulation of (2.2), take the test function

$$\varphi=-(u-k)_-\zeta^2$$

over $Q_t=K_R\times(-\lambda\,,t]$, where $-\lambda<t\le 0$.
Taking into account that

$$\frac{\partial}{\partial\tau}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)=-u^{n-1}(u-k)_-u_\tau,$$

and estimating the various terms separately, we have first

$$\begin{aligned}
-\iint_{Q_t}(u^n)_\tau\,(u-k)_-\zeta^2\,dxd\tau &= n\iint_{Q_t}\frac{\partial}{\partial\tau}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta^2\,dxd\tau\\
&\ge n\int_{K_R}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta^2(x,t)\,dx\\
&\quad- n\int_{K_R}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta^2(x,-\lambda)\,dx\\
&\quad- 2n\iint_{Q_t}\left(\int_u^k (k-s)_+ s^{n-1}ds\right)\zeta\,|\zeta_\tau|\,dxd\tau
\end{aligned}$$

From the structure conditions (2.1) and Young's inequality it follows that

$$
\begin{aligned}
&-\iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\, D\big[(u-k)_-\zeta^2\big]\, dxd\tau \\
&= -\iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\, D(u-k)_-\zeta^2\, dxd\tau - 2\iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\,(u-k)_-\,\zeta\, D\zeta\, dxd\tau \\
&\geq nmC_0 \iint_{Q_t} |u|^{nm-1}\big|D(u-k)_-\big|^2\,\zeta^2\, dxd\tau \\
&\quad - 2nmC_1 \iint_{Q_t} |u|^{nm-1}\big|D(u-k)_-\big|\,(u-k)_-\,\zeta\,|D\zeta|\, dxd\tau \\
&\geq nm\frac{C_0}{2} \iint_{Q_t} |u|^{nm-1}\big|D\big[(u-k)_-\zeta\big]\big|^2\, dxd\tau - 2nm\frac{C_1^2}{C_0}\iint_{Q_t} |u|^{nm-1}\,(u-k)_-^2\,|D\zeta|^2\, dxd\tau.
\end{aligned}
$$

Combining these estimates and taking the supremum over $t \in (-\lambda\,,0\,]$, proves the proposition. □

Proposition 2.2. (Energy estimates for sub-solutions)
Let u be a local, weak sub-solution to (2.2) in Ω_T. There exists a positive constant c, depending only upon data, such that for every cylinder $(y,s)+Q_R(\lambda)\subset\Omega_T$, every level $k\in\mathbb{R}$ and every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial K_R(y)$,

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{s-\lambda<t\leq s} \int_{K_R(y)} \left(\int_k^u (s-k)_+ s^{n-1} ds\right)\zeta^2(x,t)\,dx \\
&\quad + \iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1}|D[(u-k)_+\zeta]|^2 dxd\tau \\
&\leq c\Bigg\{ \int_{K_R(y)} \left(\int_k^u (s-k)_+ s^{n-1} ds\right)\zeta^2(x,s-\lambda)\,dx \\
&\quad + \iint_{(y,s)+Q_R(\lambda)} \left(\int_k^u (s-k)_+ s^{n-1} ds\right)|\zeta_\tau|\,dxd\tau \\
&\quad + \iint_{(y,s)+Q_R(\lambda)} |u|^{nm-1}(u-k)_+^2|D\zeta|^2 dxd\tau \Bigg\}.
\end{aligned}
\tag{2.4}
$$

Proof. The proof is analogous to the previous one, just take the test function

$$\varphi = (u-k)_+\zeta^2,$$

and observe that
$$\frac{\partial}{\partial\tau}\left(\int_k^u (s-k)_+ s^{n-1} ds\right) = u^{n-1}(u-k)_+ u_\tau.$$

□

Let us introduce the logarithmic function

$$\psi\left(H^n,(u^n-k^n)_+,\nu^n\right)=\log^+\left(\frac{H^n}{H^n-(u^n-k^n)_++\nu^n}\right),$$

where

$$H^n=\operatorname*{ess\,sup}_{(x_0,t_0)+Q_R(\lambda)}(u^n-k^n)_+,\qquad\qquad 0<\nu^n<\min\left\{1,H^n\right\},$$

and for $s>0$

$$\log^+ s=\max\{\log s,0\}.$$

Proposition 2.3. (Logarithmic estimates)
Let u be a local, weak solution to (2.2) in Ω_T. There exists a positive constant c, depending only upon data, such that for every cylinder $(y,s)+Q_R(\lambda)\subset\Omega_T$, every level $k\in\mathbb{R}$ and every non-negative, piecewise smooth cutoff function $\zeta=\zeta(x)$

$$\begin{aligned}\operatorname*{ess\,sup}_{s-\lambda<t\le s}&\int_{K_R(y)}\psi^2\left(H^n,(u^n-k^n)_+,\nu^n\right)(x,t)\,\zeta^2(x)\,dx\\ &\le\int_{K_R(y)}\psi^2\left(H^n,(u^n-k^n)_+,\nu^n\right)(x,s-\lambda)\,\zeta^2(x)\,dx\\ &\quad+c\iint_{(y,s)+Q_R(\lambda)}|u|^{n(m-1)}\,\psi\left(H^n,(u^n-k^n)_+,\nu^n\right)\,|D\zeta|^2\,dxd\tau.\end{aligned}\tag{2.5}$$

Proof. Again we assume that (y,s) coincides with the origin. Put $v=u^n$ and, in the weak formulation of (2.2), take the test function

$$\varphi=\frac{\partial\psi^2}{\partial v}\,\zeta^2=2\psi\psi'\zeta^2,$$

over $Q_t=K_R\times(-\lambda\,,t]$, where $-\lambda<t\le 0$.
By direct calculation

$$\left(\psi^2\right)''=2(1+\psi)(\psi')^2\in L^\infty_{loc}(\Omega_T),\tag{2.6}$$

which implies that such a φ is an admissible testing function.
Estimating the various terms separately, we have

$$\begin{aligned}\iint_{Q_t}v_\tau\frac{\partial\psi^2}{\partial v}\,\zeta^2dxd\tau&=\iint_{Q_t}\frac{\partial}{\partial\tau}\psi^2\,\zeta^2dxd\tau\\ &=\int_{K_R}\psi^2(x,t)\,\zeta^2(x)\,dx-\int_{K_R}\psi^2(x,-\lambda)\,\zeta^2(x)\,dx,\end{aligned}$$

using (2.6) and the structure conditions (2.1)

$$\begin{aligned}
&\iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\, D\left(\frac{\partial \psi^2}{\partial v}\,\zeta^2\right) dxd\tau \\
&\qquad = \iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\, Dv(\psi^2)''\,\zeta^2 dxd\tau + 2\iint_{Q_t} (\psi^2)'\,\zeta\,\tilde{\mathcal{A}}(x,\tau,u,Du)\, D\zeta\, dxd\tau \\
&\qquad = 2n\iint_{Q_t} u^{n-1}\tilde{\mathcal{A}}(x,\tau,u,Du)\, Du\,(1+\psi)(\psi')^2\,\zeta^2 dxd\tau \\
&\qquad\quad + 4\iint_{Q_t} \psi\,\psi'\,\zeta\,\tilde{\mathcal{A}}(x,\tau,u,Du)\, D\zeta\, dxd\tau \\
&\qquad \geq 2n^2 mC_0 \iint_{Q_t} u^{n-1}|u|^{nm-1}|D(u-k)_+|^2(1+\psi)(\psi')^2\zeta^2 dxd\tau \\
&\qquad\quad - 4nmC_1 \iint_{Q_t} |u|^{nm-1}|D(u-k)_+|\,\zeta\,|D\zeta|\,\psi\psi' dxd\tau.
\end{aligned}$$

Applying Young's inequality, we get

$$\begin{aligned}
&\iint_{Q_t} \tilde{\mathcal{A}}(x,\tau,u,Du)\, D\left(\frac{\partial \psi^2}{\partial v}\,\zeta^2\right) dxd\tau \\
&\qquad \geq 2nm(nC_0 - C_1\varepsilon^2)\iint_{Q_t} |u|^{nm-1}|u|^{n-1}|D(u-k)_+|^2\psi(\psi')^2\zeta^2 dxd\tau \\
&\qquad\quad - 2nm\,\frac{C_1}{\varepsilon^2}\iint_{Q_t} |u|^{n(m-1)}|D\zeta|^2\psi\, dxd\tau.
\end{aligned}$$

Combining these estimates, discarding the term with the gradient on the left-hand side, and taking the supremum over $t \in (-\lambda\,, 0\,]$, proves the proposition. □

2.3 Reduction of the oscillation

To obtain the Hölder regularity, we argue as usual by a reduction of oscillation procedure. Let us state the basic result.

Theorem 2.2. *Let $(y,s) \in \Omega_T$, and ρ, $\omega > 0$ such that*

$$(y,s) + Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right) \subset \Omega_T\,, \qquad \operatorname*{ess\,osc}_{(y,s)+Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)} u \leq \omega\,,$$

where

$$\theta = \omega^{\frac{1-n}{2}}.$$

Then, there exist η_*, $c_0 \in (0,1)$, depending only upon data, such that

$$\operatorname*{ess\,osc}_{\mathcal{Q}^*} u \leq \eta_*\omega\,,$$

being

$$\mathcal{Q}^* = (y,s) + Q_{\theta\rho}\left(\theta_*\rho^2\right), \qquad \theta_* = \frac{c_0}{2}\,\omega^{1-nm}.$$

Our proof of this theorem splits into two alternatives.
Let us define

$$\mu_+ \geq \operatorname*{ess\,sup}_{(y,s)+Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)} u\,, \qquad \mu_- \leq \operatorname*{ess\,inf}_{(y,s)+Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)} u\,,$$

such that $\omega = \mu_+ - \mu_-$.
Let us recall that, without loss of generality, we can assume $\mu_+ > 0$, $\mu_- < 0$ and

$$\mu_+ \geq |\mu_-|\,.$$

Indeed, otherwise just change the sign of u and work with the new function.

2.3.1 The first alternative

We distinguish two alternatives, the first of them consists in assuming

$$\left|\left\{u < \mu_- + \frac{\omega}{2}\right\} \cap \left\{(y,s) + Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right\}\right| \leq c_0 \left|Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right|, \tag{2.7}$$

being $c_0 \in (0,1)$ a constant to be specified later.

Let us prove now the following De Giorgi type lemma.

Lemma 2.1. *There exists a number $c_0 \in (0,1)$, depending only upon data, such that if (2.7) is true, then*

$$u \geq \mu_- + \frac{\omega}{4} \qquad a.e.\ in\ (y,s) + Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right). \tag{2.8}$$

Proof. Without loss of generality we may assume $(y,s) = (0,0)$ and for $k = 0,1,\ldots$, set

$$\rho_k = \rho + \frac{\rho}{2^k}, \qquad \tilde{K}_k = K_{\theta\rho_k}, \qquad \tilde{Q}_k = \tilde{K}_k \times \left(-\frac{\rho_k^2}{\omega^{nm-1}}, 0\right].$$

Let ζ_k be a piecewise smooth cutoff function in $\tilde{Q}_k$ vanishing on the parabolic boundary of $\tilde{Q}_k$ such that $0 \leq \zeta_k \leq 1$, $\zeta_k = 1$ in $\tilde{Q}_{k+1}$ and

$$|D\zeta_k| \leq \frac{2^{k+2}}{\rho}\,\omega^{\frac{n-1}{2}}, \qquad 0 \leq \zeta_{k,t} \leq \frac{2^k}{\rho^2}\,\omega^{nm-1}.$$

Consider the following levels

$$\begin{aligned} h_k &= \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{k+2}} \qquad \text{if } \mu_- \geq -\frac{\omega}{8}\,, \\ h_k &= \mu_- + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} \qquad \text{if } \mu_- < -\frac{\omega}{8}\,. \end{aligned} \tag{2.9}$$

We treat first the least favourable case in which u might be close to zero, i.e. we assume first

$$\mu_- \geq -\frac{\omega}{8}. \tag{2.10}$$

Write down the energy estimates (2.3) for $(u-h_k)_-$ over the cylinder $\tilde{Q}_k$, to get

$$\begin{aligned} &\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} \int_{\tilde{K}_k} \left(\int_u^{h_k} (h_k-s)_+ s^{n-1} ds \right) \zeta_k^2(x,t)\, dx \\ &+ \iint_{\tilde{Q}_k} |u|^{nm-1} |D[(u-h_k)_-\zeta_k]|^2 dx d\tau \\ &\leq c \left\{ \iint_{\tilde{Q}_k} \left(\int_u^{h_k} (h_k-s)_+ s^{n-1} ds \right) |\zeta_{k,\tau}|\, dx d\tau + \iint_{\tilde{Q}_k} |u|^{nm-1} (u-h_k)_-^2 |D\zeta_k|^2 dx d\tau \right\}. \end{aligned}$$

Let us introduce the truncation

$$v = \max\left(u, \frac{\omega}{2^4}\right)$$

in order to estimate the terms with the integral over $[u, h_k]$; we have

$$\int_u^{h_k} (h_k-s)_+ s^{n-1} ds \geq \int_v^{h_k} (h_k-s)_+ s^{n-1} ds \geq v^{n-1} \frac{(v-h_k)_-^2}{2} \geq \left(\frac{\omega}{2^4}\right)^{n-1} \frac{(v-h_k)_-^2}{2}. \tag{2.11}$$

As $(u-h_k)_- \leq \omega$ and $\mu_- < 0$, we have

$$\int_u^{h_k} (h_k-s)_+ s^{n-1} ds \leq h_k^{n-1} \frac{(u-h_k)_-^2}{2} \leq \frac{\omega^{n+1}}{2}. \tag{2.12}$$

By the definition of v, we obtain

$$\begin{aligned} &\iint_{\tilde{Q}_k} v^{nm-1} |D[(v-h_k)_-\zeta_k]|^2 dx d\tau \\ &\quad = \iint_{\tilde{Q}_k \cap \left\{u > \frac{\omega}{2^4}\right\}} |u|^{nm-1} |D[(u-h_k)_-\zeta_k]|^2 dx d\tau \\ &\quad\quad + \iint_{\tilde{Q}_k \cap \left\{u \leq \frac{\omega}{2^4}\right\}} \left(\frac{\omega}{2^4}\right)^{nm-1} \left(\frac{\omega}{2^4} - h_k\right)_-^2 |D\zeta_k|^2 dx d\tau \\ &\quad \leq \iint_{\tilde{Q}_k} |u|^{nm-1} |D[(u-h_k)_-\zeta_k]|^2 dx d\tau + \frac{2^{2(k+1)}}{\rho^2} \omega^{n(m+1)} |A_k|\,, \end{aligned} \tag{2.13}$$

where

$$A_k = \{u < h_k\} \cap \tilde{Q}_k.$$

Let us prove

$$A_k = \tilde{A}_k := \{v < h_k\} \cap \tilde{Q}_k. \tag{2.14}$$

The inclusion $A_k \supseteq \tilde{A}_k$ follows by the definition of v. Prove the other one: if $v = u$ there is nothing to prove; if $v = \dfrac{\omega}{2^4}$, by (2.10) we have

$$h_k = \mu_- + \frac{\omega}{4} + \frac{\omega}{2^{k+2}} \geq \frac{\omega}{8} + \frac{\omega}{2^{k+2}} \geq \frac{\omega}{2^4}.$$

Taking into account that $|u| \leq \omega$, (2.11)–(2.14) yield

$$\begin{aligned}\left(\frac{\omega}{2^4}\right)^{n-1} & \operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} \int_{\tilde{K}_k} (v-h_k)_-^2\, \zeta_k^2(x,t)\,dx \\ & + \iint_{\tilde{Q}_k} v^{nm-1} |D[(v-h_k)_-\zeta_k]|^2 dx d\tau \leq c\frac{2^{2k}}{\rho^2}\,\omega^{n(m+1)}|\tilde{A}_k|.\end{aligned}$$

and again, thanks to the definition of v, it follows

$$\begin{aligned}\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} & \int_{\tilde{K}_k} (v-h_k)_-^2\, \zeta_k^2(x,t)\,dx + \left(\frac{\omega}{2^4}\right)^{n(m-1)} \iint_{\tilde{Q}_k} |D[(v-h_k)_-\zeta_k]|^2 dx d\tau \\ & \leq c\frac{2^{2k}}{\rho^2}\,\omega^{nm+1}|\tilde{A}_k|.\end{aligned} \tag{2.15}$$

The change of variables

$$\bar{x} = x\,\theta^{-1}, \qquad\qquad \bar{t} = \omega^{nm-1}\,\tau$$

maps the cube $\tilde{K}_k$ into K_{ρ_k} and the cylinder $\tilde{Q}_k$ into $Q_k = K_{\rho_k} \times (-\rho_k^2, 0]$.
With $(\bar{x},\bar{t}) \to u(\bar{x},\bar{t})$ denoting again the transformed function, the assumption (2.7) of the lemma becomes

$$\left|\left\{v < \mu_- + \frac{\omega}{2}\right\} \cap Q_0\right| \leq c_0|Q_0|. \tag{2.16}$$

Performing such a change of variables in (2.15), we have

$$\begin{aligned}\operatorname*{ess\,sup}_{-\rho_k^2<t\leq 0} & \int_{K_{\rho_k}} (v-h_k)_-^2\, \zeta_k^2(\bar{x},t)\,d\bar{x} + \iint_{Q_k} |D[(v-h_k)_-\zeta_k]|^2 d\bar{x} d\bar{t} \\ & \leq c\frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|,\end{aligned}$$

where

$$\bar{A}_k = \{v < h_k\} \cap Q_k.$$

This implies

$$\left\|(v-h_k)_-\zeta_k\right\|^2_{V^2(Q_k)} \le c\frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|. \tag{2.17}$$

Then from Corollary 1.1 with $p=2$ and (2.17), one gets

$$\begin{aligned}\iint_{Q_{k+1}}(v-h_k)^2_-\,d\bar{x}d\bar{t} &\le \iint_{Q_k}(v-h_k)^2_-\zeta_k^2\,d\bar{x}d\bar{t}\\ &\le c\,|\{v<h_k\}\cap Q_k|^{\frac{2}{N+2}}\left\|(v-h_k)_-\zeta_k\right\|^2_{V^2(Q_k)} \le c\frac{2^{2k}}{\rho^2}\,\omega^2|\bar{A}_k|^{1+\frac{2}{N+2}};\end{aligned}$$

the left-hand side is estimated by

$$\begin{aligned}\iint_{Q_{k+1}}(v-h_k)^2_-\,d\bar{x}d\bar{t} &= \iint_{Q_{k+1}\cap\{v<h_k\}}(h_k-v)^2 d\bar{x}d\bar{t}\\ &\ge \iint_{Q_{k+1}\cap\{v<h_{k+1}\}}(h_k-v)^2 d\bar{x}d\bar{t} \ge (h_k-h_{k+1})^2|\bar{A}_{k+1}| = \left(\frac{\omega}{2^{k+3}}\right)^2|\bar{A}_{k+1}|.\end{aligned}$$

Combining the previous estimates yields

$$|\bar{A}_{k+1}| \le c\frac{2^{4k}}{\rho^2}\,|\bar{A}_k|^{1+\frac{2}{N+2}},$$

and setting

$$Y_k = \frac{|\bar{A}_k|}{|Q_k|},$$

it follows

$$Y_{k+1} \le c\,2^{4k}\,Y_k^{1+\frac{2}{N+2}}.$$

Thanks to Lemma 1.4, we deduce that Y_k tends to zero as $k\to\infty$, provided

$$Y_0 = \frac{|\{v<h_0\}\cap Q_0|}{|Q_0|} = \frac{\left|\left\{v<\mu_-+\frac{\omega}{2}\right\}\cap Q_0\right|}{|Q_0|} \le c^{-\frac{N+2}{2}}\,2^{-(N+2)^2},$$

that is (2.16) with $c_0 := c^{-\frac{N+2}{2}}\,2^{-(N+2)^2}$.
Therefore

$$v \ge \mu_- + \frac{\omega}{4} \qquad\qquad \text{a.e. in } K_\rho \times \left(-\rho^2, 0\right].$$

Returning to the variables x, t, one has

$$v \ge \mu_- + \frac{\omega}{4} \qquad\qquad \text{a.e. in } Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right); \tag{2.18}$$

this implies that $u = v$ in $Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)$ and, consequently, (2.8). In fact, by contradiction, if there were a point $(x,t) \in Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)$ such that $v(x,t) = \dfrac{\omega}{2^4}$, by (2.18) and (2.10), we would obtain

$$\frac{\omega}{2^4} \geq \mu_- + \frac{\omega}{4} \geq \frac{\omega}{8}.$$

Assume now that (2.10) is violated, that is $\mu_- < -\dfrac{\omega}{8}$; choosing the levels h_k according to (2.9), we have

$$h_k = \mu_- + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} < -\frac{\omega}{8} + \frac{\omega}{2^5} + \frac{\omega}{2^{k+5}} \leq -\frac{\omega}{2^5},$$

which is false.
Therefore on the set $\{u \leq h_k\}$

$$|u|^{nm-1} \geq \left(\frac{\omega}{2^5}\right)^{nm-1}.$$

It follows that $|u|^{nm-1}$ can be estimated above and below by ω^{nm-1} up to a constant depending only upon the data; the proof can be repeated as before, but in this case there is no need to introduce the truncated function v. □

Therefore, under assumption (2.7)

$$-\operatorname*{ess\,inf}_{Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)} u \leq -\mu_- - \frac{\omega}{4};$$

adding

$$\operatorname*{ess\,sup}_{Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)} u,$$

gives

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)} u \leq -\mu_- - \frac{\omega}{4} + \operatorname*{ess\,sup}_{Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)} u \leq \frac{3}{4}\omega.$$

2.3.2 The second alternative

Let us recall the two fundamental hypotheses, namely

$$\mu_+ > 0\,, \qquad \mu_- < 0, \qquad \mu_+ \geq |\mu_-|.$$

Throughout this new section, let us assume that (2.7) does not hold, i.e.

$$\left|\left\{u \geq \mu_- + \frac{\omega}{2}\right\} \cap \left\{(y,s) + Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right\}\right| < (1-c_0)\left|Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right|. \tag{2.19}$$

For simplicity in the following we assume $(y,s) = (0,0)$.

Lemma 2.2. *There exists a time level t^* in the interval $\left(-\frac{(2\rho)^2}{\omega^{nm-1}}, -\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}\right)$ such that*

$$\left|\left\{u(\cdot,t^*) < \mu_- + \frac{\omega}{2}\right\} \cap K_{2\theta\rho}\right| > \frac{c_0}{2}\,|K_{2\theta\rho}|. \tag{2.20}$$

This in turn implies

$$\left|\left\{u(\cdot,t^*) \geq \mu_+ - \frac{\omega}{4}\right\} \cap K_{2\theta\rho}\right| \leq \left(1 - \frac{c_0}{2}\right)\,|K_{2\theta\rho}|. \tag{2.21}$$

Proof. By contradiction, suppose that (2.20) does not hold for every t^* in the indicated range, then

$$\begin{aligned}\left|\{u < \mu_- + \frac{\omega}{2}\} \cap Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right| &= \int_{-\frac{(2\rho)^2}{\omega^{nm-1}}}^{-\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}} |\{u(\cdot,t^*) < \mu_- + \xi\omega\} \cap K_{2\theta\rho}|\,dt^* \\ &\quad + \int_{-\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}}}^{0} |\{u(\cdot,t^*) < \mu_- + \xi\omega\} \cap K_{2\theta\rho}|\,dt^* \\ &< \frac{c_0}{2}\,|K_{2\theta\rho}|\left(1 - \frac{c_0}{2}\right)\frac{(2\rho)^2}{\omega^{nm-1}} + |K_{2\theta\rho}|\frac{c_0}{2}\frac{(2\rho)^2}{\omega^{nm-1}} < c_0\left|Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right)\right|.\end{aligned}$$

This proves (2.20); (2.21) follows by the fact that (2.20) is equivalent to

$$\left|\left\{u(\cdot,t^*) \geq \mu_- + \frac{\omega}{2}\right\} \cap K_{2\theta\rho}\right| < \left(1 - \frac{c_0}{2}\right)|K_{2\theta\rho}|$$

and $\mu_- + \frac{\omega}{2} \leq \mu_+ - \frac{\omega}{4}$. □

The next lemma asserts that a property similar to (2.21) continues to hold for all time levels from t^* up to zero.

Lemma 2.3. *There exists a positive integer j^*, depending upon the data and c_0, such that*

$$\left|\left\{u(\cdot,t) > \mu_+ - \frac{\omega}{2^{j^*}}\right\} \cap K_{2\theta\rho}\right| < \left(1 - \frac{c_0^2}{4}\right)|K_{2\theta\rho}| \tag{2.22}$$

for all times $t^* < t < 0$.

Proof. Consider the logarithmic estimates (2.5) written over the cylinder $K_{2\theta\rho} \times (t^*, 0)$ for the function $(u^n - k^n)_+$ and for the level $k = \left(\mu_+^n - \left(\frac{\omega}{4}\right)^n\right)^{\frac{1}{n}}$. Notice that, thanks to our assumptions, $\mu_+ > \frac{\omega}{4}$, so $k > 0$.

The number ν in the definition of the logarithmic function is taken as $\nu = \frac{\omega}{2^{j+2}}$, where j is a positive integer to be chosen. Thus we have

$$\psi\left(H^n, (u^n - k^n)_+, \nu^n\right) = \log^+\left(\frac{H^n}{H^n - (u^n - k^n)_+ + \frac{\omega^n}{2^{(j+2)n}}}\right),$$

where

$$H^n = \operatorname*{ess\,sup}_{K_{2\theta\rho}\times(t^*,0)} \left[u^n - \left(\mu_+^n - \left(\frac{\omega}{4}\right)^n\right)\right]_+ .$$

The cutoff function $x \to \zeta(x)$ is taken such that

$$\zeta = 1 \text{ on } K_{(1-\sigma)2\theta\rho} \quad \text{for } \sigma \in (0,1), \qquad\qquad |D\zeta| \le \frac{1}{2\sigma\theta\rho}.$$

With these choices, inequality (2.5) yields

$$\int_{K_{(1-\sigma)2\theta\rho}} \psi^2(x,t)\,dx \le \int_{K_{2\theta\rho}} \psi^2(x,t^*)\,dx + c\int_{t^*}^0\int_{K_{2\theta\rho}} |u|^{n(m-1)}\,\psi\,|D\zeta|^2 dx d\tau \tag{2.23}$$

for all $t^* \le t \le 0$. Let us observe that

$$\psi \le \log\left(\frac{\frac{\omega^n}{2^{2n}}}{\frac{\omega^n}{2^{(j+2)n}}}\right) = jn\log 2.$$

To estimate the first integral on the right-hand side of (2.23), notice that ψ vanishes on the set $\{u^n < k^n\}$ and that $\mu_+^n - \left(\frac{\omega}{4}\right)^n \ge \left(\mu_+ - \frac{\omega}{4}\right)^n$; therefore by (2.21)

$$\int_{K_{2\theta\rho}} \psi^2(x,t^*)\,dx \le j^2 n^2 \log^2 2\left(1 - \frac{c_0}{2}\right)|K_{2\theta\rho}|.$$

The remaining integral is estimated in the following way

$$c\int_{t^*}^0\int_{K_{2\theta\rho}} |u|^{n(m-1)}\,\psi\,|D\zeta|^2 dx d\tau \le \frac{c}{(\sigma\theta\rho)^2}\,jn\log 2\,\frac{(2\rho)^2}{\omega^{nm-1}}\,\omega^{n(m-1)}|K_{2\theta\rho}| = \frac{c}{\sigma^2}\,jn\,|K_{2\theta\rho}|.$$

Combining the previous estimates

$$\int_{K_{(1-\sigma)2\theta\rho}} \psi^2(x,t)\,dx \le \left\{j^2 n^2\log^2 2\left(1 - \frac{c_0}{2}\right) + \frac{c}{\sigma^2}\,jn\right\}|K_{2\theta\rho}| \tag{2.24}$$

for all $t^* \le t \le 0$. The left-hand side of (2.24) is estimated below by integrating over the smaller set

$$\left\{u^n > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\};$$

on such a set, since ψ is a decreasing function of H^n, we have

$$\psi^2 \ge \log^2\left(\frac{\frac{\omega^n}{2^{2n}}}{\frac{\omega^n}{2^{(j+1)n}}}\right) = (j-1)^2 n^2 \log^2 2;$$

hence, for all $t^* \le t \le 0$, we obtain

$$\left|\left\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\} \cap K_{(1-\sigma)2\theta\rho}\right| \le \left\{\left(\frac{j}{j-1}\right)^2\left(1-\frac{c_0}{2}\right) + \frac{c}{\sigma^2 j}\right\}|K_{2\theta\rho}|.$$

On the other hand

$$\begin{aligned}
&\left|\left\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\} \cap K_{2\theta\rho}\right| \\
&\quad \le \left|\left\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\} \cap K_{(1-\sigma)2\theta\rho}\right| + |K_{2\theta\rho} \setminus K_{(1-\sigma)2\theta\rho}| \\
&\quad \le \left|\left\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\} \cap K_{(1-\sigma)2\theta\rho}\right| + N\sigma|K_{2\theta\rho}|.
\end{aligned}$$

Then

$$\left|\left\{u^n(\cdot,t) > \mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right\} \cap K_{2\theta\rho}\right| \le \left\{\left(\frac{j}{j-1}\right)^2\left(1-\frac{c_0}{2}\right) + \frac{c}{\sigma^2 j} + N\sigma\right\}|K_{2\theta\rho}|.$$

for all $t^* \le t \le 0$.
Now choose σ so small and then j so large as to obtain

$$\left|\left\{u(\cdot,t) > \left(\mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right)^{\frac{1}{n}}\right\} \cap K_{2\theta\rho}\right| \le \left(1-\frac{c_0^2}{4}\right)|K_{2\theta\rho}| \qquad \forall t^* \le t \le 0.$$

Notice that our hypotheses imply $\mu_+ \ge \dfrac{\omega}{2}$, $\mu_+ < \omega$, therefore

$$\begin{aligned}
\left(\mu_+^n - \frac{\omega^n}{2^{(j+2)n}}\right)^{\frac{1}{n}} &< \left(\mu_+^n - \frac{\mu_+^n}{2^{(j+2)n}}\right)^{\frac{1}{n}} = \mu_+\left(1 - \frac{1}{2^{(j+2)n}}\right)^{\frac{1}{n}} \\
&\le \mu_+\left(1 - \frac{1}{2^{(j+2)n}\,n}\right) \le \mu_+ - \frac{\omega}{2^{(j+2)n+1}\,n}.
\end{aligned}$$

The proof is finished once we choose j^* as the smallest integer such that

$$\mu_+ - \frac{\omega}{2^{(j+2)n+1}\,n} \le \mu_+ - \frac{\omega}{2^{j^*}}.$$

□

Corollary 2.1. *For all $j \ge j^*$ and for all times $-\dfrac{c_0}{2}\dfrac{(2\rho)^2}{\omega^{nm-1}} < t < 0$,*

$$\left|\left\{u(\cdot,t) > \mu_+ - \frac{\omega}{2^j}\right\} \cap K_{2\theta\rho}\right| < \left(1-\frac{c_0^2}{4}\right)|K_{2\theta\rho}|. \tag{2.25}$$

Motivated by Corollary 2.1, introduce the cylinder

$$Q_* = K_{2\theta\rho} \times (-\theta_*(2\rho)^2, 0], \qquad \text{with} \quad \theta_* = \frac{c_0}{2}\,\omega^{1-nm}.$$

Lemma 2.4. *For every $\nu_* \in (0,1)$, there exists a positive integer $q_* = q_*(data, \nu_*)$ such that*

$$\left|\left\{u \geq \mu_+ - \frac{\omega}{2^{j_*+q_*}}\right\} \cap Q_*\right| \leq \nu_* |Q_*|.$$

Proof. Write down the energy estimates (2.4) for the truncated functions $(u-k_j)_+$, with $k_j = \mu_+ - \frac{\omega}{2^j}$, for $j = j_*, \dots, j_* + q_*$ over the cylinder

$$\tilde{Q} = K_{4\theta\rho} \times \left(-c_0 \frac{(2\rho)^2}{\omega^{nm-1}}, 0 \right] \supset Q_*\,;$$

the cutoff function ζ is taken to be one on Q_*, vanishing on the parabolic boundary of $\tilde{Q}$ and such that

$$|D\zeta| \leq \frac{1}{\theta\rho}, \qquad 0 \leq \zeta_t \leq \frac{\omega^{nm-1}}{c_0\rho^2}.$$

Thanks to these choices, the energy estimates (2.4) take the form

$$\begin{aligned}
&\iint_{\tilde{Q}} |u|^{nm-1} |D(u-k_j)_+|^2 \zeta^2 dx d\tau \\
&\quad \leq c \left\{ \frac{\omega^{nm-1}}{c_0\rho^2} \iint_{\tilde{Q}} \left(\int_{k_j}^{u} (s-k_j)_+ s^{n-1} ds \right) dx d\tau + \frac{\omega^{n-1}}{\rho^2} \iint_{\tilde{Q}} |u|^{nm-1} (u-k_j)_+^2 dx d\tau \right\}.
\end{aligned}$$

Estimating

$$\int_{k_j}^{u} (s-k_j)_+ s^{n-1} ds \leq u^{n-1} \frac{(u-k_j)_+^2}{2} \leq \omega^{n-1} \frac{(u-k_j)_+^2}{2}, \tag{2.26}$$

and taking into account $(u-k_j)_+ \leq \frac{\omega}{2^j}$, yields

$$\iint_{\tilde{Q}} |u|^{nm-1} |D(u-k_j)_+|^2 \zeta^2 dx d\tau \leq c \left(\frac{\omega}{2^j}\right)^2 \omega^{n-1} \frac{\omega^{nm-1}}{c_0\,\rho^2} |Q_*|.$$

Notice that $u > k_j \geq \frac{\omega}{4}$: indeed the second inequality is equivalent to

$$\mu_+ \geq |\mu_-| \frac{\frac{1}{4} + \frac{1}{2^j}}{\frac{3}{4} - \frac{1}{2^j}}$$

and this is implied by our assumptions.
So we can estimate

$$\iint_{\tilde{Q}} |u|^{nm-1}|D(u-k_j)_+|^2\zeta^2 dxd\tau \geq \iint_{Q_*} |u|^{nm-1}|D(u-k_j)_+|^2 dxd\tau$$
$$\geq \left(\frac{\omega}{4}\right)^{nm-1} \iint_{Q_*} |D(u-k_j)_+|^2 dxd\tau;$$

it follows

$$\iint_{Q_*} |D(u-k_j)_+|^2 dxd\tau \leq c\left(\frac{\omega}{2^j}\right)^2 \omega^{n-1} \frac{1}{c_0\,(2\rho)^2}\,|Q_*|. \tag{2.27}$$

Next, apply the isoperimetric inequality of Lemma 1.3 to the function $u(\cdot,t)$, for t in the range $(-\theta_*(2\rho)^2, 0]$, over the cube $K_{2\theta\rho}$, and for the levels

$$k = k_j < l = k_{j+1};$$

in this way $(l-k) = \dfrac{\omega}{2^{j+1}}$.
Taking into account (2.25), this gives

$$\frac{\omega}{2^{j+1}}\,|\{u(\cdot,t) > k_{j+1}\} \cap K_{2\theta\rho}| \leq \frac{(2\theta\rho)^{N+1}}{|\{u(\cdot,t) < k_j\} \cap K_{2\theta\rho}|} \int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_{2\theta\rho}} |Du|\,dx$$
$$\leq \frac{8\theta\rho}{c_0^2} \int_{\{k_j<u(\cdot,t)<k_{j+1}\}\cap K_{2\theta\rho}} |Du|\,dx,$$

integrating in dt over the indicated interval and applying the Hölder inequality, one gets

$$\frac{\omega}{2^{j+1}}\,|A_{j+1}| \leq \frac{8\theta\rho}{c_0^2}\left(\iint_{Q_*} |D(u-k_j)_+|^2 dxdt\right)^{\frac{1}{2}} \left(|A_j| - |A_{j+1}|\right)^{\frac{1}{2}},$$

where

$$A_j = \{u > k_j\} \cap Q_*.$$

Square both sides of this inequality and estimate above the term containing $|D(u-k_j)_+|$ by inequality (2.27), to obtain

$$|A_{j+1}|^2 \leq \frac{c}{c_0^5}\,|Q_*|\,\left(|A_j| - |A_{j+1}|\right).$$

Add these recursive inequalities for $j = j_*+1, \ldots, j_*+q_*-1$, where q_* is to be chosen. Majorizing the right-hand side with the corresponding telescopic series, gives

$$(q_*-2)\,|A_{j_*+q_*}|^2 \leq \sum_{j=j_*+1}^{j_*+q_*-1} |A_{j+1}|^2 \leq \frac{c}{c_0^5}\,|Q_*|^2.$$

From this

$$|A_{j_*+q_*}| \leq \frac{1}{\sqrt{q_*-2}} \sqrt{\frac{c}{c_0^5}}\, |Q_*|.$$

The number ν_* being fixed, choose q_* from

$$\frac{1}{\sqrt{q_*-2}} \sqrt{\frac{c}{c_0^5}} = \nu_*.$$

□

Now let $\beta \in \left(0, \frac{1}{2}\right)$, $a \in (0,1)$ be fixed numbers.

Lemma 2.5. *There exists a number $c_* \in (0,1)$, depending upon the data, β, and a, such that if*

$$\left|\{u \geq \mu_+ - \beta\omega\} \cap Q_*\right| \leq c_* \left|Q_*\right|, \tag{2.28}$$

then

$$u \leq \mu_+ - a\beta\omega \qquad \textit{a.e. in } Q_{\theta\rho}\left(\theta_*\rho^2\right). \tag{2.29}$$

Proof. For $k = 0, 1, \ldots$, set

$$\rho_k = \rho + \frac{\rho}{2^k}, \qquad K_k = K_{\theta\rho_k}, \qquad Q_k = K_k \times \left(-\theta_*\rho_k^2, 0\right].$$

Let $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$ be a piecewise smooth cutoff function in Q_k such that

$$\zeta_1 = \begin{cases} 1 & \text{in } K_{k+1} \\ 0 & \text{in } \mathbb{R}^N \setminus K_k \end{cases} \qquad |D\zeta_1| \leq \frac{2^{k+2}}{\theta\rho},$$

$$\zeta_2 = \begin{cases} 1 & \text{if } t \geq -\dfrac{\rho_{k+1}^2}{\omega^{nm-1}} \\ 0 & \text{if } t < -\dfrac{\rho_k^2}{\omega^{nm-1}} \end{cases} \qquad 0 \leq \zeta_{2,t} \leq \frac{2^k}{\theta_*\rho^2}.$$

Choose the sequence of truncating levels

$$h_k = \mu_+ - \beta_k\omega, \qquad \text{where } \beta_k = a\beta + \frac{1-a}{2^k}\beta$$

and write down the energy estimates (2.4) for $(u-h_k)_+$ over the cylinder Q_k

$$\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} \int_{K_k} \left(\int_{h_k}^{u} (s-h_k)_+ s^{n-1} ds\right) \zeta^2(x,t)\, dx + \iint_{Q_k} |u|^{nm-1} |D[(u-h_k)_+\zeta]|^2 dx d\tau$$
$$\leq c\left\{ \iint_{Q_k} \left(\int_{h_k}^{u} (s-h_k)_+ s^{n-1} ds\right) |\zeta_\tau|\, dx d\tau + \iint_{Q_k} |u|^{nm-1}(u-h_k)_+^2 |D\zeta|^2 dx d\tau \right\}.$$

Let us estimate

$$
\begin{aligned}
&\int_{h_k}^{u}(s-h_k)_+ s^{n-1}ds \ge h_k^{n-1}\frac{(u-h_k)_+^2}{2},\\
&\int_{h_k}^{u}(s-h_k)_+ s^{n-1}ds \le u^{n-1}\frac{(u-h_k)_+^2}{2} \le \omega^{n-1}\frac{(u-h_k)_+^2}{2}.
\end{aligned}
\tag{2.30}
$$

Taking into account that $(u-h_k)_+ \le \beta\omega$ and the definitions of θ and θ_*, we have

$$
\begin{aligned}
\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0} \; & h_k^{n-1}\int_{K_k}\frac{(u-h_k)_+^2}{2}\,\zeta^2(x,t)\,dx + \iint_{Q_k}|u|^{nm-1}|D[(u-h_k)_+\zeta]|^2dxd\tau\\
&\le c\,(\beta\omega)^2\left\{\omega^{n-1}\frac{2^k}{\theta_*\rho^2}+\omega^{nm-1}\frac{2^{2k}}{(\theta\rho)^2}\right\}|A_k|\\
&= c\,\frac{2^{2k}}{\rho^2}\,(\beta\omega)^2\omega^{n-1}\,\omega^{nm-1}\,|A_k|,
\end{aligned}
$$

where

$$
A_k = \{u<h_k\}\cap Q_k.
$$

Now, notice that $u > h_k \ge \left(\frac{1}{2}-\beta\right)\omega$: indeed the last inequality is equivalent to

$$
\mu_+ \ge |\mu_-|\,\left(\frac{1}{2}-\beta+\beta_k\right)\left(\frac{1}{2}+\beta-\beta_k\right)^{-1}
$$

and this follows by our hypotheses.
So, we obtain

$$
\begin{aligned}
\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\le 0}\int_{K_k}(u-h_k)_+^2\,\zeta^2(x,t)\,dx &\le c\,\frac{2^{2k}}{\rho^2}\left(\frac{1}{2}-\beta\right)^{1-n}(\beta\omega)^2\,\omega^{nm-1}\,|A_k|,\\
\iint_{Q_k}|D[(u-h_k)_+\zeta]|^2dxd\tau &\le c\,\frac{2^{2k}}{\rho^2}\left(\frac{1}{2}-\beta\right)^{1-nm}(\beta\omega)^2\omega^{n-1}\,|A_k|.
\end{aligned}
\tag{2.31}
$$

By $(u-h_k)_+ \ge \dfrac{1-a}{2^{k+1}}\,\beta\omega$, applying the Hölder inequality, and Proposition 1.1, (2.31) yields

$$\frac{(1-a)^2}{2^{2(k+1)}}(\beta\omega)^2|A_{k+1}| \leq \iint_{Q_{k+1}} (u-h_k)_+^2 dx d\tau \leq \iint_{Q_k} (u-h_k)_+^2 \zeta^2 dx d\tau$$

$$\leq \left(\iint_{Q_k} [(u-h_k)_+\zeta]^{\frac{2(N+2)}{N}} dx d\tau\right)^{\frac{N}{N+2}} |A_k|^{\frac{2}{N+2}}$$

$$\leq c\left(\iint_{Q_k} |D[(u-h_k)_+\zeta]|^2 dx d\tau\right)^{\frac{N}{N+2}} \left(\operatorname*{ess\,sup}_{-\frac{\rho_k^2}{\omega^{nm-1}}<t\leq 0} \int_{K_k} [(u-h_k)_+\zeta]^2 dx\right)^{\frac{2}{N+2}} |A_k|^{\frac{2}{N+2}}$$

$$\leq c\frac{2^{2k}}{\rho^2}(\beta\omega)^2 \omega^{\frac{2(nm-1)+N(n-1)}{N+2}} \left(\frac{1}{2}-\beta\right)^{\frac{N(1-nm)+2(1-n)}{N+2}} |A_k|^{1+\frac{2}{N+2}}.$$

Therefore

$$|A_{k+1}| \leq c\,\frac{2^{4k}}{(1-a)^2\rho^2}\,\omega^{\frac{2(nm-1)+N(n-1)}{N+2}} \left(\frac{1}{2}-\beta\right)^{\frac{N(1-nm)+2(1-n)}{N+2}} |A_k|^{1+\frac{2}{N+2}}.$$

Setting

$$Y_k = \frac{|A_k|}{|Q_k|},$$

we obtain

$$Y_{k+1} \leq c\,\frac{2^{4k}}{(1-a)^2\rho^2}\,\omega^{\frac{2(nm-1)+N(n-1)}{N+2}} \left(\frac{1}{2}-\beta\right)^{\frac{N(1-nm)+2(1-n)}{N+2}} \rho^2\left(\theta^N\theta_*\right)^{\frac{2}{N+2}}\ Y_k^{1+\frac{2}{N+2}}$$

$$= c\,\frac{2^{4k}}{(1-a)^2}\left(\frac{1}{2}-\beta\right)^{\frac{N(1-nm)+2(1-n)}{N+2}} Y_k^{1+\frac{2}{N+2}}.$$

By Lemma 1.4, Y_k tends to zero as $k\to\infty$, provided

$$Y_0 = \frac{|\{u>h_0\}\cap Q_0|}{|Q_0|} = \frac{|\{u>\mu_+-\beta\omega\}\cap Q_0|}{|Q_0|}$$

$$\leq \frac{c^{-\frac{N+2}{2}}}{(1-a)^{-(N+2)}}\left(\frac{1}{2}-\beta\right)^{\frac{N(nm-1)+2(n-1)}{2}} 2^{-(N+2)^2},$$

that is (2.28) with $c_* := \dfrac{c^{-\frac{N+2}{2}}}{(1-a)^{-(N+2)}}\left(\dfrac{1}{2}-\beta\right)^{\frac{N(nm-1)+2(n-1)}{2}} 2^{-(N+2)^2}.$

This concludes the proof. □

Thanks to Lemma 2.4, we can apply Lemma 2.5 with $\beta = \dfrac{1}{2^{j_*+q_*}}$ and $a=\dfrac{1}{2}$, getting

$$u \leq \mu_+ - \frac{\omega}{2^{j_*+q_*+1}} \qquad \text{a.e. in } Q_{\theta\rho}\left(\theta_*\rho^2\right),$$

which implies

$$\operatorname*{ess\,sup}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \mu_+ - \frac{\omega}{2^{j_*+q_*+1}}.$$

Hence

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \mu_+ - \operatorname*{ess\,inf}_{Q_{\theta\rho}(\theta_*\rho^2)} u - \frac{\omega}{2^{j_*+q_*+1}} \le \omega\left(1 - \frac{1}{2^{j_*+q_*+1}}\right).$$

2.3.3 Conclusion

The two alternatives just discussed can be combined to prove Theorem 2.2.

Proof of Theorem 2.2. The concluding statement of the first alternative says that

$$\operatorname*{ess\,osc}_{Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right)} u \le \frac{3}{4}\omega;$$

analogously, the conclusion of the second alternative is that

$$\operatorname*{ess\,osc}_{\mathcal{Q}^*} u = \operatorname*{ess\,osc}_{Q_{\theta\rho}(\theta_*\rho^2)} u \le \omega\left(1 - \frac{1}{2^{j_*+q_*+1}}\right).$$

Recalling the definition of θ_*, we observe that

$$\mathcal{Q}^* = Q_{\theta\rho}\left(\theta_*\rho^2\right) \subset Q_{\theta\rho}\left(\frac{\rho^2}{\omega^{nm-1}}\right).$$

The thesis follows by defining

$$\eta_* := 1 - \frac{1}{2^{j_*+q_*+1}}.$$

□

We are now ready to show the Hölder regularity.

Proof of Theorem 2.1. We fix any $(y,s) \in \Omega_T$, and we choose R_0, $\omega_0 > 0$ such that

$$(y,s) + Q_{R_0}(R_0^2) \subset \Omega_T, \qquad \omega_0 \ge \max\left\{1, \operatorname*{ess\,osc}_{(y,s)+Q_{R_0}(R_0^2)} u\right\}.$$

Let now b, $\delta \in (0,1)$ to be chosen, and let us introduce the sequences

$$R_k := R_0 b^k, \qquad \omega_k := \omega_0\delta^k, \qquad \theta_k := \omega_k^{\frac{1-n}{2}} \qquad Q_k := (y,s) + Q_{\theta_k R_k}\left(\frac{R_k^2}{\omega_k^{nm-1}}\right),$$

for $k \in \mathbb{N}$.
The thesis follows by standard arguments once we prove that

$$Q_{k+1} \subset Q_k \subset Q_{R_0}(R_0^2) \subset \Omega_T \qquad \forall k \in \mathbb{N},$$

$$\operatorname*{ess\,osc}_{Q_k} u \leq \omega_k \, . \tag{2.32}$$

The fact that $Q_0 \subset Q_{R_0}(R_0^2)$ is a direct consequence of $\omega_0 \geq 1$, while $Q_{k+1} \subset Q_k$ is equivalent to

$$b \leq \min\left\{\delta^{\frac{n-1}{2}}, \delta^{\frac{nm-1}{2}}\right\} = \delta^{\frac{n-1}{2}} \, .$$

To prove (2.32), we will argue by induction. The validity for $k = 0$ is true by construction. Assume that (2.32) holds for k and apply Theorem 2.2 taking $\rho = \dfrac{R_k}{2}$ and $\omega = \omega_k$. Thanks to this choice

$$\theta = \theta_k \, , \qquad\qquad (y, s) + Q_{2\theta\rho}\left(\frac{(2\rho)^2}{\omega^{nm-1}}\right) = Q_k \, .$$

The assumptions of Theorem 2.2 are satisfied because (2.32) holds for k; hence, we get

$$\operatorname*{ess\,osc}_{\mathcal{Q}^*} u \leq \eta_* \omega_k \, ,$$

where in this setting

$$\mathcal{Q}^* = (y, s) + Q_{\theta_k \frac{R_k}{2}}\left(\frac{c_0}{8}\, \omega_k^{1-nm} R_k^2\right) \, .$$

This leads us to choose $\delta = \eta_* \in (0, 1)$, so that $\eta_* \omega_k = \omega_{k+1}$. It remains only to check $Q_{k+1} \subset \mathcal{Q}^*$, which by a simple calculation is equivalent to

$$b \leq \min\left\{\frac{1}{2}\,\delta^{\frac{n-1}{2}} \, , \sqrt{\frac{c_0}{8}}\,\delta^{\frac{nm-1}{2}}\right\} \, .$$

We conclude by choosing b small enough. □

Chapter 3

Harnack estimates for non-negative weak solutions to singular porous medium type equation

In this chapter we want to prove Harnack estimates for non-negative, weak solutions to (1.4)-(1.5), with $0 < m < 1$. In order to use a comparison principle (see Section 3.2 below) we have to require the following further monotonicity assumption and growth conditions, namely we assume that there exists a positive constant L such that

$$\begin{cases} \big(\mathcal{A}(x,t,z,\xi_1) - \mathcal{A}(x,t,z,\xi_2)\big) \cdot (\xi_1 - \xi_2) \geq 0, \\ |\mathcal{A}(x,t,z_1,z_1^{1-m}\xi) - \mathcal{A}(x,t,z_2,z_2^{1-m}\xi)| \leq L\,|z_1^m - z_2^m|\,\big(1+|\xi|\big) \end{cases} \tag{3.1}$$

for a.e. $(x,t) \in \Omega_T$ and all $z, z_1, z_2 \in \mathbb{R}_+$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^N$.
A class of quasi-linear porous medium type equations satisfying (3.1) is

$$u_t = \sum_{i,j} \big(a_{ij}(x,t)|u|^{m-1}\, u_{x_i}\big)_{x_j},$$

where the coefficients a_{ij} belong to $L^\infty_{loc}(\Omega_T)$ and the matrix (a_{ij}) is symmetric and elliptic in Ω_T. In particular, (3.1) is verified by the porous medium equation (1.6).

We consider u to be a non-negative, local, weak solution to the equation (1.4) with conditions (1.5) and (3.1), such that

$$u \in L^r_{loc}(\Omega_T), \text{ for some } r \geq 1 \text{ with } \lambda_r = N(m-1) + 2r > 0. \tag{3.2}$$

This assumption in turn implies that u is locally bounded (see Proposition 3.1 below).

Having fixed $(y,s)\in\Omega_T$ and $\rho>0$ with $B_\rho(y)\subset\Omega$, we define

$$\delta := \left(\gamma \fint_{B_\rho(y)} u(x,s)dx\right)^{1-m}\rho^2, \qquad \eta := \left(\frac{\fint_{B_\rho(y)} u(x,s)dx}{\left(\fint_{B_\rho(y)} u^r(x,s)dx\right)^{\frac{1}{r}}}\right)^{\frac{2r}{\lambda_r}},$$

where γ is a parameter that will be fixed in the following.

Theorem 3.1. *There exist a constant $\gamma\in(0,1)$, depending only upon the data, and two positive constants c and d, depending upon the data and r, such that if $B_{16\rho}(y)\times[s,s+\delta]\subset\Omega_T$, then*

$$\inf_{B_{4\rho}(y)\times[s+\frac{3}{4}\delta,s+(\frac{3}{4}+\frac{1}{2^6})\delta]} u \ge c\,\eta^d \sup_{B_{\frac{\rho}{2}}(y)\times[s+\frac{1}{2}\delta,s+\delta]} u.$$

Remark 3.1. *It is known (see for instance [26, 37]) that, if $m>\dfrac{N-2}{N+2}$, then every local weak solution u satisfies (3.2), therefore it is locally bounded. As a consequence, in the previous range for m, Theorem 3.1 holds true for every non-negative, local, weak solution.*

3.1 Some useful estimates

Here we state some technical results we will use in the sequel.
We start with an L^r_{loc}-L^∞_{loc} estimate one can find in [69, 56]; see also the Appendix B of [34].

Proposition 3.1. *Let u be a non-negative, local, weak super-solution to (1.4) in Ω_T and let $y\in\Omega$, $\rho>0$. There exists a positive constant c_r, depending upon the data and r, such that for every cylinder $B_{2\rho(y)}\times[2s-t,t]\subset\Omega_T$*

$$\sup_{B_\rho(y)\times[s,t]} u \le \frac{c_r}{(t-s)^{\frac{N}{\lambda_r}}}\left(\int_{B_{2\rho(y)}} u^r(x,2s-t)\,dx\right)^{\frac{2}{\lambda_r}} + c_r\left(\frac{t-s}{\rho^2}\right)^{\frac{1}{1-m}}.$$

We pass then to a sort of Harnack inequality in the L^1 topology, originally proved for the prototype porous medium equation (1.4) in [42]; for a proof in the general case see the Appendix B of [34].

Proposition 3.2. *Let u be a non-negative, local, weak super-solution to (1.4) in Ω_T and let $y\in\Omega$, $\rho>0$. There exists a positive constant c, depending upon the data, such that for every cylinder $B_{2\rho}(y)\times[s,t]\subset\Omega_T$*

$$\sup_{s<\tau<t}\fint_{B_\rho(y)} u(x,\tau)\,dx \le c\inf_{s<\tau<t}\fint_{B_{2\rho(y)}} u(x,\tau)\,dx + c\left(\frac{t-s}{\rho^2}\right)^{\frac{1}{1-m}}.$$

Let us prove now the following energy estimates.

Proposition 3.3. *Let u be a non-negative, local, weak super-solution to (1.4) in Ω_T and let $y \in \Omega$, $\rho > 0$. There exists a positive constant $c = c(data)$ such that for every cylinder $B_\rho(y) \times (t_1, t_2) \subset \Omega_T$, every level $k > 0$ and every non-negative piecewise smooth cutoff function $\zeta = \zeta(x)$ vanishing on $\partial B_\rho(y)$,*

$$
\begin{aligned}
k^{m-1} \operatorname*{ess\,sup}_{t_1<t\le t_2} \int_{B_\rho(y)} (u-k)_-^2 \zeta^2(x)\,dx + \int_{t_1}^{t_2}\int_{B_\rho(y)} |D(u^m-k^m)_-|^2 \zeta^2(x)\,dxd\tau \\
\le k^m \int_{B_\rho(y)} \Big(u(x,t_1)-k\Big)_- \zeta^2(x)\,dx + c\,k^{2m}\int_{t_1}^{t_2}\int_{B_\rho(y)} \chi_{\{u<k\}}\,|D\zeta|^2 dxd\tau.
\end{aligned}
\tag{3.3}
$$

Proof. Assume for simplicity $y = 0$. In the weak formulation (1.7), take $\varphi = (u^m - k^m)_-\zeta^2$ as test function over $Q_t = B_\rho \times (t_1, t]$, with $t_1 < t \le t_2$.
Notice that

$$
\left(\int_u^k (k^m - s^m)_+ ds\right)_\tau = -(u^m - k^m)_-\, u_\tau;
$$

therefore, looking at the first term of the weak formulation, we obtain

$$
\begin{aligned}
\iint_{Q_t} u_\tau (u^m - k^m)_- \zeta^2(x)\,dxd\tau &= -\iint_{Q_t}\left(\int_u^k (k^m-s^m)_+ds\right)_\tau \zeta^2(x)\,dxd\tau \\
&= \int_{B_\rho}\left(\int_{u(x,t_1)}^k (k^m-s^m)_+ds\right)\zeta^2(x)\,dx - \int_{B_\rho}\left(\int_{u(x,t)}^k (k^m-s^m)_+ds\right)\zeta^2(x)\,dx.
\end{aligned}
$$

By a simple calculation, one has

$$
\begin{aligned}
\int_{u(x,t_1)}^k (k^m-s^m)_+ds &\le k^m\Big(u(x,t_1)-k\Big)_-, \\
\int_{u(x,t)}^k (k^m-s^m)_+ds &\ge \frac{m}{2}\,k^{m-1}(u-k)_-^2,
\end{aligned}
$$

thus, one gets

$$
\begin{aligned}
&\iint_{Q_t} u_\tau (u^m-k^m)_-\zeta^2(x)\,dxd\tau \\
&\quad\le k^m\int_{B_\rho}\Big(u(x,t_1)-k\Big)_-\zeta^2(x)\,dx - \frac{m}{2}\,k^{m-1}\int_{B_\rho}(u-k)_-^2\zeta^2(x)\,dx.
\end{aligned}
$$

Thanks to structure conditions (1.5) and Young's inequality

$$\begin{aligned}
&\iint_{Q_t} \mathcal{A}(x,\tau,u,Du)\, D\big[(u^m-k^m)_-\zeta^2\big]\,dxd\tau \\
&\qquad = -\, m \iint_{Q_t\cap\{u<k\}} u^{m-1}\mathcal{A}(x,\tau,u,Du)\, Du\,\zeta^2 dxd\tau \\
&\qquad +2 \iint_{Q_t} (u^m-k^m)_-\, \mathcal{A}(x,\tau,u,Du)\,\zeta\, D\zeta\, dxd\tau \\
&\le -C_0 \iint_{Q_t} |D(u^m-k^m)_-|^2\zeta^2 dxd\tau + 2C_1 \iint_{Q_t} (u^m-k^m)_-\, |D(u^m-k^m)_-|\,\zeta\,|D\zeta| dxd\tau \\
&\le -\frac{C_0}{2} \iint_{Q_t} |D(u^m-k^m)_-|^2\zeta^2 dxd\tau + c \iint_{Q_t} (u^m-k^m)_-^2\, |D\zeta|^2 dxd\tau.
\end{aligned}$$

Combining these estimates and taking the supremum over $t\in(t_1,t_2]$, we get (3.3). □

One of the main points of our proof takes advantage of the following local higher integrability estimate for Du.

Proposition 3.4. *There exist two positive constant ℓ and C, that can be determined only in terms of the data and r, such that for every cylinder*

$$Q_{4R,\theta}(x_0,t_0) = B_{4R}(x_0)\times\big(t_0-\theta(4R)^2, t_0+\theta(4R)^2\big)\subset\Omega_T,$$

it holds

$$\begin{aligned}
&R^\ell\theta^{\frac{m\ell}{m-1}} ⨍\!\!⨍_{Q_{R,\theta}(x_0,t_0)} |Du^m|^{2+\ell} dxd\tau \\
&\quad \le C\max\left\{1, \left(\theta^{\frac{r}{m-1}} ⨍\!\!⨍_{Q_{4R,\theta}(x_0,t_0)} u^r\,dxd\tau\right)^{\frac{\ell\lambda_r}{m+1}}\right\} ⨍\!\!⨍_{Q_{2R,\theta}(x_0,t_0)} |Du^m|^2 dxd\tau.
\end{aligned} \tag{3.4}$$

The estimate (3.4) can be easily deduced by [39]. In fact, in the proof of [39, Theorem 1] (see step 3) the inequality

$$⨍\!\!⨍_{Q_1} |Du^m|^{2+\varepsilon} dz \le c\,k_0^{\varepsilon m} ⨍\!\!⨍_{Q_2} |Du^m|^2 dz$$

is established for

$$k_0 = \max\left\{1, \left(⨍\!\!⨍_{Q_4} u^r dz\right)^{\frac{m+1}{m\lambda_r}}\right\}.$$

Changing variables by putting

$$v = \theta^{\frac{1}{m-1}}\, u(x_0+Rx, t_0+\theta R^2 t),$$

one directly gets (3.4).

We will use the following expansion of positivity property taken from [33] (see also [34]).

Proposition 3.5. *For* $(y, s) \in \Omega_T$, $M > 0$ *and* α, $\varepsilon \in (0,1)$ *suppose that*

$$|\{u(\cdot, t) \geq M\} \cap B_\rho(y)| \geq \alpha |B_\rho(y)|,$$

for all times

$$s - \varepsilon M^{1-m} \rho^2 < t \leq s;$$

suppose moreover that

$$B_{16\rho} \times (s - \varepsilon M^{1-m}\rho^2, s] \subset \Omega_T.$$

Then there exists $\sigma \in (0,1)$ *depending upon the data,* α *and* ε*, and independent of* M*, such that*

$$u(x,t) \geq \sigma M \qquad\qquad \forall x \in B_{2\rho}(y)$$

for all times

$$s - \frac{\varepsilon}{2} M^{1-m}\rho^2 < t \leq s.$$

3.2 Comparison principle

Our proof of the Harnack estimate will make use of a comparison principle. The idea of using suitable comparison principles to show Harnack type estimates for quasi-linear parabolic equations is not new (see [19] and also [37]).
Let us introduce the boundary value problem associated to the equation (1.4) as

$$\begin{cases} u_t = \operatorname{div}\mathcal{A}(x,t,u,Du) & \text{in } \Omega \times (0,T)\,, \\ u \; = f & \text{on } \partial\Omega \times (0,T)\,, \\ u \; = u_0 & \text{in } \Omega \times \{t = 0\}\,. \end{cases} \tag{3.5}$$

We can then state the comparison principle that we are going to use.

Proposition 3.6. *Assume (1.5) and (3.1) hold, and let* u, v *be two weak solutions to the boundary value problem (3.5), with initial data* f_u, u_0 *and* f_v, v_0 *respectively. If* $f_u \leq f_v$ *on* $\partial\Omega \times (0,T)$ *and* $u_0 \leq v_0$ *in* Ω*, then* $u \leq v$ *in* Ω_T*. In particular, for every data* f *and* u_0 *there can be at most one solution of (3.5).*

We will obtain the above result as an easy consequence of the following comparison principle, given in [34, Chapter 7, Proposition 5.1 and Corollary 5.2] (one can find a proof for the $p-$laplacian in [26]). Their result considers the case of signed solutions; however, since we are only interested to the case of non-negative solutions, we restrict their result to this case. Notice that they work in a slightly different context.

Proposition 3.7. *Let us consider the boundary value problem*

$$\begin{cases} u_t &= \mathrm{div}\widehat{\mathcal{A}}(x,t,u,u^{m-1}Du) & \text{in } \Omega\times(0,T)\,, \\ u^m &= \hat{f} & \text{on } \partial\Omega\times(0,T)\,, \\ u &= u_0 & \text{in } \Omega\times\{t=0\}\,, \end{cases} \tag{3.6}$$

where

$$\begin{cases} \widehat{\mathcal{A}}(x,t,z,\xi)\cdot\xi \geq \widehat{C}_0|\xi|^2\,, \\ \big|\widehat{\mathcal{A}}(x,t,z,\xi)\big| \leq \widehat{C}_1|\xi|\,, \\ \Big(\widehat{\mathcal{A}}(x,t,z,\xi_1-\widehat{\mathcal{A}}(x,y,z,\xi_2)\Big)\cdot(\xi_1-\xi_2)\geq 0\,, \\ \big|\widehat{\mathcal{A}}(x,t,z_1,\xi)-\widehat{\mathcal{A}}(x,t,z_2,\xi)\big| \leq \widehat{L}\,\big|z_1^m-z_2^m\big|\big(1+|\xi|\big)\,. \end{cases} \tag{3.7}$$

If u and v are two weak solutions with initial data $u_0\leq v_0$ and $\hat{f}_u\leq\hat{f}_v$, then $u\leq v$ in Ω_T. In particular, the uniqueness for given boundary data follows.

Proof. Following the lines of [34], we only give here the proof of the uniqueness of solutions (if any) to the boundary value problem (3.6), since the generalization to get the comparison principle is standard.

Assume then that u and v are two different solutions to (3.6) with $u_0=v_0$ and $\hat{f}_u=\hat{f}_v$, and introduce the function

$$\psi(x,t)=u(x,t)^m-v(x,t)^m\,.$$

Let now H_ε be the Lipschitz approximation of the Heaviside function given by

$$H_\varepsilon(s):=\begin{cases} 0 & \text{for } s<0\,, \\ \frac{s}{\varepsilon} & \text{for } 0\leq s\leq\varepsilon\,, \\ 1 & \text{for } s>\varepsilon\,, \end{cases}$$

and finally set $\varphi=H_\varepsilon\circ\psi$. Using φ as a test function in (3.6) for u and for v, one obtains

$$\begin{aligned}&\int_\Omega\Big(u(\cdot,t)-v(\cdot,t)\Big)H_\varepsilon\circ\varphi-\int_\Omega\Big(u(\cdot,0)-v(\cdot,0)\Big)H_\varepsilon\circ\varphi\\ &\quad=\int_0^t\int_\Omega H_\varepsilon'(\psi)\Big(\widehat{\mathcal{A}}(x,t,v,v^{m-1}Dv)-\widehat{\mathcal{A}}(x,t,u,u^{m-1}Du)\Big)\cdot D\psi\end{aligned}$$

for a generic $t \in (0,T)$. Recalling that $u_0 = v_0$, the left-hand side equals

$$\int_\Omega \Big(u(\cdot,t) - v(\cdot,t)\Big) H_\varepsilon \circ \varphi \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \big(u(\cdot,t) - v(\cdot,t)\big)_+ .$$

Concerning the right-hand side, we can rewrite is as

$$\int_0^t \int_\Omega H'_\varepsilon(\psi)\Big(\widehat{\mathcal{A}}(x,t,v,v^{m-1}Dv) - \widehat{\mathcal{A}}(x,t,u,v^{m-1}Dv)\Big)\cdot D\psi \\ + \int_0^t \int_\Omega H'_\varepsilon(\psi)\Big(\widehat{\mathcal{A}}(x,t,u,v^{m-1}Dv) - \widehat{\mathcal{A}}(x,t,u,u^{m-1}Du)\Big)\cdot D\psi$$

Since $D\psi = m\big(u^{m-1}Du - v^{m-1}Dv\big)$, the second term is negative by (3.7), while the first term can be bounded by

$$\begin{aligned} \int_0^t \int_{\Omega\cap\{\psi\in(0,\varepsilon)\}} & H'_\varepsilon(\psi)\widehat{L}\big(u^m - v^m\big)\big(1+|v^{m-1}Dv|\big)\big|D\psi\big| \\ &\le \frac{\widehat{L}}{\varepsilon}\int_0^t \int_{\Omega\cap\{\psi\in(0,\varepsilon)\}} \psi\big(1+|v^{m-1}Dv|\big)\big|D\psi\big| \\ &\le \widehat{L}\int_0^t \int_{\Omega\cap\{\psi\in(0,\varepsilon)\}} \big(1+|v^{m-1}Dv|\big)\big|D\psi\big| \underset{\varepsilon\to 0}{\longrightarrow} 0 . \end{aligned}$$

Summarizing, we have obtained that

$$\int_\Omega \big(u(\cdot,t) - v(\cdot,t)\big)_+ = 0 ,$$

which means that $u \le v$ at time t. The proof is concluded recalling that t is generic, and exchanging the role of u and v. □

We are now ready to derive Proposition 3.6 from Proposition 3.7.

Proof of Proposition 3.6. We start defining

$$\widehat{\mathcal{A}}(x,t,z,\xi) := \mathcal{A}(x,t,z,z^{1-m}\xi) , \tag{3.8}$$

so that the solutions to the boundary value problem (3.5) coincide with those to (3.6). Therefore, the claim of Proposition 3.6 directly follows from Proposition 3.7 as soon as one verifies that (1.5) and (3.1) imply (3.7), but this is an immediate check from (3.8). □

3.3 Proof of the main result

The aim of this section is to present the proof of Theorem 3.1, which will be achieved by means of some technical results.

For the sake of simplicity, we will consider $(y, s) = (0, 0)$ and we will denote by $\Omega'_T = \Omega' \times (-s, T')$ the translated cylinder.
Recall that

$$\delta = \left(\gamma \fint_{B_\rho} u(x,0)\, dx \right)^{1-m} \rho^2, \qquad \eta = \left(\frac{\fint_{B_\rho} u(x,0)\, dx}{\left(\fint_{B_\rho} u^r(x,0)\, dx \right)^{\frac{1}{r}}} \right)^{\frac{2r}{\lambda_r}} \tag{3.9}$$

where $\lambda_r = N(m-1) + 2r > 0$, with $r \geq 1$.
Following the approach of [37], we start finding an upper bound for the supremum of u.

Proposition 3.8. *Assume $B_\rho \times [0, \delta] \subset \Omega'_T$. Then there exists a positive constant $\tilde{c}$, depending upon data and r, such that*

$$\sup_{B_{\frac{\rho}{2}} \times [\frac{\delta}{2}, \delta]} u \leq \tilde{c} \, \frac{\gamma^{\frac{N(m-1)}{\lambda_r}}}{\eta} \fint_{B_\rho} u(x,0)\, dx.$$

Proof. Thanks to the L^r-L^∞ estimate given in Proposition 3.1, there exists a positive constant c_r, depending upon the data and r, such that

$$\sup_{B_{\frac{\rho}{2}} \times [\frac{\delta}{2}, \delta]} u \leq c_r \left[\left(\frac{2}{\delta} \right)^{\frac{N}{\lambda_r}} \left(\int_{B_\rho} u^r(x,0)\, dx \right)^{\frac{2}{\lambda_r}} + \left(\frac{2\delta}{\rho^2} \right)^{\frac{1}{1-m}} \right].$$

By the expressions of δ and η in (3.9), we get

$$\begin{aligned}
\sup_{B_{\frac{\rho}{2}} \times [\frac{\delta}{2}, \delta]} u \leq & c_r \left[\left(\gamma \fint_{B_\rho} u(x,0)\, dx \right)^{\frac{N(m-1)}{\lambda_r}} \rho^{\frac{-2N}{\lambda_r}} \left(\int_{B_\rho} u^r(x,0)\, dx \right)^{\frac{2}{\lambda_r}} + \gamma \fint_{B_\rho} u(x,0)\, dx \right] \\
\leq & \tilde{c} \left[\left(\gamma \fint_{B_\rho} u(x,0)\, dx \right)^{\frac{N(m-1)}{\lambda_r}} \left(\fint_{B_\rho} u(x,0)\, dx \right)^{\frac{2r}{\lambda_r}} \eta^{-1} + \gamma \fint_{B_\rho} u(x,0)\, dx \right] \\
= & \tilde{c} \left(\frac{\gamma^{\frac{N(m-1)}{\lambda_r}}}{\eta} + \gamma \right) \fint_{B_\rho} u(x,0)\, dx \leq \tilde{c} \, \frac{\gamma^{\frac{N(m-1)}{\lambda_r}}}{\eta} \fint_{B_\rho} u(x,0)\, dx.
\end{aligned}$$

□

Suppose now that

$$B_{16\rho} \times [0, \delta] \subset \Omega'_T \tag{3.10}$$

and introduce an auxiliary function

$$v \in C\left(0, T'; L^2(B_{16\rho})\right) \qquad \text{with } |v|^m \in L^2\left(0, T'; H_0^1(B_{16\rho})\right),$$

solution of the following Cauchy-Dirichlet problem

$$\begin{cases} v_t = \mathrm{div}\mathcal{A}(x,t,u,Du) & \text{in } B_{16\rho} \times (0, T') \\ v = 0 & \text{on } \partial B_{16\rho} \times (0, T') \\ v = u\chi_{B_\rho} & \text{in } B_{16\rho} \times \{t = 0\}. \end{cases}$$

Let us notice that the function v just defined exists and is unique (see [49]).

As already observed, by the comparison principle (Proposition 3.6)

$$u \geq v \text{ in } B_{16\rho} \times (0, T'].$$

Then, once we find a lower estimate for v, the same bound holds also for u. Observe that the definitions of δ and η do not change if we put v instead of u. Moreover, since v vanishes outside $B_{16\rho}$, the previous proposition holds true even for v.

Introducing the notation

$$\kappa := \gamma^{\frac{N(m-1)}{\lambda_r}} \fint_{B_\rho} u(x, 0)\, dx, \tag{3.11}$$

by the observations just done, and keeping in mind (3.10), we have

$$\sup_{B_{8\rho} \times [\frac{\delta}{2}, \delta]} v \leq \tilde{c}\, \frac{\kappa}{\eta}\,. \tag{3.12}$$

Proposition 3.9. *There exist a constant $\gamma \in (0,1)$, depending upon the data, and two constants $\nu, \hat{c} \in (0,1)$, depending upon data and r, such that*

$$|B_{2\rho} \cap \{v(\cdot, \tau) \geq \nu\kappa\}| \geq \hat{c}\, \eta\, |B_{2\rho}|, \tag{3.13}$$

for all times $\tau \in \left[\dfrac{\delta}{2}, \delta\right]$.

Proof. By Proposition 3.2 with $s = 0$ and $t = \delta$, there exists a positive constant c, depending upon the data, such that for every cylinder $B_{2\rho} \times [0, \delta] \subset E_T$

$$\sup_{0<\tau<\delta} \fint_{B_\rho} v(x,\tau)\, dx \leq c \inf_{0<\tau<\delta} \fint_{B_{2\rho}} v(x,\tau)\, dx + c \left(\frac{\delta}{\rho^2}\right)^{\frac{1}{1-m}},$$

which, by (3.9), yields

$$
\fint_{B_\rho} v(x,0) \le \sup_{0<\tau<\delta} \fint_{B_\rho} v(x,\tau)\,dx \le c \inf_{0<\tau<\delta} \fint_{B_{2\rho}} v(x,\tau)\,dx + c\,\gamma \fint_{B_\rho} v(x,0)\,dx
$$
$$
\le c \fint_{B_{2\rho}} v(x,\tau)\,dx + c\,\gamma \fint_{B_{2\rho}} v(x,0)\,dx
$$

for all times $\tau \in [0,\delta]$.

By (3.11), choosing $\gamma \in (0,1)$ such that $c\,\gamma \le \frac{1}{2}$, and thanks to the upper estimate (3.12), we deduce

$$
\begin{aligned}
\kappa\gamma^{\frac{N(1-m)}{\lambda_r}} &= \fint_{B_\rho} v(x,0) \le 2c \fint_{B_{2\rho}} v(x,\tau)\,dx \\
&= \frac{2c}{|B_{2\rho}|}\left(\int_{B_{2\rho}\cap\{v(\cdot,\tau)<\nu\kappa\}} v(x,\tau)\,dx + \int_{B_{2\rho}\cap\{v(\cdot,\tau)\ge\nu\kappa\}} v(x,\tau)\,dx \right) \\
&\le 2c\nu\kappa + 2c\tilde{c}\,\frac{\kappa}{\eta}\,\frac{|B_{2\rho}\cap\{v(\cdot,\tau)\ge\nu\kappa\}|}{|B_{2\rho}|}
\end{aligned}
$$

with $\nu \in (0,1)$ to be chosen and for every $\tau \in \left[\frac{\delta}{2},\delta\right]$. Therefore

$$
|B_{2\rho}\cap\{v(\cdot,\tau)\ge\nu\kappa\}| \ge \frac{\left(\gamma^{\frac{N(1-m)}{\lambda_r}} - 2c\nu\right)}{2c\tilde{c}}\,\eta\,|B_{2\rho}|, \qquad \forall\,\tau \in \left[\frac{\delta}{2},\delta\right],
$$

and the proof is concluded by taking ν and γ small enough. □

Lemma 3.1. *For all times $\tau \in \left[\frac{\delta}{2},\delta\right]$, there exist $x_\tau \in B_{2\rho}$ and $\varepsilon_\tau \in (0,1)$ such that*

$$
\left| B_{r_\tau}(x_\tau) \cap \left\{ v(\cdot,\tau) \ge \frac{\nu\kappa}{2} \right\} \right| > \frac{1}{2}\,|B_{r_\tau}(x_\tau)|\,, \tag{3.14}
$$

where

$$
r_\tau = 2\,\varepsilon_\tau\rho = c\,\frac{\nu\kappa\eta^2}{\displaystyle\fint_{B_{2\rho}} |Dv(\cdot,\tau)|} \tag{3.15}
$$

and c is a positive constant depending only upon the data.

Proof. For any $\tau \in \left[\frac{\delta}{2},\delta\right]$, we want to apply the measure-theoretical Lemma 1.5 to the function $\frac{v}{\nu\kappa}$, choosing $\delta = \lambda = \frac{1}{2}$. To do so, we need to select also α, $\beta > 0$ such that (1.2) holds true. The right estimate in (1.2) coincides with (3.13) as soon as one choose

$\beta = \hat{c}\eta$, so we have to concentrate on the left one. Thanks to (3.12) we know that $v \le \dfrac{\tilde{c}\kappa}{\eta}$, thus

$$\|w\|_{L^1(B_{2\rho})} = \left\| \frac{v}{\nu\kappa} \right\|_{L^1(B_{2\rho})} \le \frac{(2\rho)^N \tilde{c}}{\nu\eta}\,.$$

On the other hand,

$$\|Dw\|_{L^1(B_{2\rho})} = \frac{1}{\nu\kappa}\int_{B_{2\rho}} |Dv| = \frac{(2\rho)^N}{\nu\kappa} \fint_{B_{2\rho}} |Dv|\,.$$

Summarizing, we can say that

$$\frac{\|w\|_{W^{1,1}(B_{2\rho})}}{\rho^{N-1}} \le \frac{\rho}{\nu}\left(\frac{2^N}{\kappa}\fint_{B_{2\rho}} |Dv| + \frac{\tilde{c}}{\eta} \right) \le c\,\frac{\rho}{\nu\kappa}\fint_{B_{2\rho}} |Dv|\,,$$

where $c > 0$ is a constant depending on the data. Concerning the last inequality, a geometric estimate shows that if it is not true, then one directly deduce (3.14).

As a consequence, the left estimate in (1.2) holds true with the choice

$$\alpha = c\,\frac{\rho}{\nu\kappa}\fint_{B_{2\rho}} |Dv|\,.$$

The constant ε of Lemma 1.5 can be explicitly evaluated as $\varepsilon = c\,\dfrac{\beta^2}{\alpha}$: this can be obtained by inspecting the proof given in [34], as pointed out in [34, Remark 3.1]. In our case, this gives

$$r_\tau = 2\varepsilon_\tau\rho = c\rho\,\frac{\beta^2}{\alpha} = c\,\frac{\eta^2\nu\kappa}{\fint_{B_{2\rho}} |Dv|}\,,$$

which is exactly (3.15). □

Now, we prove the following "time propagation of positivity" property.

Proposition 3.10. *For $\alpha \in \mathbb{N}$ sufficiently large, depending upon the data, and for every $\tau \in \left[\dfrac{\delta}{2}, \delta\right]$, it holds*

$$\left| \left\{ v(\cdot,t) \ge \frac{\nu\kappa}{2^{\alpha+1}} \right\} \cap B_{r_\tau}(x_\tau) \right| \ge \frac{1}{4}\,|B_{r_\tau}(x_\tau)|$$

for all times

$$\tau < t \le \tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_\tau^2.$$

Proof. Let us write the energy estimates (3.3) for the level $\dfrac{\nu\kappa}{2}$ over the cylinder

$$Q_\tau := B_{r_\tau}(x_\tau) \times \left(\tau, \tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_\tau^2\right], \tag{3.16}$$

for $\alpha \in \mathbb{N}$ to be chosen later. Take $\zeta = \zeta(x)$ to be a non-negative, piecewise smooth cutoff function in $B_{r_\tau}(x_\tau)$, which equals 1 on $B_{(1-a)r_\tau}(x_\tau)$ and such that $|D\zeta| \le c(ar_\tau)^{-1}$, with $a \in (0,1)$.

Thanks to (3.14), we have

$$\begin{aligned}
\left(\frac{\nu\kappa}{2}\right)^{m-1} & \int_{B_{r_\tau}(x_\tau)} \left(v - \frac{\nu\kappa}{2}\right)_-^2 \zeta^2(x)\, dx \\
&\le \frac{1}{2}\left(\frac{\nu\kappa}{2}\right)^{m+1} |B_{r_\tau}(x_\tau)| + c\left(\frac{\nu\kappa}{2}\right)^{2m} \frac{1}{a^2 r_\tau^2}\, |B_{r_\tau}(x_\tau)| \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_\tau^2 \\
&= \left(\frac{1}{2} + c\,\frac{2^{\alpha(m-1)}}{a^2}\right)\left(\frac{\nu\kappa}{2}\right)^{m+1} |B_{r_\tau}(x_\tau)|,
\end{aligned} \tag{3.17}$$

for every $t \in \left(\tau, \tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_\tau^2\right]$.

Estimate the left-hand side by integrating on the smaller set $B_{(1-a)r_\tau}(x_\tau) \cap \left\{v < \dfrac{\nu\kappa}{2^{\alpha+1}}\right\}$

$$\begin{aligned}
\left(\frac{\nu\kappa}{2}\right)^{m-1} & \int_{B_{r_\tau}(x_\tau)} \left(v - \frac{\nu\kappa}{2}\right)_-^2 \zeta^2(x)\, dx \\
&\ge \left(\frac{\nu\kappa}{2}\right)^{m-1} \int_{B_{(1-a)r_\tau}(x_\tau)\cap\{v<\frac{\nu\kappa}{2^{\alpha+1}}\}} \left(v - \frac{\nu\kappa}{2}\right)_-^2 \zeta^2(x)\, dx \\
&\ge \left(1 - \frac{1}{2^\alpha}\right)^2 \left(\frac{\nu\kappa}{2}\right)^{m+1} \left| B_{(1-a)r_\tau}(x_\tau) \cap \left\{v < \frac{\nu\kappa}{2^{\alpha+1}}\right\}\right|.
\end{aligned}$$

By the last estimate and (3.17), we get

$$\left| B_{(1-a)r_\tau}(x_\tau) \cap \left\{v < \frac{\nu\kappa}{2^{\alpha+1}}\right\}\right| \le \left(1 - \frac{1}{2^\alpha}\right)^{-2} \left(\frac{1}{2} + c\,\frac{2^{\alpha(m-1)}}{a^2}\right) |B_{r_\tau}(x_\tau)|,$$

and finally

$$\begin{aligned}
\left| B_{r_\tau}(x_\tau) \cap \left\{v < \frac{\nu\kappa}{2^{\alpha+1}}\right\}\right| &\le \left| B_{(1-a)r_\tau}(x_\tau) \cap \left\{v < \frac{\nu\kappa}{2^{\alpha+1}}\right\}\right| + Na\,|B_{r_\tau}(x_\tau)| \\
&\le \left[\left(1 - \frac{1}{2^\alpha}\right)^{-2} \left(\frac{1}{2} + c\,\frac{2^{\alpha(m-1)}}{a^2}\right) + Na\right] |B_{r_\tau}(x_\tau)|.
\end{aligned}$$

By taking a small and α large, we conclude. □

Let us introduce the following quantities

$$\mu = (3-m)\ell + 6 + \frac{r\ell\lambda_r}{m+1}, \qquad\qquad b = \mu\frac{2}{\ell}. \tag{3.18}$$

Proposition 3.11. *There exists a number $\bar{\varepsilon} \in \left(0, \frac{1}{2^6}\right)$, depending upon data and r, such that for every $\tau \in \left[\frac{3}{4}\delta, \left(\frac{3}{4} + \frac{1}{2^6}\right)\delta\right]$ there exists $s_\tau \in \left[\left(\frac{3}{4} - \frac{1}{2^6}\right)\delta, \tau - \bar{\varepsilon}\eta^b\delta\right]$, with*

$$s_\tau \le \tau \le s_\tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2. \tag{3.19}$$

Proof. Suppose, by contradiction, that for every $\varepsilon \in \left(0, \frac{1}{2^6}\right)$, there exists a number $\bar{\tau} \in \left[\frac{3}{4}\delta, \left(\frac{3}{4} + \frac{1}{2^6}\right)\delta\right]$ such that, for every

$$s \in \left[\left(\frac{3}{4} - \frac{1}{2^6}\right)\delta, \bar{\tau} - \varepsilon\eta^b\delta\right] =: [t_1, t_2],$$

$\bar{\tau}$ does not belong to the time interval (3.19). By construction, this is equivalent to

$$\bar{\tau} > s + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_s^2.$$

By the previous inequality, the definition of r_s in (3.15) and the Hölder inequality, one gets

$$\bar{\tau} - s > c\left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} \frac{(\nu\kappa\eta^2)^2}{\left(\fint_{B_{2\rho}} |Dv(x,s)|\,dx\right)^2} \ge \frac{c}{2^{(\alpha+1)(1-m)}} \frac{(\nu\kappa)^{3-m}\eta^4}{\left(\fint_{B_{2\rho}} |Dv(x,s)|^{2+\ell}\,dx\right)^{\frac{2}{2+\ell}}},$$

being ℓ the constant found in Proposition 3.4.
Therefore

$$\fint_{B_{2\rho}} |Dv(x,s)|^{2+\ell}\,dx > c\left(\frac{(\nu\kappa)^{3-m}\eta^4}{2^{(\alpha+1)(1-m)}}\,\frac{1}{\bar{\tau}-s}\right)^{\frac{2+\ell}{2}};$$

integrating on $s \in [t_1, t_2]$ and taking into account that $\bar{\tau} - t_1 \ge \frac{\delta}{2^6}$, we obtain

$$\int_{t_1}^{t_2}\fint_{B_{2\rho}} |Dv(x,s)|^{2+\ell}\,dxds > c\left(\frac{(\nu\kappa)^{3-m}\eta^4}{2^{(\alpha+1)(1-m)}}\right)^{\frac{2+\ell}{2}} \frac{2}{\ell}\left[(\varepsilon\eta^b)^{-\frac{\ell}{2}} - 2^{3\ell}\right]\delta^{-\frac{\ell}{2}}. \tag{3.20}$$

Now, by (3.9), (3.11), and the definition of λ_r in (3.2), we notice that δ can be written as

$$\delta = \left(\gamma^{1+\frac{N(1-m)}{\lambda_r}}\kappa\right)^{1-m}\rho^2 = \left(\gamma^{\frac{2r}{\lambda_r}}\kappa\right)^{1-m}\rho^2; \tag{3.21}$$

thus, putting (3.21) into (3.20), and thanks to (3.18), we have

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv(x,s)|^{2+\ell}\, dxds > c\,\kappa^{\frac{(3-m)(2+\ell)}{2}}\, \eta^{2(2+\ell)} \left[\varepsilon^{-\frac{\ell}{2}}\eta^{-\mu} - 2^{3\ell}\right] \left(\gamma^{\frac{2r}{\lambda_r}}\kappa\right)^{-\frac{(1-m)\ell}{2}} \rho^{-\ell}$$
$$\geq c\,\kappa^{\ell+3-m}\, \eta^{2(2+\ell)} \left[\varepsilon^{-\frac{\ell}{2}}\eta^{-\mu} - 2^{3\ell}\right] \rho^{-\ell}. \tag{3.22}$$

By the higher integrability result (3.4) with $(x_0, t_0) = \left(0, \frac{3}{4}\,\delta\right)$, $R = 2\rho$, $\theta = \dfrac{\delta}{4(8\rho)^2}$, and

$$s_1 := t_0 - \theta(2R)^2 = \left(\frac{3}{4} - \frac{1}{2^4}\right)\delta, \qquad s_2 := t_0 + \theta(2R)^2 = \left(\frac{3}{4} + \frac{1}{2^4}\right)\delta$$

we obtain

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv^m|^{2+\ell} dxd\tau$$
$$\leq c\frac{\theta^{\frac{m\ell}{1-m}}}{(2\rho)^\ell} \max\left\{1, \left(\theta^{\frac{r}{m-1}} \fint_{\frac{\delta}{2}}^{\delta} \fint_{B_{8\rho}} v^r dxd\tau\right)^{\frac{\ell\lambda_r}{m+1}}\right\} \int_{s_1}^{s_2} \fint_{B_{4\rho}} |Dv^m|^2 dxd\tau.$$

Notice that the interval $[t_1, t_2]$ is included in $[t_0 - \theta R^2, t_0 + \theta R^2] = \left[t_1, \left(\frac{3}{4} + \frac{1}{2^6}\right)\delta\right]$.
On the other hand, thanks to (3.12)

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv^m|^{2+\ell} dxd\tau = m^{2+\ell} \int_{t_1}^{t_2} \fint_{B_{2\rho}} v^{(m-1)(2+\ell)} |Dv|^{2+\ell}$$
$$\geq c\, m^{2+\ell} \left(\frac{\kappa}{\eta}\right)^{(m-1)(2+\ell)} \int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv|^{2+\ell}.$$

Therefore, using again (3.12), we get

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv|^{2+\ell}$$
$$\leq c\left(\frac{\kappa}{\eta}\right)^{(1-m)(2+\ell)} \frac{\theta^{\frac{m\ell}{1-m}}}{(2\rho)^\ell} \max\left\{1, \left(\theta^{\frac{r}{m-1}} \frac{\kappa^r}{\eta^r}\right)^{\frac{\ell\lambda_r}{m+1}}\right\} \int_{s_1}^{s_2} \fint_{B_{4\rho}} |Dv^m|^2 dxd\tau.$$

Taking into account (3.21), we rewrite $\theta = \dfrac{(\gamma^{\frac{2r}{\lambda_r}}\kappa)^{1-m}}{2^8}$, and the previous inequality can be rewritten as

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv|^{2+\ell}$$
$$\leq c\,\eta^{(m-1)(2+\ell)}\, \kappa^{2(1-m)+\ell}\, \frac{\gamma^{\frac{2rm\ell}{\lambda_r}}}{\rho^\ell} \max\left\{1, \left(\frac{\gamma^{-\frac{2r^2}{\lambda_r}} 2^{\frac{8r}{1-m}}}{\eta^r}\right)^{\frac{\ell\lambda_r}{m+1}}\right\} \int_{s_1}^{s_2} \fint_{B_{4\rho}} |Dv^m|^2 dxd\tau;$$

since $\dfrac{\gamma^{-\frac{2r^2}{\lambda_r}} 2^{\frac{8r}{1-m}}}{\eta^r} > 1$, we obtain

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv|^{2+\ell} \leq c\, \eta^{(m-1)(2+\ell)-\frac{r\ell\lambda_r}{m+1}}\, \kappa^{2(1-m)+\ell}\, \frac{1}{\rho^\ell} \int_{s_1}^{s_2} \fint_{B_{4\rho}} |Dv^m|^2 dx d\tau. \tag{3.23}$$

Now, we want to estimate the integral on the right-hand side of (3.23): in order to do this, let us write the energy estimates (3.3) over the cylinder $B_{8\rho} \times \left(\frac{\delta}{2}, \delta\right)$ for the level $\tilde{c}\,\frac{\kappa}{\eta}$, being $\zeta = \zeta(x)$ a non-negative, piecewise smooth cutoff function in $B_{8\rho}$ which equals 1 on $B_{4\rho}$, with $|D\zeta| \leq c\,(4\rho)^{-1}$, we obtain

$$\begin{aligned} \int_{\frac{\delta}{2}}^{\delta} \int_{B_{4\rho}} |Dv^m|^2\, dx d\tau &= \int_{\frac{\delta}{2}}^{\delta} \int_{B_{4\rho}} |D(v^m - k^m)_-|^2\, dx d\tau \\ &\leq c \left\{ \left(\frac{\kappa}{\eta}\right)^{m+1} + \left(\frac{\kappa}{\eta}\right)^{2m} \frac{1}{(4\rho)^2} \frac{\delta}{2} \right\} |B_{8\rho}|. \end{aligned}$$

By (3.21) and recalling that $\eta \leq 1$, one gets

$$\int_{\frac{\delta}{2}}^{\delta} \int_{B_{4\rho}} |Dv^m|^2\, dx d\tau \leq c\, \frac{\kappa^{m+1}}{\eta^{2m}}\, |B_{8\rho}|.$$

Thus, noticing that $[s_1, s_2] \subset \left(\frac{\delta}{2}, \delta\right)$ and relying on (3.23), one obtains

$$\int_{t_1}^{t_2} \fint_{B_{2\rho}} |Dv|^{2+\ell} \leq c\, \eta^{\ell(m-1)-2-\frac{r\ell\lambda_r}{m+1}}\, \kappa^{\ell+3-m}\, \frac{1}{\rho^\ell}.$$

Finally, putting this together with (3.22) yields

$$\varepsilon^{-\frac{\ell}{2}} \leq c + 2^{3\ell}\eta^\mu,$$

and again by $\eta \leq 1$, one gets

$$\varepsilon \geq (c + 2^{3\ell})^{-\frac{2}{\ell}},$$

which contradicts our assumption $\varepsilon < 2^{-6}$. □

Corollary 3.1. *There exists a number $\sigma \in (0,1)$, depending upon data and r, such that, for every $\tau \in \left[\frac{3}{4}\delta, \left(\frac{3}{4} + \frac{1}{2^6}\right)\delta\right]$, there exists $s_\tau < \tau$ for which $r_{s_\tau} \geq 2\sigma^{\frac{1}{2}}\eta^{\frac{b}{2}}\rho$ and*

$$\left| \left\{ v(\cdot, t) \geq \frac{\eta^{\frac{b}{1-m}}\nu\kappa}{2^{\alpha+1}} \right\} \cap B_{r_{s_\tau}}(x_{s_\tau}) \right| \geq \frac{1}{4}\, |B_{r_{s_\tau}}(x_{s_\tau})|, \tag{3.24}$$

for all

$$\tau - \sigma\eta^b \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2 < t \leq \tau.$$

Proof. Fix $\tau \in \left[\frac{3}{4}\delta, \left(\frac{3}{4}+\frac{1}{2^6}\right)\delta\right]$ and notice that, by the previous proposition, there exist $\bar{\varepsilon} \in \left(0, \frac{1}{2^6}\right)$ and $s_\tau \in \left[\left(\frac{3}{4}-\frac{1}{2^6}\right)\delta, \tau - \bar{\varepsilon}\eta^b\delta\right]$ such that (3.19) holds.
This implies

$$[\tau - \bar{\varepsilon}\eta^b\delta, \tau] \subseteq \left[s_\tau, s_\tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2\right]. \tag{3.25}$$

Thanks to (3.21), we can write

$$\delta = \left(\gamma^{\frac{2r}{\lambda_r}} \frac{2^{\alpha+1}}{\nu}\right)^{1-m} \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} \rho^2;$$

setting

$$\sigma := \frac{\bar{\varepsilon}}{4}\left(\gamma^{\frac{2r}{\lambda_r}} \frac{2^{\alpha+1}}{\nu}\right)^{1-m}$$

we get

$$\delta = \frac{4}{\bar{\varepsilon}}\sigma \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} \rho^2.$$

Let us notice that

$$\tau \le s_\tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2 \le \tau - \bar{\varepsilon}\eta^b\delta + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2;$$

then

$$r_{s_\tau}^2 \ge \bar{\varepsilon}\eta^b\delta \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{m-1} = 4\sigma\eta^b\rho^2,$$

which implies $r_{s_\tau} \ge 2\sigma^{\frac{1}{2}}\eta^{\frac{b}{2}}\rho$.
The following inclusion holds

$$\left[\tau - \sigma\eta^b \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2, \tau\right] \subseteq [\tau - \bar{\varepsilon}\eta^b\delta, \tau] :$$

indeed

$$\sigma\eta^b \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2 \le \bar{\varepsilon}\eta^b\delta \qquad \Longleftrightarrow \qquad r_{s_\tau}^2 \le 4\rho^2,$$

which holds true by the definition of r_τ (3.15).
By (3.25) and the previous inclusion, it follows

$$\left[\tau - \sigma\eta^b \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2, \tau\right] \subseteq \left[s_\tau, s_\tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2\right].$$

We conclude by Proposition 3.10, which ensures

$$\left|\left\{v(\cdot,t) \ge \frac{\nu\kappa}{2^{\alpha+1}}\right\} \cap B_{r_{s_\tau}}(x_\tau)\right| \ge \frac{1}{4}\left|B_{r_{s_\tau}}(x_\tau)\right| \qquad \forall\, t \in \left(s_\tau, s_\tau + \left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_\tau^2\right].$$

□

3.4 Conclusion

We are now ready to obtain a lower bound for the infimum of v.

Proposition 3.12. *There exist two positive constants c and d, depending only upon data and r, such that*

$$\inf_{B_{4\rho}\times[\frac{3}{4}\delta,(\frac{3}{4}+\frac{1}{2^6})\delta]} v \geq c\,\eta^{d-1}\kappa. \tag{3.26}$$

Proof. For any $\tau \in \left[\frac{3}{4}\delta, \left(\frac{3}{4}+\frac{1}{2^6}\right)\delta\right]$ fixed, there exists $s_\tau < \tau$ as in the previous corollary such that (3.24) holds for every $t \in \left(\tau - \sigma\eta^b\left(\frac{\nu\kappa}{2^{\alpha+1}}\right)^{1-m} r_{s_\tau}^2, \tau\right]$. Then apply Proposition 3.5 to the function v with $y = x_{s_\tau}$, $\rho = r_{s_\tau}$ and $M = \eta^{\frac{b}{1-m}}\frac{\nu\kappa}{2^{\alpha+1}}$. This guarantees the existence of $\xi \in (0,1)$, depending upon the data and σ, such that

$$v(x,t) \geq \xi M \qquad \forall x \in B_{2r_{s_\tau}}(x_{s_\tau}),\ \forall\, t \in \left(\tau - \frac{\sigma}{2}M^{1-m}r_{s_\tau}^2, \tau\right].$$

If we take

$$s \in \left(\tau - \frac{\sigma}{2}M^{1-m}r_{s_\tau}^2 + \sigma(\xi M)^{1-m}(2r_{s_\tau})^2, \tau\right], \tag{3.27}$$

then the following inclusion holds

$$\left(s - \sigma(\xi M)^{1-m}(2r_{s_\tau})^2, s\right] \subseteq \left(\tau - \frac{\sigma}{2}M^{1-m}r_{s_\tau}^2, \tau\right].$$

Therefore, in particular

$$v(x,t) \geq \xi M \qquad \forall x \in B_{2r_{s_\tau}}(x_{s_\tau}),\ \forall\, t \in \left(s - \sigma(\xi M)^{1-m}(2r_{s_\tau})^2, s\right].$$

So we can apply once again the expansion of positivity, obtaining

$$v(x,t) \geq \bar{\xi}\xi M \qquad \forall x \in B_{4r_{s_\tau}}(x_{s_\tau}),\ \forall\, t \in \left(s - \frac{\sigma}{2}(\xi M)^{1-m}(2r_{s_\tau})^2, s\right].$$

Denoting again ξ the smallest between $\bar{\xi}$ and ξ, and considering all the possible s in (3.27), we get

$$v(x,t) \geq \xi^2 M \qquad \forall x \in B_{4r_{s_\tau}}(x_{s_\tau}),\ \forall\, t \in \left(\tau - \frac{\sigma}{2}M^{1-m}r_{s_\tau}^2 + \frac{\sigma}{2}(\xi M)^{1-m}(2r_{s_\tau})^2, \tau\right].$$

After j iterations, one obtains

$$v(x,t) \geq \xi^j M \qquad \forall x \in B_{2^j r_{s_\tau}}(x_{s_\tau})$$

for all times

$$t \in \Big(\tau - \frac{\sigma}{2} M^{1-m} r_{s_\tau}^2 + \frac{\sigma}{2} M^{1-m} r_{s_\tau}^2 \sum_{i=1}^{j-1} (4\xi^{1-m})^i, \tau\Big]. \tag{3.28}$$

Recall that, by the previous corollary, $r_{s_\tau} \geq 2\sigma^{\frac{1}{2}}\eta^{\frac{b}{2}}\rho$. We choose j such that $6\rho \leq 2^{j+1}\sigma^{\frac{1}{2}}\eta^{\frac{b}{2}}\rho \leq 2^j r_{s_\tau}$, which is true if

$$j = \Big[\log_2(3\sigma^{-\frac{1}{2}}\eta^{-\frac{b}{2}})\Big] + 1,$$

where $[\cdot]$ denotes the integer part.

In this way $B_{4\rho} \subset B_{2^j r_{s_\tau}}(x_{s_\tau})$, as $x_{s_\tau} \in B_{2\rho}$, and

$$\xi^j \geq \xi\, \xi^{\log_2(3\sigma^{-\frac{1}{2}}\eta^{-\frac{b}{2}})} = \xi\,(3\sigma^{-\frac{1}{2}}\eta^{-\frac{b}{2}})^{\log_2 \xi} = \xi \left(\frac{\sigma^{\frac{1}{2}}}{3}\right)^{\log_2 \frac{1}{\xi}} \eta^{\frac{b}{2}\log_2 \frac{1}{\xi}}.$$

Thanks to these remarks and substituting the value of M selected before, one gets

$$v(x,t) \geq \xi \left(\frac{\sigma^{\frac{1}{2}}}{3}\right)^{\log_2 \frac{1}{\xi}} \eta^{\frac{b}{2}\log_2 \frac{1}{\xi} + \frac{b}{1-m}} \frac{\nu\kappa}{2^{\alpha+1}} \qquad \forall x \in B_{4\rho},$$

and for all times t as in (3.28).

Setting $d = \frac{b}{2}\log_2 \frac{1}{\xi} + \frac{b}{1-m} + 1$, we conclude the proof. □

We can finally conclude the proof of our main theorem.

Proof of Theorem 3.1. Thanks to the comparison principle and to (3.26), we get

$$\inf_{B_{4\rho}\times[\frac{3}{4}\delta,(\frac{3}{4}+\frac{1}{2^6})\delta]} u \geq \inf_{B_{4\rho}\times[\frac{3}{4}\delta,(\frac{3}{4}+\frac{1}{2^6})\delta]} v \geq c\,\eta^{d-1}\kappa,$$

and by Proposition 3.8

$$\sup_{B_{\frac{\rho}{2}}\times[\frac{\delta}{2},\delta]} u \leq \tilde{c}\,\frac{\kappa}{\eta}.$$

It follows

$$\inf_{B_{4\rho}\times[\frac{3}{4}\delta,(\frac{3}{4}+\frac{1}{2^6})\delta]} u \geq \frac{c}{\tilde{c}}\,\eta^d \sup_{B_{\frac{\rho}{2}}\times[\frac{\delta}{2},\delta]} u.$$

□

Part II

Improved regularity for some Dirichlet problems

Chapter 4

Existence and regularity results for solutions to non-coercive Dirichlet problems

In this part we are interested in the existence and regularity for solutions to the following Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(M(x)\,Du) = -\mathrm{div}\,\big(|u|^{\theta-1}\,u\,E(x)\big) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded set, $N > 2$, M is a symmetric elliptic matrix with measurable and bounded coefficients, E and f are measurable functions satisfying suitable summability properties and $0 < \theta < 1$.

The main difficulty of our problem is due to the non-coercivity of the nonlinear differential operator

$$u \longmapsto -\mathrm{div}\,\big(M(x)\,Du - |u|^{\theta-1}\,u\,E(x)\big)$$

In the case $\theta = 1$, and if $\mu > 0$ is sufficiently large the problem

$$\begin{cases} -\mathrm{div}\,(M(x)\,Du - u\,E(x)) + B(x)Du + \mu u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is coercive and existence and regularity of the weak solution have been studied by Stampacchia in [62, 63], assuming $|E|, |B| \in L^N(\Omega)$.

Nonlinear problems of the same type have been considered in [9, 10].

Recently, Boccardo in [7] dealt with the non-coercive problem

$$\begin{cases} -\mathrm{div}\,(M(x)\,Du) = -\mathrm{div}\,(u\,E(x)) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

establishing existence and regularity results for solutions, depending upon the regularity of the data. Our results can be regarded as an interpolation between the linear case and (4.2), corresponding to $\theta = 0$ and $\theta = 1$, respectively.

Let us state our assumptions and our results.

Throughout this last part we assume $M : \Omega \to \mathbb{R}^{N^2}$ to be a measurable matrix such that

$$\exists\, \alpha > 0 : \; M(x)\, \xi \cdot \xi \geq \alpha |\xi|^2 \qquad \text{for a.e. } x \in \Omega,\, \forall\, \xi \in \mathbb{R}^N, \tag{4.3}$$

$$\exists\, \beta > 0 : \; |M(x)| \leq \beta \qquad \text{for a.e. } x \in \Omega. \tag{4.4}$$

Let $E : \Omega \to \mathbb{R}^N$ be a vector field and $f : \Omega \to \mathbb{R}$ be a measurable function with the following summability properties

$$|E| \in L^q(\Omega), \qquad f \in L^m(\Omega), \tag{4.5}$$

where the values of q and m will be specified later and will be different in our different results.

Let us define the exponents

$$\bar{q} = \bar{q}(\theta) := \frac{2N}{N - \theta(N-2)}, \tag{4.6}$$

$$\tilde{q} = \tilde{q}(\theta, m) := \max\left\{2, \frac{mN}{N - m - \theta(N - 2m)}\right\}. \tag{4.7}$$

Remark 4.1. *Notice that* $2 < \bar{q} < N$ *and if* $\frac{2N}{N+2} \leq m < \frac{N}{2}$ *we can compare our exponents as follows*

$$2 < \bar{q}(\theta) \leq \tilde{q}(\theta, m) < N, \qquad \tilde{q}\left(\theta, \frac{2N}{N+2}\right) = \bar{q}(\theta) \qquad \forall \theta \in (0,1).$$

Moreover, in the case $m = 1$*, the exponent* $\tilde{q}$ *equals 2 if and only if* $\theta \leq \frac{1}{2}$.

Our results can be summarized as follows

- if $m > \frac{N}{2}$ and $q > N$, then $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (Theorems 4.1 and 4.5);
- if $\frac{2N}{N+2} \leq m < \frac{N}{2}$ and $q = \tilde{q}$, then $u \in H_0^1(\Omega) \cap L^{m^{**}}(\Omega)$ (Theorems 4.1 and 4.2);

- if $1 < m < \dfrac{2N}{N+2}$ and $q = \tilde{q}$, then $u \in W_0^{1,m^*}(\Omega)$ (Theorem 4.3);

- if $m = 1$ and $q = \tilde{q}$, then $u \in W_0^{1,p}(\Omega)$, $\forall p < \dfrac{N}{N-1}$ (Theorem 4.4).

Furthermore, the presence of the nonlinear term allows us to obtain existence of weak (or distributional) solutions to (4.1) under weaker assumptions on the summability of E, namely

- if $m \geq \dfrac{2N}{N+2}$ and $q = \bar{q}$, then $u \in H_0^1(\Omega)$ (Theorems 4.1);

- if $1 \leq m < \dfrac{2N}{N+2}$, $q = 2$ and $\dfrac{1}{2} < \theta < \dfrac{N}{2(N-1)}$, then $u \in W_0^{1,p}(\Omega)$, $\forall p < \dfrac{2N(1-\theta)}{N-2\theta}$ (Theorem 4.4).

4.1 Approximate problems

For any $n \in \mathbb{N}$, let us denote by T_n the usual truncation

$$T_n(s) = \begin{cases} s & \text{if } |s| \leq n \\ n\dfrac{s}{|s|} & \text{if } |s| > n \end{cases}$$

and let $G_n(s) = s - T_n(s)$.
Following the approach of [9],[10], for any $n \in \mathbb{N}$, let $u_n \in H_0^1(\Omega)$ be the weak solution of the approximate problem

$$-\mathrm{div}\,(M(x)\,Du_n) = -\mathrm{div}\left(\frac{|u_n|^{\theta-1}\,u_n}{1+\dfrac{|u_n|^\theta}{n}} \, \frac{E(x)}{1+\dfrac{|E|}{n}} \right) + f_n, \tag{4.8}$$

where $f_n = T_n(f)$.
The existence of u_n follows by Schauder fixed point theorem and, thanks to Stampacchia's regularity theorem (see [62]), u_n is bounded.

Throughout this section, we fix $n \in \mathbb{N}$ and call for simplicity w a solution of (4.8). All of the following estimates will not depend on n.

Lemma 4.1. *If $|E| \in L^2(\Omega)$ and $f \in L^1(\Omega)$, then the solution w of (4.8) satisfies*

$$\left(\int_\Omega |\log(1+|w|)|^{2^*}\right)^{\frac{2}{2^*}} \le c\left(\int_\Omega |E^2| + \int_\Omega |f|\right) \tag{4.9}$$

where c is a positive constant depending only upon α and the Sobolev constant S.

Proof. Take $\dfrac{|w|}{1+|w|}$ as test function in the weak formulation of (4.8) and use the fact that

$$1+\frac{|w|^\theta}{n} \ge 1, \qquad 1+\frac{|E|}{n} \ge 1, \qquad |f_n| \le |f|, \tag{4.10}$$

to obtain

$$\int_\Omega M(x)\,\frac{Dw\cdot Dw}{(1+|w|)^2} \le \int_\Omega |w|^\theta |E|\,\frac{|Dw|}{(1+|w|)^2} + \int_\Omega |f|\,\frac{|w|}{1+|w|}\,.$$

By (4.3), the fact that $\dfrac{|w|^\theta}{1+|w|} \le \dfrac{|w|^\theta}{(1+|w|)^\theta} \le 1$, Young inequality and $\dfrac{|w|}{1+|w|} < 1$, we get

$$\alpha\int_\Omega \frac{|Dw|^2}{(1+|w|)^2} \le \frac{1}{2\alpha}\int_\Omega |E|^2 + \frac{\alpha}{2}\int_\Omega \frac{|Dw|^2}{(1+|w|)^2} + \int_\Omega |f|\,.$$

Now, observe that

$$\frac{|Dw|}{1+|w|} = |D\log(1+|w|)|$$

and apply Sobolev inequality to find

$$\frac{\alpha}{2S^2}\left(\int_\Omega |\log(1+|w|)|^{2^*}\right)^{\frac{2}{2^*}} \le \frac{\alpha}{2}\int_\Omega |D\log(1+|w|)|^2 \le \frac{1}{2\alpha}\int_\Omega |E|^2 + \int_\Omega |f|\,.$$

□

Let us now introduce the following notation

$$A_k := \{|w| \ge k\} \cap \Omega.$$

Remark 4.2. *For any $\varepsilon > 0$, it is possible to choose k_ε such that*

$$|A_k|^{\frac{2}{2^*}} \le \varepsilon, \qquad \text{for every } k > k_\varepsilon. \tag{4.11}$$

Indeed, thanks to (4.9), we obtain

$$|\log(1+k)|^2\,|A_k|^{\frac{2}{2^*}} \le \left(\int_{A_k} |\log(1+|w|)|^{2^*}\right)^{\frac{2}{2^*}}$$
$$\le \left(\int_\Omega |\log(1+|w|)|^{2^*}\right)^{\frac{2}{2^*}} \le c\left(\int_\Omega |E^2| + \int_\Omega |f|\right).$$

4.1.1 Finite energy solutions

Here we consider the case of finite energy solutions that corresponds to taking $m \geq \dfrac{2N}{N+2}$. Let us recall that for any $\theta \in (0,1)$

$$\tilde{q}(\theta, m) \geq \bar{q}(\theta) \quad \text{if } \frac{2N}{N+2} \leq m < \frac{N}{2}, \qquad \tilde{q}\left(\theta, \frac{2N}{N+2}\right) = \bar{q}(\theta).$$

Lemma 4.2. *If $|E| \in L^2(\Omega)$ and $f \in L^1(\Omega)$, then, for every positive number k, the truncation $T_k(w)$ is bounded in $H_0^1(\Omega)$. More precisely,*

$$\int_\Omega |DT_k(w)|^2 \leq c\left(k^{2\theta}\int_\Omega |E|^2 + k\int_\Omega |f|\right), \tag{4.12}$$

where c is a positive constant depending only upon α.

Proof. Taking $T_k(w)$ as test function in the weak formulation of (4.8) and using again (4.10), we get

$$\int_\Omega M(x)\, Dw \cdot DT_k(w) \leq \int_\Omega |w|^\theta |E|\, |DT_k(w)| + \int_\Omega |f|\, |T_k(w)|.$$

Then, by applying (4.3), Young inequality and the fact that $|T_k(w)| \leq k$, one obtains

$$\alpha\int_\Omega |DT_k(w)|^2 \leq \frac{1}{2\alpha}\int_{\{|w|<k\}\cap\Omega} |w|^{2\theta}|E|^2 + \frac{\alpha}{2}\int_\Omega |DT_k(w)|^2 + k\int_\Omega |f|,$$

which implies (4.12). □

Lemma 4.3. *Assume $|E| \in L^{\bar{q}}(\Omega)$, with $\bar{q}$ as in (4.6), and $f \in L^m(\Omega)$, with $m \geq \dfrac{2N}{N+2}$. Then, for every positive number k, the truncation $G_k(w)$ is bounded in $H_0^1(\Omega)$. In particular,*

$$\int_\Omega |DG_k(w)|^2 \leq c\left(\|E\|_{\bar{q},A_k}^{\frac{2}{1-\theta}} + k^{2\theta}\int_{A_k} |E|^2 + \|f\|^2_{\frac{2N}{N+2},A_k}\right), \tag{4.13}$$

where c is a positive constant depending upon α and the Sobolev constant S.

Proof. The use of $G_k(w)$ as test function in the weak formulation of (4.8) and (4.10) yields

$$\int_\Omega M(x)\, Dw \cdot DG_k(w) \leq \int_\Omega |w|^\theta |E|\, |DG_k(w)| + \int_\Omega |f|\, |G_k(w)|.$$

Notice that

$$|w| = |G_k(w)| + k \qquad \text{in } A_k,$$

which implies the existence of a constant $\bar{c} > 1$ such that

$$|w|^\theta \leq \bar{c}\left(|G_k(w)|^\theta + k^\theta\right) \qquad \text{in } A_k. \tag{4.14}$$

Then, by (4.3), (4.14), Hölder and Young inequalities, and observing that, thanks to (4.6), $\dfrac{\theta}{2^*} + \dfrac{1}{\bar{q}} + \dfrac{1}{2} = 1$, we have

$$\begin{aligned}
\alpha \int_\Omega |DG_k(w)|^2 \leq & \bar{c} \int_\Omega |G_k(w)|^\theta |E|\,|DG_k(w)| + \bar{c}\,k^\theta \int_\Omega |E|\,|DG_k(w)| + \int_\Omega |f|\,|G_k(w)| \\
\leq & \bar{c}\,\|G_k(w)\|_{2^*}^\theta \|E\|_{\bar{q},A_k} \|DG_k(w)\|_2 + \frac{\bar{c}k^{2\theta}}{4\varepsilon} \int_{A_k} |E|^2 \\
& + \bar{c}\,\varepsilon \|DG_k(w)\|_2^2 + \|f\|_{\frac{2N}{N+2},A_k} \|G_k(w)\|_{2^*}.
\end{aligned}$$

By choosing $\varepsilon = \dfrac{\alpha}{4\bar{c}}$ and by applying Sobolev and Young inequalities with exponent $\dfrac{2}{\theta+1}$, we get

$$\begin{aligned}
\alpha & \int_\Omega |DG_k(w)|^2 \\
& \leq c \left\{ \|E\|_{\bar{q},A_k} \|DG_k(w)\|_2^{\theta+1} + k^{2\theta} \int_{A_k} |E|^2 \right\} \\
& \quad + \frac{\alpha}{4} \int_\Omega |DG_k(w)|^2 + S\,\|f\|_{\frac{2N}{N+2},A_k} \|DG_k(w)\|_2 \\
& \leq c \left\{ \|E\|_{\bar{q},A_k}^{\frac{2}{1-\theta}} + k^{2\theta} \int_{A_k} |E|^2 + \|f\|_{\frac{2N}{N+2},A_k}^2 \right\} + \frac{\alpha}{2} \int_\Omega |DG_k(w)|^2,
\end{aligned}$$

where c is a positive constant depending upon α and S.

This last estimate implies the thesis. □

Remark 4.3. *The function $\bar{q} = \bar{q}(\theta)$ defined in (4.6) is increasing with respect to θ and $2 < \bar{q} < N$, since $0 < \theta < 1$.*
Furthermore, $\bar{q}$ converges to N in the limit case $\theta \to 1$.

Corollary 4.1. *Assume $|E| \in L^{\bar{q}}(\Omega)$ and $f \in L^m(\Omega)$, with $m \geq \dfrac{2N}{N+2}$. Then a weak solution w of (4.8) is bounded in $H_0^1(\Omega)$.*

Proof. By (4.12) and (4.13)

$$\int_\Omega |Dw|^2 = \int_\Omega |DT_k(w)|^2 + \int_\Omega |DG_k(w)|^2$$
$$\leq c\left(k^{2\theta}\int_\Omega |E|^2 + k\int_\Omega |f| + \|E\|_{\tilde{q},A_k}^{\frac{2}{1-\theta}} + \|f\|^2_{\frac{2N}{N+2},A_k}\right).$$

□

Lemma 4.4. *Assume $|E| \in L^{\tilde{q}}(\Omega)$, with $\tilde{q}$ defined in (4.7), and $f \in L^m(\Omega)$, with $\frac{2N}{N+2} \leq m < \frac{N}{2}$. Then a weak solution w of (4.8) is bounded in $L^{m^{**}}(\Omega)$.*

Proof. Choose $\varphi = \dfrac{|G_k(w)|^{2(\lambda-1)}G_k(w)}{2\lambda-1}$ as test function in the weak formulation of (4.8), with $\lambda = \dfrac{m^{**}}{2^*}$, which is greater than 1 thanks to our choice of m. In this way

$$D\varphi = |G_k(w)|^{2(\lambda-1)}DG_k(w),$$

and using (4.10), we obtain

$$\int_\Omega M(x)\, Dw\, |G_k(w)|^{2(\lambda-1)}\, DG_k(w)$$
$$\leq \int_\Omega |w|^\theta |E|\, |G_k(w)|^{2(\lambda-1)}\, |DG_k(w)| + \frac{1}{2\lambda-1}\int_\Omega |f|\, |G_k(w)|^{2\lambda-1}.$$

Thanks to (4.3), (4.14), Young and Hölder inequalities

$$\alpha\int_\Omega |G_k(w)|^{2(\lambda-1)}\, |DG_k(w)|^2$$
$$\leq \bar{c}\int_\Omega |G_k(w)|^{\theta+2(\lambda-1)}\, |E|\, |DG_k(w)| + \bar{c}\, k^\theta \int_\Omega |E|\, |G_k(w)|^{2(\lambda-1)}\, |DG_k(w)|$$
$$+ \frac{1}{2\lambda-1}\int_\Omega |f|\, |G_k(w)|^{2\lambda-1}$$
$$\leq \frac{\alpha}{2}\int_\Omega |G_k(w)|^{2(\lambda-1)}\, |DG_k(w)|^2 + c\Bigg\{\int_\Omega |G_k(w)|^{2(\theta+\lambda-1)}\, |E|^2$$
$$+ k^{2\theta}\int_\Omega |G_k(w)|^{2(\lambda-1)}\, |E|^2 + \|f\|_{m,A_k}\left(\int_\Omega |G_k(w)|^{m'(2\lambda-1)}\right)^{\frac{1}{m'}}\Bigg\}.$$

By the previous estimate and by Sobolev and Hölder inequalities again, one finds

$$\begin{aligned}\left(\int_\Omega |G_k(w)|^{2^*\lambda}\right)^{\frac{2}{2^*}} \leq & S^2\int_\Omega \left|D|G_k(w)|^\lambda\right|^2 \\ \leq & c\Bigg\{\left(\int_{A_k}|E|^{\tilde q}\right)^{\frac{2}{\tilde q}}\left(\int_\Omega |G_k(w)|^{2^*\lambda}\right)^{\frac{2(\theta+\lambda-1)}{2^*\lambda}} \\ & + k^{2\theta}\left(\int_{A_k}|E|^{\tilde q}\right)^{\frac{2}{\tilde q}}\left(\int_\Omega |G_k(w)|^{2^*\lambda}\right)^{\frac{2(\lambda-1)}{2^*\lambda}}|A_k|^{1-\frac{2}{\tilde q}-\frac{2(\lambda-1)}{2^*\lambda}} \\ & + \|f\|_{m,A_k}\left(\int_\Omega |G_k(w)|^{m'(2\lambda-1)}\right)^{\frac{1}{m'}}\Bigg\}\end{aligned}$$

where c is a positive constant depending only upon α and S.
Observe that

$$\frac{2(\lambda-1)}{2^*\lambda} < \frac{2(\theta+\lambda-1)}{2^*\lambda} < \frac{2}{2^*},$$
$$\frac{1}{m'} < \frac{2}{2^*} \iff 1-\frac{1}{m} < 1-\frac{2}{N} \iff m < \frac{N}{2},$$

and also that

$$m'(2\lambda-1) = m^{**}.$$

Therefore

$$\begin{aligned}\left(\int_\Omega |G_k(w)|^{m^{**}}\right)^{\frac{2}{2^*}} \leq & c(\alpha,S)\Bigg\{\left(\int_{A_k}|E|^{\tilde q}\right)^{\frac{2}{\tilde q}}\left(\int_\Omega |G_k(w)|^{m^{**}}\right)^{\frac{2(\theta+\lambda-1)}{2^*\lambda}} \\ & + k^{2\theta}\left(\int_{A_k}|E|^{\tilde q}\right)^{\frac{2}{\tilde q}}\left(\int_\Omega |G_k(w)|^{m^{**}}\right)^{\frac{2(\lambda-1)}{2^*\lambda}}|A_k|^{1-\frac{2}{\tilde q}-\frac{2(\lambda-1)}{2^*\lambda}} \\ & + \|f\|_{m,A_k}\left(\int_\Omega |G_k(w)|^{m^{**}}\right)^{\frac{1}{m'}}\Bigg\}.\end{aligned} \tag{4.15}$$

Let us now recall the general fact that, if a is any positive number verifying the following inequality

$$a^p \leq k_1 + k_2\, a^r \qquad\qquad \text{with } p > r \text{ and } k_1, k_2 > 0,$$

then a is bounded. As a consequence, the thesis directly follows by (4.15). □

Remark 4.4. *The quantity $\tilde q = \tilde q(\theta, m)$ defined in (4.7) is an increasing function of θ and m, which converges to N when $\theta \to 1$. One can see that, under the assumptions made on m, $2 < \tilde q < N$.*

4.1.2 Infinite energy solutions

In this section we will treat the case in which

$$f \in L^m(\Omega), \qquad \text{with } 1 \le m < \frac{2N}{N+2}.$$

In this case our problem does not admit weak solutions, but distributional ones.

Let us introduce now

$$\tilde{\theta} := \frac{2N - m(N+2)}{2(N-2m)}, \tag{4.16}$$

and let us observe that $\tilde{\theta} \in (0,1)$.

Notice that, while in the case of finite energy one had $\tilde{q} > 2$, for every $\frac{2N}{N+2} \le m < \frac{N}{2}$ (see Remark 4.4), now the exponent $\tilde{q}$ satisfies

$$\tilde{q}(\theta, m) > 2 \qquad \text{if and only if} \qquad \theta > \tilde{\theta}, \tag{4.17}$$

and $\tilde{q}(\theta, m) < N$.

Lemma 4.5. *Assume $|E| \in L^{\tilde{q}}(\Omega)$, with $\tilde{q}$ as in (4.7), and $f \in L^m(\Omega)$, with $1 < m < \frac{2N}{N+2}$. Then a distributional solution w of (4.8) is bounded in $L^{m^{**}}(\Omega)$.*

Proof. Take

$$\varphi = \frac{(1 + |G_k(w)|)^{2\gamma-1} - 1}{2\gamma - 1} \operatorname{sign}(w)$$

as test function in (4.8), with

$$\gamma = \frac{m^{**}}{2^*}$$

and observe that $\frac{1}{2} < \gamma < 1$. In this way

$$D\varphi = (1 + |G_k(w)|)^{2(\gamma-1)} DG_k(w).$$

Thanks to these choices, (4.10), (4.14) and Hölder inequality

$$\begin{aligned}
&\int_\Omega M(x) \frac{Dw \cdot DG_k(w)}{(1 + |G_k(w)|)^{2(1-\gamma)}} \\
&\quad \le \int_\Omega |w|^\theta |E| \frac{DG_k(w)}{(1 + |G_k(w)|)^{2(1-\gamma)}} + \frac{1}{2\gamma - 1} \int_\Omega |f| \, (1 + |G_k(w)|)^{2\gamma-1} \\
&\quad \le \bar{c} \int_\Omega |G_k(w)|^\theta |E| \frac{DG_k(w)}{(1 + |G_k(w)|)^{2(1-\gamma)}} + \bar{c} k^\theta \int_\Omega |E| \frac{DG_k(w)}{(1 + |G_k(w)|)^{2(1-\gamma)}} \\
&\qquad + \frac{1}{2\gamma - 1} \|f\|_{m, A_k} \left(\int_\Omega (1 + |G_k(w)|)^{m'(2\gamma-1)} \right)^{\frac{1}{m'}}.
\end{aligned}$$

Now, let us apply (4.3) and Young inequality to get

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(w)|)^{2(1-\gamma)}} &\le c\Bigg\{\int_\Omega (1+|G_k(w)|)^{2(\theta+\gamma-1)}|E|^2 + k^{2\theta}\int_{A_k}|E|^2 \\ &\quad + \|f\|_{m,A_k}\left(\int_\Omega (1+|G_k(w)|)^{m'(2\gamma-1)}\right)^{\frac{1}{m'}}\Bigg\}.\end{aligned}$$

By Sobolev inequality, and observing that $m'(2\gamma-1)=2^*\gamma=m^{**}$

$$\begin{aligned}&\left(\int_\Omega \left|(1+|G_k(w)|)^{\frac{m^{**}}{2^*}}-1\right|^{2^*}\right)^{\frac{2}{2^*}} \le S^2\int_\Omega \left|D[(1+|G_k(w)|)^\gamma-1]\right|^2 \\ &\le c\left\{\int_\Omega (1+|G_k(w)|)^{2(\theta+\gamma-1)}|E|^2 + k^{2\theta}\int_{A_k}|E|^2 + \|f\|_{m,A_k}\left(\int_\Omega (1+|G_k(w)|)^{m^{**}}\right)^{\frac{1}{m'}}\right\}.\end{aligned}$$

Now, we have to take into account the sign of the exponent $\theta+\gamma-1$.

If $\theta+\gamma-1\le 0$, which happens if and only if $\theta\le\tilde\theta$, then

$$(1+|G_k(w)|)^{2(\theta+\gamma-1)}\le 1,$$

therefore

$$\begin{aligned}&\left(\int_\Omega \left|(1+|G_k(w)|)^{\frac{m^{**}}{2^*}}-1\right|^{2^*}\right)^{\frac{2}{2^*}} \\ &\qquad\le c\left\{\left(1+k^{2\theta}\right)\int_{A_k}|E|^2 + \|f\|_{m,A_k}\left(\int_\Omega (1+|G_k(w)|)^{m^{**}}\right)^{\frac{1}{m'}}\right\},\end{aligned}$$

and we conclude by observing that $\dfrac{1}{m'}<\dfrac{2}{2^*}$.

On the other hand, if $\theta+\gamma-1>0$, which happens if and only if $\theta>\tilde\theta$, we can apply Hölder inequality with exponent $\dfrac{2^*\gamma}{2(\theta+\gamma-1)}>1$ to get

$$\begin{aligned}&\left(\int_\Omega \left|(1+|G_k(w)|)^{\frac{m^{**}}{2^*}}-1\right|^{2^*}\right)^{\frac{2}{2^*}} \\ &\le c\Bigg\{\left(\int_\Omega (1+|G_k(w)|)^{2^*\gamma}\right)^{\frac{2(\gamma+\theta-1)}{2^*\gamma}}\|E\|^2_{\tilde q,A_k} \\ &\quad + k^{2\theta}\int_{A_k}|E|^2 + \|f\|_{m,A_k}\left(\int_\Omega (1+|G_k(w)|)^{m^{**}}\right)^{\frac{1}{m'}}\Bigg\},\end{aligned}$$

and we conclude just by observing that $\dfrac{2(\gamma+\theta-1)}{2^*\gamma}<\dfrac{2}{2^*}$. □

Corollary 4.2. *Assume* $|E| \in L^{\tilde{q}}(\Omega)$ *and* $f \in L^m(\Omega)$*, with* $1 < m < \dfrac{2N}{N+2}$*. Then a distributional solution* w *of (4.8) is bounded in* $W_0^{1,m^*}(\Omega)$*.*

Proof. In the proof of the previous lemma, we have just proved that $G_k(w) \in L^{2^*\gamma}$ and that

$$\int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(u)|)^{2(1-\gamma)}}$$

is bounded, with $\gamma = \dfrac{m^{**}}{2^*}$. Thanks to Hölder inequality, we have

$$\begin{aligned}\int_\Omega |DG_k(w)|^{m^*} &= \int_\Omega \frac{|DG_k(w)|^{m^*}}{(1+|G_k(w)|)^{m^*(1-\gamma)}}\,(1+|G_k(w)|)^{m^*(1-\gamma)}\\ &\le \left(\int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(u)|)^{2(1-\gamma)}}\right)^{\frac{m^*}{2}} \left(\int_\Omega (1+|G_k(w)|)^{\frac{2m^*(1-\gamma)}{2-m^*}}\right)^{\frac{2-m^*}{2}}\end{aligned}$$

and the thesis follows by observing that $\dfrac{2m^*(1-\gamma)}{2-m^*} = 2^*\gamma$. □

Remark 4.5. *Let us observe that when* $m = 1$*,* $\tilde{\theta}$ *defined in (4.16) becomes* $\dfrac{1}{2}$*.*

Lemma 4.6. *Suppose* $f \in L^1(\Omega)$*. If* $|E| \in L^{\tilde{q}}(\Omega)$*, then a distributional solution* w *of (4.8) is bounded in* $L^p(\Omega)$*, for every* $p < \dfrac{N}{N-2}$*.*
Furthermore, if $\dfrac{1}{2} < \theta < \dfrac{N}{2(N-1)}$ *and* $|E| \in L^2(\Omega)$*, then a distributional solution* w *of (4.8) is bounded in* $L^p(\Omega)$*, for every* $p \le 2^*(1-\theta)$*.*

Proof. Choose

$$\varphi = \frac{1-(1+|G_k(w)|)^{1-2\sigma}}{2\sigma-1}\operatorname{sign}(w), \quad \text{with } \frac{1}{2} < \sigma < 1,$$

as test function in (4.8), so that its gradient is

$$D\varphi = (1+|G_k(w)|)^{-2\sigma}\operatorname{sign}(w),$$

then by (4.10) and (4.14), we obtain

$$\begin{aligned}\int_\Omega M(x)\frac{Dw\cdot DG_k(w)}{(1+|G_k(w)|)^{2\sigma}} &\le \int_\Omega |w|^\theta |E|\frac{DG_k(w)}{(1+|G_k(w)|)^{2\sigma}} + \frac{1}{2\sigma-1}\int_{A_k}|f|\\ &\le \bar{c}\int_\Omega |G_k(w)|^\theta |E|\frac{DG_k(w)}{(1+|G_k(w)|)^{2\sigma}} + \bar{c}k^\theta\int_\Omega |E|\frac{DG_k(w)}{(1+|G_k(w)|)^{2\sigma}} + \frac{1}{2\sigma-1}\int_{A_k}|f|.\end{aligned}$$

Thanks to (4.3) and Young inequality, one has

$$\int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(w)|)^{2\sigma}} \le c\left(\int_\Omega \frac{|G_k(w)|^{2\theta}|E|^2}{(1+|G_k(w)|)^{2\sigma}} + k^{2\theta}\int_\Omega |E|^2 + \int_{A_k}|f|\right).$$

and applying Sobolev inequality, one gets

$$\left(\int_\Omega \left[1-(1+|G_k(w)|)^{1-\sigma}\right]^{2^*}\right)^{\frac{2}{2^*}} \le S\int_\Omega \left|D[1-(1+|G_k(w)|)^{1-\sigma}]\right|^2$$
$$\le c\left(\int_\Omega (1+|G_k(w)|)^{2(\theta-\sigma)}|E|^2 + k^{2\theta}\int_{A_k}|E|^2 + \int_{A_k}|f|\right).$$

Now, we have to distinguish two cases. When $\sigma \ge \theta$, we can estimate

$$\left(\int_\Omega \left[1-(1+|G_k(w)|)^{1-\sigma}\right]^{2^*}\right)^{\frac{2}{2^*}} \le c\left(\left(1+k^{2\theta}\right)\int_{A_k}|E|^2 + \int_{A_k}|f|\right). \tag{4.18}$$

When $\sigma < \theta$, we use Hölder inequality with exponent $\dfrac{2^*(1-\sigma)}{2(\theta-\sigma)} > 1$ to have

$$\left(\int_\Omega \left[1-(1+|G_k(w)|)^{1-\sigma}\right]^{2^*}\right)^{\frac{2}{2^*}}$$
$$\le c\left\{\left(\int_\Omega (1+|G_k(w)|)^{2^*(1-\sigma)}\right)^{\frac{2(\theta-\sigma)}{2^*(1-\sigma)}}\left(\int_{A_k}|E|^{\frac{2N(1-\sigma)}{N-2\sigma-\theta(N-2)}}\right)^{\frac{N-2\sigma-\theta(N-2)}{N(1-\sigma)}}\right. \tag{4.19}$$
$$\left. + k^{2\theta}\int_{A_k}|E|^2 + \int_{A_k}|f|\right\},$$

and we observe that $\dfrac{2(\theta-\sigma)}{2^*(1-\sigma)} < \dfrac{2}{2^*}$. Let us notice also that

$$\frac{2N(1-\sigma)}{N-2\sigma-\theta(N-2)} > 2.$$

We can then subdivide our argument. First of all, if $\theta \le \dfrac{1}{2}$, then $\tilde{q} = 2$. Therefore, assuming $|E| \in L^2(\Omega)$, for any $\sigma > \dfrac{1}{2}$ we can apply (4.18), which yields $G_k(w) \in L^p(\Omega)$, with $p = 2^*(1-\sigma)$. As a consequence, by letting $\sigma \to \dfrac{1}{2}$, we obtain that $G_k(w) \in L^p(\Omega)$, for every $p < \dfrac{N}{N-2}$.

On the other hand, if $\theta > \dfrac{1}{2}$, then $\tilde{q} > 2$, and we split our argument in the following way. If $|E| \in L^2(\Omega)$, then we apply again (4.18) choosing $\sigma = \theta$ and we obtain $G_k(w) \in$

$L^{2^*(1-\theta)}(\Omega)$. Notice that $2^*(1-\theta) > 1$ under the assumptions made on θ.
Finally, consider what happens for $|E| \in L^{\tilde{q}}(\Omega)$. If it is so, it is immediate to check that

$$\frac{2N(1-\sigma)}{N-2\sigma-\theta(N-2)} < \tilde{q} \qquad\qquad \text{for every } \sigma > \frac{1}{2}. \tag{4.20}$$

Arguing as in the first case, but using (4.19) in spite of (4.18), by letting $\sigma \to \frac{1}{2}$, we again find $G_k(w) \in L^p(\Omega)$ for every $p < \frac{N}{N-2}$. □

Remark 4.6. *Consider the case when $|E|$ has an intermediate summability, that is $|E| \in L^q(\Omega)$, with some $2 < q < \tilde{q}$. By the previous Lemma, just using $|E| \in L^2(\Omega)$, we already know that $w \in L^p(\Omega)$, for every $p \leq 2^*(1-\theta)$. However, it is possible to say something more. In fact, by the same argument of the Lemma, we obtain $w \in L^{2^*(1-\sigma)}$, for every $\frac{1}{2} < \sigma < \theta$ for which*

$$\frac{2N(1-\sigma)}{N-2\sigma-\theta(N-2)} \geq q. \tag{4.21}$$

A simple calculation ensures that (4.21) is equivalent to

$$\sigma \geq \hat{\sigma}(q,\theta) := \frac{N(2-q)+q\theta(N-2)}{2(N-q)},$$

and in turn $\hat{\sigma}$ is strictly decreasing in q, with

$$\hat{\sigma} \to \frac{1}{2} \quad \text{for } q \to \tilde{q}, \qquad\qquad \hat{\sigma} \to \theta \quad \text{for } q \to 2,$$

so that, in particular,

$$\frac{1}{2} < \hat{\sigma} < \theta.$$

Summarizing, for any $2 < q < \tilde{q}$, we can say that $|E| \in L^q(\Omega)$ implies $w \in L^{2^*(1-\hat{\sigma})}$.

Corollary 4.3. *Suppose $f \in L^1(\Omega)$. If $|E| \in L^{\tilde{q}}(\Omega)$, then a distributional solution w of (4.8) is bounded in $W_0^{1,p}(\Omega)$, for every $p < \frac{N}{N-1}$.*
Furthermore, if $\frac{1}{2} < \theta < \frac{N}{2(N-1)}$ and $|E| \in L^2(\Omega)$, then a distributional solution w of (4.8) is bounded in $W_0^{1,p}(\Omega)$, for every $p \leq p(\theta) = \frac{2N(1-\theta)}{N-2\theta}$.

Proof. When $|E| \in L^{\tilde{q}}(\Omega)$, by the proof of Lemma 4.6 we know that

$$\int_\Omega |G_k(w)|^{2^*(1-\sigma)}, \quad \int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(w)|)^{2\sigma}}$$

are bounded, for every $\frac{1}{2} < \sigma < 1$.
Take $1 < p < 2$ and apply Hölder inequality to get

$$\begin{aligned}\int_\Omega |DG_k(w)|^p &= \int_\Omega \frac{|DG_k(w)|^p}{(1+|G_k(w)|)^{p\sigma}} (1+|G_k(w)|)^{p\sigma} \\ &\leq \left(\int_\Omega \frac{|DG_k(w)|^2}{(1+|G_k(w)|)^{2\sigma}}\right)^{\frac{p}{2}} \left(\int_\Omega (1+|G_k(w)|)^{\frac{2p\sigma}{2-p}}\right)^{\frac{2-p}{2}}.\end{aligned}$$

Now, we notice that

$$\frac{2p\sigma}{2-p} = 2^*(1-\sigma) \qquad \text{if and only if} \qquad p = \frac{2N(1-\sigma)}{N-2\sigma},$$

and letting $\sigma \to \frac{1}{2}$, we conclude the proof for the case $|E| \in L^{\tilde{q}}(\Omega)$.
When $|E| \in L^2(\Omega)$, we proceed almost in the same way, just replacing σ by θ and observing that $p(\theta) > 1$ is equivalent to $\theta < \dfrac{N}{2(N-1)}$. □

4.2 Passing to the limit

In this section we will prove existence and regularity for solutions to problem (4.1).

Theorem 4.1. *Assume $|E| \in L^{\bar{q}}(\Omega)$, with $\bar{q}$ as in (4.6), and $f \in L^m(\Omega)$, with $m \geq \dfrac{2N}{N+2}$. Then there exists a weak solution $u \in H_0^1(\Omega)$ of (4.1), that is*

$$\int_\Omega M(x)DuDv = \int_\Omega |u|^{\theta-1} u\, E(x)Dv + \int_\Omega fv \qquad \forall v \in H_0^1(\Omega). \tag{4.22}$$

Proof. Let u_n be a weak solution of (4.8). Then, for every $n \in \mathbb{N}$, we can write

$$\int_\Omega M(x)Du_nDv = \int_\Omega \frac{|u_n|^{\theta-1}u_n}{1+\frac{|u_n|^\theta}{n}} \frac{E}{1+\frac{|E|}{n}} Dv + \int_\Omega f_n v \qquad \forall v \in H_0^1(\Omega). \tag{4.23}$$

We want to pass to the limit in the previous equality. We start observing that

$$\phi_n := \frac{|u_n|^{\theta-1}u_n}{1+\frac{|u_n|^\theta}{n}}$$

is bounded in $L^{\frac{2^*}{\theta}}(\Omega)$, since

$$\int_\Omega |\phi_n|^{\frac{2^*}{\theta}} \le \int_\Omega |u_n^\theta|^{\frac{2^*}{\theta}} = \int_\Omega |u_n|^{2^*} \qquad \forall n \in \mathbb{N},$$

and the integral on the right-hand side is bounded thanks to Corollary 4.1.

Now, $\dfrac{E}{1+\frac{|E|}{n}}$ converges to E in $L^{\bar{q}}(\Omega)$ and ϕ_n weakly converges to a function in $L^{\frac{2^*}{\theta}}(\Omega)$. Again by Corollary 4.1, $u_n \to u$ in $L^2(\Omega)$ and then it converges a.e. up to a subsequence. This implies $\phi_n \to |u|^{\theta-1}u$ a.e., therefore $\phi_n \rightharpoonup |u|^{\theta-1}u$ in $L^{\frac{2^*}{\theta}}(\Omega)$.
Moreover, f_n converges to f in $L^{2^{*\prime}}(\Omega)$
Finally, by the fact that $Dv \in L^2(\Omega)$ and that the exponents 2, $\bar{q}$, $\dfrac{2^*}{\theta}$ are conjugated, we can pass to the limit in (4.23) finding (4.22). □

Proceeding as in the previous theorem we can prove also the following results.

Theorem 4.2. *Assume $|E| \in L^{\tilde{q}}(\Omega)$, with $\tilde{q}$ defined in (4.7), and $f \in L^m(\Omega)$, with $\dfrac{2N}{N+2} \le m < \dfrac{N}{2}$. Then there exists a weak solution $u \in L^{m^{**}}(\Omega)$ of (4.1).*

Theorem 4.3. *Assume $|E| \in L^{\tilde{q}}(\Omega)$ and $f \in L^m(\Omega)$, with $1 < m < \dfrac{2N}{N+2}$. Then there exists a distributional solution $u \in W_0^{1,m^*}(\Omega)$ of (4.1).*

Theorem 4.4. *Suppose $f \in L^1(\Omega)$. If $|E| \in L^{\tilde{q}}(\Omega)$, then there exists a distributional solution u of (4.1), which belongs to $W_0^{1,p}(\Omega)$, for every $p < \dfrac{N}{N-1}$.*
Furthermore, if $\dfrac{1}{2} < \theta < \dfrac{N}{2(N-1)}$ and $|E| \in L^2(\Omega)$, then there exists a distributional solution u of (4.1), which belongs to $W_0^{1,p}(\Omega)$, for every $p \le p(\theta) = \dfrac{2N(1-\theta)}{N-2\theta}$.

4.3 Bounded solutions

We conclude this chapter by showing the boundedness of solutions, for which we need further regularity assumptions on E and f.
We follow the lines of [7], which use a standard technique due to Stampacchia (see [63])).

Lemma 4.7. *Let $\varphi(t) :]h_0, +\infty[\to \mathbb{R}$ be a non-negative, decreasing function such that*

$$\varphi(k) \le \frac{c}{(k-h)^\sigma}\,[\varphi(h)]^\delta \qquad \forall k > h \ge h_0,$$

being c, σ, δ positive constants. If $\delta > 1$, then

$$\varphi(h_0 + d) = 0,$$

with $d^\sigma = c\,[\varphi(h_0)]^{\delta-1}\, 2^{\frac{\sigma\delta}{\delta-1}}$.

Theorem 4.5. *Assume $|E| \in L^q(\Omega)$, with $q > N$, and $f \in L^m(\Omega)$, with $m > \dfrac{N}{2}$. Then there exists a weak solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (4.1).*

Proof. Let $u \in H_0^1(\Omega)$ be a weak solution of (4.1), which exists by Theorem 4.1 and let us take as test function

$$\psi(u) = \begin{cases} 0 & \text{if } |u| \le k \\ \dfrac{u}{1+u} - \dfrac{k}{1+k} & \text{if } u > k \\ \dfrac{u}{1-u} - \dfrac{k}{1+k} & \text{if } u < -k \end{cases}$$

to get

$$\int_{\Omega\cap\{|u|>k\}} M(x)\,\frac{Du\cdot Du}{1+|u|^2} \le \int_{\Omega\cap\{|u|>k\}} |u|^\theta |E|\,\frac{Du}{1+|u|^2} + \int_\Omega |f|\,|\psi|.$$

By (4.3), the fact that $\dfrac{|u|^\theta}{1+|u|} \le \dfrac{|u|^\theta}{(1+|u|)^\theta} \le 1$, Young inequality and $|\psi| \le 1$, one has

$$\frac{\alpha}{2}\int_{\Omega\cap\{|u|>k\}} \frac{|Du|^2}{1+|u|^2} \le \frac{1}{2\alpha}\int_{\Omega\cap\{|u|>k\}} |E|^2 + \int_{\Omega\cap\{|u|>k\}} |f|\,,$$

and putting $k = e^h - 1$, one gets

$$\int_{\Omega\cap\{\log(1+|u|)>h\}} |D\log(1+|u|)|^2 \le c\int_{\Omega\cap\{\log(1+|u|)>h\}} \big(|E|^2 + |f|\,\big).$$

Let us denote by

$$v = \log(1+|u|), \qquad\qquad g = |E|^2 + |f|,$$

and let us observe that $g \in L^m(\Omega)$, with $m > \dfrac{N}{2}$. Now, we introduce the notation

$$\bar{A}_h := \Omega \cap \{|v| > h\},$$

in this way

$$\int_\Omega |DG_h(v)|^2 = \int_{\bar{A}_h} |Dv|^2 \le c\int_{\bar{A}_h} |g|.$$

Applying Sobolev and Hölder inequalities, we get

$$\left(\int_\Omega |G_h(v)|^{2^*}\right)^{\frac{2}{2^*}} \le c\,\|g\|_{m,\bar{A}_h}\,|\bar{A}_h|^{1-\frac{1}{m}}. \tag{4.24}$$

Take $\ell > h > 0$, then

$$\bar{A}_\ell \subseteq \bar{A}_h, \qquad\qquad |G_h(v)| \ge \ell - h \quad \text{in } \bar{A}_\ell,$$

and therefore (4.24) implies

$$(\ell - h)^2 |\bar{A}_\ell|^{\frac{2}{2^*}} \le \left(\int_{\bar{A}_\ell} |G_h(v)|^{2^*}\right)^{\frac{2}{2^*}} \le c\,\|g\|_{m,\bar{A}_h}\,|\bar{A}_h|^{1-\frac{1}{m}}.$$

It follows

$$|\bar{A}_\ell| \le \frac{c}{(\ell - h)^{2^*}}\,\|g\|_{m,\bar{A}_h}^{\frac{2^*}{2}}\,|\bar{A}_h|^{\frac{2^*}{2}(1-\frac{1}{m})},$$

so we can apply Lemma 4.7 to the measure of the set $\bar{A}_h$ with $\delta = \dfrac{2^*}{2}\left(1 - \dfrac{1}{m}\right)$ which is greater than 1 if and only if $m > \dfrac{N}{2}$.

This tell us that $|v| \le d$ a.e., where d is a positive constant depending only upon $|\Omega|$, N, E, f. So u is bounded and our proof is complete. □

Remark 4.7. *We want to stress what follows. In all our results we have needed weaker assumptions on $|E|$ than those in [7] for the problem (4.2), corresponding to the limit case $\theta = 1$. Instead, in this last result we used the same hypothesis as in [7]. This is however not surprising because, when m approaches $\dfrac{N}{2}$, $\tilde{q}(\theta, m)$ converges to N for every $\theta \in (0, 1)$.*

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