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A new class of fractional Orlicz-Sobolev space and singular elliptic problems

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A R T I C L E I N F O

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A B S T R A C T

This paper seeks to introduce a new class of fractional Orlicz-Sobolev space with variable-order. We give basic properties of this space and prove some compactness results. Then, using some techniques of calculus of variations combined with theory of Musielak functions, we prove the existence of a nonnegative weak solution for a singular elliptic type problem in a fractional variable-order Orlicz-Sobolev space with homogeneous Dirichlet boundary conditions.

1. Introduction

The motivation behind this paper was the recent fundamental enrichment to the functional analysis of integrodifferential operators. We mainly refer to the pioneering contributions of Luis Caffarelli and Luis Silvestre who studied the construction of fractional Laplacian operator $(-\Delta)^s$, where $0 < s < 1$, from an extension problem to the upper half space for a specific elliptic partial differential equation [10]. This operator is considered as an extension of the previously known operator $(-\Delta)^{\frac{1}{2}}$ and contributed to treat more problems involving integrodifferential operators.

The first main objective of our work is to introduce a new fractional G-Laplacian operator and its corresponding function space.

Given an open set $\Omega \subset \mathbb{R}^n, n \geq 1$, this paper is concerned with a new type of fractional Orlicz-Sobolev space with variable order defined by

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$$W^{s(.),G}(\Omega) := \Big\{ u \in L^G(\Omega) : \iint\limits_{\Omega\times\Omega} G\Big(\frac{\lambda|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big) \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} < \infty \quad \text{for some}\, \lambda > 0 \Big\}, \tag{1.1}$$

where $s : \Omega \times \Omega \to]0,1[$ is a measurable function and G is an Orlicz function. See section 2 for more details. This space is a natural extension of the classical fractional constant-order Orlicz-Sobolev space, namely

$$W^{s,G}(\Omega) := \Big\{ u \in L^G(\Omega) : \iint\limits_{\Omega\times\Omega} G\Big(\frac{\lambda|u(x)-u(y)|}{|x-y|^{s}}\Big) \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} < \infty \quad \text{for some}\, \lambda > 0 \Big\}.$$

Also, the space given in (1.1) is an extension of fractional variable-order Sobolev space with constant exponent, namely

$$W^{s(.),p}(\Omega) := \Big\{ u \in L^p(\Omega) : \iint\limits_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps(x,y)}} \,\mathrm{d}x\mathrm{d}y < \infty \Big\}.$$

Naturally, the space $W^{s(.),G}(\Omega)$ is associated with the new fractional $G-$ Laplacian operator under the form

$$(-\Delta)^{s(.)}_G u(x) := p.v. \int\limits_{\mathbb{R}^n} G'\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big) \frac{u(x)-u(y)}{|u(x)-u(y)|} \frac{\mathrm{d}y}{|x-y|^{s(x,y)+n}},$$

where G' is the right derivative of G and p.v. stands for principal value.

The first documented appearance of the concept of fractional order was through papers exchanged between the two mathematicians Leibniz and L'Hôpital during the seventeen century in which they discussed the interpretation of the operators $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ when α is not integer. The fractional derivative and pseudodifferential operators with variable order have been discussed and studied by many mathematicians and physicists from different research areas. In [21], Hartley and Lorenzo first presented the physical motivation toward variable order operators. In particular, some diffusion processes reacting to temperature changes may be better described using variable order derivatives in a nonlocal integro-differential operator [22]. Xiang et al. [33] discussed the multiplicity of solutions for variable order fractional Laplacian problems with concave-convex nonlinearity involving variable exponent. In [7], the authors studied the existence and multiplicity results for variable-order nonlocal Choquard problems with variable exponents. Choudhuri et al. [12] proved the multiplicity of solutions for critical nonlocal degenerate Kirchhoff problems with a variable singular exponent. In 2020, Patnaik et al. [27] gave a review and applications of these type of operators in mechanical engineering, evolution differential equations, anomalous transport, variable control and mathematical modelling.

Partial differential equations driven by nonhomogeneous operators have been extensively investigated and received much attention since they can be presented as models for many physical phenomena. As examples, the field of nonlinear elasticity, plasticity and electro-rheological fluids, see [28–30]. A simple example of problem with nonhomogeneous operator which illustrates the need for more inclusive classes of function spaces than $W^{s(.),p}(\mathbb{R}^n)$ is driven by the following equation

$$p.v. \int\limits_{\mathbb{R}^n} 2q\Big(1 + \frac{|u(x)-u(y)|^2}{|x-y|^{2s(x,y)}}\Big)^{q-1} (u(x)-u(y)) \frac{\mathrm{d}y}{|x-y|^{2s(x,y)+n}} = 0 \qquad \text{in} \quad \mathbb{R}^n. \tag{1.2}$$

The energy functional associated to the Euler-Lagrange equation (1.2) is defined by

$$E(u) = \iint\limits_{\mathbb{R}^{2n}} \Big(\Big(1 + \frac{|u(x)-u(y)|^2}{|x-y|^{2s(x,y)}}\Big)^q - 1\Big) \frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}.$$

It seems to be difficult to deal with the functional E on the usual fractional variable-order Sobolev space $W^{s(.),p}(\mathbb{R}^n)$. Indeed, it is clear that

$$(1+t^2)^q - 1 \sim qt^2 \qquad t \to 0\,,$$
$$(1+t^2)^q - 1 \sim t^{2q} \qquad t \to \infty.$$

When $q \neq 1$, neither $W^{s(.),2}(\mathbb{R}^n)$ nor $W^{s(.),2q}(\mathbb{R}^n)$ includes the other. Consequently, the functional E is not well defined on neither of them. The most natural Sobolev space on which E is defined is the fractional variable-order Orlicz-Sobolev space associated with the Orlicz function $G(t) = (1+t^2)^q - 1$.

Let us comment a little more on the motivation of this space. A. Alberico et al. [2] gave sufficient structural conditions for an embedding theorem of fractional constant-order Orlicz-Sobolev space when Ω is a bounded Lipschitz domain, namely

$$\int_0^1 \Big(\frac{z}{G(z)}\Big)^{\frac{s}{n-s}}\,\mathrm{d}z < \infty \tag{1.3}$$

and

$$\int_1^\infty \Big(\frac{z}{G(z)}\Big)^{\frac{s}{n-s}}\,\mathrm{d}z = \infty. \tag{1.4}$$

Under the conditions (1.3) and (1.4), the authors detected the target space $L^{G_{\frac{n}{s}}}(\Omega)$ where $G_{\frac{n}{s}}$ is the optimal Orlicz function that plays the role of a critical function in the class of Orlicz functions. Specifically, they showed the embedding

$$W^{s,G}(\Omega) \hookrightarrow L^{G_{\frac{n}{s}}}(\Omega). \tag{1.5}$$

A natural question is to seek embedding results in the case where s is no longer constant. To this purpose for more inclusive classes of function space than $W^{s,G}(\Omega)$, we introduce the operator $(-\Delta)^{s(.)}_G$ and its corresponding Sobolev space. Certainly, some new classes of nonlinear equations will appear. For more papers on the study of problems involving $(-\Delta)^s_G$, we refer, for instance, to [4,6,8,9,31] for a list of references and results from the variational analysis and regularity theory.

Let us now describe our results in further details. In this manuscript, we define first the space given in (1.1). We will study in details its main properties. We prove structural results of these spaces. Precisely, under the condition:

There exist $a \geq 1$ such that

$$\frac{G(t)}{t} \leq a\frac{G(s)}{s} \quad \text{for all } 0 < t \leq s,$$

we show that

$$u \mapsto \inf\Big\{\lambda > 0 : \int_\Omega G(\frac{|u(x)|}{\lambda})\mathrm{d}x + \iint_{\Omega\times\Omega} G\Big(\frac{|u(x)-u(y)|}{\lambda|x-y|^{s(x,y)}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \leq 1\Big\} \tag{1.6}$$

is a quasi-additive functional in $W^{s(.),G}(\Omega)$ and the constant of quasi-additivity is smaller or equal to $8a$. Moreover, under structural conditions "almost-increasing, almost-decreasing", we prove the relation between the functional (1.6) and the modular functional

$$u \mapsto \int_{\Omega} G(|u(x)|)\mathrm{d}x + \iint_{\Omega\times\Omega} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}.$$

Also, we give criteria for which $W^{s(.),G}(\Omega)$ is separable, reflexive and uniformly convex. As far as we know, it seems to be the first time where fractional Sobolev space $W^{s(.),G}(\Omega)$ with G following "almost-increasing, almost-decreasing" structural conditions is introduced. Even in the case where s is constant, we never found it in the literature. For more information about these properties, see subsection 3.1. For a special case, see [5].

In [12], the authors established the embedding

$$W^{s(.),2}(\Omega) \hookrightarrow L^{q(.)}(\Omega), \tag{1.7}$$

where the variable exponent $q(.)$ can be close to the exponent $2^\star(.) := \frac{2n}{n-2s(x,x)}$. Specifically, the function $q(.)$ follows the conditions:

- $q(.) : \bar{\Omega} \to \mathbb{R}$ is continuous and $1 < q^- \leq q(x) \leq q^+$ for all $x \in \bar{\Omega}$.
- There exists $\epsilon = \epsilon(x) > 0$ such that

$$\sup_{y\in\Omega_{x,\epsilon}} q(y) \leq \frac{2n}{n - 2\inf_{(y,z)\in\Omega_{x,\epsilon}\times\Omega_{x,\epsilon}} s(y,z)},$$

where $\Omega_{x,\epsilon} = B_\epsilon(x) \cap \Omega$, for $x \in \Omega$.

This result extends the classical continuous embedding when the order s is constant. Our main contribution is the overall to sharp counterparts of the embeddings (1.5) and (1.7). Especially, we prove that our space $W^{s(.),G}(\Omega)$ embedded in a large space that includes a Lebesgue space with variable exponent and a Lebesgue-Orlicz space. Using interpolation arguments and a suitable auxiliary function in our conditions, we guarantee the stability of the continuous embedding (1.5) when s is constant. Also, we show that our space is related with Lebesgue-Musielak space. Under the continuity condition of the order function, we give an embedding theorem that allows us to identify one of the functions that can play the role of a critical function in a class of Musielak functions. Obviously, this space can raise delicate mathematical questions, in particular, with regard to the critical function, as well as to the question concerning the target space and whether it can be determined under the restrictive condition of the order function continuity.

Our abstract results are motivated by the existence of solutions to the following singular elliptic problem:

$$\begin{cases} (-\Delta)^{s(.)}_G u(x) = g(x)f'(x,|u|)\dfrac{u}{|u|} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.8}$$

where f' is a generalized singular term and g is a positive function. See section 4 for more details about conditions on g and f.

In the recent years, many authors have treated singular elliptic problems with a G-Laplacian operator, see [11,15], where the authors have used various methods. To the best of our knowledge, this is the first work examining singular elliptic equations by using variational arguments combined with the theory of Musielak functions and Musielak-Lebesgue spaces. Below, we describe the novelty of our equation:

1. It is considered a large class of singular term which includes the singular term of the form $f(x,t) = t^{\gamma(x)}$ where $0 < \gamma(x) < 1$.
2. Related to our approach, we are using new and optimal assumptions on singular term.
3. We are using a new general embedding theorem.
4. We are using new inhomogeneous operators.

Let us present some relevant articles in this approach where the authors used monotonicity arguments, algebraic calculations on the variable exponent combined with Hölder's inequality and some properties of the norm. In [23], Kefi-Saoudi investigated the existence of solutions for the following variable exponent elliptic equation involving the biharmonic operator, namely

$$\begin{cases} \Delta(|\Delta u|^{p(x)-2}\Delta u) = g(x)u^{-\gamma(x)} + \lambda f(x,u) & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n (n \geq 3)$, $g \in L^{\frac{p^\star(x)}{p^\star(x)+\gamma(x)-1}}(\Omega)$, $0 < \gamma < 1$ is a continuous function and f is a subcritical nonlinearity.

The existence of a solution for the fractional Laplacian operator with Kirchhoff term and critical nonlinearity has been proved by A. Fiscella [16] who studied the problem

$$\begin{cases} \left(\iint\limits_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\mathrm{d}y\right)^{\theta-1}(-\Delta)^s u = \lambda \frac{1}{u^\gamma} + u^{2_s^\star-1} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ with continuous boundary, $n > 2s$, $0 < s < 1$, $2_s^\star := \frac{2n}{n-2s}$ and $0 < \gamma < 1$.

Recently, Bahrouni-Rădulescu treated a singular double phase system with variable growth for the Baouendi-Grushin operator [3], namely

$$\begin{cases} -\Delta_{\Phi(x,y)}u + |u|^{q(z)-2}u + |u|^{p(z)-2}u = a_1(z)u^{-\gamma_1(z)} - b(z)\alpha(z)|v|^{\beta(z)}|u|^{\alpha(z)-2}u, \\ -\Delta_{\Phi(x,y)}v + |v|^{q(z)-2}v + |v|^{p(z)-2}v = a_2(z)v^{-\gamma_2(z)} - b(z)\beta(z)|u|^{\alpha(z)}|v|^{\beta(z)-2}v, \end{cases}$$

with $z = (x,y) \in \mathbb{R}^N$, $a_1, a_2, b, p, q, \beta \in C(\mathbb{R}^N, \mathbb{R})$, $\gamma_1, \gamma_2 : \mathbb{R}^N \to]0,1[$ are continuous functions such that $\gamma_1 < \gamma_2$ and $\Delta_{\Phi(x,y)}$ stands for the Baouendi-Grushin operator with variable coefficient, which is defined by

$$\Delta_{\Phi(x,y)}u = \sum_{i=1}^{n}\left(|\nabla_x u|^{\Phi(x,y)-2}u_{x_i}\right)_{x_i} + |x|^\gamma \sum_{i=1}^{m}\left(|\nabla_y u|^{\Phi(x,y)-2}u_{y_i}\right)_{y_i},$$

where $\Phi : \mathbb{R}^N \to]1,\infty[$ is a continuous function and $\mathbb{R}^N \approx \mathbb{R}^n \times \mathbb{R}^m$.

In our case, the situation is more general since we deal with a larger variety of singular terms. This forces us to use flexible algebraic calculations compatible with these functions.

Organization of paper: In the second section, we recall some properties of Musielak-Lebesgue spaces and first order Orlicz-Sobolev space. The third section is devoted to proving some properties of the new fractional variable-order Orlicz-Sobolev space and finally, in section 4, we present our existence result for problem (1.8).

2. Preliminaries

The basics on Musielak functions and Musielak-Lebesgue spaces may be found in the monographs [19, 25,26] but we recall here some necessary results. Let Ω be a measurable subset of $\mathbb{R}^n$. We denote by $L^0(\Omega)$ the set of measurable functions on Ω. We say that a quasi-normed space X is continuously embedded into a quasi-normed space Y, denoted, $X \hookrightarrow Y$, if $X \subset Y$ and there exists a constant $c > 0$ such that $||x||_Y \leq c||x||_X$ for all $x \in X$. The embedding of X into Y, denoted, $X \hookrightarrow\hookrightarrow Y$, if $X \hookrightarrow Y$ and bounded sets in X are precompact in Y.

2.1. Musielak functions

Definition 2.1.

- A function $G : \Omega \times [0, \infty[\longrightarrow [0, \infty[$ is called a weak Musielak function if the following conditions hold:

  (1) $x \mapsto G(x, |u(x)|)$ is measurable for all $u \in L^0(\Omega)$;
  (2) for a.e. $x \in \Omega$, $G(x, .)$ is continuous and strictly increasing;
  (3) There exists $a \geq 1$ such that

$$\frac{G(x,t)}{t} \leq a\frac{G(x,s)}{s}, \tag{2.1}$$

  for a.e. $x \in \Omega$ and for all $0 < t \leq s$.

- A weak Musielak function is called a Musielak function, if $G(x, .)$ is convex for a.e. $x \in \Omega$.
- A weak Musielak function is called an N-Musielak function if for a.e. $x \in \Omega$,

$$\lim_{t \to 0^+} \frac{G(x,t)}{t} = 0 \quad \text{and} \quad \lim_{t \to \infty} \frac{G(x,t)}{t} = \infty.$$

- When G is independent of the variable x, i.e. $G(x, t) = G(t)$, G is called a weak Orlicz function, Orlicz function or N-Orlicz function instead of weak Musielak function, Musielak function or N-Musielak function respectively.

Remark 2.1.

1. By continuity and assumption (2.1), we have

$$G(x, 0) = 0 \quad \textit{and} \quad \lim_{t \to \infty} G(x, t) = +\infty,$$

   for a.e. $x \in \Omega$.
2. If $G(x, .)$ is convex and $G(x, 0) = 0$ for a.e. $x \in \Omega$, then $G(x, .)$ is strictly increasing for a.e. $x \in \Omega$ and satisfies (2.1) with a=1.
3. A Musielak function can be represented as

$$G(x,t) = \int_0^t G'(x,s)\, \mathrm{d}s,$$

where $G'(x,.)$ is the right-hand derivative of $G(x,.)$. The function $G'(x,.)$ is right-continuous and nondecreasing.

Let G be a weak Musielak function. We consider the condition (A) that will be used throughout this paper.

There exist two constants $g^0 \geq g_0 > 1$ such that

$$1 < g_0 \leq \frac{tG'(x,t)}{G(x,t)} \leq g^0, \tag{A}$$

for a.e. $x \in \Omega$ and for all $t > 0$.

Under the condition (A), we have the following remarks.

Remark 2.2. [25] Let G be a weak Musielak function satisfying (A). Then, for a.e. $x \in \Omega$ we have

- $G(x, \beta t) \leq \beta^{g^0} G(x,t), \quad$ for all $t \geq 0$, $\beta > 1$;
- $G(x, \beta t) \geq \beta^{g_0} G(x,t), \quad$ for all $t \geq 0$, $\beta > 1$.

Definition 2.2. Let G be a weak Musielak function. Define $G^\star : \Omega \times [0,\infty[\to [0,\infty]$ for a.e. $x \in \Omega$, by

$$G^\star(x,s) := \sup_{t \geq 0} (st - G(x,t)) \quad s \geq 0.$$

$G^\star$ is called the complementary or conjugate function of G.

If $G^\star(x,.)$ is finite for a.e. $x \in \Omega$, it is also a Musielak function and can be represented as

$$G^\star(x,t) = \int_0^t \Big(G^\star(x,.)\Big)'(s)\mathrm{d}s,$$

with

$$\Big(G^\star(x,.)\Big)'(s) = \sup\{t \geq 0 : G'(x,t) \leq s\}.$$

By definition of $G^\star$,

$$st \leq G(x,t) + G^\star(x,s),$$

for a.e. $x \in \Omega$ and for all $s, t \geq 0$. This is called Young's inequality.

Definition 2.3. Let G be a weak Musielak function. We denote $G^{-1}(x,.) : [0,\infty[\to [0,\infty[$ for a.e. $x \in \Omega$ the inverse function of $G(x,.)$. Furthermore, we define $G^{-1} : \Omega \times [0,\infty[\to [0,\infty[$ such that:

$$G^{-1}(x,s) = t,$$

where $G(x,t) = s$.

The next proposition establishes that the conjugate function $G^\star$ satisfies also the condition (A).

Proposition 2.1. *[8] Let G be a Musielak function satisfying (A), then the following relations hold*

(1) For a.e. $x \in \Omega$, we have

$$G^\star(x, G'(x,t)) \leq (g^0 - 1)G(x,t), \quad \textit{for all} \quad t \geq 0;$$

(2) For a.e. $x \in \Omega$, $G^\star(x,.)$ satisfies

$$\frac{g^0}{g^0-1} \leq \frac{t\Big(G^\star(x,.)\Big)'(t)}{G^\star(x,t)} \leq \frac{g_0}{g_0-1}, \quad \textit{for all} \quad t \geq 0;$$

(3) For a.e. $x \in \Omega$, $G^{-1}(x,.)$ satisfies

$$\frac{1}{g^0} \leq \frac{t\Big(G^{-1}(x,.)\Big)'(t)}{G^{-1}(x,t)} \leq \frac{1}{g_0}, \quad \textit{for all} \quad t \geq 0.$$

When we study continuous and compact embeddings, we need the notion of comparison between functions. For this purpose, we have the next definition.

Definition 2.4.

(1) Let $\Phi, \Psi : \Omega \times [0,\infty[\to [0,\infty[$. Ψ is larger than Φ, denoted by $\Phi \leq \Psi$, if there exist $C > 0$ and $h \in L^1(\Omega)$ such that

$$\Phi(x, \frac{t}{C}) \leq \Psi(x,t) + h(x),$$

for a.e. $x \in \Omega$ and for all $t \geq 0$.

(2) Ψ is essentially larger than Φ, denoted by $\Phi \ll \Psi$, if for any $c > 0$

$$\lim_{t\to\infty} \Big(\sup_{x\in\bar{\Omega}} \frac{\Phi(x,ct)}{\Psi(x,t)}\Big) = 0,$$

where Ω is a bounded domain and Φ, Ψ are continuous functions on $\bar{\Omega} \times [0,\infty[$.

(3) Two functions $\Phi, \Psi : \Omega \times [0,\infty[\to [0,\infty[$ are equivalent, denoted by $\Phi \approx \Psi$, if there exists $L \geq 1$, such that

$$\frac{1}{L}\Psi(x,t) \leq \Phi(x,t) \leq L\Psi(x,t),$$

for a.e. $x \in \Omega$ and for all $t \geq 0$.

Remark 2.3. [1] It is easy to see that, $\Phi \ll \Psi$ if and only if

$$\lim_{t\to\infty} \Big(\sup_{x\in\bar{\Omega}} \frac{\Psi^{-1}(x,t)}{\Phi^{-1}(x,t)}\Big) = 0,$$

where Φ, Ψ are weak Musielak continuous functions on $\bar{\Omega} \times [0,\infty[$.

2.2. Musielak-Lebesgue spaces and first order Orlicz-Sobolev spaces

Given a weak Musielak function G and an open set $\Omega \subset \mathbb{R}^n$, we consider the Lebesgue-Musielak space $L^{G(\cdot)}(\Omega)$ defined as follows:

$$L^{G(\cdot)}(\Omega) := \Big\{ u \in L^0(\Omega) : \int_\Omega G(x, \lambda|u|)\, \mathrm{d}x < \infty \quad \text{for some } \lambda > 0 \Big\}.$$

This space is endowed with the so-called Luxemburg quasi-norm defined as:

$$\Big|\Big|u\Big|\Big|_G := \Big|\Big|u\Big|\Big|_{L^{G(\cdot)}(\Omega)} = \inf \Big\{ \lambda > 0 : \int_\Omega G(x, \frac{|u(x)|}{\lambda})\ \mathrm{d}x \leq 1 \Big\}.$$

A Hölder's type inequality holds [19]:

$$\int_\Omega u(x)v(x)\ \mathrm{d}x \leq 2\Big|\Big|u\Big|\Big|_G \Big|\Big|v\Big|\Big|_{G^\star}, \tag{2.2}$$

for all $u \in L^{G(\cdot)}(\Omega)$ and $v \in L^{G^\star(\cdot)}(\Omega)$.

Remark 2.4. If G is a weak Orlicz function, we write $L^G(\Omega)$ instead of $L^{G(\cdot)}(\Omega)$.

Also, we have the following known results.

Theorem 2.1. *[19,26] Let G be a Musielak function satisfying (A). Then $L^{G(\cdot)}(\Omega)$ is a reflexive separable Banach space.*

Theorem 2.2. *[19,26]*

1. *Let Φ, Ψ be two weak Musielak functions. If $\Phi \leq \Psi$, then $L^{\Psi(\cdot)}(\Omega)$ embedded continuously into $L^{\Phi(\cdot)}(\Omega)$.*
2. *Let Φ, Ψ be two weak Musielak functions. If $\Phi^{-1} \approx \Psi^{-1}$, then $L^{\Phi(\cdot)}(\Omega) = L^{\Psi(\cdot)}(\Omega)$.*

Proposition 2.2. *[25] Let G be a weak Musielak function satisfying (A), then*

$$\min\Big\{ \Big|\Big|u\Big|\Big|_G^{g_0}, \Big|\Big|u\Big|\Big|_G^{g^0} \Big\} \leq \int_\Omega G(x, |u(x)|)\ \mathrm{d}x \leq \max\Big\{ \Big|\Big|u\Big|\Big|_G^{g_0}, \Big|\Big|u\Big|\Big|_G^{g^0} \Big\},$$

for all $u \in L^{G(\cdot)}(\Omega)$.

Now, we recall the definition of the first order Orlicz-Sobolev space and also some continuous and compact embedding results. We define

$$W^{1,G}(\Omega) := \Big\{ u \in L^G(\Omega) : |\nabla u| \in L^G(\Omega) \quad \text{in the distribution sense} \Big\}.$$

This space is equipped with the norm

$$\Big|\Big|u\Big|\Big|_{1,G} := \Big|\Big|u\Big|\Big|_G + \Big|\Big|\nabla u\Big|\Big|_G.$$

In order for the Sobolev embedding results to hold, one needs to impose some condition on G. We consider the following assumption:

$$\int_0^1 \Big(\frac{z}{G(z)}\Big)^{\frac{1}{n-1}}\,\mathrm{d}z < \infty \quad \text{and} \quad \int_1^\infty \Big(\frac{z}{G(z)}\Big)^{\frac{1}{n-1}}\,\mathrm{d}z = \infty. \tag{2.3}$$

Under assumption (2.3), we define the optimal critical function by:

$$G_n(t) = GoH_n^{-1}(t),$$

where

$$H_n(t) = \Big(\int_0^t \Big(\frac{z}{G(z)}\Big)^{\frac{1}{n-1}}\,\mathrm{d}z\Big)^{\frac{n-1}{n}}.$$

The following fundamental Orlicz-Sobolev embedding theorem can be found in [13].

Theorem 2.3. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^n$ and G be an Orlicz function satisfying (2.3). Then, we have*

$$W^{1,G}(\Omega) \hookrightarrow L^{G_n}(\Omega).$$

Moreover, given Ψ any Orlicz function, the embedding

$$W^{1,G}(\Omega) \hookrightarrow L^{\Psi}(\Omega)$$

is compact if only if $\Psi \ll G_n$.

3. New class of fractional Sobolev spaces

In this section, we look for the properties of a new class of fractional-Sobolev spaces under optimal assumptions on generating weak Orlicz-function.

3.1. Almost increasing-almost decreasing properties

Let $G : \Omega \times [0, +\infty[\to [0, +\infty[$ be a function and Ω is a measurable subset of $\mathbb{R}^n$. We consider the following structure conditions:

$\textbf{(AInc)}_{g_0,a}$: There exist $g_0 > 0$ and $a \geq 1$ such that

$$\frac{G(x,t)}{t^{g_0}} \leq a\frac{G(x,z)}{z^{g_0}},$$

for a.e. $x \in \Omega$ and for all $0 < t < z$.

$\textbf{(ADec)}_{g^0,b}$: There exist $g^0 > 0$ and $b \geq 1$ such that

$$\frac{G(x,t)}{t^{g^0}} \leq b\frac{G(x,z)}{z^{g^0}},$$

for a.e. $x \in \Omega$ and for all $0 < z < t$.

From now on, we consider that a and b as the smallest constants characterizing the properties $(\text{AInc})_{g_0,a}$ and $(\text{ADec})_{g^0,b}$.

Remark 3.1.

(1) Every weak Musielak function satisfies $(\text{AInc})_{1,a}$ for some $a \geq 1$.
(2) Let G be a Musielak function satisfying the condition (A), then G satisfies $(\text{AInc})_{g_0,1}$ and $(\text{ADec})_{g^0,1}$.
(3) Let $G(x,.) \in C^2(]0,+\infty[)$ be a weak Orlicz function such that

$$0 < g_0 - 1 \leq \frac{tG''(x,t)}{G'(x,t)} \leq g^0 - 1 \quad \text{for all}\, t > 0.$$

Then, it is easy to see that

$$g_0 \leq \frac{tG'(x,t)}{G(x,t)} \leq g^0 \quad \text{for all}\, t > 0.$$

Conversely, if G satisfies

$$g_0 \leq \frac{tG'(x,t)}{G(x,t)} \leq g^0 \quad \text{for all}\, t > 0, \tag{3.1}$$

then $G'(x,.)$ satisfies $(\text{AInc})_{g_0-1,\frac{g^0}{g_0}}$ and $(\text{ADec})_{g^0-1,\frac{g^0}{g_0}}$. It is for this reason that we are interested in these properties.

Indeed, by (3.1), for all $t > 0$ we have

$$\frac{G(x,t)}{t^{g_0}} \leq \frac{1}{g_0}\frac{G'(x,t)}{t^{g_0-1}} \quad \text{and} \quad \frac{G'(x,t)}{t^{g_0-1}} \leq g^0 \frac{G(x,t)}{t^{g_0}}. \tag{3.2}$$

Let $0 < t < z$ then, by (3.2), we have

$$\frac{G'(x,t)}{t^{g_0-1}} \leq g^0 \frac{G(x,t)}{t^{g_0}} \leq g^0 \frac{G(x,z)}{z^{g_0}} \leq \frac{g^0}{g_0}\frac{G'(x,z)}{z^{g_0-1}},$$

therefore

$$\frac{G'(x,t)}{t^{g_0-1}} \leq \frac{g^0}{g_0}\frac{G'(x,z)}{z^{g_0-1}},$$

hence $G'(x,.)$ satisfies $(\text{AInc})_{g_0-1,\frac{g^0}{g_0}}$. Similarly, $G'(x,.)$ satisfies $(\text{ADec})_{g^0-1,\frac{g^0}{g_0}}$.
(4) If Ψ is a Musielak function satisfying $(\text{AInc})_{\psi_0,a}$ and $(\text{ADec})_{\psi^0,b}$ with $\psi_0 > 1$, then Ψ satisfies the condition (A) (see Lemma 2.2.6 in [19]).

Note that the condition (A) is used by Lieberman [24] and Simonenko [32]. The structure conditions almost increasing-almost decreasing are considered by Harjulehto-Hästö in [20] and also by Ragusa et al. in [18].

Now, we define a function that plays an important role in our hypotheses of problem (1.8).

Let $\Psi, \Phi : \Omega \times [0,\infty[\to [0,\infty[$. We define $\Psi o \Phi : \Omega \times [0,\infty[\to [0,\infty[$ by:

$$\Psi o \Phi(x,t) := \Psi(x, \Phi(x,t)).$$

If Ψ, Φ are two functions independent of the variable x, then the definition above $\Psi o \Phi$ matches with the classical composition of two functions.

Proposition 3.1.

(1) Let $\Psi, \Phi : \Omega \times [0, \infty[\to [0, \infty[$ satisfying $(AInc)_{\psi_0, a_1}$ and $(AInc)_{\phi_0, a_2}$ respectively. If for a.e. $x \in \Omega$, $\Phi(x, .)$ is strictly increasing and $\Phi(x, 0) = 0$, then $\Psi o \Phi$ satisfies $(AInc)_{\phi_0 \psi_0, a_1 a_2^{\psi_0}}$.
(2) Let $\Psi, \Phi : \Omega \times [0, \infty[\to [0, \infty[$ satisfying $(ADec)_{\psi^0, b_1}$ and $(ADec)_{\phi^0, b_2}$ respectively. If for a.e. $x \in \Omega$, $\Phi(x, .)$ is strictly increasing and $\Phi(x, 0) = 0$, then $\Psi o \Phi$ satisfies $(ADec)_{\phi^0 \psi^0, b_1 b_2^{\psi^0}}$.
(3) If Ψ, Φ are weak Musielak functions, then $\Psi o \Phi$ is a weak Musielak function.
(4) If Ψ, Φ are Musielak functions, then $\Psi o \Phi$ is a Musielak function.
(5) If Ψ, Φ are N-Musielak functions, then $\Psi o \Phi$ is an N-Musielak function.

Proof.(1) Let $0 < t < z$. Then, for a.e. $x \in \Omega$ we have

$$\begin{aligned}\frac{\Psi(x, \Phi(x, t))}{t^{\phi_0 \psi_0}} &= \frac{\Psi(x, \Phi(x, t))}{\Big(\Phi(x, t)\Big)^{\psi_0}} \frac{\Big(\Phi(x, t)\Big)^{\psi_0}}{t^{\phi_0 \psi_0}} \\ &\leq a_1 \frac{\Psi(x, \Phi(x, z))}{\Big(\Phi(x, z)\Big)^{\psi_0}} \Big(a_2 \frac{\Phi(x, z)}{z^{\phi_0}}\Big)^{\psi_0} \\ &\leq a_1 a_2^{\psi_0} \frac{\Psi(x, \Phi(x, z))}{z^{\phi_0 \psi_0}}.\end{aligned}$$

Therefore, $\Psi o \Phi$ satisfies $(\mathrm{AInc})_{\phi_0 \psi_0, a_1 a_2^{\psi_0}}$. In the same way, the other assertions are proved. □

Examples 3.1.

(1) $\Psi_1(x, t) = t^{p(x)} \Big((p(x) + 1) Log(e + t) + \frac{t}{e+t} \Big)$ satisfies $(\mathrm{AInc})_{p^-, \frac{p^+ + 1}{p^- + 1}}$ and $(\mathrm{ADec})_{p^+, \frac{p^+ + 1}{p^- + 1}}$, where p is a measurable function on Ω, $p^+ = \sup_{x \in \Omega} p(x) < \infty$ and $p^- = \inf_{x \in \Omega} p(x) > 0$.
(2) $\Psi_2(x, t) = t^p + b(x) t^p Log(e + t)$ satisfies $(\mathrm{AInc})_{p,1}$ and $(\mathrm{ADec})_{p+1,1}$, where $b \in L^\infty_+(\Omega)$.
(3) $\Psi_3(x, t) = e^t - t - 1$ is not satisfying almost decreasing property.

3.2. Fractional variable-order Orlicz-Sobolev space

Given an open set $\Omega \subset \mathbb{R}^n$ and a measurable function $s : \Omega \times \Omega \to]0, 1[$, to simplify the notations, we denote $\mathrm{d}\mu = \frac{\mathrm{d}x \mathrm{d}y}{|x-y|^n}$ the regular Borel measure on $\Omega \times \Omega$, and we define the $s(.)-$ Hölder quotient by:

$$\nabla^s u(x, y) := \tfrac{u(x) - u(y)}{|x-y|^{s(x,y)}}.$$

Remark 3.2.

(1) If $s(x, y) = s(y, x)$, then we have $\nabla^s u(x, y) = -\nabla^s u(y, x)$.
(2) $\nabla^s (u + v) = \nabla^s u + \nabla^s v$.
(3) $\nabla^s \lambda u = \lambda \nabla^s u$, for all $\lambda \in \mathbb{R}$.
(4) $\nabla^s (uv)(x, y) = u(x) \nabla^s v(x, y) + v(y) \nabla^s u(x, y)$.

Let G be a weak Orlicz function, we introduce the fractional variable-order Orlicz-Sobolev space $W^{s(.),G}(\Omega)$ as follows:

$$W^{s(.),G}(\Omega) := \Big\{u \in L^G(\Omega) : \iint\limits_{\Omega\times\Omega} G(\lambda|\nabla^s u|)\mathrm{d}\mu < \infty \quad \text{for some}\, \lambda > 0\Big\}.$$

For any $u \in W^{s(.),G}(\Omega)$, let

$$M_G(u) := \int\limits_{\Omega} G(|u|)\mathrm{d}x + \iint\limits_{\Omega\times\Omega} G(|\nabla^s u|)\mathrm{d}\mu$$

and

$$\Big|\Big|u\Big|\Big|_{s(.),G} := \inf\Big\{\lambda > 0 : M_G(\frac{u}{\lambda}) \leq 1\Big\}.$$

The next theorem proves that $\Big|\Big|.\Big|\Big|_{s(.),G}$ is a quasi-norm in $W^{s(.),G}(\Omega)$ and the constant of triangle quasi-norm is smaller or equal to $8a$, where a is the constant of $(\mathrm{AInc})_{1,a}$ property.

Theorem 3.1. *Let G be a weak Orlicz function, then we have:*

(1) $\Big|\Big|.\Big|\Big|_{s(.),G}$ *is a quasi-norm;*
(2) If G is an Orlicz function, then $\Big|\Big|.\Big|\Big|_{s(.),G}$ *is a norm.*

Proof. 1. By standard arguments, we have absolute homogeneity and positive definiteness. We prove quasi-triangle inequality. Let $u, v \in W^{s(.),G}(\Omega)$, $\alpha > \Big|\Big|u\Big|\Big|_{s(.),G}$, $\beta > \Big|\Big|v\Big|\Big|_{s(.),G}$ and $x, y \in \Omega$. By definition of $\Big|\Big|.\Big|\Big|_{s(.),G}$, we have

$$M_G(\frac{u}{\alpha}) \leq 1 \quad \text{and} \quad M_G(\frac{v}{\beta}) \leq 1.$$

Since G satisfies

$$\frac{G(t)}{t} \leq a\frac{G(z)}{z} \quad \text{for all} \quad 0 < t \leq z,$$

we get the following inequalities:

$$G(\tfrac{|u(x)|}{4a\alpha}) \leq \tfrac{1}{4}G(\tfrac{|u(x)|}{\alpha}), \quad G(\tfrac{|v(x)|}{4a\beta}) \leq \tfrac{1}{4}G(\tfrac{|v(x)|}{\beta}),$$

$$G(\tfrac{|\nabla^s u(x,y)|}{4a\alpha}) \leq \tfrac{1}{4}G(\tfrac{|\nabla^s u(x,y)|}{\alpha}) \text{ and } G(\tfrac{|\nabla^s v(x,y)|}{4a\beta}) \leq \tfrac{1}{4}G(\tfrac{|\nabla^s v(x,y)|}{\beta}).$$

Using the inequality

$$G(t_1 + t_2) \leq G(2t_1) + G(2t_2) \quad \text{for any} \quad t_1, t_2 \geq 0,$$

we obtain

$$\begin{aligned}\int_\Omega G(\frac{|u(x)+v(x)|}{8a(\alpha+\beta)})\mathrm{d}x &\le \int_\Omega G(\frac{2|u(x)|}{8a\alpha})\mathrm{d}x + \int_\Omega G(\frac{2|v(x)|}{8a\beta})\mathrm{d}x \\ &\le \frac{1}{4}\int_\Omega G(\frac{|u(x)|}{\alpha})\mathrm{d}x + \frac{1}{4}\int_\Omega G(\frac{|v(x)|}{\beta})\mathrm{d}x \\ &\le \frac{1}{2}.\end{aligned}$$

Similarly, we have

$$\iint_{\Omega\times\Omega} G(\frac{|\nabla^s(u+v)|}{8a(\alpha+\beta)})\mathrm{d}\mu \le \frac{1}{2}.$$

Thus

$$\Big|\Big|u+v\Big|\Big|_{s(.),G} \le 8a(\alpha+\beta),$$

therefore

$$\Big|\Big|u+v\Big|\Big|_{G,s(.)} \le 8a\Big(\Big|\Big|u\Big|\Big|_{G,s(.)} + \Big|\Big|v\Big|\Big|_{s(.),G}\Big).$$

This means that $\Big|\Big|.\Big|\Big|_{s(.),G}$ is a quasi-norm.

2. Assume that G is an Orlicz function. Let $\alpha > \Big|\Big|u\Big|\Big|_{s(.),G}$ and $\beta > \Big|\Big|v\Big|\Big|_{s(.),G}$. By the convexity of M_G,

$$\begin{aligned}M_G\Big(\frac{\alpha}{\alpha+\beta}\frac{u}{\alpha} + \frac{\beta}{\alpha+\beta}\frac{v}{\beta}\Big) &\le \frac{\alpha}{\alpha+\beta}M_G(\frac{u}{\alpha}) + \frac{\beta}{\alpha+\beta}M_G(\frac{v}{\beta}) \\ &\le \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1.\end{aligned}$$

Thus $\Big|\Big|u+v\Big|\Big|_{G,s(.)} \le \alpha+\beta$, which yield $\Big|\Big|u+v\Big|\Big|_{G,s(.)} \le \Big|\Big|u\Big|\Big|_{G,s(.)} + \Big|\Big|v\Big|\Big|_{G,s(.)}$, as required for (2). □

Remark 3.3.

1. On $W^{s(.),G}(\Omega)$, the following quasi-norms:

$$\begin{aligned}\Big|\Big|.\Big|\Big|^G_{s(.)} :\ & W^{s(.),G}(\Omega) \longrightarrow \mathbb{R}^+ \\ & u \ \mapsto\ ||u||_G + [u]_{s(.),G}\end{aligned},$$

$$\begin{aligned}\Big|\Big|.\Big|\Big|^{\max}_{s(.),G} :\ & W^{s(.),G}(\Omega) \longrightarrow \mathbb{R}^+ \\ & u \ \mapsto\ \max\Big(||u||_G, [u]_{s(.),G}\Big)\end{aligned}$$

and

$$\begin{aligned}\Big|\Big|.\Big|\Big|_{s(.),G} :\ & W^{s(.),G}(\Omega) \longrightarrow \mathbb{R}^+ \\ & u \ \mapsto\ \inf\Big\{\lambda>0 : M_G(\tfrac{u}{\lambda}) \le 1\Big\}\end{aligned}$$

are equivalent, where $[u]_{s(.),G} := \inf\Big\{\lambda > 0 : \iint\limits_{\Omega\times\Omega} G(\frac{|\nabla^s u|}{\lambda})\mathrm{d}\mu \leq 1\Big\}$.

2. If G is an Orlicz function, then $\Big|\Big|.\Big|\Big|^{G}_{s(.)}$ and $\Big|\Big|.\Big|\Big|^{\max}_{s(.),G}$ are also norms.

The following proposition gives the relation between the quasi-norm and the corresponding modular function M_G under the minimal assumptions on the generating weak Orlicz function G.

Proposition 3.2. *Let G be a weak Orlicz function satisfies (AInc)$_{g_0,a}$ and (ADec)$_{g^0,b}$ properties with $g^0 \geq g_0 \geq 1$. Let $u \in W^{s(.),G}(\Omega)$, then the following properties hold:*

$$\Big|\Big|u\Big|\Big|_{s(.),G} > 1 \quad \textit{implies} \quad \frac{1}{a}\Big|\Big|u\Big|\Big|^{g_0}_{s(.),G} \leq M_G(u) \leq b\Big|\Big|u\Big|\Big|^{g^0}_{s(.),G}. \tag{3.3}$$

$$\Big|\Big|u\Big|\Big|_{s(.),G} < 1 \quad \textit{implies} \quad \frac{1}{ab}\Big|\Big|u\Big|\Big|^{g^0}_{s(.),G} \leq M_G(u) \leq a\Big|\Big|u\Big|\Big|^{g_0}_{s(.),G}. \tag{3.4}$$

Proof. Let $u \in W^{s(.),G}(\Omega)$ such that $\Big|\Big|u\Big|\Big|_{s(.),G} > 1$ and $x, y \in \Omega$. By (ADec)$_{g^0,b}$ property, we have

$$\frac{G\Big(||u||_{s(.),G}\cdot\frac{|u(x)|}{||u||_{s(.),G}}\Big)}{|u(x)|^{g^0}} \leq b\frac{G\Big(\frac{|u(x)|}{||u||_{s(.),G}}\Big)}{\Big(\frac{|u(x)|}{||u||_{s(.),G}}\Big)^{g^0}}$$

and

$$\frac{G\Big(||u||_{s(.),G}\cdot\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}}\Big)}{|\nabla^s u(x,y)|^{g^0}} \leq b\frac{G\Big(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}}\Big)}{\Big(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}}\Big)^{g^0}}.$$

Hence, we have

$$\begin{aligned}
&\int\limits_\Omega G(|u(x)|)\mathrm{d}x + \int\limits_\Omega\int\limits_\Omega G(|\nabla^s u(x,y)|)\mathrm{d}\mu \\
&\leq b\Big(\int\limits_\Omega G(\frac{|u(x)|}{||u||_{s(.),G}})\mathrm{d}x + \int\limits_\Omega\int\limits_\Omega G(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}})\mathrm{d}\mu\Big)\Big|\Big|u\Big|\Big|^{g^0}_{s(.),G}.
\end{aligned}$$

Therefore

$$M_G(u) \leq b\Big|\Big|u\Big|\Big|^{g^0}_{s(.),G}.$$

Let $\alpha \in \Big]1, \Big|\Big|u\Big|\Big|_{s(.),G}\Big[$ and $x, y \in \Omega$. By (AInc)$_{g_0,a}$ property, we have

$$\frac{G\Big(\frac{|u(x)|}{\alpha}\Big)}{\Big(\frac{|u(x)|}{\alpha}\Big)^{g_0}} \leq a\frac{G\Big(|u(x)|\Big)}{\Big(|u(x)|\Big)^{g_0}}$$

and

$$\frac{G\Big(\frac{|\nabla^s u(x,y)|}{\alpha}\Big)}{\Big(\frac{|\nabla^s u(x,y)|}{\alpha}\Big)^{g_0}} \leq a\frac{G\Big(|\nabla^s u(x,y)|\Big)}{\Big(|\nabla^s u(x,y)|\Big)^{g_0}}.$$

Hence

$$\begin{aligned}&\int\limits_{\Omega} G(|u(x)|)\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(|\nabla^s u(x,y)|)\mathrm{d}\mu \\ &\geq \alpha^{g_0}\frac{1}{a}\Big(\int\limits_{\Omega} G(\frac{|u(x)|}{\alpha})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{\alpha})\mathrm{d}\mu\Big).\end{aligned}$$

Since $\alpha < \Big|\Big|u\Big|\Big|_{s(.),G}$, then we obtain

$$\int\limits_{\Omega} G(\frac{|u(x)|}{\alpha})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{\alpha})\mathrm{d}\mu > 1,$$

therefore

$$\int\limits_{\Omega} G(|u(x)|)\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(|\nabla^s u(x,y)|)\mathrm{d}\mu \geq \frac{1}{a}\alpha^{g_0}.$$

It follows that, letting $\alpha \nearrow \Big|\Big|u\Big|\Big|_{s(.),G}$, we deduce (3.3).

Let $u \in W^{s(.),G}(\Omega)$ such that $\Big|\Big|u\Big|\Big|_{s(.),G} < 1$. By $(\mathrm{AInc})_{g_0,a}$ property, we have

$$\frac{G\Big(|u(x)|\Big)}{\Big(|u(x)|\Big)^{g_0}} \leq a\frac{G\Big(\frac{|u(x)|}{||u||_{s(.),G}}\Big)}{\Big(\frac{|u(x)|}{||u||_{s(.),G}}\Big)^{g_0}}$$

and

$$\frac{G\Big(|\nabla^s u(x,y)|\Big)}{\Big(|\nabla^s u(x,y)|\Big)^{g_0}} \leq a\frac{G\Big(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}}\Big)}{\Big(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}}\Big)^{g_0}}.$$

Using the definition of the quasi-norm, we obtain

$$M_G(u) \leq a\Big|\Big|u\Big|\Big|^{g_0}_{s(.),G}\Big(\int\limits_{\Omega} G(\frac{|u(x)|}{||u||_{s(.),G}})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{||u||_{s(.),G}})\mathrm{d}\mu\Big),$$

hence

$$M_G(u) \leq a\Big|\Big|u\Big|\Big|^{g_0}_{s(.),G}.$$

Now, let $\beta \in \Big]0, \Big|\Big|u\Big|\Big|_{s(.),G}\Big[$ with $\Big|\Big|u\Big|\Big|_{s(.),G} < 1$. Since $\beta < 1$, then by $(\mathrm{ADec})_{g^0,b}$ property we have

$$\frac{G\Big(|u(x)|\Big)}{\Big(|u(x)|\Big)^{g^0}} \geq \frac{1}{b}\frac{G\Big(\frac{|u(x)|}{\beta}\Big)}{\Big(\frac{|u(x)|}{\beta}\Big)^{g^0}}$$

and

$$\frac{G\Big(|\nabla^s u(x,y)|\Big)}{\Big(|\nabla^s u(x,y)|\Big)^{g^0}} \geq \frac{1}{b}\frac{G\Big(\frac{|\nabla^s u(x,y)|}{\beta}\Big)}{\Big(\frac{|\nabla^s u(x,y)|}{\beta}\Big)^{g^0}}.$$

It follows that

$$\int\limits_{\Omega} G(|u(x)|)\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(|\nabla^s u(x,y)|)\mathrm{d}\mu \tag{3.5}$$

$$\geq \frac{1}{b}\beta^{g^0}\Big(\int\limits_{\Omega} G(\frac{|u(x)|}{\beta})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{\beta})\mathrm{d}\mu\Big). \tag{3.6}$$

If we take $v = \frac{u}{\beta}$, we obtain

$$\Big|\Big|v\Big|\Big|_{s(.),G} = \Big|\Big|\frac{u}{\beta}\Big|\Big|_{s(.),G}.$$

Using (3.3), we find

$$\int\limits_{\Omega} G(\frac{|u(x)|}{\beta})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{\beta})\mathrm{d}\mu \geq \frac{1}{a}\Big|\Big|v\Big|\Big|^{g_0}_{s(.),G} \geq \frac{1}{a}. \tag{3.7}$$

Combining (3.6) and (3.7), we deduce that

$$\int\limits_{\Omega} G(\frac{|u(x)|}{\beta})\mathrm{d}x + \int\limits_{\Omega}\int\limits_{\Omega} G(\frac{|\nabla^s u(x,y)|}{\beta})\mathrm{d}\mu \geq \frac{1}{ab}\beta^{g^0}.$$

Letting $\beta \nearrow \Big|\Big|u\Big|\Big|_{s(.),G}$, we obtain (3.4). $\square$

The following theorem gives criteria for which fractional variable-order Orlicz-Sobolev space is reflexive, uniformly convex and separable.

Theorem 3.2. *Let G be a weak Orlicz function.*

(1) If G satisfies $(ADec)_{g^0,b}$ property, then $\Big(W^{s(.),G}(\Omega), \Big|\Big|.\Big|\Big|_{G,s(.)}\Big)$ is a quasi-Banach space.
(2) If G satisfies $(ADec)_{g^0,b}$ property with $g^0 > 1$, then $\Big(W^{s(.),G}(\Omega), \Big|\Big|.\Big|\Big|_{G,s(.)}\Big)$ is separable.
(3) If G satisfies $(AInc)_{g_0,a}$ with $g_0 > 1$ and $(ADec)_{g^0,b}$, then $W^{s(.),G}(\Omega)$ is uniformly convex and reflexive.

Proof. (1) By Remark 3.3, we know that $\Big|\Big|.\Big|\Big|_{s(.),G}$ and $\Big|\Big|.\Big|\Big|^{s(.)}_{G}$ are equivalent norm on $W^{s(.),G}(\Omega)$.

Same as of the proof of the Proposition 3.2, we have

$$\min\Big(\frac{1}{a}[u]^{g_0}_{s(.),G}, \frac{1}{ab}[u]^{g^0}_{s(.),G}\Big) \leq \iint\limits_{\Omega\times\Omega} G(|\nabla^s u|)\mathrm{d}\mu \leq \max\Big(a[u]^{g_0}_{s(.),G}, b[u]^{g^0}_{s(.),G}\Big).$$

Let $\{u_m\}_{m\geq 1} \subset W^{s(.),G}(\Omega)$ be a Cauchy sequence. Then, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\Big|\Big|u_m - u_j\Big|\Big|^{G}_{s(.)} < \epsilon, \quad \text{for all}\, m, j > N, \tag{3.8}$$

which yield that

$$\Big|\Big|u_m - u_j\Big|\Big|_{G} \leq \Big|\Big|u_m - u_j\Big|\Big|^{G}_{s(.)} < \epsilon.$$

Apply the completeness of $L^G(\Omega)$ to a find a $u \in L^G(\Omega)$ such that $u_m \to u$ in $L^G(\Omega)$ as $m \to \infty$. Then, there exists a subsequence $\{u_{m_k}\} \subset W^{s(.),G}(\Omega)$ such that $u_{m_k} \to u$ almost everywhere in Ω. As a result, the Fatou's lemma and the inequality (3.8) with $\epsilon = 1$ imply that

$$\begin{aligned}
\iint\limits_{\Omega\times\Omega} G(|\nabla^s u|)\mathrm{d}\mu &\leq \liminf_{k\to\infty} \iint\limits_{\Omega\times\Omega} G(|\nabla^s u_{m_k}|)\mathrm{d}\mu \\
&\leq C \liminf_{k\to\infty} \max\Big([u_{m_k} - u_j]^{g^0}_{s(.),G} + [u_j]^{g^0}_{s(.),G}, [u_{m_k} - u_j]_{s(.),G} + [u_j]_{s(.),G}\Big) \\
&\leq C\Big(1 + \max\Big([u_j]^{g^0}_{s(.),G}, [u_j]_{s(.),G}\Big) < \infty,
\end{aligned}$$

and

$$\begin{aligned}
\Big|\Big|u\Big|\Big|_G &\leq C \liminf_{k\to\infty}\Big(\Big|\Big|u_{m_k} - u_j\Big|\Big|_G + \Big|\Big|u_j\Big|\Big|_G\Big) \\
&\leq C\Big(1 + \Big|\Big|u_j\Big|\Big|_G\Big) < \infty
\end{aligned}$$

for any fixed $j > 0$, where $C(a, b) > 0$. Therefore, $u \in W^{s(.),G}(\Omega)$. Then, the Fatou's lemma leads to

$$\Big|\Big|u_m - u\Big|\Big|^{G}_{s(.)} \leq \liminf_{k\to\infty} \Big|\Big|u_m - u_{m_k}\Big|\Big|^{G}_{s(.)} < \epsilon,$$

i.e., $u_m \to u$ in $W^{s(.),G}(\Omega)$ as $m \to \infty$.

Let

$$\mathcal{Q} : W^{s(.),G}(\Omega) \to L^G(\Omega) \times L^G\Big(\Omega \times \Omega, \mathrm{d}\mu\Big)$$

defined by

$$\mathcal{Q}(u) = \Big(u, \tfrac{u(x)-u(y)}{|x-y|^{s(x,y)}}\Big).$$

By the mapping $\mathcal{Q}$, the space $W^{s(.),G}(\Omega)$ is a closed subspace of $L^G(\Omega) \times L^G\Big(\Omega \times \Omega, \mathrm{d}\mu\Big)$.

(2) By Theorem (3.5.2) in [19], $L^G(\Omega)$ and $L^G\Big(\Omega\times\Omega, \mathrm{d}\mu\Big)$ are separable since G satisfies $(\mathrm{ADec})_{g^0,b}$ property with $g^0 > 1$. Thus, $W^{s(.),G}(\Omega)$ is separable.

(3) Uniform convexity and reflexivity depend only on the space, hence by Corollary (3.6.7) in [19], $L^G(\Omega)$ and $L^G\Big(\Omega \times \Omega, \mathrm{d}\mu\Big)$ are uniformly convex and reflexive spaces since G satisfies $(\mathrm{AInc})_{g_0,a}$ with $g_0 > 1$ and $(\mathrm{ADec})_{g^0,b}$ properties. Therefore, $W^{s(.),G}(\Omega)$ is uniformly convex and reflexive. □

3.3. Embedding theorems

Let Ω be a bounded domain with Lipshitz boundary, $s : \mathbb{R}^n \times \mathbb{R}^n \to]0,1[$ be a measurable function and G be an Orlicz function satisfying (A). This subsection is devoted to the embedding results of the new fractional Orlicz-Sobolev space $W^{s(.),G}(\Omega)$.

Theorem 3.3. *Let Ψ be a Musielak function. Suppose there exists K disjoint open sets Ω_i of Ω with Lipshitz boundary such that $\bar{\Omega} \subset \left(\bigcup_{i=1}^{i=K} \Omega_i\right) \cup N$, $|N| = 0$ and for all $i = 1, 2, ...K$*

$$\int_1^\infty \left(\frac{z}{G(z)}\right)^{\frac{s_i}{n-s_i}} \mathrm{d}z = \infty \quad \textit{and} \quad \int_0^1 \left(\frac{z}{G(z)}\right)^{\frac{s_i}{n-s_i}} \mathrm{d}z < \infty,$$

where

$$s_i = \inf_{(x,y)\in\Omega_i\times\Omega_i} s(x, y).$$

Set

$$\Psi_i^+(t) := \sup_{x\in\Omega_i} \Psi(x, t) < \infty \quad \textit{for all} t \geqslant 0$$

and

$$G_{\frac{n}{s_i}}(t) = GoH_{\frac{n}{s_i}}^{-1}(t),$$

where

$$H_{\frac{n}{s_i}}(t) = \Big(\int_0^t \left(\frac{z}{G(z)}\right)^{\frac{s_i}{n-s_i}} \mathrm{d}z\Big)^{\frac{n-s_i}{n}}.$$

If $\Psi_i^+ \leq G_{\frac{n}{s_i}}$ for all $i = 1, ...K$, then there is a continuous embedding of $W^{s(.),G}(\Omega)$ into $L^{\Psi(.)}(\Omega)$.

Moreover, if $\Psi_i^+ \ll G_{\frac{n}{s_i}}$ for all $i = 1, ...K$, then the embedding

$$W^{s(.),G}(\Omega) \hookrightarrow L^{\Psi(.)}(\Omega)$$

is compact.

Now, we state another version of embedding results whose proof is similar to the previous theorem.

Theorem 3.4. *Ψ is an N-Musielak function and $n \geq 2$. Suppose that G is an N-Orlicz function and there exists K disjoint of open sets Ω_i of Ω with Lipshitz boundary such that $\bar{\Omega} \subset \left(\bigcup_{i=1}^{i=K} \Omega_i\right) \cup N$, $|N| = 0$ and for all $i = 1, 2, ...K$*

$$\int_0^1 \frac{G^{-1}(z)}{z^{\frac{n+s_i}{n}}} \mathrm{d}z < \infty \quad \textit{and} \quad \int_0^\infty \frac{G^{-1}(z)}{z^{\frac{n+s_i}{n}}} \mathrm{d}z = \infty,$$

where

$$s_i = \inf_{(x,y)\in\Omega_i\times\Omega_i} s(x,y).$$

Set

$$\Psi_i^+(t) := \sup_{x\in\Omega_i} \Psi(x,t) < \infty \quad \textit{for all} t \geqslant 0$$

and

$$G_{\star,s_i} \textit{ the N-Orlicz function defined by their inverse by}$$
$$t \mapsto \int_0^t \frac{G^{-1}(z)}{z^{\frac{n+s_i}{n}}} \mathrm{d}z.$$

If $\Psi_i^+ \leq G_{\star,s_i}$ for all $i = 1, ...K$, then there is a continuous embedding of $W^{s(.),G}(\Omega)$ into $L^{\Psi(.)}(\Omega)$.

Moreover, if $\Psi_i^+ \ll G_{\star,s_i}$ for all $i = 1, ...K$, then the embedding

$$W^{s(.),G}(\Omega) \hookrightarrow L^{\Psi(.)}(\Omega)$$

is compact.

Remark 3.4.

1. Under the additional restrictive condition (A), Theorem 3.3 is a sharper version of Theorem (8.1) in [2] when the order function s is constant.
2. When the order function s is constant, Theorem 3.4 reduces to the Theorem (1.2) in [6].
3. When $G(t) = t^2$, Theorems 3.3 and 3.4 reduce to the Theorem (2.3) in [12]. Moreover, continuity condition for the order function s is not required in our Theorems. Clearly, our results also generalizes Theorem (2.3) in [12] for any $p > 1$ if we consider $G(t) = t^p$. We state the result in this case:

   Let $\Omega \subset \mathbb{R}^n$ be a Lipshitz bounded domain and $\alpha : \bar{\Omega} \to [1, \alpha^+]$ be a continuous function. Suppose there exists K disjoint of open sets of Ω with Lipshitz boundary such that $\bar{\Omega} \subset \left(\cup_{i=1}^{i=K} \Omega_i\right) \cup N$, $|N| = 0$ and for all $i = 1, 2, ...K$

$$\sup_{y\in\Omega_i} \{\alpha(y)\} \leq \frac{np}{n - s_i p},$$

   where $s_i = \inf_{(x,y)\in\Omega_i\times\Omega_i} s(x,y)$. Then, we have

$$W^{s(.),p}(\Omega) \hookrightarrow L^{\alpha(.)}(\Omega).$$

In the next theorem, we state another version of compactness results under the continuity condition in the order function, this result allows us to identify one of the function that can plays the role of a critical function in the class of Musielak functions. Moreover The analogous result for Musielak Sobolev spaces was proved by P. Harjulehto and P. Hästö in Theorem (6.3.8) [19].

Theorem 3.5. *Ψ be a weak Musielak function, $n \geq 2$ and p be a continuous function on $\bar{\Omega}$. Suppose that*

1. s is continuous on $\bar{\Omega} \times \bar{\Omega}$;

2. $g^0 s(x,x) < n$ *for all* $x \in \bar{\Omega}$*;*
3. $\Psi^{-1} \approx t^{-p(.)} G^{-1}$, *where* $p(x) \in \left]0, \frac{s(x,x)}{n}\right[$ *for all* $x \in \bar{\Omega}$.

Then, the embedding

$$W^{s(.),G}(\Omega) \hookrightarrow L^{\Psi(.)}(\Omega)$$

is compact.

Remark 3.5.

1. The condition of the continuity of the order function s in Theorem 3.5 is introduce to enlarge the interval of compactness.
2. If $G(t) = t^p$ and $q \in C(\bar{\Omega})$ verify

$$1 < q(x) < p^{\star}(x) = \frac{np}{n - ps(x,x)} \quad \text{for all}\, x \in \bar{\Omega},$$

   we obtain the embedding

$$W^{s(.),p}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega).$$

   The function $(x,t) \mapsto G_{\star,p}(x,t) := t^{\frac{np}{n-ps(x,x)}}$ plays the role of a critical function in this case. For a special case, see Theorem (2.1) in [33].
3. Let $q \in C(\bar{\Omega})$ such that

$$1 < q(x) < \frac{g_0 n}{n - g_0 s(x,x)} \quad \text{for all}\, x \in \bar{\Omega}.$$

   Then, we have

$$W^{s(.),G}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega).$$

   Question: Let Φ be a weak Musielak function such that

$$\Phi^{-1} \approx t^{-\frac{s(.)}{n}} G^{-1}.$$

   Is $W^{s(.),G}(\Omega)$ continuously embedded into $L^{\Phi(.)}(\Omega)$?

Here and in the remainder of the proof, C denotes a positive constant independent of $u \in W^{s(.),G}(\Omega)$ and may vary from line to line.

Lemma 3.1. *Let* Ω *be either a bounded or an unbounded open subset of* $\mathbb{R}^n$*. Then, we have*

$$W^{s^+,G}(\Omega) \hookrightarrow W^{s(.),G}(\Omega) \hookrightarrow W^{s^-,G}(\Omega),$$

where $0 < s^- := \inf_{(x,y)\in\Omega\times\Omega} s(x,y)$ *and* $s^+ := \sup_{(x,y)\in\Omega\times\Omega} s(x,y) < 1$.

Proof. Let $u \in W^{s(.),G}(\Omega) \setminus \{0\}$. Then we have

$$\begin{aligned}
&\int_\Omega \int_\Omega G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad = \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} + \int_\Omega \int_{\Omega\cap\{|x-y|<1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^-}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} + \int_\Omega \int_{\Omega\cap\{|x-y|<1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^+}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^-}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} + \int_\Omega \int_\Omega G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^+}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}.
\end{aligned}$$

We claim that

$$\int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^-}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \leq C\int_\Omega G(|u(x)|)\,\mathrm{d}x.$$

Since $t \mapsto \frac{G(t)}{t}$ is increasing, then we have

$$\begin{aligned}
&\int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)|}{|x-y|^{s^-}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} \frac{G(|u(x)|)}{|x-y|^{s^-}}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq C\int_\Omega G(|u(x)|)\,\mathrm{d}x,
\end{aligned}$$

where we used the fact that $\displaystyle\int_{|z|\geq 1} \frac{1}{|z|^{n+s^-}}\,\mathrm{d}z$ exists.

By condition (A), we have

$$\begin{aligned}
&\int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^-}}\Big)\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} \Big[G\Big(\frac{2|u(x)|}{|x-y|^{s^-}}\Big) + G\Big(\frac{2|u(y)|}{|x-y|^{s^-}}\Big)\Big]\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq 2^{g^0}\int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} \Big[G\Big(\frac{|u(x)|}{|x-y|^{s^-}}\Big) + G\Big(\frac{|u(y)|}{|x-y|^{s^-}}\Big)\Big]\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n} \\
&\quad \leq C\int_\Omega G(|u(x)|)\,\mathrm{d}x.
\end{aligned}$$

Thus, the claim is valid. So, we obtain

$$\int_\Omega \int_\Omega G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s(x,y)}}\Big)\,\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}$$
$$\leq C\Big(\int_\Omega G(|u(x)|)\,\mathrm{d}x + \int_\Omega \int_\Omega G\Big(\frac{|u(x)-u(y)|}{|x-y|^{s^+}}\Big)\,\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^n}\Big).$$

Therefore

$$W^{s^+,G}(\Omega) \hookrightarrow W^{s(.),G}(\Omega).$$

Similarly, $W^{s(.),G}(\Omega)$ is continuously embedded into $W^{s^-,G}(\Omega)$. $\square$

3.4. Proof of Theorem 3.3

Claim 1: The functions Ψ_i^+ are Orlicz functions.

It is clear Ψ_i^+ is convex, $\Psi_i^+(0) = 0$ and Ψ_i^+ is continuous on $]0, \infty[$. By convexity and the fact that $\Psi_i^+(0) = 0$, we have

$$\Psi_i^+(t) \leq \frac{t}{s}\Psi_i^+(s) \quad \text{for all} \quad 0 < t < s,$$

therefore

$$\lim_{t\to 0^+} \Psi_i^+(t) = 0 = \Psi_i^+(0).$$

It is also continuous in 0.

Claim 2: $W^{s(.),G}(\Omega_i)$ is continuously embedded into $L^{\Psi(.)}(\Omega_i)$.

Indeed, by Lemma 3.1, we have

$$W^{s(.),G}(\Omega_i) \hookrightarrow W^{s_i,G}(\Omega_i).$$

By Theorem (8.1) in [2], we obtain

$$W^{s_i,G}(\Omega_i) \hookrightarrow L^{G_{\frac{n}{s_i}}}(\Omega_i).$$

Moreover, as Ψ_i^+ is an Orlicz function and $\Psi_i^+ \leqslant G_{\frac{n}{s_i}}$, then by Theorem 2.2 it follows that

$$L^{G_{\frac{n}{s_i}}}(\Omega_i) \hookrightarrow L^{\Psi_i^+}(\Omega_i).$$

On the other hand, by definition of Ψ_i^+, we infer that

$$L^{\Psi_i^+}(\Omega_i) \hookrightarrow L^{\Psi(.)}(\Omega_i).$$

Combining the previous embeddings we deduce the embedding

$$W^{s(.),G}(\Omega_i) \hookrightarrow L^{\Psi(.)}(\Omega_i).$$

By claim (2), there exists a constant C_i such that

$$\Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega_i)} \leq C_i \Big|\Big|u\Big|\Big|_{W^{s(.),G}(\Omega_i)}.$$

Note that

$$|u| = \sum_{i=1}^{i=K} |u|\chi_{\Omega_i},$$

then we obtain

$$\Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega)} \leq \sum_{i=1}^{i=K} \Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega_i)}.$$

Thus, we have

$$\Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega)} \leq \sum_{i=1}^{i=K} \Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega_i)} \leq \sum_{i=1}^{i=K} C_i \Big|\Big|u\Big|\Big|_{W^{s(.),G}(\Omega_i)}.$$

Therefore

$$\Big|\Big|u\Big|\Big|_{L^{\Psi(.)}(\Omega)} \leq C \Big|\Big|u\Big|\Big|_{W^{s(.),G}(\Omega)}.$$

So we find that

$$W^{s(.),G}(\Omega) \hookrightarrow L^{\Psi(.)}(\Omega).$$

If $\Psi_i^+ \ll G_{\frac{n}{s_i}}$, using Theorem (9.1) in [2] and proceeding as in the previous proof, we get that the embedding

$$W^{s(.),G}(\Omega_i) \hookrightarrow L^{\Psi(.)}(\Omega_i)$$

is compact. The compactness of this embedding in Ω can be established by extracting a suitable convergent sub-sequence in $L^{\Psi(.)}(\Omega_i)$ for each $i = 1, ...K$ from a bounded sequence $\{u_k\}$ in $W^{s(.),G}(\Omega)$.

3.5. Proof of Theorem 3.5

Let $q \in C(\bar{\Omega})$ such that $0 < q(x) \leq \frac{s(x,x)}{n}$ for all $x \in \bar{\Omega}$. Under assumptions from Theorem 3.5, we have

$$\int_0^1 \frac{G^{-1}(z)}{z^{1+q(x)}} \mathrm{d}z < \infty \quad \text{and} \quad \int_1^\infty \frac{G^{-1}(z)}{z^{1+q(x)}} \mathrm{d}z = \infty.$$

Hence, we define the auxiliary N-Musielak function $G_{q(.)}$ associated to q, by

$$G_{q(.)}^{-1}(x,t) := \int_0^t \frac{G^{-1}(z)}{z^{1+q(x)}} \mathrm{d}z.$$

Claim 1: We have

$$t^{-p(.)}G^{-1} \approx G^{-1}_{p(.)}.$$

Indeed, let $x \in \bar{\Omega}$ and $\beta > 1$. By Proposition 2.1, we have

$$\beta^{\frac{1}{g^0}} G^{-1}(t) \leq G^{-1}(\beta t) \leq \beta^{\frac{1}{g_0}} G^{-1}(t) \quad \text{for all}\, t > 0. \tag{3.9}$$

Hence, by (3.9) and the definition of the function $G^{-1}_{p(.)}(x,.)$, we get

$$\beta^{\frac{1}{g^0}}.\beta^{-p(x)} G^{-1}_{p(.)}(x,t) \leq G^{-1}_{p(.)}(x,\beta t) \leq \beta^{\frac{1}{g_0}}.\beta^{-p(x)} G^{-1}_{p(.)}(x,t),$$

for all $x \in \bar{\Omega}$ and $t > 0$.

This implies that

$$\frac{\beta^{\frac{1}{g^0}-p(x)}-1}{\beta-1} G^{-1}_{p(.)}(x,t) \leq \frac{G^{-1}_{p(.)}(x,\beta t)-G^{-1}_{p(.)}(x,t)}{\beta-1} \leq \frac{\beta^{\frac{1}{g_0}-p(x)}-1}{\beta-1} G^{-1}_{p(.)}(x,t).$$

Letting $\beta \to 1^+$ we obtain

$$p_0 \leq \frac{t\Big(G^{-1}_{p(.)}(x,.)\Big)'(t)}{G^{-1}_{p(.)}(x,t)} \leq p^0,$$

where

$$0 < p^0 = \frac{1-g_0 \min_{x\in\bar{\Omega}} p(x)}{g_0} < 1 \quad \text{and} \quad 0 < p_0 = \frac{1-g^0 \max_{x\in\bar{\Omega}} p(x)}{g^0} < 1.$$

This implies that

$$p^0 G^{-1}_{p(.)}(x,t) \leq t^{-p(x)} G^{-1}(t) \leq p_0 G^{-1}_{p(.)}(x,t) \quad \text{for all}\, x \in \bar{\Omega}\, \text{and}\, t > 0.$$

Claim 2: If $q(x) > p(x)$ for all $x \in \bar{\Omega}$, then $L^{G_{q(.)}}(\Omega)$ is continuously embedded into $L^{G_{p(.)}}(\Omega)$.

Indeed, it is clear that

$$\lim_{t\to+\infty} \sup_{x\in\bar{\Omega}} \frac{t^{-q(x)}G^{-1}(t)}{t^{-p(x)}G^{-1}(t)} = 0.$$

Using claim 1, as

$$t^{-p(.)}G^{-1} \approx G^{-1}_{p(.)} \quad \text{and} \quad t^{-q(.)}G^{-1} \approx G^{-1}_{q(.)},$$

we infer that

$$\lim_{t\to+\infty} \sup_{x\in\bar{\Omega}} \frac{G^{-1}_{q(.)}(x,t)}{G^{-1}_{p(.)}(x,t)} = 0.$$

Hence, we have

$$\lim_{t\to+\infty}\sup_{x\in\bar{\Omega}}\frac{G_{p(.)}(x,kt)}{G_{q(.)}(x,t)}=0,$$

for any $k>0$. Therefore, we obtain $L^{G_{q(.)}}(\Omega)\hookrightarrow L^{G_{p(.)}}(\Omega)$.

As Ω is a bounded domain and p,s are continuous functions on $\bar{\Omega}$, there exists a constant $c>0$ such that

$$\frac{s(x,x)}{n}-p(x)\geq c>0,$$

for all $x\in\bar{\Omega}$. Hence, the finite covering theorem yields that there exist a positive constant $\epsilon(s,p)>0$ and a finite family of disjoint open bounded domains with Lipschitz boundary $\{\Omega_i : i=1,......K\}$ such that $\bar{\Omega}\subseteq\Big(\cup_{i=1}^{K}\Omega_i\Big)\cup N$, $\Big|N\Big|$=0 and $\mathrm{diam}(\Omega_i)<\epsilon$ that verify

$$\tfrac{s(y,z)}{n}-p(x)\geq\tfrac{c}{2}>0,$$

for any $x\in\Omega_i$ and $(y,z)\in\Omega_i\times\Omega_i$.

Set

$$s_i:=\min_{(y,z)\in\Omega_i\times\Omega_i}s(y,z),\quad p_i:=\max_{x\in\Omega_i}p(x)\quad\text{and}\quad q_i:=p_i+\frac{c}{3}.$$

Then

$$p(x)<q_i<\frac{s_i}{n}\quad\text{for all } x\in\Omega_i.$$

Furthermore, using Theorem (1.2) in [6], we conclude that

$$W^{s_i,G}(\Omega_i)\hookrightarrow\hookrightarrow L^{G_{q_i}}(\Omega_i).\tag{3.10}$$

By claim 2, we note that

$$L^{G_{q_i}}(\Omega_i)\hookrightarrow L^{G_{p(.)}}(\Omega_i).\tag{3.11}$$

Applying Lemma 3.1 on Ω_i, it follows that

$$W^{s(.),G}(\Omega_i)\hookrightarrow W^{s_i,G}(\Omega_i).\tag{3.12}$$

Therefore by (3.10), (3.11) and (3.12), we deduce that

$$W^{s(.),G}(\Omega_i)\hookrightarrow\hookrightarrow L^{G_{p(.)}}(\Omega_i).$$

Finally, proceeding as in the previous theorem, we obtain

$$W^{s(.),G}(\Omega)\hookrightarrow\hookrightarrow L^{\Psi(.)}(\Omega).$$

4. Singular problem

4.1. Statement of the problem

In this section, we prove our main existence result of a nonnegative weak solution. Recall that the problem under consideration is the following one

$$\begin{cases} (-\Delta)_G^{s(.)} u(x) = g(x) f'(x, |u|) \dfrac{u}{|u|} & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{4.1}$$

where Ω is a bounded domain with Lipschitz boundary.

We will work in the closed linear subspace

$$W_0^{s(.),G}(\Omega) := \Big\{ u \in W^{s(.),G}(\mathbb{R}^n) : u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \Big\}.$$

This space is equipped with the norm

$$||u||_{s(.),G} := ||u||_G + [u]_{s(.),G},$$

where

$$[u]_{s(.),G} := \inf \Big\{ \lambda > 0 : \iint_{\mathbb{R}^{2n}} G(\frac{|\nabla^s u|}{\lambda}) \mathrm{d}\mu \leq 1 \Big\}.$$

Remark 4.1.

- By Lemma 3.1 and Poincaré inequality (see Theorem (6.1) in [8]), it is easy to see that the semi-norm $[u]_{s(.),G}$ becomes a norm in $W_0^{s(.),G}(\Omega)$. Moreover, the norms $[u]_{s(.),G}$ and $||.||_{s(.),G}$ are equivalent.
- In what follows, we will use the norm $[u]_{s(.),G}$ which will be denoted by $||.||_{s(.),G}$ again.
- $C_c^2(\Omega) \subset W_0^{s^+,G}(\Omega)$.
- $W_0^{s^+,G}(\Omega) \subset W_0^{s(.),G}(\Omega)$. (Lemma 3.1)

Now, we are in position to state our conditions. G and Ψ satisfying the compactness conditions of Theorem 3.3 and also Ψ satisfies the condition (A). We suppose the following assumptions on s, g and f.

(H_s): s is a measurable function in $\mathbb{R}^n \times \mathbb{R}^n$ such that

(i) $s(x, y) = s(y, x)$ for a.e. $(x, y) \in \mathbb{R}^{2n}$;
(ii) $0 < s^- \leq s(x, y) \leq s^+ < 1$ for a.e. $(x, y) \in \mathbb{R}^{2n}$;

(H_f): $f : \Omega \times [0, \infty[\to [0, \infty[$ such that

(i) for a.e. $x \in \Omega$, $f(x, 0) = 0$ and $f(x, .) \in C^1(]0, \infty[)$ is strictly increasing;
(ii) $x \mapsto f(x, |u(x)|)$ is measurable for all measurable function u;
(iii) There exist $0 < f_0 < f^0 < 1$ and $a, b \geq 1$ such that

$$f(x, t) t^{-f_0} \leq a f(x, z) z^{-f_0},$$

for a.e. $x \in \Omega$ and $t, z > 0$ with $t < z$ and

$$f(x,t)t^{-f^0} \leq b f(x,z) z^{-f^0},$$

for a.e. $x \in \Omega$ and $t, z > 0$ with $t > z$;

(H_g): $g : \Omega \to \mathbb{R}^+$ is a measurable function verifying

$$g \in L^{(\Psi o f^{-1})^\star}(\Omega).$$

Remarks 4.1.

- Assumption (H_f) confirms the existence of singularity at $t = 0$. Indeed, let $x \in \Omega$, then we have

$$\tfrac{f(x,t)-f(x,0)}{t-0} = \tfrac{f(x,t)}{t} \geq \tfrac{1}{b}\tfrac{t^{f^0}}{t} f(x,1) = \tfrac{1}{b}\tfrac{1}{t^{1-f^0}} f(x,1),$$

  for all $0 < t < 1$. Since $0 < f^0 < 1$, we have a singularity in the right hand side term of equation (4.1).
- f^{-1} is a weak Musielak function. Indeed, let $x \in \Omega$ and $0 < \alpha < \beta$ with $\alpha = f(x,t)$ and $\beta = f(x,z)$ for some $0 < t < z$. Then, we have

$$\frac{f^{-1}(x,\alpha)}{\alpha} = \frac{f^{-1}(x, f(x,t))}{f(x,t)} = \frac{1}{f(x,t)t^{-f^0}} t^{1-f^0}$$

  and

$$\frac{f^{-1}(x,\beta)}{\beta} = \frac{f^{-1}(x, f(x,z))}{f(x,z)} = \frac{1}{f(x,z)z^{-f^0}} z^{1-f^0}.$$

  By assumption (H_f), we have

$$\frac{1}{f(x,z)z^{-f^0}} \geq \frac{1}{b}\frac{1}{f(x,t)t^{-f^0}},$$

  therefore

$$\frac{f^{-1}(x,\beta)}{\beta} \geq \frac{1}{b}\frac{f^{-1}(x,\alpha)}{\alpha}\left(\frac{z}{t}\right)^{1-f^0} \geq \frac{1}{b}\frac{f^{-1}(x,\alpha)}{\alpha},$$

  hence

$$\frac{f^{-1}(x,\alpha)}{\alpha} \leq b\frac{f^{-1}(x,\beta)}{\beta}.$$

  By Lemma (2.5.12) in [19], $x \mapsto f^{-1}(x, |u(x)|)$ is measurable for all $u \in L^0(\Omega)$, therefore f^{-1} is a weak Musielak function. By Proposition 3.1, $\Psi o f^{-1}$ is a weak Musielak function. Hence, the condition (H_g) makes sense.

Definition 4.1. We say that $u_0 \in W_0^{s(.),G}(\Omega)$ is a weak solution of (4.1) if

$$g(.)f'(., |u_0|)\varphi \in L^1(\Omega)$$

and

$$\iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi \,\mathrm{d}\mu - \int_\Omega g(x)f'(x,|u_0|)\frac{u_0}{|u_0|}\varphi \,\mathrm{d}x = 0,$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$.

Theorem 4.1. *Under assumptions (H_s), (H_f) and (H_g), (4.1) has a nontrivial weak solution $u_0 \in W_0^{s(.),G}(\Omega)$.*

Problem (4.1) has a variational structure and $E : W_0^{s(.),G}(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \iint_{\mathbb{R}^{2n}} G(|\nabla^{s(.)}u|)\,\mathrm{d}\mu - \int_\Omega g(x)f(x,|u|)\,\mathrm{d}x$$

is the energy functional associated to (4.1). Because of the presence of a singular term in (4.1), the functional E is not differentiable on $W_0^{s(.),G}(\Omega)$ in the Fréchet sense.

The next proposition is an extension of a result obtained by J. Giacomoni et al. in [17] and D. Edmunds et al. in [14] which is useful to verify some properties related to our existence result.

Proposition 4.1. *Let Ω be either a bounded or an unbounded measurable subset of $\mathbb{R}^n$, Φ a weak Musielak function and $S : \Omega \times [0,\infty[\to [0,\infty[$ satisfying $(AInc)_{m_0,a_1}$ and $(ADec)_{m^0,a_2}$. Suppose that $\Phi o S$ is a weak Musielak function and $x \mapsto S(x,|u(x)|)$ is measurable for all $u \in L^0(\Omega)$. If $u \in L^{\Phi o S}(\Omega)$, then $S(.,|u|) \in L^\Phi(\Omega)$, and we have*

$$\Big|\Big|S(.,|u|)\Big|\Big|_\Phi \le a_1\Big|\Big|u\Big|\Big|_{\Phi o S}^{m_0}, \quad \text{if} \quad \Big|\Big|u\Big|\Big|_{\Phi o S} \le 1;$$

$$\Big|\Big|S(.,|u|)\Big|\Big|_\Phi \le a_2\Big|\Big|u\Big|\Big|_{\Phi o S}^{m^0}, \quad \text{if} \quad \Big|\Big|u\Big|\Big|_{\Phi o S} \ge 1;$$

$$\Big|\Big|S(.,|u|)\Big|\Big|_\Phi \ge \frac{1}{a_2^{\frac{m^0}{m_0}}a_1^{\frac{m^0}{m_0}}}\Big|\Big|u\Big|\Big|_{\Phi o S}^{m^0}, \quad \text{if} \quad \Big|\Big|S(.,|u|)\Big|\Big|_\Phi \le 1;$$

$$\Big|\Big|S(.,|u|)\Big|\Big|_\Phi \ge \frac{1}{a_1}\Big|\Big|u\Big|\Big|_{\Phi o S}^{m_0}, \quad \text{if} \quad \Big|\Big|S(.,|u|)\Big|\Big|_\Phi \ge 1.$$

Proof. If $\Big|\Big|u\Big|\Big|_{\Phi o S} \le 1$, then by $(\mathrm{AInc})_{m_0,a_1}$ property, we have

$$\begin{aligned} 1 &\ge \int_\Omega \Phi\Big(x, S(x, \frac{|u(x)|}{||u||_{\Phi o S}})\Big)\mathrm{d}x \\ &\ge \int_\Omega \Phi\Big(x, \frac{1}{a_1||u||_{\Phi o S}^{m_0}}S(x,|u(x)|)\Big)\mathrm{d}x. \end{aligned}$$

Therefore

$$\Big|\Big|S(.,|u|)\Big|\Big|_\Phi \le a_1\Big|\Big|u\Big|\Big|_{\Phi o S}^{m_0}.$$

Suppose $\left|\left|u\right|\right|_{\Phi oS} \geq 1$. Then by $(\mathrm{ADec})_{m^0,a_2}$ property, we have

$$\begin{aligned} 1 &\geq \int_\Omega \Phi\Big(x, S(x, \frac{|u(x)|}{||u||_{\Phi oS}})\Big)\mathrm{d}x \\ &\geq \int_\Omega \Phi\Big(x, \frac{1}{a_2||u||_{\Phi oS}^{m^0}} S(x,|u(x)|)\Big)\mathrm{d}x. \end{aligned}$$

Hence

$$\left|\left|S(.,|u|)\right|\right|_{\Phi} \leq a_2 \left|\left|u\right|\right|_{\Phi oS}^{m^0}.$$

Now, suppose that $\left|\left|S(.,|u|)\right|\right|_{\Phi} \leq 1$. Then by $(\mathrm{AInc})_{m_0,a_1}$ and $(\mathrm{ADec})_{m^0,a_2}$ properties, we have

$$\begin{aligned} 1 &\geq \int_\Omega \Phi\Big(x, \frac{S(x,|u(x)|)}{||S(.,|u|)||_{\Phi}}\Big)\mathrm{d}x \\ &\geq \int_\Omega \Phi\Big(x, \frac{1}{a_2} S(x, \frac{|u(x)|}{||S(.,|u|)||_{\Phi}^{\frac{1}{m^0}}})\Big)\mathrm{d}x \\ &\geq \int_\Omega \Phi\Big(x, S(x, \frac{|u(x)|}{(a_1a_2)^{\frac{1}{m_0}}||S(.,|u|)||_{\Phi}^{\frac{1}{m^0}}})\Big)\mathrm{d}x. \end{aligned}$$

It follows that

$$\left|\left|u\right|\right|_{\Phi oS} \leq (a_1a_2)^{\frac{1}{m_0}} \left|\left|S(.,|u|)\right|\right|_{\Phi}^{\frac{1}{m^0}},$$

therefore

$$(\tfrac{1}{a_1a_2})^{\frac{m^0}{m_0}} \left|\left|u\right|\right|_{\Phi oS}^{m^0} \leq \left|\left|S(.,|u|)\right|\right|_{\Phi}.$$

If $\left|\left|S(.,|u|)\right|\right|_{\Phi} \geq 1$, by $(\mathrm{AInc})_{m_0,a_1}$ property, we have

$$\begin{aligned} 1 &\geq \int_\Omega \Phi\Big(x, \frac{S(x,|u(x)|)}{||S(.,|u|)||_{\Phi}}\Big)\mathrm{d}x \\ &\geq \int_\Omega \Phi\Big(x, S(x, \frac{|u(x)|}{a_1^{\frac{1}{m_0}}||S(.,|u|)||_{\Phi}^{\frac{1}{m_0}}})\Big)\mathrm{d}x. \end{aligned}$$

Thus, we obtain

$$\left|\left|u\right|\right|_{\Phi oS} \leq a_1^{\frac{1}{m_0}} \left|\left|S(.,|u|)\right|\right|_{\Phi}^{\frac{1}{m_0}}$$

and

$$\frac{1}{a_1}\left|\left|u\right|\right|_{\Phi oS}^{m_0} \leq \left|\left|S(.,|u|)\right|\right|_{\Phi}. \quad \square$$

Remark 4.2.

1. Let Ω be either a bounded or an unbounded measurable subset of $\mathbb{R}^n$, Φ be a weak Musielak function and $S:\Omega\times[0,\infty[\to[0,\infty[$ satisfying $(\mathrm{AInc})_{m_0,1}$ and $(\mathrm{ADec})_{m^0,1}$. Suppose that ΦoS is a weak Musielak function and $x\mapsto S(x,|u(x)|)$ is measurable for all $u\in L^0(\Omega)$. If $u\in L^{\Phi oS}(\Omega)$, then $S(.,|u|)\in L^{\Phi}(\Omega)$, and we have

$$\Big|\Big|u\Big|\Big|_{\Phi oS}^{m^0}\le\Big|\Big|S(.,|u|)\Big|\Big|_{\Phi}\le\Big|\Big|u\Big|\Big|_{\Phi oS}^{m_0},\quad \text{if}\quad \Big|\Big|u\Big|\Big|_{\Phi oS}\le 1;$$
$$\Big|\Big|u\Big|\Big|_{\Phi oS}^{m_0}\le\Big|\Big|S(.,|u|)\Big|\Big|_{\Phi}\le\Big|\Big|u\Big|\Big|_{\Phi oS}^{m^0},\quad \text{if}\quad \Big|\Big|u\Big|\Big|_{\Phi oS}\ge 1.$$

2. In [14,17], it is established that if p, q are real measurable functions in Ω such that $0<p^-\le p(x)\le p^+$, $1\le q(x)<\infty$, $1\le p(x)q(x)<\infty$ for a.e. $x\in\Omega$ and $u\in L^{p(x)q(x)}(\Omega)$, then

$$\Big|\Big|u\Big|\Big|_{p(x)q(x)}^{p^+}\le\Big|\Big||u|^{p(.)}\Big|\Big|_{q(x)}\le\Big|\Big|u\Big|\Big|_{p(x)q(x)}^{p^-}\quad \text{if}\quad \Big|\Big|u\Big|\Big|_{p(x)q(x)}\le 1;$$
$$\Big|\Big|u\Big|\Big|_{p(x)q(x)}^{p^-}\le\Big|\Big||u|^{p(.)}\Big|\Big|_{q(x)}\le\Big|\Big|u\Big|\Big|_{p(x)q(x)}^{p^+}\quad \text{if}\quad \Big|\Big|u\Big|\Big|_{p(x)q(x)}\ge 1.$$

Proposition 4.1 generalizes this result. Indeed, set $S(x,t)=t^{p(x)}$ and $\Phi(x,t)=t^{q(x)}$. It is clear that S satisfies $(\mathrm{AInc})_{p^-,1}$, $(\mathrm{ADec})_{p^+,1}$ properties, and $\Phi(x,t)=t^{q(x)}$ and $\Phi oS(x,t)=t^{p(x)q(x)}$ are weak Musielak functions.

4.2. Proof of the main result

Lemma 4.1. *Under assumptions (H_s), (H_f) and (H_g), the functional E is coercive on $W_0^{s(.),G}(\Omega)$.*

Proof. Let $u\in W_0^{s(.),G}(\Omega)$ with $\Big|\Big|u\Big|\Big|_{s(.),G}>1$. Then, from Proposition 3.2, we have

$$\iint_{\mathbb{R}^{2n}} G(|\nabla^s u|)\,\mathrm{d}\mu\ge\Big|\Big|u\Big|\Big|_{s(.),G}^{g_0}. \tag{4.2}$$

Since $u\in L^{\Psi(.)}(\Omega)$, then by Proposition 4.1 and Hölder's inequality, we get

$$\int_\Omega g(x)f(x,|u|)\,\mathrm{d}x\leqslant\Big|\Big|g\Big|\Big|_{(\Psi of^{-1})^\star}\Big|\Big|f(.,|u|)\Big|\Big|_{\Psi of^{-1}}.$$

Again, by Proposition 4.1, we have

$$\begin{aligned}\int_\Omega g(x)f(x,|u|)\,\mathrm{d}x&\leqslant\Big|\Big|g\Big|\Big|_{(\Psi of^{-1})^\star}\max\Big(a\Big|\Big|u\Big|\Big|_{\Psi of^{-1}of}^{f_0},b\Big|\Big|u\Big|\Big|_{\Psi of^{-1}of}^{f^0}\Big)\\&\leqslant\Big|\Big|g\Big|\Big|_{(\Psi of^{-1})^\star}\max\Big(a\Big|\Big|u\Big|\Big|_{\Psi}^{f_0},b\Big|\Big|u\Big|\Big|_{\Psi}^{f^0}\Big).\end{aligned}$$

Then, we obtain from Theorem 3.3 that

$$\int_{\Omega} g(x) f(x,|u|)\,\mathrm{d}x \leqslant C \Big|\Big|g\Big|\Big|_{(\Psi o f^{-1})^{\star}} \Big|\Big|u\Big|\Big|_{s(.),G}^{f^0}. \tag{4.3}$$

Finally, combining (4.2) and (4.3), we obtain

$$\begin{aligned} E(u) &= \iint_{\mathbb{R}^{2n}} G(|\nabla^s u|)\,\mathrm{d}\mu - \int_{\Omega} g(x) f(x,|u|)\,\mathrm{d}x \\ &\geq \Big|\Big|u\Big|\Big|_{s(.),G}^{g_0} - C\Big|\Big|g\Big|\Big|_{(\Psi o f^{-1})^{\star}} \Big|\Big|u\Big|\Big|_{s(.),G}^{f^0}. \end{aligned}$$

Since $0 < f^0 < 1 < g_0$, we infer that $E(u) \to \infty$ as $\Big|\Big|u\Big|\Big|_{s(.),G} \to \infty$ and we conclude that E is coercive on $W_0^{s(.),G}(\Omega)$. $\square$

Lemma 4.2. *Under assumptions (H_s), (H_f) and (H_g), there exists $v_0 \in W_0^{s(.),G}(\Omega)$ such that $E(tv_0) < 0$ for $t > 0$ small enough.*

Proof. Let $v \in C_0^{\infty}(\Omega) \setminus \{0\}$ and $0 < t < 1$. Hence, by conditions (A) and (H_f), we have

$$G(|\nabla^s tv|) \leq t^{g_0} G(|\nabla^s v|)$$

and

$$f(x, t|v|) \geq \frac{1}{b} t^{f^0} f(x,|v|).$$

Therefore

$$\begin{aligned} E(tv) &= \iint_{\mathbb{R}^{2n}} G(|\nabla^s tv|)\,\mathrm{d}\mu - \int_{\Omega} g(x) f(x,t|v|)\,\mathrm{d}x \\ &\leq t^{g_0} \iint_{\mathbb{R}^{2n}} G(|\nabla^s v|)\,\mathrm{d}\mu - \frac{1}{b} t^{f^0} \int_{\Omega} g(x) f(x,|v|)\,\mathrm{d}x. \end{aligned}$$

Consequently, $E(tv) < 0$ for all $0 < t < t_0^{\frac{1}{g_0 - f^0}}$, where

$$t_0 < \min\left(1, \frac{\int_{\Omega} g(x) f(x,|v|)\,\mathrm{d}x}{b \iint_{\mathbb{R}^{2n}} G(|\nabla^s v|)\,\mathrm{d}\mu}\right). \quad \square$$

Lemma 4.3. *Let $J : W_0^{s(.),G}(\Omega) \to \mathbb{R}$ defined by*

$$J(u) := \iint_{\mathbb{R}^{2n}} G(|\nabla^s u|)\,\mathrm{d}\mu.$$

Then, we have

1. $J \in C^1(W_0^{s(.),G}(\Omega))$ *and*

$$\Big\langle J'(u), \varphi \Big\rangle = \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u|)\frac{\nabla^s u}{|\nabla^s u|}\nabla^s \varphi \, \mathrm{d}\mu, \quad \textit{for all} \quad \varphi \in W_0^{s(.),G}(\Omega).$$

2. J *is sequentially weakly lower semicontinuous.*

Proof. By standard arguments, it is easy to see that

$$\Big\langle J'(u), \varphi \Big\rangle = \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u|)\frac{\nabla^s u}{|\nabla^s u|}\nabla^s \varphi \, \mathrm{d}\mu,$$

for all $u, \varphi \in W_0^{s(.),G}(\Omega)$. It suffices to prove that $J' : W_0^{s(.),G}(\Omega) \to \Big(W_0^{s(.),G}(\Omega)\Big)^\star$ is continuous.

Let $(u_k)_k \subset W_0^{s(.),G}(\Omega)$ with $u_k \to u$ strongly in $W_0^{s(.),G}(\Omega)$, then

$$\nabla^s u_k \longrightarrow \nabla^s u \quad \text{in} \quad L^G(\mathbb{R}^{2n}, \mathrm{d}\mu).$$

By dominated convergence theorem, there exists a subsequence $\nabla^s u_{k_j}$ and a function T in $L^G(\mathbb{R}^{2n}, \mathrm{d}\mu)$ such that

$$\nabla^s u_{k_j}(x,y) \to \nabla^s u(x,y) \quad \text{for} \quad \text{a.e} \quad (x,y) \in \mathbb{R}^{2n}$$

and

$$|\nabla^s u_{k_j}(x,y)| \leq |T(x,y)| \quad \text{for} \quad \text{a.e} \quad (x,y) \in \mathbb{R}^{2n}.$$

Hence

$$G'(|\nabla^s u_{k_j}(x,y)|)\frac{\nabla^s u_{k_j}(x,y)}{|\nabla^s u_{k_j}(x,y)|} \to G'(|\nabla^s u(x,y)|)\frac{\nabla^s u(x,y)}{|\nabla^s u(x,y)|} \quad \text{for} \quad \text{a.e.} \quad (x,y) \in \mathbb{R}^{2n}$$

and

$$|G'(|\nabla^s u_{k_j}(x,y)|)\frac{\nabla^s u_{k_j}(x,y)}{|\nabla^s u_{k_j}(x,y)|}| \leq |G'(|T(x,y)|)| \quad \text{for} \quad \text{a.e.} \quad (x,y) \in \mathbb{R}^{2n}.$$

By Proposition 2.1, we have

$$G^\star(G'(|T(x,y)|)) \leq (g^0 - 1)G(|T(x,y)|).$$

It follows that

$$G' o |T| \in L^{G^\star}(\mathbb{R}^{2n}, \mathrm{d}\mu).$$

Using Hölder's inequality, we get

$$\begin{aligned} &\Big|\iint\limits_{\mathbb{R}^{2n}} \Big(G'(|\nabla^s u_{k_j}|)\frac{\nabla^s u_{k_j}}{|\nabla^s u_{k_j}|} - G'(|\nabla^s u|)\frac{\nabla^s u}{|\nabla^s u|}\Big)\nabla^s \varphi \, \mathrm{d}\mu\Big| \\ &\qquad \leq 2\Big|\Big|(G'(|\nabla^s u_{k_j}|)\frac{\nabla^s u_{k_j}}{|\nabla^s u_{k_j}|} - G'(|\nabla^s u|)\frac{\nabla^s u}{|\nabla^s u|}\Big)\Big|\Big|_{G^\star}\Big|\Big|\varphi\Big|\Big|_{G,s}. \end{aligned}$$

Then, by dominated convergence theorem, we obtain

$$J'(u_{k_j}) \to J'(u) \quad \text{in} \quad \left(W_0^{s(.),G}(\Omega)\right)^{\star}.$$

(2) On one hand, by (1), we have $J \in C^1(W_0^{s(.),G}(\Omega))$ and

$$\left\langle J'(u), \varphi \right\rangle = \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u|) \frac{\nabla^s u}{|\nabla^s u|} \nabla^s \varphi \, \mathrm{d}\mu,$$

for all $u, \varphi \in W_0^{s(.),G}(\Omega)$. On the other hand, J is a convex functional since G is. Let $(u_k)_k \subset W_0^{s(.),G}(\Omega)$ with

$$u_k \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad W_0^{s(.),G}(\Omega).$$

By convexity of J, we have

$$J(u_k) \geq J(u) + \left\langle J'(u), u_k - u \right\rangle.$$

Therefore, we obtain

$$J(u) \leq \liminf_k J(u_k).$$

This implies that J is weakly lower semicontinuous. $\square$

Lemma 4.4. *Under assumptions (H_s), (H_f) and (H_g), there exists $u_0 \in W_0^{s(.),G}(\Omega)$ such that*

$$E(u_0) = \inf_{u \in W_0^{s(.),G}(\Omega)} E(u) = m.$$

Proof. Let $(u_k)_k \subset W_0^{s(.),G}(\Omega)$ be a minimizing sequence for m, that is such that

$$\lim_{k \to \infty} E(u_k) = m.$$

Since E is coercive, $(u_k)_k \subset W_0^{s(.),G}(\Omega)$ is bounded in $W_0^{s(.),G}(\Omega)$. As $W_0^{s(.),G}(\Omega)$ is reflexive, therefore, up to a subsequence, there exists $u_0 \in W_0^{s(.),G}(\Omega)$ such that

$$\begin{aligned} u_k \rightharpoonup u_0 &\quad \text{weakly in} \quad W_0^{s(.),G}(\Omega), \\ u_k \to u_0 &\quad \text{strongly in} \quad L^{\Psi(.)}(\Omega), \\ u_k(x) \to u_0(x) &\qquad \text{for a.e. } x \in \Omega. \end{aligned} \tag{4.4}$$

By Lemma 4.3, we have

$$\iint\limits_{\mathbb{R}^{2n}} G(|\nabla^s u_0|) \, \mathrm{d}\mu \leq \liminf_{k \to \infty} \iint\limits_{\mathbb{R}^{2n}} G(|\nabla^s u_k|) \, \mathrm{d}\mu. \tag{4.5}$$

We claim that

$$\int\limits_{\Omega} g(x) f(x, |u_0(x)|) \, \mathrm{d}x = \lim_{k \to \infty} \int\limits_{\Omega} g(x) f(x, |u_k(x)|) \, \mathrm{d}x. \tag{4.6}$$

It is clear that

$$\lim_{k\to\infty} g(x)f(x,|u_k|)\,\mathrm{d}x = g(x)f(x,|u_0|) \qquad \text{for a.e. } x \in \Omega.$$

According to Vitali's theorem it suffices to show that the family

$$\Big\{g(x)f(x,|u_k|),\, k\in\mathbb{N}\Big\}$$

is uniformly absolutely continuous, which means:

Given $\epsilon>0$ there exists $\eta>0$, such that, if $|\Omega'|<\eta$, then $\displaystyle\int_{\Omega'} g(x)f(x,|u_k|)\,\mathrm{d}x<\epsilon$, for all k.

Let $\epsilon>0$, there exists $\gamma,\eta>0$ such that

$$\Big|\Big|g\Big|\Big|^{\gamma}_{\Omega',(\Psi o f^{-1})^{\star}} \le \int_{\Omega'} \Big(\Psi o f^{-1}\Big)^{\star}(|g(x)|)\,\mathrm{d}x \le \epsilon^{\gamma},$$

for every $\Omega'\subset\Omega$ with $|\Omega'|<\eta$. On the other hand, by Proposition 4.1 and Hölder's inequality, one has

$$\begin{aligned}\int_{\Omega'} g(x)f(x,|u_k|)\,\mathrm{d}x &\le \Big|\Big|g\Big|\Big|_{\Omega',(\Psi o f^{-1})^{\star}}\Big|\Big|f(.,|u_k|)\Big|\Big|_{\Omega',\Psi o f^{-1}}\\ &\le \max\Big(b\Big|\Big|u_k\Big|\Big|^{f^0}_{\Psi}, a\Big|\Big|u_k\Big|\Big|^{f_0}_{\Psi}\Big)\epsilon.\end{aligned}$$

Finally, the fact that $\Big|\Big|u_k\Big|\Big|_{\Psi}$ is bounded, implies that the claim is valid.

Hence, by (4.5) and (4.6), we deduce that

$$m \le E(u_0) \le \liminf_{k\to\infty} E(u_k) = m.$$

Consequently

$$E(u_0) = m. \quad \square$$

By Lemma 4.2 and the fact that u_0 is a global minimum, we have

$$m = E(u_0) < 0 = E(0),$$

therefore $u_0 \ne 0$.

It is clear that

$$\Big|\nabla^s|u_0|\Big| \le |\nabla^s u_0|,$$

hence

$$|u_0| \in W_0^{s(.),\Phi}(\Omega).$$

Therefore

$$E(|u_0|) \leq E(u_0) = \inf_{u \in W_0^{s(.),\Phi}(\Omega)} E(u),$$

this implies that

$$E(|u_0|) = E(u_0).$$

Thus, we can suppose that $u_0 \geq 0$.

Now, we are in position to prove Theorem 4.1. Let $\varphi \geq 0$ and $t > 0$, we have

$$\begin{aligned} 0 &\leq \liminf_{t \to 0} \frac{E(u_0 + t\varphi) - E(u_0)}{t} \\ &\leq \iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|) \frac{\nabla^s u_0}{|\nabla^s u_0|} \nabla^s \varphi \, \mathrm{d}\mu \\ &- \limsup_{t \to 0} \int_\Omega g(x) \frac{f(x, u_0 + t\varphi) - f(x, u_0)}{t} \, \mathrm{d}x, \end{aligned}$$

hence

$$\begin{aligned} &\limsup_{t \to 0} \int_\Omega g(x) \frac{f(x, u_0 + t\varphi) - f(x, u_0)}{t} \, \mathrm{d}x \\ &\qquad \leq \iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|) \frac{\nabla^s u_0}{|\nabla^s u_0|} \nabla^s \varphi \, \mathrm{d}\mu. \end{aligned}$$

By the mean value theorem, there exists $\epsilon \in]0, 1[$ such that

$$\int_\Omega g(x) \frac{f(x, u_0 + t\varphi) - f(x, u_0)}{t} \, \mathrm{d}x = \int_\Omega g(x) f'(x, u_0 + t\epsilon\varphi) \varphi \, \mathrm{d}x.$$

Since $\varphi \geq 0$, by Fatou's lemma, we get

$$\begin{aligned} &\limsup_{t \to 0} \int_\Omega g(x) \frac{f(x, u_0 + t\varphi) - f(x, u_0)}{t} \, \mathrm{d}x \\ &\qquad \geq \liminf_{t \to 0} \int_\Omega g(x) \frac{f(x, u_0 + t\varphi) - f(x, u_0)}{t} \, \mathrm{d}x \\ &\qquad = \liminf_{t \to 0} \int_\Omega g(x) f'(x, u_0 + t\epsilon\varphi) \varphi \, \mathrm{d}x \geq \int_\Omega g(x) f'(x, u_0) \varphi \, \mathrm{d}x. \end{aligned}$$

Therefore

$$\iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|) \frac{\nabla^s u_0}{|\nabla^s u_0|} \nabla^s \varphi \, \mathrm{d}\mu - \int_\Omega g(x) f'(x, u_0) \varphi \, \mathrm{d}x \geq 0, \tag{4.7}$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$ with $\varphi \geq 0$. Hence, we get

$$0 \leq \int_\Omega g(x) f'(x, u_0) |\varphi| \, \mathrm{d}x \leq \iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|) \frac{\nabla^s u_0}{|\nabla^s u_0|} \nabla^s |\varphi| \, \mathrm{d}\mu < \infty$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$. Therefore

$$g(.)f'(., u_0)\varphi \in L^1(\Omega),$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$.

Now, let $\varphi \in W_0^{s(.),G}(\Omega)$ and consider the following sets

$$\begin{aligned}
\Omega_\epsilon &:= \Big\{x \in \mathbb{R}^n : u_0 + \epsilon\varphi \leq 0\Big\},\\
\Omega^\epsilon &:= \Big\{x \in \mathbb{R}^n : u_0 + \epsilon\varphi > 0\Big\},\\
A_{u_0}^+ &:= \Big\{(x, y) \in \mathbb{R}^{2n} : \nabla^s u_0(x, y) > 0\Big\},\\
A_{u_0}^- &:= \Big\{(x, y) \in \mathbb{R}^{2n} : \nabla^s u_0(x, y) \leq 0\Big\}.
\end{aligned}$$

We define also the following functions

$$\begin{aligned}
\varphi_\epsilon &:= u_0 + \epsilon\varphi,\\
\varphi_\epsilon^+ &:= \max\Big(u_0 + \epsilon\varphi, 0\Big),\\
\varphi_\epsilon^- &:= \max\Big(-(u_0 + \epsilon\varphi), 0\Big).
\end{aligned}$$

It is clear that

$$\Big|\nabla^s\varphi_\epsilon(x, y)\Big| \geq \Big|\nabla^s\varphi_\epsilon^+(x, y)\Big| \quad \text{a.e.} \quad (x, y) \in \mathbb{R}^{2n}.$$

Hence, $\varphi_\epsilon^+, \varphi_\epsilon^- \in W_0^{s(.),G}(\Omega)$. Choosing φ_ϵ^+ in (4.7), one has

$$\begin{aligned}
0 &\leq \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^+ \,\mathrm{d}\mu - \int\limits_\Omega g(x)f'(x, u_0)\varphi_\epsilon^+ \,\mathrm{d}x\\
&= \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon \,\mathrm{d}\mu + \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^- \,\mathrm{d}\mu\\
&- \Big(\int\limits_\Omega - \int\limits_{\Omega_\epsilon}\Big)\Big(g(x)f'(x, u_0)\varphi_\epsilon \,\mathrm{d}x\Big)\\
&= \Big(\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)|\nabla^s u_0| \,\mathrm{d}\mu - \int\limits_\Omega g(x)f'(x, u_0)u_0 \,\mathrm{d}x\Big)\\
&+ \epsilon\Big(\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi \,\mathrm{d}\mu - \int\limits_\Omega g(x)f'(x, u_0)\varphi \,\mathrm{d}x\Big)\\
&+ \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^- \,\mathrm{d}\mu + \int\limits_{\Omega_\epsilon} g(x)f'(x, u_0)\varphi_\epsilon \,\mathrm{d}x.
\end{aligned}$$

If we define $h : [-\epsilon, \epsilon] \to \mathbb{R}$ as

$$h(t) := E((1 + t)u_0).$$

Hence $h(0) = \inf_{t\in[-\epsilon,\epsilon]} h(t)$, from where $h'(0) = 0$. It is straightforward to see that u_0 satisfies

$$\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)|\nabla^s u_0|\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)u_0\,\mathrm{d}x = 0.$$

Therefore, we get

$$\begin{aligned} 0 \leq &\epsilon\Big(\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)\varphi\,\mathrm{d}x\Big) \\ &+ \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^-\,\mathrm{d}\mu. \end{aligned}$$

Thus, dividing by ϵ, we obtain

$$\begin{aligned} 0 \leq &\Big(\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)\varphi\,\mathrm{d}x\Big) \\ &+ \frac{1}{\epsilon}\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^-\,\mathrm{d}\mu. \end{aligned}$$

We now claim that there holds

$$0 \leq \iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)\varphi\,\mathrm{d}x.$$

The claim will follow if we prove that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^-\,\mathrm{d}\mu = 0. \tag{4.8}$$

Let

$$\mathcal{N}_\epsilon(x,y) = G'(|\nabla^s u_0(x,y)|)\frac{\nabla^s u_0(x,y)}{|\nabla^s u_0(x,y)|}\nabla^s\varphi_\epsilon^-(x,y)$$

and

$$\mathcal{N}(x,y) = G'(|\nabla^s u_0(x,y)|)\frac{\nabla^s u_0(x,y)}{|\nabla^s u_0(x,y)|}\nabla^s\varphi(x,y).$$

It is clear that

$$\mathcal{N}_\epsilon(x,y) = \mathcal{N}_\epsilon(y,x).$$

Hence, we have

$$\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi_\epsilon^-\,\mathrm{d}\mu = I_1 + 2I_2,$$

where

$$I_1 := \int_{\Omega_\epsilon}\int_{\Omega_\epsilon} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu \quad \text{and} \quad I_2 := \int_{\Omega_\epsilon}\int_{\mathbb{R}^n\setminus\Omega_\epsilon} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu.$$

It is clear that, if $I_1 + 2I_2 \leq 0$, then

$$\iint_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu - \int_{\Omega} g(x)f'(x,u_0)\varphi\,\mathrm{d}x \geq 0.$$

Without loss of generality, we may assume that $I_1 + 2I_2 \geq 0$.

Estimate of I_1:

$$I_1 = -\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} G'(|\nabla^s u_0|)\frac{|\nabla^s u_0|^2}{|\nabla^s u_0|}\,\mathrm{d}\mu - \epsilon\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu.$$

It is clear that

$$\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} G'(|\nabla^s u_0|)\frac{|\nabla^s u_0|^2}{|\nabla^s u_0|}\,\mathrm{d}\mu \geq 0,$$

therefore

$$I_1 \leq -\epsilon\int_{\Omega_\epsilon}\int_{\Omega_\epsilon} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu.$$

Estimate of I_2:

Consider

$$Z^+_{u_0} := \Big(\Omega_\epsilon \times (\mathbb{R}^n\setminus\Omega_\epsilon)\Big)\cap A^+_{u_0}$$

and

$$Z^-_{u_0} := \Big(\Omega_\epsilon \times (\mathbb{R}^n\setminus\Omega_\epsilon)\Big)\cap A^-_{u_0}.$$

We have

$$\iint_{Z^+_{u_0}} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu = \iint_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\Big(\frac{-(u_0+\epsilon\varphi)(x)}{|x-y|^{s(x,y)}}\Big)\,\mathrm{d}\mu.$$

Let $(x,y)\in Z^+_{u_0}$, then $\nabla^s u_0(x,y)\geq 0$ and $u_0(y)+\epsilon\varphi(y)\geq 0$, which implies that

$$\begin{aligned}\iint_{Z^+_{u_0}} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu \leq &\iint_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\Big(\frac{-(u_0+\epsilon\varphi)(x)}{|x-y|^{s(x,y)}}\Big)\,\mathrm{d}\mu\\ &+\iint_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\Big(\frac{(u_0+\epsilon\varphi)(y)}{|x-y|^{s(x,y)}}\Big)\,\mathrm{d}\mu.\end{aligned}$$

Hence

$$\begin{aligned}\iint\limits_{Z^+_{u_0}} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu &\le -\iint\limits_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{|\nabla^s u_0|^2}{|\nabla^s u_0|}\,\mathrm{d}\mu - \epsilon\iint\limits_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu\\ &\le -\epsilon\iint\limits_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu.\end{aligned}$$

On the other hand

$$\iint\limits_{Z^-_{u_0}} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu = \iint\limits_{Z^-_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\Big(\frac{-(u_0+\epsilon\varphi)(x)}{|x-y|^{s(x,y)}}\Big)\,\mathrm{d}\mu \le 0.$$

Finally, we get

$$I_2 = \int\limits_{\Omega_\epsilon}\int\limits_{\mathbb{R}^n\setminus\Omega_\epsilon} \mathcal{N}_\epsilon(x,y)\,\mathrm{d}\mu \le -\epsilon\iint\limits_{Z^+_{u_0}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu.$$

Collecting the previous estimations of I_1 and I_2, we obtain

$$\begin{aligned}\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi^-_\epsilon\,\mathrm{d}\mu &\le -\epsilon\Big(\int\limits_{\Omega_\epsilon}\int\limits_{\Omega_\epsilon}\mathcal{N}(x,y)\,\mathrm{d}\mu + 2\int\limits_{\Omega_\epsilon}\int\limits_{\mathbb{R}^n\setminus\Omega_\epsilon}\mathcal{N}(x,y)\,\mathrm{d}\mu\Big)\\ &\le 2\epsilon\int\limits_{\Omega_\epsilon}\int\limits_{\mathbb{R}^n}|\mathcal{N}(x,y)|\,\mathrm{d}\mu.\end{aligned}$$

By Proposition 2.1, it is easy to see that

$$G'(|\nabla^s u_0|)\in L^{G^\star}(\mathbb{R}^{2n},\mathrm{d}\mu),$$

and invoking Hölder's inequality, we deduce that

$$\mathcal{N}\in L^1(\mathbb{R}^{2n},\mathrm{d}\mu).$$

Hence, for any $r>0$ there exists R_r sufficiently large such that

$$\iint\limits_{(\operatorname{supp}\varphi)\times(\mathbb{R}^n\setminus B_{R_r})} |\mathcal{N}(x,y)|\,\mathrm{d}\mu < \frac{r}{2}.$$

Also, by the definition of Ω_ϵ, we have $\Omega_\epsilon\subset\operatorname{supp}\varphi$ and $\Big|\Omega_\epsilon\times B_{R_r}\Big|\to 0$ as $\epsilon\to 0^+$. Thus, since $\mathcal{N}\in L^1(\mathbb{R}^{2n},\mathrm{d}\mu)$ there exist $\eta_r>0$ and $\epsilon_r>0$ such that for any $\epsilon\in]0,\epsilon_r[$,

$$\Big|\Omega_\epsilon\times B_{R_r}\Big| < \eta_r \quad\text{and}\quad \iint\limits_{\Omega_\epsilon\times B_{R_r}} |\mathcal{N}(x,y)|\,\mathrm{d}\mu < \frac{r}{2}.$$

Therefore, for any $\epsilon\in]0,\epsilon_r[$,

$$\iint\limits_{\Omega_\epsilon\times\mathbb{R}^n} |\mathcal{N}(x,y)|\,\mathrm{d}\mu < r,$$

from which we get

$$\lim_{\epsilon\to 0^+} \iint\limits_{\Omega_\epsilon\times\mathbb{R}^n} |\mathcal{N}(x,y)|\,\mathrm{d}\mu = 0.$$

Therefore, (4.8) is valid. By the arbitrariness of φ, we have

$$\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s(-\varphi)\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)(-\varphi)\,\mathrm{d}x \geq 0,$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$.

Consequently

$$\iint\limits_{\mathbb{R}^{2n}} G'(|\nabla^s u_0|)\frac{\nabla^s u_0}{|\nabla^s u_0|}\nabla^s\varphi\,\mathrm{d}\mu - \int\limits_{\Omega} g(x)f'(x,u_0)\varphi\,\mathrm{d}x = 0,$$

for all $\varphi \in W_0^{s(.),G}(\Omega)$.

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