Decidability of the satisfiability problem for Boolean set theory with the unordered Cartesian product operator

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We give a positive solution to the decidability problem for the fragment of set theory, dubbed BST⊗, consisting of quantifier-free formulae involving the Boolean set operators of union, intersection, and set difference, along with the unordered Cartesian product operator ⊗ (where $s \otimes t := \big\{\{u, v\} \mid u \in s \wedge v \in t\big\}$), and the equality predicate, but no membership. Specifically, we provide nondeterministic exponential decision procedures for both the ordinary and the finite satisfiability problems for BST⊗. We expect that these decision procedures can be adapted for the standard Cartesian product and, with added technicalities, to the cases involving membership, providing a solution to a longstanding problem in computable set theory.

CCS Concepts: • **Theory of computation → Automated reasoning**.

Additional Key Words and Phrases: Satisfiability problem and decision procedures · Unordered Cartesian product · NEXPTIME satisfiability tests · Computable Set Theory.

INTRODUCTION

The decision problem in set theory has been studied quite thoroughly in the last decades, giving rise to the field of Computable Set Theory [CFO89]. The initial goal was the mechanical formalization of mathematics with a proof verifier based on the set-theoretic formalism [OS02, COSU03, OCPS06, SCO11], but soon a foundational interest aimed at the identification of the boundary in set theory between the decidable and the undecidable became more and more compelling.

The precursor fragment of set theory investigated for decidability was MLS, which stands for Multi-Level Syllogistic. MLS consists of the quantifier-free formulae of set theory involving only the Boolean set operators ∪, ∩, \ and the relators = and ∈, besides set variables (assumed to be existentially quantified). The satisfiability problem (s.p., briefly) for MLS, namely the problem of establishing whether or not any given MLS-formula is satisfiable by some set assignment, has been proved to be decidable in the seminal paper [FOS80], and its NP-completeness has later been established in [COP90]. Following that, several extensions of MLS with various combinations of the set operators $\{\cdot\}$ (singleton), pow (power set), $\bigcup$ (unary union), $\bigcap$ (unary intersection), rk (rank), etc., and of the set predicates rank comparison, cardinality comparison, finiteness, etc., have been also proved decidable over the years.[1]

However, the decidability problem for the extension MLS× of MLS with the Cartesian product ×,[2] proposed by the first author since the middle 80s, soon appeared to be very challenging and resisted several efforts to

[1]The monographs [CFO89, COP01, SCO11, OPT17, CU18] provide a rather comprehensive account.

[2]For definiteness, we may assume that the Cartesian product is expressed in terms of Kuratowski's ordered pairs $(u, v) := \{\{u\}, \{u, v\}\}$. Hence, $s \times t := \{(u, v) \mid u \in s, v \in t\}$, for any sets s and t.

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find a solution, either positive or negative. As a matter of fact, for long it was not even excluded that the s.p. for MLS× were undecidable (in particular, when restricted to finite models—*finite s.p.*) and that some tricky reduction of the well-celebrated Hilbert's Tenth problem (H10, for short) to the s.p. for MLS× was lurking around. We recall that H10, posed by David Hilbert at the beginning of last century [Hil02] asks for a uniform procedure that can determine in a finite number of steps whether any given Diophantine polynomial equation with integral coefficients is solvable in integers. In 1970, it was shown that no algorithmic procedure exists for H10, as result of the combined efforts of M. Davis, H. Putnam, J. Robinson, and Yu. Matiyasevich (DPRM theorem, see [Rob52, DPR61, Mat70]).

It was deemed reasonable that the union of disjoint sets and the Cartesian product might somehow play the roles of integer addition and multiplication in H10, respectively, in consideration of the fact that $|s \cup t| = |s| + |t|$, for any disjoint sets s and t, and $|s \times t| = |s| \cdot |t|$, for any sets s and t. In fact, such an observation is at the base of the proof of the undecidability of the s.p. for MLS× when it is extended with cardinality comparison, namely the two-place predicate $|\cdot| \leqslant |\cdot|$ where $|s| \leqslant |t|$ holds if and only if the cardinality of s does not exceed that of t (see [CCP90] and [COP20]).

Attempts to solve the s.p. for MLS× helped shape the development of computable set theory and led to the introduction of the powerful technique of *formative processes*,[3] which has been the main tool in the highly technical solutions to the decision problems for the extension MLSSP of MLS with the power set and the singleton operators [COU02] and the extension MLSSPF with the finiteness predicate too [CU14].

In this paper, we provide an algorithmic solution to the s.p., both unrestricted and restricted to (hereditarily) finite models, for the fragment of set theory dubbed BST⊗, which is closely related to MLS×. The fragment BST⊗ (which stands for Boolean Set Theory with the *unordered Cartesian product* ⊗) is obtained by dropping the membership predicate ∈ from MLS× and replacing the (ordered) Cartesian product operator × with its unordered variant ⊗, where, for any sets s and t, $s \otimes t$ is the set of all unordered pairs $\{u, v\}$ for which $u \in s$ and $v \in t$, namely

$$s \otimes t := \big\{\{u, v\} \mid u \in s \wedge v \in t\big\}.$$

Notice that none of the above two changes affects the aforementioned connection with H10. The reason why we chose to address here the case of BST⊗ rather than the one of MLS× is that in doing so we can get rid of irrelevant features that would only make our analysis much more technical. In fact, we are very confident that the s.p. for the fragments BST× and MLS× admit an algorithmic solution too.[4]

For both variants of the s.p. for BST⊗ (unrestricted and restricted to finite models), we shall provide nondeterministic exponential decision procedures. These will be expressed in terms of the existence of a special graph, called ⊗-*graph*, enjoying a certain connectivity property of *accessibility*. Given a BST⊗-formula Φ to be tested for satisfiability, in the case of the ordinary s.p. it will be enough to require that a candidate accessible ⊗-graph *fulfills* Φ, in a sense that will be defined in due course. In the case of the finite s.p. it will be additionally required that the ⊗-graph admits also a sort of topological order. In both cases, it will be shown that, when satisfied, these conditions (which are also necessary) ensure that the ⊗-graph can be used as a sort of flow graph that allows one to build a model for Φ in denumerably many steps, in the case of the ordinary s.p., or in a bounded finite number of steps, in the case of the (hereditarily) finite s.p. Such a construction procedure can be regarded as a simplified use of the formative processes mentioned before.

As shown in [Schw78, CCS90], the finite s.p. for the extension of MLS with cardinality comparison can be reduced to purely existential Presburger arithmetic, which is known to be NP-complete (see [Sca84]). On the other hand, when either BST× or BST⊗ is enriched with cardinality comparison, the *finite* s.p. for the resulting extensions becomes undecidable, as sketched in Section 1.6.1, since H10 would be reducible to it, much as proved

[3] See [CU18] for a quite accessible introduction.
[4] Of course, BST× is Boolean Set Theory with Cartesian product.

in [CCP90] and [COP20] for MLS⊗ and MLS×. This is clear evidence that the decision problem for both BST× and BST⊗ is very close to the edge of decidability.

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The paper is organized as follows. In Section 1, we introduce the precise syntax and semantics of the fragment of our interest BST⊗ and of other related theories. In particular, semantics is presented in terms of satisfying partitions, and it is shown that this approach leads easily to the decidability of the purely Boolean subset BST of BST⊗. We also provide a sketch of the proof of the undecidability of the finite s.p. for the extension of BST⊗ with cardinality comparison. Subsequently, in Section 2, we introduce the central notion of accessible ⊗-graphs, together with that of fullfilment of a BST⊗-formula by an accessible ⊗-graph, and we prove that any satisfiable BST⊗-formula is fulfilled by a suitable accessible ⊗-graph. We also prove that the existence of an accessible ⊗-graph $\mathcal{G}$ fulfilling a given BST-formula Φ is sufficient for the satisfiability of Φ, by describing in details a construction process that uses the ⊗-graph $\mathcal{G}$ as a kind of flow graph that allows one to build a model for Φ in denumerably many steps. The main definitions and proofs of the section are illustrated in detail with the help of a running example in five parts. Afterward, in Section 3, we introduce the notion of ordered ⊗-graphs and prove that the existence of an ordered ⊗-graph fulfilling a given BST⊗-formula Φ is a necessary and sufficient condition for Φ to be (hereditarily) finitely satisfiable. Finally, in Section 4, we briefly discuss some plans for future research.

1 PRELIMINARIES

We briefly review some significant fragments of computable set theory, providing their syntax and semantics. Then we illustrate the notion of *satisfiability by partition* in the simplified case in which the unordered Cartesian product ⊗ is not present, in view of its generalization to the complete case of BST⊗. We also recall the definition of some complexity classes relevant to our decision procedures. Finally, we sketch the proof of the undecidability of the s.p. for the extension of BST⊗ with the cardinality comparison predicate, providing evidence that the decision problem for BST⊗ (and BST×) is at the verge of decidability.

1.1 A glimpse to computable set theory

The quantifier-free fragments of set theory, whose decision problem has been actively investigated during the last decades, involve, among others and in various combinations, the following set operators and predicates:

- the Boolean set operators of union "∪", intersection "∩", and set difference "\";
- the singleton "{·}" and finite enumerations "{·, . . . , ·}" operators;
- the Cartesian product "×" and its unordered variant "⊗";
- the powerset "pow", unary union "⋃", and unary intersection "⋂" operators;
- set equality "=", inclusion "⊆", membership "∈", finiteness "*Finite*(·)", and cardinality comparison "$|\cdot| \leqslant |\cdot|$".

The fragment of primary importance for the present paper is BST⊗, namely the quantifier-free propositional closure of atoms of the following types:

$$x = y \cup z, \quad x = y \cap z, \quad x = y \setminus z, \quad x \subseteq y, \quad x = y \otimes z,$$

where x, y, z stand for (existentially quantified) set variables.

Particularly relevant to our research are also the fragments MLS (Multi-Level Syllogistics) and BST (Boolean Set Theory), and their extensions MLS×, MLS⊗, and BST× with the Cartesian product × and its unordered variant ⊗, for which we provide precise definitions, for completeness sake:

– MLS is the propositional combination of literals of the forms

$$x = y \cup z, \quad x = y \cap z, \quad x = y \setminus z, \quad x \subseteq y, \quad x \in y;$$

– BST is the propositional combination of literals of the forms

$$x = y \cup z, \quad x = y \cap z, \quad x = y \setminus z, \quad x \subseteq y;$$

– MLS$^{\times}$ is the extension of MLS with literals of the form $x = y \times z$;
– MLS$^{\otimes}$ is the extension of MLS with literals of the form $x = y \otimes z$;
– BST$^{\times}$ is the extension of BST with literals of the form $x = y \times z$.

Given a formula Φ in any of the above fragments, we denote by $\mathrm{Vars}(\Phi)$ the collection of all the set variables occurring in Φ.

1.2 Semantics

The standard way to define the semantics of fragments of set theory is through *set assignments*. In the following, we define accurately the semantics of MLS and BST, and their extensions with literals of the forms $x = y \times z$ and $x = y \otimes z$.

A SET ASSIGNMENT M is any map from a collection V of set variables (called the VARIABLE-DOMAIN OF M and denoted $\mathrm{dom}(M)$) into the von Neumann universe $\boldsymbol{\mathcal{V}}$ of all well-founded sets.

We recall that $\boldsymbol{\mathcal{V}}$ is a cumulative hierarchy constructed in stages by transfinite recursion over the class On of all ordinals. Specifically, $\boldsymbol{\mathcal{V}} := \bigcup_{\alpha \in On} \mathcal{V}_\alpha$ where, recursively, $\mathcal{V}_\alpha := \bigcup_{\beta<\alpha} \mathrm{pow}(\mathcal{V}_\beta)$, for every $\alpha \in On$, with $\mathrm{pow}(\cdot)$ denoting the powerset operator. Based on that construction, we can readily define the RANK of any well-founded set $s \in \boldsymbol{\mathcal{V}}$, denoted $\mathrm{rk}\ s$, as the least ordinal α such that $s \subseteq \mathcal{V}_\alpha$. The collection of the sets of finite rank, hence belonging to $\mathcal{V}_\alpha$ for some finite ordinal α, forms the set HF of the HEREDITARILY FINITE SETS. Thus, $\mathrm{HF} = \mathcal{V}_\omega$, where ω is the first limit ordinal, namely the smallest non-null ordinal with no immediate predecessor.

Given a set assignment M and a set of variables $W \subseteq V$, where $V := \mathrm{dom}(M)$, we put $MW := \{Mv \mid v \in W\}$. The SET-DOMAIN OF M is defined as the set $\bigcup MV = \bigcup_{v \in V} Mv$. A set assignment M is FINITE (resp., HEREDITARILY FINITE), if so is its set-domain.

The operators in the fragments of our interest are interpreted according to their usual semantics. Thus, given a set assignment M, for any $x, y, z \in \mathrm{dom}(M)$ we put:

$$M(x \star y) := Mx \star My, \tag{1}$$

where $\star \in \{\cup, \cap, \setminus, \times, \otimes\}$ and where we have $s \times t := \{(u, v) \mid u \in s, v \in t\}$ (with $(u, v) := \{\{u\}, \{u, v\}\}$ the *ordered Kuratowski pair*) and $s \otimes t := \{\{u, v\} \mid u \in s, v \in t\}$, for any sets s and t.[5]

Finally, any set assignment M is extended to a **true**/**false** interpretation of all the formulae of our fragments, over the variables in $\mathrm{dom}(M)$, by putting

$$\begin{aligned}
M(x = y \star z) = \mathbf{true} \quad &\overset{Def}{\longleftrightarrow} \quad Mx = M(y \star z),\\
M(x = y) = \mathbf{true} \quad &\overset{Def}{\longleftrightarrow} \quad Mx = My,\\
M(x \subseteq y) = \mathbf{true} \quad &\overset{Def}{\longleftrightarrow} \quad Mx \subseteq My,\\
M(x \in y) = \mathbf{true} \quad &\overset{Def}{\longleftrightarrow} \quad Mx \in My,\\
M(\mathit{Finite}(x)) = \mathbf{true} \quad &\overset{Def}{\longleftrightarrow} \quad Mx \text{ is finite},
\end{aligned}$$

[5]We did not bother to use different symbols in (1) for the operator and the operation: hence, the '$\star$' in the *definiendum* stands for a set operator, whereas the '$\star$' in the *definiens* stands for the corresponding set operation.

for all $x, y, z \in \mathrm{dom}(M)$ and $\star \in \{\cup, \cap, \setminus, \times, \otimes\}$, and recursively

$$M(\neg\Phi) := \neg M\Phi, \qquad M(\Phi \wedge \Psi) := M\Phi \wedge M\Psi,$$
$$M(\Phi \vee \Psi) := M\Phi \vee M\Psi, \qquad M(\Phi \to \Psi) := M\Phi \to M\Psi, \qquad \text{etc.},$$

for all formulae Φ and Ψ such that $\mathrm{Vars}(\Phi), \mathrm{Vars}(\Psi) \subseteq \mathrm{dom}(M)$ and all the propositional connectives such as $\neg$, $\wedge$, $\vee$, $\to$, etc.

Given a formula Φ, a set assignment M over $\mathrm{Vars}(\Phi)$ is said to SATISFY Φ if $M\Phi =$ **true** holds, in which case we also write $M \models \Phi$ and say that M is a MODEL for Φ. If Φ has a model, we say that Φ is SATISFIABLE; otherwise, we say that Φ is UNSATISFIABLE. If $M \models \Phi$ and M is finite (resp., hereditarily finite), then Φ is FINITELY SATISFIABLE (resp., HEREDITARILY FINITELY SATISFIABLE).

For a quantifier-free fragment $\mathcal{S}$ of set theory (such as MLS, BST, and their extensions), the SATISFIABILITY PROBLEM (s.p.) for $\mathcal{S}$ is the problem of establishing whether or not any given $\mathcal{S}$-formula is satisfiable by some set assignment. If there is an algorithmic test $\mathcal{A}_\mathcal{S}$ that can answer all of its instances, the s.p. for $\mathcal{S}$ is said to be DECIDABLE and the test $\mathcal{A}_\mathcal{S}$ is called a DECISION PROCEDURE for $\mathcal{S}$; otherwise it is UNDECIDABLE. Likewise, if there is an algorithmic test $\mathcal{A}'_\mathcal{S}$ that can answer all the positive (resp., negative) instances, the satisfiability (resp., unsatisfiability) problem for $\mathcal{S}$ is said to be SEMI-DECIDABLE and the test $\mathcal{A}'_\mathcal{S}$ is called a SEMI-DECISION TEST for $\mathcal{S}$.

By restricting oneself to (hereditarily) finite set assignments, one can define in the obvious way the (HEREDITARILY) FINITE SATISFIABILITY PROBLEM for $\mathcal{S}$.

Any two $\mathcal{S}$-formulae Φ and Ψ are (FINITELY) EQUISATISFIABLE when Φ is (finitely) satisfiable if and only if so is Ψ, possibly by different models.

The s.p. for MLS has been solved in [FOS80], whereas as of now the satisfiability problems for MLS$\times$ and MLS$\otimes$ are still open.

Since BST is a subfragment of MLS, the same decision procedure for MLS works also for BST. Recently, an alternative decision procedure for the s.p. for BST in terms of *partition assignments* has been presented in [CU18, Sect. 2.1.3]. We shall review a version of it in Section 1.5, slightly adapted to our needs in preparation for the main results of the paper, namely the design of algorithmic solutions to the s.p.—both unrestricted and finite—for BST$\otimes$.

For the sake of completeness, we list some relevant fragments of set theory whose decision problem has already been solved over the years (after each acronym, we list the operators and relators present in the fragment and the appropriate references to the literature):

MLS:	$\cup, \cap, \setminus, \subseteq, =, \in$	(cf. [FOS80])
MLSS:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}$	(cf. [FOS80])
MLSU:	$\cup, \cap, \setminus, \subseteq, =, \in, \bigcup(\cdot)$	(cf. [CFS87])
MLSSI:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}, \bigcap(\cdot)$	(cf. [CC89])
MLSP:	$\cup, \cap, \setminus, \subseteq, =, \in, \mathrm{pow}(\cdot)$	(cf. [CFS85])
MLSSP:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}, \mathrm{pow}(\cdot)$	(cf. [Can91] and [COU02])
MLSSPF:	$\cup, \cap, \setminus, \subseteq, =, \in, \{\cdot\}, \mathrm{pow}(\cdot), \mathit{Finite}(\cdot)$	(cf. [CU14]) .

The interested reader can find an extensive treatment of these and other results in [CFO89, COP01, SCO11].

We also mention a recent research aimed at identifying useful 'small' fragments of set theory, all of them easily expressible in MLS, endowed with polynomial-time decision procedures.

More specifically, we analyzed the 2040 fragments of the collection of all conjunctions of literals of the following types

$$s = \emptyset, \qquad s \neq \emptyset, \qquad s \cap t = \emptyset, \qquad s \cap t \neq \emptyset,$$
$$s \subseteq t, \qquad s \not\subseteq t, \qquad s = t, \qquad s \neq t,$$

where s and t stand for terms that can involve set variables and the Boolean set operators $\cup$, intersection $\cap$, and set difference $\setminus$. After identifying the 18 *minimal* NP-complete fragments and the 5 *maximal* polynomial fragments, it was proved in [CDMO21] that 1278 of these fragments are NP-complete and the remaining 762 are polynomial (of degree at most 4). Some notable cases admitting cubic decision procedures have been reported in [CMO21] and in [CMO23].

We also analyzed in [CMO20] the fragments of the collection of all the conjunctions of literals of the two forms

$$s \in t \qquad \text{and} \qquad s \notin t,$$

where as above s and t stand for terms that can involve set variables and the Boolean set operators $\cup$, intersection $\cap$, and set difference $\setminus$. The maximal polynomial and the minimal NP-complete fragments have been identified and some notable non-maximal polynomial fragments have been explored in depth.

1.3 Satisfiability by partitions

Satisfiability by set assignments is equivalent to the notion of satisfiability by partitions, introduced in [CU18, Sect. 2.1.3], which we review next.

A PARTITION is a set of pairwise disjoint non-null sets, called the BLOCKS of the partition. The union $\bigcup\Sigma$ of a partition Σ is its DOMAIN.

Let V be a finite set of set variables and Σ a partition. Also, let $\mathfrak{I}\colon V \to \mathrm{pow}(\Sigma)$ be any map. In a very natural way, the map $\mathfrak{I}$ induces a set assignment $M_{\mathfrak{I}}$ over V defined by:

$$M_{\mathfrak{I}} v := \bigcup \mathfrak{I}(v), \qquad \text{for } v \in V.$$

We refer to the map $\mathfrak{I}$ (or to the pair $(\Sigma, \mathfrak{I})$, when we want also to emphasize the partition Σ) as a PARTITION ASSIGNMENT.

DEFINITION 1. Let Σ be a partition and $\mathfrak{I}\colon V \to \mathrm{pow}(\Sigma)$ be a partition assignment over a finite set V of set variables. Given a BST⊗-formula Φ such that $\mathrm{Vars}(\Phi) \subseteq V$, we say that $\mathfrak{I}$ SATISFIES Φ, and write $\mathfrak{I} \models \Phi$, when the set assignment $M_{\mathfrak{I}}$ induced by $\mathfrak{I}$ satisfies Φ (equivalently, one may say that Σ SATISFIES Φ VIA THE MAP $\mathfrak{I}$, and write $\Sigma, \mathfrak{I} \models \Phi$, if we want to emphasize the partition Σ). We say that Σ SATISFIES Φ, and write $\Sigma \models \Phi$, if Σ satisfies Φ via some map $\mathfrak{I}\colon V \to \mathrm{pow}(\Sigma)$. □

The following result can be proved immediately.

LEMMA 1.1. *If a* BST⊗*-formula is satisfied by a partition Σ, then it is satisfied by any partition $\overline{\Sigma}$ that includes Σ as a subset, namely such that $\Sigma \subseteq \overline{\Sigma}$.*

Plainly, a BST⊗-formula Φ satisfied by some partition is satisfied by a set assignment. Indeed, if $\Sigma \models \Phi$, then $\Sigma, \mathfrak{I} \models \Phi$ for some map $\mathfrak{I}\colon V \to \mathrm{pow}(\Sigma)$, and therefore $M_{\mathfrak{I}} \models \Phi$. The converse holds too. In fact, let us assume that $M \models \Phi$, for some set assignment M over the set $V = \mathrm{Vars}(\Phi)$ of the set variables occurring in Φ, and let Σ_M be the EULER-VENN PARTITION induced by M, namely

$$\Sigma_M := \left\{ \bigcap MV' \setminus \left(\bigcup M(V \setminus V') \right) \mid \emptyset \neq V' \subseteq V \right\} \setminus \{\emptyset\}.$$

Hence, we have:

- $(\forall \sigma \in \Sigma_M)(\forall v \in V)(\sigma \cap Mv = \emptyset \vee \sigma \subseteq Mv)$,

- $(\forall \sigma, \sigma' \in \Sigma_M)\big((\forall v \in V)(\sigma \subseteq Mv \iff \sigma' \subseteq Mv) \iff \sigma = \sigma'\big)$, and
- $\bigcup \Sigma_M = \bigcup MV$.

Let $\mathfrak{I}_M \colon V \to \mathrm{pow}(\Sigma_M)$ be the map defined by

$$\mathfrak{I}_M(v) \coloneqq \{\sigma \in \Sigma_M \mid \sigma \subseteq Mv\}, \qquad \text{for } v \in V.$$

It is an easy matter to check that the set assignment induced by $\mathfrak{I}_M$ is just M. Thus $\Sigma_M, \mathfrak{I}_M \models \Phi$, and therefore $\Sigma_M \models \Phi$, proving that Φ is satisfied by some partition, in fact by the Euler-Venn partition induced by M, whose size is at most $2^{|V|} - 1$.

Thus, the notion of satisfiability by set assignments and that of satisfiability by partitions coincide.

As a by-product of Lemma 1.1 and of the above considerations, we also have:

Lemma 1.2. *Every* BST⊗*-formula* Φ *with n distinct variables is satisfiable if and only if it is satisfied by some partition with* $2^n - 1$ *blocks.*

1.4 Normalization of BST⊗-formulae

By applying disjoint normal form and the simplification rules illustrated in [CU18], the satisfiability problem for BST⊗ can be reduced to the satisfiability problem for normalized conjunctions of BST⊗, namely conjunctions of BST⊗-literals of the following restricted types:

$$x = y \cup z, \quad x = y \setminus z, \quad x = y \otimes z, \quad x \neq y, \tag{2}$$

where x, y, z stand for set variables. Indeed, it is enough to observe that:

- $x \subseteq y$ is equivalent to $x = x \cap y$;
- the terms $y \cap z$ and $y \setminus (y \setminus z)$ are equivalent, so an atom of the form $x = y \cap z$ is equisatisfiable with the conjunction $x = y \setminus y' \wedge y' = y \setminus z$, where y' stands for any fresh set variable;
- each negative literal of the form $x \neq y \star z$ (with $\star \in \{\cup, \cap, \setminus, \otimes\}$) is equisatisfiable with the conjunction $x' = y \star z \wedge x' \neq x$, where x' stands for any fresh set variable.

We shall refer to literals of the form $x = y \otimes z$ as ⊗-literals, and similarly for the other types of literals in (2).

1.5 The Boolean case

In order to allow a smooth transition to the treatment of the s.p. for BST⊗-formulae, it is useful to preliminarily review the restricted case (solved in [CU18, Sect. 2.3]) of BST-conjunctions, namely conjunctions of Boolean literals of the forms

$$x = y \cup z, \quad x = y \setminus z, \quad x \neq y,$$

where ⊗-literals are not present.

Specifically, we prove next that the satisfiability status of any BST-conjunction by a given partition Σ depends solely on the size of Σ, and not on the internal structure of its blocks. Though in this case we could restrict ourselves to the special case in which all the blocks are singletons, we prefer to proceed in full generality to allow for a more natural generalization to the case of BST⊗-conjunctions of our interest in Sections 2 and 3.

Lemma 1.3. *Let* Σ *be a partition and let* $\mathfrak{I} \colon V \to \mathrm{pow}(\Sigma)$ *be a partition assignment over a (finite) set of variables* V*. Then, for all* $x, y, z \in V$ *and* $\star \in \{\cup, \setminus\}$*, we have:*

(a) $\mathfrak{I} \models \big(x = y \star z\big) \iff \mathfrak{I}(x) = \mathfrak{I}(y) \star \mathfrak{I}(z)$,
(b) $\mathfrak{I} \models \big(x \neq y\big) \iff \mathfrak{I}(x) \neq \mathfrak{I}(y)$.

Proof. It is enough to observe that since Σ is a partition (and therefore its blocks are nonempty and mutually disjoint), for all $x, y, z \in V$ and $\star \in \{\cup, \setminus\}$ we have:

$$\begin{aligned} \mathfrak{I} \models (x = y \star z) &\iff \bigcup \mathfrak{I}(x) = \bigcup \mathfrak{I}(y) \star \bigcup \mathfrak{I}(z) \\ &\iff \bigcup \mathfrak{I}(x) = \bigcup \big(\mathfrak{I}(y) \star \mathfrak{I}(z)\big) \\ &\iff \mathfrak{I}(x) = \mathfrak{I}(y) \star \mathfrak{I}(z) \end{aligned}$$

and

$$\begin{aligned} \mathfrak{I} \models (x \neq y) &\iff \bigcup \mathfrak{I}(x) \neq \bigcup \mathfrak{I}(y) \\ &\iff \mathfrak{I}(x) \neq \mathfrak{I}(y). \end{aligned}$$

□

Satisfiability of BST-conjunctions can be expressed in purely combinatorial terms by means of *fulfilling maps*.

Definition 1.4. Let Φ be a BST-conjunction, and let $\mathfrak{F} \colon V \to \mathrm{pow}(\mathrm{pow}^+(V))$ be any map, where $V := \mathrm{Vars}(\Phi)$ and $\mathrm{pow}^+(V) := \mathrm{pow}(V) \setminus \{\emptyset\}$. We say that the map $\mathfrak{F}$ fulfills Φ provided that:

(a) $\mathfrak{F}(x) = \mathfrak{F}(y) \star \mathfrak{F}(z)$, for each conjunct $x = y \star z$ in Φ, with $\star \in \{\cup, \setminus\}$;
(b) $\mathfrak{F}(x) \neq \mathfrak{F}(y)$, for each conjunct $x \neq y$ in Φ.

A map $\mathfrak{F}$ satisfying conditions (a) and (b) above is called a fulfilling map for Φ. □

In Section 2.2, the definition of fulfilling maps in the context of the s.p. for BST⊗ will be strengthened so as to take into account also ⊗-literals.

Lemma 1.5. *Any satisfiable* BST*-conjunction admits some fulfilling map.*

Proof. Let Φ be a satisfiable BST-conjunction and let Σ be a partition satisfying Φ via a certain map $\mathfrak{I} \colon V \to \mathrm{pow}(\Sigma)$, where $V := \mathrm{Vars}(\Phi)$. For each $\sigma \in \Sigma$, we put:

$$V_\sigma := \{v \in V \mid \sigma \in \mathfrak{I}(v)\}.$$

Let us define the map $\mathfrak{F}_\mathfrak{I} \colon V \to \mathrm{pow}(\mathrm{pow}^+(V))$ by putting

$$\mathfrak{F}_\mathfrak{I}(x) := \{V_\sigma \mid \sigma \in \mathfrak{I}(x)\}, \qquad \text{for } x \in V.$$

Preliminarily, we observe that

$$V_\sigma \in \mathfrak{F}_\mathfrak{I}(x) \iff \sigma \in \mathfrak{I}(x), \qquad \text{for } \sigma \in \Sigma \text{ and } x \in V. \tag{3}$$

Indeed, if $V_\sigma \in \mathfrak{F}_\mathfrak{I}(x)$, then $V_\sigma = V_{\sigma'}$, for some $\sigma' \in \mathfrak{I}(x)$. But then, since $\sigma \in \mathfrak{I}(v) \iff \sigma' \in \mathfrak{I}(v)$ for all $v \in V$, we have $\sigma \in \mathfrak{I}(x)$.

Thus, for every conjunct $x = y \star z$ in Φ (with $\star \in \{\cup, \setminus\}$), we have:

$$\begin{aligned} \mathfrak{F}_\mathfrak{I}(x) &= \{V_\sigma \mid \sigma \in \mathfrak{I}(x)\} \\ &= \{V_\sigma \mid \sigma \in \mathfrak{I}(y) \star \mathfrak{I}(z)\} && \text{(by Lemma 1.3(a))} \\ &= \{V_\sigma \mid \sigma \in \mathfrak{I}(y)\} \star \{V_\sigma \mid \sigma \in \mathfrak{I}(z)\} && \text{(by (3))} \\ &= \mathfrak{F}_\mathfrak{I}(y) \star \mathfrak{F}_\mathfrak{I}(z). \end{aligned}$$

Likewise, for every conjunct $x \neq y$ in Φ, by Lemma 1.3(b) and (3) we have:

$$\mathfrak{F}_\mathfrak{I}(x) = \{V_\sigma \mid \sigma \in \mathfrak{I}(x)\} \neq \{V_\sigma \mid \sigma \in \mathfrak{I}(y)\} = \mathfrak{F}_\mathfrak{I}(y).$$

Hence, the map $\mathfrak{F}_\mathfrak{I}$ fulfills Φ. □

Lemma 1.6. *If a* BST*-conjunction with n distinct variables admits a fulfilling map, then it is satisfied by* every *partition of size* $2^n - 1$.

Proof. Let Φ be a satisfiable BST-conjunction with n distinct variables fulfilled by a map $\mathfrak{F}\colon V \to \mathrm{pow}(\mathrm{pow}^+(V))$, where $V := \mathrm{Vars}(\Phi)$, and let $\Sigma_{\mathfrak{F}}$ be any partition of size $2^n - 1$. We prove that the partition $\Sigma_{\mathfrak{F}}$ satisfies Φ. Thus, let $\beta\colon \mathrm{pow}^+(V) \twoheadrightarrow \Sigma_{\mathfrak{F}}$ be any bijection from $\mathrm{pow}^+(V)$ onto $\Sigma_{\mathfrak{F}}$ and define the map $\mathfrak{I}_{\mathfrak{F}}\colon V \to \mathrm{pow}(\Sigma_{\mathfrak{F}})$ by setting

$$\mathfrak{I}_{\mathfrak{F}}(x) := \beta[\mathfrak{F}(x)], \qquad \text{for } x \in V.$$

Then, for every literal $x = y \star z$ in Φ (with $\star \in \{\cup, \setminus\}$), we have:

$$\mathfrak{I}_{\mathfrak{F}}(x) \;=\; \beta[\mathfrak{F}(x)] \;=\; \beta[\mathfrak{F}(y) \star \mathfrak{F}(z)] \;=\; \beta[\mathfrak{F}(y)] \star \beta[\mathfrak{F}(z)] \;=\; \mathfrak{I}_{\mathfrak{F}}(y) \star \mathfrak{I}_{\mathfrak{F}}(z).$$

Hence, by Lemma 1.3(a), $\mathfrak{I}_{\mathfrak{F}} \models \big(x = y \star z\big)$.

Similarly, for every conjunct $x \neq y$ in Φ, we have

$$\mathfrak{I}_{\mathfrak{F}}(x) = \beta[\mathfrak{F}(x)] \neq \beta[\mathfrak{F}(y)] = \mathfrak{I}_{\mathfrak{F}}(y),$$

proving that, by Lemma 1.3(b), $\mathfrak{I}_{\mathfrak{F}} \models \big(x \neq y\big)$.

Thus, in conclusion, we have $\mathfrak{I}_{\mathfrak{F}} \models \Phi$, and so the partition $\Sigma_{\mathfrak{F}}$ satisfies Φ. □

Immediate consequences of the preceding lemma are the following results.

Lemma 1.7. *A* BST*-conjunction is satisfiable if and only if it admits a fulfilling map.*

Lemma 1.8. *A* BST*-conjunction with n distinct variables is satisfiable if and only if it is satisfied by* any *partition of size* $2^n - 1$.

In the light of [CU18, Lemma 2.36, p. 42] and Lemma 1.2, Lemmas 1.8 and 1.7 can be strengthened as follows:

Lemma 1.9. *A* BST*-conjunction involving n distinct variables is satisfiable if and only if it is satisfied by* any *partition of size* $n - 1$.

Lemma 1.10. *A* BST*-conjunction over a set V of variables is satisfiable if and only if it is fulfilled by a map* $\mathfrak{F}\colon V \to \mathrm{pow}(\mathrm{pow}^+(V))$ *such that* $|\bigcup \mathfrak{F}[V]| \leqslant |V| - 1$.

The previous two lemmas readily yield that the satisfiability problem for BST-conjunctions can be solved in nondeterministic polynomial time, namely it belongs to the class NP.

As shown in [CDMO19], the satisfiability problem for conjunctions of Boolean literals of the form $t_1 \neq t_2$, where t_1 and t_2 are set terms involving only variables and the set difference operator '$\setminus$', is NP-complete. Therefore, we have:

Lemma 1.11. *The satisfiability problem for* BST*-conjunctions is* NP*-complete.*

1.5.1 Some complexity classes.

We recall that NP (nondeterministic polynomial time) is the set of decision problems whose positive instances are solvable in polynomial time by a nondeterministic Turing machine. Equivalently, NP is the set of all decision problems whose positive instances can be certified in deterministic polynomial time.

Some of the decision problems in NP are harder than the others, in the sense that any deterministic polynomial-time solution for them would yield a deterministic polynomial-time solution to all the decision problems in NP. These are the so-called NP-complete problems. A notable example of an NP-complete problem is the propositional satisfiability problem (SAT).

Similarly, NEXPTIME (nondeterministic exponential time) is the set of decision problems whose positive instances can be solved in exponential time by a nondeterministic Turing machine.

1.6 Dealing also with literals of type $x = y \otimes z$: the unary unordered Cartesian product

It is convenient to introduce a unary variant of the unordered Cartesian product, in analogy with the binary set operators $\cup$ and $\cap$, which are equipped with the following unary variants:

$$\bigcup S \coloneqq \{t \mid t \in s, \text{ for some } s \in S\} \qquad \text{and} \qquad \bigcap S \coloneqq \{t \mid t \in s, \text{ for all } s \in S\}.$$

Specifically, for all sets s and t (not necessarily distinct), we put

$$\otimes\{s, t\} = s \otimes t. \tag{4}$$

The definition of the operator $\otimes(\cdot)$ is well-given, as the binary unordered product is commutative, namely $s \otimes t = t \otimes s$ holds.

Remark 1.12. Though in this paper the operator $\otimes(\cdot)$ will be used only with arguments of the form $\{s, t\}$, for the sake of completeness we extend it to any set S by putting

$$\otimes S \coloneqq \Big\{\{u, v\} \mid u, v \in \bigcup S \text{ and } \{u, v\} \cap s \neq \emptyset \text{ for all } s \in S\Big\}. \tag{5}$$

Plainly, (5) agrees with (4).

In the extended form, the operator $\otimes(\cdot)$ is a variant of the INTERSECTING POWER SET OPERATOR pow^*, where for any set S

$$\mathrm{pow}^*(S) \coloneqq \Big\{t \subseteq \bigcup S \mid t \cap s \neq \emptyset, \text{ for every } s \in S\Big\},$$

introduced in [Can91] in connection with the solution of the satisfiability problem for a fragment of set theory involving the power set and the singleton operators.[6] Specifically, for any set S it holds that $\otimes S = \mathrm{pow}^*_{\{1,2\}}(S)$, where, more in general, for every $I \subseteq \mathbb{N}$ we have

$$\mathrm{pow}^*_I(S) \coloneqq \Big\{t \in \mathrm{pow}^*(S) \mid |t| \in I\Big\}.$$

Likewise, we put

$$\mathrm{pow}_I(S) \coloneqq \Big\{t \in \mathrm{pow}^*(S) \mid |t| \in I\Big\}.$$

Thus, for any $I \subseteq \mathbb{N}$,

- $\mathrm{pow}^*_I(S)$ is the set of all the members of $\mathrm{pow}^*(S)$ whose cardinality belongs to I; and
- $\mathrm{pow}_I(S)$ is the set of all the subsets of S whose cardinality belongs to I. □

Given any partition Σ, it is an easy matter to check that the $\otimes(\cdot)$ operator is injective over $\mathrm{pow}_{\{1,2\}}(\Sigma)$,[7] namely the following equivalence holds

$$\otimes B = \otimes B' \qquad \Longleftrightarrow \qquad B = B' \tag{6}$$

for all $B, B' \in \mathrm{pow}_{\{1,2\}}(\Sigma)$. In fact, the $\otimes(\cdot)$ operator enjoys a stronger injectivity property, as stated in the following lemma.

LEMMA 1.13 (STRONG INJECTIVITY). *Given any partition Σ, for all $B, B' \in \mathrm{pow}_{\{1,2\}}(\Sigma)$, the following equivalence holds:*

$$\otimes B \cap \otimes B' \neq \emptyset \qquad \Longleftrightarrow \qquad B = B'.$$

[6] It is worth noticing that several properties of the operator pow^* are listed in [CU18, pp. 16–20].
[7] According to the Remark 1.12, $\mathrm{pow}_{\{1,2\}}(\Sigma)$ is the set of all the subsets of Σ that contain either one or two elements.

Proof. Let Σ, B, and B' be as in the hypothesis. Plainly, if $B = B'$ then $\otimes B \cap \otimes B' = \otimes B \neq \emptyset$.

For the converse implication, it is convenient to specify the members of B and B'. So, let $B = \{\beta_1, \beta_2\}$ and $B' = \{\beta'_1, \beta'_2\}$, for suitable blocks $\beta_1, \beta_2, \beta'_1, \beta'_2$ of Σ, where it is not excluded that $\beta_1 = \beta_2$ and/or $\beta'_1 = \beta'_2$ may hold. Let us assume that $\otimes B \cap \otimes B' \neq \emptyset$ holds, that is, $(\beta_1 \otimes \beta_2) \cap (\beta'_1 \otimes \beta'_2) \neq \emptyset$, and let $\{s, t\} \in (\beta_1 \otimes \beta_2) \cap (\beta'_1 \otimes \beta'_2)$. Then, for some $i_0, j_0 \in \{1, 2\}$ we have

$$s \in \beta_{i_0},\ t \in \beta_{3-i_0} \quad \text{and} \quad s \in \beta'_{j_0},\ t \in \beta'_{3-j_0}.$$

Thus, $\beta_{i_0} = \beta'_{j_0}$ and $\beta_{3-i_0} = \beta'_{3-j_0}$ hold, and therefore

$$\{\beta_1, \beta_2\} \ = \ \{\beta_{i_0}, \beta_{3-i_0}\} \ = \ \{\beta'_{j_0}, \beta'_{3-j_0}\} \ = \ \{\beta'_1, \beta'_2\},$$

completing the proof of the lemma. □

As proved in the following lemma, the injectivity property (6) can be readily generalized to the image map $\otimes[\cdot]$ induced by the operator $\otimes(\cdot)$ over the set $\mathrm{pow}(\mathrm{pow}_{\{1,2\}}(\Sigma))$, where

$$\otimes[\mathcal{B}] \coloneqq \{\otimes B \mid B \in \mathcal{B}\}, \tag{7}$$

for all $\mathcal{B} \in \mathrm{pow}(\mathrm{pow}_{\{1,2\}}(\Sigma))$.[8]

Lemma 1.14. *Let Σ be a partition. For all $\mathcal{B}, \mathcal{B}' \subseteq \mathrm{pow}_{\{1,2\}}(\Sigma)$, we have*

$$\bigcup \otimes[\mathcal{B}] = \bigcup \otimes[\mathcal{B}'] \quad \Longleftrightarrow \quad \mathcal{B} = \mathcal{B}'.$$

Proof. As in the hypotheses, let Σ be any partition, and let $\mathcal{B}, \mathcal{B}' \subseteq \mathrm{pow}_{\{1,2\}}(\Sigma)$. If $\mathcal{B} = \mathcal{B}'$, then we immediately have $\bigcup \otimes[\mathcal{B}] = \bigcup \otimes[\mathcal{B}']$. On the other hand, if $\mathcal{B} \neq \mathcal{B}'$ and assuming without loss of generality that $\mathcal{B} \not\subseteq \mathcal{B}'$, we can pick $\{b_1, b_2\} \in \mathcal{B} \setminus \mathcal{B}'$. Then $\otimes\{b_1, b_2\} \in \otimes[\mathcal{B}]$ and so $\otimes\{b_1, b_2\} \subseteq \bigcup \otimes[\mathcal{B}]$. We claim that $\otimes\{b_1, b_2\} \not\subseteq \bigcup \otimes[\mathcal{B}']$. Indeed, if for a contradiction $\otimes\{b_1, b_2\} \subseteq \bigcup \otimes[\mathcal{B}']$, then there would exist

$$\{\beta_1, \beta_2\} \in \otimes\{b_1, b_2\} \cap \otimes\{b'_1, b'_2\},$$

for some $\{b'_1, b'_2\} \in \mathcal{B}'$. Hence,

$$\beta_1 \in b_i \cap b'_j \quad \text{and} \quad \beta_2 \in b_{3-i} \cap b'_{3-j}$$

would hold for some $1 \leqslant i,\ j \leqslant 2$. By recalling that b_1, b_2, b'_1, b'_2 are blocks of the partition Σ, we must have $b_i = b'_j$ and $b_{3-i} = b'_{3-j}$, and therefore $\{b_1, b_2\} = \{b'_1, b'_2\} \in \mathcal{B}'$, which contradicts our assumption $\{b_1, b_2\} \notin \mathcal{B}'$. Thus, $\otimes\{b_1, b_2\} \not\subseteq \bigcup \otimes[\mathcal{B}']$ holds as claimed, and therefore we must have $\bigcup \otimes[\mathcal{B}] \neq \bigcup \otimes[\mathcal{B}']$, completing the proof of the lemma. □

The unordered Cartesian operator $\otimes$ enjoys the following distributive property.

Lemma 1.15 (Distributivity). *For all sets S and T, the following identity holds:*

$$\bigcup S \otimes \bigcup T = \bigcup\{s \otimes t \mid s \in S,\ t \in T\}. \tag{8}$$

Proof. Let $u \in \bigcup S \otimes \bigcup T$. Then $u = \{u', u''\}$ for some $u' \in \bigcup S$ and $u'' \in \bigcup T$. Hence, $u' \in \bar{s}$ and $u'' \in \bar{t}$ for some $\bar{s} \in S$ and $\bar{t} \in T$, so that $u = \{u', u''\} \in \bar{s} \otimes \bar{t} \subseteq \bigcup\{s \otimes t \mid s \in S,\ t \in T\}$. Thus,

$$\bigcup S \otimes \bigcup T \subseteq \bigcup\{s \otimes t \mid s \in S,\ t \in T\}. \tag{9}$$

For the converse inclusion, let $u \in \bigcup\{s \otimes t \mid s \in S,\ t \in T\}$. Then $u \in \bar{s} \otimes \bar{t}$, for some $\bar{s} \in S$ and $\bar{t} \in T$, and therefore $u = \{u', u''\}$, for some $u' \in \bar{s}$ and $u'' \in \bar{t}$. Since $\bar{s} \subseteq \bigcup S$ and $\bar{t} \subseteq \bigcup T$, then $u \in \bigcup S \otimes \bigcup T$, and so

$$\bigcup\{s \otimes t \mid s \in S,\ t \in T\} \subseteq \bigcup S \otimes \bigcup T.$$

Together with the converse inclusion (9), the latter yields (8), completing the proof of the lemma. □

[8]We warn the reader that in the rest of the paper we will often use terms of the form $\otimes(\cdot)$ and $\otimes[\cdot]$, whose precise meanings are defined by (4) and (7), respectively.

Before proving our main decidability results for BST⊗-formulae, we sketch the proof (based on [CCP90] and [COP20]) that the extension $\mathrm{BST}\otimes_{\leqslant}$ of BST⊗ with cardinality comparison $|\cdot| \leqslant |\cdot|$ has an undecidable *finite* s.p., by way of a reduction of the celebrated Hilbert's Tenth problem (H10).

1.6.1 Undecidability of the finite s.p. for $\mathrm{BST}\otimes_{\leqslant}$.
The reduction of H10 to the finite s.p. for $\mathrm{BST}\otimes_{\leqslant}$ is based on the fact that, under finiteness, the following relations (empty set, equinumerosity, product and sum of cardinalities, and unitary cardinality) are expressible by $\mathrm{BST}\otimes_{\leqslant}$-formulae:

(i) **empty set:** $x = \emptyset \ \equiv_{\mathrm{Def}} \ x = x \setminus x$;
(ii) **equinumerosity:** $|x| = |y| \ \equiv_{\mathrm{Def}} \ |x| \leqslant |y| \ \wedge \ |y| \leqslant |x|$;
(iii) **product of cardinalities:** $|z| = |x| \cdot |y| \ \equiv_{\mathrm{Def}} \ |z| = |x \otimes y'| \ \wedge \ |y'| = |y| \ \wedge \ y' \cap x = \emptyset$;
(iv) **sum of cardinalities:** $|z| = |x| + |y| \ \equiv_{\mathrm{Def}} \ |z| = |x \cup y'| \ \wedge \ |y'| = |y| \ \wedge \ y' \cap x = \emptyset$;
(v) **unitary cardinality:** $|z| = 1 \ \equiv_{\mathrm{Def}} \ |z| = |z \otimes z| \ \wedge \ z \neq \emptyset$.

By iterating (iii), one can express $|z| = |x|^k$, for any integer constant $k \geqslant 1$, and more in general $|z| = |x_1|^{k_1} \cdot \ldots \cdot |x_r|^{k_r}$, for any integer constants $k_1, \ldots, k_r \geqslant 1$ where $r \geqslant 1$. Since by (v) one can express $|z| = 1$, it is also possible to express $|z| = M(|x_1|, \ldots, |x_r|)$ for any monic monomial $M(X_1, \ldots, X_r)$, even of degree 0. Finally, by iterating (iv), one can express $|z| = N(|x_1|, \ldots, |x_r|)$, for any monomial $N(X_1, \ldots, X_r)$ with any positive integral coefficient, and more in general $|z| = P(|x_1|, \ldots, |x_r|)$, for any polynomial $P(X_1, \ldots, X_r)$ with positive integral coefficients. Thus, given a Diophantine equation

$$D(x_1, \ldots, x_r) = 0, \tag{10}$$

letting $D^+(x_1, \ldots, x_r)$ be the sum of the positive monomials in $D(x_1, \ldots, x_r)$ and $-D^-(x_1, \ldots, x_r)$ the sum of the negative monomials in $D(x_1, \ldots, x_r)$, equation (10) admits a nonnegative integral solution if and only if the corresponding $\mathrm{BST}\otimes_{\leqslant}$-formula

$$|z| = D^+(|x_1|, \ldots, |x_r|) \ \wedge \ |z| = D^-(|x_1|, \ldots, |x_r|) \tag{11}$$

is finitely satisfiable, namely it is satisfied by a finite set assignment over the variables $z, x_1, \ldots, x_r$, plus all the auxiliary set variables implicit in (11).

For instance, the Diophantine equation $X^3 + Y^2 - 3Z - 2 = 0$ admits a nonnegative integral solution if and only if the $\mathrm{BST}\otimes_{\leqslant}$-formula $\bigwedge_{i=1}^{7} \varphi_i$ is finitely satisfiable, where

$\varphi_1 \ \equiv_{\mathrm{Def}} \ |x_2'| = |x' \otimes x| \wedge |x_3| = |x'' \otimes x_2'| \wedge |x'| = |x| \wedge x' \cap x = \emptyset \wedge |x''| = |x| \wedge x'' \cap x_2' = \emptyset$;
$\varphi_2 \ \equiv_{\mathrm{Def}} \ |y_2| = |y' \otimes y| \wedge |y'| = |y| \wedge y' \cap y = \emptyset$;
$\varphi_3 \ \equiv_{\mathrm{Def}} \ |z_2'| = |z' \otimes z| \wedge |z_3| = |z'' \cup z_2'| \wedge |z'| = |z| \wedge z' \cap z = \emptyset \wedge |z''| = |z| \wedge z'' \cap z_2' = \emptyset$;
$\varphi_4 \ \equiv_{\mathrm{Def}} \ |w_2| = |w'' \cup w'| \wedge |w' \otimes w'| = |w'| \wedge w' \neq \emptyset \wedge |w''| = |w'| \wedge w'' \cap w' = \emptyset$;
$\varphi_5 \ \equiv_{\mathrm{Def}} \ |d^+| = |y_2' \cup x_3| \wedge |y_2'| = |y_2| \wedge y_2' \cap x_3 = \emptyset$;
$\varphi_6 \ \equiv_{\mathrm{Def}} \ |d^-| = |w_2' \cup z_3| \wedge |w_2'| = |w_2| \wedge w_2' \cap z_3 = \emptyset$;
$\varphi_7 \ \equiv_{\mathrm{Def}} \ |d^+| = |d^-|$.

Indeed,

φ_1	is equisatisfiable with	$\lvert x_3\rvert = \lvert x\rvert^3$
φ_2	is equisatisfiable with	$\lvert y_2\rvert = \lvert y\rvert^2$
φ_3	is equisatisfiable with	$\lvert z_3\rvert = 3 \cdot \lvert z\rvert$
$\varphi_1 \wedge \varphi_2 \wedge \varphi_5$	is equisatisfiable with	$\lvert d^+\rvert = \lvert x\rvert^3 + \lvert y\rvert^2$
$\varphi_3 \wedge \varphi_4 \wedge \varphi_6$	is finitely equisatisfiable with	$\lvert d^-\rvert = 3 \cdot \lvert z\rvert + 2$
$\bigwedge_{i=1}^{7} \varphi_i$	is finitely equisatisfiable with	$\lvert x\rvert^3 + \lvert y\rvert^2 - 3 \cdot \lvert z\rvert - 2 = 0$.

In view of the undecidability of H10 (cf. [Mat70]), we have the following result:

Theorem 1.16. *The finite s.p. for* $\mathsf{BST}\otimes_{\leqslant}$ *is undecidable.*

2 THE ORDINARY SATISFIABILITY PROBLEM FOR BST⊗-CONJUNCTIONS

Results like those contained in Lemmas 1.8 and 1.9 cannot hold for BST⊗, since literals of type $x = y \otimes z$ put constraints on the internal structure of certain blocks in any partition Σ that satisfies them. Roughly speaking, these are the ⊗-*blocks*, for which we provide next a general characterization that is independent of any particular BST⊗-conjunction. Their collection will form a subpartition of Σ to be denoted $\Sigma_\otimes$.

Specifically, given any fixed ⊗-literal $x = y \otimes z$ satisfied by Σ, the subpartition $\Sigma_\otimes$ will contain exactly all the blocks α in $\mathfrak{I}(x)$, for each partition assignment $\mathfrak{I}\colon \{x, y, z\} \to \mathrm{pow}(\Sigma)$ that satisfies $x = y \otimes z$, namely such that it holds that

$$\bigcup \mathfrak{I}(x) = \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z).$$

It will also be convenient to define a companion collection $\Pi_\otimes \subseteq \Sigma \otimes \Sigma$ of unordered pairs of blocks that includes exactly all the unordered pairs $\{\beta, \gamma\}$ in $\mathfrak{I}(y) \otimes \mathfrak{I}(z)$, for each partition assignment $\mathfrak{I}\colon \{x, y, z\} \to \mathrm{pow}(\Sigma)$ that satisfies $x = y \otimes z$.

Definition 2.1. Given a partition Σ, a subpartition Σ^* of Σ is a ⊗-subpartition if $\bigcup \Sigma^* = \bigcup \otimes[\mathcal{B}]$, for some $\mathcal{B} \subseteq \Sigma \otimes \Sigma$.

We denote by $\Sigma_\otimes$ the largest ⊗-subpartition of Σ and we refer to its elements as the ⊗-blocks of Σ, whereas the remaining blocks are called source blocks. We also denote by $\Pi_\otimes$ *the* subset of $\Sigma \otimes \Sigma$ such that $\bigcup \Sigma_\otimes = \bigcup \otimes[\Pi_\otimes]$. □

Remark 2.2. The definitions of $\Sigma_\otimes$ and $\Pi_\otimes$ are well-given. Concerning $\Sigma_\otimes$, it is enough to show that the set of the ⊗-subpartitions of Σ is closed under union. Thus, let $\{\Sigma_i \mid i \in I\}$ be any set of ⊗-subpartitions of Σ, and let $\{\mathcal{B}_i \mid i \in I\}$ be a corresponding collection of subsets of $\Sigma \otimes \Sigma$ such that $\bigcup \Sigma_i = \bigcup \otimes[\mathcal{B}_i]$, for $i \in I$. Then, we plainly have:

$$\bigcup \bigcup_{i\in I} \Sigma_i \;=\; \bigcup_{i\in I} \bigcup \Sigma_i \;=\; \bigcup_{i\in I} \bigcup \otimes[\mathcal{B}_i] \;=\; \bigcup \bigcup_{i\in I} \otimes[\mathcal{B}_i] \;=\; \bigcup \otimes[\bigcup_{i\in I} \mathcal{B}_i],$$

proving that $\bigcup_{i\in I} \Sigma_i$ is a ⊗-subpartition of Σ, as $\bigcup_{i\in I} \Sigma_i \subseteq \Sigma$ and $\bigcup_{i\in I} \mathcal{B}_i \subseteq \Sigma \otimes \Sigma$. Hence,

$$\Sigma_\otimes \coloneqq \bigcup \big\{\Sigma^* \mid \Sigma^* \text{ is a } \otimes\text{-subpartition of } \Sigma\big\}$$

is the largest ⊗-subpartition of Σ.

Having shown that $\Sigma_\otimes$ is the largest ⊗-subpartition of Σ, then, in particular, $\bigcup \Sigma_\otimes = \bigcup \otimes[\mathcal{B}]$ for some $\mathcal{B} \subseteq \Sigma \otimes \Sigma$. In addition, such $\mathcal{B}$ is unique, by Lemma 1.14. Hence, the set $\Pi_\otimes$ is well-defined as well. □

We illustrate Definition 2.1 with the following example.

Running Example (Part I). Let us consider the partition $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, where

$$\sigma_1 = \big\{\{\emptyset\}\big\}, \quad \sigma_2 = (\mathsf{HF} \otimes \mathsf{HF}) \setminus \sigma_1, \quad \sigma_3 = \mathsf{HF} \setminus (\sigma_1 \cup \sigma_2), \quad \sigma_4 = \big\{\{\emptyset, \mathsf{HF}\}\big\}$$

(we recall that HF is the set of the hereditarily finite sets; see Section 1.2).

We have:

– $\{\sigma_1, \sigma_2\}$ is a ⊗-subpartition of Σ. Indeed, by preliminarily observing that

$$\sigma_1 \cup \sigma_2 = \mathsf{HF} \otimes \mathsf{HF} \qquad \text{and} \qquad \bigcup \{\sigma_1, \sigma_2, \sigma_3\} = \sigma_1 \cup \sigma_2 \cup \sigma_3 = \mathsf{HF},$$

we have

$$\begin{aligned}\sigma_1 \cup \sigma_2 &= \mathsf{HF} \otimes \mathsf{HF} \\ &= \bigcup\{\sigma_1, \sigma_2, \sigma_3\} \otimes \bigcup\{\sigma_1, \sigma_2, \sigma_3\} \\ &= \bigcup\big\{\otimes\{\sigma_i, \sigma_j\} \mid 1 \leqslant i \leqslant j \leqslant 3\big\} \qquad \text{(by Lemma 1.15)} \\ &= \bigcup\otimes[\{\sigma_1, \sigma_2, \sigma_3\} \otimes \{\sigma_1, \sigma_2, \sigma_3\}].\end{aligned}$$

As obviously $\otimes[\{\sigma_1, \sigma_2, \sigma_3\} \otimes \{\sigma_1, \sigma_2, \sigma_3\}] \subseteq \Sigma \otimes \Sigma$, it readily follows that $\{\sigma_1, \sigma_2\}$ is a $\otimes$-subpartition of Σ.

- σ_3 does not belong to any $\otimes$-subpartition of Σ. It is enough to observe that σ_3 contains elements that are not unordered pairs, such as $\big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\big\}$;
- σ_4 does not belong to any $\otimes$-subpartition of Σ. Indeed, since the set HF does not belong to any σ_i, it follows that $\sigma_4 \not\subseteq \bigcup\otimes\big[\big(\bigcup_{1\leqslant i\leqslant 4}\sigma_i\big) \otimes \big(\bigcup_{1\leqslant i\leqslant 4}\sigma_i\big)\big]$.

Hence, $\{\sigma_1, \sigma_2\}$ is the largest $\otimes$-subpartition of Σ. We can therefore conclude that $\Sigma_\otimes = \{\sigma_1, \sigma_2\}$, namely σ_1 and σ_2 are the $\otimes$-blocks of Σ, and $\Pi_\otimes = \{\sigma_1, \sigma_2, \sigma_3\} \otimes \{\sigma_1, \sigma_2, \sigma_3\}$. □

As a consequence of Lemma 1.8 (resp., Lemma 1.9), to test whether a given BST-conjunction Φ with n distinct variables is satisfiable, it is enough to pick any partition with $2^n - 1$ (resp., $n - 1$) blocks and check whether it satisfies Φ.

Instead, as we shall see, in the case of BST$\otimes$-conjunctions with n distinct variables, rather than checking a single partition for satisfiability, one would have to test a whole set of doubly exponential size of partitions with $2^n - 1$ blocks. Remarkably, the partitions in such a set can be conveniently described by special graphs, called $\otimes$-*graphs*, which enjoy a particular connectivity property termed *accessibility*.

Given a BST$\otimes$-conjunction Φ to be tested for satisfiability, in the case of the ordinary s.p. it will be enough to find an accessible $\otimes$-graph which *fulfills* Φ, in the sense that will be soon made precise, whereas for the (hereditarily) finite s.p., besides accessibility and fulfillability, it will be additionally requested that the $\otimes$-graphs admit a "weak" topological order.

Both for the ordinary s.p. and for the (hereditarily) finite s.p., such an approach will yield nondeterministic exponential decision procedures in the number of distinct variables of the input formula.

Next, we provide precise definitions of the notions mentioned above. We begin with $\otimes$-graphs.

2.1 $\otimes$-graphs

The mere characterization of the largest $\otimes$-subpartition $\Sigma_\otimes$ of a given partition Σ and of its accompanying set $\Pi_\otimes \subseteq \Sigma \otimes \Sigma$ such that

$$\bigcup\Sigma_\otimes = \bigcup\otimes[\Pi_\otimes] \tag{12}$$

is not sufficient for our decidability purposes. We need also to characterize the flow of the unordered pairs from the (unordered Cartesian product of the) members of $\Pi_\otimes$ to the members of $\Sigma_\otimes$. From (12), it follows that every member p of any set $\otimes B$, for $B \in \Pi_\otimes$, belongs to some block σ_p in $\Sigma_\otimes$.[9] Such additional information is gathered in the following directed bipartite graph $\mathbb{G}_\Sigma$ (called $\otimes$-*graph*), whose parts are Σ and $\Pi_\otimes$,[10] and whose edges are

(i) $\langle\sigma, B\rangle$, for each $\sigma \in B \in \Pi_\otimes$;
(ii) $\langle B, \sigma\rangle$, for each $B \in \Pi_\otimes$ and $\sigma \in \{\sigma_p \mid p \in \otimes B\}$,
namely for each $B \in \Pi_\otimes$ and for each block $\sigma \in \Sigma_\otimes$ such that

$$\sigma \cap \otimes B \neq \emptyset. \tag{13}$$

[9]And, conversely, every member p of any block σ in $\Sigma_\otimes$ belongs to some $\otimes B$, for $B \in \Pi_\otimes$ (and therefore it is an unordered pair).
[10]In the present introductory overview, we assume that the sets Σ and $\Pi_\otimes$ are disjoint. The upcomoing Definition (2.3) will take care also of the case left out now.

It turns out that the information contained in the graph $\mathbb{G}_\Sigma$ is enough for our purposes. Indeed, it can be shown that if Σ and Σ' are any partitions whose corresponding graphs $\mathbb{G}_\Sigma$ and $\mathbb{G}_{\Sigma'}$, constructed as outlined above, are isomorphic, then Σ and Σ' satisfy the very same BST⊗-formulae, namely

$$\Sigma \models \Phi \quad \Longleftrightarrow \quad \Sigma' \models \Phi$$

holds, for every BST⊗-formula Φ.

In more formal terms, ⊗-graphs are defined as follows.

Definition 2.3 (⊗-graphs). A ⊗-GRAPH $\mathcal{G}$ is a directed bipartite graph whose set of vertices comprises two disjoint parts: a set of PLACES $\mathcal{P}$, such that $\mathcal{P} \cap (\mathcal{P} \otimes \mathcal{P}) = \emptyset$, and a set of ⊗-NODES $\mathcal{N}$, where $\mathcal{N} \subseteq \mathcal{P} \otimes \mathcal{P}$.[11] The edges issuing from each place q are exactly all pairs $\langle q, B\rangle$ such that $q \in B \in \mathcal{N}$: these are the MEMBERSHIP EDGES. The remaining edges of $\mathcal{G}$, called DISTRIBUTION EDGES, go from ⊗-nodes to places. When there is an edge $\langle B, q\rangle$ from a ⊗-node B to a place q, we say that q is a TARGET of B. Every ⊗-node must have at least one target. A place that is a target of some ⊗-node is a ⊗-PLACE. The map $\mathcal{T}$over $\mathcal{N}$ defined by

$$\mathcal{T}(B) \coloneqq \{q \in \mathcal{P} \mid q \text{ is a target of } B\}, \quad \text{for } B \in \mathcal{N},$$

is the TARGET MAP of $\mathcal{G}$, hence we have $\mathcal{T} \colon \mathcal{N} \to \mathrm{pow}^+(\mathcal{P})$. Plainly, a ⊗-graph $\mathcal{G}$ is fully characterized by the set $\mathcal{P}$ of its places and its target map $\mathcal{T}$, since the sets of ⊗-nodes of $\mathcal{G}$ is expressible as $\mathrm{dom}(\mathcal{T})$. When convenient, we shall explicitly write $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T})$ for a ⊗-graph with set of places $\mathcal{P}$, set of ⊗-nodes $\mathcal{N}$, and target map $\mathcal{T}$. The SIZE of a ⊗-graph is the cardinality of its set of places. □

Next, we illustrate how to construct the ⊗-graph $\mathcal{G}_\Sigma$ INDUCED BY A GIVEN A PARTITION Σ. Let $\Sigma_\otimes$ be the largest ⊗-subpartition of Σ, and let $\Pi_\otimes \subseteq \Sigma \otimes \Sigma$ be such that $\bigcup \Sigma_\otimes = \bigcup \otimes[\Pi_\otimes]$.

To begin with, we select a set of places $\mathcal{P}_\Sigma$, of the same cardinality as Σ and such that $\mathcal{P}_\Sigma$ and $\mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma$ are disjoint. Places q in $\mathcal{P}_\Sigma$ are intended to be an abstract representation of the blocks of Σ via a bijection $q \mapsto q^{(\bullet)}$ from $\mathcal{P}_\Sigma$ onto Σ. Then we define the set $\mathcal{N}_\Sigma$ of the ⊗-nodes of $\mathcal{G}_\Sigma$ as the collection of the unordered pairs $\{p, q\}$ of places such that $\{p^{(\bullet)}, q^{(\bullet)}\} \in \Pi_\otimes$, that is, $\mathcal{N}_\Sigma \coloneqq \{B \in \mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma \mid B^{(\bullet)} \in \Pi_\otimes\}$, where the bijection $^{(\bullet)}$ has naturally been extended to any set $B \in \mathcal{P}_\Sigma \otimes \mathcal{P}_\Sigma$ by putting $B^{(\bullet)} \coloneqq \{q^{(\bullet)} \mid q \in B\}$.[12] At this point, we define the vertex set of $\mathcal{G}_\Sigma$ as the union $\mathcal{P}_\Sigma \cup \mathcal{N}_\Sigma$. The disjoint sets $\mathcal{P}_\Sigma$ and $\mathcal{N}_\Sigma$ will form the parts of the bipartite graph $\mathcal{G}_\Sigma$ we are after.

Having defined the vertex set of $\mathcal{G}_\Sigma$, next we describe its edge set. The edges issuing from each place q are exactly all pairs $\langle q, B\rangle$ such that $q \in B \in \mathcal{N}_\Sigma$ (membership edges of $\mathcal{G}_\Sigma$). The remaining edges of $\mathcal{G}_\Sigma$ go from ⊗-nodes to places (these are the distribution edges of $\mathcal{G}_\Sigma$). Specifically, for a ⊗-node B and a place q of $\mathcal{G}_\Sigma$, there is an edge $\langle B, q\rangle$ in $\mathcal{G}_\Sigma$ exactly when

$$q^{(\bullet)} \cap \otimes B^{(\bullet)} \neq \emptyset, \tag{14}$$

in agreement with (13), namely when there is a "flow" of unordered pairs from $\otimes B^{(\bullet)}$ to $q^{(\bullet)}$ (through the edge $\langle B, q\rangle$).

Only places q corresponding to ⊗-blocks $q^{(\bullet)}$ of Σ (namely the blocks in $\mathcal{P}_\Sigma$), hence called ⊗-PLACES and whose collection is denoted by $\mathcal{P}_{\Sigma,\otimes}$, can have incoming edges. This is a consequence of the following lemma.

LEMMA 2.4. *A place q of the ⊗-graph $\mathcal{G}_\Sigma = (\mathcal{P}_\Sigma, \mathcal{N}_\Sigma, \mathcal{T}_\Sigma)$ induced by Σ is a ⊗-place if and only if $q^{(\bullet)}$ is a ⊗-block.*

PROOF. If q is a ⊗-place, then it has some incoming edge $\langle B, q\rangle$ in $\mathcal{G}_\Sigma$, for some $B \in \mathcal{N}_\Sigma$. Hence $B^{(\bullet)} \in \Pi_\otimes$ and $q^{(\bullet)} \cap \otimes B^{(\bullet)} \neq \emptyset$ hold, and therefore

$$\emptyset \;\neq\; q^{(\bullet)} \cap \otimes B^{(\bullet)} \;\subseteq\; q^{(\bullet)} \cap \bigcup \otimes[\Pi_\otimes] \;=\; q^{(\bullet)} \cap \bigcup \Sigma_\otimes.$$

[11] The terms *place* and *node* have been originally introduced in [FOS80] and [CFS85], respectively.

[12] Hence, ⊗-nodes $\{p, q\}$ in $\mathcal{N}_\Sigma$ are intended to represent the unordered pairs $\{p^{(\bullet)}, q^{(\bullet)}\}$ of the blocks represented by their places.

Thus, $q^{(\bullet)} \cap \bigcup \Sigma_{\otimes} \neq \emptyset$, so that $q^{(\bullet)} \in \Sigma_{\otimes}$, namely $q^{(\bullet)}$ is a $\otimes$-block.

Conversely, if $q^{(\bullet)}$ is a $\otimes$-block, then $q^{(\bullet)} \in \Sigma_{\otimes}$, and therefore

$$q^{(\bullet)} \cap \bigcup \otimes[\Pi_{\otimes}] \;=\; q^{(\bullet)} \cap \bigcup \Sigma_{\otimes} \;=\; q^{(\bullet)} \neq \emptyset.$$

Thus, $q^{(\bullet)} \cap \otimes B^{(\bullet)} \neq \emptyset$, for some $B^{(\bullet)} \in \Pi_{\otimes}$. Hence, B is a $\otimes$-node of $\mathcal{G}_{\Sigma}$ and, by (14), $\langle B, q\rangle$ is a distribution edge of $\mathcal{G}_{\Sigma}$, proving that the place q has incoming edges and so it is a $\otimes$-place. □

Similarly, every $\otimes$-node of $\mathcal{G}_{\Sigma}$ must have some outgoing edges. Indeed, if B is any $\otimes$-node of $\mathcal{G}_{\Sigma}$, then $B^{(\bullet)} \in \Pi_{\otimes}$, and since

$$\emptyset \;\neq\; \otimes B^{(\bullet)} \;\subseteq\; \bigcup \otimes[\Pi_{\otimes}] \;=\; \bigcup \Sigma_{\otimes},$$

we have $\otimes B^{(\bullet)} \cap q^{(\bullet)} \neq \emptyset$ for some $q^{(\bullet)} \in \Sigma_{\otimes}$, so that, by (14), the $\otimes$-place q is a target of the $\otimes$-node B.

The target map $\mathcal{T}_{\Sigma}$ of $\mathcal{G}_{\Sigma}$ is plainly defined by

$$\mathcal{T}_{\Sigma}(B) \coloneqq \{q \in \mathcal{P}_{\Sigma,\otimes} \mid q^{(\bullet)} \cap \otimes B^{(\bullet)} \neq \emptyset\}, \quad \text{for } B \in \mathcal{N}_{\Sigma}.$$

Running Example (Part II). We construct the $\otimes$-graph $\mathcal{G}_{\Sigma}$ induced by the partition $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ in Part I, where

$$\sigma_1 = \big\{\{\emptyset\}\big\}, \quad \sigma_2 = (\mathsf{HF} \otimes \mathsf{HF}) \setminus \sigma_1, \quad \sigma_3 = \mathsf{HF} \setminus (\sigma_1 \cup \sigma_2), \quad \sigma_4 = \big\{\{\emptyset, \mathsf{HF}\}\big\}.$$

Let $\mathcal{P}_{\Sigma} \coloneqq \{q_1, q_2, q_3, q_4\}$, where the q_i's are pairwise distinct and none of them is an unordered pair, so that $\mathcal{P}_{\Sigma} \cap (\mathcal{P}_{\Sigma} \otimes \mathcal{P}_{\Sigma}) = \emptyset$. Also let $q \mapsto q^{(\bullet)}$ be the bijection from $\mathcal{P}_{\Sigma}$ onto Σ such that $q_i^{(\bullet)} = \sigma_i$, for $i = 1, 2, 3, 4$. Recalling from Part I that $\Sigma_{\otimes} = \{\sigma_1, \sigma_2\}$ and $\Pi_{\otimes} = \{\sigma_1, \sigma_2, \sigma_3\} \otimes \{\sigma_1, \sigma_2, \sigma_3\}$, the set of the $\otimes$-places of $\mathcal{G}_{\Sigma}$ is $\mathcal{P}_{\Sigma,\otimes} = \{q_1, q_2\}$ and the set of the $\otimes$-nodes of $\mathcal{G}_{\Sigma}$ is $\mathcal{N}_{\Sigma} = \{q_1, q_2, q_3\} \otimes \{q_1, q_2, q_3\}$. The membership edges of $\mathcal{G}_{\Sigma}$ are

$$\begin{array}{ccccc} \langle q_1, \{q_1\}\rangle, & \langle q_2, \{q_2\}\rangle, & \langle q_3, \{q_3\}\rangle, & \langle q_1, \{q_1, q_2\}\rangle, & \langle q_2, \{q_1, q_2\}\rangle, \\ \langle q_1, \{q_1, q_3\}\rangle, & \langle q_3, \{q_1, q_3\}\rangle, & \langle q_2, \{q_2, q_3\}\rangle, & \langle q_3, \{q_2, q_3\}\rangle; & \end{array}$$

and the distribution egdes of $\mathcal{G}_{\Sigma}$ are

$$\langle \{q_1\}, q_2\rangle, \quad \langle \{q_1, q_2\}, q_2\rangle \quad \langle \{q_1, q_3\}, q_2\rangle \quad \langle \{q_2, q_3\}, q_2\rangle, \quad \langle \{q_3\}, q_1\rangle, \quad \langle \{q_3\}, q_2\rangle.$$

Finally, the target map $\mathcal{T}_{\Sigma}$ is:

$$\begin{array}{ccc} \mathcal{T}_{\Sigma}(\{q_1\}) = \{q_2\}, & \mathcal{T}_{\Sigma}(\{q_1, q_2\}) = \{q_2\}, & \mathcal{T}_{\Sigma}(\{q_1, q_3\}) = \{q_2\}, \\ \mathcal{T}_{\Sigma}(\{q_2, q_3\}) = \{q_2\}, & \mathcal{T}_{\Sigma}(\{q_3\}) = \{q_1, q_2\}. & \end{array}$$

The $\otimes$-graph $\mathcal{G}_{\Sigma}$ of the partition Σ is shown in Fig. 1. Distribution edges are in black, whereas membership edges are grayed out. Notice that the graph $\mathcal{G}_{\Sigma}$ is cyclic, due to the presence of the "clover" centered in the place q_2, and contains an isolated vertex, namely place q_4. □

2.1.1 *Accessible $\otimes$-graphs.*

Only *accessible* $\otimes$-graphs are relevant for our decidability results.

Definition 2.5 (Accessible $\otimes$-graphs). A place of a $\otimes$-graph $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T})$ is a source place if it has no incoming edges. The remaining places, namely those with incoming edges, are called $\otimes$-places. We denote by $\mathcal{P}_{\otimes}$ the set of the $\otimes$-places of $\mathcal{G}$.

A place of $\mathcal{G}$ is accessible (from the source places of $\mathcal{G}$) if either it is a source place or, recursively, it is the target of some node of $\mathcal{G}$ whose places are *all* accessible from the source places of $\mathcal{G}$. Finally, a $\otimes$-graph is accessible when all its places are accessible.[13] □

[13]Thus, a $\otimes$-graph with no source places is trivially not accessible.

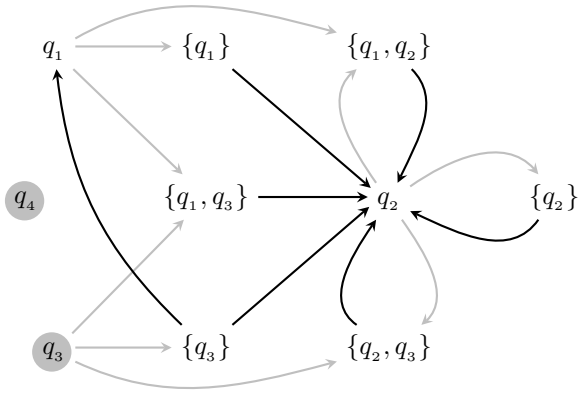

Fig. 1. The ⊗-graph $\mathcal{G}_\Sigma$ of the partition Σ in Part II of our running example. Distribution edges are shown in black, whereas membership edges have been grayed out. Source places are in a gray field.

The following result holds.

Lemma 2.6. *The ⊗-graph $\mathcal{G}_\Sigma$ induced by a given partition Σ is accessible.*

Proof. Let $\mathcal{G}_\Sigma = (\mathcal{P}_\Sigma, \mathcal{N}_\Sigma, \mathcal{T}_\Sigma)$ be the ⊗-graph induced by the partition Σ via a given bijection $q \mapsto q^{(\bullet)}$ from $\mathcal{P}_\Sigma$ onto Σ.

For a contradiction, let us assume that $\mathcal{G}_\Sigma$ is not accessible. Among the non-accessible places of $\mathcal{G}_\Sigma$, we select a place $q \in \mathcal{P}_\Sigma$ whose corresponding block $q^{(\bullet)} \in \Sigma$ contains an element s of smallest rank. Plainly, q must be a ⊗-place, because otherwise it would be a source place, which is trivially accessible. Thus, by Lemma 2.4, $q^{(\bullet)}$ must be a ⊗-block, and therefore $q^{(\bullet)} \subseteq \bigcup \Sigma_\otimes = \bigcup \otimes[\Pi_\otimes]$, , where we recall that $\Pi_\otimes$ is the subset of $\Sigma \otimes \Sigma$ such that $\bigcup \Sigma_\otimes = \bigcup \otimes[\Pi_\otimes]$ (see Definition 2.1), and so $s \in \bigcup \otimes[\Pi_\otimes]$. Hence, $s = \{s_1, s_2\} \in \otimes B^{(\bullet)}$, for some ⊗-node $B = \{q_1, q_2\}$ such that $s_1 \in q_1^{(\bullet)}$ and $s_2 \in q_2^{(\bullet)}$, and therefore $q \in \mathcal{T}_\Sigma(B)$. Since $q_1^{(\bullet)}$ and $q_2^{(\bullet)}$ contain elements of rank strictly less than the rank of s, the places q_1 and q_2 must be accessible. Thus, after all, the place q would be one of the targets of a node whose places are both accessible, and therefore it would be accessible, contradicting our assumption. Hence, $\mathcal{G}_\Sigma$ is accessible. □

Running Example (Part III). We check that the ⊗-graph $\mathcal{G}_\Sigma$ in Fig. 1, induced by the partition Σ in Part I, is accessible, as it should be in view of Lemma 2.6. Indeed,

- q_3 and q_4 are source places, so they are accessible;
- the place q_1 is accessible, because of the edge $\langle \{q_3\}, q_1 \rangle$; and
- the place q_2 is accessible, because of the edge $\langle \{q_3\}, q_2 \rangle$ (but also thanks to the edges $\langle \{q_1\}, q_2 \rangle$ and $\langle \{q_1, q_3\}, q_2 \rangle$). □

2.2 Fulfillment by an accessible ⊗-graph

Our next task is to figure out which additional properties, besides accessibility, are enjoyed by the ⊗-graph $\mathcal{G}_\Sigma$ induced by a partition Σ (via a certain bijection $q \mapsto q^{(\bullet)}$ from $\mathcal{P}_\Sigma$ onto Σ) that satisfies a given BST⊗-conjunction Φ.

Thus, let us assume that Σ satisfies a conjunction Φ via a partition assignment $\mathfrak{I}\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\Sigma)$. Our sought-after properties will pertain to the abstraction $\mathfrak{F}_\Sigma\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P}_\Sigma)$ of the map $\mathfrak{I}$, which is defined by

$$\mathfrak{F}_\Sigma(x) \coloneqq \{q \in \mathcal{P}_\Sigma \mid q^{(\bullet)} \in \mathfrak{I}(x)\}, \qquad \text{for } x \in \mathrm{Vars}(\Phi). \tag{15}$$

For each literal in Φ, we shall derive suitable effective conditions on the map $\mathfrak{F}_\Sigma$ that are necessary for the satisfiability of Φ. Subsequently, in Section 2.3, we shall prove that such conditions are also sufficient for the satisfiability of Φ, thereby proving that the s.p. for BST$\otimes$ can be solved algorithmically.

To begin with, we show that, for every Boolean literal $x = y \star z$ in Φ (with $\star \in \{\cup, \setminus\}$), we have

(a) $\mathfrak{F}_\Sigma(x) = \mathfrak{F}_\Sigma(y) \star \mathfrak{F}_\Sigma(z)$.

Indeed, since $\mathfrak{I} \models x = y \star z$, by Lemma 1.3 we have $\mathfrak{I}(x) = \mathfrak{I}(y) \star \mathfrak{I}(z)$, and therefore by (15) we obtain $\mathfrak{F}_\Sigma(x) = \mathfrak{F}_\Sigma(y) \star \mathfrak{F}_\Sigma(z)$.

Similarly, for every literal $x \neq y$ in Φ, we have

(b) $\mathfrak{F}_\Sigma(x) \neq \mathfrak{F}_\Sigma(y)$.

Indeed, in this case we have $\mathfrak{I} \models x \neq y$, which yields $\mathfrak{F}_\Sigma(x) \neq \mathfrak{F}_\Sigma(y)$, again by (15) and Lemma 1.3.

The situation for literals $x = y \otimes z$ is more complex. Specifically, we prove that the following conditions hold for each such literal in Φ:

(c_1) $\{v, \zeta\} \in \mathcal{N}_\Sigma$ and $\emptyset \neq \mathcal{T}_\Sigma(\{v, \zeta\}) \subseteq \mathfrak{F}_\Sigma(x)$, for all $v \in \mathfrak{F}_\Sigma(y)$ and $\zeta \in \mathfrak{F}_\Sigma(z)$;
(c_2) $\mathfrak{F}_\Sigma(x) \subseteq \bigcup \mathcal{T}_\Sigma[\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z)]$;
(c_3) $\bigcup \mathcal{T}_\Sigma[\mathcal{N}_\Sigma \setminus (\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z))] \cap \mathfrak{F}_\Sigma(x) = \emptyset$.

Concerning (c_1), since $\mathfrak{I} \models x = y \otimes z$, then

$$\bigcup \mathfrak{I}(x) = \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z) \tag{16}$$

holds. Hence, $\mathfrak{I}(x)$ is a $\otimes$-subpartition of Σ and so $\mathfrak{I}(x) \subseteq \Sigma_\otimes$. Indeed,

$$\begin{aligned} \bigcup \mathfrak{I}(x) &= \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z) && \text{(by (16))} \\ &= \bigcup \{\sigma \otimes \tau \mid \sigma \in \mathfrak{I}(y), \tau \in \mathfrak{I}(z)\} && \text{(by Lemma 1.15)} \\ &= \bigcup \otimes[\mathfrak{I}(y) \otimes \mathfrak{I}(z)]. \end{aligned}$$

Next, let $v \in \mathfrak{F}_\Sigma(y)$ and $\zeta \in \mathfrak{F}_\Sigma(z)$. Then $v^{(\bullet)} \subseteq \bigcup \mathfrak{I}(y)$ and $\zeta^{(\bullet)} \subseteq \bigcup \mathfrak{I}(z)$, and consequently, by (16), $v^{(\bullet)} \otimes \zeta^{(\bullet)} \subseteq \bigcup \mathfrak{I}(x)$. In addition, since $\emptyset \neq v^{(\bullet)} \otimes \zeta^{(\bullet)} \subseteq \bigcup \mathfrak{I}(x)$, there exists some $q^{(\bullet)} \in \mathfrak{I}(x)$ such that $q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset$. Conversely, if $q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset$ for some $q \in \mathcal{P}_\Sigma$, then

$$q^{(\bullet)} \cap \bigcup \mathfrak{I}(x) = q^{(\bullet)} \cap \big(\bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z)\big) \neq \emptyset.$$

Hence $q^{(\bullet)} \cap \bigcup \mathfrak{I}(x) \subseteq \Sigma_\otimes$, namely $q^{(\bullet)}$ is a $\otimes$-block, and so q is a $\otimes$-place. Thus,

$$\emptyset \neq \big\{q^{(\bullet)} \in \Sigma \mid q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset\big\} \subseteq \mathfrak{I}(x),$$

which yields

$$\emptyset \neq \big\{q \in \mathcal{P}_\Sigma \mid q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset\big\} \subseteq \mathfrak{F}_\Sigma(x) \cap \mathcal{P}_{\Sigma,\otimes},$$

and therefore

$$\{v, \zeta\} \in \mathcal{N}_\Sigma \quad \text{ and } \quad \emptyset \neq \mathcal{T}_\Sigma(\{v, \zeta\}) = \big\{q \in \mathcal{P}_{\Sigma,\otimes} \mid q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset\big\} \subseteq \mathfrak{F}_\Sigma(x).$$

As for (c_2), let $q \in \mathfrak{F}_\Sigma(x)$, so that $q^{(\bullet)} \subseteq \bigcup \mathfrak{I}(x)$. Let $s \in q^{(\bullet)}$. Hence, by (16), $s \in \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z)$, and therefore $s \in v^{(\bullet)} \otimes \zeta^{(\bullet)}$, for some $v, \zeta \in \mathcal{P}_\Sigma$ such that $v^{(\bullet)} \in \mathfrak{I}(y)$ and $\zeta^{(\bullet)} \in \mathfrak{I}(z)$. Thus, $v \in \mathfrak{F}_\Sigma(y)$ and $\zeta \in \mathfrak{F}_\Sigma(z)$. Since $q^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset$ and $q^{(\bullet)} \in \Sigma_\otimes$, we have $q \in \mathcal{T}_\Sigma(\{v, \zeta\}) \subseteq \bigcup \mathcal{T}_\Sigma[\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z)]$. Hence, we have

$$\mathfrak{F}_\Sigma(x) \subseteq \bigcup \mathcal{T}_\Sigma[\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z)].$$

Finally, concerning (c_3), let us assume for a contradiction that there exists some $q \in \bigcup \mathcal{T}_\Sigma[\mathcal{N}_\Sigma \setminus (\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z))] \cap \mathfrak{F}_\Sigma(x)$. Hence, $q \in \mathcal{T}_\Sigma(A)$, for some $A \in \mathcal{N}_\Sigma \setminus (\mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z))$. Since $q^{(\bullet)} \cap \otimes A^{(\bullet)} \neq \emptyset$ and $q^{(\bullet)} \subseteq \bigcup \mathfrak{I}(x)$, by (16) we have $\otimes A^{(\bullet)} \cap (\bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z)) \neq \emptyset$, and so $\otimes A^{(\bullet)} \cap (v^{(\bullet)} \otimes \zeta^{(\bullet)}) \neq \emptyset$, for some $v^{(\bullet)} \in \mathfrak{I}(y)$ and $\zeta^{(\bullet)} \in \mathfrak{I}(z)$. Thus, by Lemma 1.13 we have $A^{(\bullet)} = \{v^{(\bullet)}, \zeta^{(\bullet)}\}$, and therefore $A = \{v, \zeta\} \in \mathfrak{F}_\Sigma(y) \otimes \mathfrak{F}_\Sigma(z)$, which is a contradiction, thus proving (c_3).

We can recap what we have just established by saying that the accessible $\otimes$-graph induced by any partition Σ satisfying a given BST$\otimes$-conjunction Φ *fulfills* Φ, according to the following definition.

Definition 2.7 (Fulfillment by an accessible $\otimes$-graph). An accessible $\otimes$-graph $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T})$ FULFILLS a given BST$\otimes$-conjunction Φ provided that there exists a map $\mathfrak{F} \colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P})$ (called a $\mathcal{G}$-FULFILLING MAP FOR Φ) such that the following conditions are satisfied:

(a) $\mathfrak{F}(x) = \mathfrak{F}(y) \star \mathfrak{F}(z)$, for every conjunct $x = y \star z$ in Φ, where $\star \in \{\cup, \setminus\}$;
(b) $\mathfrak{F}(x) \neq \mathfrak{F}(y)$, for every conjunct $x \neq y$ in Φ;
(c) for every conjunct $x = y \otimes z$ in Φ,
(c_1) $\{v, \zeta\} \in \mathcal{N}$ and $\emptyset \neq \mathcal{T}(\{v, \zeta\}) \subseteq \mathfrak{F}(x)$, for all $v \in \mathfrak{F}(y)$ and $\zeta \in \mathfrak{F}(z)$;
(c_2) $\mathfrak{F}(x) \subseteq \bigcup \mathcal{T}[\mathfrak{F}(y) \otimes \mathfrak{F}(z)]$;
(c_3) $\bigcup \mathcal{T}[\mathcal{N} \setminus (\mathfrak{F}(y) \otimes \mathfrak{F}(z))] \cap \mathfrak{F}(x) = \emptyset$. □

In summary, we have proved the following result:

LEMMA 2.8. *The accessible $\otimes$-graph induced by a partition satisfying a given* BST$\otimes$*-conjunction Φ fulfills Φ.*

As an immediate consequence, we have:

COROLLARY 2.9. *A satisfiable* BST$\otimes$*-conjunction with n variables is fulfilled by an accessible $\otimes$-graph of size (at most) $2^n - 1$.*

PROOF. Let Φ be a satisfiable BST$\otimes$-conjunction with n variables. As stated in Lemma 1.2, Φ is satisfied by a partition Σ with exactly $2^n - 1$ blocks. Thus, the $\otimes$-graph $\mathcal{G}_\Sigma$ induced by Σ has size $2^n - 1$ and, by Lemmas 2.6 and 2.8, it is accessible and fulfills Φ. □

RUNNING EXAMPLE (PART IV). Let Φ be the following BST$\otimes$-conjunction:

$$y = y \setminus y \quad \wedge \quad v = w \setminus x \quad \wedge \quad v \neq y \quad \wedge \quad x \neq y \quad \wedge \quad y = z \setminus x \quad \wedge \quad z = x \otimes x,$$

and let again $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be the partition in our running example, with

$$\sigma_1 = \big\{\{\emptyset\}\big\}, \quad \sigma_2 = (\mathsf{HF} \otimes \mathsf{HF}) \setminus \sigma_1, \quad \sigma_3 = \mathsf{HF} \setminus (\sigma_1 \cup \sigma_2), \quad \sigma_4 = \big\{\{\emptyset, \mathsf{HF}\}\big\}.$$

Let $\mathfrak{I} \colon V \to \mathrm{pow}(\Sigma)$ be the following partition assignment over $V := \{v, w, x, y, z\}$:

$$\mathfrak{I}(v) = \{\sigma_4\}, \quad \mathfrak{I}(w) = \{\sigma_2, \sigma_4\}, \quad \mathfrak{I}(x) = \{\sigma_1, \sigma_2, \sigma_3\}, \quad \mathfrak{I}(y) = \emptyset, \quad \mathfrak{I}(z) = \{\sigma_1, \sigma_2\}.$$

Also, let $M_\mathfrak{I}$ be the set assignment over V induced by $\mathfrak{I}$:

$$M_\mathfrak{I} v = \big\{\{\emptyset, \mathsf{HF}\}\big\}, \qquad M_\mathfrak{I} w = \big((\mathsf{HF} \otimes \mathsf{HF}) \setminus \{\{\emptyset\}\}\big) \cup \big\{\{\emptyset, \mathsf{HF}\}\big\}, \qquad M_\mathfrak{I} x = \mathsf{HF},$$
$$M_\mathfrak{I} y = \emptyset, \qquad M_\mathfrak{I} z = \mathsf{HF} \otimes \mathsf{HF}.$$

It it an easy matter to check that $M_\mathfrak{I} \models \Phi$. Hence, $\Sigma, \mathfrak{I} \models \Phi$, namely the partition Σ satisfies Φ via the map $\mathfrak{I}$.

Let $\mathcal{G}_\Sigma$ be the $\otimes$-graph induced by Σ via the bijection $q \mapsto q^{(\bullet)}$ from $\mathcal{P}_\Sigma$ onto Σ defined in Part II (see Fig. 1). The partition assignment $\mathfrak{I}$ induces the following map $\mathfrak{F}_\Sigma\colon \{v, w, x, y, z\} \to \mathrm{pow}(\mathcal{P}_\Sigma)$:

$$\mathfrak{F}_\Sigma(v) = \{q_4\}, \quad \mathfrak{F}_\Sigma(w) = \{q_2, q_4\}, \quad \mathfrak{F}_\Sigma(x) = \{q_1, q_2, q_3\}, \quad \mathfrak{F}_\Sigma(y) = \emptyset, \quad \mathfrak{F}_\Sigma(z) = \{q_1, q_2\}.$$

Next we check that the $\otimes$-graph $\mathcal{G}_\Sigma$ fulfills the conjunction Φ via the map $\mathfrak{F}_\Sigma$. Plainly, we have:

$$\mathfrak{F}_\Sigma(v) = \mathfrak{F}_\Sigma(w) \setminus \mathfrak{F}_\Sigma(x), \qquad \mathfrak{F}_\Sigma(v) \neq \mathfrak{F}_\Sigma(y) \neq \mathfrak{F}_\Sigma(x), \qquad \mathfrak{F}_\Sigma(y) = \mathfrak{F}_\Sigma(z) \setminus \mathfrak{F}_\Sigma(x).$$

hold. Hence the map $\mathfrak{F}_\Sigma$ indeed satisfies conditions (a) and (b) of Definition 2.7.

Next we check that also condition (c) of Definition 2.7 is satisfied.

We have just one $\otimes$-literal in Φ, namely $z = x \otimes x$. Recalling from Part II that

$$\mathcal{T}_\Sigma(\{q_1\}) = \{q_2\}, \qquad \mathcal{T}_\Sigma(\{q_1, q_2\}) = \{q_2\}, \qquad \mathcal{T}_\Sigma(\{q_1, q_3\}) = \{q_2\},$$
$$\mathcal{T}_\Sigma(\{q_2, q_3\}) = \{q_2\}, \qquad \mathcal{T}_\Sigma(\{q_3\}) = \{q_1, q_2\},$$

it is immediate to check that

$$\{v, \zeta\} \in \mathcal{N}_\Sigma \quad \text{and} \quad \emptyset \neq \{q_2\} \subseteq \mathcal{T}_\Sigma(\{v, \zeta\}) \subseteq \{q_1, q_2\} = \mathfrak{F}_\Sigma(z),$$

for all $v, \zeta \in \mathfrak{F}_\Sigma(x)$, so the sub-condition ($c_1$) of Definition 2.7 is verified.

Concerning the sub-condition (c_2), we have to check that $\mathfrak{F}_\Sigma(z) \subseteq \bigcup \mathcal{T}_\Sigma\big[\mathfrak{F}_\Sigma(x) \otimes \mathfrak{F}_\Sigma(x)\big]$ holds. Plainly, $\mathcal{T}_\Sigma\big[\mathfrak{F}_\Sigma(x) \otimes \mathfrak{F}_\Sigma(x)\big] = \big\{\{q_2\}, \{q_1, q_2\}\big\}$, and so

$$\mathfrak{F}_\Sigma(z) = \{q_1, q_2\} = \bigcup \mathcal{T}_\Sigma\big[\mathfrak{F}_\Sigma(x) \otimes \mathfrak{F}_\Sigma(x)\big].$$

Hence, also sub-condition (c_2) is satisfied.

Finally, we observe that sub-condition (c_3) is vacuously true, since $\mathcal{N}_\Sigma \setminus \big(\mathfrak{F}_\Sigma(x) \otimes \mathfrak{F}_\Sigma(x)\big) = \emptyset$. Thus condition (c) of Definition 2.7 is satisfied too, and therefore the $\otimes$-graph $\mathcal{G}_\Sigma$ indeed fulfills Φ via the map $\mathfrak{F}_\Sigma$. □

2.3 Construction process

Corollary 2.9 provides a necessary condition for a BST⊗-conjunction Φ with n distinct variables to be satisfiable. We also observe that such condition can be effectively checked, as there is a bounded number of accessible $\otimes$-graphs of size $2^n - 1$ and for each of them one can effectively check whether it fulfills the given BST⊗-conjunction Φ.

We prove next that the condition in Corollary 2.9 is not only necessary but also sufficient for a BST⊗-conjunction Φ with n distinct variables to be satisfiable, thereby showing that the s.p. for BST⊗-formulae is decidable.

Lemma 2.10. *If a* BST⊗*-conjunction is fulfilled by an accessible* $\otimes$*-graph, then it is satisfiable.*

Proof. Let $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T})$ be an accessible $\otimes$-graph, and let us assume that $\mathcal{G}$ fulfills a given BST⊗-conjunction Φ via the map $\mathfrak{F}\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P})$.

To each place $q \in \mathcal{P}$, we associate a set $q^{(\bullet)}$, initially empty. Then, by suitably exploiting the $\otimes$-graph $\mathcal{G}$ as a kind of flow graph, we shall show that the sets $q^{(\bullet)}$ can be monotonically extended by a (possibly infinite) *construction process* (comprising a finite *initialization phase* and a subsequent, possibly infinite, *stabilization phase*) in such a way that the following properties hold:

(P1) After each step, the sets $q^{(\bullet)}$ are pairwise disjoint.
(P2) At the end of the initialization phase all the $q^{(\bullet)}$'s are nonempty (and pairwise disjoint). Thus, after each step in the subsequent stabilization phase, the sets $q^{(\bullet)}$, with $q \in \mathcal{P}$, form a partition equinumerous with $\mathcal{P}$.

(P3) After each step in the stabilization phase, the inclusion

$$q^{(\bullet)} \subseteq \bigcup \{ \otimes A^{(\bullet)} \mid A \in \mathcal{T}^{-1}(q) \}$$

holds, for each $\otimes$-place $q \in \mathcal{P}_\otimes$, where, as before, we are using the notation $B^{(\bullet)} := \{p^{(\bullet)} \mid p \in B\}$ for $B \in \mathcal{N}$.

(P4) At the end of the construction process, we have

$$\otimes A^{(\bullet)} \subseteq \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\},$$

for each $\otimes$-node A.[14]

Subsequently, we shall prove that the properties (P1)–(P4) together with the conditions (a)–(c) of Definition 2.7, characterizing the fulfilling $\otimes$-graph $\mathcal{G}$, allow one to show that the partition $\{q^{(\bullet)} \mid q \in \mathcal{P}\}$ resulting from the above construction process satisfies our conjunction Φ.

The initialization and stabilization phases of our construction process consist of the following steps.

Initialization phase:

(I_1) To begin with, let $\{\overline{q} \mid q \in \mathcal{P} \setminus \mathcal{P}_\otimes\}$ be any partition equinumerous with the set $\mathcal{P} \setminus \mathcal{P}_\otimes$ of the source places of $\mathcal{G}$, where each block $\overline{q}$, for $q \in \mathcal{P} \setminus \mathcal{P}_\otimes$, is a hereditarily finite set of cardinality (at least) $\max(2\,|\mathcal{P}_\otimes|, 1)$ and whose members all have cardinality *strictly* greater than 2,[15] and put

$$q^{(\bullet)} := \begin{cases} \overline{q} & \text{if } q \in \mathcal{P} \setminus \mathcal{P}_\otimes \\ \emptyset & \text{if } q \in \mathcal{P}_\otimes. \end{cases}$$

We say that a place $q \in \mathcal{P}$ has already been *initialized* when $q^{(\bullet)} \neq \emptyset$. Likewise, a $\otimes$-node $A \in \mathcal{N}_\otimes$ has been *initialized* when its places have all been initialized. During the initialization phase, an initialized $\otimes$-node $A \in \mathcal{N}_\otimes$ is said to be READY if it has some target that has not been yet initialized.

(I_2) While there are places in $\mathcal{P}$ not yet initialized, pick any ready node $A \in \mathcal{N}$ and distribute *evenly* all the members of $\otimes A^{(\bullet)}$ among all of its targets.

More precisely, for a ready node A with k distinct targets $q_1, \ldots, q_k$, the set $\otimes A^{(\bullet)}$ is partitioned into k blocks $\Delta_1, \ldots, \Delta_k$ such that

$$\left\lfloor \left| \otimes A^{(\bullet)} \right| / k \right\rfloor \leqslant |\Delta_1|, \ldots, |\Delta_1| \leqslant \left\lfloor \left| \otimes A^{(\bullet)} \right| / k \right\rfloor + 1$$

and then the following k assignments

$$q_1^{(\bullet)} := q_1^{(\bullet)} \cup \Delta_1\,, \quad \ldots\,, \quad q_k^{(\bullet)} := q_k^{(\bullet)} \cup \Delta_k$$

are executed.

The accessibility of $\mathcal{G}$ guarantees that the while-loop (I_2) terminates in a finite number of iterations.

At the end of the initialization phase all the $q^{(\bullet)}$'s are nonempty, so property (P2) holds. Indeed, if there were no $\otimes$-places, then all places would be initialized just after step (I_1), and so all the $q^{(\bullet)}$'s would be nonempty sets. On the other hand, if $|\mathcal{P}_\otimes| > 0$, then at the end of the while-loop (I_2) we shall have $|q^{(\bullet)}| \geqslant 2\,|\mathcal{P}_\otimes|$, for each $q \in \mathcal{P}$.

[14] Should the construction process involve denumerably many steps, the final values of the $q^{(\bullet)}$'s are to be intended as limit of the sequences of their values after each step in the stabilization phase.

[15] For the present case concerning the ordinary satisfiability problem, we could have allowed that the $\overline{q}$'s were all infinite sets, rather than hereditarily finite sets. However, we chose to enforce hereditarily finiteness of the $\overline{q}$'s even in the current case in order that the initialization phase would coincide with that for the hereditarily finite satisfiability case to be addressed in the next section.

This follows just from the initialization step (I_1), for all source places $q \in \mathcal{P} \setminus \mathcal{P}_\otimes$. Otherwise, by induction, we have $|q^{(\bullet)}| \geqslant 2\,|\mathcal{P}_\otimes|$, for every q in a ready node $A \in \mathcal{N}$, and therefore

$$|\otimes A^{(\bullet)}| \geqslant \binom{2\,|\mathcal{P}_\otimes|}{2} + 2\,|\mathcal{P}_\otimes| = |\mathcal{P}_\otimes| \cdot (2\,|\mathcal{P}_\otimes| + 1).$$

Hence, each of the $|\mathcal{T}(A)| \leqslant |\mathcal{P}_\otimes|$ sets $t^{(\bullet)}$, for $t \in \mathcal{T}(A)$, will receive at least $2\,|\mathcal{P}_\otimes|+1$ elements by the distribution step relative to the node A.

Concerning property (P1), we observe that at each distribution step, only elements of cardinality 1 or 2 are added to the sets $q^{(\bullet)}$. Therefore, the disjointness of the sets $q_1^{(\bullet)}$ and $q_2^{(\bullet)}$, for any two distinct places $q_1, q_2 \in \mathcal{P}$ such that at least one of them is a source place, will be guaranteed. Indeed, if both q_1 and q_2 are source places, then $q_1^{(\bullet)} \cap q_2^{(\bullet)} = \overline{q}_1 \cap \overline{q}_2 = \emptyset$. On the other hand, if only one of them is a source node, say q_1, then since $q_1^{(\bullet)} = \overline{q}_1$ contains only members of cardinality strictly greater than 2 whereas, by step (I_2), all the members of $q_2^{(\bullet)}$ have cardinality less than or equal to 2, it follows that even in this case we have $q_1^{(\bullet)} \cap q_2^{(\bullet)} = \emptyset$. Finally, for any two distinct places q_1 and q_2, none of which is a source node, we observe that if they have been initialized by a distribution step applied to the same node $A \in \mathcal{N}$, we trivially have $q_1^{(\bullet)} \cap q_2^{(\bullet)} = \emptyset$. On the other hand, if q_i is initialized by distributing over a $\otimes$-node A_i, where $i = 1, 2$ and $A_1 \neq A_2$, we can easily show using induction on the number of distribution steps and Lemma 1.13 that $A_1^{(\bullet)} \neq A_2^{(\bullet)}$ and therefore $q_1^{(\bullet)} \cap q_2^{(\bullet)} \subseteq \otimes A_1^{(\bullet)} \cap \otimes A_2^{(\bullet)} = \emptyset$.

Stabilization phase: During the stabilization phase, a $\otimes$-node $A \in \mathcal{N}_\otimes$ is RIPE if

$$\otimes A^{(\bullet)} \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\} \neq \emptyset.$$

We execute the following (possibly infinite) loop:

(S_1) While there are ripe $\otimes$-nodes, pick any of them, say $A \in \mathcal{N}$, and distribute all the members of $\otimes A^{(\bullet)} \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}$ (namely the members of $\otimes A^{(\bullet)}$ that have not been distributed yet) among its targets howsoever.

The fairness condition that one must comply with is the following:

> once a $\otimes$-node becomes ripe during the stabilization phase, it must be picked for distribution within a finite number of iterations of the while-loop (S_1).

A way to enforce such condition consists, for instance, in maintaining all ripe $\otimes$-nodes in a queue Q, picking always the $\otimes$-node to be used in a distribution step from the front of Q and adding the $\otimes$-nodes that have just become ripe to the back of Q, unless they are already in Q.

By induction on $n \in \mathbb{N}$, it can be shown that properties (P1) and (P3) will hold just after the n-th iteration of the while-loop (S_1) of the stabilization phase, and that property (P4) will hold at the end of the stabilization phase, in case of termination.

Instead, when the stabilization phase runs for denumerably many steps, the final partition $\mathcal{P}^{(\bullet)}$ is to be intended as the limit of the partial partitions constructed after each step of the stabilization phase. Specifically, for each place $q \in \mathcal{P}$, we let $q^{(i)}$ be the value of $q^{(\bullet)}$ just after the i-th iteration of (S_1). Plainly, we have

$$q^{(i)} \subseteq q^{(i+1)}, \quad \text{for } i \in \mathbb{N}. \tag{17}$$

Then we put

$$q^{(\bullet)} := \bigcup_{i \in \mathbb{N}} q^{(i)}, \quad \text{for } q \in \mathcal{P} \tag{18}$$

(notation overloading should not be a problem).

By way of illustration, we prove that property (P4) holds for the partition $\mathcal{P}^{(\bullet)} = \{q^{(\bullet)} \mid q \in \mathcal{P}\}$, when the $q^{(\bullet)}$'s are defined by (18). To this purpose, let $A \in \mathcal{N}$ be such that $\mathcal{T}(A) \neq \emptyset$, and assume for contradiction that

$$\otimes A^{(\bullet)} \not\subseteq \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}.$$

Let s be any element in $\otimes A^{(\bullet)} \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}$, and let $i \in \mathbb{N}$ be the smallest index such that $s \in \otimes A^{(i)}$, where $A^{(i)} \coloneqq \{q^{(i)} \mid q \in A\}$. Since $s \in \otimes A^{(i)} \setminus \bigcup \{q^{(i)} \mid q \in \mathcal{T}(A)\}$, the node A must have been ripe just after the i-th iteration of (S_1). Therefore, by the fairness condition, the node A will be picked for distribution in a finite number of steps, say k, after the i-th step, so that we have

$$\begin{aligned} \otimes A^{(i)} &\subseteq \otimes A^{(i+k)} && \text{(by (17))} \\ &\subseteq \bigcup \{q^{(i+k+1)} \mid q \in \mathcal{T}(A)\} \\ &\subseteq \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}, \end{aligned}$$

and therefore $s \in \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}$, which is a contradiction. Thus, property (P4) holds also when the construction process takes a denumerable number of steps.

Next, we show that the final partition $\mathcal{P}^{(\bullet)} = \{q^{(\bullet)} \mid q \in \mathcal{P}\}$ satisfies Φ. In particular, we prove that the partition assignment $\mathfrak{I} \colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P}^{(\bullet)})$ defined by

$$\mathfrak{I}(x) \coloneqq \{q^{(\bullet)} \mid q \in \mathfrak{F}(x)\}, \quad \text{for } x \in \mathrm{Vars}(\Phi),$$

satisfies Φ, where we recall that $\mathfrak{F}$ is the $\mathcal{G}$-fulfilling map for Φ.

Since $\mathfrak{F}$ is a $\mathcal{G}$-fulfilling map for Φ, then

- for every literal $x = y \star z$ in Φ, with $\star \in \{\cup, \setminus\}$, we have $\mathfrak{F}(x) = \mathfrak{F}(y) \star \mathfrak{F}(z)$, so that $\mathfrak{I}(x) = \mathfrak{I}(y) \star \mathfrak{I}(z)$ holds; and
- for every literal $x \neq y$ in Φ, we have $\mathfrak{F}(x) \neq \mathfrak{F}(y)$, so that $\mathfrak{I}(x) \neq \mathfrak{I}(y)$ holds.

Thus, by Lemma 1.3, the partition assignment $\mathfrak{I}$ satisfies all Boolean literals in Φ of the following types

$$x = y \cup z, \quad x = y \setminus z, \quad x \neq y.$$

Next, let $x = y \otimes z$ be a conjunct of Φ. We prove separately that the following inclusions hold:

$$\bigcup \mathfrak{I}(x) \subseteq \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z) \tag{19}$$

$$\bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z) \subseteq \bigcup \mathfrak{I}(x). \tag{20}$$

Concerning (19), let $q^{(\bullet)} \subseteq \bigcup \mathfrak{I}(x)$. Then $q^{(\bullet)} \in \mathfrak{I}(x)$, so that $q \in \mathfrak{F}(x)$. By (c_2) of Definition 2.7, q cannot be a source place. Hence, by (P3), we have:

$$q^{(\bullet)} \subseteq \bigcup \{\otimes A^{(\bullet)} \mid A \in \mathcal{T}^{-1}(q)\}.$$

We show now that

$$\mathcal{T}^{-1}(q) \subseteq \mathfrak{F}(y) \otimes \mathfrak{F}(z). \tag{21}$$

Let $A \in \mathcal{T}^{-1}(q)$ (so that $q \in \mathcal{T}(A)$), and for contradiction assume that $A \notin \mathfrak{F}(y) \otimes \mathfrak{F}(z)$. Then, by ($c_3$) of Definition 2.7, we have $\mathcal{T}(A) \cap \mathfrak{F}(x) = \emptyset$, contradicting $q \in \mathcal{T}(A) \cap \mathfrak{F}(x)$. Thus, $A \in \mathfrak{F}(y) \otimes \mathfrak{F}(z)$, proving (21).

Hence, we have:

$$\begin{aligned} q^{(\bullet)} &\subseteq \bigcup\{\otimes A^{(\bullet)} \mid A \in \mathcal{T}^{-1}(q)\} \\ &\subseteq \bigcup\{\otimes A^{(\bullet)} \mid A \in \mathfrak{F}(y) \otimes \mathfrak{F}(z)\} \\ &= \bigcup\{\otimes A^{(\bullet)} \mid A^{(\bullet)} \in \mathfrak{I}(y) \otimes \mathfrak{I}(z)\} \\ &= \bigcup \mathfrak{I}(y) \otimes \bigcup \mathfrak{I}(z) \qquad \text{(by Lemma 1.15),} \end{aligned}$$

and therefore the inclusion (19) holds.

Concerning the inclusion (20), let $s \in \bigcup\mathfrak{I}(y) \otimes \bigcup\mathfrak{I}(z)$. Hence, $s \in q_1^{(\bullet)} \otimes q_2^{(\bullet)} = \otimes\{q_1^{(\bullet)}, q_2^{(\bullet)}\}$, for some $q_1 \in \mathfrak{F}(y)$ and $q_2 \in \mathfrak{F}(z)$. From (c_1), we have $\emptyset \neq \mathcal{T}(\{q_1, q_2\}) \subseteq \mathfrak{F}(x)$. Thus, by (P4),

$$\otimes\{q_1^{(\bullet)}, q_2^{(\bullet)}\} \;\subseteq\; \bigcup\{q^{(\bullet)} \mid q \in \mathcal{T}(\{q_1, q_2\})\} \;\subseteq\; \bigcup\{q^{(\bullet)} \mid q \in \mathfrak{F}(x)\} \;=\; \bigcup\mathfrak{I}(x),$$

and therefore $s \in \bigcup\mathfrak{I}(x)$, proving (20).

Thus, the partition assignment $\mathfrak{I}$ satisfies also all the $\otimes$-conjuncts of Φ, and in turn the final partition $\mathcal{P}^{(\bullet)}$ satisfies the conjunction Φ. □

Running Example (Part V–final). Let Φ be the BST$\otimes$-conjunction in Part IV and $\mathcal{G}_\Sigma$ the accessible $\otimes$-graph in Fig. 1 (see also Part II). As already checked before, $\mathcal{G}_\Sigma$ fulfills Φ.

We illustrate now how the construction process applied to the $\otimes$-graph $\mathcal{G}_\Sigma$ allows us to generate a partition (not necessarily equal to Σ) that satisfies Φ.

Thus, let

$$\overline{q}_1 := \overline{q}_2 := \emptyset, \qquad \overline{q}_3 := \{a_1, a_2, a_3, a_4\}, \qquad \overline{q}_4 := \{b_1, b_2, b_3, b_4\},$$

where the a_i's and the b_i's are pairwise distinct hereditarily finite sets of cardinality at least 2, and put

$$\mathbf{q}_i^{(\bullet)} := \overline{q}_i, \qquad \text{for } i = 1, 2, 3, 4.$$

The sets $\mathbf{q}_i^{(\bullet)}$'s have been printed in boldface to distinguish them from the sets $q_i^{(\bullet)}$'s in Part II. Notice also that

$$|\overline{q}_2| = |\overline{q}_2| = \max(2|\mathcal{P}_{\Sigma,\otimes}|, 1) = 4,$$

where $\mathcal{P}_{\Sigma,\otimes}$ is the set of $\otimes$-places of $\mathcal{G}_\Sigma$.

After these initial assignments, the places q_3 and q_4 are *initialized* and the $\otimes$-node $\{q_3\}$, which has two targets (namely q_1 and q_2), is *ready*.

The set $\otimes\mathbf{q}_3^{(\bullet)}$ has 10 elements, so 5 of these will be assigned to $\mathbf{q}_1^{(\bullet)}$ and the remaining 5 to $\mathbf{q}_2^{(\bullet)}$.

At this point, all places of $\mathcal{G}_\Sigma$ are *initialized*, so the initialization phase terminates and the stabilization phase stars, with all the $\otimes$-nodes of $\mathcal{G}_\Sigma$ but $\{q_3\}$ *ripe*. Notice that all of these $\otimes$-nodes have as target just the $\otimes$-place q_2. Hence, $\mathbf{q}_2^{(\bullet)}$ is the only set that will receive new elements at each step the stabilization phase, and so after the execution of each such step the $\otimes$-nodes $\{q_1, q_2\}$, $\{q_2\}$, and $\{q_2, q_3\}$ will always be *ripe*. Thus, the stabilization phase never stops, taking denumerably many steps and yielding a final limit partition $\{\mathbf{q}_1^{(\bullet)}, \mathbf{q}_2^{(\bullet)}, \mathbf{q}_3^{(\bullet)}, \mathbf{q}_4^{(\bullet)}\}$ (distinct from Σ and where $\mathbf{q}_2^{(\bullet)}$ is infinite) that satisfies Φ. This can be proved by induction on the number of iterations of the stabilization loop.

In order to facilitate the understanding of what has been said, we present the configuration of the queue Q of the ripe $\otimes$-nodes after the first six steps of the stabilization phase (see Table 1). The queue Q is depicted with its front on the left and its back on the right.

- Step 0 shows the configuration of Q when the stabilization phase starts, namely at the end of the initialization phase.

Stabilization step	Queue Q	Selected $\otimes$-node
0	$\{q_1\}, \{q_1, q_3\}, \{q_1, q_2\}, \{q_2\}, \{q_2, q_3\}$	—
1	$\{q_1, q_3\}, \{q_1, q_2\}, \{q_2\}, \{q_2, q_3\}$	$\{q_1\}$
2	$\{q_1, q_2\}, \{q_2\}, \{q_2, q_3\}$	$\{q_1, q_3\}$
3	$\{q_2\}, \{q_2, q_3\}, \{q_1, q_2\}$	$\{q_1, q_2\}$
4	$\{q_2, q_3\}, \{q_1, q_2\}, \{q_2\}$	$\{q_2\}$
5	$\{q_1, q_2\}, \{q_2\}, \{q_2, q_3\}$	$\{q_2, q_3\}$
6	$\{q_2\}, \{q_2, q_3\}, \{q_1, q_2\}$	$\{q_1, q_2\}$
⋮	⋮	⋮

Table 1. Configuration of the queue Q of the ripe $\otimes$-nodes after the first six steps of the stabilization phase. The front of the queue is on the left and the back is on the right.

- During step 1, the ripe node $\{q_1\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_1\}$ leaves the queue and no other ripe node enters it, since the nodes $\{q_1, q_2\}$, $\{q_2\}$, $\{q_2, q_3\}$ are already in queue. So the configuration of the queue at the end of step 1 is
$$\{q_1, q_3\}, \quad \{q_1, q_2\}, \quad \{q_2\}, \quad \{q_2, q_3\},$$
as reported in the table.
- During step 2, the ripe node $\{q_1, q_3\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_1, q_3\}$ leaves the queue and no other ripe node enters Q, since the nodes $\{q_1, q_2\}$, $\{q_2\}$, $\{q_2, q_3\}$ are already in Q.
- During step 3, the ripe node $\{q_1, q_2\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_1, q_2\}$ leaves the queue and it reenters it, since it became ripe again (nothing is done for the other nodes $\{q_2\}$ and $\{q_2, q_3\}$, as they are already in the queue).
- During step 4, the ripe node $\{q_2\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_2\}$ leaves the queue and it immediately reenters it, since it became ripe again (nothing is done for the other nodes $\{q_2, q_3\}$ and $\{q_1, q_2\}$, as they are already in the queue).
- During step 5, the ripe node $\{q_2, q_3\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_2, q_3\}$ leaves the queue and it immediately reenters it, since it became ripe again (nothing is done for the other nodes $\{q_1, q_2\}$ and $\{q_2\}$, since they are already in the queue).
- During step 6, the ripe node $\{q_1, q_2\}$ is selected for distribution, so fresh elements are added to the block corresponding to its lonely target q_2. Thus, $\{q_1, q_2\}$ leaves the queue and it immediately reenters it, since it became ripe again (nothing is done for the other nodes $\{q_2\}$ and $\{q_2, q_3\}$, since they are already in the queue).
- We observe that the queue configuration Q at the end of step 6 is identical to that at the end of step 3. As a consequence, the queue will endlessly loop through the configurations at the end of steps 3, 4, and 5 in sequence. □

By combining Lemmas 2.8 and 2.10 and Corollary 2.9, we obtain:

Theorem 2.11. *A* BST⊗*-conjunction with n variables is satisfiable if and only if it is fulfilled by an accessible* ⊗*-graph of size (at most)* $2^n - 1$.

The preceding theorem is at the base of the following trivial decision procedure for BST⊗:

procedure BST⊗-satisfiability-test(Φ);
1. $n := |\mathrm{Vars}(\Phi)|$;
2. **for** each ⊗-graph $\mathcal{G}$ with $2^n - 1$ places **do**
3. **if** $\mathcal{G}$ is accessible and fulfills Φ **then**
4. **return** "Φ is satisfiable";
5. **return** "Φ is unsatisfiable";

end procedure;

Concerning the complexity of the above procedure, we observe that, given a BST⊗-conjunction Φ of size m and with n distinct variables, we have:

- the size of a ⊗-graph with $2^n - 1$ places is $O(8^n)$;
- the size of any candidate fulfilling map over a set of n variables is $O(n2^n)$ and the time needed to check whether it is actually a $\mathcal{G}$-fulfilling map for Φ, for a given ⊗-graph $\mathcal{G}$ with $2^n - 1$ places, is $O(m8^n)$.

Hence, for a BST⊗-conjunction Φ with n distinct variables the procedure BST⊗-satisfiability-test has a nondeterministic $O(m8^n)$-time complexity. Thus, we have:

Theorem 2.12. *The satisfiability problem for* BST⊗*-conjunctions belongs to the complexity class* NEXPTIME.

The above result can be easily generalized to BST⊗-formulae that are not necessarily conjunctions.

Theorem 2.13. *The satisfiability problem for* BST⊗*-formulae belongs to the complexity class* NEXPTIME.

As we have seen, the formula Φ in Part IV of our running example, namely

$$y = y \setminus y \quad \wedge \quad v = w \setminus x \quad \wedge \quad v \neq y \quad \wedge \quad x \neq y \quad \wedge \quad y = z \setminus x \quad \wedge \quad z = x \otimes x,$$

admits several distinct infinite models: the one induced by the partition Σ and the map $\mathfrak{I}$ in Part IV of our running example and the infinitely many ones corresponding to different ways to instantiate the construction process based on the fulfilling ⊗-graph $\mathcal{G}_\Sigma$ (and possibly on other fulfilling ⊗-graphs).

A question is therefore in order: Does our formula Φ admit also any finite model?

The answer is negative. Indeed, if M is any model for Φ, then $My = My \setminus My$, and so $My = \emptyset$. Hence, $My = Mz \setminus Mx$ implies $Mz \subseteq Mx$, and since $Mz = Mx \otimes Mx$, we have $Mx \otimes Mx \subseteq Mx$. Finally, $Mx \neq My$ implies $Mx \neq \emptyset$, so $s \in Mx$ for some set s. But then, iteratively, the infinitely many sets

$$\{s\},\ \{\{s\}\},\ \{\{\{s\}\}\},\ \ldots$$

must all belong to Mx, proving that Mx must be infinite.

It is therefore worthwhile to investigate the finite s.p. for BST⊗, which we do in the next section.

3 THE FINITE AND THE HEREDITARILY FINITE SATISFIABILITY PROBLEMS FOR BST⊗

Let Φ be a finitely satisfiable BST⊗-conjunction, and let now Σ be a partition with *finite* domain $\bigcup\Sigma$ that satisfies Φ via some partition assignment $\mathfrak{I}\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\Sigma)$. Also, let $\mathcal{G}_\Sigma = (\mathcal{P}_\Sigma, \mathcal{N}_\Sigma, \mathcal{T}_\Sigma)$ be the ⊗-graph induced by Σ via a given bijection $q \mapsto q^{(\bullet)}$. As argued just before Lemma 2.6, the graph $\mathcal{G}_\Sigma$ is ⊗-accessible and fulfills Φ via the map $\mathfrak{F}_\Sigma\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P}_\Sigma)$ induced by $\mathfrak{I}$ and defined by

$$\mathfrak{F}_\Sigma(x) := \{q \in \mathcal{P}_\Sigma \mid q^{(\bullet)} \in \mathfrak{I}(x)\}, \qquad \text{for } x \in \mathrm{Vars}(\Phi)$$

(so, $\mathfrak{F}_\Sigma$ is a $\mathcal{G}_\Sigma$-fulfilling map for Φ).

We shall see that the finiteness of $\bigcup\Sigma$ yields a weak kind of acyclicity for the induced $\otimes$-graph $\mathcal{G}_\Sigma$, which is expressed in terms of a restricted form of topological order.

Definition 3.1. A TOPOLOGICAL $\otimes$-ORDER of a $\otimes$-graph $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T})$ is any total order $\prec$ over its set of places $\mathcal{P}$ such that

$$\max_{\prec} A \;\prec\; \max_{\prec} \mathcal{T}(A), \tag{22}$$

for every $\otimes$-node A of $\mathcal{G}$.

We write $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T}, \prec)$ for a $\otimes$-graph $(\mathcal{P}, \mathcal{N}, \mathcal{T})$ endowed with a topological $\otimes$-order $\prec$, and we refer to it as a (TOPOLOGICALLY) $\otimes$-ORDERED GRAPH. □

Notice that a $\otimes$-ordered graph need not be acyclic. On the other hand, any acyclic $\otimes$-graph admits a topological order of its vertices and therefore a topological $\otimes$-order, as can be easily checked. In this sense, topological $\otimes$-orders are less demanding than ordinary topological orders.

Later we shall also see that, together with fulfillability and accessibility, the existence of a topological $\otimes$-order is sufficient for a BST$\otimes$-conjunction to be hereditarily finitely satisfiable, thereby showing that the finite and the hereditarily finite satisfiability problems for BST$\otimes$ are equivalent.

To start with, we show, as announced before, that the induced $\otimes$-graph $\mathcal{G}_\Sigma$ admits a topological $\otimes$-order. Thus, let $\prec_\Sigma$ be any total order over $\mathcal{P}_\Sigma$ that refines the partial order induced by the rank function, namely such that

$$\mathsf{rk}\, p^{(\bullet)} < \mathsf{rk}\, q^{(\bullet)} \quad\Longrightarrow\quad p \prec_\Sigma q, \quad \text{for } p, q \in \mathcal{P}_\Sigma.$$

We prove that (22) holds for $\prec_\Sigma$, namely $\prec_\Sigma$ is a topological $\otimes$-order of $\mathcal{G}_\Sigma$. So, let A be any $\otimes$-node of $\mathcal{G}_\Sigma$. For each $q \in A$, we select an $s_q \in q^{(\bullet)}$ of maximal rank, which exists since $q^{(\bullet)}$ is finite, and put $s_A \coloneqq \{s_q \mid q \in A\}$. Let q_A be the target of A such that $s_A \in q_A^{(\bullet)}$ (plainly, such a target exists, since $s_A \in \otimes A^{(\bullet)} \subseteq \bigcup\Sigma_\otimes$). Hence, for each $q \in A$ we have

$$\mathsf{rk}\, q^{(\bullet)} \leqslant \mathsf{rk}\, s_A < \mathsf{rk}\, q_A^{(\bullet)},$$

so that $q \prec_\Sigma q_A$ holds. But then

$$\max_{\prec_\Sigma} A \;\prec_\Sigma\; q_A \;\;\underset{=}{\leq_\Sigma}\;\; \max_{\prec_\Sigma} \mathcal{T}(A),$$

showing that $\prec_\Sigma$ is a topological $\otimes$-order of $\mathcal{G}_\Sigma$.

Summing up, we have proved that:

LEMMA 3.2. *A finitely satisfiable* BST$\otimes$*-conjunction is fulfilled by an accessible ordered $\otimes$-graph.*

Next, we prove that if a BST$\otimes$-conjunction Φ is fulfilled by an accessible ordered $\otimes$-graph, then it is satisfiable by a *hereditarily* finite model.

Thus, let $\mathcal{G} = (\mathcal{P}, \mathcal{N}, \mathcal{T}, \prec)$ be an accessible ordered $\otimes$-graph that fulfills Φ via a map $\mathfrak{F}\colon \mathrm{Vars}(\Phi) \to \mathrm{pow}(\mathcal{P})$, and let $\preccurlyeq$ be the *total preorder* induced by $\prec$ over $\mathcal{N}$, defined by

$$A \preccurlyeq B \;\overset{Def}{\longleftrightarrow}\; \max_{\prec} A \;\underset{=}{\leq}\; \max_{\prec} B,$$

for all $A, B \in \mathcal{N}$.

Much the same construction process described at depth in the proof of Lemma 2.10 concerning the ordinary s.p. for BST$\otimes$ will allow us to build a hereditarily finite model for Φ.

Specifically, the initialization phase of our new construction process coincides with that of the old construction process, and therefore consists in the steps (I_1) and (I_2) seen previously. Instead, the old stabilization loop (S_1) is replaced by the following one:

(S_1') While there are ripe $\otimes$-nodes, pick any $\preccurlyeq$-minimal ripe $\otimes$-node, say $A \in \mathcal{N}_\otimes$, and assign all members of $\otimes A^{(\bullet)} \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}$ to the block $q_A^{(\bullet)}$ such that $q_A = \max_\prec \mathcal{T}(A)$, namely execute the assignment

$$q_A^{(\bullet)} := \left(q_A^{(\bullet)} \cup \otimes A^{(\bullet)}\right) \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}.$$

(As before, during the stabilization phase a $\otimes$-node A is *ripe* if the set $\otimes A^{(\bullet)} \setminus \bigcup \{q^{(\bullet)} \mid q \in \mathcal{T}(A)\}$ is nonempty.)

We prove that the while-loop (S_1') can be executed at most $|\mathcal{N}_\otimes|$ times. Thus, let

$$A_1,\ A_2, \ldots,\ A_k, \ldots \tag{23}$$

be the sequence of the $\otimes$-nodes picked for distribution during the execution of the loop (S_1'). It is enough to show that the $\otimes$-nodes in the sequence (23) are pairwise distinct. To this end, we first prove that we have

$$A_1 \preccurlyeq A_2 \preccurlyeq \ldots \preccurlyeq A_k \preccurlyeq \ldots \tag{24}$$

Arguing by contradiction, assume that (24) does not hold, and let $\ell \in \mathbb{N}$ be the least index such that we have

$$A_\ell \not\preccurlyeq A_{\ell+1}, \tag{25}$$

so that $A_{\ell+1} \preccurlyeq A_\ell$ must hold, since the preorder $\preccurlyeq$ is total. Plainly, at the ℓ-th iteration of (S_1'), the node $A_{\ell+1}$ cannot be ripe, as otherwise it would have been chosen at step ℓ in place of A_ℓ. So, the target $q_{A_\ell} = \max_\prec \mathcal{T}(A_\ell)$ of A_ℓ must belong to $A_{\ell+1}$, and therefore

$$\max_\prec A_\ell \quad \prec \quad \max_\prec \mathcal{T}(A_\ell) \quad = \quad q_{A_\ell} \quad \preceq \quad \max_\prec A_{\ell+1}$$

must hold, yielding $A_\ell \preccurlyeq A_{\ell+1}$ which contradicts (25).

In what follows, for any node $A \in \mathcal{N}$ we shall denote by $A^{(i)}$ the value of the set $A^{(\bullet)}$ (associated with A) just before the i-th iteration of the loop (S_1').

We are now ready to prove that the nodes in the sequence (23) are pairwise distinct. Arguing by contradiction, suppose that $A_i = A_j$, with $i < j$. Then $A_i^{(i)} \neq A_i^{(j)}$, so that at least one place q in A_i must be the $\prec$-maximum target of some $\otimes$-node, say A_t (with $i \leqslant t \leqslant j-1$), in the sequence $A_i, \ldots, A_{j-1}$. But then we would have:

$$\max_\prec A_i \quad \preceq \quad \max_\prec A_t \quad \prec \quad q_{A_t} \quad \preceq \quad \max_\prec A_i,$$

which is a contradiction. Therefore, the while-loop (S_1') must terminate in at most $|\mathcal{N}_\otimes|$ iterations.

Thus, at the end of the construction process under consideration, all sets $q^{(\bullet)}$ with $q \in \mathcal{P}$ are plainly hereditarily finite, and since the loop (S_1') is a particular instance (which is guaranteed to terminate) of the loop (S_1), then the partition $\{q^{(\bullet)} \mid q \in \mathcal{P}\}$ resulting from the above construction process satisfies our conjunction Φ, just as argued in the proof of Lemma 2.10.

In conclusion, we have:

Lemma 3.3. *A* BST⊗*-conjunction fulfilled by an accessible ordered $\otimes$-graph is satisfiable by a* hereditarily *finite model.*

From Lemmas 3.2 and 3.3 and Corollary 2.9, we deduce:

Theorem 3.4. *The finite and the hereditarily finite satisfiability problems for* BST⊗*-conjunctions are equivalent. In addition, any* BST⊗*-conjunction with n variables is (hereditarily) finitely satisfiable if and only if it is fulfilled by an accessible ordered $\otimes$-graph of size (at most) $2^n - 1$.*

The preceding theorem justifies the following trivial decision procedure for the (hereditarily) finite satisfiability problem for BST⊗:

procedure BST⊗-finite-satisfiability-test(Φ);
1. $n := |\mathrm{Vars}(\Phi)|$;
2. **for** each ⊗-graph $\mathcal{G}$ with $2^n - 1$ places **do**
3. **if** $\mathcal{G}$ is ⊗-ordered, accessible and fulfills Φ **then**
4. **return** "Φ is (hereditarily) finitely satisfiable";
5. **return** "Φ is not (hereditarily) finitely satisfiable";

end procedure;

Much as in the previous section, we can deduce that:

Theorem 3.5. *The (hereditarily) finite satisfiability problem for* BST⊗*-conjunctions belongs to the complexity class* NEXPTIME.

4 CONCLUDING REMARKS

In this paper, we provided an algorithmic solution to the s.p. for the slightly simplified variant BST⊗ of MLS×, whose decision problem has been a long-standing open problem in computable set theory. BST⊗ differs from MLS× in that membership has been dropped and the Cartesian product has been replaced by its unordered variant ⊗. Specifically, we proved that both the ordinary s.p. and the (hereditarily) finite s.p. for BST⊗ are in NEXPTIME. Despite the simplifications made in moving from MLS× to BST⊗, the s.p. for BST⊗ remains fully representative of the combinatorial difficulties due to the presence of the Cartesian product operator.

We expect that the technique introduced in this paper, based on ⊗-graphs and fulfilling maps, may be adapted to ascertain the decidability of various extensions of BST with operators belonging to a specific class, which includes, among others, the (ordered) Cartesian product × and the power set operator and its 'siblings' $\mathrm{pow}^*_I(\cdot)$ and $\mathrm{pow}_I(\cdot)$.

Finally, we are very confident that the decidability result for BST⊗ can be adapted to MLS×, though at the cost of several technicalities, and we intend to carry on such a generalization in the future.

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