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*Research article*

# Existence of solutions to some quasilinear degenerate elliptic systems with right hand side in a Marcinkiewicz space[†]

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**Abstract:** We prove the existence of a solution to a quasilinear system of degenerate equations, when the datum is in a Marcinkiewicz space. The main assumption asks the off-diagonal coefficients to have support in the union of a geometric progression of squares.

**Keywords:** degenerate elliptic system; Marcinkiewicz space; existence of solution

## 1. Introduction

In this article we consider the following quasilinear boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,u(x))Du(x)) = f(x), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, $f, u : \Omega \to \mathbb{R}^N$, with $N \geq 2$, and $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{N^2 n^2}$ is a matrix valued function whose entries are $a^{\alpha,\beta}_{i,j}(x,u)$ with $i, j \in \{1,\ldots,n\}$ and $\alpha, \beta \in \{1,\ldots,N\}$. Therefore the first line in (1.1) is a system of $N$ equations of the form

$$-\sum_{i=1}^{n} D_i \left( \sum_{j=1}^{n} \sum_{\beta=1}^{N} a_{i,j}^{\alpha,\beta}(x,u) D_j u^\beta \right) = f^\alpha \qquad \alpha = 1, \dots, N. \tag{1.2}$$

For the treated problem there is an extensive literature in the scalar case $N = 1$.

In particular, for the existence of a suitably defined solution, the Reader can refer to the papers [9, 10, 14, 58, 69] while, relatively to uniqueness and *a priori* estimates, we can quote respectively the papers [67] and [3]. For what concerns the regularity of a solution we cite the works [35, 38]. Moreover, similar conclusions for the nonlinear case can be found in [2, 8, 12] and for the anisotropic case in [4, 36]. Subsequently the aforementioned results have been extended to the operator with lower order terms too (see also [15–17, 20, 51]). In this context one can also see [7, 11, 12, 24, 25, 30]. Furthermore, in [37] the right hand side appears in divergence form, that is $f = -\text{div}F$ and in [1] the biharmonic operator is studied.

For further regularity results concerning elliptic operators the Reader is invited to refer to the foundamental works [5, 6, 26–28, 41–48, 64] and the survey [65].

As it is shown by the De Giorgi's counterexample [29], see also [39, 40, 49, 60, 61], the good regularity properties obtained in the scalar case can not be in general extended to the vectorial one, unless new structural assumptions are introduced.

An existence result of bounded weak solution for nonlinear degenerate elliptic systems is obtained in [55], using a componentwise coercivity condition. In several other papers, conditions on the support of the off-diagonal coefficients $a_{i,j}^{\alpha,\beta}(x)$ have been used to address different problems. Let us mention that a maximum principle result is obtained in [66] where the assumption is $a_{i,j}^{\alpha,\beta}(x, y) = 0$ for $\alpha \neq \beta$ when $y^\alpha$ is large and in [52] where different shapes of support are considered. Hölder continuity of the solutions is proved in [70] for a tridiagonal system, $a_{i,j}^{\alpha,\beta} = 0$ for $\beta > \alpha$. $L^\infty$ regularity results are obtained in [53] for an oblique type of support for the coefficients and in [54] for a butterfly support. Measure data problems are faced in [56] and [57] where the support of $a_{i,j}^{\alpha,\beta}(x, y)$ is contained in squares along the $y^\alpha = \pm y^\beta$ diagonals.

These kind of assumptions on the coefficients have been recently employed also to deal with degenerate elliptic systems. In this context there are results on problem (1.1) when the datum $f \in L^m$, which extend the ones contained in [14] for the scalar case. Namely, in [31] the existence of a bounded solution is proved when $m > \frac{n}{2}$, assuming a butterfly support for the off-diagonal coefficients; moreover in [32] the case of a datum $f$ with an intermediate grade of integrability ($m < \frac{n}{2}$) is treated, thanks to an appropriate choice of the support for the off-diagonal coefficient.

In this paper we extend to the degenerate vectorial problem (1.1) an existence result concerning degenerate scalar operators, with the datum $f$ in a suitable Marcinkiewicz space, contained in [14, 62, 63] (see also [21, 22]). Since we are dealing with the vectorial case the support of the coefficients is required to have a particular structure. In Section 2 we give the precise notions of degenerate ellipticity and Marcinkiewicz spaces, see respectively $(\mathcal{A}_2)$ and definition 2.1, while the assumption on the shape of the support of the coefficients is stated in $(\mathcal{A}_3)$.

Also in this context the extension to the vectorial case of the known result in the scalar one is not obvious. Indeed, starting from De Giorgi's counterexample, it is possible to construct an example of an elliptic system with datum $f \in L^p$ for every $p < n$, whose unique solution is unbounded and has low integrability, see [31] for details on the counterexample.

When dealing with systems of $N$ equations, like (1.2), whose coefficients are only measurable with respect to $x$, little is known. Most articles are devoted to study existence or regularity of solutions of systems with right hand side $f^\alpha \in L^m$, either when $m$ is large, or when $m$ is small. When $m$ is large, namely $m > \frac{n}{2}$, existence of bounded solutions is obtained in [31].

When $m$ is small, namely $m = 1$, or even when $f^\alpha$ is a measure, existence of solutions have been studied for general systems

$$-\sum_{i=1}^{n} D_i \left(A_i^\alpha(x, u, Du)\right) = f^\alpha, \qquad \alpha = 1, \dots, N, \tag{1.3}$$

under structure conditions on $A_i^\alpha$. Namely, in [33] and [34], authors assume that

$$0 \leq \sum_{i=1}^{n} A_i^\alpha(x, y, \xi) \left((Id - b \times b)\,\xi\right)_i^\alpha \tag{1.4}$$

for every $b \in \mathbb{R}^N$ with $|b| \leq 1$. On the other hand, in [71], the author assumes the componentwise sign condition

$$0 \leq \sum_{i=1}^{n} A_i^\alpha(x, y, \xi) \xi_i^\alpha \tag{1.5}$$

for every $\alpha = 1, \dots, N$. When $N = 2$, (1.4) implies (1.5): it is enough to take first $b = (1, 0)$, then $b = (0, 1)$. Note that, in the present paper, we address the quasilinear case

$$A_i^\alpha(x, y, \xi) = \sum_{\beta=1}^{N} \sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y) \xi_j^\beta; \tag{1.6}$$

in this case, as far as one off-diagonal coefficient $a_{\tilde{i},\tilde{j}}^{\tilde{\alpha},\tilde{\beta}}(x, y)$ is non zero, then (1.5) is no longer true: it is enough to take $\alpha = \tilde{\alpha}$, $\xi_j^\beta = 0$ if $\beta \notin \{\tilde{\alpha}, \tilde{\beta}\}$, $\xi_i^{\tilde{\alpha}} = 0$ if $i \neq \tilde{i}$, $\xi_{\tilde{i}}^{\tilde{\alpha}} = 1$, $\xi_j^{\tilde{\beta}} = 0$ if $j \neq \tilde{j}$, $\xi_{\tilde{j}}^{\tilde{\beta}} = t \frac{a_{\tilde{i},\tilde{j}}^{\tilde{\alpha},\tilde{\beta}}(x,y)}{\left|a_{\tilde{i},\tilde{j}}^{\tilde{\alpha},\tilde{\beta}}(x,y)\right|^2}$ with $t \to -\infty$. When $N = 2$, failure of (1.5) implies failure of (1.4). We recall that the study of quasilinear systems (1.2) with $f^\alpha \in L^1$ is contained in [57] under the assumption that the support of off-diagonal coefficients is contained in a sequence of squares with side lenght $r$ along the diagonals of the $y^\alpha - y^\beta$ plane.

Concerning existence and regularity of suitable defined solutions of linear ellitptic systems

$$-\sum_{i=1}^{n} D_i \left(A_i^\alpha(x) Du\right) = f^\alpha, \qquad \alpha = 1, \dots, N,$$

with VMO coefficients and datum $f = (f^\alpha)$ in a Lebesgue space $L^\gamma$ with $\gamma \in (1, \frac{2n}{n+2}]$ (i.e., below the duality exponent) or in a suitable Lorentz-Morrey space one can refer also to [50]. While if $f$ belongs to the natural dual Lebesgue space but the linear operator in not coercive due to the presence of a lower order term, called "drift term",

$$-\sum_{i=1}^{n} D_i \left[A_i^\alpha(x) Du - E_i^\alpha(x) u\right] = f^\alpha(x)$$

then existence and regularity results can be found in [19]. The above result has been extended to non linear operator under the so-called Landes condition (similar to (1.4)) with datum being in $L^1$ or in $L^{\frac{2n}{n+2}}$ respectively in [18, 23].

In the present work we address the existence of a regular solution to (1.2) when $f^\alpha$ has an intermediate degree of integrability, namely, $f^\alpha \in M^m$ with $1 < \frac{2n}{n+2-\theta(n-2)} < m < \frac{n}{2}$ and $M^m$ is the Marcinkiewicz space. In this case, the higher degree of integrability of the right hand side $f^\alpha$ allows us to weaken the condition on the support of off-diagonal coefficients.

In the next section we present assumptions on the coefficients and on the datum $f$ and our result. In Section 3 we consider a sequence of approximating non degenerate problems and we prove estimates on their weak solutions; then, with a limit procedure, we get the result for our problem.

## 2. Assumptions and result

For all $i, j \in \{1, \dots, n\}$ and all $\alpha, \beta \in \{1, \dots, N\}$ we assume that $a_{i,j}^{\alpha,\beta} : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions:

($\mathcal{A}_0$) $x \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x, y)$ is continuous;

($\mathcal{A}_1$) (boundedness of all the coefficients) there exists $c > 0$ such that

$$|a_{i,j}^{\alpha,\beta}(x, y)| \leq c$$

for almost every $x \in \Omega$ and for all $y \in \mathbb{R}^N$;

($\mathcal{A}_2$) (degenerate ellipticity of all the coefficients) there exist constants $\nu > 0$ and $\theta \in (0, 1)$ such that

$$\sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta}(x, y)\xi_i^\alpha \xi_j^\beta \geq \nu \sum_{\alpha=1}^{N} \frac{|\xi^\alpha|^2}{(1 + |y^\alpha|)^\theta},$$

for almost every $x \in \Omega$, for all $y \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{N\times n}$;

($\mathcal{A}_3$) (support of off-diagonal coefficients) there exists $L_0 \geq 1$ such that ($\mathcal{A}'_3$) and ($\mathcal{A}''_3$) hold, where

($\mathcal{A}'_3$) (support of off-diagonal coefficients contained in a central square) if $a_{i,j}^{\alpha,\beta}(x, y) \neq 0$ and $0 \leq |y^\alpha| < L_0$, then it holds also $0 \leq |y^\beta| < L_0$;

($\mathcal{A}''_3$) (support of off-diagonal coefficients contained in the union of a geometric progression of squares) if $a_{i,j}^{\alpha,\beta}(x, y) \neq 0$ and there exists $t \in \mathbb{N} \cup \{0\}$ such that $2^t L_0 \leq |y^\alpha| < 2^{t+1} L_0$, then it holds also $2^t L_0 \leq |y^\beta| < 2^{t+1} L_0$.

Let us remark that from assumption ($\mathcal{A}_2$) it follows that we have degeneracy in the $\alpha$ equation when $u^\alpha$ is large. In [13] is treated for $N = 2$ the case in which degeneracy in the $\alpha$ equation arises when $u^\beta$ is large, with $\beta \neq \alpha$.

Note that ($\mathcal{A}'_3$) and ($\mathcal{A}''_3$) are always fulfilled when $\alpha = \beta$. On the contrary, when $\alpha \neq \beta$, ($\mathcal{A}_3$) forces the support of $a_{i,j}^{\alpha,\beta}(x, y)$ to be contained in the union of infinite squares along the diagonals, see grey region in Figure 1.

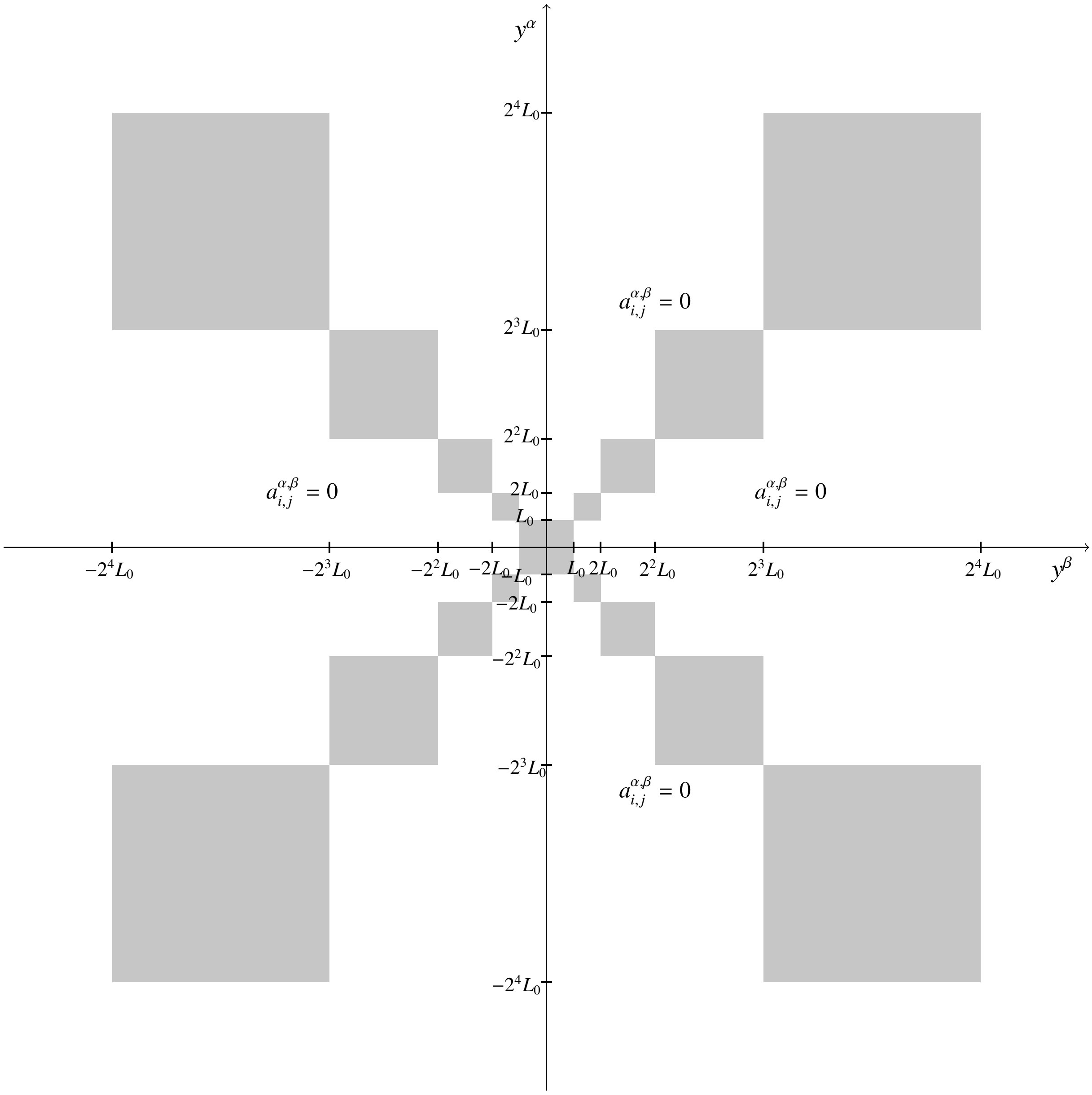


**Figure 1.** Assumption $(\mathcal{A}_3)$.

On $f$ we assume that it belongs to the Marcinkiewicz space $M^m(\Omega, \mathbb{R}^N)$, with

$$\frac{2n}{n+2-\theta(n-2)} < m < \frac{n}{2}.$$

For the convenience of the Reader, we recall the definition of Marcinkiewicz spaces, also known as weak Lebesgue spaces.

**Definition 2.1.** *Let $m$ be a positive number. We say that a measurable function $f : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^m(\Omega, \mathbb{R})$ if there exists a positive constant $c$ sucht that*

$$|\{x \in \Omega : |f(x)| > t\}| < \frac{c}{t^m}, \quad \forall t > 0; \tag{2.1}$$

*in such a case we set*

$$M_m(f,\Omega) = (\inf\{c > 0 \text{ such that } (2.1) \text{ holds }\})^{\frac{1}{m}}.$$

$M^m(\Omega,\mathbb{R}^N)$ *is the space of functions* $f = (f^1,\dots,f^N)$ *such that* $f^i \in M^m(\Omega,\mathbb{R})$ *for each i. Moreover* $M_m(f) = \sum_{\alpha=1}^N M_m(f^\alpha)$.

We recall some properties on Marcinkiewicz spaces:

$$L^m(\Omega) \subset M^m(\Omega) \subset L^{m-\varepsilon}(\Omega),\ \ \forall\, m > 1,\ \forall\, 0 < \varepsilon \le m-1 \tag{2.2}$$

and

$$\int_E |f|dx \le M_m(f,\Omega)|E|^{1-\frac{1}{m}}, \quad \forall f \in M^m(\Omega), \quad \forall E \subset \Omega. \tag{2.3}$$

For more details on Marcinkiewicz space see [10, 68].

Let us explicitly remark that, being $0 < \theta < 1$, from (2.2) it follows that

$$f \in L^{\frac{2n}{n+2}}(\Omega), \quad \forall f \in M^m(\Omega) \text{ with } m > \frac{2n}{n+2-\theta(n-2)}. \tag{2.4}$$

Under our set of assumptions we prove the following theorem:

**Theorem 2.1.** *Assume* $(\mathcal{A}_0)$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$, *with* $n \ge 3$. *If* $f \in M^m(\Omega,\mathbb{R}^N)$, *with* $\frac{2n}{n+2-\theta(n-2)} < m < \frac{n}{2}$, *then there exists* $u \in W_0^{1,2}(\Omega,\mathbb{R}^N) \cap M^r(\Omega,\mathbb{R}^N)$, *with*

$$r = \frac{nm(1-\theta)}{n-2m}, \tag{2.5}$$

*weak solution of the problem* (1.1)*, that is such that*

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x,u(x)) D_j u^\beta(x) D_i\varphi^\alpha(x)dx = \int_\Omega \sum_{\alpha=1}^N f^\alpha(x)\varphi^\alpha(x)dx \tag{2.6}$$

*for all* $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$.

## 3. Approximation and estimates

We set for all $k \in \mathbb{N}$

$$\tilde{a}_{i,j,k}^{\alpha,\beta}(x,y) = a_{i,j}^{\alpha,\beta}(x,y) + \frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j}$$

with

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j. \end{cases}$$

We consider the following family of approximating problems

$$\begin{cases} -\sum_{i=1}^n D_i\left(\sum_{j=1}^n \sum_{\beta=1}^N \tilde{a}_{i,j,k}^{\alpha,\beta}(x,u_k) D_j u_k^\beta\right) = f^\alpha, & x \in \Omega \\ u_k = 0, & x \in \partial\Omega. \end{cases} \tag{$\tilde{\mathcal{P}}_k$}$$

We want to show the existence of a weak solution for each problem $(\tilde{\mathcal{P}}_k)$, that is a function $u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k}^{\alpha,\beta}(x, u_k(x)) D_j u_k^\beta(x) D_i \varphi^\alpha(x) dx = \int_\Omega \sum_{\alpha=1}^N f^\alpha(x) \varphi^\alpha(x) dx \tag{3.1}$$

for all $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

Let us first show some properties of the coefficients $\tilde{a}_{i,j,k}^{\alpha,\beta}$. From assumption $(\mathcal{A}_1)$ it follows that

$$|\tilde{a}_{i,j,k}^{\alpha,\beta}(x, y)| \leq c + 1. \tag{$\tilde{\mathcal{A}}_1$}$$

Using assumption $(\mathcal{A}_2)$ we have the following non degenerate ellipticity condition

$$\begin{aligned} &\sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \tilde{a}_{i,j,k}^{\alpha,\beta}(x, y) \xi_i^\alpha \xi_j^\beta = \\ &= \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, y) \xi_i^\alpha \xi_j^\beta + \frac{1}{k} \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \delta_{\alpha\beta} \delta_{i,j} \xi_i^\alpha \xi_j^\beta \geq \\ &\geq \nu \sum_{\alpha=1}^N \frac{|\xi^\alpha|^2}{(1 + |y^\alpha|)^\theta} + \frac{1}{k}|\xi|^2. \end{aligned} \tag{$\tilde{\mathcal{A}}_2$}$$

Now let us show that for all $f \in M^m(\Omega, \mathbb{R}^N)$, with $m > \frac{2n}{n+2-\theta(n-2)}$, the linear operator

$$\begin{aligned} F : W_0^{1,2}(\Omega, \mathbb{R}^N) &\to \mathbb{R} \\ v \quad &\mapsto \int_\Omega \sum_{\alpha=1}^N f^\alpha(x) v^\alpha(x) dx \end{aligned}$$

is continuous. Indeed, using Hölder inequality, (2.4) and Sobolev embedding, we have for a suitable constant $C > 0$

$$\begin{aligned} |F(v)| = \left| \int_\Omega \sum_{\alpha=1}^N f^\alpha(x) v^\alpha(x) dx \right| &\leq \sum_{\alpha=1}^N \int_\Omega |f^\alpha(x) v^\alpha(x)| dx \leq \\ &\leq \sum_{\alpha=1}^N \|f^\alpha\|_{L^{\frac{2n}{n+2}}} \|v^\alpha\|_{L^{\frac{2n}{n-2}}} \leq \\ &\leq C \sum_{\alpha=1}^N \|f^\alpha\|_{L^{\frac{2n}{n+2}}} \|v\|_{W_0^{1,2}(\Omega, \mathbb{R}^N)} \end{aligned}$$

and the continuity of $F$ is proved. Therefore we can apply the surjectivity result of Leray-Lions, see [59], and we have the existence of a weak solution $u_k$ for the problem $(\tilde{\mathcal{P}}_k)$, that is, there exists $u_k \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ such that (3.1) holds true for every $\varphi \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.

In the next Lemma 3.1, arguing as in [14], we prove that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega, \mathbb{R}^N) \cap M^r(\Omega, \mathbb{R}^N)$. We first recall the following elementary inequalities that will be used in the proof of Lemma 3.1. We have

$$\sum_{\alpha=1}^{M} a_\alpha^p \le M\left(\sum_{\alpha=1}^{M} a_\alpha\right)^p, \tag{3.2}$$

$$\left(\sum_{\alpha=1}^{M} a_\alpha\right)^p \le M^p \sum_{\alpha=1}^{M}(a_\alpha)^p, \tag{3.3}$$

$$\sum_{\alpha=1}^{M}(a_\alpha\, b_\alpha) \le \left(\sum_{\alpha=1}^{M} a_\alpha\right)\left(\sum_{\alpha=1}^{M} b_\alpha\right), \tag{3.4}$$

provided $a_\alpha, b_\alpha \ge 0$ for all $\alpha \in \{1, \dots, M\}$ and $p > 0$.

**Lemma 3.1.** *Assume that $f \in M^m(\Omega, \mathbb{R}^N)$ with $\frac{2n}{n+2-\theta(n-2)} < m < \frac{n}{2}$ and let $u_k$ be a weak solution of $(\tilde{\mathcal{P}}_k)$. Then the sequences $\|u_k\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)}$ and $M_r(u_k, \Omega)$, with $r$ given in (2.5), are bounded by a positive constant which depends only on $L_0$, $\theta$, $m$, $n$, $N$, $\nu$, $|\Omega|$ and $M_m(f, \Omega)$.*

*Proof.* For any $t \in \mathbb{N} \cup \{0\}$ and for $L_0 \ge 1$ given by assumption $(\mathcal{A}_3)$, we define the following functions

$$G_{2^tL_0}(s) = \begin{cases} 0 & \text{if } |s| \le 2^t L_0 \\ s - 2^t L_0 \frac{s}{|s|} & \text{if } |s| > 2^t L_0 \end{cases}$$

and

$$T_{2^tL_0}(s) = \begin{cases} s & \text{if } -2^t L_0 \le s \le 2^t L_0 \\ 2^t L_0 & \text{if } s > 2^t L_0 \\ -2^t L_0 & \text{if } s < -2^t L_0. \end{cases}$$

We consider as test function in (3.1) the function $\varphi_t \in W_0^{1,2}(\Omega, \mathbb{R}^N)$ defined as

$$\varphi_t = (\varphi_t^1, \dots, \varphi_t^N) = (T_{2^tL_0}(G_{2^tL_0}(u_k^1)), \dots, T_{2^tL_0}(G_{2^tL_0}(u_k^N))). \tag{3.5}$$

We introduce the sets

$$A^\alpha_{k,2^tL_0} = \{x \in \Omega : |u_k^\alpha| \ge 2^t L_0\} \ \text{ and } \ B^\alpha_{k,2^tL_0} = \{x \in \Omega : 2^t L_0 \le |u_k^\alpha| < 2^{t+1} L_0\}.$$

For all $\alpha \in \{1, \dots, N\}$ we have

$$\operatorname{supp} \varphi_t^\alpha \subset A^\alpha_{k,2^tL_0}, \quad |\varphi_t^\alpha| \le 2^t L_0 \quad \text{ and } \quad D_i \varphi_t^\alpha = D_i u_k^\alpha \, \mathbb{I}_{B^\alpha_{k,2^tL_0}}, \tag{3.6}$$

where $\mathbb{I}_B(x) = 1$ if $x \in B$ and $\mathbb{I}_B(x) = 0$ otherwise. Moreover, using $(\mathcal{A}''_3)$, we have

$$a^{\alpha,\beta}_{i,j}(x, u_k(x))\mathbb{I}_{B^\alpha_{k,2^tL_0}}(x) = a^{\alpha,\beta}_{i,j}(x, u_k(x))\mathbb{I}_{B^\alpha_{k,2^tL_0}}(x)\mathbb{I}_{B^\beta_{k,2^tL_0}}(x). \tag{3.7}$$

Indeed, if $a^{\alpha,\beta}_{i,j}(x, u_k(x)) = 0$ or $x \notin B^\alpha_{k,2^tL_0}$, then the (3.7) is obvious. If $a^{\alpha,\beta}_{i,j}(x, u_k(x)) \ne 0$ and $x \in B^\alpha_{k,2^tL_0}$, that is $2^t L_0 \le |u_k^\alpha| < 2^{t+1} L_0$, then for $(\mathcal{A}''_3)$ we have $2^t L_0 \le |u_k^\beta| < 2^{t+1} L_0$ so that $x \in B^\beta_{k,2^tL_0}$.

From (3.6), (3.7) and $(\mathcal{A}_2)$ we have

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}_{i,j,k}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_i\varphi_t^{\alpha}(x)=\\
&=\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}(a_{i,j}^{\alpha,\beta}(x,u_k(x))+\frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j})D_ju_k^{\beta}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B^{\alpha}_{k,2^tL_0}}(x)=\\
&=\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)\mathbb{I}_{B^{\beta}_{k,2^tL_0}}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B^{\alpha}_{k,2^tL_0}}(x)+\\
&\qquad+\sum_{\alpha=1}^{N}\sum_{i=1}^{n}\frac{1}{k}|D_iu_k^{\alpha}(x)|^2\mathbb{I}_{B^{\alpha}_{k,2^tL_0}}\geq\\
&\geq\nu\sum_{\alpha=1}^{N}\frac{|Du_k^{\alpha}(x)\mathbb{I}_{B^{\alpha}_{k,2^tL_0}}(x)|^2}{(1+|u_k^{\alpha}(x)|)^{\theta}}.
\end{aligned}
\tag{3.8}
$$

Then, replacing in the left side of (3.1) the test function (3.5) and using (3.8), we get

$$
\begin{aligned}
&\int_{\Omega}\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}_{i,j,k}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_i\varphi_t^{\alpha}(x)dx\geq\\
&\geq\nu\sum_{\alpha=1}^{N}\int_{B^{\alpha}_{k,2^tL_0}}\frac{|Du_k^{\alpha}(x)|^2}{(1+|u_k^{\alpha}(x)|)^{\theta}}dx\geq\nu\sum_{\alpha=1}^{N}\int_{B^{\alpha}_{k,2^tL_0}}\frac{|Du_k^{\alpha}(x)|^2}{(1+2^{t+1}L_0)^{\theta}}dx\\
&=\frac{\nu}{(1+2^{t+1}L_0)^{\theta}}\sum_{\alpha=1}^{N}\int_{B^{\alpha}_{k,2^tL_0}}|Du_k^{\alpha}(x)|^2dx.
\end{aligned}
\tag{3.9}
$$

Combining (3.9) with (3.1), we get

$$
\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{B^{\alpha}_{k,2^tL_0}}|Du_k^{\alpha}(x)|^2dx\leq\frac{(1+2^{t+1}L_0)^{\theta}}{\nu}\int_{\Omega}\sum_{\alpha=1}^{N}f^{\alpha}(x)\varphi_t^{\alpha}(x)dx=\\
&=\frac{(1+2^{t+1}L_0)^{\theta}}{\nu}\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}}f^{\alpha}(x)\varphi_t^{\alpha}(x)dx.
\end{aligned}
\tag{3.10}
$$

Using Sobolev's embedding and (3.10) we have

$$
\begin{aligned}
&\left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{2}{2^*}} = \left(\int_\Omega |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \leq \\
&\leq C_S \int_\Omega |D\varphi^\alpha_t(x)|^2 dx = C_S \int_\Omega \sum_{i=1}^n |D_i\varphi^\alpha_t(x)|^2 dx = \\
&= C_S \int_\Omega \sum_{i=1}^n |D_i u^\alpha_k(x)\mathbb{I}_{B^\alpha_{k,2^tL_0}}|^2 dx = C_S \int_{B^\alpha_{k,2^tL_0}} \sum_{i=1}^n |D_i u^\alpha_k(x)|^2 dx = \\
&= C_S \int_{B^\alpha_{k,2^tL_0}} |Du^\alpha_k(x)|^2 dx,
\end{aligned}
\tag{3.11}
$$

where $C_S$ is the Sobolev embedding constant. Summing on $\alpha$ in (3.11) and using (3.10), we have

$$
\begin{aligned}
&\sum_{\alpha=1}^N \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \leq C_S \sum_{\alpha=1}^N \int_{B^\alpha_{k,2^tL_0}} |Du^\alpha_k(x)|^2 dx \leq \\
&\leq C_S \frac{(1+2^{t+1}L_0)^\theta}{\nu} \sum_{\alpha=1}^N \int_{A^\alpha_{k,2^tL_0}} f^\alpha(x)\varphi^\alpha_t(x) dx.
\end{aligned}
\tag{3.12}
$$

From (2.4) we have $f \in L^{\frac{2n}{n+2}}(\Omega)$ and, by Sobolev immersion, we have also $\varphi_t \in L^{2^*} = L^{\frac{2n}{n-2}}$. Then, using the Hölder inequality with exponents $\frac{2n}{n+2}$ and $\frac{2n}{n-2} = 2^*$ and applying (2.3) to the function $|f^\alpha|^{\frac{2n}{n+2}} \in M^{\frac{(n+2)m}{2n}}(\Omega)$, we deduce for all $\alpha = 1, \ldots, n$

$$
\begin{aligned}
&\int_{A^\alpha_{k,2^tL_0}} f^\alpha(x)\varphi^\alpha_t(x) dx \leq \\
&\leq \left(\int_{A^\alpha_{k,2^tL_0}} |f^\alpha(x)|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{1}{2^*}} \leq \\
&\leq M_m(f^\alpha,\Omega)|A^\alpha_{k,2^tL_0}|^{\frac{n+2}{2n}(1-\frac{2n}{(n+2)m})} \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{1}{2^*}}.
\end{aligned}
\tag{3.13}
$$

From (3.12) and (3.13) it follows that

$$
\begin{aligned}
&\sum_{\alpha=1}^N \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \leq \\
&\leq C_S \frac{(1+2^{t+1}L_0)^\theta}{\nu} \sum_{\alpha=1}^N M_m(f^\alpha,\Omega)|A^\alpha_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}} \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{1}{2^*}} \leq \\
&\leq C_1(2^tL_0)^\theta \sum_{\alpha=1}^N |A^\alpha_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}} \left(\int_{A^\alpha_{k,2^tL_0}} |\varphi^\alpha_t(x)|^{2^*} dx\right)^{\frac{1}{2^*}},
\end{aligned}
$$

where $C_1$ is a constant depending only on $C_s$, $\nu$, $M_m(f,\Omega)$.

Now, using last inequality and (3.3), (3.4), (3.2), we have

$$\begin{aligned}
&\left(\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{2}{2^*}} \le N^{\frac{2}{2^*}}\sum_{\alpha=1}^{N}\left(\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{2}{2^*}} \le\\
&\le N^{\frac{2}{2^*}}C_1(2^tL_0)^{\theta}\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}\left(\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} \le\\
&\le N^{\frac{2}{2^*}}C_1(2^tL_0)^{\theta}\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}\right)\left[\sum_{\alpha=1}^{N}\left(\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}}\right] \le\\
&\le N^{1+\frac{2}{2^*}}C_1(2^tL_0)^{\theta}\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}\right)\left(\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}};
\end{aligned}$$

Therefore

$$\left(\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} \le N^{1+\frac{2}{2^*}}C_1(2^tL_0)^{\theta}\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}. \tag{3.14}$$

Since $\varphi^{\alpha}_t = T_{2^tL_0}(G_{2^tL_0}(u^{\alpha}_k))$, for all $t\in\mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}
&\left(\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} \ge \left(\int_{A^{\alpha}_{k,2^{t+1}L_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} =\\
&=\left(\int_{A^{\alpha}_{k,2^{t+1}L_0}}(2^tL_0)^{2^*}dx\right)^{\frac{1}{2^*}} = 2^tL_0|A^{\alpha}_{k,2^{t+1}L_0}|^{\frac{1}{2^*}};
\end{aligned}$$

Then, summing on $\alpha$ and using (3.2) we have

$$\begin{aligned}
&\left(\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} \ge \frac{1}{N}\sum_{\alpha=1}^{N}\left(\int_{A^{\alpha}_{k,2^tL_0}}|\varphi^{\alpha}_t(x)|^{2^*}dx\right)^{\frac{1}{2^*}} \ge\\
&\ge \frac{1}{N}\sum_{\alpha=1}^{N}2^tL_0|A^{\alpha}_{k,2^{t+1}L_0}|^{\frac{1}{2^*}}.
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) it follows that

$$\frac{1}{N}\sum_{\alpha=1}^{N}2^tL_0|A^{\alpha}_{k,2^{t+1}L_0}|^{\frac{1}{2^*}} \le N^{1+\frac{2}{2^*}}C_1(2^tL_0)^{\theta}\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}$$

and then

$$\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^{t+1}L_0}|^{\frac{1}{2^*}} \le \frac{1}{(2^tL_0)^{1-\theta}}N^{2+\frac{2}{2^*}}C_1\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}}.$$

From the last inequality and using (3.3) and (3.2) we have

$$\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^{t+1}L_0}|\right)^{\frac{1}{2^*}} \le N^{\frac{1}{2^*}}\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^{t+1}L_0}|^{\frac{1}{2^*}} \le$$
$$\le \frac{1}{(2^tL_0)^{1-\theta}}N^{2+\frac{3}{2^*}}C_1\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|^{\frac{mn+2m-2n}{2nm}} \le$$
$$\le \frac{1}{(2^tL_0)^{1-\theta}}N^{3+\frac{3}{2^*}}C_1\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|\right)^{\frac{mn+2m-2n}{2nm}};$$

Therefore

$$\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^{t+1}L_0}| \le \frac{C_2}{(2^tL_0)^{(1-\theta)2^*}}\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|\right)^{\frac{mn+2m-2n}{m(n-2)}}, \tag{3.16}$$

where $C_2$ is a positive constant depending only on $N$, $n$, $C_S$, $\nu$ and $M_m(f,\Omega)$.

Let us set

$$\gamma = \frac{mn+2m-2n}{m(n-2)} \in (0,1)$$

and let us remark that for $r$ given in (2.5) the following equality holds

$$r-(1-\theta)2^* = r\gamma. \tag{3.17}$$

Now, for all $h>0$ and for $r$ given in (2.5), let us define

$$\rho(h) = h^r\sum_{\alpha=1}^{N}|A^{\alpha}_{k,h}|. \tag{3.18}$$

For all $t\in\mathbb{N}\cup\{0\}$, it follows from (3.16) that

$$\rho(2^{t+1}L_0) = (2^{t+1}L_0)^r\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^{t+1}L_0}| \le$$
$$\le 2^r(2^tL_0)^r\frac{C_2}{(2^tL_0)^{(1-\theta)2^*}}\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|\right)^{\frac{mn+2m-2n}{m(n-2)}} =$$
$$= 2^rC_2(2^tL_0)^{r\gamma}\left(\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|\right)^{\gamma} =$$
$$= 2^rC_2\left((2^tL_0)^r\sum_{\alpha=1}^{N}|A^{\alpha}_{k,2^tL_0}|\right)^{\gamma} = 2^rC_2[\rho(2^tL_0)]^{\gamma}.$$

Therefore we obtain that there exists a constant $C_3 = \max(1, 2^rC_2) \ge 1$, depending only on $N$, $n$, $C_S$, $\nu$, $M_m(f,\Omega)$, $\theta$ and $m$ such that, for all $t\in\mathbb{N}\cup\{0\}$, we have

$$\rho(2^{t+1}L_0) \le C_3[\rho(2^tL_0)]^{\gamma}$$

and, arguing by induction, it follows that

$$\rho(2^s L_0) \le C_3^{\sum_{h=0}^{s-1}\gamma^h}[\rho(L_0)]^{\gamma^s} \le C_3^{\sum_{h=0}^{+\infty}\gamma^h}[\rho(L_0)]^{\gamma^s}, \quad \forall s \in \mathbb{N}. \tag{3.19}$$

Being $\gamma < 1$ and $\rho(L_0) \ge 0$, the elementary inequality

$$[\rho(L_0)]^{\gamma^s} \le 1 + \rho(L_0), \quad \forall s \in \mathbb{N} \tag{3.20}$$

holds. Using the notation $C_4 = C_3^{\sum_{h=0}^{+\infty}\gamma^h} = C_3^{\frac{1}{1-\gamma}}$ and putting together (3.19) and (3.20), we have

$$\rho(2^s L_0) \le C_4(1 + \rho(L_0)), \quad \forall s \in \mathbb{N}, \tag{3.21}$$

where $C_4 \ge 1$ is a constant depending only on $N$, $n$, $C_S$, $\nu$, $M_m(f,\Omega)$, $\theta$ and $m$.

Using (3.21), we want to prove that there exists a constant $C_5$ depending only on $N$, $n$, $C_S$, $\nu$, $M_m(f,\Omega)$, $\theta$, $m$, $L_0$ and $|\Omega|$ such that

$$\rho(h) = h^r \sum_{\alpha=1}^{N} |A^\alpha_{k,h}| \le C_5, \quad \forall h \ge L_0. \tag{3.22}$$

Indeed, for $h \in [L_0, 2L_0]$, we have

$$\rho(h) = h^r \sum_{\alpha=1}^{N} |A^\alpha_{k,h}| \le (2L_0)^r \sum_{\alpha=1}^{N} |\Omega| = (2L_0)^r N|\Omega|. \tag{3.23}$$

For all $h \ge 2L_0$ there exists $s \in \mathbb{N}$ and $w \in [L_0, 2L_0)$ such that $h = 2^s w$. Then, using (3.21) and (3.23), we have for all $h \ge 2L_0$

$$\begin{aligned} \rho(h) &= \rho(2^s w) = (2^s w)^r \sum_{\alpha=1}^{N} |A^\alpha_{k,2^s w}| \le \\ &\le (2^{s+1}L_0)^r \sum_{\alpha=1}^{N} |A^\alpha_{k,2^s L_0}| = 2^r (2^s L_0)^r \sum_{\alpha=1}^{N} |A^\alpha_{k,2^s L_0}| = \\ &= 2^r \rho(2^s L_0) \le 2^r C_4(1 + \rho(L_0)) \le \\ &\le 2^r C_4(1 + (2L_0)^r N|\Omega|) := C_5. \end{aligned} \tag{3.24}$$

From (3.23) and (3.24) follows (3.22).

For all $h \ge L_0$, using (3.22), we have

$$\sum_{\alpha=1}^{N} |\{x \in \Omega : |u^\alpha_k| > h\}| \le \sum_{\alpha=1}^{N} |A^\alpha_{k,h}| = \frac{\rho(h)}{h^r} \le \frac{C_5}{h^r}; \tag{3.25}$$

for $h \in (0, L_0)$ we have

$$\sum_{\alpha=1}^{N} |\{x \in \Omega : |u^\alpha_k| > h\}| \le \sum_{\alpha=1}^{N} |\Omega| = N\Omega \le \frac{N|\Omega|L_0^r}{L_0^r} < \frac{N|\Omega|L_0^r}{h^r}. \tag{3.26}$$

Then, setting $C_6 = \max(C_5, N|\Omega|L_0^r)$, from (3.25) and (3.26), we get

$$\sum_{\alpha=1}^{N} |\{x \in \Omega : |u_k^\alpha| > h\}| \leq \frac{C_6}{h^r}, \quad \forall h > 0,$$

proving the boundness of the sequence $\{u_k\}_{k\in\mathbb{N}}$ in $M^r(\Omega)$.

It remains to prove that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega, \mathbb{R}^N)$.

From (3.10), for all $t \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \int_{\Omega} |Du_k^\alpha(x)|^2 dx = \sum_{\alpha=1}^{N} \int_{\{|u_k^\alpha|<L_0\}} |Du_k^\alpha(x)|^2 dx + \sum_{\alpha=1}^{N} \int_{\{|u_k^\alpha|\geq L_0\}} |Du_k^\alpha(x)|^2 dx = \\
&= \sum_{\alpha=1}^{N} \int_{\{|u_k^\alpha|<L_0\}} |Du_k^\alpha(x)|^2 dx + \sum_{t=0}^{+\infty} \sum_{\alpha=1}^{N} \int_{\{2^t L_0 \leq |u_k^\alpha| < 2^{t+1} L_0\}} |Du_k^\alpha(x)|^2 dx = \\
&= \sum_{\alpha=1}^{N} \int_{\{|u_k^\alpha|<L_0\}} |Du_k^\alpha(x)|^2 dx + \sum_{t=0}^{+\infty} \sum_{\alpha=1}^{N} \int_{B^\alpha_{k,2^t L_0}} |Du_k^\alpha(x)|^2 dx \leq \\
&\leq \sum_{\alpha=1}^{N} \int_{\{|u_k^\alpha|<L_0\}} |Du_k^\alpha(x)|^2 dx + \sum_{t=0}^{+\infty} \frac{(1 + 2^{t+1} L_0)^\theta}{\nu} \sum_{\alpha=1}^{N} \int_{A^\alpha_{k,2^t L_0}} f^\alpha(x) \varphi_t^\alpha(x) dx.
\end{aligned} \tag{3.27}$$

Now we estimate the right hand side of (3.27).

Observing that $|\varphi_t^\alpha(x)| \leq 2^t L_0$, for all $x$ and for all $t \in \mathbb{N} \cup \{0\}$, and using (2.3), we have

$$\begin{aligned}
&\int_{A^\alpha_{k,2^t L_0}} f^\alpha(x) \varphi_t^\alpha(x) dx \leq 2^t L_0 \int_{A^\alpha_{k,2^t L_0}} |f^\alpha(x)| dx \leq \\
&\leq 2^t L_0 M_m(f^\alpha, \Omega) |A^\alpha_{k,2^t L_0}|^{1-\frac{1}{m}} \leq 2^t L_0 M_m(f, \Omega) |A^\alpha_{k,2^t L_0}|^{1-\frac{1}{m}}.
\end{aligned}$$

Summing on $\alpha = 1, .., N$ the previous inequality, by (3.2), the definition of $\rho$ in (3.18) and (3.22), we get

$$\begin{aligned}
&\sum_{\alpha=1}^{N} \int_{A^\alpha_{k,2^t L_0}} f^\alpha(x) \varphi_t^\alpha(x) dx \leq 2^t L_0 M_m(f, \Omega) \sum_{\alpha=1}^{N} |A^\alpha_{k,2^t L_0}|^{1-\frac{1}{m}} \leq \\
&\leq 2^t L_0 M_m(f, \Omega) N \Big( \sum_{\alpha=1}^{N} |A^\alpha_{k,2^t L_0}| \Big)^{1-\frac{1}{m}} = 2^t L_0 M_m(f, \Omega) N \Big( \frac{\rho(2^t L_0)}{(2^t L_0)^r} \Big)^{1-\frac{1}{m}} \leq \\
&\leq C_5^{1-\frac{1}{m}} 2^t L_0 M_m(f, \Omega) N (2^{-tr})^{1-\frac{1}{m}} L_0^{-r(1-\frac{1}{m})} = \\
&= C_5^{1-\frac{1}{m}} M_m(f, \Omega) N (2^{1-r(1-\frac{1}{m})})^t L_0^{1-r(1-\frac{1}{m})}.
\end{aligned}$$

From this inequality it follows that

$$
\begin{aligned}
&\sum_{t=0}^{+\infty}\frac{(1+2^{t+1}L_0)^{\theta}}{\nu}\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}} f^{\alpha}(x)\varphi^{\alpha}_t(x)dx \leq\\
&\leq \sum_{t=0}^{+\infty}\frac{(1+2^{t+1}L_0)^{\theta}}{\nu}C_5^{1-\frac{1}{m}}M_m(f,\Omega)N(2^{1-r(1-\frac{1}{m})})^tL_0^{1-r(1-\frac{1}{m})}=\\
&=\frac{C_5^{1-\frac{1}{m}}M_m(f,\Omega)NL_0^{1-r(1-\frac{1}{m})}}{\nu}\sum_{t=0}^{+\infty}(1+2^{t+1}L_0)^{\theta}(2^{1-r(1-\frac{1}{m})})^t\leq\\
&\leq\frac{C_5^{1-\frac{1}{m}}M_m(f,\Omega)NL_0^{1-r(1-\frac{1}{m})}}{\nu}\sum_{t=0}^{+\infty}(2^{t+2}L_0)^{\theta}(2^{1-r(1-\frac{1}{m})})^t=\\
&=\frac{C_5^{1-\frac{1}{m}}M_m(f,\Omega)NL_0^{1-r(1-\frac{1}{m})}}{\nu}\sum_{t=0}^{+\infty}2^{2\theta}2^{t\theta}L_0^{\theta}(2^{1-r(1-\frac{1}{m})})^t=\\
&=\frac{C_5^{1-\frac{1}{m}}M_m(f,\Omega)N2^{2\theta}L_0^{1-r(1-\frac{1}{m})+\theta}}{\nu}\sum_{t=0}^{+\infty}(2^{\theta+1-r(1-\frac{1}{m})})^t.
\end{aligned}
\tag{3.28}
$$

Since $\frac{2n}{n+2-\theta(n-2)}<m<\frac{n}{2}$, it results that $\theta+1-r(1-\frac{1}{m})<0$ and the series in the right side of the last inequality converges; we have

$$
\sum_{t=0}^{+\infty}\frac{(1+2^{t+1}L_0)^{\theta}}{\nu}\sum_{\alpha=1}^{N}\int_{A^{\alpha}_{k,2^tL_0}} f^{\alpha}(x)\varphi^{\alpha}_t(x)dx \leq C_7, \tag{3.29}
$$

where $C_7$ is a positive constant depending only on $n, N, m, \theta, \nu, M_m(f,\Omega), |\Omega|, C_S, L_0$.

Now, let us prove that $\sum_{\alpha=1}^{N}\int_{\{|u^{\alpha}_k|<L_0\}}|Du^{\alpha}_k(x)|^2dx$ is bounded.

To this aim we use $\psi=(\psi^1,...,\psi^N)=(T_{L_0}(u^1_k),...,T_{L_0}(u^N_k))$ as a test function in the weakly formulation (3.1) of problem $(\tilde{P}_k)$. Observing that

$$
D_i\psi^{\alpha}=D_iu^{\alpha}_k\mathbb{I}_{B^{\alpha}_{k,0}}(x)
$$

where $B_{k,0}^{\alpha}=\{x\in\Omega:0\le|u_k^{\alpha}(x)|<L_0\}$, we have

$$\begin{aligned}
&\int_\Omega\sum_{\alpha=1}^{N}f^{\alpha}(x)\psi^{\alpha}(x)dx=\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}_{i,j,k}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_i\psi^{\alpha}(x)dx=\\
&=\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}(a_{i,j}^{\alpha,\beta}(x,u_k(x))+\frac{1}{k}\delta_{\alpha,\beta}\delta_{i,j})D_ju_k^{\beta}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B_{k,0}^{\alpha}}dx=\\
&=\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B_{k,0}^{\alpha}}dx+\\
&+\int_\Omega\sum_{\alpha=1}^{N}\sum_{i=1}^{n}\frac{1}{k}|D_iu_k^{\alpha}(x)|^2\,\mathbb{I}_{B_{k,0}^{\alpha}}dx\ge\\
&\ge\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B_{k,0}^{\alpha}}dx.
\end{aligned}\tag{3.30}$$

From $(\mathcal{A}_3')$, we get

$$a_{i,j}^{\alpha,\beta}(x,u_k(x))\mathbb{I}_{B_{k,0}^{\alpha}}=a_{i,j}^{\alpha,\beta}(x,u_k(x))\mathbb{I}_{B_{k,0}^{\alpha}}(x)\mathbb{I}_{B_{k,0}^{\beta}}(x).\tag{3.31}$$

Combining (3.30), (3.31) and $(\mathcal{A}_2)$, we deduce that

$$\begin{aligned}
&\int_\Omega\sum_{\alpha=1}^{N}f^{\alpha}(x)\psi^{\alpha}(x)dx\ge\\
&\ge\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B_{k,0}^{\alpha}}dx=\\
&=\int_\Omega\sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a_{i,j}^{\alpha,\beta}(x,u_k(x))D_ju_k^{\beta}(x)\mathbb{I}_{B_{k,0}^{\beta}}(x)D_iu_k^{\alpha}(x)\mathbb{I}_{B_{k,0}^{\alpha}}(x)dx\ge\\
&\ge\int_\Omega\nu\sum_{\alpha=1}^{N}\frac{|Du_k^{\alpha}(x)|^2\mathbb{I}_{B_{k,0}^{\alpha}}(x)}{(1+|u_k^{\alpha}(x)|)^{\theta}}dx\ge\frac{\nu}{(1+L_0)^{\theta}}\sum_{\alpha=1}^{N}\int_{B_{k,0}^{\alpha}}|Du_k^{\alpha}|^2dx.
\end{aligned}\tag{3.32}$$

From (3.32) and (2.3), it follows that

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{B_{k,0}^{\alpha}}|Du_k^{\alpha}|^2dx\le\frac{(1+L_0)^{\theta}}{\nu}\int_\Omega\sum_{\alpha=1}^{N}f^{\alpha}(x)\psi^{\alpha}(x)dx\le\\
&\le\frac{(1+L_0)^{\theta}L_0}{\nu}\sum_{\alpha=1}^{N}\int_\Omega|f^{\alpha}(x)|dx\le\\
&\le\frac{(1+L_0)^{\theta}L_0}{\nu}\sum_{\alpha=1}^{N}M_m(f,\Omega)|\Omega|^{1-\frac{1}{m}}=\\
&=\frac{(1+L_0)^{\theta}L_0N}{\nu}M_m(f,\Omega)|\Omega|^{1-\frac{1}{m}}.
\end{aligned}\tag{3.33}$$

Therefore, combining (3.27), (3.29) and (3.33), we have

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{\Omega}|Du_k^\alpha(x)|^2dx \leq\\
&\leq \sum_{\alpha=1}^{N}\int_{\{|u_k^\alpha|<L_0\}}|Du_k^\alpha(x)|^2dx + \sum_{t=0}^{+\infty}\frac{(1+2^{t+1}L_0)^\theta}{\nu}\sum_{\alpha=1}^{N}\int_{A^\alpha_{k,2^tL_0}} f^\alpha(x)\varphi_t^\alpha(x)dx \leq\\
&\leq C_7 + \frac{(1+L_0)^\theta L_0 N}{\nu}M_m(f,\Omega)|\Omega|^{1-\frac{1}{m}} := C_8,
\end{aligned}$$

where $C_8$ is a positive constant depending only on $n, N, m, \theta, \nu, M_m(f,\Omega), |\Omega|, C_S, L_0,$ and the boundedness of $u_k$ in $W_0^{1,2}(\Omega,\mathbb{R}^{\mathbb{N}})$ is proved.

□

**Proof of Theorem 2.1**

*Proof.* Let $u_k$ be a solution of $(\tilde{P}_k)$. Lemma 3.1 states that the sequence of $\{u_k\}$ is uniformly bounded in $M^r(\Omega,\mathbb{R}^N)$ and in $W_0^{1,2}(\Omega,\mathbb{R}^N)$. Then there exists a positive constant $C$ such that $M_r(u_k,\Omega) \leq C$ and $\|u_k\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \leq C$ for all $k \in \mathbb{N}$. Being $\{u_k\}$ bounded in $W_0^{1,2}(\Omega,\mathbb{R}^N)$ there exists a subsequence $\{u_{k_\lambda}\}$ weakly converging in $W_0^{1,2}(\Omega,\mathbb{R}^N)$ to a function $u \in W_0^{1,2}(\Omega,\mathbb{R}^N)$. Moreover, by Rellich-Kondrachov embedding Theorem, Sobolev space $W_0^{1,2}(\Omega,\mathbb{R}^N)$ is compactly embedded in $L^2(\Omega,\mathbb{R}^N)$; then, there exists a subsequence, not relabeled, also in the sequel, strongly converging to $u$ in $L^2$. From $L^2$ convergence we get pointwise convergence almost everywhere, up to a further subsequence. Briefly we write

$$\begin{aligned}
&u_{k_\lambda} \rightharpoonup u \quad \text{in } W_0^{1,2}(\Omega),\\
&u_{k_\lambda} \to u \quad \text{in } L^2(\Omega),\\
&u_{k_\lambda}(x) \to u(x) \quad \text{almost everywhere in } \Omega,\\
&M_r(u_k,\Omega) \leq C, \quad \|u_{k_\lambda}\|_{W_0^{1,2}(\Omega,\mathbb{R}^N)} \leq C.
\end{aligned} \tag{3.34}$$

Now, we pass to the limit as $\lambda \to +\infty$, in the weak formulation of problem $(\tilde{P}_k)$, written when $k = k_\lambda$, to prove that $u$ solves problem (2.6). More precisely, we verify that for all $\varphi \in W_0^{1,2}(\Omega,\mathbb{R}^N)$

$$\begin{aligned}
&\lim_{\lambda\to+\infty}\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}^{\alpha,\beta}_{i,j,k_\lambda}(x,u_{k_\lambda}(x))D_ju^\beta_{k_\lambda}(x)D_i\varphi^\alpha(x)dx =\\
&\qquad = \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a^{\alpha,\beta}_{i,j}(x,u(x))D_ju^\beta(x)D_i\varphi^\alpha(x)dx.
\end{aligned}$$

To this aim, we estimate

$$\begin{aligned}
&\Bigg|\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}\tilde{a}^{\alpha,\beta}_{i,j,k_\lambda}(x,u_{k_\lambda}(x))D_ju^\beta_{k_\lambda}(x)D_i\varphi^\alpha(x)dx+\\
&\qquad - \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n}a^{\alpha,\beta}_{i,j}(x,u(x))D_ju^\beta(x)D_i\varphi^\alpha(x)dx\Bigg| \leq
\end{aligned}$$

$$\leq \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n [a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))] D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx \right| +$$
$$+ \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x, u(x)) [D_j u_{k_\lambda}^\beta(x) - D_j u^\beta(x)] D_i \varphi^\alpha(x) dx \right| +$$
$$+ \left| \int_\Omega \sum_{\alpha=1}^N \sum_{i=1}^n \frac{1}{k_\lambda} D_i u_{k_\lambda}^\alpha(x) D_i \varphi^\alpha(x) dx \right| :=$$
$$= I_{k_\lambda} + II_{k_\lambda} + III_{k_\lambda}.$$

We obtain the result by proving that $I_{k_\lambda}, II_{k_\lambda}, III_{k_\lambda}$ tend to zero as $\lambda \to +\infty$. We start to estimate $I_{k_\lambda}$. Using Hölder inequality and boundedness of the sequence $\{u_{k_\lambda}\}$ in $W_0^{1,2}(\Omega)$ we have

$$\begin{aligned} I_{k_\lambda} &= \left| \int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n [a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))] D_j u_{k_\lambda}^\beta(x) D_i \varphi^\alpha(x) dx \right| \leq \\ &\leq \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \left( \int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}} \|D_j u_{k_\lambda}^\beta\|_{L^2} \leq \qquad (3.35) \\ &\leq C \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n \left( \int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

For any $i, j = 1, \dots, n$ and for any $\alpha, \beta = 1, \dots, N$, using pointwise convergence in (3.34) and continuity of functions $y \to a_{i,j}^{\alpha,\beta}(x, y)$ we have that

$$|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 \to 0 \quad \text{as } \lambda \to +\infty;$$

moreover from $(\mathcal{A}_1)$ we get

$$\begin{aligned} &|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 \leq \\ &\leq (|a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x))| + |a_{i,j}^{\alpha,\beta}(x, u(x))|)^2 |D_i \varphi^\alpha(x)|^2 \leq \\ &\leq (c + c)^2 |D_i \varphi^\alpha(x)|^2 \in L^1(\Omega); \end{aligned}$$

therefore, by dominated convergence theorem, we obtain that

$$\left( \int_\Omega |a_{i,j}^{\alpha,\beta}(x, u_{k_\lambda}(x)) - a_{i,j}^{\alpha,\beta}(x, u(x))|^2 |D_i \varphi^\alpha(x)|^2 dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } \lambda \to +\infty.$$

The above limit and (3.35) imply that $I_{k_\lambda}$ tends to zero as $\lambda \to +\infty$.

Observing that $u_{k_\lambda}^\beta \rightharpoonup u^\beta$ in $W_0^{1,2}(\Omega)$, for any $i, j = 1, \dots, n$ and for any $\alpha, \beta = 1, \dots, N$, we have

$$\int_\Omega a_{i,j}^{\alpha,\beta}(x, u(x)) [D_j u_{k_\lambda}^\beta(x) - D_j u^\beta(x)] D_i \varphi^\alpha(x) dx \to 0 \quad \text{as } \lambda \to +\infty,$$

hence $II_{k_\lambda}$ tends to zero as $\lambda \to +\infty$.

Using Hölder inequality and (3.34), it results that

$$III_{k_\lambda} = \left| \int_\Omega \sum_{\alpha=1}^{N} \sum_{i=1}^{n} \frac{1}{k_\lambda} D_i u^\alpha_{k_\lambda}(x) D_i \varphi^\alpha(x) dx \right| \leq \frac{1}{k_\lambda} \sum_{\alpha=1}^{N} \sum_{i=1}^{n} ||D_i u^\alpha_{k_\lambda}||_{L^2} ||D_i \varphi^\alpha||_{L^2} \leq$$
$$\leq \frac{1}{k_\lambda} nNC ||\varphi||_{W^{1,2}_0(\Omega,R^N)}.$$

Passing to the limit as $\lambda \to +\infty$, we obtain that $III_{k_\lambda}$ tends to zero and the proof is completed.

□

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## Conflict of interest

The authors declare no conflict of interest.

## References

1. S. Aouaoui, Solutions to quasilinear equations of $N$-biharmonic type with degenerate coercivity, *Electronic Journal of Differential Equations*, **2014** (2014), 228.
2. A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti, Existence results for nonlinear elliptic equations with degenerate coercivity, *Annali di Matematica*, **182** (2003), 53–79. https://doi.org/10.1007/s10231-002-0056-y
3. A. Alvino, V. Ferone, G. Trombetti, A priori estimates for a class of non uniformly elliptic equations, *Atti Sem. Mat. Fis. Univ. Modena*, **46** (1998), 381–391.
4. H. Ayadi, F. Mokhtari, Nonlinear anisotropic elliptic equations with variable exponents and degenerate coercivity, *Electronic Journal of Differential Equations*, **2018** (2018), 45.
5. P. Baroni, M. Colombo, G. Mingione, Regularity for general functionals with double phase, *Calc. Var.*, **57** (2018), 62. https://doi.org/10.1007/s00526-018-1332-z
6. L. Beck, G. Mingione, Lipschitz bounds and nonuniform ellipticity, *Commun. Pure Appl. Math.*, **73** (2020), 944–1034, https://doi.org/10.1002/cpa.21880
7. L. Boccardo, Quasilinear elliptic equations with natural growth terms: the regularizing effect of the lower order terms, *J. Nonlinear Convex Anal.*, **7** (2006), 355–365.
8. L. Boccardo, Marcinkiewicz estimates for solutions of some elliptic problems with nonregular data, *Annali di Matematica*, **188** (2009), 591–601. https://doi.org/10.1007/s10231-008-0090-5

9. L. Boccardo, H. Brézis, Some remarks on a class of elliptic equations with degenerate coercivity, *Boll. Unione Mat. Ital.*, **6** (2003), 521–530.

10. L. Boccardo, G. Croce, *Elliptic partial differential equations: Existence and regularity of distributional solutions*, Berlin: De Gruyter, 2013. https://doi.org/10.1515/9783110315424

11. L. Boccardo, G. Croce, L. Orsina, Existence of solutions for some noncoercive elliptic problems involving derivatives of nonlinear terms, *Differential Equations & Applications*, **4** (2012), 3–9. https://doi.org/10.7153/dea-04-02

12. L. Boccardo, G. Croce, L. Orsina, Nonlinear degenerate elliptic problems with $W_0^{1,1}(\Omega)$ solutions, *Manuscripta Math.*, **137** (2012), 419–439. https://doi.org/10.1007/s00229-011-0473-6

13. L. Boccardo, G. Croce, C. Tanteri, An elliptic system with degenerate coercivity, *Rend. Mat. Appl. (7)*, **36** (2015), 1–9.

14. L. Boccardo, A. Dall'Aglio, L. Orsina, Existence and regularity results for some elliptic equations with degenerate coercivity, *Atti Sem. Mat. Fis. Univ. Modena*, **46** (1998), 51–81.

15. P. Cianci, G. R. Cirmi, S. D'Asero, S. Leonardi, Morrey estimates for solutions of singular quadratic nonlinear equations, *Annali di Matematica*, **196** (2017), 1739–1758. https://doi.org/10.1007/s10231-017-0636-5

16. G. R. Cirmi, S. D'Asero, S. Leonardi, Fourth-order nonlinear elliptic equations with lower order term and natural growth conditions, *Nonlinear Anal. Theor.*, **108** (2014), 66–86. https://doi.org/10.1016/j.na.2014.05.014

17. G. R. Cirmi, S. D'Asero, S. Leonardi, Morrey estimates for a class of elliptic equations with drift term, *Adv. Nonlinear Anal.*, **9** (2020), 1333–1350. https://doi.org/10.1515/anona-2020-0055

18. G. R. Cirmi, S. D'Asero, S. Leonardi, On the existence of weak solutions to a class of nonlinear elliptic systems with drift term, *J. Math. Anal. Appl.*, **491** (2020), 124370. https://doi.org/10.1016/j.jmaa.2020.124370

19. G. R. Cirmi, S. D'Asero, S. Leonardi, Morrey estimates for a class of noncoercive elliptic systems with VMO-coefficients, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, **32** (2021), 317–334. https://doi.org/10.4171/RLM/938

20. G. R. Cirmi, S. D'Asero, S. Leonardi, M. M. Porzio, Local regularity results for solutions of linear elliptic equations with drift term, *Adv. Calc. Var.*, **15** (2022), 19–32. https://doi.org/10.1515/acv-2019-0048

21. G. R. Cirmi, S. Leonardi, Regularity results for the gradient of solutions linear elliptic equations with $L^{1,\lambda}$ data, *Annali di Matematica*, **185** (2006), 537–553. http://doi.org/10.1007/s10231-005-0167-3

22. G. R. Cirmi, S. Leonardi, Regularity results for solutions of nonlinear elliptic equations with $L^{1,\lambda}$ data, *Nonlinear Anal. Theor.*, **69** (2008), 230–244. https://doi.org/10.1016/j.na.2007.05.014

23. G. R. Cirmi, S. Leonardi, Higher differentiability for the solutions of nonlinear elliptic systems with lower-order terms and $L^{1,\theta}$-data, *Annali di Matematica*, **193** (2014), 115–131. https://doi.org/10.1007/s10231-012-0269-7

24. G. Croce, An elliptic problem with degenerate coercivity and a singular quadratic gradient lower order term, *Discrete Contin. Dyn. Syst. S*, **5** (2012), 507–530. https://doi.org/10.3934/dcdss.2012.5.507

25. G. Croce, The regularizing effects of some lower order terms in an elliptic equation with degenerate coercivity, *Rendiconti di Matematica, Serie VII*, **27** (2007), 299–314.

26. G. Cupini, F. Giannetti, R. Giova, A. Passarelli di Napoli, Regularity results for vectorial minimizers of a class of degenerate convex integrals, *J. Differ. Equations*, **265** (2018), 4375–4416. https://doi.org/10.1016/j.jde.2018.06.010

27. G. Cupini, P. Marcellini, E. Mascolo, Regularity of minimizers under limit growth conditions, *Nonlinear Anal. Theor.*, **153** (2017), 294–310. https://doi.org/10.1016/j.na.2016.06.002

28. C. De Filippis, G. Mingione, Lipschitz bounds and nonautonomous integrals, *Arch. Rational Mech. Anal.*, **242** (2021), 973–1057. https://doi.org/10.1007/s00205-021-01698-5

29. E. De Giorgi, Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Un. Mat. Ital. Ser. IV*, **1** (1968), 135–137.

30. F. Della Pietra, G. Di Blasio, Comparison, existence and regularity results for a class of non-uniformly elliptic equations, *Differential Equations & Applications*, **2** (2010), 79–103. https://doi.org/10.7153/dea-02-07

31. P. Di Gironimo, F. Leonetti, M. Macrì, P. V. Petricca, Existence of bounded solutions for some quasilinear degenerate elliptic systems, *Minimax Theory and its Applications*, **6** (2021), 321–340.

32. P. Di Gironimo, F. Leonetti, M. Macrì, P. V. Petricca, Existence of solutions to some quasilinear degenerate elliptic systems when the datum has an intermediate degree of integrability, *Complex Var. Elliptic Equ.*, in press. https://doi.org/10.1080/17476933.2022.2069753

33. G. Dolzmann, N. Hungerbuhler, S. Muller, Non-linear elliptic systems with measure-valued right hand side, *Math. Z.*, **226** (1997), 545–574. https://doi.org/10.1007/PL00004354

34. V. Ferone, N. Fusco, VMO solutions of the $N$-Laplacian with measure data, *Comptes Rendus de l'Académie des Sciences Series I-Mathematics*, **325** (1997), 365–370. https://doi.org/10.1016/S0764-4442(97)85618-2

35. H. Gao, M. Huang, W. Ren, Regularity for entropy solutions to degenerate elliptic equations, *J. Math. Anal. Appl.*, **491** (2020), 124251. https://doi.org/10.1016/j.jmaa.2020.124251

36. H. Gao, F. Leonetti, W. Ren, Regularity for anisotropic elliptic equations with degenerate coercivity, *Nonlinear Anal.*, **187** (2019), 493–505. https://doi.org/10.1016/j.na.2019.06.017

37. D. Giachetti, M. M. Porzio, Existence results for some nonuniformly elliptic equations with irregular data, *J. Math. Anal. Appl.*, **257** (2001), 100–130. https://doi.org/10.1006/jmaa.2000.7324

38. D. Giachetti, M. M. Porzio, Elliptic equations with degenerate coercivity: gradient regularity, *Acta Math. Sinica*, **19** (2003), 349–370. https://doi.org/10.1007/s10114-002-0235-1

39. W. Hao, S. Leonardi, J. Nečas, An example of irregular solution to a nonlinear Euler-Lagrange elliptic system with real analytic coefficients, *Annali della Scuola Normale Superiore di Pisa (IV)*, **23** (1996), 57–67.

40. W. Hao, S. Leonardi, M. Steinhauer, Examples of discontinuous, divergence-free solutions to elliptic variational problems, *Comment. Math. Univ. Carolin.*, **36** (1995), 511–517.

41. J. Kristensen, G. Mingione, The singular set of $\omega$-minima, *Arch. Rational Mech. Anal.*, **177** (2005), 93–114. https://doi.org/10.1007/s00205-005-0361-x

42. J. Kristensen, G. Mingione, The singular set of minima of integral functionals, *Arch. Rational Mech. Anal.*, **180** (2006), 331–398. https://doi.org/10.1007/s00205-005-0402-5

43. J. Kristensen, G. Mingione, Boundary regularity in variational problems, *Arch. Rational Mech. Anal.*, **198** (2010), 369–455. https://doi.org/10.1007/s00205-010-0294-x

44. T. Kuusi, G. Mingione, Universal potential estimates, *J. Funct. Anal.*, **262** (2012), 4205–4269. https://doi.org/10.1016/j.jfa.2012.02.018

45. T. Kuusi, G. Mingione, Linear potentials in nonlinear potential theory, *Arch. Rational Mech. Anal.*, **207** (2013), 215–246. https://doi.org/10.1007/s00205-012-0562-z

46. T. Kuusi, G. Mingione, Guide to nonlinear potential estimates, *Bull. Math. Sci.*, **4** (2014), 1–82. https://doi.org/10.1007/s13373-013-0048-9

47. T. Kuusi, G. Mingione, The Wolff gradient bound for degenerate parabolic equations, *J. Eur. Math. Soc.*, **16** (2014), 835–892. https://doi.org/10.4171/JEMS/449

48. T. Kuusi, G. Mingione, Vectorial nonlinear potential theory, *J. Eur. Math. Soc.*, **20** (2018), 929–1004. https://doi.org/10.4171/JEMS/780

49. S. Leonardi, On constants of some regularity theorems. De Giorgi's type counterexample, *Math. Nachr.*, **192** (1998), 191–204. https://doi.org/10.1002/mana.19981920111

50. S. Leonardi, Gradient estimates below duality exponent for a class of linear elliptic systems, *Nonlinear Differ. Equ. Appl.*, **18** (2011), 237–254. http://doi.org/10.1007/s00030-010-0093-y

51. S. Leonardi, Morrey estimates for some classes of elliptic equations with a lower order term, *Nonlinear Anal.*, **177** (2018), 611–627. https://doi.org/10.1016/j.na.2018.05.010

52. S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, V. Staicu, Maximum principles for some quasilinear elliptic systems, *Nonlinear Anal.*, **194** (2020), 111377. https://doi.org/10.1016/j.na.2018.11.004

53. S. Leonardi, F. Leonetti, C. Pignotti, E. Rocha, V. Staicu, Local boundedness for weak solutions to some quasilinear elliptic systems, *Minimax Theory and its Applications*, **6** (2021), 365–378.

54. S. Leonardi, F. Leonetti, E. Rocha, V. Staicu, Butterfly support for off diagonal coefficients and boundedness of solutions to quasilinear elliptic systems, *Adv. Nonlinear Anal.*, **11** (2022), 672–683. https://doi.org/10.1515/anona-2021-0205

55. F. Leonetti, P. V. Petricca, Existence of bounded solutions to some nonlinear degenerate elliptic systems, *Discrete Contin. Dyn. Syst. B*, **11** (2009), 191–203. https://doi.org/10.3934/dcdsb.2009.11.191

56. F. Leonetti, E. Rocha, V. Staicu, Quasilinear elliptic systems with measure data, *Nonlinear Anal. Theor.*, **154** (2017), 210–224. https://doi.org/10.1016/j.na.2016.04.002

57. F. Leonetti, E. Rocha, V. Staicu, Smallness and cancellation in some elliptic systems with measure data, *J. Math. Anal. Appl.*, **465** (2018), 885–902. https://doi.org/10.1016/j.jmaa.2018.05.047

58. F. Leonetti, R. Schianchi, A remark on some degenerate elliptic problems, *Ann. Univ. Ferrara*, **44** (1998), 123–128. https://doi.org/10.1007/BF02828019

59. J. Leray, J. L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bulletin de la Société Mathématique de France*, **93** (1965), 97–107. https://doi.org/10.24033/bsmf.1617

60. V. G. Maz'ja, Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients, *Funct. Anal. Appl.*, **2** (1968), 230–234. https://doi.org/10.1007/BF01076124

61. G. Mingione, Regularity of minima: An invitation to the dark side of the calculus of variations, *Appl. Math.*, **51** (2006), 355–426. https://doi.org/10.1007/s10778-006-0110-3

62. G. Mingione, The Calderón-Zygmund theory for elliptic problems with measure data, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, **6** (2007), 195–261. https://doi.org/10.2422/2036-2145.2007.2.01

63. G. Mingione, Gradient estimates below the duality exponent, *Math. Ann.*, **346** (2010), 571–627. https://doi.org/10.1007/s00208-009-0411-z

64. G. Mingione, Gradient potential estimates, *J. Eur. Math. Soc.*, **13** (2011), 459–486. https://doi.org/10.4171/JEMS/258

65. G. Mingione, G. Palatucci, Developments and perspectives in nonlinear potential theory, *Nonlinear Anal.*, **194** (2020), 111452. https://doi.org/10.1016/j.na.2019.02.006

66. J. Nečas, J. Stará, Principio di massimo per i sistemi ellittici quasi-lineari non diagonali, *Boll. Un. Mat. Ital. Ser. IV*, **6** (1972), 1–10.

67. A. Porretta, Uniqueness and homogenization for a class of non coercive operators in divergence form, *Atti Sem. Mat. Fis. Univ. Modena*, **46** (1998), 915–936.

68. E. Stein, G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.

69. C. Trombetti, Existence and regularity for a class of non-uniformly elliptic equations in two dimensions, *Differential Integral Equations*, **13** (2000), 687–706.

70. Z. Q. Yan, Everywhere regularity for solutions to quasilinear elliptic systems of triangular form, In: *Partial differential equations*, Berlin: Springer, 1988, 255–261. https://doi.org/10.1007/BFb0082938

71. S. Zhou, A note on nonlinear elliptic systems involving measures, *Electronic Journal of Differential Equations*, **2000** (2000), 08.