University of Catania

Department of Mathematics and Computer Scienze

PhD in Mathematics and Computer Science XXXII Cylce

Doctoral Thesis in Mathematics

SSD: MAT/05

ELLIPTIC PROBLEMS INVOLVING NONLINEARITIES INDEFINITE IN SIGN

Author:

Luca Furnari

Supervisor:

Prof. Giovanni Anello

Head of PhD School:

Prof. Giovanni Russo

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the Partner Institution

University of Messina

Department of Mathematical and Computer Sciences,

Physical Sciences and Earth Sciences

To my two Grandmothers Maria

"Reserve your right to think,

for even to think wrongly is better than not to think at all."

Ipazia

"Logic will get you from A to B.

Imagination will get you everywhere."

Albert Einstein

Acknowledgements

With regards to these years of Ph.D. course, I would like to thank all the people who have been close to me. In particular, my family members who have always supported me in my choices.

I also want to thank my supervisor, Prof G. Anello, whose help was essential for my research work.

I would like to take this opportunity to thank all my friends. Not only my old friends, but also my colleagues who accompanied me during degree and Ph.D. years and have become dear friends.

Contents

Introduction

In this thesis we present the main results obtained during the PhD course. It is structured in three chapters: Chapter I is devoted to the preliminaries; Chapter II contains the main results and their proofs; Chapter III contains some open problems issued by the main results.

We will deal with two elliptic equations with zero Dirichlet boundary condition. In particular:

1. the first equation involves the well known Laplacian operator defined by:

$$-\Delta \stackrel{\text{def}}{=} -\text{div}(\nabla(\cdot));$$

2. the second equation involves the following non-local operator of Kirchhoff type:

$$-\left(a + b\int_{\Omega} |\nabla(\cdot)|^2\, dx\right)\Delta.$$

Before summarizing the topics mentioned in the above discussion, we recall some applications, especially in mathematical physics, of Dirichlet problems involving the differential operators just introduced. Here, we consider a generic nonlinearity $f(x,u)$ but, in the two problems we will expose in this thesis, the nonlinearity has a specific form which is a model of physical phenomena with a stronger absorbtion than diffusion.

Boundary-value problems of type

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)$$

have been studied intensively during the last 40 years. Thay arise in application as a stationary model for population dynamics, chemical reaction, combustion, etc... , and positive solutions are, in many situations, the only relevant ones.

Here, Ω is a bounded domain, "the Laplacian is used to model diffusion"and the "nonlinearity f represents the reaction term".

Solutions of problem above can be interpreted as stationary solutions to the associated parabolic problem.

Existence and other properties of solutions have been studied using various methods (sub- and super-solutions, critical-point theory, bifurcation) and combinations of them. Usually, the nonlinear reaction term is smooth and, in many cases, satisfies the conditions $f(x,0) \geq 0$.

Note that when $f(0) = 0$, then $u(x) \equiv 0$ is a solution (the "trivial one"). Obviously, the relevant interest is the study of positive solution or nonnegative and nonzero solutions.

Using variational method (see §1.3), under suitable assumptions on f, a solution is found as the global minimum of the "energetic functional" defined by

$$I(u) = \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left(\int_0^{u(x)} f(x,t) \, dt \right) dx.$$

This point of view was introduced by Riemann in 1851 and it is known as Dirichlet Principle. The fundamental idea is to interpret a Dirichlet problem as a differential problem consisting in finding the solution of the so called Euler-Lagrange equation

$$I'(u) = 0, \tag{1}$$

where I' is the differential of the functional I (see §1.2). Usually, one looks for global minimum points (or local minumim points) and not for global maximum points of I. This is due to the properties that the functional I which, in general, is only sequentially weakly lower semicontinuous .

Of course, not all the boundary value problems can be reformulated in the form (1). When this occurs, one says that the problem has a variational structure.

From a physical point of view, the functional I can represent, for example, the total energy of a homogeneous and isotropic thermal (or electrical) conductor occupying the region Ω of the space and under the influence of distributed sources of heat (or electric fields) .

The non-local general problem of Kirchhoff type:

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 \, dx)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{$P_{a,b}$}$$

is related to the stationary analogue of the hyperbolic equation

$$u_{tt} - \left(a + b \int_\Omega |\nabla u|^2 \, dx \right) \Delta u = f(x,u) \quad \text{in } \Omega, \tag{2}$$

where Δ is the Laplacian operator defined above. Equation (2) is a general version of the Kirchhoff equation

$$\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L u_x^2 \, dx \right) u_{xx} = 0, \tag{3}$$

presented by Kirchhoff in [13]. This equation is an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The parameters in equation (3) have the following meanings: E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of cross-section, and P_0 is the initial tension. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations.

On the other hand, nonlocal boundary value problems like problem $(P_{a,b})$ model several physical and biological system where u describe a process which depends on the average of itself, as for example, the population density.

This problem has also been studied here using variational methods (§2.3).

In the first chapter we introduce some basic notations and classical results which are the preliminaries to the development of the thesis. In general, the solutions of the Euler-Lagrange equation $I(u) = 0$ are not classical solutions of differential equations. They are usually called weak solutions and belongs to particular functional spaces, called Sobolev spaces, which are the natural spaces to work with the variational methods.

The preliminaries contain some standard differentiability results for real functionals and some other results about the existence of global minima of these latter. Furthermore, we will also introduce the well known Palais-Smale Condition which is one of the key assumptions of the celebrated Mountain Pass Theorem by Ambrosetti Rabinowitz [3]. This last result ensures the existence of solutions of the Euler-Lagrange equation which are not local minima of the energy functional, so that it represents an useful tool for establishing multiplicity results.

In the second chapter, we will present our main results. They concern the existence of solutions for a class of elliptic problems.

We will first study the following non autonomous elliptic problem:

$$\begin{cases} -\Delta u = \alpha(x)u^{s-1} - \mu\beta(x)u^{r-1}, & \text{in} \quad \Omega \\ u \geq 0, & \text{in} \quad \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \qquad (P_\mu)$$

Here, $\mu \in \mathbb{R}$ is a parameter, $\alpha, \beta : \Omega \to \mathbb{R}$ are two measurable weight functions, and r, s are two exponents such that $s \in]1, 2[$ and $r \in]1, s[$.

We will establish, via minimax methods, a multiplicity result under suitable summability conditions on the weight functions α, β.

Due to the condition $1 < r < s < 2$, it happens that when $\beta(x)$ is positive, the reaction term $f(x, u) = \alpha(x)u^{s-1} - \mu\beta(x)u^{r-1}$ is negative and has a sublinear growth for u small, so that it is a model for physical phenomena with a stronger absorption than diffusion. In our case, this situation depends explicitly on spatial variable x, namely the problem is non-autonomous.

Problem (P_μ) was first addressed by Hernández, Mancebo and Vega in [12], who considered the case $\alpha = \beta \equiv 1$, that is the autonomous problem

$$\begin{cases} -\Delta u = u^{s-1} - \mu u^{r-1}, & \text{in} \quad \Omega \\ u \geq 0, & \text{in} \quad \Omega \qquad (P_{1,\mu}) \\ u_{|\partial\Omega} = 0. \end{cases}$$

They proved the following result (cf. Theorem 3.13 of [12]) which is reported here in a equivalent form which can be easily obtained by a scaling argument

Theorem A *Let $s \in]0,2[$ and $r \in]0,s[$. Then, there exists a positive constant μ_0 such that the problem $(P_{1,\mu})$ admits at least a positive solution if $\mu \in]0,\mu_0[$; no positive solution if $\mu > \mu_0$.*

The result of Hernández, Mancebo and Vega was successively improved in the nonsingular case in [6] and partially extended to the quasi-linear case in [7]. In particular, in [6] the author proved that

Theorem A' *Let $s \in]1,2[$, $r \in]1,s[$ and μ_0 be as in Theorem A. Then, for each $\mu \in]0,\mu_0[$, there exist at least two solutions, one of which is a local minimum of the energy functional associated to the problem and is positive in Ω; if $\mu = \mu_0$, there exists at least a non zero solution, and if $\mu > \mu_0$, there are no positive solutions.*

To deal with problem (P_μ), we use a similar idea as in [7], consisting of the following steps:

- we find a global minimum point of the functional I_μ with negative energy, (lemma 2.2.4);
- we find another local minimum point of I_μ, by assuming two sets of independent hypotheses. In particular we find that

– either $u = 0$ is a local minimum point, (Lemma 2.2.5);

– or there exists a local minimum point in any neighborhood of 0, (Lemma 2.2.7);

- we apply the Mountain Pass Theorem.

Next we study the following non-local autonomous version of (P_μ) involving the Kirchhoff equation:

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\, dx\right)\Delta u = \lambda u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_{\lambda,\mu})$$

Here, s, r, a, b, λ, μ are positive real numbers with $s \in]1, \min\{4, 2^*\}[$ and $r \in]1, s[$.

We will find a relation involving the above parameters which ensure the existence of at least two solution for $(P_{\lambda,\mu})$.

Note that $b = 0$, $a = 1$, $\lambda = 1$ and $1 < r < s < 2$, the problem is local and gives back problem $(P_{1,\mu})$. Then, as seen before, there exist $\mu_0 > 0$ such that the Theorem A is applies.

Actually, letting $a > 0$ and using a scaling argument, we can reformulate Theorem A as follows:

Theorem B *For each $a > 0$, $s \in]0, 2[$, $r \in]0, s[$, $\mu > 0$, problem $(P_{\lambda,\mu})$, with $b = 0$, admits a solution if $\lambda > \left(\frac{\mu}{\mu_0}\right)^{\frac{2-s}{2-r}} a^{\frac{s-r}{r-2}}$, and no solution if $0 < \lambda < \left(\frac{\mu}{\mu_0}\right)^{\frac{2-s}{2-r}} a^{\frac{s-r}{r-2}}$.*

Indeed, for $s \in]0,2[$, $r \in]0,s[$ and $\lambda, \mu > 0$, if we put $\mu' := \frac{\mu}{\lambda}\left(\frac{a}{\lambda}\right)^{\frac{s-r}{2-s}}$, we have the following equivalence:

v is a solution of problem $(P_{1,\mu'})$, with $a = 1$ and $b = 0$,

if and only if

$u := \left(\frac{\lambda}{a}\right)^{\frac{1}{2-s}} v$ is a solution of problem $(P_{\lambda,\mu})$, with $a > 0$ and $b = 0$.

Moreover, one has

$$\mu' \in]0, \mu_0[\quad \text{if and only if} \quad \lambda > \left(\frac{\mu}{\mu_0}\right)^{\frac{2-s}{2-r}} a^{\frac{s-r}{r-2}}.$$

Therefore, Theorem B is equivalent to Theorem A (which in turn is equivalent to Theorem 3.13 of [12]).

Our goal is to extend Theorem B to the Kirchhoff problem $(P_{\lambda,\mu})$, that is to the case $b > 0$. We will consider the non singular case (that is r, $s > 1$) and we will see that the upper bound $s < 2$ can be weakened by assuming a lower bound for λ in the case $s > 2$. We point out that the presence of the nonlocal term $\int_\Omega |\nabla u|^2 dx$ makes the sub-supersolution method somewhat hard to be applied to show, as in [12], that, for a fixed $\lambda > 0$, the set of parameters μ such that a positive solution for problem $(P_{\lambda,\mu})$ exists is exactly an interval. Thus, this remains an open question in the nonlocal case.

Chapter 1

Preliminaries

In this chapter we introduce the basic notions and results, useful for the development of the thesis. In particular, we will outline the functional spaces we will work with, and we will recall some regularity properties of nonlinear functionals as well as some of the main variational results.

We start by fixing our notations. For a measurable set $E \subset \mathbb{R}^N$ ($N \geq 1$) and $1 \leq p \leq +\infty$, we denote by $(L^p(E, \mathbb{R}^M), \|\cdot\|_{L^p(E,\mathbb{R}^M)})$ the Banach space of measurable functions $u : E \to \mathbb{R}^M$ ($M \geq 1$) for which the quantity

$$\|u\|_{L^p(E,\mathbb{R}^M)} := \begin{cases} \left(\int_E |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \\ \operatorname*{ess\,sup}_{x \in E} |u(x)| & \text{if } p = +\infty \end{cases}$$

is finite. Here the symbol $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^M$, which coincides with the absolute value if $M = 1$. When there is no ambiguity, we abbreviate $L^p(E) = L^p(E, \mathbb{R}^M)$ and we will often write $\|\cdot\|_p$ to mean $\|\cdot\|_{L^p(E,\mathbb{R}^M)}$.

Definition 1.0.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set. We say that an open set Ω' in $\mathbb{R}^N$ is strongly included in Ω and we write $\Omega' \subset\subset \Omega$ if $\overline{\Omega'} \subset \Omega$ and $\overline{\Omega'}$ is compact.*

We set

- $C_0^\infty(\Omega) = \{\varphi \in C^\infty(\Omega) : \varphi \text{ has a compact support}\}$, which we call the test function space;
- $L^1_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : \text{for all } K \subset \Omega \text{ compact, } f \in L^1(K)\}$, which is the space of locally integrable functions on Ω;

We recall that a function $u : \Omega \to \mathbb{R}$ is Hölder continuous with exponent $\alpha \in (0, 1]$ (α-Hölder continuous in short) in Ω if there exists a constant $C > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in \Omega.$$

We also set

- $C^{0,\alpha}(\overline{\Omega}) := \{u : \Omega \to \mathbb{R} : u \text{ is bounded and } \alpha\text{-Hölder continuous in } \Omega\}$.

The space $C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha \leq 1$, is a Banach space with the norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x\in\Omega} |u(x)| + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

1.1 Sobolev Spaces

Sobolev spaces are the main tool in the modern approach to study nonlinear boundary value problems. Their name is due to the Russian mathematician Sergei Lvovich Sobolev.

1.1.1 The spaces $W^{1,p}$ and $W_0^{1,p}$

Definition 1.1.1 (Weak derivative)**.** *Let* $u, v_1, v_2, \dots, v_N \in L^1_{loc}(\Omega)$ *and* $\varphi \in C_0^\infty(\Omega)$. *We say that* v_i *is the weak (or distributional) derivative of* u*, with respect to the* i*-th variable* x_i*, if*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega v_i \varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega), \tag{1.1}$$

where the symbol $\frac{\partial}{\partial x_i}$ *denotes the classical derivative.*

Remark 1. *Following the literature, we use the same notation to indicate the weak and classical partial derivatives of a function. When* $u \in W^{1,p}(\Omega)$*, unless otherwise specified,* $\frac{\partial u}{\partial x_i}$ *is the weak partial derivative of* u*.*

Definition 1.1.2. *If* u *has weak derivatives* $\frac{\partial u}{\partial x_i}$ *for each* $i = 1, ..., N$*, we call weak gradient of* u*, the vector* $\nabla u := (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N})$*, where* $\frac{\partial u}{\partial x_i}$ *is the weak derivative of* u *for* $i = 1, \dots, N$*.*

Obviously the integrals in (1.1) makes sense. In fact, if $u \in L^1_{loc}(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$ then $u\varphi$ is summable in Ω and one has:

$$\int_\Omega u(x)\varphi(x)\, dx = \int_{\text{supp}\varphi} u(x)\varphi(x)\, dx.$$

Remark 2. *If* u *is smooth enough to have a classical continuous derivative, then we can integrate by parts and conclude that the classical derivative coincides with the weak one. Of course, the weak derivative may exists without having the existence of the classical derivative. Moreover, the weak derivative, being an element of* $L^1_{loc}(\Omega)$*, is defined up to a Lebesgue-null set, and it is unique.*

Definition 1.1.3 (Sobolev space)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open set and* $1 \leq p \leq +\infty$*. The Sobolev space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega) \ \ \text{for all } i = 1, \ldots, N\}, \tag{1.2}$$

being $\frac{\partial u}{\partial x_i}$ *the weak partial derivative.*

Remark 3. *When* $p = 0$*, we set* $W^{0,p}(\Omega) = L^p(\Omega)$*. Clearly, if* $u \in W^{1,p}(\Omega)$ *then* $|\nabla u| \in L^p(\Omega)$*.*

For all $1 \leq p \leq \infty$, we consider the space $W^{1,p}(\Omega)$ endowed with the norm (called *Sobolev norm*)

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p. \tag{1.3}$$

Remark 4. *For* $1 \leq p < \infty$*, sometimes the equivalent norm*

$$\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}}$$

is used.

Property 1.1.1. *Let* $\Omega \subset \mathbb{R}^N$ *be an open set. Then the space* $(W^{1,p}(\Omega), \|\cdot\|_{W^{1,p}(\Omega)})$ *is a:*

1. *Banach space if* $1 \leq p \leq +\infty$*;*

2. *separable space if* $1 \leq p < +\infty$*;*

3. *uniformly convex space if* $1 < p < +\infty$*.*

From *3.* of the above property and the Milman-Pettis theorem, one has that $W^{1,p}(\Omega)$ is a reflexive space for $1 < p < +\infty$.

We give a characterization of $W^{1,p}(\Omega)$ in terms of difference quotients. Results of this type are often useful in the regularity theory for partial differential equations. Moreover, they provide characterizations that do not involve derivatives and thus they can be used to extend the definition of Sobolev spaces to more abstract setting.

We recall two known results often useful in the context of Sobolev spaces, also used in the proof of characterization (which we will not do).

Theorem 1.1.2 (Absolute continuity on lines)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p < \infty$. A function $u \in L^p(\Omega)$ belongs to the space $W^{1,p}(\Omega)$ if and only if it has a representative $\overline{u}$ that is absolutely continuous on $\mathcal{L}^{N-1}$-a.e. line segments of Ω that are parallel to the coordinate axses and whose first-order (classical) partial derivatives belong to $L^p(\Omega)$. Moreover the (classical) partial derivatives of $\overline{u}$ agree $\mathcal{L}^N$-a.e. with the weak derivatives of u.*

Theorem 1.1.3 (Compactness)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 < p < \infty$. Assume that $\{u_n\} \subset W^{1,p}(\Omega)$ is bounded. Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in W^{1,p}(\Omega)$ such that $u_{n_k} \rightharpoonup u$ in $W^{1,p}(\Omega)$.*

Let $\Omega \subset \mathbb{R}^N$ be an open set and for ever $i = 1, \dots, N$ and $h > 0$, let

$$\Omega_{h,i} := \{x \in \Omega : x + te_i \in \Omega \text{ for all } 0 < t \leq h\}.$$

We mention the following further characterization of $W^{1,p}(\Omega)$:

Theorem 1.1.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W^{1,p}(\Omega)$, $1 \leq p < +\infty$. Then for every $i = i, \ldots, N$ and $h > 0$,*

$$\int_{\Omega_{h,i}} \frac{|u(x + he_i) - u(x)|^p}{h^p}\, dx \leq \int_{\Omega} \left| \frac{\partial u}{\partial x_i}(x) \right|^p dx \tag{1.4}$$

and

$$\lim_{h \to 0^+} \left(\int_{\Omega_{h,i}} \frac{|u(x + he_i) - u(x)|^p}{h^p}\, dx \right)^{\frac{1}{p}} = \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_i}(x) \right|^p dx \right)^{\frac{1}{p}}. \tag{1.5}$$

Conversely, if $u \in L^p(\Omega)$, $1 < p < \infty$, is such that

$$\liminf_{h \to 0^+} \left(\int_{\Omega_{h,i}} \frac{|u(x + he_i) - u(x)|^p}{h^p}\, dx \right)^{\frac{1}{p}} < \infty \tag{1.6}$$

for every $i = 1, \ldots, N$, then $u \in W^{1,p}(\Omega)$.

When $p = 1$, in the first part of the statement of the theorem, $W^{1,1}(\Omega)$ should be replaced by the space of function of bounded variation $BV(\Omega)$, that is the space of functions in $L^1(\Omega)$ whose weak derivatives are bounded Radon measures.

Property 1.1.5. *When $p = 2$, the Sobolev space $W^{1,2}(\Omega)$, also denoted by $H^1(\Omega)$, equipped with the inner product*

$$\begin{aligned} (u, v)_{W^{1,2}(\Omega)} &= (\nabla u \cdot \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} \\ &= \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} uv dx \end{aligned}$$

is a Hilbert space. The norm endowed by the inner product is

$$\|u\|_{W^{1,2}(\Omega)} = \sqrt{(u, u)} = \left(\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} u^2 dx \right)^{\frac{1}{2}},$$

which is an equivalent norm to (1.3).

When we work with Dirichlet problem associated to elliptic equations, the boundary condition means that the Sobolev spaces defined above are not the optimal ones. For this reason we introduce the subspace $W_0^{1,p}(\Omega)$ of $W^{1,p}(\Omega)$ as follows:

Definition 1.1.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $1 \leq p \leq \infty$. The space $W_0^{1,p}(\Omega)$ is defined as the closure of the space $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ (with respect to the norm-topology of $W^{1,p}(\Omega)$).*

The space $W_0^{1,p}(\Omega)$ can be equipped with the $W^{1,p}(\Omega)$-norm and it is a separable Banach space. Moreover, if $1 < p < \infty$, the space $W_0^{1,p}(\Omega)$ is reflexive.

Remark 5. *Since $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$, we have*

$$W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$$

Moreover, if $\Omega \subset \mathbb{R}^N$, in general, $W_0^{1,p}(\Omega) \subsetneq W^{1,p}(\Omega)$.

As done for $W^{1,p}(\Omega)$, we want to give a characterization of $W_0^{1,p}(\Omega)$. We first give the notions of regularity for the domain Ω.

Definition 1.1.5. *Let $\Omega \subset \mathbb{R}^N$ be an open set. The boundary $\partial\Omega$ of Ω is called of class C (or regular) if:*

(i) $\partial\Omega = \partial(\overline{\Omega})$,

(ii) *for each point $x_0 \in \partial\Omega$ there exist a neighborhood A of x_0, local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$, with $y = 0$ at $x = x_0$, and a function $f \in C\left(\overline{Q_{N-1}(0,r)}\right)$, $r > 0$, such that*

$$\partial\Omega \cap A = \{(y', f(y')) : y' \in Q_{N-1}(0,r)\},$$

where Q_{N-1} is the $(N-1)$-dimensional cube.

Definition 1.1.6. *The boundary $\partial\Omega$ of an open and bounded set $\Omega \subset \mathbb{R}^N$ is uniformly Lipschitz if there exists $\varepsilon, L > 0$, $M \in \mathbb{N}$, and a locally finite countable open cover $\{\Omega_n\}$ of $\partial\Omega$ such that*

(i) *if $x \in \partial\Omega$, then $B(x, \varepsilon) \subset \Omega_n$ for some $n \in \mathbb{N}$,*

(ii) *no point of $\mathbb{R}^N$ is contained in more than M of the Ω_n's,*

(iii) *for each n there exist local coordinates $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and a Lipschitz function $f : \mathbb{R}^{N-1} \to \mathbb{R}$ (both depending on n), with $Lip(f) \leq L$, such that*

$$\Omega_n \cap \Omega = \Omega_n \cap \{(y', y_n) \in \mathbb{R}^{N-1} \times \mathbb{R} : y_n > f(y')\}.$$

Studying the spaces $W_0^{1,p}(\Omega)$, an important but delicate argument is the trace theory. Since a function $u \in W^{1,p}(\Omega)$ is defined almost everywhere in Ω and the measure of $\partial\Omega$ is zero, the restriction of u to $\partial\Omega$ in the classical sense has no meaning. For this reason, to give a meaningful definition of restriction, we need to introduce the concept of *trace operator*. We set

$$\mathbb{R}^N_+ = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > 0\}$$

and

$$\mathcal{X}_p = \{u \in L^1_{loc}(\mathbb{R}^N_+) : \nabla u \in L^p(\mathbb{R}^N_+, \mathbb{R}^N) \text{ and } u \text{ vanishing at infinity}\},$$

with $1 < p < N$.

The following theorem defines the trace operator "Tr"and some of its properties.

Theorem 1.1.6. *There exist a linear operator*

$$Tr : \mathcal{X}_p \to L^{\frac{p(N-1)}{N-p}}(\mathbb{R}^{N-1}) \tag{1.7}$$

and a constant $C = C(N,p) > 0$ *such that*

(i) $Tr(u)(x') = u(x',0)$ *for all* $u \in \mathcal{X}_p \cap C\left(\overline{\mathbb{R}^N_+}\right)$,

(ii) *for all* $u \in \mathcal{X}_p$,

$$\left(\int_{\mathbb{R}^{N-1}} |Tr(u)(x')|^{\frac{p(N-1)}{N-p}} dx'\right)^{\frac{N-p}{p(N-1)}} \leq C \left(\int_{\mathbb{R}^N_+} |\nabla u(x)|^p dx\right)^{\frac{1}{p}},$$

(iii) *for all* $\varphi \in C^1_0(\mathbb{R}^N)$, $u \in \mathcal{X}_p$ *and* $i = 1, \ldots, N$

$$\int_{\mathbb{R}^N_+} u \frac{\partial \varphi}{\partial x_i} dx = -\int_{\mathbb{R}^N_+} \varphi \frac{\partial u}{\partial x_i} dx + \int_{\mathbb{R}^{N-1}} \varphi Tr(u) \nu_i dx',$$

where $\nu = -e_N$.

It is shown that the (ii) also holds in the case $p = 1$.

Remark 6. *Note that* $W^{1,p}(\Omega) \subset \mathcal{X}_p$, *and so the linear operator* Tr, *restricted to* $W^{1,p}(\Omega)$, *satisfies* $(i) - (iii)$. *In particular, it is continuous in* $W^{1,p}(\Omega)$.

We can now characterize the functions of $W^{1,p}_0(\Omega)$ as the subspace of functions in $W^{1,p}(\Omega)$ with trace zero.

Theorem 1.1.7. *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 2$ *be an open set whose boundary* $\partial\Omega$ *is uniformly Lipschitz, let* $1 \leq p < \infty$, *and let* $u \in W^{1,p}(\Omega)$. *Then* $Tr(u) = 0$ *if and only if* $u \in W^{1,p}_0(\Omega)$.

The following result provides an equivalent norm in $W_0^{1,p}(\Omega)$ for a large class of domains Ω.

Theorem 1.1.8 (Poincaré's inequality in $W_0^{1,p}(\Omega)$)**.** *Assume that the open set $\Omega \subset \mathbb{R}^N$ has finite width, that is, it lies between two parallel hyperplanes, and let $1 \leq p < \infty$. Then for all $u \in W_0^{1,p}(\Omega)$,*

$$\int_\Omega |u(x)|^p dx \leq \frac{d^p}{p} \int_\Omega |\nabla u(x)|^p dx \tag{1.8}$$

where d is the distance between the two hyperplanes.

Thus if the open set $\Omega \subset \mathbb{R}^N$ has finite width, then the inequality (1.8) says that the expression $\|\nabla u\|_p$ (which is a seminorm in $W^{1,p}(\Omega)$) is a norm on $W_0^{1,p}(\Omega)$ which is equivalent to the norm (1.3).

Remark 7. *In general, if Ω is an open bounded domain, we can say that there exist a constant $C = C(\Omega, p) > 0$ such that*

$$\int_\Omega |u(x)|^p dx \leq C \int_\Omega |\nabla u(x)|^p dx. \tag{1.9}$$

If we consider the space $W_0^{1,2}(\Omega)$ (also denoted by $H_0^1(\Omega)$), then the bilinear form

$$(u, v) := \int_\Omega \nabla u \cdot \nabla v dx$$

is a inner product that induces the norm $\|\nabla u\|_2$, and it is equivalent to the norm (1.3).

From now on, we will set, for simplicity, $\| \cdot \| := \| \cdot \|_{W_0^{1,2}(\Omega)}$.

Finally, we recall the following useful property

Property 1.1.9. *Let* $u \in W^{1,2}(\Omega)$*. Then*

1. $|u| \in W^{1,2}(\Omega)$*;*

2. *if we set* $u_+(x) = \max\{u(x), 0\}$ *and* $u_-(x) = \max\{-u(x), 0\}$*, the functions* u_+ *and* u_- *belong to* $W^{1,2}(\Omega)$*.*

1.1.2 Embeddings and Sobolev Inequalities

In general terms "Sobolev inequality"has come to mean an estimation of lower order derivatives of a function in terms of its higher order derivatives. Such estimates, valid for all functions in certain classes, are a standard tool in proving existence and regularity of weak solutions of partial differential equations, and, more in general, in the calculus of variations, in geometric measure theory and in many others branches of analysis. The goal is to discover embeddings of various Sobolev space in other space.

Remark 8. *The inequalities (1.8) and (1.9) are examples of Sobolev inequalities.*

In order to deal with embeddings theorems, we need to put forward two important definitions.

Definition 1.1.7. *Let* X *and* Y *two Banach spaces. We say that* X *is embedded continuously in* Y *(briefly* $X \hookrightarrow Y$*) if*

1. $X \subseteq Y$*;*

2. the canonical injection $j : X \to Y$ is a continuous linear operator. This means that there exists a constant $C > 0$ such that $\|j(u)\|_Y \leq C\|u\|_X$, which one writes $\|u\|_Y \leq C\|u\|_X$, for all $u \in X$.

Definition 1.1.8. *Let X and Y Banach space. We say that X is embedded compactly in Y if X is embedded continuously in Y and the canonical injection j is a compact operator, namely for ever bounded subset A in X, the set $j(A)$ is relatively compact in Y.*

The Sobolev inequalities strongly depends on the relation between the exponent p and the dimension N. For this reason we distinguish the cases: $1 \leq p < N$, $p = N$ and $N < p \leq \infty$.

Embeddings: $1 \leq p < N$

One of the most well-known Sobolev embeddings theorem is the following.

Theorem 1.1.10 (Sobolev-Gagliardo-Nirenberg's embeddings theorem)**.** *Let $1 \leq p < N$. Then there exists a constant $C = C(N, p) > 0$ such that for every function $u \in W^{1,p}(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p dx \right)^{\frac{1}{p}}, \tag{1.10}$$

for each $q \in [p, p^*]$, where $p^* := \frac{pN}{N-p}$ is the so called Sobolev critical exponent. In particular, $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$, for each $q \in [p, p^*]$ (note that $p^* > p$).

Definition 1.1.9. *Given $1 \leq p < \infty$, an open set $\Omega \subset \mathbb{R}^N$ is called an extension domain for the Sobolev space $W^{1,p}(\Omega)$ if there exists a continuous linear operator*

$$\mathcal{E} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$$

with the property that for all $u \in W^{1,p}(\Omega)$,

$$\mathcal{E}(u)(x) = u(x) \quad \textit{for a.e. } x \in \Omega.$$

Theorem 1.1.11. *Let $1 \leq p < N$ and let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{1,p}(\Omega)$. Then there exists a constant $C = C(p, N, \Omega) > 0$ such that*

$$\|u\|_q \leq C\|u\|_{W^{1,p}(\Omega)} \tag{1.11}$$

for all $q \leq p \leq p^*$ and $u \in W^{1,p}(\Omega)$.

If $p \leq q < p^*$ and Ω is an extension domain for $W^{1,p}(\Omega)$ with finite measure, then the embedding

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

is actually compact, as stated from the next result.

Theorem 1.1.12 (Rellich-Kondrachov)**.** *Let $1 \leq p < N$ and let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{1,p}(\Omega)$ with finite measure. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be a bounded sequence. Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in L^{p^*}(\Omega)$ such that $u_{n_k} \to u$ in $L^q(\Omega)$ for all $1 \leq q < p^*$.*

In particular, Rellich-Kondrachov theorem applies in the case of bounded domains with boundary of class C.

Embeddings: $p = N$

For $p = N$, we still have embeddings of type

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

that is, inequality of the type

$$\|u\|_{L^q(\mathbb{R}^N)} \leq c\|u\|_{W^{1,N}(\mathbb{R}^N)}.$$

We begin by observing that when $p \to N$ then $p^* \to \infty$, and so one would be tempted to say that if $u \in W^{1,N}(\mathbb{R}^N)$, than $u \in L^\infty(\mathbb{R}^N)$. For $N = 1$ this is true since if $u \in W^{1,1}(\mathbb{R}^N)$, it admits a representative locally absolutely continuous in $\mathbb{R}$, which is also bounded in $\mathbb{R}$. For $N > 1$ this is not the case, indeed, for instance, the function

$$u(x) := \log\left(\log\left(1 + \frac{1}{|x|}\right)\right) \quad x \in B(0,1) \setminus \{0\},$$

belongs to $W^{1,1}(B(0,1))$ but not to $L^\infty(B(0,1))$.

We have the following result.

Theorem 1.1.13. *The space $W^{1,N}(\mathbb{R}^N)$ is continuously embedded in the space $L^q(\mathbb{R}^N)$ for all $N \leq q < \infty$*

Theorem 1.1.14. *Let $\Omega \subset \mathbb{R}^N$ be an extension domain for $W^{1,N}(\Omega)$. Then,*

(i) *There is a constant $C = C(N, \Omega) > 0$ such that*

$$\|u\|_q \leq C\|u\|_{W^{1,p}(\Omega)}$$

for all $N \leq q < \infty$.

(ii) *if Ω has finite measure, the embedding*

$$W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$$

is compact for all $1 \leq q < \infty$.

Embeddings: $N < p \leq \infty$

The next theorem show that if $p > N$, a function $u \in W^{1,p}(\mathbb{R}^N)$ has a representative in the space $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$.

Theorem 1.1.15 (Morrey)**.** *Let $p > N$. Then the space $W^{1,p}(\mathbb{R}^N)$ is continuously embedded in $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$. Moreover, if $u \in W^{1,p}(\mathbb{R}^N)$ and $\overline{u}$ is its representative in $C^{0,1-\frac{N}{p}}(\mathbb{R}^N)$, then*

$$\lim_{|x|\to\infty} \overline{u}(x) = 0.$$

Remark 9. *From the Morrey theorem, in particular from his proof, it can be seen that if $u \in W^{1,p}(\mathbb{R}^N)$ from some $N < p \leq \infty$, then a representative $\overline{u}$ di u is Hölder continuous with exponent $1 - \frac{N}{p}$ and there is a constant $C = C(N,p)$ such that*

$$|\overline{u}(x) - \overline{u}(y)| \leq C|x-y|^{1-\frac{N}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N,\mathbb{R}^N)}$$

for all $x, y \in \mathbb{R}^N$.

The Rellich-Kondrachof Theorem (1.1.12) is still valid

Theorem 1.1.16 (Rellich-Kondrachof, $p > N$)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded extension domain for $W^{1,p}(\Omega)$ and let $p > N$. Then, for all $0 < \alpha < 1 - \frac{N}{p}$ the* *embedding*

$$W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$$

is compact.

Since we will mainly deal with the space $W^{1,2}(\Omega)$, we prefer to explicitly state the above results for this space.

Theorem 1.1.17. *Let $\Omega \subset \mathbb{R}^N$ be an bounded open set, with $N \geq 3$. Then*

$$W_0^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$$

for every $q \in [1, 2^*]$, with $2^* := \frac{2N}{N-2}$ as already defined above. Moreover, the embeddings is compact if and only if $q \in [1, 2^*[$.

In the case $\Omega = \mathbb{R}^N$ one has

Theorem 1.1.18. *Let $N \geq 3$. Then*

$$W^{1,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$$

for every $q \in [2, 2^*]$ and the embeddings is never compact.

In particular, for an arbitrary domain Ω, the following inequality holds

$$\|u\|_q \leq C\|u\| \text{ for all } u \in W_0^{1,2}(\Omega)$$

where C is a constant independent of u.

1.2 Basics of Differential Calculus for Real Functionals

We present a short review of the main definitions and results concerning the differential calculus for real functionals defined on a Banach space.

Definition 1.2.1. *Let X be a normed space and $U \subset X$. A functional I on U is an application*

$$I : U \to \mathbb{R}.$$

Given a normed space X, we denote by X^* its *topological dual*, namely

$$X^* := \{A : X \to \mathbb{R} : A \text{ is a continuous linear functional}\}.$$

We recall that X^* is always a Banach space, if endowed with the norm

$$\|A\|_{X^*} = \sup_{\substack{u \in X \\ \|u\|_X = 1}} |A(u)|.$$

We present the two principal definitions of differentiability and their main properties.

1.2.1 Fréchet differenziability

Definition 1.2.2. *Let X a normed space, U an open subset of X and let $I : U \to \mathbb{R}$ be a functional. We say that I is Fréchet differentiable at $u \in U$ if there exists $A \in X^*$ such that*

$$\lim_{\|v\| \to 0} \frac{I(u+v) - I(u) - Av}{\|v\|} = 0. \tag{1.12}$$

In the following, unless otherwise specified, the concept of differentiability will be meant in the sense of Fréchet differentiability.

By definition of Fréchet differentiability we have that

$$I(u+v) - I(u) = Av + 0(\|v\|)$$

as $\|v\| \to 0$ for some $A \in X^*$, namely the increment $I(u+v) - I(u)$ is linear in v, up to higher order quantity. This implies the following property

Property 1.2.1. *Let X a normed space, U an open subset of X and let $I : U \to \mathbb{R}$ be a functional Fréchet differentiable at $u \in U$. Then I is continuous in u.*

Property 1.2.2. *Let X a normed space, U an open subset of X and let $I : U \to \mathbb{R}$ be a functional Fréchet differentiable at $u \in U$. Then, there is a unique $A \in X^*$ satisfying definition (1.2.2).*

Proof. Let A and B be two different elements of X^* that satisfy (1.12), then plainly

$$\lim_{\|v\| \to 0} \frac{(A-B)v}{\|v\|} = 0,$$

so that, if $u \in X$ and $\|u\| = 1$,

$$(A-B)u = \lim_{t \to 0^+} \frac{(A-B)(tu)}{t} = 0,$$

which means $A = B$. □

Definition 1.2.3. *Let X a normed space, U an open subset of X and let $I : U \to \mathbb{R}$ be a functional differentiable at $u \in U$. The unique element of X^* such that (1.12)* *holds is called the Fréchet differential of I at u, and it is denoted by $I'(u)$ or by $dI(u)$. We thus have*

$$I(u+v) = I(u) + I'(u)(v) + o(\|v\|)$$

as $\|v\| \to 0$. If the functional I is differentiable at every $u \in U$, we say that I is differentiable on U. The map $I' : U \to X^*$ that send $u \in U$ to $I'(u) \in X^*$ is called the Fréchet derivative of I and in general is a nonlinear map. Moreover, if I' is continuous from U to X^* we say that I is of class C^1 on U and we write $I \in C^1(U)$.

A particular but very important case is that of real functionals defined on a Hilbert space H with scalar product $(\cdot, \cdot)$. We recall the following result

Theorem 1.2.3 (Riesz)**.** *Let H be a Hilbert space, and let H^* be its topological dual. Then for every $f \in H^*$ there exists a unique $u_f \in H$ such that*

$$f(v) = (u_f, v) \quad \textit{for all } v \in H.$$

Moreover, $\|u_f\|_H = \|f\|_{H^*}$. The linear application $R : H^* \to H$ that sends f to u_f is called the Riesz isomorphism.

For the previous theorem (1.2.3), thanks to the Riesz isomorphism, the linear functional on H^* can be represented by the scalar product in H, in the sense that for every $A \in H^*$ there exists a unique $RA \in H$ such that

$$A(u) = (RA, u) \quad \text{for every } u \in H.$$

Definition 1.2.4. *Let H be a Hilbert space, $U \subseteq H$ an open set and let $R : H^* \to H$ be the Riesz isomorphism. Assume that the functional $I : U \to \mathbb{R}$ is differentiable at u. The element $RI'(u) \in H$ is called gradient of I at u and it is denoted by $\nabla I(u)$; therefore*

$$I'(u)v = (\nabla I(u), v) \quad \text{for every } v \in H.$$

The following proposition collects the properties of the functional Fréchet differentiable.

Proposition 1.2.4. *Let X a normed space, U an open subset of X and let $I, J : U \to \mathbb{R}$ be two functionals. Assume that I and J are differentiable at $u \in U$. Then the following properties hold:*

1. *if a and b are real numbers, $aI + bJ$ is differentiable at u and*

$$(aI + bJ)'(u) = aI'(u) + bJ'(u);$$

2. *the product IJ is differentiable at u and*

$$(IJ)'(u) = I'(u)J(u) + I(u)J'(u);$$

3. *if $\gamma : \mathbb{R} \to U$is differentiable at t_0 and $u = \gamma(t_0)$, then the composition $\eta : \mathbb{R} \to \mathbb{R}$ defined by $\eta(t) = I(\gamma(t))$ is differentiable at t_0 and*

$$\eta'(t_0) = I'(u)\gamma'(t_0);$$

4. *if $A \subseteq \mathbb{R}$ is an open set, $f : A \to \mathbb{R}$ is differentiable at $I(u) \in A$, then the composition $K(u) = f(I(u))$ is defined in an open neighborhood V of u, is*

differentiable at u *and*

$$K'(u) = f'(I(u))I'(u).$$

1.2.2 Gâteaux differentiability

We now introduce a second notion of differentiability weaker than the Fréchet-differentiability. This notion is easier to verify than the previous one, so it is very useful for applications.

Definition 1.2.5. *Let* X *be a normed space,* $U \subseteq X$ *an open set and let* $I : U \to \mathbb{R}$ *be a functional. We say that* I *is Gâteaux differentiable at* $u \in U$ *if there exists* $A \in X^*$ *such that, for all* $v \in X$*,*

$$\lim_{t \to 0} \frac{I(u+tv) - I(u)}{t} = Av. \tag{1.13}$$

If I *is Gâteaux differentiable at* u*, there is only one* $A \in X^*$ *satisfying (1.13). It is called Gâteaux differential of* I *at* u *and it is denoted by* $I'_G(u)$*.*

Remark 10. *By the definitions (1.2.2) and (1.2.5) is obviously that if* I *is Fréchet differentiable at* u*, then it is also Gâteaux differentiable and* $I'(u) = I'_G(u)$*. In general, the reverse is not true.*

As for the notion of the Fréchet differentiability, if the functional I is Gâteaux differentiable at every u of an open set $U \subset X$, we say that I is Gâteaux differentiable on U. The (generally nonlinear) map $I'_G : U \to X^*$ that sends $u \in U$ to $I'_G(u) \in X^*$ is called the Gâteaux derivative of I.

Proposition 1.2.5. *Let X be a normed space, $U \subseteq X$ an open set and let $I : U \to \mathbb{R}$ be a functional which is Gâteaux differentiable functional on U. Given $u, v \in U$ such that the segment $[u, v] = \{tu + (1-t)v : t \in [0,1]\} \subseteq U$. Then, one has*

$$|I(u) - I(v)| \leq \sup_{w \in [u,v]} I'_G(w) \|u - v\|.$$

The following classical result gives a condition in order to a Gâteaux-differentiable functional is also of Fréchet-differentialble.

Proposition 1.2.6. *Let X be a normed space, $U \subseteq X$ an open set and let $I : U \to \mathbb{R}$ be a functional which is Gâteaux differentiable functional on U. Suppose that I'_G is continuous at $u \in U$. Then I is also Fréchet differentiable at u.*

Proof. We consider the functional $R : U \to \mathbb{R}$ defined by

$$R(h) := I(u + h) - I(u) - I'_G(u)h.$$

Plainly, R is Gâteaux differentiable in the ball B_ε, with $\varepsilon > 0$ small enough, and

$$R'_G(h) : k \to I'_G(u + h)k - I'_G(u)k. \tag{1.14}$$

Applying the proposition (1.2.5), with $[u, v] = [0, h]$, being $R(0) = 0$, we find

$$|R(h)| \leq \sup_{0 \leq t \leq 1} \|R'_G(th)\| \|h\|. \tag{1.15}$$

From (1.14), with th instead h, we deduce

$$\|R'_G(th)\| = \|I'_G(u + th) - I'_G(u)\|.$$

Substituting in (1.15), we find

$$|R(h)| \leq \sup_{0 \leq t \leq 1} \|I'_G(u + th) - I'_G(u)\| \|h\|.$$

Since I'_G is continuous

$$\sup_{0\leq t\leq 1} \|I'_G(u+th) - I'_G(u)\| \to 0 \ \text{ as } \|h\| \to 0$$

and therefore $R(h) = o(\|h\|)$.

Thus

$$I'_G(u)h = I(u+h) - I(u) - o(\|h\|)$$

namely

$$I'_G(u)h = I'(u)h.$$

□

Remark 11. *The importance of this proposition lies in the fact that it is often technically easier to compute the Gâteaux derivative and then prove that it is continuous, rather then proving directly the Fréchet differentiability.*

We conclude this section with the definitions of critical points and critical levels, which are fundamental concepts for the development of the thesis.

Definition 1.2.6. *Let X be a Banach space, $U \subseteq X$ an open set and assume that $I : U \to \mathbb{R}$ is functional differentiable in U. A critical point of I is a point $u \in U$ such that*

$$I'(u) = 0.$$

Since $I'(u)$ is an element of the dual space X^*, the equation $I'(u) = 0$ is equivalent to $I'(u)(v) = 0$, for all $v \in X$.

If $I'(u) = 0$ and $I(u) = c$, we say that u is a *critical point for I at level c.* If for some $c \in \mathbb{R}$ the set $I^{-1}(c) \subset X$ contains at least a critical point, we say that c is a *critical level* for I.

The equation $I'(u) = 0$ is called the *Euler, or Euler-Lagrange equation* associated to the functional I.

1.3 Variational Method

One of the main advantages of extending the class of solutions of a partial differential equation from classical solutions with continuous derivatives to weak solutions with weak derivatives is that often it is easier to prove the existence of this latter kind of solutions. Once the existence of weak solutions is established, one may then studying their properties, such as uniqueness and regularity, and, in some cases, proving, under appropriate assumptions, that the weak solutions are, actually, classical solutions.

There is often considerable freedom in how one defines a weak solution of a partial differential equation; for example, the function space to which a solution is required to belong is not given a priori by the partial differential equation itself. Typically, we look for a weak formulation that reduces to the classical formulation under appropriate smoothness assumptions and which is amenable to a mathematical analysis; the notion of solution and space to which solutions belong are dictated by the available estimates and analysis.

1.3.1 Weak Solutions and critical point

Let $\Omega \subset \mathbb{R}^N$ an bounded and open set with regular boundary $\partial\Omega$.

Let us consider the Dirichlet problem for the Laplacian with homogeneous boundary conditions

$$\begin{cases} -\Delta u = f(x, u(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P)$$

with f a continuous function on $\overline{\Omega} \times \mathbb{R}$.

Definition 1.3.1. *A function $u : \overline{\Omega} \to \mathbb{R}$ is a classical solution of Problem (P) if $u \in C^2(\overline{\Omega})$ and satisfies (P) for every $x \in \overline{\Omega}$.*

To introduce the definition of a weak solution, we need to make some observations.

Let $\varphi \in C_0^\infty(\Omega)$. We multiply the equation of (P) by φ and integrate the result over Ω one has

$$-\int_\Omega \Delta u \varphi \, dx = \int_\Omega f(x, u)\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

By the Green's formula and noting that $\varphi_{|\partial\Omega} = 0$, since φ has compact support, we obtain that if u is a classical solution, then

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega f(x, u)\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

We observe that this equation makes sense even if u is not C^2; for example C^1 suffices. Therefore, the regularity requirements on u and φ can still be weakened very much. Indeed for the integrals to be finite it is enough that $u, \varphi \in L^2(\Omega)$ and so do $\frac{\partial u}{\partial x_i}$ and $\frac{\partial \varphi}{\partial x_i}$, for every $i = 1, \dots, N$.

This motivates the definitions of weak solution.

Definition 1.3.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that $f(x,u)v \in L^1(\Omega)$, for each $u, v \in W_0^{1,2}(\Omega)$, and with the following growth condition*

$$\operatorname*{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^q} < +\infty$$

where $q \in]0, 2^*[$.

A weak solution of Problem (P) is a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x, u(x)) v(x)\, dx, \quad \textit{for every } v \in W_0^{1,2}(\Omega).$$

Remark 12. *A classical solution is also a weak solutions. Moreover, it is easy to prove that if u is a weak solution and $u \in C^2(\overline{\Omega})$, then u is a classical solution.*

The following definition is essential for the discussion, as it is closely linked to weak solutions.

Definition 1.3.3 (Energy Functional)**.** *Let $I : W_0^{1,2}(\Omega) \to \mathbb{R}$ be a real functional. We say I is the "energy functional" associated to problem (P), if*

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\, dx - \int_\Omega \left(\int_0^{u(x)} f(x,t)\, dt \right) dx, \quad for\ each\ u \in W_0^{1,2}(\Omega) \tag{1.16}$$

The next result gives a sufficient condition for the differentiability of the energy functional associate to problem (P).

Theorem 1.3.1. *Let $\Omega \subset \mathbb{R}^N$, be a bounded open set. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

- $\operatorname*{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^q} < +\infty$, *for some* $q \in [0, 2^* - 1]$, *if* $N \geq 3$;
- $\operatorname*{ess\,sup}_{(x,t)\in\Omega\times\mathbb{R}} \frac{|f(x,t)|}{1+|t|^q} < +\infty$, *for some* $q \in [0, +\infty[$, *if* $N = 2$;
- $\sup_{|t|\leq r} |f(\cdot, t)| \in L^1(\Omega)$, *for all* $r > 0$, *if* $N = 1$.

Then, the functional I introduced in (1.16) is well defined and differentiable in $W_0^{1,2}(\Omega)$, and one has

$$I'(u)(v) = \int_\Omega \nabla u(x)\nabla v(x)dx - \int_\Omega f(x, u(x))v(x), \quad \text{for all } u, v \in W_0^{1,2}(\Omega). \tag{1.17}$$

Moreover, if $q < 2^*$ when $N \geq 3$, the functional

$$u \in W_0^{1,2}(\Omega) \longrightarrow \int_\Omega \left(\int_0^{u(x)} f(x,t)\, dt \right) dx$$

is sequentially weakly continuous. Thus, in particular, the functional I is sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$.

Now the connection between weak solutions and critical points is evident: comparing Definition (1.3.2) and (1.17), one sees that

u is a weak solution of problem (P) if and only if u is a critical point of the functional I.

The correspondence between weak solutions and critical point of functionals outlined above is valid of course for more general nonlinear problems.

Remark 13. *The growth conditions imposed on* f *in Theorem (1.3.1) are essentially to have that the energy functional is well-defined on* $W_0^{1,2}(\Omega)$*. Indeed, if* $f(x,\cdot)$ *grows faster that* $|t|^{2^*-1}$*, then* $F(x,t) = \int_0^t f(x,s)ds$ *grows faster than* $|t|^{2^*}$*; therefore, since* $W_0^{1,2}(\Omega)$ *is not embedded in* $L^p(\Omega)$ *for* $p > 2^*$*, the function* $F(x,u(x))$ *might be not in* $L^1(\Omega)$*, for some* $u \in W_0^{1,2}(\Omega)$*.*

Convex Functional and Minimum Theorems

The typical functionals I whose critical points give rise to weak solutions of differential equations are integral functionals that normally contain a term involving the gradient of u, in many cases the integral of some power of $|\nabla u|$. This term is bounded below because it is nonnegative, but in general it is not bounded above. In addiction, a term of this type is not sequentially weakly continuous but only sequentially weakly lower semicontinuous. Thus if a functional contains such a term, it may be perhaps minimized, but probably not maximized.

This is why the most "natural" critical points of differentiable integral functionals are often its global (or local) minima.
A sometimes relevant assumption in looking for global minima is the convexity of I.

Definition 1.3.4. *A functional* $I : X \to \mathbb{R}$ *on a vector space* X *is called convex if, for every* $u, v \in X$ *and every real* $t \in [0,1]$*, one has*

$$I(tu + (1-t)v) \leq tI(u) + (1-t)I(v).$$

The functional is strictly convex if, for every $u, v \in X$*,* $u \neq v$*, and every real*

$t \in (0, 1)$, *one has*

$$I(tu + (1-t)v) < tI(u) + (1-t)I(v).$$

Finally, we say that I is (strictly) concave if $-I$ is (strictly) convex.

Theorem 1.3.2. *Let $I : X \to \mathbb{R}$ be a continuous convex functional on a normed space X. Then I is (sequentially) weakly lower semicontinuous. In particular, for every sequence $\{u_k\}_{k\in\mathbb{N}} \subset X$ converging weakly to $u \in X$, we have*

$$I(u) \leq \liminf_{k\to\infty} I(u_k).$$

Remark 14. *The norm of a normed space is an example of a continuous convex functional.*

A continuous convex functional need not have a minimum, even if it is bounded below. To have a minimum we need to introduce the concept of coercivity.

Definition 1.3.5. *A functional $I : X \to \mathbb{R}$ on a normed space X is called coercive if*

$$\lim_{\|u\|_X \to +\infty} I(u) = +\infty.$$

The following is one of the fundamental result in nonlinear analysis.

Theorem 1.3.3. *Let X be a reflexive Banach space and let $I : X \to \mathbb{R}$ be a continuous, convex and coercive functional. Then I has a global minimum point.*

Proof. Let $m = \inf_{u\in X} I(u)$ and let $\{u_k\}_{k\in\mathbb{N}} \subset X$ be a minimizing sequence. Coercivity implies that $\{u_k\}_{k\in\mathbb{N}}$ is bounded. Since X is reflexive, by the Kakutani

Theorem we can extract from $\{u_k\}_{k\in\mathbb{N}}$ a subsequence, still denoted u_k, such that u_k converges weakly to some $u \in X$. By theorem (1.3.2) we then obtain

$$I(u) \leq \liminf_{k\to\infty} I(u_k) = m.$$

Therefore $I(u) = m$ and u is a global minimum for I. □

We observe that the convexity assumption is only used to deduce weak lower semicontinuity from continuity (via theorem (1.3.2)). A more general statement is thus the following version of the Weierstrass Theorem:

Theorem 1.3.4 (Weierstrass)**.** *Let X a reflexive Banach space and let $I : X \to \mathbb{R}$ be a weakly lower semicontinuous and coercive functional. Then I has a global minimum point.*

The strict convexity is related to uniqueness properties.

Theorem 1.3.5. *Let X vector space and let $I : X \to \mathbb{R}$ be a strictly convex functional. Then I has at most one minimum point in X.*

Proof. Assume that I has two different global minima u_1 and u_2 in X. By strict convexity,

$$\begin{aligned}\min_{u\in X} I(u) \quad &\leq I\left(\frac{u_1+u_2}{2}\right) < \frac{1}{2}I(u_1) + \frac{1}{2}I(u_2) = \frac{1}{2}\min_{u\in X} I(u) + \frac{1}{2}\min_{u\in X} I(u)\\ &= \min_{u\in X} I(u),\end{aligned}$$

a contradiction. □

As a direct consequence of the previous result one has

Theorem 1.3.6. *Let X be a normed space and let $I : X \to \mathbb{R}$ be strictly convex and differentiable. Then I has at most one critical point in X.*

1.4 Minimax Metod: The Mountain Pass Theorem

In the preceding section we have seen that (sequential) weak lower semicontinuity and coercivity of a functional J on a reflexive Banach space X suffice to guarantee the existence of a minimizer of J.

To prove the existence of saddle points we will now strengthen the regularity hypothesis on I and, in general, we will require I to be of class $C^1(X)$. Moreover, we will impose a certain compactness assumption on J, the so called "Palais-Smale condition". At first, we recall a classical result in finite dimensions.

Theorem 1.4.1. *Let $J \in C^1(\mathbb{R}^N)$ be a coercive functional having two distinct strict relative minima u_1 and u_2. Then J possesses a critical point u_3 which is not a relative minimizer of J, and hence distinct from u_1, u_2. Moreover, u_3 satisfies*

$$J(u_3) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J(u), \tag{1.18}$$

where

$$\Gamma = \{\gamma \in C^0([0,1], \mathbb{R}^N) : \quad \gamma(0) = u_1, \quad \gamma(1) = u_2\}$$

is the class of "paths" connecting u_1 and u_2.

A point satisfying the identity (1.18) is called "a saddle point" of J. The previous result is sometimes called the finite dimensional "Mountain Pass Theorem".

In the infinite dimensional case things are a bit more complicated. In fact, the previous result is no longer valid if J is defined in a infinite dimensional normed space. Indeed, in this case, under the same assumption on J, saddle points in general need not exist, unless a certain compactness property holds.

Definition 1.4.1. *Let X be a normed space and let $J : X \to \mathbb{R}$ be a differentiable functional. A sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq X$ such that*

$$\{J(u_k)\}_k \qquad \textit{is bounded (in } \mathbb{R}\textit{) and}$$
$$J'(u_k) \to 0 \qquad \textit{(in } X'\textit{) as } k \to \infty,$$

is called a Palais-Smale sequence for J.

Let $c \in \mathbb{R}$. If

$$J(u_k) \to c \qquad \textit{(in } \mathbb{R}\textit{) and}$$
$$J'(u_k) \to 0 \qquad \textit{(in } X'\textit{) as } k \to \infty,$$

then $\{u_k\}_k$ is called a Palais-Smale sequence for J at level c. In this case c is called a Palis-Smale level for J.

Remark 15. *In a Hilbert space H we can identify the differential with the gradient via the inner product. Therefore the second property of a Palais-Smale sequence reads*

$$\nabla J(u_k) \to 0 \textit{ (in } H\textit{) as } k \to \infty.$$

As we will see, the convergence of Palais–Smale sequences is crucial in proving the existence of saddle points. This fact leads to the following definition

Definition 1.4.2 ((PS) condition)**.** *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a differentiable functional. We say that J satisfies the Palais-Smale condition* *(shortly: J satisfies (PS)) if every Palais-Smale sequence for J has a converging subsequence. We say that J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly: J satisfies $(PS)_c$) if every Palais-Smale sequence at level c has a converging subsequence.*

A bounded below functional satisfying the (PS) condition always admits global minima, as stated by the following theorem

Theorem 1.4.2. *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a differentiable functional bounded below in X and satisfies (PS). Then, there exists $u_0 \in X$ such that*

$$J(u_0) = \min_{u \in X} J(u) \quad \textit{and} \quad J'(u_0) = 0.$$

The prototype of a differentiable functional with (PS) condition is that of a functional $J : \mathbb{R}^N \to \mathbb{R}$ with continuous first partial derivatives and satisfying coercivity condition $J(u) \to +\infty$ as $\|u\| \to +\infty$.

Another important class of functionals satisfying (PS) is provided by the following proposition [see Example 38.25 of [18]]

Property 1.4.3. *Let X be a Banach space and let $A, C : X \to \mathbb{R}$ be Gâteaux differentiable functionals. Suppose that*

(i) $A(u) + C(u) \to +\infty$ *as* $\|u\| \to +\infty$*;*

(ii) $A' : X \to X^*$ *has a continuous inverse operator, and* $C' : X \to X^*$ *is compact.*

Then, $A + C$ satisfies the (PS) condition.

Now, we present one of the most celebrated result in critical point theory, the Mountain Pass Theorem by Ambrosetti - Rabinowitz [3]. This result have been generalized in several directions, but very frequently it is still used in their original form. We give below two versions of this result (see also [10] and [16] for other versions).

Lemma 1.4.4 (Mountain Pass Lemma)**.** *Let X be a Banach space and let $J : X \to \mathbb{R}$ be a continuously differentiable functional. Assume that there exists two distinct points $u_0\,, u_1 \in X$ and a number $r \in]0, \|u_0 - u_1\|_X[$ such that*

$$\inf_{\|u-u_0\|=r} J(u) > \max\{J(u_0),\ J(u_1)\}.$$

Moreover, let

$$c = \inf_{u\in\Gamma(u_0,u_1)} \sup_{t\in[0,1]} J(u(t)),$$

where $\Gamma(u_0, u_1) = \{u \in C^0([0, 1], X) : \quad u(0) = u_0, \quad u(1) = u_1\}$.

Then, for all $\varepsilon > 0$ there exists $u_\varepsilon \in X$ such that

$$\begin{cases} c \le J(u_\varepsilon) \le c + \varepsilon \\ \|J'(u_\varepsilon)\|_{X^*} \le \varepsilon. \end{cases} \tag{1.19}$$

Theorem 1.4.5 (Mountain Pass Theorem)**.** *Under the same hypotheses of Lemma 1.4.4, assume, in addiction, that J satisfies the (PS) condition. Then, there exist $\hat{u} \in X$ such that:*

$$J(\hat{u}) = c \quad and \quad J'(\hat{u}) = 0_{X^*}.$$

Proof. By the Lemma (1.4.4), for all $n \in \mathbb{N}$, there exists $u_n \in X$, such that:

$$c \le J(u_n) \le c + \frac{1}{n}; \tag{1.20}$$

$$\|J'(u_n)\|_{X^*} \leq \frac{1}{n}. \tag{1.21}$$

By (1.20) it follows that

$$\lim_{n\to+\infty} J(u_n) = c, \tag{1.22}$$

while, by (1.21) one has

$$\lim_{n\to+\infty} \|J'(u_n)\|_{X^*} = 0,$$

that is

$$\lim_{n\to+\infty} J'(u_n) = 0_{X^*}. \tag{1.23}$$

Therefore, $\{u_n\}_n$ is a Palais-Smale sequence for J. Since J satisfies the (PS) condition, there exists a subsequence $\{u_{n_k}\}_k$ of $\{u_n\}_n$ and $\hat{u} \in X$ such that

$$\lim_{k\to+\infty} u_{n_k} = \hat{u}.$$

By the continuity of J and J' in X and taking into account (1.22) and (1.23), one has

$$J(\hat{u}) = \lim_{k\to+\infty} J(u_{n_k}) = c,$$
$$J'(\hat{u}) = \lim_{k\to+\infty} J'(u_{n_k}) = 0_{X^*}.$$

□

The most used form in applications is the following.

Theorem 1.4.6 (Mountain Pass Theorem)**.** *Let H be a Hilbert space, and let $J \in C^1(H)$ satisfying $J(0) = 0$. Assume that there exist a positive numbers ρ and α such that*

1. $J(u) \geq \alpha$ if $\|u\| = \rho$;

2. There exists $v \in H$ such that $\|v\| > \rho$ and $J(v) < \alpha$.

Then there exists a Palais-Smale sequence for J at a level $c \geq \alpha$. If J satisfies $(PS)_c$, then there exists a critical point at level c.

Remark 16. *The term "mountain pass"is justified by the geometrical properties of the graph of J. When a functional J satisfies the conditions* 1. *and* 2. *we say that J has the "Mountain Pass Geometry".*

1.5 Maximum Principle

We conclude this first chapter of premises with a basic tool for proving existence of positive solutions to elliptic boundary value problem, that is the "Hopf Maximum Principle".

The strong maximum principle is based on the Hopf boundary maximum principle.

We consider the elliptic operator $-\Delta + c(x)$. The following lemma demonstrates that a super-solution u cannot attain its minimum at an interior point of a connected region at all, unless u is constant. This statement is the *strong maximum principle*, which depends on the following subtle analysis of the outer normal derivative $\frac{\partial u}{\partial \nu}$ at a boundary maximum point.

Lemma 1.5.1 (Hopf's Lemma)**.** *Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and*

$$c \equiv 0 \quad in\ \Omega.$$

Suppose further

$$-\Delta u + c \geq 0 \quad in\ \Omega,$$

and there exists a point $x^0 \in \partial\Omega$ *such that*

$$u(x^0) < u(x) \quad for\ all\ x \in \Omega.$$

Assume finally that Ω *satisfies the interior ball condition at* x^0*; that is, there exists an open ball* $B \subset \Omega$ *with* $x^0 \in \partial B$.

(i) Then

$$\frac{\partial u}{\partial \nu}(x^0) < 0,$$

where ν *is the outer unit normal to* B *at* x^0.

(ii) If

$$c \geq 0 \quad in\ \Omega,$$

the same conclusion holds provided

$$u(x^0) \leq 0.$$

Remark 17. *The importance of (i) is the strict inequality: that* $\frac{\partial u}{\partial \nu}(x_0) \leq 0$ *is obvious. Note that the interior ball condition automatically holds if* $\partial\Omega$ *is* C^2.

Hopf's Lemma is the main technical tool for demonstrating the following theorem

Theorem 1.5.2 (Strong maximum principle)**.** *Assume* $u \in C^2(\Omega) \cap C(\overline{\Omega})$ *and*

$$c \equiv 0 \quad in\ \Omega.$$

Suppose also Ω *is connected, open and bounded.*

(i) If

$$-\Delta u + c \leq 0 \quad in\ \Omega$$

and u *attains its maximum over* $\overline{\Omega}$ *at an interior point, then*

$$u \text{ is constant within } \Omega.$$

(ii) Similarly, if

$$-\Delta u + c \geq 0 \quad in\ \Omega$$

and u *attains its minimum over* $\overline{\Omega}$ *at in interior point, the*

$$u \text{ is constant within } \Omega.$$

A classical version of the maximum principle (see B.4 Theorem of [17]) is the following

Theorem 1.5.3. *Under the same previous hypotheses, suppose that* u *satisfies*

$$-\Delta u + c \geq 0 \ in\ \Omega,\ and\ u \geq 0\ on\ \partial\Omega.$$

Moreover, suppose there exists $h \in C^2(\Omega) \cup C^0(\overline{\Omega})$ *such that*

$$-\Delta h + c \geq 0\ in\ \Omega,\ and\ h > 0\ on\ \Omega.$$

Then, either $u > 0$ *in* Ω*, or* $u = \beta h$ *for some* $\beta \leq 0$*.*

As for the Laplacian (i.e. when the coefficient $c \equiv 0$ in Ω), one has the following:

Corollary 1.5.4. *Let* $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy*

$$-\Delta u \geq 0 \quad in\ \Omega.$$

Suppose that $u \geq 0$ *on* Ω. *Then*

- *either* $u > 0$ *in* Ω;
- *or* $u \equiv 0$ *in* Ω.

Remark 18. *It is show that the previous corollary is still valid if the coefficient* $c \in C^\alpha(\overline{\Omega})$.

Chapter 2

Two Elliptic Problems involving nonlinearities indefinite in sign

In this chapter, we study two elliptic equations with Dirichlet condition. In particular, we will establish multiplicity results of nonnegative and nonzero solutions.

2.1 Notations

Throughout this chapter $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is nonempty open bounded set with regular boundary $\partial\Omega$ (1.1.5). We will make use of the following notations, already introduced in the first Chapter:

- for $p \geq 1$, $\|\cdot\|_p := \left(\int_\Omega |\cdot|^p dx\right)^{\frac{1}{p}}$ is the standard norm in the space $L^p(\Omega)$;
- $\|\cdot\|_\infty = \operatorname{ess\,sup}_\Omega |u|$ is the standard norm in the space $L^\infty(\Omega)$.
- we denote by $\|\cdot\| \overset{\text{def}}{=} \|\nabla(\cdot)\|_2$ the standard norm in the space $W_0^{1,2}(\Omega)$.

- for $p \in [1, 2^*]$,

$$c_p \stackrel{\text{def}}{=} \sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\|u\|_p}{\|u\|}$$

is the best constant for the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$, with $2^* = \frac{2N}{N-2}$. In particular,

$$\lambda_1 = c_2^{-2},$$

is the first eigenvalue of the Laplacian on Ω.

- $\|\cdot\|_{C^1(\overline{\Omega})}$ is the standard norm in the space $C^1(\overline{\Omega})$.

- For $\alpha \in]0,1[$, $\|\cdot\|_{C^{1,\alpha}(\overline{\Omega})}$ is the standard norm in the space $C^{1,\alpha}(\overline{\Omega})$.

- Given a function $h : \Omega \to \mathbb{R}$, the symbols h_+, h_- denotes the functions defined by

$$h_+(x) \stackrel{\text{def}}{=} \max\{h(x), 0\}, \quad h_-(x) \stackrel{\text{def}}{=} \max\{-h(x), 0\}, \quad x \in \Omega.$$

Remark 19. *In the sequel, when we write $u = 0$ in $\partial\Omega$ we mean in the sense of the trace (1.1.6). In literature it is usual to abuse the notation and write only $u_{|\partial\Omega} = 0$, omitting that it is in the sense of the trace operator.*

2.2 A Multiplicity result for non-Autonomous Sublinear Elliptic Problem

We first study a local problem involving two measurable weight functions $\alpha, \beta : \Omega \to \mathbb{R}$. Let s and r be two positive numbers such that $s \in]1, 2[$ and $r \in]1, s[$. We deal with the following non autonomous elliptic problem

$$\begin{cases} -\Delta u = \alpha(x)u^{s-1} - \mu\beta(x)u^{r-1}, & \text{in} \quad \Omega \\ u \geq 0, & \text{in} \quad \Omega \\ u_{|\partial\Omega} = 0 \end{cases} \qquad (P_\mu)$$

where $\mu \in \mathbb{R}$ is a parameter. We will establish, via minimax methods, a multiplicity result under suitable summability conditions on the weight functions α, β.

Solutions to problem (P_μ) will be understood in the weak sense, that is

Definition 2.2.1. *A weak solution of problem (P_μ) is a function $u \in W_0^{1,2}(\Omega)$, with $u \geq 0$ a.e. in Ω, satisfying the equation*

$$\int_\Omega \big(\nabla u(x)\nabla\varphi(x) - \alpha(x)u(x)^{s-1}\varphi(x) + \mu\beta(x)u(x)^{r-1}\varphi(x)\big)\, dx = 0,$$

for all $\varphi \in W_0^{1,2}(\Omega)$. A weak solution u to problem (P_μ) is said positive if $u > 0$ a.e. in Ω.

The next proposition gives suitable summability conditions on α, β in order to the energy functional associated to (P_μ) is (well defined and) differentiable. The proof follows standard arguments and we omit it for sake of brevity.

Property 2.2.1. *Assume $\alpha \in L^{\frac{2^*}{2^*-s}}(\Omega)$ and $\beta \in L^{\frac{2^*}{2^*-r}}(\Omega)$. Then, the functional*

$$I_\mu(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{s}\int_\Omega \alpha(x)(u_+)^s\, dx + \frac{\mu}{r}\int_\Omega \beta(x)(u_+)^r\, dx, \quad u \in W_0^{1,2}(\Omega), \tag{2.1}$$

is well defined and Gâteaux differentiable in Ω for each $\mu \in \mathbb{R}$.

Assuming a higher summability on α, β we also get the sequential weak lower semicontinuity of I_μ, as stated by the following proposition. Its proof is a direct consequence of the Rellich-Kondrachov compact embeddings.

Property 2.2.2. *If $\alpha \in L^q(\Omega)$ and $\beta \in L^m(\Omega)$, for some $q > \frac{2^*}{2^*-s}$ and $m > \frac{2^*}{2^*-r}$, the functional I_μ turns out to be sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$.*

Of course, I_μ is strongly continuous in $W_0^{1,2}(\Omega)$ as well. Moreover, note that, if $u \in W_0^{1,2}(\Omega)$ is a critical point of I, one has

$$I'_\mu(u)(v) = \int_\Omega \big(\nabla u(x)\nabla v(x) - \alpha(x)u_+(x)^{s-1}v(x) + \mu\beta(x)u_+(x)^{r-1}v(x)\big)\, dx$$

for all $v \in W_0^{1,2}(\Omega)$. Testing with $v = u_-$, we get $\|u_-\| = 0$ which means that u is nonnegative. Then, recalling (2.2.1), the critical points of I_μ are exactly the weak solutions of (P_μ).

2.2.1 Existence results

Let us start by considering the functional associated with the unperturbed problem (P_0).

For $\mu = 0$, the functional I_μ takes the form

$$I_\mu(u) = I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u_+(x)^s\, dx, \quad u \in W_0^{1,2}(\Omega).$$

Our first result gives a sufficient condition (which is also necessary, as it can be easily checked) on the weigh function α to get a global minimum point u_0 of I_0 with negative energy.

Lemma 2.2.3. *Let $q > \frac{2^*}{2^*-s}$ and $\alpha \in L^q(\Omega)$, with $\operatorname{ess\,sup}_\Omega \alpha > 0$. Then, there exists $u_0 \in W_0^{1,2}(\Omega)$ such that*

$$I_0(u_0) = \inf_{u \in W_0^{1,2}(\Omega)} I_0(u) < 0.$$

Moreover, one has $\|u_0\| = \left(\sup_{\|u\|=1} \int_\Omega \alpha(x) u_+(x)^s dx \right)^{\frac{1}{2-s}}$.

Proof. At first, observe that

$$\begin{aligned}
\inf_{u \in W_0^{1,2}(\Omega)} I_0(u) &= \inf_{\sigma>0} \inf_{\|u\|=\sigma} \left(\frac{1}{2}\|u\|^2 - \frac{1}{s} \int_\Omega \alpha(x) u_+^s \, dx \right) \\
&= \inf_{\sigma>0} \left(\frac{1}{2}\sigma^2 + \inf_{\|u\|=\sigma} \left(-\frac{1}{s} \int_\Omega \alpha(x) u_+^s \, dx \right) \right) \\
&= \inf_{\sigma>0} \left(\frac{1}{2}\sigma^2 - \frac{1}{s} \sup_{\|u\|=\sigma} \int_\Omega \alpha(x) u_+^s \, dx \right) \\
&= \inf_{\sigma>0} \left(\frac{1}{2}\sigma^2 - \frac{\sigma^s}{s} \sup_{\|u\|=1} \int_\Omega \alpha(x) u_+^s \, dx \right)
\end{aligned}$$

Then, since $s \in]1,2[$, one has $\inf_{u \in W_0^{1,2}(\Omega)} I_0(u) < 0$, if and only if

$$\sup_{\|u\|=1} \int_\Omega \alpha(x) u_+^s \, dx > 0. \tag{2.2}$$

Let us to show that the condition $\operatorname{ess\,sup}_\Omega \alpha > 0$, which is equivalent to $\alpha_+ \neq 0$, implies (2.2). To this end, note that if $\alpha_+ \neq 0$, one has

$$\int_\Omega \alpha_+(x) \alpha_+(x)^{\frac{s}{2^*-s}} \, dx = \int_\Omega \alpha_+(x)^{\frac{2^*}{2^*-s}} \, dx > 0. \tag{2.3}$$

Moreover, the functional

$$J(u) \stackrel{\text{def}}{=} \int_\Omega \alpha(x) u_+(x)^s dx, \quad u \in L^{2^*}(\Omega)$$

is (well defined and) continuous in $L^{2^*}(\Omega)$. Since $\alpha_+(x)^{\frac{1}{2^*-s}} \in L^{2^*}(\Omega)$, we can evaluate J at $u = \alpha_+(x)^{\frac{1}{2^*-s}}$. Thanks to (2.3), one has

$$J(\alpha_+^{\frac{1}{2^*-s}}) = \int_\Omega \alpha_+(x)^{\frac{2^*}{2^*-s}} \, dx > 0.$$

Then, using the density of $W_0^{1,2}(\Omega)$ in $L^{2^*}(\Omega)$ and (2.3), we can find a function $v \in W_0^{1,2}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \alpha_+(x)v_+(x)dx > 0.$$

Therefore,

$$\sup_{\|u\|=1} \int_\Omega \alpha_+(x)u_+(x)dx \geq \int_\Omega \alpha_+(x)\frac{v_+(x)}{\|v\|}dx > 0$$

which proves (2.2).

Finally, note that being I_0 coercive and sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$, there exists a global minimum point $u_0 \in W_0^{1,2}(\Omega)$ of I_0 in $W_0^{1,2}(\Omega)$. Since u_0 is a critical point of I_0, one has

$$I_0'(u_0)(u_0) = \|u_0\|^2 - \int_\Omega \alpha(x)u_0(x)^s dx = 0.$$

Thus, if we consider the Nehari manifold $\mathcal{N}_0$ of I_0, defined by

$$\mathcal{N}_0 = \left\{u \in W_0^{1,2}(\Omega) \setminus \{0\} : \|u\|^2 = \int_\Omega \alpha(x)u(x)_+^s dx\right\} = \left\{\left(\frac{\int_\Omega \alpha(x)u(x)_+^s dx}{\|u\|^2}\right)^{\frac{1}{2-s}} u : u \in W_0^{1,2}(\Omega) \setminus \{0\}, \int_\Omega \alpha(x)u(x)_+^s dx \geq 0\right\},$$

we get

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{s}\right)\|u_0\|^2 = I_0(u_0) = \\
&\inf_{u \in W_0^{1,2}(\Omega)} I_0(u) = \left(\frac{1}{2} - \frac{1}{s}\right)\sup_{u \in \mathcal{N}_0}\|u\|^2 = \\
&\left(\frac{1}{2} - \frac{1}{s}\right)\left[\sup_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \|u\|^{-1}\left(\int_\Omega \alpha(x)u(x)_+^s dx\right)^{\frac{1}{s}}\right]^{\frac{2s}{2-s}} = \\
&\left(\frac{1}{2} - \frac{1}{s}\right)\left(\sup_{\|u\|=1}\int_\Omega \alpha(x)u(x)_+^s dx\right)^{\frac{2}{2-s}},
\end{aligned}$$

that is $\|u_0\| = \left(\sup_{\|u\|=1} \int_\Omega \alpha(x)u(x)_+^s dx\right)^{\frac{1}{2-s}}$ which concludes the proof. □

Thanks to the previous Lemma 2.2.3, we can find, for μ sufficiently small, a global minimum point of I_μ with negative energy. More precisely, we will prove the following lemma

Lemma 2.2.4. *Let* $q > \frac{2^*}{2^*-s}$, $m > \frac{2^*}{2^*-r}$, $\alpha \in L^q(\Omega)$, *with* $\operatorname{ess\,sup}_\Omega \alpha > 0$, *and* $\beta \in L^m(\Omega)$. *Then, there exist* $\mu_0, \rho, c_0 \in]0, +\infty[$ *such that, for each* $\mu \in [-\mu_0, \mu_0]$, *there exists a (nonnegative) function* $u_\mu \in W_0^{1,2}(\Omega)$ *satisfying*

$$I_\mu(u_\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u) \leq -\rho < 0, \tag{2.4}$$

$$\|u_\mu\| \geq c_0. \tag{2.5}$$

Proof. For each $\mu > 0$, the functional I_μ is coercive e sequentially weakly lower semicontinuous. Hence, there exists $u_\mu \in W_0^{1,2}(\Omega)$ (with u_μ nonnegative in Ω) such that

$$I_\mu(u_\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u).$$

Now, consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(\mu) = \inf_{u \in W_0^{1,2}(\Omega)} I_\mu(u), \quad \text{for each } \mu \in \mathbb{R}.$$

Since g is the inferior envelope of affine functions, then g is concave in $\mathbb{R}$, and thus continuous there as well. Moreover, by Lemma 2.2.3, one has $g(0) < 0$. Thus, if we fix $\rho \in]g(0), 0[$, there exists $\mu_0 > 0$ such that $g(\mu) < \rho$, for each $\mu \in [-\mu_0, \mu_0]$. Therefore, (2.4) holds.

It remains to prove (2.5). Let $\mu \in [-\mu_0, \mu_0]$ and let $u_\mu \in W_0^{1,2}(\Omega)$ be satisfying (2.4). Using the Sobolev embedding theorems, we can find two constants $c_1, c_2 > 0$ (depending only on $N, s, r, \Omega, \mu_0, \alpha, \beta,$) such that

$$
\begin{aligned}
-\rho \geq I_\mu(u_\mu) = &\\
&\frac{1}{2}\|u_\mu\|^2 - \frac{1}{s}\int_\Omega \alpha(x) u_\mu^s dx + \frac{\mu}{r}\int_\Omega \beta(x) u_\mu^r dx \geq\\
&\frac{1}{2}\|u_\mu\|^2 - \frac{1}{s}\int_\Omega |\alpha(x)| u_\mu^s dx - \frac{\mu_0}{r}\int_\Omega |\beta(x)| u_\mu^r dx \geq\\
&\frac{1}{2}\|u_\mu\|^2 - c_1\|u_\mu\|^s - c_2\|u_\mu\|^r \geq\\
&\frac{1}{4}\|u_\mu\|^2 - \varepsilon_0\|u_\mu\|
\end{aligned}
$$

where $\varepsilon_0 = \max_{t>0}(c_1 t^{s-1} + c_2 t^{r-1} - \frac{1}{4}t) > 0$. The inequality $\frac{1}{4}\|u_\mu\|^2 - \varepsilon_0\|u_\mu\| + \rho < 0$ entails $\varepsilon_0^2 > \rho$ and

$$\|u_\mu\| \geq 2(\varepsilon_0 - \sqrt{\varepsilon_0^2 - \rho}) := c_0 > 0.$$

This concludes the proof. □

2.2.2 Multiplicity results

Our aim is now to see if there are additional conditions on the weight functions α, β under which we get a mountain pass geometry for the functional I_μ, at least for μ small enough. The next two lemmas give such conditions.

Let us consider two disjoint sets $\Omega_{\alpha_+}, \Omega_{\alpha_-} \subset \Omega$ such that $\Omega_{\alpha_+} \cup \Omega_{\alpha_-} = \Omega$, $\alpha(x) > 0$ for almost all $x \in \Omega_{\alpha_+}$, and $\alpha(x) \leq 0$ for almost all $x \in \Omega_{\alpha_-}$;

Lemma 2.2.5. *Let α, β be as in Lemma 2.2.4. Assume that*

(*i*) $\lambda_{\alpha\beta} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega_{\alpha_+}} \dfrac{\beta(x)}{\alpha(x)} > 0$;

(ii) $\alpha_+ \in L^q(\Omega)$, *for some* $q > \frac{N}{2}$.

Then, for each $\mu > 0$, *there exists* $\sigma_0 > 0$ *such that*

$$\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{4}\sigma^2 > 0, \quad \text{for all} \quad \sigma \in]0, \sigma_0[\tag{2.6}$$

In particular, $u = 0$ *is a (strict) local minimum point for the functional* I_μ.

Proof. Let $q > \frac{N}{2}$ be such that $\alpha_+ \in L^q(\Omega)$ and let $\mu > 0$. We may assume $q < \infty$. Let us denote $q' = \frac{q}{q-1}$. Then, $q' < \frac{N}{N-2}$ and

$$p_0 \stackrel{\text{def}}{=} \frac{2^*}{q'} \in]2, 2^*[.$$

By an easy calculation we see that for almost all $x \in \Omega_{\alpha+}$, one has $\alpha(x) > 0$ and

$$\sup_{t>0} \left\{ \frac{\frac{1}{s}\alpha(x)t^s - \frac{\mu}{r}\beta(x)t^r}{t^{p_0}} \right\} = \left(\frac{\mu}{r} \frac{\beta(x)}{\alpha(x)} \right)^{-\frac{p_0-s}{s-r}} \left(\frac{s-r}{p_0-s} \right) \alpha(x) \leq M\, \alpha(x) \tag{2.7}$$

where

$$M = \lambda_{\alpha\beta}^{-\frac{p_0-s}{s-r}} \left(\frac{r}{\mu} \right)^{\frac{p_0-s}{s-r}} \left(\frac{s-r}{p_0-s} \right) > 0.$$

Moreover, by the Hölder inequality, we see that $\alpha \cdot u^{p_0} \in L^1(\Omega)$, for each $u \in W_0^{1,2}(\Omega)$. Thus, keeping in mind (2.7), for $\sigma > 0$ one has:

$$
\begin{aligned}
&\inf_{\|u\|=\sigma} I_\mu(u)\\
&\qquad = \inf_{\|u\|=\sigma}\left(\frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x)u^s\,dx + \frac{\mu}{r}\int_\Omega \beta(x)u^r\,dx\right)\\
&\qquad = \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma}\left(\int_\Omega \frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r\,dx\right)\\
&\qquad \geq \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma}\int_{\Omega_{\alpha_+}}\left(\frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r\right)dx\\
&\qquad \geq \frac{1}{2}\sigma^2 - M\sup_{\|u\|=\sigma}\int_{\Omega_{\alpha_+}}\alpha(x)u^{p_0}dx\\
&\qquad \geq \frac{1}{2}\sigma^2 - M\sup_{\|u\|=\sigma}\left(\int_{\Omega_{\alpha_+}}\alpha(x)^q dx\right)^{\frac{1}{q}}\left(\int_\Omega u^{2^*}dx\right)^{\frac{1}{q'}}\\
&\qquad \geq \frac{1}{2}\sigma^2 - Mc_{2^*}^{\frac{2^*}{q'}}\sup_{\|u\|=\sigma}\left(\int_{\Omega_{\alpha_+}}\alpha(x)^q dx\right)^{\frac{1}{q}}\|u\|^{p_0}\\
&\qquad = \frac{1}{2}\sigma^2 - Mc_{2^*}^{\frac{2^*}{q'}}\left(\int_{\Omega_{\alpha_+}}\alpha(x)^q dx\right)^{\frac{1}{q}}\sigma^{p_0}
\end{aligned}
$$

Since $p_0 > 2$, one has

$$
\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{4}\sigma^2 > 0 = I_\mu(0),
$$

provided that σ is small enough. This concludes the proof. □

Property 2.2.6. *Under the assumption of Lemmas* 2.2.4 *and* 2.2.5*, the energy functional* I_μ *has the mountain pass geometry for all* $\mu \in]0, \mu_0]$.

Proof. To view this, just observe that:

1. from Lemma 2.2.5 $\inf_{\|u\|=\sigma} I_\mu(u) > 0$ for all $\sigma \in]0, \sigma_0[$;

2. from Lemma 2.2.4 there exists $u_\mu \in W_0^{1,2}(\Omega)$ such that $\|u_\mu\| > c_0$ and $I_\mu(u_\mu) < 0$.

Then, choosing $\sigma \in]0, \min\{c_0, \sigma_0\}[$, we realize that I_μ has the mountain pass geometry for all $\mu \in]0, \mu_0]$. □

The mountain pass geometry for I_μ derives from the fact that the inequality (2.6) of the Lemma (2.2.5) says that I_μ has a strict local minimum at $u = 0$. This property of I_μ can be also obtained under a different assumption, as stated by the next lemma.

Lemma 2.2.7. *Let $\alpha, \beta, c_0, \mu_0$ be as in Lemma 2.2.4. Assume that*

(i) $\lambda_{\alpha\beta} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega_{\alpha_+}} \dfrac{\beta(x)}{\alpha(x)} > 0$;

(ii) $\liminf_{k\to+\infty} \left(k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha(x)\geq k} \alpha(x)^{\frac{2^*}{2^*-s}} dx \right) = 0$

Then, for each $\mu \in]0, \mu_0]$, there exists $\sigma \in]0, c_0[$ such that

$$\inf_{\|u\|=\sigma} I_\mu(u) \geq \frac{1}{6}\sigma^2 > 0. \tag{2.8}$$

Proof. Similarly as in the proof of Lemma 2.2.5, we see that for almost all $x \in \Omega_{\alpha_+}$, one has

$$\sup_{t>0} \left\{ \frac{\frac{1}{s}\alpha(x)t^s - \frac{\mu}{r}\beta(x)t^r}{t^{2^*}} \right\} = \left(\frac{\mu}{r}\frac{\beta(x)}{\alpha(x)} \right)^{-\frac{2^*-s}{s-r}} \left(\frac{s-r}{2^*-s} \right) \alpha(x) \leq M\,\alpha(x) \tag{2.9}$$

where

$$M = \lambda_{\alpha\beta}^{-\frac{2^*-s}{s-r}} \left(\frac{r}{\mu} \right)^{\frac{2^*-s}{s-r}} \left(\frac{s-r}{2^*-s} \right) > 0.$$

Keeping in mind (2.9) and using the Hölder's inequality, for $K, \sigma > 0$ one has:

$$
\begin{aligned}
&\inf_{\|u\|=\sigma} I_\mu(u) \\
&= \inf_{\|u\|=\sigma} \left(\frac{1}{2}\|u\|^2 - \frac{1}{s}\int_\Omega \alpha(x) u^s \, dx + \frac{\mu}{r}\int_\Omega \beta(x) u^r \, dx \right) \\
&= \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \left(\int_\Omega \frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r \, dx \right) \\
&\geq \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \left[\int_{0<\alpha(x)\leq k} \left(\frac{1}{s}\alpha(x)u^s - \frac{\mu}{r}\beta(x)u^r \right) dx + \frac{1}{s}\int_{\alpha(x)\geq k} \alpha(x)u^s \, dx \right] \\
&\geq \frac{1}{2}\sigma^2 - \sup_{\|u\|=\sigma} \int_{0<\alpha(x)\leq k} M\alpha(x)u^{2^*} \, dx - \frac{1}{s}\sup_{\|u\|=\sigma} \int_{\alpha(x)\geq k} \alpha(x)u^s \, dx \\
&\geq \frac{1}{2}\sigma^2 - Mkc_{2^*}^{2^*}\sigma^{2^*} - \frac{c_{2^*}^s}{s}\left(\int_{\alpha(x)\geq k} \alpha(x)^{\frac{2^*}{2^*-s}} \, dx \right)^{\frac{2^*-s}{2^*}} \sigma^s
\end{aligned} \tag{2.10}
$$

Now, thanks to assumption $ii)$, we can find $k_0 > 0$ such that

- $(6Mc_{2^*}^{2^*})^{\frac{1}{2^*-2}} k_0^{-\frac{1}{2^*-2}} < c_0,$

- $k_0^{\frac{1}{2^*-2}} \left(\int_{\alpha(x)\geq k_0} \alpha^{\frac{2^*}{2^*-s}} \right)^{\frac{2^*-s}{2^*(2-s)}} \leq (6Mc_{2^*}^{2^*})^{\frac{1}{2-2^*}} \left(\frac{6}{s} c_{2^*}^s \right)^{\frac{1}{s-2}}.$

Hence, we can fix $\sigma > 0$ such that

$$
\left(\frac{6}{s} c_{2^*}^s \right)^{\frac{1}{2-s}} \left(\int_{\alpha(x)\geq k_0} \alpha^{\frac{2^*}{2^*-s}} \right)^{\frac{(2^*-s)}{2^*(2-s)}} \leq \sigma \leq (6Mc_{2^*}^{2^*})^{\frac{1}{2-2^*}} k_0^{\frac{1}{2-2^*}} < c_0
$$

We deduce that

$$
\begin{aligned}
&\inf_{\|u\|=\sigma} I_\mu(u) \\
&\geq \frac{1}{6}\sigma^2 + \left(\frac{1}{6}\sigma^2 - Mk_0 c_q^q \sigma^q \right) + \left[\frac{1}{6}\sigma^2 - \frac{c_{2^*}^s}{s} \left(\int_{\alpha\geq k} \alpha(x)^{\frac{2^*}{2^*-s}} \, dx \right)^{\frac{2^*-s}{2^*}} \sigma^s \right] \\
&\geq \frac{1}{6}\sigma^2 > 0
\end{aligned}
$$

□

As an easy consequence of the above lemmas we obtain the following multiplicity result:

Theorem 2.2.8. *Let* $r, s \in]1, 2[$*, with* $r < s$*,* $q > \frac{2^*}{2^*-s}$*,* $m > \frac{2^*}{2^*-r}$*,* $\alpha \in L^q(\Omega)$*, and* $\beta \in L^m(\Omega)$*. Assume that*

a) $\operatorname{ess\,sup}_\Omega \alpha > 0$.

Then, there exists $\mu_0 > 0$ *such that problem* (P_μ) *admits at least a nonzero solution for each* $\mu \in [-\mu_0, \mu_0]$.

If, in addiction, α, β *satisfy*

b) $\beta(x) \geq 0$ *for a.e.* $x \in \Omega$*;*

c) $\lambda_{\alpha\beta} \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x \in \Omega_{\alpha_+}} \dfrac{\beta(x)}{\alpha(x)} > 0$*;*

d) either $\alpha_+ \in L^p(\Omega)$ *for some* $p > \frac{N}{2}$*,*

or $\displaystyle \liminf_{k \to +\infty} \left(k^{\frac{2^*}{2^*-2} \frac{2-s}{2^*-s}} \int_{\alpha(x) \geq k} \alpha(x)^{\frac{2^*}{2^*-s}} dx \right) = 0,$

then, for each $\mu \in]0, \mu_0]$*, admits at least two nonzero solutions.*

Proof. Under assumption *a*), the first part of the thesis follows directly from Lemma 2.2.4. Under the additional assumptions *b*), *c*) and *d*), Property 2.2.6 says that I_μ satisfies the mountain pass geometry for all $\mu \in]0, \mu_0]$. By Property 1.4.3 it is easy to realize that I_μ also satisfies the Palais-Smale condition. Then, there exists a second solution of mountain pass type. This second solution is nonzero in view of inequalities (2.6), (2.8). □

2.2.3 Remark on the assumption (ii) of the Lemma 2.2.7

A sufficient condition for the validity of assumption (ii) of the Lemma 2.2.7 can be given in terms of the symmetric-decreasing rearrangement α_+^* of the function α_+, defined by

$$\alpha_+^*(x) = \int_0^\infty \chi_{\{\alpha_+ > t\}^*}(x)dt, \quad \text{for all} \quad x \in \Omega^*,$$

where Ω^* is the open ball centered at 0 with the same measure as that of Ω, $\{\alpha_+ > t\}^*$ is the open ball centered at 0 with the same measure as that of $\{y \in \Omega : \alpha_+(y) > t\}$, and $\chi_{\{\alpha_+ > t\}^*}$ is the characteristic function of $\{\alpha_+ > t\}^*$ (see [15], page 80 for furher details). More precisely, let us to show that if

$$\lim_{|x|\to 0} |x|^2 \alpha_+^*(x) := 0$$

then, condition (ii) of Lemma 2.2.7 holds. Let $\varepsilon > 0$ and let $\delta_\varepsilon > 0$ be such that

$$|x|^2 \alpha_+^*(x) < \varepsilon, \quad \text{for each } x \in \mathbb{R}^N \text{ such that } |x| \le \delta_\varepsilon.$$

Put $k_\varepsilon = \delta_\varepsilon^{-2}\varepsilon$ and let $k \in \mathbb{R}$ be such that $k \ge k_\varepsilon$. Then, for each $x \in \mathbb{R}^N$ such that $\varepsilon|x|^{-2} \ge k$ one has $|x| \le \delta_\varepsilon$, and thus $|x|^2\alpha_+^*(x) < \varepsilon$. Consequently, after noticing that $\frac{N}{2} = \frac{2^*}{2^*-2}$, one has

$$\begin{aligned}
&\int_{\alpha_+^* \ge k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}} dx \\
&\le \varepsilon \int_{\varepsilon|x|^{-2} \ge k} (\varepsilon|x|^{-2})^{\frac{2^*}{2^*-s}} dx \\
&\le \varepsilon^{\frac{2^*}{2^*-s}} \int_0^{\sqrt{\frac{\varepsilon}{k}}} t^{N-1-\frac{22^*}{2^*-s}} dt \\
&= \left(N - \frac{22^*}{2^*-s}\right)^{-1} \varepsilon^{\frac{2^*}{2^*-s}} \varepsilon^{N-\frac{2^*}{2^*-s}} k^{\frac{N}{2}-\frac{2^*}{2^*-s}} \\
&= 2\frac{2^*-2}{2^*}\frac{2^*-s}{2-s}\varepsilon^N k^{\frac{2^*}{2^*-2}\frac{s-2}{2^*-s}}
\end{aligned}$$

Therefore, for $k \geq k_\varepsilon$, one has

$$k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha_+^* \geq k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}} dx < 2\frac{2^*-2}{2^*}\frac{2^*-s}{2-s}\varepsilon^N.$$

This means that

$$\lim_{k\to+\infty} k^{\frac{2^*}{2^*-2}\frac{2-s}{2^*-s}} \int_{\alpha_+^* \geq k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}} dx = 0. \tag{2.11}$$

Then, the validity of assumption $ii)$ of Lemma 2.2.7 follows by the identity (see [15], pag 81)

$$\int_{\alpha_+^* \geq k} \alpha_+^*(x)^{\frac{2^*}{2^*-s}} dx = \int_{\alpha_+ \geq k} \alpha_+(x)^{\frac{2^*}{2^*-s}} dx = \int_{\alpha \geq k} \alpha(x)^{\frac{2^*}{2^*-s}} dx.$$

It is worth of noticing that, by similar arguments, one can show that the limit (2.11) implies $\lim_{|x|\to 0} |x|^2 \alpha_+^*(x) := 0$, but this latter might not hold if in (2.11) "lim"is replaced by "liminf".

2.3 Positive solutions to a Kirchhoff problem

We now pass to study a non-local parametric problem of Kirchhoff type, involving a nonlinearity similar to that of problem (P_μ) introduced in the previous chapter.

Let s, r, a, $b \in \mathbb{R}$ be real numbers with $a > 0$, $b > 0$, and $1 < r < s < \min\{4, 2^*\}$. We consider the following non-local Kirchhoff problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \, dx\right) \Delta u = \lambda u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$P_{\lambda,\mu}$}$$

on varying of the parameters $\lambda, \mu > 0$. We will establish, making use of some standard regularity results, the existence of a positive solution. Then, a multiplicity result is also established via minimax methods.

As before, solutions to problem $(P_{\lambda,\mu})$ will be understood in the weak sense, that is

Definition 2.3.1. *A weak solution of problem $(P_{\lambda,\mu})$ is a function $u \in W_0^{1,2}(\Omega)$, with $u > 0$ a.e. in Ω, satisfying the equation*

$$\left(a + b\|u\|^2\right) \int_\Omega \nabla u \nabla v \, dx - \int_\Omega (\lambda u^{s-1} - \mu u^{r-1}) v \, dx = 0 \tag{2.12}$$

for all $v \in W_0^{1,2}(\Omega)$.

The energy functional $I_{\lambda,\mu} : W_0^{1,2}(\Omega) \to \mathbb{R}$ associated to the problem $(P_{\lambda,\mu})$ is defined by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\left(a + \frac{b}{2}\int_\Omega |\nabla u|^2 \, dx\right) \int_\Omega |\nabla u|^2 \, dx - \int_\Omega \left(\frac{\lambda}{s} u_+^s - \frac{\mu}{r} u_+^r\right) dx, \tag{2.13}$$

$u \in W_0^{1,2}(\Omega)$.
With some abuse of notation, we will write $\|u\|_p$ in place of $\|u_+\|_p$ $(p \geq 1)$. With this notation, we can rewrite $I_{\lambda,\mu}$ as follows:

$$I_{\lambda,\mu}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{s}\|u\|_s^s + \frac{\mu}{r}\|u\|_r^r.$$

By classical compact embeddings theorems (e.g. 1.1.12) we have the following property:

Property 2.3.1. *Let s, r, a, $b \in \mathbb{R}$ be real numbers with $a > 0$, $b > 0$, and $1 < r < s < \min\{4, 2^*\}$. Then the functional*

$$u \in W_0^{1,2}(\Omega) \to \int_\Omega \left(\frac{\lambda}{s} u_+^s - \frac{\mu}{r} u_+^r\right) dx \tag{2.14}$$

is of class C^1 and sequentially weakly continuous, and the functional

$$u \in W_0^{1,2}(\Omega) \to \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 \tag{2.15}$$

is of class C^1 and sequentially weakly lower semicontinuous. Therefore, the functional $I_{\lambda,\mu}$ is of class C^1 and sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$.

Of course, $I_{\lambda,\mu}$ is strongly continuous in $W_0^{1,2}(\Omega)$ as well. Moreover, one us

$$I'_{\lambda,\mu}(u)(v) = \left(a + b\|u\|^2\right)\int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega \left(\lambda u_+^{s-1} - \mu u_+^{r-1}\right) v \, dx, \tag{2.16}$$

for all $v \in W_0^{1,2}(\Omega)$. From (2.12), it is an easy matter to see that the positive critical point of $I_{\lambda,\mu}$ are exactly the weak solutions of $(P_{\lambda,\mu})$.

Following the same arguments as in Chapter 1, we see that any $u \in W_0^{1,2}(\Omega)$ such that $I'_{\lambda,\mu}(u) = 0$ is nonnegative and, by classical regularity results (see e.g. Theorem A.5 and Lemma B.3 of [17]), it follows that $u \in W^{2,q} \cap W_0^{1,2}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$ and $q < \infty$.

In particular, a weak solution of problem $(P_{\lambda,\mu})$ is nonnegative and belongs to the space $C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.

Finally, we introduce two definitions, useful for demonstrating the two main results regarding the problem $(P_{\lambda,\mu})$.

Consider the set

$$\mathcal{P} = \left\{ u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega \right\}, \tag{2.17}$$

with ν the outer normal on $\partial\Omega$, which is the interior of the positive cone of $C_0^{1,\alpha}(\overline{\Omega})$, and the sets

$$\mathcal{S} = \{(\lambda,\mu) \in]0,+\infty[\times[0,+\infty[: \quad (P_{\lambda,\mu}) \text{ has a solution belonging to } \mathcal{P}\};$$

$$\mathcal{S}_\lambda = \{\mu > 0 : \{\lambda\} \times [0,\mu[\subset \mathcal{S}\}, \quad \text{for each } \lambda > 0.$$

Also, put

$$\lambda_{a,b} = \begin{cases} 0 & if \;\; 1 < s < 2, \\ a\lambda_1 & if \;\; s = 2, \\ \frac{s}{c_s^s}\left(\frac{b}{2s-4}\right)^{\frac{s-2}{2}}\left(\frac{a}{4-s}\right)^{\frac{4-s}{2}} & if \;\; 2 < s < \min\{4, 2^*\}. \end{cases} \tag{2.18}$$

2.3.1 Existence of Positive Solution

As for the previous problem, we first consider the unperturbed problem $(P_{\lambda,0})$, whose associated energy functional is

$$I_{\lambda,0}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{s}\|u\|_s^s, \quad u \in W_0^{1,2}(\Omega).$$

The following lemma gives a necessary and sufficient conditions, involving the constant $\lambda_{a,b}$ defined above, to get a global minimum point of $I_{\lambda,0}$ with negative energy.

Lemma 2.3.2. *Let $\lambda_{a,b}$ be as in* (2.18)*. The functional $I_{\lambda,0}$ admits a global minimum in $W_0^{1,2}(\Omega)$ and one has*

$$\inf_{W_0^{1,2}(\Omega)} I_{\lambda,0} < 0 \quad if \text{ and only } if \quad \lambda > \lambda_{a,b}. \tag{2.19}$$

Moreover, any nonzero global minimum point of $I_{\lambda,0}$ belongs to $\mathcal{P}$.

Proof. Let $\lambda > 0$. Since

$$I_{\lambda,0}(u) \geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{c_s^s}{s}\|u\|^s, \quad \text{for every } u \in W_0^{1,2}(\Omega),$$

and $s < 4$, it follows

$$\lim_{\|u\| \to +\infty} I_{\lambda,0}(u) = +\infty.$$

Then, $I_{\lambda,0}$ admits a global minimum point in $W_0^{1,2}(\Omega)$. Let us to show that

$$\inf_{W_0^{1,2}(\Omega)} I_{\lambda,0} < 0 \quad \text{if and only if} \quad \lambda > \lambda_{a,b}$$

We have

$$\begin{aligned}
\inf_{u \in W_0^{1,2}(\Omega)} I_{\lambda,0}(u) \qquad &= \inf_{\sigma>0} \inf_{\|u\|=\sigma} \left(\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{s}\int_\Omega u_+^s \, dx \right) \\
&= \inf_{\sigma>0} \left(\frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - \frac{\lambda}{s} \sup_{\|u\|=\sigma} \int_\Omega u_+^s \, dx \right) \\
&= \inf_{\sigma>0} \left(\frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - \frac{\lambda c_s^s}{s}\sigma^s \right) = \\
&\inf_{\sigma>0} \left[\sigma^s \left(\frac{a}{2}\sigma^{2-s} + \frac{b}{4}\sigma^{4-s} - \frac{\lambda}{s}c_s^s \right) \right].
\end{aligned}$$

Since $s \in]1,4[$, an elementary calculation shows that

$$\inf_{\sigma \in]0,+\infty[} \left(\frac{a}{2}\sigma^{2-s} + \frac{b}{4}\sigma^{4-s} - \frac{\lambda c_s^s}{s} \right) < 0 \quad \text{if and only if} \quad \lambda > \lambda_{a,b}.$$

Hence,

$$\inf_{u \in W_0^{1,2}(\Omega)} I_{\lambda,0}(u) < 0 \quad \text{if and only if} \quad \lambda > \lambda_{a,b}.$$

To conclude the proof, let $u_0 \in W_0^{1,2}(\Omega)$ be a nonzero global minimum point of I_0. Then, u_0 is a critical point of I_0. Therefore, as already observed, u_0 is nonnegative. Since u_0 is also non-zero, by the Hopf's Boundary Lemma we infer $u_0 \in \mathcal{P}$. $\square$

For each $\lambda, \mu > 0$, the next lemma allows us to find a non-zero global minimum point of functional $I_{\lambda,\mu}$. Furthermore, it provides a convergence property of these minimum points.

Lemma 2.3.3. *For each $\lambda, \mu > 0$, the functional $I_{\lambda,\mu}$ admits a global minimum point in $W_0^{1,2}(\Omega)$. Moreover, for each fixed $\lambda > 0$, each sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset]0, +\infty[$, with $\mu_n \to 0$, and each sequence $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ such that u_n is a global minimum point of I_{λ,μ_n}, there exists a global minimum point u_0 of $I_{\lambda,0}$ such that (up to a subsequence)*

$$\lim_{n\to+\infty} \|u_n - u_0\|_{C^1(\overline{\Omega})} = 0 \tag{2.20}$$

Proof. Let $\lambda, \mu > 0$. The proof of the existence of a global minimum for $I_{\lambda,\mu}$ is the same as that for $I_{\lambda,0}$ in Lemma 2.3.3. Now, fix $\lambda > 0$ and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $]0, +\infty[$ such that $\mu_n \to 0$. Also, for each $n \in \mathbb{N}$, let $\{u_n\}_{n\in\mathbb{N}}$ be a global minimum point of I_{λ,μ_n}. One has

$$\begin{aligned} 0 = \;& I'_{\lambda,\mu_n}(u_n)(u_n) \\ &= a\|u_n\|^2 + b\|u_n\|^4 - \lambda\|u_n\|_s^s + \mu_n\|u_n\|_r^r \\ &\geq a\|u_n\|^2 + b\|u_n\|^4 - \lambda c_s^s\|u_n\|^s, \qquad \text{for each } n \in \mathbb{N}. \end{aligned}$$

Since $s < 4$, the previous inequality says that $\{u_n\}_{n\in\mathbb{N}}$ is bounded $W_0^{1,2}(\Omega)$. Therefore, it weakly converges to some $u_0 \in W_0^{1,2}(\Omega)$. Consequently, by the lower semicontinuity of $I_{\lambda,0}$ and the fact that $\mu_n\|u_n\|_r^r \to 0$, we get

$$\liminf_{n\to+\infty} I_{\lambda,\mu_n}(u_n) = \liminf_{n\to+\infty} \left(I_{\lambda,0}(u_n) + \frac{\mu_n}{r}\|u_n\|_r^r\right) \geq I_{\lambda,0}(u_0) \tag{2.21}$$

Now, consider the function

$$\mu \in \mathbb{R} \to \inf_{u \in W_0^{1,2}(\Omega)} I_{\lambda,\mu}(u).$$

This function is the lower envelop of a family of linear functions, thus it is convex in $\mathbb{R}$ and, in particular, continuous there. Therefore, in view of (2.21), one has

$$\inf_{u \in W_0^{1,2}(\Omega)} I_{\lambda,0}(u) = \lim_{n \to +\infty} \inf_{u \in W_0^{1,2}(\Omega)} I_{\lambda,\mu_n}(u) = \lim_{n \to +\infty} I_{\lambda,\mu_n}(u_n) \geq I_{\lambda,0}(u_0).$$

This means that u_0 is a global minimum point of $I_{\lambda,0}$. Finally, since u_n is a weak solution to problem (P_{λ,μ_n}), then $u_n \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in]0,1[$. Moreover, recalling that $s < 2^*$, if we fix

$$\sigma \in]\max\{2, s\}, 2^*[\quad \text{and} \quad q \in \left] \frac{2^*}{2^* - 2}, \frac{2^*}{\sigma - 2} \right[,$$

(note that $\frac{2^*}{2^*-2} = \frac{N}{2}$), then there exists a constant C (independent of n) such that

$$\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \left[\frac{1}{a + b\|u_n\|^2} \left(\int_\Omega \left|\lambda u_n^{s-1} - \mu_n u_n^{r-1}\right|^q dx \right)^{\frac{1}{q}} + \|u_n\|_q \right], \quad (2.22)$$

for all $n \in \mathbb{N}$ (see Theorem 8.2' of [2] for instance). Now, since $\sigma > \max\{2, s\}$, from (2.22), we infer that, for some constant C_1 independent of $n \in \mathbb{N}$, the following inequality holds

$$\begin{aligned} &\|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \\ &\qquad \leq C_1 \left[1 + \left(\int_\Omega |u_n|^{q(\sigma-1)} dx \right)^{\frac{1}{q}} \right] \\ &\qquad \leq C_1 \left[1 + K_n \left(\int_\Omega |u_n|^{2^*} dx \right)^{\min\left\{\frac{\sigma-1}{2^*}, \frac{1}{q}\right\}} \right], \quad \text{for all } \ n \in \mathbb{N} \end{aligned}$$

where

$$K_n = m(\Omega)^{\frac{2^* - q(\sigma-1)}{2^* q}}, \quad \text{if} \quad q(\sigma - 1) \le 2^*,$$

and

$$K_n = \|u_n\|_{C^{1,\alpha}(\overline{\Omega})}^{\frac{q(\sigma-1)-2^*}{q}} \quad \text{if} \quad q(\sigma - 1) > 2^*.$$

Observe also that, being $q < \frac{2^*}{\sigma-2}$, one has $\frac{q(\sigma-1)-2^*}{q} < 1$. Therefore, the previous inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,2}(\Omega)$ (thus in $L^{2^*}(\Omega)$ as well), imply the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $C^{1,\alpha}(\overline{\Omega})$. By the Ascoli-Arzelà Theorem, we infer that, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges in $C^1(\overline{\Omega})$. Recalling that $u_n \to u_0$ weakly in $W_0^{1,2}(\Omega)$, we finally deduce that $u_n \to u_0$ in $C^1(\overline{\Omega})$, that is the limit (2.20) holds. □

Exploiting the fact that $\mathcal{P}$ is an open subset of $C_0^{1,\alpha}(\overline{\Omega})$, we will are able to prove the existence of global minimum of $I_{\lambda,\mu}$ belonging to $\mathcal{P}$, at least for μ small enough.

Lemma 2.3.4. *Let $\lambda > \lambda_{a,b}$. Then, there exists $\mu(\lambda) > 0$ such that, for each $\mu \in [0, \mu(\lambda)[$ and each global minimum point u_μ of $I_{\lambda,\mu}$, one has $u_\mu \in \mathcal{P}$.*

Proof. Arguing by contradiction, assume that there exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of positive numbers, with $\mu_n \to 0$, and a global minimum point u_n of I_{λ,μ_n}, such that $u_n \notin \mathcal{P}$. By Lemma 2.3.3, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ converges in the $C^1(\overline{\Omega})$ topology to a global minimum point u_0 of $I_{\lambda,0}$. Since $\lambda > \lambda_{a,b}$, we know, by Lemma

2.3.2, that $u_0 \in \mathcal{P}$. Hence, being $\mathcal{P}$ an open set in $C^1(\overline{\Omega})$, we infer that $u_n \in \mathcal{P}$, for each $n \in \mathbb{N}$ big enough, a contradiction. $\square$

Now, consider the unique solution $u_s \in \mathcal{P}$ of the problem

$$\begin{cases} -\Delta u = u^{s-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and put $K_s = \max_{\overline{\Omega}} u_s$. We have the following lemma

Lemma 2.3.5. *For each $(\lambda, \mu) \in \mathcal{S}$ one has $\mu \leq \lambda^{\frac{2-r}{(2-s)(s-r)}} K_s^{s-r}$.*

Proof. Let $(\lambda, \mu) \in \mathcal{S}$ and let $u_{\lambda,\mu} \in \mathcal{P}$ be a solution to problem $(P_{\lambda,\mu})$. Then, we see that it cannot be

$$\lambda u_{\lambda,\mu}(x)^{s-1} - \mu u_{\lambda,\mu}(x)^{r-1} \leq 0, \quad \text{for each} \quad x \in \Omega,$$

for, otherwise, $u_{\lambda,\mu}$ should be identically zero, by the Maximum Principle. Therefore,

$$\max_{\overline{\Omega}} u_{\lambda,\mu} \geq \left(\frac{\mu}{\lambda}\right)^{\frac{1}{s-r}}. \tag{2.23}$$

Moreover, $u_{\lambda,\mu}$ turns out to be a subsolution of problem

$$\begin{cases} -\Delta u = \frac{\lambda}{a} u^{s-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and that the function $\left(\frac{\lambda}{a}\right)^{\frac{1}{2-s}} u_s$ is a solution of the same problem. Thus, by comparison results (see Lemma 3.3 of [4] for instance), we get

$$u_{\lambda,\mu}(x) \leq \left(\frac{\lambda}{a}\right)^{\frac{1}{2-s}} u_s(x) \leq \left(\frac{\lambda}{a}\right)^{\frac{1}{2-s}} K_s, \quad \text{for each} \quad x \in \Omega.$$

Consequently, from (2.23), it follows

$$\left(\frac{\mu}{\lambda}\right)^{\frac{1}{s-r}} \leq \left(\frac{\lambda}{a}\right)^{\frac{1}{2-s}} K_s$$

which is equivalent to $\mu \leq \lambda^{\frac{2-r}{(2-s)(s-r)}} K_s^{s-r}$. □

A direct consequence of the previous two lemmas is the following theorem that gives some information on the structure of the set of pairs of parameters (λ, μ) such that the problem $(P_{\lambda,\mu})$ have a positive solutions.

Theorem 2.3.6. *Let $r, s \in]1, \min\{4, 2^*\}[$, with $r < s$, and let $\lambda > \lambda_{a,b}$. Then, $\mathcal{S}_\lambda$ is a nonempty and bounded set.*

Proof. The fact that S_λ is nonempty follows from Lemma 2.3.4. The boundedness of S_λ follows from Lemma 2.3.5. □

2.3.2 Multiplicity result

A multiplicity result of nonnegative solutions for problem $(P_{\lambda,\mu})$ can be obtained following the same technique as for problem (P_μ) previously discussed. Indeed, we are going to show that the energy functional $I_{\lambda,\mu}$ defined in (2.13) has a mountain pass geometry and satisfies the Palais-Smale condition.

Lemma 2.3.7. *Let $\lambda, \mu > 0$. There exists $\sigma_0 > 0$ such that, for all $\sigma \in]0, \sigma_0[$, one has:*

$$\inf_{\|u\|=\sigma} I_{\lambda,\mu}(u) > 0. \tag{2.24}$$

Proof. Fix $q \in]2, \min\{4, 2^*\}[$. By an easy calculation one has

$$\sup_{t>0} \left\{ \frac{\frac{\lambda}{s}t^s - \frac{\mu}{r}t^r}{t^q} \right\} = \left(\frac{r}{\mu}\right)^{\frac{q-s}{s-r}} \left(\frac{\lambda(q-s)}{s(q-r)}\right)^{\frac{q-r}{s-r}} \left(\frac{s-r}{q-s}\right) := M > 0$$

Put

$$\sigma_0 \stackrel{\text{def}}{=} \left(\frac{a}{2Mc_q^q}\right)^{\frac{1}{q-2}} > 0$$

and let $\sigma \in]0, \sigma_0[$. Keeping in mind the definitions of M, σ_0, we get

$$\begin{aligned}
\inf_{\|u\|=\sigma} I_\mu(u) &= \inf_{\|u\|=\sigma} \left[\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \int_\Omega \left(\frac{\lambda}{s}u_+^s - \frac{\mu}{r}u_+^r\right) dx\right] \\
&= \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - \sup_{\|u\|=\sigma} \int_\Omega \left(\frac{\lambda}{s}u_+^s - \frac{\mu}{r}u_+^r\right) dx \\
&\geq \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - \sup_{\|u\|=\sigma} \int_\Omega M u_+^q \, dx \\
&= \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - M\sigma^q \sup_{\|u\|=1} \|u\|_q^q \\
&= \frac{a}{2}\sigma^2 + \frac{b}{4}\sigma^4 - Mc_q^q\sigma^q \\
&> \sigma^2 \left(\frac{a}{2} - Mc_q^q\sigma^{q-2}\right) > 0.
\end{aligned}$$

□

Lemma 2.3.8. *Let r, s, λ be as in Theorem 2.3.6, and let $\mu \in]0, \mu(\lambda)[$, where $\mu(\lambda)$ is as in Lemma 2.3.4. Then, $I_{\lambda,\mu}$ has the mountain pass geometry.*

Proof. The proof is as that of Lemma 2.2.6. □

Lemma 2.3.9. *Let $\lambda, \mu > 0$. Then, the functional $I_{\lambda,\mu}$ satisfies the Palais-Smale condition.*

Proof. Define $\Phi, \Psi : W_0^{1,2}(\Omega) \to \mathbb{R}$ as follows

$$\Phi(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4, \quad \Psi(u) = \int_\Omega \left(\frac{\lambda}{s}u_+^s - \frac{\mu}{r}u_+^r\right) dx$$

for all $u \in W_0^{1,2}(\Omega)$. Then,

$$I_{\lambda,\mu}(u) = \Phi(u) - \Psi(u).$$

Let us to show that Φ' is invertible with continuous inverse. To this end, denote by K the inverse of the function

$$t \in [0, +\infty[\longrightarrow at + bt^3.$$

From the continuity of K and $K(0) = 0$, we have $\frac{K(\|u\|)}{\|u\|}u \in W_0^{1,2}(\Omega)$, for all $u \in W_0^{1,2}(\Omega) \setminus \{0\}$. Moreover, if we consider the operator $T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ defined by

$$T(u) = \begin{cases} \frac{K(\|u\|)}{\|u\|}u & \text{if } W_0^{1,2}(\Omega) \setminus \{0\} \\ 0 & \text{if } \ u = 0, \end{cases}$$

then T is continuous in $W_0^{1,2}(\Omega)$ and, for each $u \in W_0^{1,2}(\Omega) \setminus \{0\}$, one has

$$\begin{aligned} T(\Phi'(u)) = T((a + b\|u\|^2)u) \qquad &= \frac{K((a + b\|u\|^2)\|u\|)}{(a + b\|u\|^2)\|u\|}(a + b\|u\|^2)u \\ &= \frac{\|u\|}{(a + b\|u\|^2)\|u\|}(a + b\|u\|^2)u = u. \end{aligned}$$

Thus, T is a continuous inverse of Φ'. By compact embedding results, we also know that Ψ' is continuous and compact. Conclusion follows by Property 1.4.3.

□

As a consequence of the above lemmas, we obtained the following multiplicity result:

Theorem 2.3.10. *Let r, s, λ, μ be as in Lemma 2.3.8. Then, there exist a positive solution and a further nonzero and nonnegative solution of the problem*

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u = \lambda u^{s-1} - \mu u^{r-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. By Lemma 2.3.4, we know that $I_{\lambda,\mu}$ admits a global minimum point $u_\mu \in \mathcal{P}$, which is a solution in $\mathcal{P}$ of the problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u = \lambda u^{s-1} - \mu u^{r-1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Therefore, thanks to Lemma 2.3.8 and 2.3.9, we can apply the Mountain Pass Theorem 1.4.6 (or [10], [16]) to get a second nonzero and nonnegative solution v_μ of this problem. □

Chapter 3

Open problems and research perspectives

We point out two open problems issued from Theorem 2.2.8 and Theorem 2.3.10.

Let us consider the particular but significant case of Theorem 2.2.8 where the domain Ω contains the origin and the weight functions α, β have the form

$$\alpha(x) = \beta(x) = |x|^\eta$$

with $\eta \in \mathbb{R}$. It is an easy matter to see that all the assumptions $a), b), c), d)$ of Theorem 2.2.8 are satisfied if and only if $\eta > -2$. Thus, it is of interest to investigate the limit case $\eta = -2$ whose corresponding expression of the weight functions is $\alpha(x) = \beta(x) = |x|^{-2}$. The function $|\cdot|^{-2}$ represents what is known in literature as Calogero potential.

Clearly, one of the natural way to study problem (P_μ) in this case is to consider the following approximating problem

$$\begin{cases} -\Delta u = \frac{1}{|x|^{2-\varepsilon_n}}\left(u^{s-1} - \mu u^{r-1}\right), & \text{in} \quad \Omega \\ u \geq 0, & \text{in} \quad \Omega, \\ u_{|\partial\Omega} = 0, \end{cases} \qquad (P_{\mu,n})$$

where $\{\varepsilon_n\}$ is a sequence of positive numbers converging to 0. Indeed, for each $n \in \mathbb{N}$, one has $|\cdot|^{-(2-\varepsilon_n)} \in L^{\frac{N}{2}}(\Omega)$ and $\frac{N}{2} > \frac{2^*}{2^*-s} > \frac{2^*}{2^*-r}$, therefore, reasoning as in Lemmas 2.2.3 and 2.2.4, we can infer that there exist $\mu_0, \rho, c_0 \in]0, +\infty[$ such that, for each $n \in \mathbb{N}$ and each $\mu \in]0, \mu_0[$, there exists a solution $u_{n,\mu}$ of problem $(P_{\mu,n})$ satisfying

$$\frac{1}{2}\|u_{n,\mu}\|^2 - \int_\Omega \frac{1}{|x|^{2-\varepsilon_n}}\left(\frac{1}{s}u_{n,\mu}(x)^s - \frac{\mu}{r}u_{n,\mu}(x)^r\right) dx \leq -\rho < 0,$$
$$\|u_{n,\mu}\|^2 \geq c_0.$$

From the first inequality we deduce that the sequence $\{u_{n,\mu}\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$, thus it weakly converges in $W_0^{1,2}(\Omega)$ to a function u_μ which satisfies

$$\frac{1}{2}\|u_\mu\|^2 - \int_\Omega \frac{1}{|x|^2}\left(\frac{1}{s}u_\mu(x)^s - \frac{\mu}{r}u_\mu(x)^r\right) dx \leq -\rho < 0,$$

and is a nonzero solution of the problem

$$\begin{cases} -\Delta u = \frac{1}{|x|^2}\left(u^{s-1} - \mu u^{r-1}\right), & \text{in} \quad \Omega \\ u \geq 0, & \text{in} \quad \Omega, \\ u_{|\partial\Omega} = 0, \end{cases} \qquad (P_\mu^*)$$

Applying Lemma 2.2.5 or Lemma 2.2.7 we can also infer the existence of a solution $v_{n,\mu}$ of mountain pass type of problem $(P_{\mu,n})$. As before, we have that the sequence

$\{v_{n,\mu}\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,2}(\Omega)$. This is an easy consequence of the identity

$$\|v_{n,\mu}\|^2 - \int_\Omega \frac{1}{|x|^{2-\varepsilon_n}} \left(v_{n,\mu}(x)^s - \mu v_{n,\mu}(x)^r\right) dx = 0.$$

Therefore, $v_{n,\mu}$ weakly converges to a solution $v_\mu \in W_0^{1,2}(\Omega)$ of (P_μ^*). However, in this case, we have no sufficient information to conclude that v_μ is nonzero.

We finally observe that when $0 \in \Omega$, one has $|\cdot|^{-2} \in L^q(\Omega)$ for each $q \in [1, \frac{N}{2}[$ and $|\cdot|^{-2} \notin L^{\frac{N}{2}}(\Omega)$. This means that the Moser iteration scheme cannot be applied to derive the boundedness of a nonzero solution to (P_μ^*).

Then, it is quite natural to pose the following open questions:

a) Is the solution v_μ nonzero in Ω?.

b) Is a nonzero solution to (P_μ^*) necessarily bounded? If not, what is the growth rate of a nonzero solution near 0?

Concerning the last question we think that its investigation can be addressed using the method introduced in the recent paper [11] where a similar topic was considered.

A further question (already mentioned in the Introduction) we propose as research perspective comes from a comparison of Theorem 2.3.6 and the results obtained in [12] for the local case of problem $(P_{\lambda,\mu})$ (corresponding to $b = 0$). More precisely, in [12], the authors used a sub-super solution method to establish that, for each $\lambda > 0$, the set $\mathcal{S}_\lambda$ of all $\mu > 0$ such that $(P_{\lambda,\mu})$ admits a positive solution is exactly an interval. In the nonlocal case, due to the presence of the nonlocal term $b\displaystyle\int_\Omega |\nabla u|^2dx$, this method does not seem to work and the question to known whether or not $\mathcal{S}_\lambda$ is an interval remains an open problem.

We think that, at least in the case $s\in]1,2[$, a possible way to address this question could be the following: let $\lambda>0$ and let $\mu'>0$ such that problem $(P_{\lambda,\mu'})$ admits a positive solution, say $u_{\mu'}$. Consider the function $f:\Omega\times\mathbb{R}\to\mathbb{R}$ defined by

$$f(x,t)=\begin{cases}\lambda u_{\mu'}(x)^{s-1}-\mu u_{\mu'}(x)^{r-1}, & \text{if } x\in\Omega \text{ and } t\le u_{\mu'}(x),\\ \lambda t^{s-1}-\mu t^{r-1}, & \text{if } x\in\Omega \text{ and } t> u_{\mu'}(x).\end{cases}$$

Clearly, $f\in C^0(\overline{\Omega}\times\mathbb{R})$. Consider also the functional $I_1:W_0^{1,2}(\Omega)\to\mathbb{R}$, defined by

$$I_1(u)=\frac{1}{2}(a+b\|u_{\mu'}\|^2)\|u\|^2-\int_\Omega F(x,u(x))dx, \tag{3.1}$$

where

$$F(x,\xi)=\int_0^\xi f(x,t)dt, \qquad \text{for each } \ (x,\xi)\in\Omega\times\mathbb{R}.$$

If $s<2$, the functional I_1 is coercive and admits a nonzero global minimum u_1. Similarly, we can consider the sequence of functionals $\{I_n\}_{n\in\mathbb{N}}$ recursively defined by

$$I_{n+1}(u)=\frac{1}{2}(a+b\|u_n\|^2)\|u\|^2-\int_\Omega F(x,u(x))dx, \quad u\in W_0^{1,2}(\Omega),$$

where u_n is a global minimum point of the functional I_n, and , for $n=1$, u_1 is the global minimum point of the functional I_1 defined in (3.1). By a standard regularity argument, we can see that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in the space

$C^{1,\alpha}(\overline{\Omega})$ (for some $\alpha \in]0,1[$) and this means that, up to a subsequence, u_n strongly converges in $C^1(\overline{\Omega})$ to a function u_μ which turns out a solution of the problem

$$\begin{cases} -(a+b\|u\|^2)\,\Delta u = f(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases}$$

At this point, exploiting the properties of the sequence $\{u_n\}$, the goal is to see whether u_μ satisfies the inequality $u_\mu(x) \geq u_{\mu'}(x)$, for all $x \in \Omega$. If this fact was true, then u_μ solves problem (P_μ) and this would allow to conclude that $\mathcal{S}_\lambda$ is an interval.

Bibliography

[1] R. Adams, J. Fournier, *Sobolev Spaces*, Elsevier Ltd, 2003.

[2] S. Agmon, *The L^p approach to the Dirichlet problem*, Ann. Scuola Norm. Sup. Pisa, **13** (1959) 405–448.

[3] A. Ambrosetti, P. H. Rabinowitz *Dual Variational Methods in Critical Point Theory and Applications*, J. Functional Anal. **14** (1973), 349-381 .

[4] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effetcts of concave and convex nonlinearities in some elliptic problems*, J. Functional Anal. **122**, (1994) 519–543.

[5] A. Ambrosetti, G. Prodi, *A primer of Nonlinear Analysis*, Cambridge University Press, 1995.

[6] G. Anello, *Existence and multiplicity of nonnegative solutions for semilinear elliptic problems involving nonlinearities indefinite in sign*, Nonlinear Analysis **75**, (2012), 6103-6107.

[7] G. Anello, *Multiplicity and asymptotic behavior of nonnegative solutions for elliptic problems involving nonlinearities indefinite in sign*, Nonlinear Analysis **75**, (2012), 3618-3628.

[8] M. Badiale, E. Serra, *Semilinear Elliptic Equations for Beginners*, Springer, 2000.

[9] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

[10] N. Ghoussoub, D. Preiss, *A general mountain pass principle for locating and classifying critical points*, Annales de l'I. H. P., section C, tome **6**, n 5 (1989), p. 321-330.

[11] M. Grossi, A. Stehlick, *Regularity and asymptotic approach to semilinear elliptic equation with singular potential*, Indiana Univ. Math. J., **67** (2018), 2313–2335.

[12] J. Hernandez, F.J. Mancebo, J.M. Vega, *Positive solutions for singular nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh, **137**(A) (2007) 41–62.

[13] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[14] G. Leoni, *A First Course in Sobolev Spaces*, American Mathematical Society, Providence, Rhode Island, 2000.

[15] E. H. Lieb, M. Loss, *Analysis*, American Mathematical Society, Providence (2001)

[16] P. Pucci, J. Serrin, *A mountain pass theorem*, J. Differ. Equations **60** (1985), 142-149.

[17] M. Struwe, *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer 2000.

[18] E. Zeidler, *Nonlinear functional analysis and its applications*, vol III, Springer-Verlag, 1985.