BOUNDARY REGULARITY FOR STRONGLY DEGENERATE OPERATORS OF GRUSHIN TYPE

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ABSTRACT. We prove Harnack inequality and global regularity results for weak solutions of quasilinear degenerate equations driven by operators of Grushin type with natural growth. Degeneracy is a power of a strong A_∞ weight. Regularity results are achieved under minimal assumptions on the lower order coefficients.

1. INTRODUCTION

In recent decades regularity for elliptic PDEs became more and more important both in theoretical and in applied Mathematics. This paper contributes towards a complete regularity theory concerning solutions of degenerate elliptic equations under minimal assumptions on the coefficients.

Concerning the study of the regularity properties of solutions of quasilinear elliptic equations of the form

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 \tag{1.1}$$

we recall the classical works [18, 23, 24], where regularity under L^p assumptions took its final form.

The first paper where non-L^p-conditions appear, is [20], where (1.1) has been investigated under somewhat simplified structure although the right hand side is allowed to be a measure in some Morrey space. Following De Giorgi method, as adapted by Ladyzhenskaya and Ural'tseva in [18] to quasilinear equations, Hölder continuity of the weak solutions is proved.

Improvements of [20] can be found in [17, 19] where some L^p assumptions are weakened or replaced by Morrey space assumptions. Equation (1.1) has been investigated in [25] assuming Stummel-Kato type hypotheses on the lower order coefficients. This is a kind of generalization of the [1, 4] to quasilinear elliptic equations.

We now turn on degenerate elliptic equations. Many regularity results for weak solutions of elliptic equations have been generalized to the Carnot Caratheodory (CC) spaces. In such spaces the metric is generated by sub-unit curves associated to a system of non-commuting vector fields. In this direction we quote [3, 5, 6, 11, 12],

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where the above mentioned results are obtained in the subelliptic setting through Morrey and Stummel-Kato type assumptions.

Almost at the same time a parallel investigation has been performed in [13, 8], where degeneracy of operator is due to a suitable power of a strong A_∞. There a weighted version of Stummel-Kato class has been defined to obtain the continuity of weak solutions.

Strong A_∞ weights have been introduced by David and Semmes in [7] for different purposes and it has been found useful in several problems related to geometric measure theory and quasiconformal mappings. In [8, 13] a strong A_∞ weighted Stummel-Kato class has been defined and the continuity of weak solutions has been obtained.

The first attempt to consider together the two types of degeneracy described above, has been exploited by Franchi, Gutierrez and Wheeden in [14, 15] where they proved Harnack inequality for positive weak solutions of equation

$$\operatorname{div}(w(z)\nabla_\lambda u(z)) = 0$$

where w is a power of a strong A_∞ weight and $\nabla_\lambda u = (\nabla_x u(z), \lambda(x)\nabla_y u(z))$. We remark that equations studied in [14, 15] do not contain lower order terms.

In this article we extend results in [14, 15] to a more general equation. Namely, in $\Omega \subset \mathbb{R}^N = \mathbb{R}^n_x \times \mathbb{R}^m_y$, we consider the quasilinear elliptic equation in divergence form

$$\operatorname{div} A(z, u, \nabla_\lambda u) + B(z, u, \nabla_\lambda u) = 0 \tag{1.2}$$

where A and B are measurable functions satisfying suitable structure conditions of the form

$$
\begin{aligned}
|A(z,u,\xi)| &\leq aw(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b(z)|u|^{p-1} + e(z) \\
|B(z,u,\xi)| &\leq b_0 w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} + b_1(z)[|\xi_x|^2 \\
&\quad + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d(z)|u|^{p-1} + f(z) \\
\xi \cdot A(z,u,\xi) &\geq w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} - d_1(z)|u|^p - g(z)
\end{aligned}
\tag{1.3}
$$

where $w = v^{1-\frac{p}{N}}$, $1 < p < N$, v is a strong A_∞ weight and λ is a suitable function we make precise later.

We briefly describe the content of this paper. In Section 2 we recall the definition of strong A_∞ weight and the related function spaces. In Section 3 we give the definitions of Stummel-Kato and Morrey classes introduced in [9, 10] (see e.g. [21, 22]) and state Fefferman type inequality which will allows us to control the integral arising from the coefficients in (1.3). In Section 4, following Trudinger pattern (see [24]), we obtain Harnack inequality for non negative weak solutions of equation (1.2) and, as consequence, the regularity of weak solutions. In particular, we prove continuity results under Stummel-Kato type assumptions, and Hölder continuity result under Morrey type assumptions. Finally, in Section 5 we use Trudinger technique to prove Harnack inequality up to the boundary.

2. Strong A_∞ weights and function spaces

We denote by $z = (x, y)$ a point in $\mathbb{R}^N$, with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, $n + m = N$. We assume that there exists a function $\lambda(x)$ defined on $\mathbb{R}^n$ such that

(H1) $\lambda = \lambda(x)$ is a continuous nonnegative function vanishing only at a finite number of points;

(H2) λ^n is a strong A_∞ weight (see Definition 2.2);

(H3) λ satisfies an infinite order reverse Hölder inequality, i.e. for any $x_0 \in \mathbb{R}^n$, $r > 0$ we have

$$\fint_{|x-x_0|<r} \lambda(x)dz \sim \max_{|x-x_0|<r} \lambda(x) \,.$$

If $u : \mathbb{R}^N \to \mathbb{R}$ is a differentiable function a.e. in $\mathbb{R}^N$, we put $\nabla_\lambda u(z) = (\nabla_x u(z), \lambda(x)\nabla_y u(z))$, so that $|\nabla_\lambda u|^2 = |\nabla_x u|^2 + \lambda^2(x)|\nabla_y u|^2$. We define the Carnot-Carathéodory metric (C-C metric) ρ in $\mathbb{R}^N$ with respect to ∇_λ in the following way. An absolutely continuous curve $\gamma : [0,T] \to \mathbb{R}^N$ is said to be sub-unit if for any $z = (x,y) \in \mathbb{R}^N$,

$$\langle \gamma'(t), z \rangle^2 \le |x|^2 + \lambda(\gamma(t))|y|^2$$

for a.e. $t \in [0,T]$. If $z_1, z_2 \in \mathbb{R}^N$ we put

$$\rho(z_1, z_2) = \inf\Big\{ T \ge 0 : \exists \text{ a sub-unit curve } \gamma : [0,T] \to \mathbb{R}^N,$$

$$\text{such that } \gamma(0) = z_1 \text{ and } \gamma(T) = z_2 \Big\}.$$

We denote by $B = B_r = B(z_0, r)$ the C-C metric ball centered at z_0 of radius r. Now we define the A_q and the strong A_∞ weights (see [14, 15]).

Definition 2.1. Let $q > 1$ and let v be a nonnegative locally integrable function in $\mathbb{R}^N$. We say that v is a weight of the Muckenhoupt class A_q if

$$\sup_B \Big(\frac{1}{|B|} \int_B v(z)\,dz \Big)\Big(\frac{1}{|B|} \int_B [v(z)]^{\frac{-1}{q-1}}\,dz \Big)^{q-1} \equiv C_0 < +\infty \,,$$

where the supremum is taken over all Carnot-Carathéodory metric balls B in $\mathbb{R}^N$. The number C_0 is called the A_q constant of v.

Definition 2.2. Let v be an A_q weight for some $q > 1$. If z_1, z_2 belong to $\mathbb{R}^N$, put

$$\delta(z_1, z_2) = \inf \Big(\int_B v(z)\lambda^{\frac{m}{N-1}}(z)dz \Big)^{1/N} \,,$$

where the infimum is taken over the balls B such that $z_1, z_2 \in B$.

If $\gamma : [0,T] \to \mathbb{R}^N$ is a continuous curve, we define the v-lenght of γ as

$$l(\gamma) = \liminf_{|\sigma| \to 0} \sum_{i=0}^{p-1} \delta(\gamma(t_{i+1}), \gamma(t_i)) \,,$$

where $\sigma = \{t_0, \ldots, t_p\}$ is a partition of $[0,T]$, and we define a distance $d(z_1, z_2)$ as the infimum of the v- lengths of sub-unit curves connecting z_1 and z_2. If there exist positive constants c_1 and c_2 such that

$$c_1\delta(z_1, z_2) \le d(z_1, z_2) \le c_2\delta(z_1, z_2) \,,$$

we say that v is a strong A_∞ weight for the metric ρ.

An example of strong A_∞ weight is the function $v(z) = \rho(z, z_0)^\alpha$ with $\alpha \ge 0$ and $z_0 \in \mathbb{R}^N$ (see [15]).

Using strong A_∞ weights we define Lebesgue and Sobolev classes.

Definition 2.3. Let v be a strong A_∞ weight and $w = v^{1-p/N}$, $1 \le p < N$, $\Omega \subset \mathbb{R}^N$. For any $u \in C_0^\infty(\Omega)$ we set

$$\|u\|_{L_v^p(\Omega)} = \Big(\int_\Omega |u(z)|^p w(z) \, dz \Big)^{1/p}.$$

We define $L_v^p(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the above norm.

For $u \in C^\infty(\Omega)$ we set

$$\|u\|_{H_v^{1,p}(\Omega)} = \Big(\int_\Omega |u(z)|^p w(z) \, dz \Big)^{1/p} + \Big(\int_\Omega |\nabla_\lambda u(z)|^p w(z) \, dz \Big)^{1/p}. \qquad (2.1)$$

We define $H_{0,v}^{1,p}(\Omega)$ to be the completion of $C_0^\infty(\Omega)$ with respect to the norm (2.1) and $H_v^{1,p}(\Omega)$ to be the completion of $C^\infty(\Omega)$ with respect to the same norm.

Now to recall the Sobolev embedding theorem and the representation formula proved in [15], Theorem I, we need another assumption on strong A_∞ weights.

A strong A_∞ weight v satisfies the local boundedness condition near the zeros of λ if the following condition holds

> if $\lambda(x_1) = 0$, then $v(x,y)$ is bounded as $x \to x_1$ uniformly in y, for y in every bounded set. $\qquad (2.2)$

Theorem 2.4. *Let $1 < p < N$ and v be a strong A_∞. If there exists a strong A_∞ weight w satisfying (Z_λ) such that $v^{1-p/N} w^{-(N-1)/N}$ belongs to A_p with respect to the (doubling) measure $w^{(N-1)/N} dz$, then there exists a constant $q > p$ such that*

$$\Big(\fint_{B(z_0,r)} |g - \mu|^q v^{1-p/N} dz \Big)^{1/q} \le Cr \Big(\fint_{B(z_0,r)} |\nabla_\lambda g|^p v^{1-p/N} dz \Big)^{1/p} \qquad (2.3)$$

for any Lipschitz continuous function g, where μ can be chosen to be the $v^{1-p/N}$-average of g over $B(z_0,r)$.

Remark 2.5. We stress that if we take the weights $v = w = \rho^\alpha(0,z)$ and the function $\lambda = |x|^\sigma$, $(\alpha, \sigma > 0)$, the assumptions of Theorem 2.4 are satisfied (see also [14]).

3. Stummel-Kato type classes

In this Section we recall a representation formula (see [15, Corollary 3.2]) to define Stummel - Kato type classes.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, v a strong A_∞ weight satisfying (2.2) and u a compactly supported smooth function in a metric ball $B = B_R \subset \Omega$. Then there exists c independent of u such that*

$$|u(z)| \le c \int_B |\nabla_\lambda u(\xi)| v^{1-\frac{1}{N}}(\xi) k(z,\xi) d\xi$$

for almost all $z \in B$, where

$$k(z,\xi) = \Big(\int_{B_{\rho(z,\xi)}(z)} v(\zeta) \lambda^{\frac{m}{N-1}}(\zeta) d\zeta \Big)^{\frac{1-N}{N}}.$$

Now we give the definition of Stummel-Kato and Morrey classes.

Definition 3.2. Let V be a locally integrable function in Ω, $r > 0$, and $p \in]1, N[$. Let v be a strong A_∞ weight. We set $w(z) \equiv v^{1-\frac{p}{N}}(z)$ and

$$\phi(V; r) \equiv \sup_{z \in \Omega} \left(\int_{B(z,r)} k(z, \xi) v(\xi) \left(\int_\Omega |V(\zeta)| k(\zeta, \xi) w(\zeta) d\zeta \right)^{\frac{1}{p-1}} d\xi \right)^{p-1}.$$

We say that V belongs to the class $\tilde{S}_v(\Omega)$ if $\phi(V; r)$ is just a bounded function in a neighborhood of the origin.

If, moreover $\lim_{r \to 0} \phi(V; r) = 0$ then we say that V belongs to the Stummel-Kato class $S_v(\Omega)$.

If there exists $\rho > 0$ such that

$$\int_0^\rho \frac{\phi(V; t)}{t} dt < +\infty,$$

then we say that the function V belongs to the class $S_v'(\Omega)$.

If there exist c and $\sigma > 0$ such that $\phi(V; r) \leq cr^\sigma$ then we say that the function V belongs to the Morrey class $L_v^{1,\sigma}(\Omega)$.

Remark 3.3. If $v(z) = 1$, $\lambda(x) = 1$ and $p = 2$ the previous definitions give back the classical Stummel-Kato class (see [1]) and Morrey space $L^{1,n-2+\sigma}$ for some $\sigma > 0$.

Now we state a Fefferman type inequality related to Stummel-Kato classes and its corollary.

Theorem 3.4. Let v be a strong A_∞ weight satisfying (Z_λ) and $1 < p < N$. If V belongs to the class $\tilde{S}_v(\Omega)$, then there exists a constant c such that $\forall u \in C_0^\infty(\Omega)$

$$\left(\int_B |V(z)||u(z)|^p w(z) dz \right)^{1/p} \leq c\phi^{1/p}(V; 2R) \left(\int_B |\nabla_\lambda u(z)|^p w dz \right)^{1/p},$$

where $w(z) \equiv v^{1-\frac{p}{N}}(z)$ and R is the radius of a metric ball B, containing the support of u.

For a proof of the above theorem, see [9, Theorem 2.3].

Corollary 3.5. Let $1 < p < N$ and v be a strong A_∞ weight satisfying (2.2). Let V belongs to the class $S_v(\Omega)$. For any $\varepsilon > 0$, there exists $K(\varepsilon)$ such that

$$\int_B |V(z)||u(z)|^p w(z) dz \leq \varepsilon \int_B |\nabla_\lambda u(z)|^p w(z) dz + K(\varepsilon) \int_B |u(z)|^p w(z) dz,$$

for all $u \in C_0^\infty(\Omega)$, where $w(z) = v(z)^{1-\frac{p}{N}}$,

$$K(\varepsilon) \sim \frac{\sigma}{\left[\phi^{-1}(V; \varepsilon) \right]^{N+p}}$$

and ϕ^{-1} denotes the inverse function of ϕ.

For a proof of the above colollary, see [9, Corollary 2.1].

4. Harnack inequality

In this section we prove a weak Harnack inequality for non negative weak solutions of the equation

$$\operatorname{div} A(z, u, \nabla_\lambda u) + B(z, u, \nabla_\lambda u) = 0. \tag{4.1}$$

First wee recall what we mean by weak solution of (4.1).

Definition 4.1. A function $u \in H_v^{1,p}(\Omega)$ is a local weak subsolution (supersolution) of equation (4.1) in Ω if

$$\int_\Omega A(z, u(z), \nabla_\lambda u(z)) \cdot \nabla \varphi \, dz - \int_\Omega B(z, u(z), \nabla_\lambda u(z)) \varphi \, dz \leq 0 \quad (\geq 0) \qquad (4.2)$$

for every non negative $\varphi \in H_{0,v}^{1,p}(\Omega)$. A function u is a weak solution if it is both super and sub solution.

We require the functions $A(z, u, \xi)$ and $B(z, u, \xi)$ to be measurable functions satisfying the following structural conditions

$$\begin{aligned}
|A(z, u, \xi)| &\leq aw(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b(z)|u|^{p-1} + e(z) \\
|B(z, u, \xi)| &\leq b_0 w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} + b_1(z)[|\xi_x|^2 \\
&\quad + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d(z)|u|^{p-1} + f(z) \\
\xi \cdot A(z, u, \xi) &\geq w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} - d_1(z)|u|^p - g(z) \,.
\end{aligned} \qquad (4.3)$$

where $1 < p < N$, $w = v^{1 - \frac{p}{N}}$ and v is a strong A_∞ weight.

We show that locally bounded weak solutions satisfy a Harnack inequality and, as a consequence, some regularity properties. We shall make the following assumptions on the lower order terms to ensure the continuity of local weak solutions:

$$a, b_0 \in \mathbb{R}, \quad \left(\frac{b}{w}\right)^{\frac{p}{p-1}}, \quad \left(\frac{b_1}{w}\right)^p, \frac{d}{w}, \frac{d_1}{w}, \left(\frac{e}{w}\right)^{\frac{p}{p-1}}, \frac{f}{w}, \frac{g}{w} \in S_v'(\Omega) \,. \qquad (4.4)$$

Theorem 4.2. *Let u be a non negative weak supersolution of equation (4.1) in Ω satisfying (4.3) and (4.4). Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \leq M$ in B_{3r}. Then there exists c depending on n, M, a, b_0, p and the weight v such that*

$$w^{-1}(B_{2r}) \int_{B_{2r}} u \, wdz \leq c \Big\{ \inf_{B_r} u + h(r) \Big\}$$

where

$$h(r) = \left[\phi\left(\left(\frac{e}{w}\right)^{\frac{p}{p-1}}; r \right) + \phi\left(\frac{g}{w}; r\right) \right]^{1/p} + \left[\phi\left(\frac{f}{w}; r\right) \right]^{\frac{1}{p-1}} \,.$$

Proof. We simplify the structure assumptions by setting $u_h = u + h(r)$. We obtain

$$\begin{aligned}
|A(z, u, \xi)| &\leq aw(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b_2(z)|u_h|^{p-1} \\
|B(z, u, \xi)| &\leq b_0 w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} + b_1(z)[|\xi_x|^2 \\
&\quad + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d_2(z)|u_h|^{p-1} \\
\xi \cdot A(z, u, \xi) &\geq w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} - d_3(z)|u_h|^p
\end{aligned} \qquad (4.5)$$

where $b_2 = b + h(r)^{1-p}e$, $d_2 = d + h(r)^{1-p}f$, and $d_3 = d_1 + h(r)^{-p}g$.

It is easy to check that b_2, d_2, and d_3 satisfy the same assumptions of b, d d_1.

We take $\varphi = \eta^p u_h^\beta e^{-b_0 u_h}$, $\beta < 0$, as test function in (4.2), (see [18]), where $\eta \in C_0^1(B_{3r})$, $\eta \geq 0$. We obtain

$$\begin{aligned}
&\int_{B_{3r}} \eta^p e^{-b_0 u_h}(b_0 u_h^\beta + |\beta| u_h^{\beta-1}) \nabla_\lambda u_h \cdot A dz \\
&- p \int_{B_{3r}} u_h^\beta \eta^{p-1} e^{-b_0 u_h} \nabla_\lambda \eta \cdot A dz + \int_{B_{3r}} \eta^p u_h^\beta e^{-b_0 u_h} B dz \leq 0 \,.
\end{aligned}$$

The previous inequality and the structure assumptions (4.5) yield

$$\int_{B_{3r}} e^{-b_0 u_h} \eta^p (b_0 u_h^\beta + |\beta| u_h^{\beta-1}) |\nabla_\lambda u_h|^p w \, dz$$

$$\leq \int_{B_{3r}} e^{-b_0 u_h} \eta^p (b_0 u_h^\beta + |\beta| u_h^{\beta-1}) (\nabla_\lambda u_h \cdot A + d_3 |u_h|^p) dz$$

$$\leq p \int_{B_{3r}} u_h^\beta \eta^{p-1} e^{-b_0 u_h} \nabla_\lambda \eta \cdot A \, dz - \int_{B_{3r}} \eta^p u_h^\beta e^{-b_0 u_h} B \, dz$$

$$+ \int_{B_{3r}} e^{-b_0 u_h} \eta^p (b_0 u_h^\beta + |\beta| u_h^{\beta-1}) d_3 |u_h|^p \, dz$$

$$\leq p \int_{B_{3r}} u_h^\beta \eta^{p-1} e^{-b_0 u_h} \nabla_\lambda \eta \cdot A \, dz$$

$$+ \int_{B_{3r}} \eta^p u_h^\beta e^{-b_0 u_h} (b_0 |\nabla_\lambda u_h|^p w + b_1 |\nabla_\lambda u_h|^{p-1} + d_2 |u_h|^{p-1}) dz$$

$$+ \int_{B_{3r}} e^{-b_0 u_h} \eta^p (b_0 u_h^\beta + |\beta| u_h^{\beta-1}) d_3 |u_h|^p \, dz \, .$$

By Young inequality and boundedness of u_h in B_{3r} we obtain

$$|\beta| \int_{B_{3r}} \eta^p u_h^{\beta-1} |\nabla_\lambda u_h|^p w \, dz$$

$$\leq cp \int_{B_{3r}} u_h^\beta \eta^{p-1} \nabla_\lambda \eta \cdot A \, dz$$

$$+ c \int_{B_{3r}} \eta^p u_h^\beta (b_1 |\nabla_\lambda u_h|^{p-1} + d_2 |u_h|^{p-1}) dz$$

$$+ c \int_{B_{3r}} \eta^p (b_0 u_h^\beta + |\beta| u_h^{\beta-1}) d_3 |u_h|^p \, dz$$

$$\leq c \int_{B_{3r}} \Big\{ p u_h^\beta \eta^{p-1} |\nabla_\lambda \eta| (aw |\nabla_\lambda u_h|^{p-1} + b_2 |u_h|^{p-1})$$

$$+ \eta^p u_h^\beta b_1 |\nabla_\lambda u_h|^{p-1} + \eta^p u_h^{\beta+p-1} d_2$$

$$+ \eta^p b_0 u_h^{\beta+p} d_3 + |\beta| \eta^p u_h^{\beta+p-1} d_2 \Big\} dz$$

$$\leq c \int_{B_{3r}} \Big\{ p u_h^\beta \eta^{p-1} |\nabla_\lambda \eta| aw |\nabla_\lambda u_h|^{p-1} + p u_h^{\beta+p-1} \eta^{p-1} |\nabla_\lambda \eta| b_2$$

$$+ \eta^p u_h^\beta b_1 |\nabla_\lambda u_h|^{p-1} + (1 + |\beta|) \eta^p u_h^{\beta+p-1} d_2 + \eta^p b_0 u_h^{\beta+p} d_3 \Big\} dz \, .$$

Then

$$|\beta| \int_{B_{3r}} \eta^p u_h^{\beta-1} |\nabla_\lambda u_h|^p w \, dz$$

$$\leq c(b_0, M, p) \int_{B_{3r}} \Big\{ u_h^\beta \eta^{p-1} |\nabla_\lambda \eta| |a| |\nabla_\lambda u_h|^{p-1} w \, dz$$

$$+ \epsilon \eta^p u_h^{\beta-1} |\nabla_\lambda u_h|^p w + c(\epsilon) \eta^p \frac{b_1^p}{w^{p-1}} u_h^{\beta+p-1}$$

$$+ \eta^{p-1} |\nabla_\lambda \eta| u_h^{\beta+p-1} b_2 + (1 + |\beta|) \eta^p u_h^{\beta+p-1} d_2 + \eta^p u_h^{\beta+p} d_3 \Big\} dz$$

$$\leq c(b_0, M, a, p) \int_{B_{3r}} \left\{ u_h^{\beta} \eta^{p-1} |\nabla_\lambda \eta| |\nabla_\lambda u_h|^{p-1} w \, dz \right.$$

$$+ \epsilon \eta^p u_h^{\beta-1} |\nabla_\lambda u_h|^p w + c(\epsilon) \eta^p \frac{b_1^p}{w^{p-1}} u_h^{\beta+p-1}$$

$$+ u_h^{\beta+p-1} |\nabla_\lambda \eta|^p w + \eta^p u_h^{\beta+p-1} \frac{b_2^{\frac{p}{p-1}}}{w^{\frac{1}{p-1}}}$$

$$\left. + (1+|\beta|) \eta^p u_h^{\beta+p-1} d_2 + \eta^p u_h^{\beta+p-1} d_3 \right\} dz \,.$$

Setting $V = \dfrac{b_2^{\frac{p}{p-1}}}{w^{\frac{1}{p-1}}} + \dfrac{b_1^p}{w^{p-1}} + d_2 + d_3$, we obtain

$$\int_{B_{3r}} \eta^p u_h^{\beta-1} |\nabla_\lambda u_h|^p w \, dz$$
$$\leq c(1+|\beta|^{-1})^p \int_{B_{3r}} \left\{ |\nabla_\lambda \eta|^p u_h^{\beta+p-1} w + V \eta^p u_h^{\beta+p-1} \right\} dz \,. \tag{4.6}$$

Now we set

$$\mathbb{U}(x) = \begin{cases} u_h^q(x) \text{ where } pq = p+\beta-1 & \text{if } \beta \neq 1-p \\ \log u_h(x) & \text{if } \beta = 1-p \end{cases}$$

by (4.6) we have

$$\int_{B_{3r}} \eta^p |\nabla_\lambda \mathbb{U}|^p w \, dz$$
$$\leq c|q|^p (1+|\beta|^{-1})^p \left\{ \int_{B_{3r}} |\nabla_\lambda \eta|^p \mathbb{U}^p w \, dz + \int_{B_{3r}} V \eta^p \mathbb{U}^p \, dz \right\}, \quad \beta \neq 1-p, \tag{4.7}$$

while

$$\int_{B_{3r}} \eta^p |\nabla_\lambda \mathbb{U}|^p w \, dz \leq c \left\{ \int_{B_{3r}} |\nabla_\lambda \eta|^p w \, dz + \int_{B_{3r}} V \eta^p \, dz \right\} \tag{4.8}$$

if $\beta = 1-p$.

Let us start with the case $\beta = 1-p$. By Theorem 3.4 we have

$$\int_{B_{3r}} V \eta^p \, dz \leq c \phi \left(\frac{V}{w}; \operatorname{diam} \Omega \right) \int_{B_{3r}} |\nabla_\lambda \eta|^p w \, dz \,,$$

and from (4.8),

$$\int_{B_{3r}} \eta^p |\nabla_\lambda \mathbb{U}|^p w \, dz \leq c \int_{B_{3r}} |\nabla_\lambda \eta|^p w \, dz \,.$$

Let B_h be a ball contained in B_{2r}. Choosing $\eta(x)$ so that $\eta = 1$ in B_h, $0 \leq \eta \leq 1$ in $B_{3r} \setminus B_h$ and $|\nabla_\lambda \eta| \leq \dfrac{3}{h}$, we obtain

$$\|\nabla_\lambda \mathbb{U}\|_{L_v^p(B_h)} \leq c \frac{w(B_h)^{1/p}}{h} \,.$$

By Theorem 2.4 and John-Nirenberg lemma [2], there exist two positive constants p_0 and c, such that

$$\left(\fint_{B_{2r}} e^{p_0 \mathbb{U}} w \, dz \right)^{1/p_0} \left(\fint_{B_{2r}} e^{-p_0 \mathbb{U}} w \, dz \right)^{1/p_0} \leq c \,. \tag{4.9}$$

Let us consider the family of seminorms

$$\Phi(p, \rho) = \Big(\int_{B_\rho} |u_h|^p w \, dz \Big)^{1/p}, \quad p \neq 0.$$

By (4.9) we have

$$\frac{1}{w(B_{2r})^{1/p_0}} \Phi(p_0, 2r) \leq c \, w(B_{2r})^{1/p_0} \Phi(-p_0, 2r).$$

In the case (4.7) by Corollary 3.5 we obtain

$$\int_{B_{3r}} |\nabla_\lambda \mathbb{U}|^p \eta^p w \, dz \leq c \Big\{ (|q|^p + 1)\Big(1 + \frac{1}{|\beta|}\Big)^p \int_{B_{3r}} |\nabla_\lambda \eta|^p \mathbb{U}^p w \, dz$$
$$+ \Big[\frac{1}{\phi^{-1}\big(\frac{V}{w}; |q|^{-p}(1 + \frac{1}{|\beta|})^{-p}\big)} \Big]^{n+p} \int_{B_{3r}} \eta^p \mathbb{U}^p w \, dz \Big\}.$$

From (2.3), setting $k = q/p$, we have

$$\Big(\int_{B_{3r}} |\eta \mathbb{U}|^{kp} w \, dz \Big)^{1/k} \leq c \, w(B)^{\frac{1}{k}-1} \Big\{ (|q|^p + 2)\Big(1 + \frac{1}{|\beta|}\Big)^p \int_{B_{3r}} |\nabla_\lambda \eta|^p \mathbb{U}^p w \, dz$$
$$+ \Big[\frac{1}{\phi^{-1}\big(\frac{V}{w}; |q|^{-p}\big(1 + \frac{1}{|\beta|}\big)^{-p}\big)} \Big]^{n+p} \int_{B_{3r}} \eta^p \mathbb{U}^p w \, dz \Big\}$$

where c is a positive constant independent of w.

Now we choose the function η. Let r_1 and r_2 be real numbers such that $r \leq r_1 < r_2 \leq 2r$ and let the function η be chosen so that $\eta(z) = 1$ in B_{r_1}, $0 \leq \eta \leq 1$ in B_{r_2}, $\eta(z) = 0$ outside B_{r_2}, $|\nabla_\lambda \eta| \leq \frac{c}{r_2 - r_1}$ for some fixed constant c. we have

$$\Big(\int_{B_{r_1}} \mathbb{U}^{kp} w \, dz \Big)^{1/k} \leq c \, w(B)^{\frac{1}{k}-1} \frac{1}{(r_2 - r_1)^p}(|q|^p + 2)\Big(1 + \frac{1}{|\beta|}\Big)^p$$
$$\times \Big[\frac{1}{\phi^{-1}\big(\frac{V}{w}; |q|^{-p}\big(1 + \frac{1}{|\beta|}\big)^{-p}\big)} \Big]^{n+p} \int_{B_{r_2}} \mathbb{U}^p w \, dz.$$

Setting $\gamma = pq = p + \beta - 1$ and recalling that $\mathbb{U}(z) = u_h^q(z)$, we obtain

$$\Phi(k\gamma, r_1) \geq c^{1/\gamma} w(B)^{\frac{1}{\gamma}(\frac{1}{k}-1)}(|q|^p + 2)^{1/\gamma}$$
$$\times \Big[\frac{1}{\phi^{-1}\big(\frac{V}{w}; |q|^{-p}\big)} \Big]^{\frac{n+p}{\gamma}} \frac{1}{(r_2 - r_1)^{1/p}} \Phi(\gamma, r_2), \tag{4.10}$$

for negative γ. This is the inequality we are going to iterate. If $\gamma_i = k^i p_0$ and $r_i = r + \frac{r}{2^i}$, $i = 1, 2, \dots$ iteration of (4.10) and use of [9, Lemma 2.4] yield

$$\Phi(-\infty, r) \geq c(p, a, \phi_{\frac{V}{w}}, \operatorname{diam}\Omega) \omega(B_r)^{\frac{1}{p_0}} \Phi(-p_0, 2r).$$

Therefore by Hölder inequality,

$$\Phi(p_0', 2r) \leq \Phi(p_0, 2r) w(B_r)^{\frac{1}{p_0'} - \frac{1}{p_0}}, \quad p_0' \leq p_0.$$

So we obtain

$$w^{-1}(B_{2r}) \Phi(1, 2r) \leq c \Phi(-\infty, r)$$

where $c \equiv c(p, a, \phi_{\frac{V}{w}}, \operatorname{diam}\Omega)$ and the result follows. $\square$

We obtain a weak Harnack inequality for weak subsolutions in a similar way of Theorem 4.2.

Theorem 4.3. *Let u be a non negative weak subsolution of equation* (4.1) *in Ω satisfying* (4.3) *and* (4.4). *Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \le M$ in B_{3r}. Then there exists c depending on n, M, a, b_0, p and the weight v such that*

$$\sup_{B_r} u \le c \Big\{ w^{-1}(B_{2r}) \int_{B_{2r}} u \, w dz + h(r) \Big\}$$

where

$$h(r) = \Big[\phi\Big(\big(\frac{e}{w}\big)^{\frac{p}{p-1}}; 3r \Big) + \phi\Big(\frac{g}{w}; 3r\Big) \Big]^{1/p} + \Big[\phi\Big(\frac{f}{w}; 3r\Big) \Big]^{\frac{1}{p-1}} .$$

If we take a non negative weak solution, we can put together the two previous results.

Theorem 4.4. *Let u be a non negative weak solution of equation* (4.1) *in Ω satisfying* (4.3) *and* (4.4). *Let B_r be a ball such that $B_{3r} \Subset \Omega$ and let M be a constant such that $u \le M$ in B_{3r}. Then there exists c depending on n, M, a, b_0, p and the weight v such that*

$$\sup_{B_r} u \le c \big\{ \inf_{B_r} u + h(r) \big\} ,$$

where

$$h(r) = \Big[\phi\Big(\big(\frac{e}{w}\big)^{\frac{p}{p-1}}; 3r \Big) + \phi\Big(\frac{g}{w}; 3r\Big) \Big]^{1/p} + \Big[\phi\Big(\frac{f}{w}; 3r\Big) \Big]^{\frac{1}{p-1}} .$$

Now, as a simple consequence of Harnack inequality, we obtain some regularity results for weak solutions of (4.1). The proof is an immediate consequence of Harnack inequality so we omit it.

Theorem 4.5. *Let u be a locally bounded weak solution of equation* (4.1) *in Ω satisfying* (4.3) *and* (4.4). *Then u is continuous in Ω.*

If we assume more restrictive assumptions on the lower order terms we obtain the following refinement of the previous one.

Theorem 4.6. *Let u be a locally bounded weak solution of equation* (4.1) *in Ω satisfying* (4.3) *and*

$$a, b_0 \in \mathbb{R}, \quad \big(\frac{b}{w}\big)^{\frac{p}{p-1}}, \big(\frac{b_1}{w}\big)^p, \frac{d}{w}, \frac{d_1}{w}, \big(\frac{e}{w}\big)^{\frac{p}{p-1}}, \frac{f}{w}, \frac{g}{w} \in L_v^{1,\sigma}(\Omega), \quad \sigma > 0 .$$

Then u is locally Hölder continuous in Ω.

5. Boundary Harnack inequality

Our next step is to show a Harnack inequality near the boundary of Ω for weak supersolutions and subsolutions to the equation

$$\operatorname{div} A(z, u, \nabla_\lambda u) + B(z, u, \nabla_\lambda u) = 0 , \tag{5.1}$$

with the structural conditions

$$|A(z, u, \xi)| \le aw(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + b(z)|u|^{p-1} + e(z)$$
$$|B(z, u, \xi)| \le b_0 w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} + b_1(z)[|\xi_x|^2$$
$$+ \lambda^2(x)|\xi_y|^2]^{\frac{p-1}{2}} + d(z)|u|^{p-1} + f(z) \tag{5.2}$$
$$\xi \cdot A(z, u, \xi) \ge w(z)[|\xi_x|^2 + \lambda^2(x)|\xi_y|^2]^{p/2} - d_1(z)|u|^p - g(z)$$

where $1 < p < N$, $w = v^{1-\frac{p}{N}}$ and v is a strong A_∞ weight and

$$a, b_0 \in \mathbb{R}, \quad \left(\frac{b}{w}\right)^{\frac{p}{p-1}}, \left(\frac{b_1}{w}\right)^p, \frac{d}{w}, \frac{d_1}{w}, \left(\frac{e}{w}\right)^{\frac{p}{p-1}}, \frac{f}{w}, \frac{g}{w} \in S'_v(\Omega). \tag{5.3}$$

Let S be a subset of $\partial\Omega$ and u be a function on Ω. We say that $u \leq M$ on S if for all $\epsilon > 0$ there exists a neighborhood N of S such that $u(x) \leq M + \epsilon$ for a.e. $x \in N \cap \Omega$. In this way we can define $\inf_S u$, $\sup_S u$ and $\operatorname{osc}_S u$.

Now, let B_r be a ball centered at $x_0 \in \partial\Omega$ and $u \in H_v^{1,p}(\Omega \cap B_{4r}, w)$ we set

$$\tilde{u}(x) = \begin{cases} \min\{u, m\} & \text{if } x \in \Omega \cap B_{4r} \\ m & \text{if } x \in \mathbb{R}^n \setminus (\Omega \cap B_{4r}) \end{cases}$$

where $m = \inf_{\partial\Omega \cap B_{4r}} u$. Moreover, we define $b = 0$, $d = 0$, $d_1 = 0$, $e = 0$, $f = 0$, $g = 0$ outside Ω.

Theorem 5.1. *Let* $u \in H_v^{1,p}(\Omega \cap B_{4r})$ *be a weak non negative supersolution of* (5.1) *in* $\Omega \cap B_{4r}$. *Assume* (5.2) *and* (5.3). *Let* M *be a constant such that* $u \leq M$ *on* $\Omega \cap B_{4r}$. *Then there exists* c *depending on* n, M, a, b_0, p *and the weight* v *such that*

$$w^{-1}(B_{2r}) \int_{B_{2r}} \tilde{u} \, w dz$$
$$\leq c \left\{ \inf_{B_r} \tilde{u} + \phi^{1/p} \left[\left(\frac{e}{w}\right)^{\frac{p}{p-1}}; r \right] + \phi^{\frac{1}{p-1}} \left(\frac{f}{w}; r\right) + \phi^{1/p} \left(\frac{g}{w}; r\right) \right\}. \tag{5.4}$$

Proof. Set

$$h = \phi^{1/p} \left[\left(\frac{e}{w}\right)^{\frac{p}{p-1}}; r \right] + \phi^{\frac{1}{p-1}} \left(\frac{f}{w}; r\right) + \phi^{1/p} \left(\frac{g}{w}; r\right)$$

and $\tilde{v} = \tilde{u} + h$. Let $\eta \in C_0^1(B_{3r})$ and $\eta \geq 0$. For $\beta < 0$ we define $\varphi(z) = \eta^p[\tilde{v}^\beta - (m+h)^\beta]e^{-|b_0|\tilde{v}} \in H_{0,v}^{1,p}(B_{3r})$. From (5.2), we obtain in the support of φ,

$$|A(z, u, \nabla_\lambda u)| \leq a w(z) |\nabla_\lambda u|^{p-1} + b_2(z)|\tilde{v}|^{p-1}$$
$$|B(z, u, \nabla_\lambda u)| \leq b_0 w(z)|\nabla_\lambda u|^p + b_1(z)|\nabla_\lambda u|^{p-1} + d_2(z)|\tilde{v}|^{p-1} \tag{5.5}$$
$$\xi \cdot A(z, u, \nabla_\lambda u) \geq w(z)|\nabla_\lambda u|^p - d_3(z)|\tilde{v}|^p$$

where $b_2(z) = b(z) + \frac{e(z)}{h^{p-1}}$, $d_2(z) = d + \frac{f(z)}{h^{p-1}}$, and $d_3(z) = d_1(z) + \frac{g(z)}{h^p}$.

Since, for any $0 < \rho < 3r$,

$$\phi\left(\left(\frac{b_2}{w}\right)^{\frac{p}{p-1}}; \rho\right) \leq \phi\left(\left(\frac{b}{w}\right)^{\frac{p}{p-1}}; \rho\right) + \frac{1}{h^p}\phi\left(\left(\frac{e}{w}\right)^{\frac{p}{p-1}}; \rho\right),$$
$$\phi\left(\frac{d_2}{w}; \rho\right) \leq \phi\left(\frac{d}{w}; \rho\right) + \frac{1}{h^{p-1}}\phi\left(\frac{f}{w}; \rho\right),$$
$$\phi\left(\frac{d_3}{w}; \rho\right) \leq \phi\left(\frac{d_1}{w}; \rho\right) + \frac{1}{h^p}\phi\left(\frac{g}{w}; \rho\right)$$

we obtain

$$\left(\frac{b_2}{w}\right)^{\frac{p}{p-1}}, \frac{d_2}{w}, \frac{d_3}{w} \in S'_v(B_{3r}).$$

Since u is a supersolution of (5.1) we have

$$\int_{B_{3r}} \eta^p e^{-|b_0|\tilde{v}} \{|\beta|v^{\beta-1} + b_0[\tilde{v}^\beta - (m+h)^\beta]\} A(z, u, \nabla_\lambda u) \cdot \nabla_\lambda u \, dz$$
$$+ \int_{B_{3r}} B(z, u, \nabla_\lambda u)\varphi dz$$

$$\leq p \int_{B_{3r}} \eta^{p-1} A(z, u, \nabla_\lambda u) \cdot \nabla_\lambda \eta [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz$$

Using (5.5) we obtain

$$\int_{B_{3r}} (w|\nabla_\lambda u|^p d_3 \tilde{v}^p) \eta^p e^{-|b_0|\tilde{v}} \{|\beta|\tilde{v}^{\beta-1} + b_0 [\tilde{v}^\beta - (m+h)^\beta]\} dz$$

$$\leq p \int_{B_{3r}} (aw|\nabla_\lambda u|^{p-1} + b_0 \tilde{v}^{p-1}) \eta^{p-1} |\nabla_\lambda \eta| [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz$$

$$+ \int_{B_{3r}} (b_0 w |\nabla_\lambda u|^p + b_1 |\nabla_\lambda u|^{p-1} + d_2 \tilde{v}^{p-1}) \eta^p [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz$$

from which

$$\int_{B_{3r}} |\beta| |\nabla_\lambda u|^p \eta^p e^{-|b_0|\tilde{v}} \tilde{v}^{\beta-1} w \, dx$$

$$\leq |\beta| \int_{B_{3r}} d_3 \eta^p \tilde{v}^{p+\beta-1} e^{-|b_0|\tilde{v}} dz + b_0 \int_{B_{3r}} d_3 \eta^p \tilde{v}^p [v^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}}$$

$$+ p \int_{B_{3r}} a\eta^{p-1} |\nabla_\lambda \eta| |\nabla_\lambda u|^{p-1} [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} w dz$$

$$+ p \int_{B_{3r}} b_2 \tilde{v}^{p-1} \eta^{p-1} |\nabla_\lambda \eta| [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz$$

$$+ \int_{B_{3r}} b_1 \eta^p |\nabla_\lambda u|^{p-1} [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz$$

$$+ \int_{B_{3r}} d_2 \tilde{v}^{p-1} \eta^p [\tilde{v}^\beta - (m+h)^\beta] e^{-|b_0|\tilde{v}} dz \,.$$

Then

$$\int_{B_{3r}} |\beta| |\nabla_\lambda u|^p \eta^p \tilde{v}^{\beta-1} w \, dx$$

$$\leq c|\beta| \int_{B_{3r}} d_3 \eta^p \tilde{v}^{p+\beta-1} dz + c \int_{B_{3r}} d_3 \eta^p \tilde{v}^p [\tilde{v}^\beta - (m+h)^\beta]$$

$$+ c \int_{B_{3r}} a\eta^{p-1} |\nabla_\lambda \eta| |\nabla_\lambda u|^{p-1} [\tilde{v}^\beta - (m+h)^\beta] w dz$$

$$+ c \int_{B_{3r}} b_2 \tilde{v}^{p-1} \eta^{p-1} |\nabla_\lambda \eta| [\tilde{v}^\beta - (m+h)^\beta] dz$$

$$+ c \int_{B_{3r}} b_1 \eta^p |\nabla_\lambda u|^{p-1} [\tilde{v}^\beta - (m+h)^\beta] dz$$

$$+ c \int_{B_{3r}} d_2 \tilde{v}^{p-1} \eta^p [\tilde{v}^\beta - (m+h)^\beta] dz \,.$$

Since $\tilde{v}^\beta - (m+h)^\beta \leq \tilde{v}^\beta$ the proof follows as in the proof of Theorem 4.2. $\qquad\square$

In a similar way, let B_r be a ball centered at $x_0 \in \partial\Omega$ and $u \in H_v^{1,p}(\Omega \cap B_{4r})$, we set

$$\overline{u}(x) = \begin{cases} \max\{u, M\} & \text{if } x \in \Omega \cap B_{4r} \\ M & \text{if } x \in \mathbb{R}^n \setminus (\Omega \cap B_{4r}) \end{cases}$$

where $M = \sup_{\partial\Omega \cap B_{4r}} u$.

Theorem 5.2. *Let $u \in H_v^{1,p}(\Omega \cap B_{4r})$ be a weak non negative subsolution of* (5.1) *in $\Omega \cap B_{4r}$. Assume* (5.2) *and* (5.3). *Let M be a constant such that $u \leq M$ on $\Omega \cap B_{4r}$. Then there exists c depending on n, M, a, b_0, p and the weight v such that*

$$\sup_{B_r} \overline{u} \leq c \Big\{ w^{-1}(B_{2r}) \int_{B_{2r}} \overline{u}w \, dz + \phi^{1/p}\Big[\Big(\frac{e}{w}\Big)^{\frac{p}{p-1}}; r\Big]$$

$$+ \phi^{\frac{1}{p-1}}\Big(\frac{f}{w}; r\Big) + \phi^{1/p}\Big(\frac{g}{w}; r\Big) \Big\}.$$

To obtain regularity up to the boundary of the domain we need some geometric assumptions.

Definition 5.3. *Let Ω be a domain in $\mathbb{R}^N$ and $z_0 \in \partial\Omega$. Let v be a strong A_∞ weight and $w = v^{1-\frac{p}{N}}$, $1 < p < N$. We say that w satisfies the condition A_v at z_0 if there exist positive constants R_0 and A such that*

$$\frac{w(B_r(z_0) \setminus \Omega)}{w(B_r(z_0))} \geq A, \quad 0 < r < R_0.$$

We say that Ω satisfies the condition A_v if it satisfies the condition at any point.

In the case $v = 1$ the A_v condition gives back the outer sphere condition. Using the geometric assumption A_v we give an estimate for the oscillation of solutions near the boundary.

Theorem 5.4. *Let Ω be a bounded open set satisfying the A_v condition at $z_0 \in \partial\Omega$. Let u be a locally bounded weak solution of equation* (5.1) *in Ω satisfying* (5.2) *and* (5.3). *Then there exists $R_0 > 0$ such that for any ball $B_r(z_0)$, with $0 < r < R_0$ and $\mu \in (0,1)$ we have*

$$\operatorname{osc}_{B_r \cap \Omega} u \leq c\Big[\Big(\frac{r}{R_0}\Big)^\alpha \operatorname{osc}_{B_{R_0} \cap \Omega} u + \operatorname{osc}_{B_{r^\mu R_0^{1-\mu}} \cap \partial\Omega} u + \overline{h}(r^\mu R_0^{1-\mu})\Big],$$

where c and α are positive constants and $\overline{h}$ is an infinitesimal function.

Proof. For $\rho > 0$ set $M(\rho) = \sup_{B_\rho \cap \Omega} u$ and $m(\rho) = \inf_{B_\rho \cap \Omega} u$, with $B_\rho = B_\rho(z_0)$. Let $0 < r \leq R_0/4$ the function $M(4r) - u$ is solution of

$$\operatorname{div} \tilde{A}(z, \overline{u}, \nabla_\lambda \overline{u}) = \tilde{B}(z, \overline{u}, \nabla_\lambda \overline{u}),$$

where

$$\tilde{A}(z, \overline{u}, \overline{\xi}) = A(z, M(4r) - \overline{u}, -\overline{\xi}),$$
$$\tilde{B}(z, \overline{u}, \overline{\xi}) = B(z, M(4r) - \overline{u}, -\overline{\xi}).$$

Moreover $\tilde{A}$ and $\tilde{B}$ satisfy

$$|\tilde{A}(z, \overline{u}, \overline{\xi})| \leq aw(z)[|\overline{\xi}_x|^2 + \lambda^2(x)|\overline{\xi}_y|^2]^{\frac{p-1}{2}} + \overline{b}|\overline{u}|^{p-1} + \overline{e},$$
$$|\tilde{B}(z, \overline{u}, \overline{\xi})| \leq b_0[|\overline{\xi}_x|^2 + \lambda^2(x)|\overline{\xi}_y|^2]^{\frac{p-1}{2}} + b_1[|\overline{\xi}_x|^2 + \lambda^2(x)|\overline{\xi}_y|^2]^{p/2}$$
$$+ \overline{d}|\overline{u}|^{p-1} + \overline{f},$$
$$\overline{\xi} \cdot \tilde{A}(z, \overline{u}, \overline{\xi}) \geq w(z)[|\overline{\xi}_x|^2 + \lambda^2(x)|\overline{\xi}_y|^2]^{p/2} - \overline{d}_1|\overline{u}|^p - \overline{g},$$

where

$$\overline{b}(z) = 2^p b(z), \quad \overline{d}(z) = 2^p d(z), \quad \overline{d}_1(z) = 2^p d_1(z),$$
$$\overline{e}(z) = 2^p b(z)M(4r)^{p-1} + e(z), \quad \overline{f}(z) = 2^p d(z)M(4r)^{p-1} + f(z),$$

$$\overline{g}(z) = 2^p d_1(z) M(4r)^{p-1} + g(z),$$

$$\left(\frac{\overline{b}}{w}\right)^{p/p-1}, \left(\frac{\overline{d}}{w}\right), \left(\frac{\overline{d_1}}{w}\right), \left(\frac{\overline{e}}{w}\right)^{p/p-1}, \frac{\overline{f}}{w}, \frac{\overline{g}}{w} \in S'_v(\Omega).$$

Then by (5.4) and condition A_v, we have

$$
\begin{aligned}
M(4r) - M &\leq \frac{w(B_{2r} \setminus \Omega)}{Aw(B_{2r})}[M(4r) - M] \\
&= \frac{1}{Aw(B_{2r})} \int_{B_{2r} \setminus \Omega} [M(4r) - M]w\, dx \\
&\leq \frac{1}{Aw(B_{2r})} \int_{B_{2r} \setminus \Omega} [\widetilde{M(4r) - u}]w\, dx \\
&\leq c[\inf_{B_r \cap \Omega}(M(4r) - u) + \overline{h}(r)] \\
&\leq c[M(4r) - M(r) + \overline{h}(r)].
\end{aligned}
\tag{5.6}
$$

where $M = \sup_{B_{4r} \cap \partial\Omega} u$, $m = \inf_{B_{4r} \cap \partial\Omega} u$ and

$$\overline{h}(r) = \phi^{1/p}\left[\left(\frac{\overline{e}}{w}\right)^{\frac{p}{p-1}}; r\right] + \phi^{\frac{1}{p-1}}\left(\frac{\overline{f}}{w}; r\right) + \phi^{1/p}\left(\frac{\overline{g}}{w}; r\right).$$

In a similar way, for $u - m(4r)$,

$$m - m(4r) \leq c[m(r) - m(4r) + \overline{h}(r)].\tag{5.7}$$

From (5.6) and (5.7) we obtain, for $\theta < 1$

$$M(r) - m(r) \leq \theta[M(4r) - m(4r)] + M - m + c\overline{h}(r),$$

from which applying [24, Lemma 8.23] (see also [16]) we obtain the result. □

As consequences of the previous Theorem we obtain the following corollary.

Corollary 5.5. *Let Ω be a bounded open set satisfying condition A_v in every $x_0 \in \partial\Omega$. Let u be a locally bounded weak solution of equation (5.1) in Ω satisfying (5.2) and (5.3). Let $u = \varphi$ on $\partial\Omega$. If φ is continuous in $\partial\Omega$ then u is continuous in $\overline{\Omega}$.*

Now we refine our assumptions on lower order terms. If we assume the coefficients in a suitable Morrey space we obtain Hölder continuity of the solution.

Corollary 5.6. *Let Ω be a bounded open set satisfying the A_v condition in every $z_0 \in \partial\Omega$. Let u be a locally bounded weak solution of equation (5.1) in Ω satisfying (5.2) and*

$$a, b_0 \in \mathbb{R}, \quad \left(\frac{b}{w}\right)^{\frac{p}{p-1}}, \left(\frac{b_1}{w}\right)^p, \frac{d}{w}, \frac{d_1}{w}, \left(\frac{e}{w}\right)^{\frac{p}{p-1}}, \frac{f}{w}, \frac{g}{w} \in L_v^{1,p-\epsilon}(\Omega),\tag{5.8}$$

with $0 < \epsilon < p$. Let $u = \varphi$ on $\partial\Omega$. If φ is Hölder continuous in $\partial\Omega$, then u is Hölder continuous in $\overline{\Omega}$.

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