Università degli Studi Di Catania

Tesi di Dottorato in Matematica Pura e Applicata
(XXVI ciclo)

Cohen-Macaulayness of tower sets and Betti Weak Lefschetz Property

Giuseppe Favacchio

Supervisore
Chia.mo Prof.
Alfio Ragusa

ANNO ACCADEMICO 2012-2013

Acknowledgments

I would like to express my deep thanks to my supervisor Prof. Alfio Ragusa for his patience, encouragement and continuous guidance. He supported me during the time of my studies sharing a lot of his knowledge and experience. I am also grateful to Prof. Giuseppe Zappalà for inspiring discussions and hints.

Contents

Introduction

Monomial ideals and monomial algebras play an important role in Commutative Algebra. This is also because many questions about ideals can be more easily studied via some corresponding monomial ideals.

For instance, in Gröbner basis theory to any ideal in a standard polynomial ring one associates the initial ideal, with respect to some monomial ordering, and this permits to reduce several questions to an investigation of combinatorial nature.

Monomial algebras are a basic tool in dimension theory. First of all, for any monomial ordering, a homogeneous ideal and its initial ideal have the same Hilbert function. Moreover, the graded Betti numbers of any homogeneous ideal in the polynomial ring are smaller than or equal to the graded Betti numbers of its initial ideal. Using this result Sturmfels, see [Stu], provided a proof of the fact that determinantal ideals are Cohen-Macaulay.

A peculiar monomial ideal is the initial ideal, with respect the Rev-Lex ordering, which arises after a generic linear transformation γ of the coordinates. This monomial ideal is called the *generic initial ideal* and written $\text{gin}(I)$. Galligo in [Ga], Bayer and Stillman in [BS] observed that generic initial ideals preserve many homological properties of the starting ideal I, for instance I is arithmetically Cohen-Macaulay if and only if $\text{gin}(I)$ is arithmetically Cohen-Macaulay and they have the same regularity, $\text{reg}(I) = \text{reg}(\text{gin}(I))$.

The generic initial ideal also has a fundamental role in the investigation of many algebraic properties, for instance, as showed by Wiebe in [Wi], given I homogeneous ideal of a standard polynomial ring R, we have R/I has the Weak Lefschetz property iff $R/\text{gin}(I)$ has the Weak Lefschetz property.

Moreover, the polarization process allows to move from monomial ideals to squarefree monomial ideals. This technique, first used by Hartshorne in [Ha], became a fundamental tool in the study of monomial ideals after the Hochster's article [Ho]. Via this process many homological properties are preserved as the height, the Cohen-Macaulayness, the projective dimension and the graded Betti numbers. The systematic study of squarefree monomial ideals began with the works of Stanley [St2] and Reisner [Re].

Squarefree monomial ideals can be seen as a bridge between commutative algebra and combinatorics. We can associate to a given simplicial complex a squarefree monomial ideal in several different ways. In the study of the Cohen-Macaulay property a particular squarefree monomial ideal has a crucial role: the Stanley-Reisner ideal.

The result known as the Reisner's criterion, uses the reduced homologies to establish when the Cohen-Macaulay property holds for the Stanley-Reisner ideal. Namely, a simplicial complex Σ is Cohen-Macaulay over k if and only if, for all faces F of Σ, including the empty face, and for all $i < \dim_k \operatorname{link}_\Sigma F$, one has

$$\tilde{H}_i(\operatorname{link}_\Sigma F; k) = 0.$$

In Chapter 1 of this thesis we collected some basic definitions and theorems which will be used in the rest of the work. In Chapter 2 we deal with the Cohen-Macaulay property for monomial squarefree ideals of codimension two.

Given a finite set $\mathcal{N} := \{x_1, \ldots, x_N\}$ and $S \subseteq C_{2,\mathcal{N}}$, we associate to S an ideal I_S of the standard polynomial ring $k[\mathcal{N}] := k[x_1, \ldots, x_N]$. We define I_S as the intersection of all the prime ideals generated by the element in S, i.e.

$$I_S := \bigcap_{s \in S} \mathfrak{p}_s,$$

where, if $s = \{x_a, x_b\}$, $\mathfrak{p}_s = \mathfrak{p}_{\{x_a, x_b\}}$ is the prime ideal $(x_a, x_b) \subseteq k[\mathcal{N}]$.

I_S is the Stanley-Reisner ideal of the simplicial complex $\langle \mathcal{N} \setminus s | s \in S \rangle$. The ideal I_S is a squarefree monomial ideal of $k[\mathcal{N}]$. The aim of Chapter 2 is to look under which conditions we have S Cohen-Macaulay, i.e. I_S Cohen-Macaulay. In this first part of the thesis we give three ways to characterize the Cohen-Macaulay property on such a S.

In Section 2.3 we introduce, for subsets of $\mathcal{N}$, the notion of *self-covered* in S, see Definition 2.2.5, this allows us to give a first characterization of the Cohen-Macaulay squarefree monomial ideals of height two. We can characterize them by looking at the minimal primes in their primary decomposition.

Theorem 2.3.10 Let $S \subseteq C_{2,\mathcal{N}}$, denoted by $S(x_a) := \{s \in S \mid x_a \notin s\}$ and by $S_{x_a} := \{x_b \in \mathcal{N} \mid \{x_a, x_b\} \in S\}$, then the following are equivalent:

1. S is Cohen-Macaulay;
2. for any $x_a \in \mathcal{N}$, $S(x_a)$ is Cohen-Macaulay and S_{x_a} is self-covered in $S(x_a)$;
3. there exists $x_a \in \mathcal{N}$ such that $S(x_a)$ is Cohen-Macaulay and S_{x_a} is self-covered in $S(x_a)$.

In Section 2.4 we characterize the Cohen-Macaulay squarefree monomial ideals just looking at the minimal prime ideals of height 2 not in $\mathrm{Min}(I_S)$. In other words, given $S \subseteq C_{2,\mathcal{N}}$, we relate the Cohen-Macaulay property to a condition on $\overline{S} := C_{2,\mathcal{N}} \setminus S$.

Definition 2.4.2 Let $V \subseteq C_{2,\mathcal{N}}$, we say that V contains a ***r*-cycle** if there exists $W \subseteq V$ of the type

$$W = \{\{x_{a_1}, x_{a_2}\}, \{x_{a_2}, x_{a_3}\}, \ldots, \{x_{a_{r-1}}, x_{a_r}\}, \{x_{a_r}, x_{a_1}\}\},$$

and W does not contain properly a s-cycle. We say that a r-cycle W is **minimal** in V if for any $v \in V \setminus W$ we have

$$v \not\subseteq \{x_{a_1}, x_{a_2}, x_{a_3}, \ldots, x_{a_{r-1}}, x_{a_r}\}.$$

We prove the following theorem.

Theorem 2.4.11 S is Cohen-Macaulay if and only if $\overline{S}$ does not contain minimal r-cycles, for any $r \geq 4$.

Many recent papers deal with special configurations of linear subvarieties of projective spaces which raised up to Cohen-Macaulay varieties, for instance partial intersections studied in [RZ6], k-configurations studied in [GHS], star configurations studied in [GHM]. In [FRZ2] the authors, introducing the notion of tower sets, generalize all this configurations in such a way to preserve the Cohen-Macaulayness. The last part of Chapter 2, Section 2.5, also contained in [FRZ2], describes a special configuration which in some sense characterizes the Cohen-Macaulyness.

Let $\mathcal{N}$ be a finite set and $T \subseteq D_{2,\mathcal{N}} := \mathcal{N} \times \mathcal{N} \setminus \{(a,a) \mid a \in \mathcal{N}\}$, we denote by $\pi_1(T)$ the set

$$\pi_1(T) := \{i \in \mathcal{N} | (i,j) \in T \text{ for some } j\}.$$

Definition 2.5.1 Let $\mathcal{N}$ be a finite set and $T \subseteq D_{2,\mathcal{N}}$. We say that T is a **tower set** if we can order the elements in $\pi_1(T)$, $a_1 < \cdots < a_t$, such that if $(a_i, b) \in T$ then $(a_{i+1}, b) \in T$.

In Definition 2.5.9, we introduce the notion of **g-tower set** for a subset of $D_{2,\mathcal{N}}$ and analogously, see Definition 2.5.11, we introduce the notion of **g-towerizable set** for a subset of $C_{2,\mathcal{N}}$.

In Theorem 2.5.12 we show that this configuration preserve the Cohen-Macaulayness.

Theorem 2.5.12 Let $S \subseteq C_{2,\mathcal{N}}$ be a g-towerizable set then S is CM.

In the last part of Section 2.5 we will “reverse”this theorem to describe as the g-tower sets characterize the Cohen-Macaulyness.

Definition 2.5.13 Let $S \subseteq C_{2,\{x_1,\ldots,x_N\}}$ and $S' \subseteq C_{2,\{y_1,\ldots,y_{N'}\}}$ we will say that S' is a restriction of S if there exists a homomorphism

$$\nu : k[y_1, \ldots, y_{N'}] \to k[x_1, \ldots, x_N]$$

for which we have

$$I_S = \nu(I_{S'}).$$

We prove the following.

Theorem 2.5.29 $S \subseteq C_{2,\mathcal{N}}$ is CM iff there exists S' a restriction of S which is a g-towerizable set.

In the study of algebraic invariants of Cohen-Macaulay ideals, by the Artinian reduction, many questions can be investigated on Artinian graded algebras. In this context, especially in the study of Hilbert function and graded Betti numbers of a graded algebra A, an important tool is the multiplicative map by a form of degree d, $\times f : A_i \to A_{d+i}$. The case $d = 1$, i.e. the multiplication by a linear form, leads to the study of the Weak Lefschetz Algebras. In the second part of this thesis we pursue the study of such algebras.

An Artinian WL algebra A has, in some sense, a “generic” linear quotient $A/\ell A$. So, it seems totally natural to study these algebras which arise as generic linear quotient of WL Artinian graded algebras and try to understand what they inherit from the starting algebra. Such an investigation is similar to what one does in Algebraic Geometry when one studies the generic hyperplane section of a projective variety. It is well known that if one starts from an arithmetically Cohen-Macaulay variety of dimension > 0 then the generic hyperplane section, since is done by a regular element, has the same graded Betti numbers of the starting variety and consequently its Hilbert function is just the first difference of the Hilbert function of the variety. In the case of the generic linear quotient of a WL Artinian graded algebra A the question is not so simple, since the form ℓ is no more a regular element. So, while it is easy to see that its Hilbert function is just the positive part of the first difference of the Hilbert function of A, the question is more tricky for the graded Betti numbers.

Let $R = k[x_1, \ldots, x_c]$ be the standard polynomial ring over an algebraically closed field k of characteristic zero. Let M be a graded k-module, $M = \oplus_i M_i$.

M is said to have the **Weak Lefschetz Property**, WLP for short, if there is a linear form $\ell \in R_1$ such that the linear map given by the multiplication by ℓ

$$\times\ell : M_i \to M_{i+1}$$

has maximal rank, for every integer i (such a linear form will be called a **WL form**). A module (algebra) with the WLP will be call **Weak Lefschetz module (algebra)**.

In Chapter 3 we give some basic properties about Weak Lefschetz modules and algebras. It also includes an overview and recent developments of the Weak Lefschetz property.

Let $A = R/I$ be a standard graded k-algebra, $A = \oplus_i A_i$, and $\ell \in R_1$. Denoted by $\varphi_{\ell,i} : A_i \to A_{i+1}$ the linear map (as k-vector spaces) obtained by the multiplication by ℓ, we have, for every integer i, the following exact sequence

$$0 \to \operatorname{Ker}\varphi_{\ell,i} \to A_i \xrightarrow{\varphi_{\ell,i}} A_{i+1} \to (A/\ell A)_{i+1} \to 0.$$

Therefore the Hilbert function of the linear quotient of A is given by

$$H_{A/\ell A}(i+1) = \Delta H_A(i+1) + \dim_k \operatorname{Ker}\varphi_{\ell,i}.$$

This allows us to give an equivalent definition of WLP for an algebra just looking at its generic linear quotient.

Proposition 3.1.4 Let A be an Artinian standard graded algebra. The following are equivalent

i) A has the WLP;

ii) there is an element $\ell \in R_1$ such that $H_{A/\ell A} = \Delta H_A^+$.

So, we can study the WLP for an Artinian algebra A just looking at the *good behavior* of the generic quotient with respect to the Hilbert function.

The aim of Chapter 4, contained in the paper [FRZ1], is to extend this *good behavior* with respect to the graded Betti numbers. In this sense we generalize the WLP to the β-WLP.

In order to do this, in Section 4.1 we study the Hilbert function and the graded Betti numbers for "generic" linear quotients of Artinian standard graded algebras, especially in the case of Weak Lefschetz algebras.

Let A be an Artinian algebra we denote by $\mathcal{H}_A$ the set of the Hilbert functions of the linear quotients of A when ℓ varies in R.

$$\mathcal{H}_A := \{H_{A/\ell A} \mid \ell \in R_1\};$$

Since we have a natural surjection $A \to A/\ell A$ we see that $H_{A/\ell A} \leq H_A$ for every $\ell \in R_1$. Moreover, since A is Artinian, $\mathcal{H}_A$ only have a finite number of elements,

$$\mathcal{H}_A = \{H_1, H_2, \ldots, H_r\}$$

Now we define

$$S_{A,H_i} := \{[\ell] \in \mathbb{P}_k(R_1) \mid H_{A/\ell A} = H_i\}.$$

So we have that set-theoretically

$$\mathbb{P}_k(R_1) = S_{A,H_1} \cup \cdots \cup S_{A,H_r}.$$

Observe that $\{S_{A,H}\}_H$ is a partition of $\mathbb{P}_k(R_1)$. Since $\mathcal{H}_A$ is a finite set, there exists u such that S_{A,H_u} contains a non empty open subset $U \subseteq \mathbb{P}_k(R_1)$ and there is only one element in $\mathcal{H}_A$ with such a property.

Definition 4.1.1 With the above notation we say that $A/\ell A$ has the generic Hilbert function with respect to A iff $[\ell] \in S_{H_u}$. In this case $H_u = H_{A/\ell A}$ will be called the Hilbert function of the generic linear section of A and will be denoted by H_A^{gen}.

Moreover we have the following

Proposition 4.1.3 Let $A = R/I$ be an Artinian standard graded R-algebra. The poset $\mathcal{H}_A$ has only one minimal element, precisely H_A^{gen}.

Because of previous discussions we set

$$S^{gen} = S_{A,H_A^{gen}}$$

Analogously to what we did before, we define the set of the graded Betti numbers of the linear quotients of A when ℓ varies in R.

$$\mathcal{B}_A = \{\beta(A/\ell A) \mid [\ell] \in S^{gen}\}.$$

By a well known result due to Bigatti, Hulett, independently in characteristic zero, and by Pardue, later on any characteristic, the set $\mathcal{B}_A$ is finite, hence

$$\mathcal{B}_A = \{\beta_1, \ldots, \beta_r\}.$$

Now, we set

$$Z_{\beta_i} := \{[\ell] \in S^{gen} \mid \beta_{A/\ell A} = \beta_i\}.$$

Therefore we have a finite partition of S^{gen},

$$S^{gen} = Z_{\beta_1} \cup \cdots \cup Z_{\beta_r}.$$

Consequently, as before, there exists $v \in \{1, \ldots, r\}$ such that Z_{β_v} contains a nonempty open subset V of S^{gen}.

Definition 4.1.4 With the above notation we say that $A/\ell A$ has the generic Betti sequence with respect to A iff $[\ell] \in Z_{\beta_v}$. In this case $\beta(A/\ell A)$ will be called the Betti sequence of the generic linear section of A and will be denoted by β_A^{gen}.

Moreover, similarly as before, we have the following

Proposition 4.1.5 Let $A = R/I$ be an Artinian standard graded R-algebra. The poset $\mathcal{B}_A$ has only one minimal element, precisely β_A^{gen}.

We showed as H_A^{gen} and β_A^{gen} have a similar behavior. In the same way as H_A^{gen} is the only minimal element in $\mathcal{H}_A$, β_A^{gen} is the only minimal element in $\mathcal{B}_A$. The lowest value can be reached by $H_{A/\ell A}$ is ΔH^+ and when this happen A is a WL algebra.

the aim of section 4.2 is to search which conditions have to be required on β_A^{gen} to obtain the analogue situation. In order to do this we will study the Betti sequences of the linear quotients of Artinian standard graded algebras which have the Weak Lefschetz property.

Let $A = R/I$ be a Weak Lefschetz Artinian algebra and $\ell \in R_1$ a WL form for A. We want to study the graded Betti numbers, $\overline{\beta}_{ij}(\overline{A})$, of the algebra

$$\overline{A} := A/\ell A \cong R/(I + (\ell))$$

as a $\overline{R}$-algebra, $\overline{R} := R/\ell$.

We set

$$t := \max\{j \mid \Delta H_A(j) > 0\} \tag{1}$$

and $J := I_{\leq t}$, the ideal generated by the elements of I with degree less than or equal to t. We consider the following commutative diagram:

$$\begin{array}{ccc} \left(R/I\right)_t & \xrightarrow{\psi} & \left(R/J\right)_{t+1} \\ & \overset{\varphi}{\searrow} & \downarrow^{p} \\ & & \left(R/I\right)_{t+1} \end{array}$$

where the maps ψ and φ are both the multiplication by ℓ (hence $\varphi = \varphi_{\ell,t}$ and p is the natural map).

Since the multiplication map by ℓ is injective up to t and since the Hilbert function of $A/\ell A$ is zero for degrees $> t$, we only have to check at degree $t + 1$ to determine the degree of all the minimal generators of $I + (\ell)$. The map ψ will be crucial in our investigation, as shows the following theorem.

Theorem 4.2.8

$$\beta_{0\ t+1}(A) - \overline{\beta}_{0\ t+1}(\overline{A}) = \dim_k(\operatorname{Ker}\varphi) - \dim_k(\operatorname{Ker}\psi).$$

Moreover,

Proposition 4.2.9 With the above notation

i) ψ is surjective iff $\overline{\beta}_{0\ t+1}(\overline{A}) = 0$;

ii) ψ is injective iff $\overline{\beta}_{0\ t+1}(\overline{A}) = \beta_{0\ t+1}(A) + \Delta H_A(t+1)$.

Now, next goal will be to determine the graded Betti numbers of $\overline{A}$. It is important to note that, from a numerical point of view, this computation was made by several authors. See for instance Lemma 8.3 in [MN2] where the authors determined the value of $\overline{\beta}_{i,j}(\overline{A})$ for $i+j<t$. Now we assume a qualitative point of view and we study a minimal free resolution of $\overline{A}$. We will see that this has a strong connection with a minimal free resolution of A.

Let us consider a graded minimal free resolution of A as a R-module

$$F_\bullet : 0 \to F_{c-1} \to \cdots \to F_i \stackrel{d_i}{\to} F_{i-1} \to \cdots \to F_0 \to R \to A \to 0$$

and a graded minimal free resolution of $\overline{A}$ as a $\overline{R}$-module

$$G_\bullet : 0 \to G_{c-2} \to \cdots \to G_i \stackrel{d'_i}{\to} G_{i-1} \to \cdots \to G_0 \to \overline{R} \to \overline{A} \to 0.$$

Let $\pi_\bullet : F_\bullet \to G_\bullet$ a lifting of the natural map of R-modules $\pi : A \to \overline{A}$.

Theorem 4.2.12 With the above notation, for every $i \geq 0$, let

$$\{\gamma_{i1}, \ldots, \gamma_{i\beta_i}\}, \quad \deg \gamma_{i1} \leq \ldots \leq \deg \gamma_{i\beta_i},$$

be a minimal set of generators for $\operatorname{Im} d_i$, and $u_i := |\{j \mid \deg \gamma_{ij} \leq t+i\}|$. If $u_i > 0$ then $\{\pi_{i-1}(\gamma_{i1}), \ldots, \pi_{i-1}(\gamma_{iu_i})\}$ can be completed to a minimal set of generators for $\operatorname{Im} d'_i$ with elements of degree $\geq t+i$.

In particular, from this we get

Corollary 4.2.16 $\beta_{1\ t+1}(A) + \dim_k(\operatorname{Ker}\psi) = \overline{\beta}_{1\ t+1}(\overline{A})$.

Proposition 4.2.17 $\beta_{i\ t+i}(A) = \overline{\beta}_{i\ t+i}(\overline{A})$ for every $i \geq 0$ iff ψ is injective.

Collecting the previous results we can give a description of the graded Betti numbers of $\overline{A}$.

$$\overline{\beta}_{ij}(\overline{A}) = \begin{cases} \beta_{ij}(A) & \text{if } j \leq t+i-1 \\ \beta_{ij}(A) + m_i & \text{if } j = t+i \\ \sum_{h \geq i+1} (-1)^{h+i+1} \beta_{h\ j}(A) + (-1)^{i+1} \Delta^{c-1} \Delta H_A^+(j) + m_{i+1} & \text{if } j = t+i+1 \\ 0 & \text{if } j > t+i+1 \end{cases} \quad (4.3)$$

where $m_i \geq 0$ and in particular $m_0 = 0$ and $m_1 = \dim_k \ker\psi$. If $c = 3$ the graded Betti numbers of $\overline{A}$ are determined by $\dim_k \operatorname{Ker}\psi$.

Definition 4.3.2 We say that $A = R/I$ has the *Betti Weak Lefschetz Property*, briefly β-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ is injective.

An equivalent version of this definition can be given looking at the graded Betti numbers of the algebra

Proposition 4.3.3 Let A be a standard graded R-algebra. The following are equivalent

1. A has the β-WLP and ℓ is a β-WL form;
2. The graded Betti numbers of $A/\ell A$ are determined by (4.3) with $m_i = 0$ for every i.

Harima Migliore Nagel Watanabe, proved in Theorem 3.20 in [HMNW], that if H is a Weak Lefschetz sequence then the set

$$\mathcal{B}_H^{\mathrm{WL}} = \{\beta_A \mid H_A = H \text{ and } A \text{ has the WLP}\}$$

admits exactly one maximal element, say, β^H. Because of this result we get

Proposition 4.3.4 Let H be a Weak Lefschetz sequence and let $A = R/I$ be an Artinian graded algebra with $H_A = H$ such that A has the WLP. If $\beta_{0\ t+1}(A) = \beta^H_{0\ t+1}$ then A has the β-WLP.

It is known that if H is the Hilbert function of an Artinian Gorenstein standard graded R-algebra of codimension 3 and $\vartheta - 3$ is its socle degree then the set of the Gorenstein Betti sequences compatible with H

$$\mathcal{G}_H = \{\beta_A \mid H_A = H \text{ and } A \text{ is a Gorenstein Algebra}\}$$

has only one maximal element β^{max} and only one minimal element β^{min} (see [RZ4] Proposition 3.7 and Remark 3.8). Recently, Ragusa and Zappalà proved in [RZ2] that there exists a Gorenstein Betti sequence $\gamma^H \in \mathcal{G}_H$, such that every Artinian Gorenstein standard graded R-algebra with Betti sequence in grater than or equal to γ^H has the WLP (see Corollary 2.7 in [RZ2]). We recall that

$$\gamma^H_{0i} = \begin{cases} \beta^{\max}_{0i} & \text{for } i = t+1, \vartheta - t - 1 \\ \beta^{\min}_{0i} & \text{otherwise} \end{cases}.$$

Actually in the next proposition we can improve this result.

Proposition 4.3.6 Let H be the Hilbert function of an Artinian Gorenstein standard graded R-algebra of codimension 3. Then every R-algebra A with Betti sequence $\beta_A \in \mathcal{G}_H$ and $\beta_A \geq \gamma^H$ has the β-WLP.

Let $A = R/I$ be a complete intersection Artinian graded standard k-algebra which have the Weak Lefschetz property. Let $I = (g_1, \ldots, g_c)$, with $\deg g_i \leq \deg g_{i+1}$ for $1 \leq i \leq c-1$. For such an algebra it is easy to study the β-WLP.

Proposition 4.4.1 Let A be as above and t be as defined in 1 then

1. If $\deg g_c > t$ then A has the β-WLP.
2. If $\deg g_c \leq t$ and $\Delta H_A(t+1) = 0$ then A has the β-WLP.
3. If $\deg g_c \leq t$ and $\Delta H_A(t+1) \neq 0$ then A has not the β-WLP.

The item 3 of the previous proposition in particular says that ψ_ℓ is not injective but still it has maximal rank. This suggests to give a weaker form of the Definition 4.3.2.

Definition 4.4.2 We say that $A = R/I$ has the *generators Weak Lefschetz Property*, briefly β_0-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ has maximal rank.

Part I

Monomial ideals and the Cohen-Macaulay property

Chapter 1

Monomial ideals, notation and basic facts

In this chapter we introduce notation, give basic definitions and recall some well-known results about modules and standard algebras. In Section 1.1 we recall some basic aspects of the Cohen-Macaulay property. A detailed exposition of the fundamental facts in Section 1.2 can be found in the books [Ei1] and [Ei2].

In Section 1.3 and Section 1.4 we recall some basic facts about monomial ideals and we describe combinatoric properties of the squarefree ideals which arise with a simplicial complex. For a more exhaustive discussion of these issues see the book [HH].

1.1 Cohen-Macaulay Rings

1.1.1 Height and Krull dimension

Let R be a Noetherian ring and $P \subseteq R$ a prime ideal. We say that the length of the chain of prime ideals

$$P_0 \subset \cdots \subset P_{r-1} \subset P_r$$

is r if the chain cannot be refined i.e. for any prime ideal Q such that $P_j \subset Q \subset P_{j+1}$ we have $Q = P_j$ or $Q = P_{j+1}$.

The **height** of a prime ideal $P \subseteq R$, written $\operatorname{ht} P$, is the supremum of lengths of the chains of primes contained in P. If $I \subseteq R$ is any ideal, then

$$\operatorname{ht} I = \min\{\operatorname{ht} P | I \subseteq P \text{ and } P \in \operatorname{Spec}(R)\}.$$

We define the **Krull-dimension** (or simply the **dimension**) of a ring R, written $\dim R$, as the supremum of the heights of the prime ideals in R.

Note that $\dim(R_P) = \operatorname{ht} P$, where R_P is the localization of R at P.

Now let R be a ring, and $I \subseteq R$ an ideal, the dimension of I, $\dim I$, is defined as the Krull dimension of R/I.

The **codimension** of I, written $\operatorname{codim} I$, is the difference

$$\dim R - \dim R/I.$$

1.1.2 Regular sequences and depth

Let R be a ring and M a R-module. A sequence of elements in R, $z_1, \ldots, z_d$, is called a **regular sequence**, or a ***M*-sequence**, if z_1 is a non-zero divisor on M and

$$z_i \text{ is a non-zero divisor on } M/(z_1, \ldots, z_{i-1})M$$

for $i = 2, \ldots, d$.

A R-regular sequence is simply called a **regular sequence**. That is, $z_1, \ldots, z_d$ is a regular sequence if z_1 is a non-zero-divisor in R, z_2 is a non-zero-divisor in the ring $R/(z_1)$, and so on.

In geometric language, if X is an affine scheme and $z_1, \ldots, z_d$ is a regular sequence in the ring of regular functions on X, then we say that the closed subscheme $V(z_1, \ldots, z_d) \subseteq X$ is a **complete intersection** subscheme of X.

Let R be a Noetherian ring, I an ideal in R, and M a finitely generated R-module. The **depth of *I* on *M***, written $\operatorname{depth}_R(I, M)$ or just $\operatorname{depth}(I, M)$, is the supremum of the lengths of all M-regular sequences of elements of I. When $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module, the **depth of *M***, written $\operatorname{depth}_R(M)$ or just $\operatorname{depth}(M)$, is $\operatorname{depth}_R(\mathfrak{m}, M)$; it is the supremum of the lengths of all M-regular sequences in the maximal ideal $\mathfrak{m}$ of R. In particular, the depth of a Noetherian local ring R is the depth of R as a R-module.

For a Noetherian local ring R, the depth of a nonzero finitely generated R-module M is at most the Krull dimension of M, that is

$$\dim M := \dim R / \operatorname{Ann}_R M$$

where $\operatorname{Ann}_R M$ is the kernel of the natural map $R \to \operatorname{End}_R(M)$ of R into the ring of R-linear endomorphisms of M.

1.1.3 Maximal regular sequences and grade

Let R be a Noetherian ring and M a R-module. If $z_1, \ldots, z_n$ is a M-sequence then we have a strictly ascendant sequence of ideals

$$(z_1) \subset (z_1, z_2) \subset (z_1, z_2, z_3) \subset \cdots \subset (z_1, z_2, \ldots, z_n).$$

A M-regular sequence $\{z_1, z_2, \dots, z_n\} \subseteq I$ is called a **maximal *M*-sequence** if any

$$y \in I \setminus \{0\} \text{ is a zero divisor in } M/(z_1, z_2, \dots, z_n)M.$$

By Noetherianity a M-sequence can be extended to a maximal one. The following theorem due to Rees shows that, in a local ring, all maximal M-sequences in an ideal I with $M \neq IM$ have same length. This is called **grade of I** with respect to M and denoted by $\text{grade}(I, M)$.

Theorem 1.1.1 (Rees)**.** *Let R be a Noetherian local ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length n given by*

$$n = \min\{i : \text{Ext}^i_R(R/I, M) \neq 0\}.$$

An ideal $I \subseteq R$ is called a **complete intersection ideal** if it is generated by a R-sequence. If I is a complete intersection ideal, the algebra R/I is called **complete intersection algebra**.

The following lemma describes some relations among the invariants so far defined

Lemma 1.1.2. *If M is a module over a local ring R and z is a regular element of M, then*

- $\text{depth}(M/zM) = \text{depth}(M) - 1$;
- $\dim(M/zM) = \dim(M) - 1$.

The depth is always bounded above by the Krull dimension. Equality provides some interesting conditions.

1.1.4 Cohen-Macaulay property

Let $(R, \mathfrak{m})$ be a Noetherian local ring. A nonzero finite R-module M is called **Cohen-Macaulay** if

$$\text{depth}(M) = \dim(M).$$

If R is a Cohen-Macaulay R-module then R is called a **Cohen-Macaulay ring**, i.e. there exists a regular sequence as long as its dimension.

More generally a ring is called Cohen-Macaulay if all of its localizations at prime ideals are Cohen-Macaulay.

Let $I \subseteq R$ be an ideal, we say that I is a **Cohen-Macaulay ideal**, for short **CM**, if R/I is Cohen-Macaulay.

They are so named in honor of the mathematicians Francis Sowerby Macaulay and Irvin Sol Cohen. They proved *the unmixedness theorem* during different periods and in two different cases.

1.1.5 The unmixedness theorem

An ideal I of a Noetherian ring R is called **unmixed** if all its associated prime ideals have the same height. The **unmixedness theorem** is said to hold for a ring R if every ideal I generated by $\mathrm{ht}(I)$ elements is unmixed.

Macaulay proved in 1916 the unmixedness theorem for polynomial rings; Cohen proved in 1946 the unmixedness theorem for formal power series rings. These facts explain the nomenclature Cohen-Macaulay only together with the next theorem.

Theorem 1.1.3 ([BH], Theorem 2.1.6)**.** *A Noetherian ring R is Cohen-Macaulay if and only if every ideal I generated by* $\mathrm{ht}(I)$ *elements is unmixed.*

1.2 Backgrounds of homological algebra

From now on $R := k[x_1, \dots, x_c]$ denotes the standard graded polynomial ring over a field of characteristic zero. We denote by $\mathfrak{m} := (x_1, \dots, x_c)$ the homogeneous maximal ideal in R. All ideals considered will be homogeneous and all modules will be finitely generated and graded.

1.2.1 Free resolutions and graded Betti numbers

Let $M = \bigoplus_d M_d$ be a R-module whose d-th graded component is M_d. Because M is finitely generated, each M_d is a finite-dimensional k-vector space, and we define the **Hilbert function of *M*** to be

$$H_M(d) = \dim_k M_d.$$

For any graded module M, we denote by $M(a)$ the module M shifted by a, i.e.

$$M(a)_d = M_{a+d}.$$

Let $\{m_1, \dots, m_r\}$ be a system of homogeneous generators of M with degree respectively $a_1, \dots, a_r$. We define a map ε from the graded free module $F_0 = \oplus_i R(-a_i)$ onto M by sending e_i, the standard element of its basis, to m_i.

The map ε is a **graded map**, i.e. ε is a degree-preserving map. The module $\operatorname{Ker}\varepsilon \subseteq F_0$ is also a finitely generated module. So, there is again a graded free R-module F_1 and a graded epimorphism $F_1 \to \operatorname{Ker}(\varepsilon)$, whose kernel is a submodule of F_1. Composing $F_1 \to \operatorname{Ker}(\varepsilon)$ with the inclusion map $\operatorname{Ker}(\varepsilon) \to F_0$, we get a homomorphism $d_1 : F_1 \to F_0$ such that

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

is exact. Proceeding in this way one constructs an exact sequence called a **graded free resolution of *M***

$$\cdots \to F_i \xrightarrow{d_i} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0.$$

Since each d_i preserves degree, we get an exact sequence of finite-dimensional vector spaces just taking the degree d of each module in the sequence, so we have

$$H_M(d) = \sum_i (-1)^i H_{F_i}(d).$$

Let M be a R-module. An exact complex

$$\mathbb{F}_\bullet : \quad \cdots \to F_i \xrightarrow{d_i} \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0$$

of finitely generated free R-modules is called a **minimal (graded) free resolution of *M***, if for each i, $\operatorname{Im} d_i \subseteq \mathfrak{m}F_{i-1}$.

A minimal free resolution always there exists, precisely we can get it just repeating the construction above choosing at each step a minimal system of homogeneous generators of $\operatorname{Ker}(d_{i-1}) = \operatorname{Im}(d_i)$. In a free resolution the module $\operatorname{Ker}(d_{i-1}) = \operatorname{Im}(d_i)$ is called the ***i*-th syzygy module of *M*** and its elements are called ***i*-th syzygies**. If $\mathbb{F}_\bullet$ is a minimal free resolution of M, where the morphisms are graded as described above, we can write for each i

$$F_i = \bigoplus_j R(-j)^{\beta_{ij}}$$

for some positive integers β_{ij}.

Any two minimal free resolutions $\mathbb{F}_\bullet$ and $\mathbb{G}_\bullet$ of M are isomorphic. In the sense that there exists a graded isomorphism of complexes $\varphi : \mathbb{F}_\bullet \to \mathbb{G}_\bullet$. In this case we will write

$$\mathbb{F}_\bullet \cong \mathbb{G}_\bullet.$$

Therefore the numbers β_{ij} are invariants for M as a graded R-module, so, given a minimal free resolution, we can write $\beta_{ij}(M) := \beta_{ij}$. We can also obtain them as the homological invariants

$$\beta_{ij}(M) = \dim_k \operatorname{Tor}_i(M,k)_j$$

since $\operatorname{Tor}_i^R(M,k) = H_i(\mathbb{F}_\bullet \otimes k) = F_i \otimes k = F_i/\mathfrak{m}F_i$, where k is seen as the graded R-module $k \cong R/\mathfrak{m}$, see [Ei2] Proposition 1.7.

The numbers $\beta_{ij}(M)$ are called the **graded Betti numbers** of M. Given an integer i, we set $\beta_i(M) := \operatorname{rank} F_i = \sum_j \beta_{ij}(M)$. The number $\beta_i(M)$ is called the **i-th Betti number** of M.

Suppose M has a finite free resolution, then the maximal number i with $\mathrm{Tor}_i(M, k) \neq 0$, i.e. $\beta_i(M) \neq 0$, is called the **projective dimension of *M*,** and denoted by proj-dim(M), so

$$\text{proj-dim}(M) = \max\{i|\beta_i(M) \neq 0\}.$$

The AuslanderBuchsbaum formula, introduced by Auslander and Buchsbaum (1957, Theorem 3.7 [AB]), describes a relation among the projective dimension and the depth.

Theorem 1.2.1 (Auslander-Buchsbaum)**.** *Let* $(S, \mathfrak{m})$ *be a Noetherian local ring, and* $M \neq 0$ *a finite* S*-module. If* proj-dim $M < \infty$ *, then*

$$\text{proj-dim}(M) + \text{depth}(M) = \text{depth}(S).$$

In particular for a R-module M, let i be an integer such that $\beta_i(M) \neq 0$, we have $i \leq \text{proj-dim}(M) \leq c$.

For a R-module M the graded Betti numbers are a finer invariant than the Hilbert function. Namely, if $\{\beta_{ij}\}$ are the graded Betti numbers for a R-module M, let $B_j = \sum_i (-1)^i \beta_{ij}$, then, see Corollary 1.10 in [Ei2],

$$H_M(d) = \sum_j B_j \binom{c+d-j-1}{c-1}. \tag{1.1}$$

Moreover, only the values of the B_j's can be recursively deduced from the function $H_M(d)$ via the formula

$$B_j = H_M(j) - \sum_{\{k|k<j\}} B_k \binom{c+j-k-1}{c-1}. \tag{1.2}$$

Interesting questions arise about what sequences of integers could be Hilbert functions of suitable algebras.

1.2.2 Hilbert functions of graded algebras

Let h be a natural number, for any integer d we can write h uniquely as

$$h_{(d)} := \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}, \quad \text{where} \quad k_d > k_{d-1} > \cdots > k_1 \geq 0.$$

This expression is called the d-**binomial expansion** of h. Now let $a, b \in \mathbb{Z}$, we set

$$(h_{(d)})_b^a := \binom{k_d + a}{d+b} + \binom{k_{d-1} + a}{d-1+b} + \cdots + \binom{k_1 + a}{1+b}.$$

We say that $h = (h_0, h_1, h_2, \ldots)$ is a ***O*-sequence** if $h_{d+1} \leq (h_{(d)})^{+1}_{+1}, \ \ d \geq 0$.

Macaulay gave a characterization, also pointed out by Stanley in [St1], on the maximal grow for the Hilbert function of a **standard *k*-algebra** R/I, where I is a homogeneous ideal of the polynomial ring R.

Theorem 1.2.2 (Macaulay [Ma1]). *Let $I \subseteq R = k[x_1, x_2, \ldots, x_c]$ be a homogeneous ideal, then there is a monomial ideal L such that $H_{R/L} = H_{R/I}$. Furthermore, let $H = (h_0, h_1, h_2, \ldots)$ be a sequence of numbers, then the following are equivalent*

i) H is the Hilbert function of some standard graded algebra;

ii) H is a O-sequence.

Given H a O-sequence we can easily see that there can be more than one ideal $I \subseteq R$ such that the Hilbert function of R/I is H. We can distinguish such ideals looking at their graded Betti numbers.

So, a question we can ask is: given a O-sequence H, what sets of numbers are graded Betti numbers for an algebra A with $H_A = H$? The question is still open. Introducing the order $\{\beta_{ij}(A)\} \leq \{\beta_{ij}(A')\}$ if and only if $\beta_{ij}(A) \leq \beta_{ij}(A')$ for all i and j, an important result due to Bigatti, Hulett and Pardue, show that there is an upper bound for the poset of all graded Betti numbers compatible with H. This in particular says that there are only a finite number of possibilities.

1.2.3 Maximal Betti numbers and cancellations

We introduce the **lexicographic order** on the monomials of R, precisely if $x_1^{a_1} \cdots x_c^{a_c}, x_1^{b_1} \cdots x_c^{b_c} \in R_d$ we say that

$$x_1^{a_1} \cdots x_c^{a_c} <_{lex} x_1^{b_1} \cdots x_c^{b_c}$$

if and only if $a_1 < b_1$ or $a_1 = b_1$ and there exists i such that $a_i < b_i$, and $a_j = b_j$ when $j < i$.

The lexicographical order is a well-ordering relation.

A monomial ideal L is a **lex-segment ideal** if given $m_1, m_2 \in R_d$ such that $m_1 \in L$ and $m_2 >_{lex} m_1$ then $m_2 \in L$. So, L_d is a k-vector space generated by the largest, with respect to the lexicographic order, monomials for each degree d. If $L \subseteq R$ is a lex-segment ideal, the algebra R/L is called **lex-segment algebra**.

Macaulay, proving the first part of Theorem 1.2.2, showed that, given a Hilbert function of an algebra R/I, if in each degree d we exclude the smallest $h_d = H_{R/I}(d)$ monomials, with respect to the lexicographic order, the remaining monomials are a k-base for a lex-segment ideal L. Furthermore

$$H_{R/L}(d) = h_d, \ d \geq 0.$$

Therefore, given a Hilbert function H, there exists a unique lex segment ideal L_H such that $H_{R/L_H} = H$, for this reason such an ideal is called the **lex-segment ideal associated to** $\boldsymbol{H}$.

The graded Betti numbers of the lex-segment ideal are the upper bound we talked about before.

Theorem 1.2.3 (Bigatti [Bi], Hulett [Hu], Pardue [Pa]). *Let I be a homogeneous ideal of R, and let $H = H_{R/I}$. Then*

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/L_H), \quad \textit{for any} \quad i, j.$$

Since the integers $\beta_{ij}(R/L_H)$ only depend by H we will denote them with $\hat{\beta}_{ij}(H)$.

Given a O-sequence H, Theorem 1.2.3 implies that, all the graded Betti numbers for an algebra with Hilbert function H must be obtained from $\hat{\beta}_{ij}(H)$ by suitable variations. To preserve the Hilbert function, as we showed in the formula 1.2, we must preserve the value of the B_j's. So the only alterations allowed are the so called *cancellations* that now we describe.

Given a sequence of positive integers $\{\beta_{ij}\}$, we define a **cancellation** the following procedure: fix an index j, and choose i and i' such that one of the numbers is odd and the other is even; then replace β_{ij} and $\beta_{i'j}$ respectively by $\beta_{ij} - 1$ and by $\beta_{i'j} - 1$. In particular we get a **consecutive cancellation** when $|i' - i| = 1$.

The terminology is justified by the fact that we delete elements from the multi-sets

$$\{a_{ij} | \text{ degree of minimal generators of the i-th syzygy module}\}_i,$$

and the cancellations are consecutive when we consider graded Betti numbers of consecutive homological degrees.

By the result in [Pe], the graded Betti numbers of an algebra A having Hilbert function H are obtained from $\hat{\beta}_{ij}(H)$ only by consecutive cancellations. Note that this is obvious for $c = 3$.

Campanella, see [Ca], proved that for O-sequences of codimension two, $H = (1, 2, h_2, \ldots)$, all the cancellations from the maximal Betti numbers are graded Betti numbers of some algebra. Therefore given H, a O-sequence of codimension two, the set $\{\{\beta_{ij}(A)\} | H_A = H\}$ also admits a minimal element.

In higher codimension consecutive cancellations could be not allowed.

Example 1.2.4. Let $R = k[x, y, z]$ and

$$I = (x^3, x^2y, x^2z, y^4) + \mathfrak{m}^5.$$

Then $A = R/I$ has Hilbert function $H_A = (1, 3, 6, 7, 8, 0, \dots)$ and minimal free resolution

$$0 \to \begin{array}{c} R(-5) \\ \oplus \\ R(-7)^8 \end{array} \to \begin{array}{c} R(-4)^3 \\ \oplus \\ R(-6)^{17} \end{array} \to \begin{array}{c} R(-3)^3 \\ \oplus \\ R(-4) \\ \oplus \\ R(-5)^8 \end{array} \to A \to R \to 0.$$

So the only cancellation we could achieve is between $\beta_{0,4}$ and $\beta_{1,4}$. But, suppose $J \subseteq R$ has minimal free resolution

$$0 \to \begin{array}{c} R(-5) \\ \oplus \\ R(-7)^8 \end{array} \to \begin{array}{c} R(-4)^2 \\ \oplus \\ R(-6)^{17} \end{array} \stackrel{d_1}{\to} \begin{array}{c} R(-3)^3 \\ \oplus \\ R(-5)^8 \end{array} \stackrel{d_0}{\to} R/J \to R \to 0$$

then, let $\{g_1, g_2, g_3\}$ be a k-base of J_3 and $\{\varphi_1, \varphi_2\}$ a k-base of the vector space $\ker(d_1)_4 \subseteq R(-4)^2 \oplus R(-6)^{17}$. We have

$$\begin{cases} a_1g_1 + a_2g_2 + a_3g_3 &= 0 \\ b_1g_1 + b_2g_2 + b_3g_3 &= 0 \\ \ell_1\varphi_1 + \ell_2\varphi_2 &= 0 \end{cases}$$

where $\ell_1, \ell_2 \in R_1$,

$$d_1(\varphi_1) = (a_1, a_2, a_3, 0, \dots, 0) \quad \text{and} \quad d_1(\varphi_2) = (b_1, b_2, b_3, 0, \dots, 0).$$

By $\ell_1 d_1(\varphi_1) + \ell_2 d_2(\varphi_2) = 0$ we have $a_i\ell_1 + b_i\ell_2 = 0$, for $i = 1, 2, 3$. Note that $\ell_1 \notin (\ell_2)$ and $a_i, b_i \in R_1$, so we have $b_i \in (\ell_1)$, for $i = 1, 2, 3$. This is a contradiction because φ_2 is part of a minimal system of generators for the first syzygies module.

Of course, Theorem 1.2.3 raises the dual question about the existence of only one minimal element in the poset of all the graded Betti numbers compatible with a given O-sequence. Many examples show that there could be more than one minimal element, see for instance [Ri] and [RZ1].

The following example makes use of Proposition 2.3. in [RZ3], where there is a characterization of all the Hilbert functions of complete intersection ideals which have the only one minimal element required.

Example 1.2.5. Let $R = k[x, y, z]$ and consider the O-sequence

$$H = (1, 3, 4, 4, 3, 1, 0, \ldots).$$

Let

$$I = (x^2, y^2, z^4) \subseteq R$$

and

$$J = (x^2, xy + xz, y^3, y^2z^3, z^5) \subseteq R$$

One can check that $H_{R/I} = H_{R/J} = H$, but

$$0 \rightarrow R(-8) \rightarrow \begin{matrix} R(-4) \\ \oplus \\ R(-6)^2 \end{matrix} \rightarrow \begin{matrix} R(-2)^2 \\ \oplus \\ R(-4) \end{matrix} \rightarrow R/I \rightarrow R \rightarrow 0.$$

and

$$0 \rightarrow \begin{matrix} R(-6) \\ \oplus \\ R(-7) \\ \oplus \\ R(-8) \end{matrix} \rightarrow \begin{matrix} R(-3) \\ \oplus \\ R(-5)^2 \\ \oplus \\ R(-6)^3 \\ \oplus \\ R(-7) \end{matrix} \rightarrow \begin{matrix} R(-2)^2 \\ \oplus \\ R(-3) \\ \oplus \\ R(-5)^2 \end{matrix} \rightarrow R/J \rightarrow R \rightarrow 0.$$

Numerically there could be only one minimal set of graded Betti numbers. But if this is the case, we have an algebra with such a resolution

$$0 \rightarrow R(-8) \rightarrow R(-6)^2 \rightarrow R(-2)^2$$

and this is not allowed since, for instance, we miss a syzygy of degree ≤ 4.

An algebra A with a finite number of non-zero entries in H_A is called **Artinian**. Of course, there are many different ways to define an Artinian algebra, in this context we looked at the finite dimension property of the Artinian algebras.

1.2.4 Artinian reduction

Let $B = R/I$ be a standard k-algebra. We recall that an algebra is Cohen-Macaulay if the length of the longest regular sequence is its the Krull dimension. If B is a Cohen-Macaulay k-algebra of Krull dimension d, then, see Chapter 1 in [Mi], there exists a maximal regular sequence in B consisting of forms having degree 1, say $\{\ell_1, \ldots, \ell_d.\}$ Therefore $A = B/(\ell_1, \ldots, \ell_d)$ is an Artinian algebra. This process is called **Artinian reduction**.

Now, we can write A as

$$A = R/(\ell_1, \ldots, \ell_d, I) \cong k[x_1, \ldots, x_{c-d}]/J,$$

where $J \subseteq k[x_1, \ldots, x_{c-d}]$ is isomorphic to $\bar{I} \subseteq R/(\ell_1, \ldots, \ell_d)$. If we set $\tilde{R} = k[x_1, \ldots, x_{c-d}]$, A has a minimal free resolution, as $\tilde{R}$-module, like the following

$$\mathbb{H}_\bullet : \quad 0 \to F_{c-d} \to \cdots \to F_1 \to F_0 \to \tilde{R} \to A \to 0$$

Moreover, by Theorem 1.3.6 in [Mi], the graded Betti numbers of B (as a R-module) are the same as the graded Betti numbers of A (as a $\tilde{R}$-module).

We can also compute H_A, the Hilbert function of A, from H_B.

Recall that if $H = (h_0, h_1, h_2, \ldots, h_s, \ldots)$ is a sequence of numbers, **the first difference of *H*** is the sequence

$$\Delta H = (h_0, h_1 - h_0, h_2 - h_1, \ldots, h_s - h_{s-1}, \ldots).$$

Recursively the ***n*-th difference of *H***, $\Delta^n H$, is the first difference of the ***n-1*-th difference of *H***, i.e.

$$\Delta^n H = (h_0, \Delta^{n-1}H(1) - \Delta^{n-1}H(2), \ldots, \Delta^{n-1}H(s-1) - \Delta^{n-1}H(s), \ldots).$$

It is easy to see that if A is the Artinian reduction of B then

$$H_A = \Delta^d H_B.$$

If A is an Artinian algebra then the finite sequence of positive integers in its Hilbert function, $h = (1, h_1, \ldots, h_s)$, is called the ***h*-vector** of A.

For Artinian algebras of codimension $c = h_1$ the formula 1.1 on page 6 can be write as

$$\sum_i (-1)^i \beta_{ij}(A) = -\Delta^c H(j).$$

1.2.5 Betti sequences

Let k be an infinite field and $R = k[x_1, \ldots, x_c]$ the polynomial ring in c variables. Let $A = R/I$ be an Artinian standard graded R-algebra (of codimension c) with minimal free resolution (as a R-module)

$$0 \to \bigoplus_j R(-j)^{\beta_{c-1\ j}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1j}} \to \bigoplus_j R(-j)^{\beta_{0j}} \to R \to A \to 0.$$

We denote the set of the i-th graded Betti numbers of A with $[\beta_i(A)] := \{\beta_{ij}(A)\}_j$, with this setting we call

$$\beta_A := ([\beta_0(A)], [\beta_1(A)], \ldots, [\beta_{c-1}(A)])$$

the **graded Betti sequence of A**.

Especially in the examples, when $\beta_{ij}(A) \neq 0$, we use for short the notation $j^{\beta_{ij}(A)}$ to indicate that there are β_{ij} independent i-th syzygies of degree j of A.

The algebras A for which $[\beta_{c-1}(A)] = \{\theta^{\beta_{c-1,\theta}}\}, \gamma := \beta_{c-1,\theta} \geq 1$ are referred to in the literature as **level algebras of type** γ. An algebra A is said to be a **Gorenstein Algebra** if it is a level algebra of type 1.

1.2.6 Gorenstein Algebras

Let $H = (1, h_0, h_1, \ldots, h_s, 0, \ldots)$ be a O-sequence, H is called a **SI-sequence** if $h_i = h_s - i$ (it is symmetric) and its first half,

$$H' = (1, h_0, h_1, \ldots, h_{\lfloor \frac{s}{2} \rfloor}, 0, \ldots),$$

is **differentiable**, i.e. if $\Delta H'$ is a O-sequence.

The study of the possible Hilbert functions of an Artinian Gorenstein algebra, also called **Gorenstein h-vectors**, is a central problem in commutative algebra. Stanley and Iarrobino conjectured that, in any codimension, a h-vector is a Gorenstein vector if and only if it is a SI-sequence. A Gorenstein h-vector, by duality, is symmetric, moreover, Migliore-Nagel, in [MN1], and Cho-Iarrobino, in [CI], show that a SI-sequence is a Gorenstein h-vector in any codimension. But the conjecture is false. Initially, Stanley, [St1], gave an example in codimension 13 that not all Gorenstein h-vectors are unimodal. Later, with the works of Bernstein, Iarrobino, Boij, Laksov, see [BI], [Bo], [BL], it has been proved that the conjecture is false in codimension ≥ 5, where there are Gorenstein h-vectors, with $h_1 \geq 5$, that are not SI-sequences.

Macaulay proved that in codimension 2 all Gorenstein h-vectors are SI-sequences, see [Ma2]. Indeed a Gorenstein algebra $A = R/I$ of codimension two A is a complete intersection algebra. So the Gorenstein h-vectors, with $h_1 = 2$, are all of the form

$$H_A = (1, 2, 3, \ldots, a-1, a, \underbrace{a, \ldots, a, a}_{\text{optional}}, a-1, \ldots, 3, 2, 1),$$

which are clearly SI-sequences.

Stanley proved, using the Buchsbaum-Eisenbud structure theorem, that the conjecture is true in codimension 3, see [St1]. In codimension 4, the conjecture is still open, we do not know if a Gorenstein h-vector, $h = (1, 4, h_2, \ldots, h_s)$, is a SI-sequence or at least unimodal. Iarrobino and Srinivasan in [IS] show that, if $h_2 \leq 7$, then h must be a SI-sequence.

There is a fundamental result which characterizes the graded Betti numbers of Gorenstein algebras in codimension 3.

Let $A = R/I$ be an Artinian Gorenstein algebra of codimension 3 with minimal free resolution (as R-module)

$$0 \to R(-\theta) \to \bigoplus_j R(-j)^{\beta_{1j}} \to \bigoplus_j R(-j)^{\beta_{0j}} \to R \to A \to 0.$$

All the possibilities that occur for the sets $[\beta_{0j}(A)]$ and $[\beta_{1j}(A)]$ were found by Diesel, see [Di], and described in [RZ4]. Precisely these results follows by the following proposition.

Proposition 1.2.6 ([RZ4], Proposition 1.1)**.** *Given $2m+1$ integers $d_0 \leq d_1 \leq \ldots \leq d_{2m}$, there exists a Gorenstein Algebra $A = R/I$ with I minimally generated in these degree if and only if*

$$\theta = \frac{\sum_{i=0}^{2m} d_i}{m}$$

is an integer and $\theta > d_i + d_{2m+1-i}$, for $i = 1, \ldots, m$.

In particular given the generators' degrees we can compute all the graded Betti numbers since the only second syzygy occurs in degree θ and the first syzygies occur in degree $\theta - d_i$, for $i = 0, \ldots, 2m$.

Moreover given a Gorenstein h-vector there exists one maximal set of graded Betti numbers of a Gorenstein algebra whose Hilbert function is h. Proposition 1.2.6 also implies that if $\{\beta_{ij}\}$ are the maximal graded Betti numbers of a Gorenstein algebra A of codimension 3, we must do an even number of cancellations to get a new set of graded Betti numbers, precisely if j_1, j_2 are such that $j_1 + j_2 = \theta$, $\beta_{ij_1} \neq 0$ and $\beta_{ij_2} \neq 0$ then

$$\{\beta_{ij} | j \neq j_1, j_2 \text{ and } i = 0, 1\} \cup \{\beta_{ij_1} - 1, \beta_{ij_2} - 1 | i = 0, 1\}$$

are graded Betti numbers of some Gorenstein algebra. On the other hand, all the Betti sequences of a Gorenstein algebra can be obtained via this numerical procedure.

1.3 Monomial ideals

It is well known that an ideal $I \subseteq R$ is called a **monomial ideal** if it is generated by monomials. We use the notation $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_c^{a_c}$ to indicate a monomial in $R = k[x_1, \ldots, x_c]$, i.e. a product of the variables each of them with a non negative exponent. If $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$, are monomials then

$$\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}.$$

The importance of the monomial ideals was here pointed out for example in Theorem 1.2.2 and Theorem 1.2.3, now we study some basic properties of these ideals.

The set of monomials of R is a k-basis of R. Therefore any polynomial $f \in R$ can be written uniquely as a k-linear combination of monomials.

If I is a monomial ideal then the quotient ring R/I is called a **monomial ring**.

Monomial ideals can be characterized by the following proposition.

Proposition 1.3.1 (1.1.3.[HH]). *Let $I \subseteq R$ be an ideal. The following conditions are equivalent:*

a) I is a monomial ideal;

b) for all $f \in R$ one has: $f \in I$ if and only if each monomial in f belongs to I.

A similar property holds for an ideal of the graded polynomial ring R, i.e., I is a homogeneous ideal if and only if, whenever $f \in I$, all homogeneous components of f belong to I.

When we use monomial ideals we often take a minimal system of monomial generators of I, it is called a **monomial basis** of I, and written $\mathcal{G}(I)$. The next proposition guarantees that it is unique.

Proposition 1.3.2. *Each monomial ideal has a unique minimal monomial set of generators. More precisely, let $\mathcal{G}(I)$ denote the set of monomials in I which are minimal with respect to divisibility. Then $\mathcal{G}(I)$ is the unique minimal set of monomial generators.*

Proof. Let $G_1 = \{u_1, \ldots, u_r\}$ and $G_2 = \{v_1, \ldots, v_s\}$ be two minimal sets of monomial generators for the monomial ideal I. Since $u_i \in I$, there exists $v_j \in G_2$ such that $u_i = \lambda v_j$ for some monomial λ. The same argument is true for v_j, so there exists $u_k \in G_1$ and a monomial μ such that $v_j = \mu u_k$. Therefore $u_i = \lambda\mu u_k$. Since G_1 is a minimal set of generators of I, we conclude that $k = i$ and $\lambda\mu = 1$. In particular, $\lambda = \mu = 1$ hence $u_i \in G_2$. This shows that $G_1 \subseteq G_2$. The same argument holds inverting the roles of G_1 and G_2, so we get $G_2 = G_1$. □

1.3.1 Squarefree ideals and polarization process

A monomial $\mathbf{x}^{\mathbf{a}}$ is said to be **squarefree** if $a_1, \ldots, a_c \in \{0, 1\}$. A monomial ideal I is said to be **squarefree monomial ideal** if it is generated by squarefree monomials.

An immediate property occurs for squarefree monomial ideals.

Proposition 1.3.3. *Let $I \subseteq R$ be a squarefree monomial ideal and let M be a squarefree monomial in R. Then if F is a monomial such that $F \in (M)$ and $F \in I : (M)$ then $F \in I$.*

Proof. By hypothesis $F = MG$, for some monomial G. Moreover, $MF = M^2G \in I$, so, since I is a squarefree monomial ideal, we have $MG \in I$. □

In the monomial case prime ideals are easy to describe. Indeed, a monomial ideal is a prime ideal only if it is of the form $(x_{i_1}, \ldots, x_{i_k})$.

Squarefree monomial ideals admit a special primary decomposition, as show the next proposition.

Proposition 1.3.4 (1.3.4. [HH]). *A squarefree monomial ideal is intersection of monomial prime ideals.*

More precisely, denoted by $\mathrm{Min}(I)$ the set of all the minimal prime ideals containing I, we have:

Proposition 1.3.5 (1.3.6.[HH]). *Let $I \subseteq R$ be a squarefree monomial ideal, then*

$$I = \bigcap_{\mathfrak{p} \in \mathrm{Min}(I)} \mathfrak{p}.$$

If we are interested in homological properties of a monomial ideal, we can assume that it is squarefree without loss of generality, this is allowed by the so-called *polarization process*. Polarization is a deformation that assigns to an arbitrary monomial ideal a squarefree monomial ideal in a new set of variables. The polarization process is based on the following theorem.

Theorem 1.3.6 (1.6.1.[HH]). *Let $I \subseteq R = k[x_1, \ldots, x_c]$ be a monomial ideal and $A = R/I$ be the monomial algebra associated. Then, there exists a squarefree monomial ideal J in the polynomial ring $R' := k[x_1, \ldots, x_{c'}]$, where $c' \geq c$ and a regular sequence of homogeneous elements of degree one, say $\{\ell_1, \ldots, \ell_{c'-c}\}$, such that, denoted by $B = R'/J$, we have*

$$A \cong B/(\ell_1, \ldots, \ell_{c'-c}).$$

In the process described above, $J \subseteq R'$ is called a polarization of $I \subseteq R$. By polarization process, many questions concerning monomial ideals can be reduced to squarefree monomial ideals.

Corollary 1.3.7 (1.6.3.[HH])**.** *Let $I \subseteq R$ be a monomial ideal and $J \subseteq R'$ one of its polarizations. Then*

1. $\operatorname{ht}(I) = \operatorname{ht}(J)$;
2. $\text{proj-dim}\, R/I = \text{proj-dim}\, R'/J$;
3. *R/I is Cohen-Macaulay iff R'/J is Cohen-Macaulay;*
4. *R/I is Gorenstein iff R'/J is Gorenstein;*
5. *$\beta_{ij}(R/I) = \beta_{ij}(R'/J)$ for all i and j.*

1.4 Simplicial Complexes

The aim of this section is to recall some basic definitions about the simplicial complexes and describe combinatoric properties of the squarefree ideals which arise from a simplicial complex. See [HH], for a more detailed exposition of these issues.

Let $V = \{x_1, \dots, x_c\}$ be a finite set. A **simplicial complex** Δ on V, the **vertex set**, is a set of subsets of V closed under inclusion. Sometimes is also required that $\{x_i\} \in \Delta$ for all i. An element $F \in \Delta$ is called **face** and its dimension is $\dim F := |F| - 1$. The dimension of Δ is the maximum among the dimension of the faces. A zero dimensional face is called **vertex**, a maximal face (under inclusion) is called **facet**.

A simplicial complex Δ is determined by the set of all its facets $\mathcal{F}(\Delta)$, if $\mathcal{F}(\Delta) = \{F_1, \dots, F_r\}$, we will write

$$\Delta = \langle F_1, \dots, F_r \rangle .$$

F is called **nonface** of Δ if $F \subseteq V$ and $F \notin \Delta$, we denote by $\mathcal{N}(\Delta)$ the set of all minimal nonfaces in Δ.

1.4.1 Stanley-Reisner ideal and the Alexander Dual

Let $S := k[V] = k[x_1, \dots, x_c]$ be the polynomial ring, on a field of characteristic zero, with the standard gradation. For each $F \subseteq V$ we set

$$\mathbf{x}_F = \prod_{x_i \in F} x_i$$

and

$$\mathfrak{p}_F = (x_i | x_i \in F).$$

Let Δ be a simplicial complex on V, we can associate to Δ a squarefree monomial ideal in several ways. The **Stanley-Reisner ideal** of Δ is

$$I_\Delta := (\mathbf{x}_F | F \in \mathcal{N}(\Delta)).$$

I_Δ is a squarefree monomial ideal minimally generated by the monomial $\mathbf{x}_F$ with $F \notin \Delta$.

The **facet ideal** of Δ is the ideal generated by the squarefree monomials in correspondence with the facets of Δ

$$I(\Delta) = (\mathbf{x}_F | F \in \mathcal{F}(\Delta)).$$

The **Alexander Dual** of a simplicial complex Δ is defined by

$$\Delta^\vee := \{V \setminus F | F \notin \Delta\}.$$

Since $\Delta^\vee$ is a simplicial complex, see Lemma 1.5.2 in [HH], it is generated by $\{V \setminus F | F \in \mathcal{N}(\Delta)\}$.

For a subset $F \subseteq V$ we set $\overline{F} = V \setminus F$ and for a simplicial complex Δ let

$$\overline{\Delta} = \left\langle \overline{F} | F \in \mathcal{F}(\Delta) \right\rangle.$$

The next lemma connects all these objects just defined.

Lemma 1.4.1 (1.5.3, 1.5.4 [HH])**.**

$$I_{\Delta^\vee} = I(\overline{\Delta}) \qquad \textit{and} \qquad I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} (\mathfrak{p}_{\overline{F}}).$$

The equalities in Lemma 1.4.1 give us a method to compute $\mathcal{G}(I_{\Delta^\vee})$. Let $I_\Delta = (\mathfrak{p}_{F_1}) \cap \ldots \cap (\mathfrak{p}_{F_m})$ the primary decomposition of I_Δ, then

$$\mathcal{G}(I_{\Delta^\vee}) = \{x_{F_1}, \ldots, x_{F_m}\}.$$

1.4.2 A combinatorial approach to Cohen-Macaulay property

For a face $F = \{v_{i_1}, \ldots, v_{i_q}\}$, we define an oriented q-face $[v_{i_1}, \ldots, v_{i_q}]$ as the set of q-tuples obtained by $[v_{i_1}, \ldots, v_{i_q}]$ after an even permutation of the indexes. So, two orderings of indexes are equivalent if they differ with an even permutation. We denote by $C_q(\Delta)$ the free R-module, $R := k[x_1, \ldots, x_c]$, generated by all oriented q-faces modulo $[v_{i_1}, \dots, v_{i_q}] + [v_{i_2}, \dots, v_{i_q}, v_{i_1}]$. We define a k-linear map

$$d_q : C_q(\Delta) \to C_{q-1}(\Delta)$$

by

$$d_q([v_{i_1}, \dots, v_{i_q}]) = \sum_{j=1}^{q} (-1)^{j-1} [v_{i_1}, \dots, \hat{v}_{i_j}, \dots, v_{i_q}]$$

where $\hat{v}_{i_j}$ means that v_{i_j} is erased from $[v_{i_1}, \dots, v_{i_q}]$.

The q-th homology of the complex

$$0 \to C_{\dim(\Delta)}(\Delta) \to \cdots \to C_1(\Delta) \to C_0(\Delta) \to k \to 0,$$

is called the **q-th reduced simplicial homology of** Δ, and denoted by $\tilde{H}_q(\Delta; k) = \ker d_q / \operatorname{Im} d_{q+1}$.

The k-algebra $k[\Delta] := R/I_\Delta$ is called the **Stanley-Reisner ring** of Δ. The Stanley-Reisner ring is a basic tool in the field of combinatorial commutative algebra. Its properties were investigated by Richard Stanley, Melvin Hochster, and Gerald Reisner in the early 1970s. We say that Δ is **Cohen-Macaulay over k** if $k[\Delta]$ is Cohen-Macaulay. In his thesis, in 1974, Gerald Reisner gave a complete characterization of Cohen-Macaulay complexes. This was soon followed up by more precise homological results about face rings due to Melvin Hochster. Stanley had the idea to prove several conjectures in combinatorial algebra by translating them into problems of commutative algebra, so as to use homological techniques. This was the origin of the rapidly developing of combinatorial commutative algebra.

If Δ is Cohen-Macaulay, all the minimal prime ideals of I_Δ have the same height. Therefore the facets of Δ, that by Lemma 1.4.1 correspond to the minimal prime ideals of I_Δ, have the same dimension (we say Δ is **pure**).

The next results due to Hochster and Reisner describe homological properties of the Stanley-Reisner ideal. We need to introduce some notation.

Let Δ be a simplicial complex on V. For a face $F \in \Delta$ we call **link** of F in Δ the complex

$$\operatorname{link}_\Delta F := \{G \in \Delta | F \cup G \in \Delta, F \cap G = \emptyset\}.$$

Theorem 1.4.2. *([Ho]) Let Δ be a simplicial complex on $V = \{x_1, \dots, x_c\}$, then the graded Betti numbers of Δ, $\beta_{ij}(I_\Delta)$, can be computed by the formula*

$$\beta_{ij}(I_\Delta) = \sum_{F \in \Delta^\vee,\ |F| = c-j} \dim_k \tilde{H}_{i-1}(\operatorname{link}_{\Delta^\vee} F; k).$$

A criterion for the Cohen-Macaulay property of the Stanley-Reisner ring is due to Reisner, see [Re].

Theorem 1.4.3 (Reisner)**.** *A simplicial complex* Δ *is Cohen-Macaulay over* k *iff, for all* $F \in \Delta$*, including the empty set, we have*

$$\tilde{H}_i(\operatorname{link}_\Delta F; k) = 0 \quad \textit{for all} \quad i < \dim \operatorname{link}_\Delta F.$$

The next corollary of Reisner's Theorem will be useful inthe next chapter.

Corollary 1.4.4. *Let* Δ *be a Cohen-Macaulay simplicial complex and* $F \in \Delta$*. Then* $\operatorname{link}_\Delta F$ *is Cohen-Macaulay.*

Proof. Let $G \in \operatorname{link}_\Delta F$ we have

$$\operatorname{link}_{\operatorname{link}_\Delta F} G = \operatorname{link}_\Delta(F \cup G).$$

So the proof follows by Reisner's Theorem. □

Chapter 2

Characterization of height 2 Cohen-Macaulay squarefree monomial ideals

In this chapter we study CM squarefree monomial ideals of height 2. In Section 2.1 we describe an exact sequence that will be an important tool in the study of the Cohen-Macaulyness. Moreover we recall the Hilbert-Burch Theorem which characterizes the structure of the Cohen-Macaulay ideals in codimension 2. Lastly, we introduce some basic notation of this chapter and we prove some preliminary results. In Section 2.3 and Section 2.4, we find two characterizations for the Cohen-Macaulyness. In Section 2.5, using the results of the previous sections, we describe the special configuration, introduced and studied in [FRZ2], that the minimal primes of a CM squarefree monomial ideals of height 2 assume.

2.1 Preliminary results

An important tool in the study of the CM property comes from an exact sequence that we will refer as Mayer-Vietoris exact sequence. For a more detailed study of the Mayer-Vietoris sequence see for instance Chapter 3, Section 25 in [Mu].

2.1.1 Mayer-Vietoris exact sequence

Let R be a ring and let $M, N \subseteq P$ be R-modules. Then we have the following short exact sequence of modules, where the maps are the obvious ones,

$$0 \to M \cap N \longrightarrow M \oplus N \longrightarrow M + N \to 0.$$

In the next section, this sequence will be called **Mayer-Vietoris sequence**.

As a consequence of the exactness, if M, N and $M \cap N$ are Cohen-Macaulay modules of projective dimension c then proj-dim$(M+N) \leq c+1$. This follows because the mapping cone resolution of $M+N$ is a free resolution (not necessarily minimal) of $M+N$ of length $c+1$.

On the other hand if M, N are Cohen-Macaulay modules of projective dimension c and $M+N$ is a Cohen-Macaulay module of projective dimension $c+1$ then also $M \cap N$ is Cohen-Macaulay.

This directly follows by the construction of the mapping cone resolution. Namely, let

$$\mathbb{G}_\bullet : 0 \to G_c \to \cdots \to G_1 \to M \oplus N \to 0$$

a minimal free resolution of $M \oplus N$. If $M \cap N$ is not Cohen-Macaulay and

$$\mathbb{F}_\bullet : \cdots \to F_{c+1} \to F_c \to \cdots \to F_1 \to M \cap N \to 0$$

is an its minimal free resolution, then the mapping cone resolution of $M+N$ is

$$\mathbb{H}_\bullet : \cdots \to F_{c+1} \to F_c \to F_{c-1} \oplus G_c \to \cdots \to F_1 \oplus G_2 \to G_1 \to M+N \to 0$$

that contradicts the Cohen-Macaulay property on $M+N$ since, from the minimality of $\mathbb{F}_\bullet$, in the resolution $\mathbb{H}_\bullet$ no cancellation is allowed between F_{c+1} and F_c.

We now recall a basic theorem for Cohen-Macaulay ideals of codimension 2, the Hilbert-Burch Theorem.

2.1.2 The Hilbert-Burch Theorem

The Hilbert-Burch Theorem shows that ideals of a local ring with a minimal free resolution of length 1 are *determinantal ideals*.

David Hilbert in 1890, see [Hi], proved a version of this theorem for polynomial ring, furthermore, Lindsay Burch in 1968, see [Bu], proved it in a more general case. Here we give the statement as given by Eisenbud in [Ei2], see Theorem 3.2.

Let U be a $\ell \times m$ matrix over a local ring R and let $0 < m \leq \ell$. For $t = 1, \ldots, m$ we denote by $I_t(U)$ the ideal generated by the determinants of $t \times t$ submatrices. Moreover, we set $I_t(U) = R$ for $t \leq 0$ and $I_t(U) = 0$ for $t > m$.

If $\varphi : F \to G$ is a homomorphism of finite free R-modules, then φ is given by a matrix U with respect to bases of F and G. Although U depends on the choice of the base, the ideal $I_t(U)$ only depends on φ. Therefore we may set $I_t(\varphi) = I_t(U)$.

Let R be a Noetherian ring. A non-zero finite R-module M is **perfect module** if $\text{proj-dim}\, M = \text{grade}\, M$. An ideal I is called **perfect ideal** if R/I is a perfect module.

Theorem 2.1.1 (Hilbert-Burch)**.** *Let R be a Noetherian ring, and let I be an ideal with a free resolution of length 1*

$$\mathbb{F}: \; 0 \to R^m \xrightarrow{\varphi} R^{m+1} \to I \to 0.$$

Then there exists a R-regular element a such that $I = aI_m(\varphi)$. If I is projective, then $I = (a)$, and if $\text{proj-dim}\, I = 1$ *then $I_m(\varphi)$ is perfect of grade 2. Conversely, if $\varphi : R^n \to R^{n+1}$ is a R-linear map with grade $I_n(\varphi) \geq 2$, then $I = I_n(\varphi)$ has free resolution $\mathbb{F}$.*

2.2 Cohen-Macaulay squarefree monomial ideals

In order to give a combinatorial characterization of squarefree monomial ideals of height 2, in this section we introduce terminology and some technical facts.

Let $\mathcal{N} := \{x_1, \ldots, x_N\}$ be a finite set, we consider the set of all subsets of cardinality 2 of $\mathcal{N}$, i.e.

$$C_{2,\mathcal{N}} := \{\{x_a, x_b\} | x_a, x_b \in \mathcal{N}, x_a \neq x_b\}.$$

Given a set $S \subseteq C_{2,\mathcal{N}}$, we associate to S an ideal I_S of the standard polynomial ring $k[\mathcal{N}] := k[x_1, \ldots, x_N]$. We define I_S as the intersection of all the prime ideals generated by the element in S, i.e.

$$I_S := \bigcap_{s \in S} \mathfrak{p}_s,$$

where, if $s = \{x_a, x_b\}$, $\mathfrak{p}_s = \mathfrak{p}_{\{x_a, x_b\}}$ is the prime ideal $(x_a, x_b) \subseteq k[\mathcal{N}]$.

If $S = \emptyset$ we assume for convention that $I_S = (0)$.

The ideal I_S is a squarefree monomial ideal of $k[\mathcal{N}]$. The aim of this chapter is to look under which conditions on S we have I_S Cohen-Macaulay. For this reason, we say that a set $S \subseteq C_{2,\mathcal{N}}$ is **Cohen-Macaulay**, for short **CM**, iff I_S is CM.

Remark 2.2.1. Note that I_S is the Stanley-Reisner ideal of the simplicial complex

$$\langle \mathcal{N} \setminus s | s \in S \rangle .$$

We start with a very useful lemma.

Lemma 2.2.2. *If I_S is CM then $(I_S : x_i)$ is CM, for any $x_i \in \mathcal{N}$.*

Proof. It follow by Corollary 1.4.4 just using Remark 2.2.1. □

For a given $x_a \in \mathcal{N}$, we will use the following notation

$$S(x_a) := \{\{x_i, x_j\} \in S | x_a \notin \{x_i, x_j\}\} \subseteq C_{2,\mathcal{N}}.$$

$S(x_a)$ is the set of the elements in S which not contain x_a. The following lemma makes clear the link between $I_{S(x_a)}$ and I_S.

Lemma 2.2.3. *Let $x_a \in \mathcal{N}$ then*

$$(I_S : x_a) = I_{S(x_a)}.$$

Proof. Let F be a monomial and suppose $x_a F \in I_S$. Take $\{x_i, x_j\} \in S(x_a)$ then $x_a F \in (x_i, x_j)$. Since $x_a \notin \{x_i, x_j\}$, we have $F \in (x_i, x_j)$, hence $F \in I_{S(x_a)}$. On the other hand, let $F \in I_{S(x_a)}$ be a monomial, it is trivial to observe that $x_a F \in (x_i, x_j)$ for all $\{x_i, x_j\} \in S$. □

Corollary 2.2.4. *If $S \subseteq C_{2,\mathcal{N}}$ is CM then $S(x_a)$ is CM, for any $x_a \in \mathcal{N}$.*

Proof. It follows from Lemmas 2.2.2 and 2.2.3. □

Definition 2.2.5. Let $S \subseteq C_{2,\mathcal{N}}$ and $x_a \in \mathcal{N}$ we define the set

$$E_S(x_a) := \{\{x_i, x_j\} \in S | \{x_i, x_a\} \notin S \text{ and } \{x_j, x_a\} \notin S\}.$$

We say that x_a is ***S*-covered**, or **covered in *S***, by $W \subseteq \mathcal{N}$ iff for any $\{x_i, x_j\} \in E_S(x_a)$ we have $\{x_i, x_j\} \cap W \neq \emptyset$. Let $\Delta \subseteq \mathcal{N}$ we say that Δ is **self-covered in *S*** iff we can write Δ as

$$\Delta = \{x_{a_1}, \ldots, x_{a_n}\}$$

and x_{a_h} is S-covered by $\{x_{a_{h+1}}, \ldots, x_{a_n}\}$ for any h.

From now on, when we say that a set $\Delta = \{x_{a_1}, \ldots, x_{a_n}\}$ is self-covered, we will mean that the x_{a_h}'s are in the right order. So, by definition, for all $\{x_i, x_j\} \in E_S(x_{a_h})$ we have x_i or x_j belongs to $\{x_{a_{h+1}}, \ldots, x_{a_n}\}$ for any h.

Remark 2.2.6. Let $\Delta = \{x_{a_1}, \ldots, x_{a_n}\}$ be self-covered in S then, since x_{a_n} is S-covered by the empty set, we have $E_S(x_{a_n}) = \emptyset$.

Proposition 2.2.7. *Let $\Delta = \{x_{a_1}, \ldots, x_{a_n}\}$ be self-covered in S then the set $\{x_{a_1}, \ldots, x_{a_{n-1}}\}$ is self-covered in $S(x_{a_n})$. Therefore $E_{S(x_{a_n})}(x_{a_{n-1}}) = \emptyset$.*

Proof. If $E_{S(x_{a_n})}(x_{a_i}) = \emptyset$, x_{a_i} is trivially covered in $S(x_{a_n})$ by the set $\{x_{a_{i+1}}, \ldots, x_{a_{n-1}}\}$. Let $\{x_a, x_b\} \in E_{S(x_{a_n})}(x_{a_i})$ thus in particular $\{x_a, x_b\} \in E_S(x_{a_i})$ so $\{x_a, x_b\} \cap \{x_{a_{i+1}}, \ldots, x_{a_n}\} \neq \emptyset$. Since $\{x_a, x_b\} \in S(x_{a_n})$ we get $\{x_a, x_b\} \cap \{x_{a_{i+1}}, \ldots, x_{a_{n-1}}\} \neq \emptyset$. □

Let $W \subseteq \mathcal{N}$ we denote by

$$P_W := \prod_{x_i \in W} x_i \in k[\mathcal{N}]$$

the monomial given by the product of the variables in correspondence with elements of W. If $W = \emptyset$ for convenience we set $P_W := 0$.

2.3 A first characterization of CM: looking at $\mathrm{Min}(I_S)$

Proposition 2.3.1. *Let $S \subseteq C_{2,\mathcal{N}}$ and $\Delta \subseteq \mathcal{N}$. If Δ is self-covered in S then $I_S + (P_\Delta)$ is equidimensional.*

Proof. Let $x_a \in \Delta$ and $\{x_r, x_s\} \in S$ then $I_S + (P_\Delta) \subseteq (x_a, x_r, x_s)$. If $\{x_a, x_r\} \notin S$ and $\{x_a, x_s\} \notin S$ then $\{x_r, x_s\} \in E_S(x_a)$. Since Δ is self-covered in S, we can assume $x_r \in \Delta$ so $I_S + (P_\Delta) \subseteq (x_r, x_s)$, and we are done. □

Corollary 2.3.2. *Let $x_a \in \mathcal{N}$ such that $E_S(x_a) = \emptyset$, then x_a divides all the generators of I_S except one i.e. $I_S + (x_a) = (F, x_a)$.*

Proof. By Proposition 2.3.1, since the set $\{x_a\}$ is self-covered in S, $I_S + (x_a)$ is equidimensional. Thus, any minimal prime $\mathfrak{p} \in \mathrm{Min}(I_S + (x_a))$ is of the type $\mathfrak{p} = (x_a, y)$, for some $y \in \mathcal{N}$. Let $F \in \mathcal{G}(I_S + (x_a))$ be a minimal monomial generator of $I_S + (x_a)$ such that $F \notin (x_a)$ then

$$F = \prod_{\{y:(x_a,y) \in \mathrm{Min}(I_S + (x_a))\}} y.$$

Hence $I_S + (x_a) = (F, x_a)$. □

Given $x_a \in \mathcal{N}$ we set

$$S_{x_a} := \{x_i \in \mathcal{N} | \{x_a, x_i\} \in S\} \subseteq \mathcal{N}.$$

This set provides an useful decomposition of I_S as the next lemma shows.

Lemma 2.3.3. *Let $S \subseteq C_{2,\mathcal{N}}$ and let $x_a \in \mathcal{N}$ be such that $S_{x_a} \neq \emptyset$, then*

$$I_S = (I_S : x_a) \cap (x_a, P_{S_{x_a}}).$$

Proof. From $S = S(x_a) \cup \{\{x_a, x_i\} \in S\}$ and $S_{x_a} \neq \emptyset$, we easily have

$$I_S = I_{S(x_a)} \cap \bigcap_{x_i \in S_{x_a}} (x_a, x_i),$$

so it follows by Lemma 2.2.3. □

Proposition 2.3.4. *Let $S \subseteq C_{2,\mathcal{N}}$ and let $y \in \mathcal{N}$. If $I_{S(y)} + (P_{S_y})$ is a height 2 CM ideal then I_S is a height 2 CM ideal.*

Proof. Denoted by $W := S_y$, by Lemma 2.3.3 we get the following Mayer-Vietoris exact sequence

$$0 \to I_S \to I_{S(y)} \oplus (y, P_W) \to I_S + (y, P_W) \to 0.$$

Since y is a regular element in $k[\mathcal{N}]/(I_{S(y)} + (P_W))$ and since, by hypothesis, $I_{S(y)} + (P_W)$ is a height 2 CM ideal, $I_{S(y)} + (y, P_W)$ is a height 3 CM ideal. Moreover, observe that (y, P_W) is a height 2 CM ideal, using the mapping cone argument, as showed in Section 2.1.1, we get I_S is a height 2 CM ideal. □

Corollary 2.3.5. *Let $S \subseteq C_{2,\mathcal{N}}$ and $y \in \mathcal{N}$. Let $I_{S(y)}$ be a CM ideal and $P_{S_y} \in I_S$ then I_S is a height 2 CM.*

Proof. It follows just using Proposition 2.3.4. □

The next theorem give a sufficient condition on S to be CM.

Theorem 2.3.6. *Let $S \subseteq C_{2,\mathcal{N}}$ and $y \in \mathcal{N}$. If $S(y)$ is CM and S_y is self-covered in $S(y)$ then S is CM.*

Proof. Let $\Delta := S_y = \{x_{a_1}, \ldots, x_{a_n}\}$ and $S' := S(y)$. By Proposition 2.3.4 it is enough to prove that $I_{S'} + (P_\Delta)$ is a height 2 CM ideal. First observe that

$$I_{S'} + (P_\Delta) = ((I_{S'} : x_{a_n}) + (x_{a_1} \cdots x_{a_{n-1}})) \cap (I_{S'}, x_{a_n}).$$

In fact, if F is a monomial in $I_{S'} + (P_\Delta)$ we have $F \in I_{S'}$ or $F \in (P_\Delta)$ hence $F \in ((I_{S'} : x_{a_n}) + (x_{a_1} \cdots x_{a_{n-1}})) \cap (I_{S'}, x_{a_n})$. On the other hand, let $F \in ((I_{S'} : x_{a_n}) + (x_{a_1} \cdots x_{a_{n-1}})) \cap (I_{S'}, x_{a_n})$, and $F \notin I_{S'}$, we have either $F \in (x_{a_n})$ and $x_{a_n} F \in I_{S'}$, thus, by Proposition 1.3.3, $F \in I_{S'}$, or $F \in (x_{a_n})$ and $F \in (x_{a_1} \cdots x_{a_{n-1}})$ and then $F \in (P_\Delta)$.

Furthermore by Corollary 2.3.2 $(I_{S'}, x_{a_n}) = (f, x_{a_n})$, for some $f \in \mathcal{G}(I_{S'})$, so applying again Proposition 2.3.4 we have that I_S is CM if

$$((I_{S'} : x_{a_n}) + (x_{a_1} \cdots x_{a_{n-1}})) + (f) = (I_{S'} : x_{a_n}) + (x_{a_1} \cdots x_{a_{n-1}})$$

is a height 2 CM ideal. Now, by Lemma 2.2.3 we have $(I_{S'} : x_{a_n}) = I_{S'(x_{a_n})}$ and by Proposition 2.2.7 $\{x_{a_1}, \dots, x_{a_{n-1}}\}$ is self-covered in $S'(x_{a_n})$, so we can repeat the same argument as above. In this way we reduce the question to prove that

$$(((I_{S'} : x_{a_n}) : x_{a_{n-1}}) : \cdots : x_{a_1})$$

is CM, but this follows by Lemma 2.2.2 since by hypothesis $S' = S(y)$ is CM. □

In order to prove the vice versa of the Theorem 2.3.6 we need the following crucial lemma. We skip the proof here. We will prove it in Section 2.5.1 on page 39.

Lemma 2.3.7. *Let $S \subseteq C_{2,\mathcal{N}}$ be CM and let $\Delta \subseteq \mathcal{N}$. If $I_S + (P_\Delta)$ is CM then there exists $x_a \in \Delta$ such that $E_S(x_a) = \emptyset$ i.e. $I_S + (x_a) = (f, x_a)$ for some $f \in I_S$.*

This lemma allows us to prove the next result.

Theorem 2.3.8. *Let $S \subseteq C_{2,\mathcal{N}}$ be CM and let $\Delta \subseteq \mathcal{N}$. If $I_S + (P_\Delta)$ is CM then Δ is self-covered in S.*

Proof. If $\Delta = \emptyset$ the statement is trivial, so we assume $|\Delta| = n > 0$. Since $I_S + (P_\Delta)$ is CM, by Lemma 2.3.7 there exists $x_{a_n} \in \Delta$, such that $E_S(x_a) = \emptyset$, and so $I_S + (x_{a_n}) = (f, x_{a_n})$. Now we observe that

$$\big(I_S + (P_\Delta)\big) : (x_{a_n}) = (I_S : x_{a_n}) + (P_{\Delta \setminus \{x_{a_n}\}})$$

that is CM by Lemma 2.2.2. Therefore if $n = 1$ we are done, otherwise, by Lemma 2.3.7, there exists $x_{a_{n-1}} \in \Delta \setminus \{x_{a_n}\}$ such that $E_{S_{a_n}}(x_{a_{n-1}}) = \emptyset$ i.e. $E_S(x_{a_{n-1}})$ is S-covered by $\{x_{a_n}\}$. By repeating this argument the statement of the theorem follows. □

Now we are in position to prove the following theorem.

Theorem 2.3.9. *Let $S \subseteq C_{2,\mathcal{N}}$ be CM and $y \in \mathcal{N}$, then S_y is self-covered in $S(y)$.*

Proof. If $S(y) = \emptyset$ then for all $x_a \in S_y$ we have $E_S(x_a) = \emptyset$. Otherwise the statement follows from Theorem 2.3.8 because, as showed in Section 2.1.1, we have $\text{proj-dim}(I_{S(y)} + (y, P_{S_y})) \leq 3$ hence $\text{proj-dim}(I_{S(y)} + (P_{S_y})) \leq 2$, and so

$$2 \leq \text{ht}(I_{S(y)} + (P_{S_y})) \leq \text{proj-dim}(I_{S(y)} + (P_{S_y})) \leq 2.$$

□

Collecting the previous results we get the main theorem of this section. We give a characterization of the CM squarefree monomial ideals of height two just looking at the minimal primes in their primary decomposition.

Theorem 2.3.10. *Let $S \subseteq C_{2,\mathcal{N}}$, then the following are equivalent:*

1. *S is Cohen-Macaulay;*
2. *for any $x_a \in \mathcal{N}$, $S(x_a)$ is Cohen-Macaulay and S_{x_a} is self-covered in $S(x_a)$;*
3. *there exists $x_a \in \mathcal{N}$ such that $S(x_a)$ is Cohen-Macaulay and S_{x_a} is self-covered in $S(x_a)$.*

Proof. If S is Cohen-Macaulay then, by Lemma 2.2.2, for any $x_a \in \mathcal{N}$, $S(x_a)$ is Cohen-Macaulay and, by Theorem 2.3.9, S_{x_a} is self-covered in S. Now, let suppose there exists $x_a \in \mathcal{N}$ such that $S(x_a)$ is CM and S_{x_a} is self-covered in S, then by Theorem 2.3.6, S is CM. □

Example 2.3.11. Let consider the set

$$S = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\}.$$

$S(x_3) = \{\{x_1, x_2\}\}$ is trivially Cohen-Macaulay. Moreover, note that $S_{x_3} = \{x_2, x_4\}$ is self-covered in $S(x_3) = \{\{x_1, x_2\}\}$, so, by Theorem 2.3.10, S is Cohen-Macaulay.

Example 2.3.12. Let

$$S = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}\}.$$

We have that $S(x_5) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\}$ is CM by example 2.3.11. Take now $S_{x_5} = \{x_1, x_4\}$, since $E_{S(x_5)}(x_1)$ and $E_{S(x_5)}(x_4)$ are non empty sets we conclude that S_{x_5} is not self-covered in $S(x_5)$, and hence S is not CM.

Example 2.3.13. Let

$$S = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}, \{x_5, x_3\}\}.$$

We have that $S(x_5) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\}$ is CM by example 2.3.11. Now we get $S_{x_5} = \{x_1, x_3, x_4\}$, and so $E_{S(x_5)}(x_1) = \{\{x_3, x_4\}\}$, $E_{S(x_5)}(x_3) = \emptyset$, and $E_{S(x_5)}(x_4) = \{\{x_1, x_2\}\}$. It is easy to check that $S_{x_5} = \{x_4, x_1, x_3\}$ is self-covered in $S(x_5)$. By Theorem 2.3.10, S is Cohen-Macaulay.

2.4 A second characterization: looking outside $\mathrm{Min}(I_S)$

In this section we characterize the Cohen-Macaulay squarefree monomial ideals of height two giving a condition on the set of the minimal prime ideals of height 2 not in $\mathrm{Min}(I_S)$. In other words, given $S \subseteq C_{2,\mathcal{N}}$, we relate the CM property to a condition on $\overline{S} := C_{2,\mathcal{N}} \setminus S$.

Remark 2.4.1. Note that, given $S \subseteq C_{2,\mathcal{N}}$, the minimal monomial generators of I_S are strongly linked with $\overline{S}$, since it is to check

$$\overline{S} = \{\{x_a, x_b\} \in C_{2,\mathcal{N}} | F \notin (x_a, x_b), \text{ for some } \ F \in \mathcal{G}(I_S)\}.$$

Definition 2.4.2. Let $V \subseteq C_{2,\mathcal{N}}$, we say that V contains a ***r*-cycle** if there exists $W \subseteq V$ of the type

$$W = \{\{x_{a_1}, x_{a_2}\}, \{x_{a_2}, x_{a_3}\}, \dots, \{x_{a_{r-1}}, x_{a_r}\}, \{x_{a_r}, x_{a_1}\}\},$$

and W does not contain properly a s-cycle. We say that a r-cycle W is **minimal** in V if for any $v \in V \setminus W$ we have

$$v \not\subseteq \{x_{a_1}, x_{a_2}, x_{a_3}, \dots, x_{a_{r-1}}, x_{a_r}\}.$$

The following theorem give a necessary condition for a squarefree monomial ideal of height two to be CM.

Theorem 2.4.3. *Let $S \subseteq C_{2,\mathcal{N}}$ be Cohen-Macaulay then $\overline{S}$ contains no minimal r-cycle for any $r \geq 4$.*

Proof. Let $V := \{\{x_{a_1}, x_{a_2}\}, \{x_{a_2}, x_{a_3}\}, \dots, \{x_{a_{r-1}}, x_{a_r}\}, \{x_{a_r}, x_{a_1}\}\}$ be a minimal r-cycle contained in $\overline{S}$, $r \geq 4$. After applying recursively Lemma 2.2.2, we get that $S' := S \cap C_{2,\{x_{a_1}, x_{a_2}, \dots, x_{a_r}\}}$ is Cohen-Macaulay and

$$\overline{S'} := C_{2,\{x_{a_1}, x_{a_2}, \dots, x_{a_r}\}} \setminus S' = V.$$

Thus, by Theorem 2.3.10, $S'_{x_{a_1}} := \{x_{a_3}, \dots, x_{a_{r-1}}\}$ is self-covered in $S'(x_{a_1})$. But, since $r \geq 4$, for any $x_{a_i} \in S'_{x_{a_1}}$ we have

$$\{x_{a_{i-1}}, x_{a_{i+1}}\} \in E_{S'(x_{a_1})}(x_{a_i}).$$

Then, since the set $E_{S'(x_{a_1})}(x_b)$ is not empty, for any $x_b \in S'_{x_{a_1}}$, by remark 2.2.6 we get a contradiction. □

The next goal of this section is to prove the vice versa of the above theorem. We start with a remark.

Remark 2.4.4. Let $S \subseteq C_{2,\mathcal{N}}$ and $y \in \mathcal{N}$. If $C_{2,\mathcal{N}\setminus\{y\}} \setminus S(y)$ contains a minimal r-cycle, then also $\overline{S}$ contains a minimal r-cycle. This follows by $S = S(y) \cup \{\{y, x_a\} \in S\}$, because

$$\overline{S} = (C_{2,\mathcal{N}} \setminus S(y)) \cap (C_{2,\mathcal{N}} \setminus \{\{y, x_a\} \in S\}) \supset C_{2,\mathcal{N}\setminus\{y\}} \setminus S(y).$$

Let $S \subseteq C_{2,\mathcal{N}}$ be not CM and $y \in \mathcal{N}$, if $S(y)$ is not CM, in order to prove that $\overline{S}$ contains a minimal r-cycle, for some $r \geq 4$, Remark 2.4.4 allows us to prove the assertion for $S(y) \cap C_{2,\mathcal{N}\setminus\{y\}}$. Thus, repeating this argument, after renaming, we can suppose that $S \subseteq C_{2,\mathcal{N}}$ is not CM and, for any $x_i \in \mathcal{N}$, $S(x_i)$ is CM.

In order to define a "minimal" covering we recall the Definition 2.2.5. We say that x_a is covered in S by $W \subseteq \mathcal{N}$ iff for any $\{x_i, x_j\} \in E_S(x_a)$ we have $\{x_i, x_j\} \cap W \neq \emptyset$.

Definition 2.4.5. Let $S \subseteq C_{2,\mathcal{N}}$, we say that $W \subseteq \mathcal{N}$ is a **minimal covering** for x_a iff $E_S(x_a)$ is S-covered by W and $E_S(x_a)$ is not covered by $W \setminus \{x_b\}$, for any $x_b \in W$. Given $x_a \in \mathcal{N}$, we denote by Γ_{x_a} the set of all the minimal coverings for x_a.

Let $x_i \in \mathcal{N}$, for short we denote by $\mathcal{G}(I_S)|_{x_i}$ the set of all monomials divisible by x_i.

Lemma 2.4.6. *Let $S \subseteq C_{2,\mathcal{N}}$ and $x_i \in \mathcal{N}$, then*

$$\mathcal{G}(I_S) = \mathcal{G}(I_S)|_{x_i} \cup \bigcup_{\Delta \in \Gamma_{x_i}} \{P_{S_{x_i}} \cdot P_\Delta\}.$$

Proof. Let $F \in \mathcal{G}(I_S)$ and $F \notin (x_i)$ then $F = P_{S_{x_i}} \cdot P_Q$, for some $Q \subseteq \mathcal{N}$. Therefore, since F is a minimal generator for I_S, Q is a minimal covering for x_i. The other inclusion follows similarly. □

Lemma 2.4.7. *Let $S \subseteq C_{2,\mathcal{N}}$ be not CM and let $E_S(x_a) = \emptyset$. Then $S(x_a)$ is not CM.*

Proof. By Lemma 2.3.3, we have $I_S = (I_S : x_a) \cap (x_a, P_{S_{x_a}})$. Since $E_S(x_a) = \emptyset$, by Lemma 2.4.6, we get $P_{S_{x_a}} \in \mathcal{G}(I_S)$. If $(I_S : x_a)$ is CM, we get a contradiction since, by Lemma 2.3.5, I_S is not CM. □

Lemma 2.4.8. *Let $S \subseteq C_{2,\mathcal{N}}$ be not CM and let $S(y)$ be CM for any $y \in \mathcal{N}$. Then for any $x_a, x_b \in \mathcal{N}$, with $x_a \neq x_b$, we have $S_{x_a} \neq S_{x_b}$.*

Proof. If $S_{x_a} = S_{x_b}$ for $x_a \neq x_b$, then $P_{S_{x_a}} = P_{S_{x_b}}$ and $\Gamma_{x_a} = \Gamma_{x_b}$. Therefore $\mathcal{G}(I_S)|x_a = \mathcal{G}(I_S)|x_b$, so $F \in (x_b)$ if and only if $F \in (x_a)$. Hence, by Lemma 2.4.6, I_S and $I_{S(x_a)}$ have the same number of minimal generators and, one can easily checks, also the same number of minimal first syzygies. But I_S is CM and $I_S(x_a)$ is not CM. □

Theorem 2.4.9. *If $S \subseteq C_{2,\mathcal{N}}$ is not Cohen-Macaulay then $\overline{S}$ contains a minimal r-cycle, for some $r \geq 4$.*

Proof. By Remark 2.4.4 we can suppose $I_S : x_i$ to be CM for each $x_i \in \mathcal{N}$. By Lemma 2.4.7 $E_S(x_i) \neq \emptyset$ for any $x_i \in \mathcal{N}$. Let $\{x_1, x_3\} \in E_S(x_2)$, i.e. $\{x_1, x_2\}, \{x_2, x_3\} \in \overline{S}$ and $\{x_1, x_3\} \in S$. Now we show that there exists $x_4 \in \mathcal{N}$ such that $\{x_2, x_4\} \in E_S(x_3)$, certainly such an element will be different from x_1. By Lemma 2.4.6 we have

$$\mathcal{G}(I_S) = \mathcal{G}(I_S)|_{x_2} \cup \bigcup_{\Delta \in \Gamma_{x_2}} \{P_{S_{x_2}} \cdot P_\Delta\} =$$

$$= \mathcal{G}(I_S)|_{x_3} \cup \bigcup_{\Delta \in \Gamma_{x_3}} \{P_{S_{x_3}} \cdot P_\Delta\}.$$

By Lemma 2.4.8 we can take $G \in \mathcal{G}(I_S)$ such that $G \in (x_2)$ and $G \notin (x_3)$. Then, there exists $\Delta_3 \in \Gamma_{x_3}$ such that $G = \prod_{x_i \in \Delta_3} x_i \cdot P_{S_{x_3}}$. Since $P_{S_{x_3}} \notin (x_2)$ then we get $x_2 \in \Delta_3$, i.e. there is $x_4 \in \mathcal{N}$ such that $\{x_2, x_4\} \in E_S(x_3)$. Repeating this argument on $E_S(x_4)$ we find $x_a \in \mathcal{N}$ such that $\{x_3, x_a\} \in E_S(x_4)$, if $x_a = x_1$ we are done, otherwise we go on until we have $\{x_1, \ldots, x_r\}$, for some $r > 3$, such that

$$\{x_1, x_2\}, \{x_2, x_3\} \; \ldots, \{x_{r-1}, x_r\}, \{x_1, x_r\} \in \overline{S},$$

and $\{x_i, x_{i+2}\} \notin \overline{S}$ (the indexes are supposed to be "mod r"). Therefore there is at least a minimal 4-cycle in $\overline{S}$. □

Corollary 2.4.10. *Let $S \subseteq C_{2,\mathcal{N}}$ be a not CM ideal such that $S(x_i)$ is CM for any x_i then $\overline{S}$ is a minimal r-cycle.*

Proof. By Theorem 2.4.9 $\overline{S}$ contains a minimal r-cycle but not $S(x_i)$, for each x_i. So each x_i appears in that cycle. □

The following theorem resumes the main result of this section.

Theorem 2.4.11. *S is Cohen-Macaulay if and only if $\overline{S}$ does not contain minimal r-cycles, for any $r \geq 4$.*

Proof. The sufficient part is the Theorem 2.4.3. The necessary part follows by Lemma 2.4.4 and Theorem 2.4.9. □

Definition 2.4.12. We say S to be **connected** if for any $v_1, v_2 \in S$ there exists $w \in S$ such that $v_1 \cap w \neq \emptyset$ and $v_2 \cap w \neq \emptyset$. In this case we say that v_1 and v_2 are **connected**.

Remark 2.4.13. Note that if S is not connected then S fails to be Cohen-Macaulay. In fact let S be not connected and suppose $\{x_1, x_2\}, \{x_3, x_4\} \in S$ and

$$\{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\} \notin S.$$

Therefore $\overline{S}$ contains a minimal 4-cycle and, by Theorem 2.4.3, S is not Cohen-Macaulay.

Example 2.4.14. If S contains a minimal 5-cycle then S fails to be Cohen-Macaulay. Let

$$\{x_1, x_2\}, \{x_2, x_3\}\ \{x_3, x_4\}\ , \{x_4, x_5\}, \{x_5, x_1\}$$

be a minimal 5-cycle in S, then

$$\{x_1, x_3\}, \{x_3, x_5\}\ \{x_5, x_2\}, \{x_2, x_4\}, \{x_4, x_1\}$$

is a minimal 5-cycle in $\overline{S}$.

Example 2.4.15. If $|\mathcal{G}(I_S)| \leq 3$ then I_S is CM. Let $\mathcal{G}(I_S) = \{F_1, F_2, F_3\}$ and, for $i = 1, 2, 3$, let $W_i := \{x_j \in \mathcal{N} | F_i \notin (x_j)\}$. By Remark 2.4.1, we have $\overline{S} = C_{2,W_1} \cup C_{2,W_2} \cup C_{2,W_3}$, so $\overline{S}$ does not contain a minimal r-cycle, $r \geq 4$.

2.5 Some general configurations with the CM property

Many recent papers deal with special configurations of linear subvarieties of projective spaces which raised up to Cohen-Macaulay varieties, for instance partial intersections studied in [RZ6], k-configurations studied in [GHS], star configurations studied in [GHM]. In [FRZ2] the authors, introducing the notion of tower sets, generalize all this configurations in such a way to preserve the Cohen-Macaulayness. In this section we study height two Cohen-Macaulay squarefree monomial ideals which arise from special configurations of their minimal primes.

Theorem 2.3.10 and Theorem 2.4.11 characterize the monomial ideals I_S which are CM looking at $\text{Min}(I_S)$. In this section we will show that, when S is Cohen-Macaulay, the elements of S assume a special configuration. Such a configuration, introduced and investigated in [FRZ2], will be called **g-tower set**.

2.5.1 Tower sets in codimension 2

All the definitions in this section are given for the height 2 case, for a general discussion see Section 1 and 2 in [FRZ2].

Let $\mathcal{N}$ be a finite set and $T \subseteq D_{2,\mathcal{N}} := \mathcal{N} \times \mathcal{N} \setminus \{(a,a) \mid a \in \mathcal{N}\}$, we denote by $\pi_1(T)$ and $\pi_2(T)$ the sets

$$\pi_1(T) := \{i \in \mathcal{N} | (i,j) \in T \text{ for some } j\},$$

and

$$\pi_2(T) := \{j \in \mathcal{N} | (i,j) \in T \text{ for some } i\}.$$

Definition 2.5.1. Let $\mathcal{N}$ be a finite set and $T \subseteq D_{2,\mathcal{N}}$. We say that T is a **tower set** if we can order the elements in $\pi_1(T)$, say $a_1 < \cdots < a_t$, such that if $(a_i, b) \in T$ then $(a_{i+1}, b) \in T$.

Let $T \subseteq D_{2,\mathcal{N}}$, we denote by $T_{(i,\bullet)}$ and $T_{(\bullet,j)}$, where $i, j \in \mathcal{N}$, the sets

$$T_{(i,\bullet)} := \{j \in \mathcal{N} | (i,j) \in T\},$$

$$T_{(\bullet,j)} := \{i \in \mathcal{N} | (i,j) \in T\}.$$

Remark 2.5.2. Note that, by Definition 2.5.1, in a tower set T if $i, j \in \pi_1(T)$ and $i > j$ then $T_{(i,\bullet)} \supseteq T_{(j,\bullet)}$.

The following proposition provides another equivalent version of Definition 2.5.1.

Proposition 2.5.3. *Let $T \subseteq \mathcal{D}_{2,\mathcal{N}}$. T is a tower set iff for all (a,b) and $(\alpha,\beta) \in T$ we have $(a,\beta) \in T$ or $(\alpha,b) \in T$.*

Proof. If T is a tower set the condition follows by definition. Vice versa we must order the elements of $\pi_1(T)$. Given $a, \alpha \in \mathcal{N}$ we compare $T_{(a,\bullet)}$ with $T_{(\alpha,\bullet)}$. If there is an element $b \in T_{(a,\bullet)} \setminus T_{(\alpha,\bullet)}$ then for every $\beta \in T_{(\alpha,\bullet)}$ we have $(a,\beta) \in T$, so $T_{(\alpha,\bullet)} \subseteq T_{(a,\bullet)}$ and then we put $a > \alpha$. (Similarly we get $\alpha > a$.) If $T_{(a,\bullet)} = T_{(\alpha,\bullet)}$ the choice of the greater element is not influent. In this way we have that T is a tower set. $\square$

We will need to make use of some results of the previous sections, so we associate to T a subset of $C_{2,\mathcal{N}}$ defining a natural map

$$\varphi : \mathcal{D}_{2,\mathcal{N}} \to C_{2,\mathcal{N}}$$

such that

$$\varphi\big((a,b)\big) := \{a,b\}.$$

Moreover, given $S \subseteq C_{2,\mathcal{N}}$, we define the set $\mathcal{T}_S := \{T \subseteq \mathcal{D}_{2,\mathcal{N}} \mid \varphi(T) = S, |T| = |S|\}$.

Definition 2.5.4. We say that S is a **towerizable set** iff there exists a tower set $T \in \mathcal{T}_S$.

Proposition 2.5.5. *$C_{2,\mathcal{N}}$ is a towerizable set.*

Proof. Let consider on $\mathcal{N} := \{x_1, \dots, x_N\}$ the order given by the indexes. Let $T := \{(x_a, x_b) | a > b\}$ and $\pi_1(T)$ with the order induced by $\mathcal{N}$. It is easy to check that $T \in \mathcal{T}_S$ and T is a tower set. □

Theorem 2.5.6. *Let S be a towerizable set then S is CM, i.e I_S is CM.*

Proof. By hypothesis there exists a tower set $T \in \mathcal{T}_S$. Let $x_1 < x_2 < \cdots < x_r$ be the order on $\pi_1(T)$, then $S(x_1)$ is a towerizable set and, for any $y \in S_{x_1}$ and $x_j \in \pi_1(T)$, we have $(x_j, y) \in T$, hence $E_S(y) = \emptyset$. The statement can be proved by Theorem 2.3.10 using an inductive argument on $|\pi_1(T)|$. □

The next example shows that there are Cohen-Macaulay sets which are not towerizable.

Example 2.5.7. Let $\mathcal{N} = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and

$$S = \big\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_4, x_6\}, \{x_1, x_4\}, \{x_1, x_6\}\big\}.$$

Then a simple computation shows that

$$I_S = (\ x_1x_3x_6,\ x_1x_4x_5,\ x_1x_4x_6,\ x_2x_4x_6\ )$$

is the determinantal ideal generated by the order 3 minors of the Hilbert-Burch matrix

$$\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & x_3 & x_5 \\ 0 & x_4 & 0 \\ 0 & 0 & x_6 \end{pmatrix},$$

so S is aCM. Let us suppose that S is towerizable. Then there exists a tower set $T \in \mathcal{T}_S$. Of course $|T| = 6$ and there is not a variable x_k such that the ideal (x_k) contains 4 of the 6 minimal primes of I_S. Consequently, $|\pi_2(T)| \leq 3$ and for every $y \in \pi_2(T)$ we have $|T_{(\bullet,y)}| \leq 3$, so only three possibilities could occur

1) $\pi_1(T) = \{a, b\}$ with $|T_{(a,\bullet)}| = 3$ and $|T_{(b,\bullet)}| = 3$;

2) $\pi_1(T) = \{a, b, c\}$ with $|T_{(a,\bullet)}| = 2$, $|T_{(b,\bullet)}| = 2$ and $|T_{(c,\bullet)}| = 2$;

3) $\pi_1(T) = \{a, b, c\}$ with $|T_{(a,\bullet)}| = 3$, $|T_{(b,\bullet)}| = 2$ and $|T_{(c,\bullet)}| = 1$.

The first two cases cannot occur since I_S does not contain monomials of degree two.

Therefore, by item 3), $T_{(a,\bullet)} \supset T_{(b,\bullet)} \supset T_{(c,\bullet)}$ and $T_{(a,\bullet)} = \{h_1, h_2, h_3\}$, $T_{(b,\bullet)} = \{h_1, h_2\}$, $T_{(c,\bullet)} = \{h_1\}$ for some h_i's and thus

$$T = \big\{(h_1, a), (h_2, a), (h_3, a), (h_1, b), (h_2, b), (h_1, c)\big\}.$$

But x_2, x_3 and x_5 belong each to one only element of S whereas in T there are only two such elements, precisely h_3 and c.

Therefore it is interesting to describe more special configurations which are CM.

2.5.2 Generalized tower sets in codimension 2

In order to introduce a new configuration we need some preliminary notation.

Let $T \subseteq D_{2,\mathcal{N}}$ be a tower set and $h \in \pi_1(T) \cap \pi_2(T)$ then, using Definition 2.2.5, we set

$$E_T(h) := \{v \in T \mid \varphi(v) \in E_{\varphi(T)}(h)\}$$

and

$$F_T(h) := \pi_1(E_T(h)).$$

The following proposition provides an equivalent representation of $F_T(h)$ which directly looks at the tower set T.

Proposition 2.5.8. *With the above notation*

$$F_T(h) = \{i \in \pi_1(T) | (i, h) \notin T \text{ and } T_{(i,\bullet)} \supsetneq T_{(h,\bullet)}\} \subseteq \mathcal{N}.$$

Proof. Let $(i, j) \in E_T(h)$, then $\{i, h\} \notin \varphi(T)$ and $\{j, h\} \notin \varphi(T)$. Since T is a tower and $T_{(h,\bullet)} \not\supseteq T_{(i,\bullet)}$ we have $T_{(h,\bullet)} \subsetneq T_{(i,\bullet)}$. On the other hand let $W := \{i \in \pi_1(T) | (i, h) \notin T \text{ and } T_{(i,\bullet)} \supsetneq T_{(h,\bullet)}\}$ and let $i \in W$. If $j \in T_{(i,\bullet)} \setminus T_{(h,\bullet)}$, we will prove that $(i, j) \in E_T(h)$. Since $j \notin T_{(h,\bullet)}$, we have $(h, j) \notin T$. Since $i \in W$, we have that $(i, h) \notin T$. From $T_{(i,\bullet)} \supsetneq T_{(h,\bullet)}$ we get $(h, i) \notin T$. If $(j, h) \in T$ we get $T_{(j,\bullet)} \supsetneq T_{(i,\bullet)} \ni j$. Hence we have $(i, j) \in E_T(h)$. □

Note that if $i \in F_T(h)$ then $i > h$.

Let $T \subset D_{2,\mathcal{N}}$, we say that T is **connected** iff $\varphi(T)$ is connected (see Definition 2.4.12).

Now we are in position to "generalize" the notion of tower set.

Definition 2.5.9. Let $\tilde{T} \subset D_{2,\mathcal{N}}$, we say that $\tilde{T}$ is a **g-tower set** if

1) $\tilde{T}$ is connected;

2) $\tilde{T} = T \cup T'$ where T is a tower set and for every $(i,j) \in T'$ we have $i \in \pi_1(T) \cap \pi_2(T)$ and $j \notin \pi_1(T) \cup \pi_2(T)$,

Moreover, for every $j \in \pi_2(T')$, let $T'_{(\bullet,j)} = \{i_1, \ldots, i_t\}$ where $i_h < i_{h+1}$, then

3) $F_T(i_h) \subseteq \{i_{h+1}, \ldots, i_t\}$, for every i_h.

Remark 2.5.10. Item 3) in Definition 2.5.9 implies that $T'_{(\bullet,j)}$ is self-covered in $\varphi(T)$. On the other hand if $W \subseteq \pi_1(T)$ is self-covered in $\varphi(T)$ then condition 3) holds for W.

Definition 2.5.11. We say that $S \subseteq C_{2,\mathcal{N}}$ is a **g-towerizable set**, iff there exists $\tilde{T} \in \mathcal{T}_S$ such that $\tilde{T}$ is a g-tower set.

Let $W \subseteq D_{2,\mathcal{N}}$ and $h \in \mathcal{N}$, we will use the following notation

$$W(h) := \{v \in W | h \notin \varphi(v)\}.$$

Theorem 2.5.12. *Let $S \subseteq C_{2,\mathcal{N}}$ be a g-towerizable set then S is CM.*

Proof. Let $\tilde{T} = T \cup T'$ a g-tower set such that $\tilde{T} \in \mathcal{T}_S$. We proceed by induction on $|\pi_2(T')|$. If $\pi_2(T') = \emptyset$ then S is a towerizable set, so the statement follows by Theorem 2.5.6. If $\pi_2(T') = \{j_1\}$ then $S(j_1)$ is a towerizable set and, by Remark 2.5.10, $S_{j_1} = T'_{(\bullet,j_1)}$ is self-covered in $S(j_1)$, so the statement follows by Theorem 2.3.10. Let $\pi_2(T') = \{j_1, \ldots, j_p\}$ and let $p \geq 2$. Assume by contradiction S is not CM, so by Theorem 2.4.11 we have that $\overline{S} := \{z \in C_{2,\mathcal{N}} | z \notin S\}$ contains a minimal r-cycle V, for some $r \geq 4$. Note that $\{j_a, j_b\} \in \overline{S}$, for all $a \neq b \in \{1, \ldots, p\}$. Moreover, by inductive hypothesis $S(j_i)$ is CM for any $j_i \in \{j_1, \ldots, j_p\}$. Thus, $j_i \in v$ for some $v \in V$. So, since $\{j_a, j_b\} \in \overline{S}$ for all $a \neq b$, if $p \geq 3$ we get V is not minimal. Finally let $\pi_2(T') = \{j_1, j_2\}$ and $V = \{\{j_1, j_2\}, \{j_2, i_3\}, \{i_3, i_4\}, \ldots, \{i_r, j_1\}\}$. Since S is connected, by Remark 2.4.13, we get $r \geq 5$, therefore $\{i_3, i_r\} \in S$. Let us suppose $(i_r, i_3) \in T$, analogously we proceed if $(i_3, i_r) \in T$. We claim that

$$T_{(i_r,\bullet)} \supsetneq T_{(i_{r-1},\bullet)} \supsetneq \cdots \supsetneq T_{(i_3,\bullet)}.$$

By the claim we have $i_r \in F_T(i_{r-1})$ and then, by definition of g-tower, $(i_r, j_1) \in \tilde{T}$, so we get a contradiction.

To prove the claim it is enough to observe that if $(i_a, i_b) \in T$ and $a - b > 2$ then $(i_{a-1}, i_b) \in T$, and $(i_a, i_{b+1}) \in T$. This follows by $T_{(i_a,\bullet)} \supsetneq T_{(i_b,\bullet)}$ and therefore $(i_b, i_{a-1}) \notin T$. On the other hand if $(i_{b+1}, i_a) \in T$, then $T_{(i_{b+1},\bullet)} \supsetneq T_{(i_a,\bullet)} \ni i_b$, that is a contradiction.

□

In the last part of this section we will "reverse" the theorem to describe as the g-tower sets characterize the Cohen-Macaulyness.

Definition 2.5.13. Let $S \subseteq C_{2,\{x_1,\ldots,x_N\}}$ and $S' \subseteq C_{2,\{y_1,\ldots,y_{N'}\}}$, with $N \geq N'$, we will say that S' is a restriction of S if exists a set of monomials $\{H_1, \ldots, H_{N'}\} \subset k[x_1, \ldots, x_N]$ such that via the homomorphism

$$\nu : k[y_1, \ldots, y_{N'}] \to k[x_1, \ldots, x_N]$$

for which $\nu(y_i) = H_i$, we have

$$I_S = \nu(I_{S'}).$$

We will prove the following.

Theorem 2.5.14. *Let $S \subseteq C_{2,\mathcal{N}}$ be CM. Then there exists a restriction of S which is a g-towerizable set.*

In order to prove the Theorem 2.5.14, we proceed in few steps.

In the first step we make use of a peculiar Hilbert-Burch matrix for I_S.

Step 1

Lemma 2.5.15. *Let $S \subseteq C_{2,\mathcal{N}}$, then I_S admits a Hilbert-Burch matrix of the form*

$$\begin{pmatrix}
M_{0,1} & 0 & \ldots & 0 & 0 & \ldots & & & & & & \ldots \\
D_1 & M_{1,2} & \ldots & M_{1,\alpha_1} & 0 & \ldots & & & & & & \ldots \\
0 & D_2 & 0 & 0 & M_{2,\alpha_1+1} & \ldots & M_{2,\alpha_2} & 0 & \ldots & & & \ldots \\
0 & 0 & D_3 & 0 & 0 & \ldots & 0 & M_{3,\alpha_2+1} & \ldots & M_{3,\alpha_3} & 0 & \ldots \\
 & & & \ddots & & & & & & & & \\
 & & & & & & & & & & & \\
0 & \ldots & & & & & & & & & \ldots & D_n
\end{pmatrix}$$

where we enumerate the rows from 0 to n, the rank of the matrix, and the columns from 1 to n. So, D_i is in the position (i, i) and $M_{i,j}$, where $i < j$, is the only other non zero entry in the column j.

Proof. Take $\mathcal{G}(I_S)$ the set of minimal monomial generators for I_S, then the first syzygy module is minimally generated by a set Φ of n elements acting each only on two generators. Moreover there are at least two generators of I_S on which only one syzygy acts. Let F_0 be one of these generators and let ϕ_1 be the syzygy acting on F_0 and let F_1 the other generator on which ϕ_1 acts. Now we call $\phi_2, \ldots, \phi_{\alpha_1}$ all the other syzygies in Φ acting respectively on F_1 and $F_2, \ldots, F_{\alpha_1} \in \mathcal{G}(I_S)$. Iterating this procedure we get our matrix. $\square$

A matrix of the type as in Lemma 2.5.15 will be called a Hilbert-Burch matrix of **standard form** for I_S.

Given $\mathcal{M} = (m_{ij})$, a Hilbert-Burch matrix of standard form for I_S of size $(n+1) \times n$, a natural map σ arises,

$$\sigma : \{1, \dots, n\} \to \{0, \dots, n-1\},$$

precisely $\sigma(j)$ is the only integer i less than j such that $m_{ij} \neq 0$. So, we can denote the entries $M_{i,j} = M_{\sigma(j),j}$ in $\mathcal{M}$, just by M_j. Note that $\sigma(1) = 0$, $\sigma(2) = 1$ and, for $j > 2$, $\sigma(j) \geq \sigma(j-1) > 0$.

Let $i \in \{1, \dots, n\}$, we denote with $\mu(i)$ the set

$$\mu(i) := \{i, \sigma(i), \sigma^2(i), \dots, \sigma^h(i)\},$$

where h is the only integer such that $\sigma^h(i) = 1$. For short we will write $j \notin \mu(i)$ if $j \in \{1, \dots, n\} \setminus \mu(i)$.

We denote by F_i the determinant of the matrix obtained by removing the row i for $0 \leq i \leq n$. By the Hilbert-Burch theorem, up to sign, we have

$$\mathcal{G}(I_S) = \{F_0, \dots, F_n\}.$$

Note that $F_0 = D_0 \cdots D_n$. In the following proposition will compute all the other generators.

Proposition 2.5.16. *For any $i \in \{1, \dots, n\}$, with the above notation, we have*

$$F_i = \prod_{j \in \mu(i)} M_j \cdot \prod_{h \notin \mu(i)} D_h.$$

Proof. Let $i \in \{1, \dots, n\}$, and let H be the square matrix obtained from $\mathcal{M}$ by removing the row containing D_i. Since M_i is the only entry in the i-th column of H, we compute the determinant by using the Laplace expansion along its i-th column. Thus, up to sign, $F_i = M_i G_1$, where G_1 is the determinant of the matrix H_1 obtained from H by removing the row $\sigma(i)$ and the i-th column. Note that $M_{\sigma(i)}$ is the only entry in the $\sigma(i)$-th column of H_1, hence $F_i = M_i M_{\sigma(i)} G_2$, where G_2 is the determinant of the matrix H_2 obtained from H_1 by removing the row $\sigma^2(i)$ and the $\sigma(i)$-th column. So, by iterating this computation, we get $F_i = \prod_{j \in \mu(i)} M_j \cdot G'$, where G' is the determinant of the matrix H' obtained from H by deleting the rows $\sigma(j)$ and the columns j, for all $j \in \mu(i)$. Finally, we observe that H' is an upper triangular matrix, therefore $G' = \prod_{h \notin \mu(i)} D_h$. □

The above proposition allow us to give a simple proof of Lemma 2.3.7.

Proof of Lemma 2.3.7. Let $J := I_S + P_\Delta$ be a CM squarefree monomial ideal of height 2 and let $\mathcal{M}_J$ be a standard form Hilbert-Burch matrix, as in Lemma 2.5.15. Let $F \in \mathcal{G}(J)$ be a minimal monomial generator for J such that $P_\Delta \in (F)$. Since either $F \in (D_n)$ or $F \in (M_1)$, we have either $\{x_j | D_n \in (x_j)\} \subseteq \Delta$ or $\{x_j | M_1 \in (x_j)\} \subseteq \Delta$. In both cases the statement is proved because D_n and M_1 divide all the generators of I_S except one. □

Using Proposition 2.5.16, in the next step we define a suitable set $S_\mathcal{M}$ and we prove that it is a restriction of S.

Step 2

Let $\mathcal{M}$ be a Hilbert-Burch matrix of standard form for I_S. Let $n := \operatorname{rank} \mathcal{M}$, we denote by $\mathcal{N}_\mathcal{M}$ a set of cardinality $2n$,

$$\mathcal{N}_\mathcal{M} := \{y_1, \ldots, y_n, z_1, \ldots, z_n\}.$$

Let

$$\nu : k[\mathcal{N}_\mathcal{M}] \to k[\mathcal{N}]$$

be the homomorphism such that $\nu(y_i) = D_i$ and $\nu(z_i) = M_i$, we set

$$S_\mathcal{M} := \{\{a, b\} \in C_{2,\mathcal{N}_\mathcal{M}} | I_S \subseteq (\nu(a), \nu(b))\}.$$

We claim that $S_\mathcal{M}$ is a g-towerizable set which is a restriction of S, as required in Theorem 2.5.14.

First of all we have to prove the following proposition.

Proposition 2.5.17. *With the above notation, $S_\mathcal{M}$ is a restriction of S.*

Proof. We prove that

$$I_S = \bigcap_{\{a,b\} \in S_\mathcal{M}} (\nu(a), \nu(b)) = \nu(I_{S_\mathcal{M}}).$$

Trivially $I_S \subseteq \nu(I_{S_\mathcal{M}})$. On the other hand let $f \in I_{S_\mathcal{M}}$ be a monomial. Let us consider the set

$$\{0\} \cup \{i | f \in (z_i) \text{ and } f \notin (y_i)\}.$$

Let r the maximum of this set, we will show that $\nu(f) \in (F_r)$. Since $\{z_j, y_j\} \in S_\mathcal{M}$ for any j, we have that $f \in (y_j)$ for any $j > r$. So if $r = 0$ or $r = 1$ we are done. So we can suppose $r > 1$. If $\nu(f) \notin (F_r)$, we have $f \notin (z_h)$ for some $h \in \mu(r)$, so from Proposition 2.5.16 we get $\{z_h, y_r\} \in S_\mathcal{M}$, and hence $f \in (y_r)$ that contradicts the definition of r. □

Since the minimal set of monomial generators only depends on the positions of the non-zero entries in $\mathcal{M}$ we have

$$\mathcal{G}(I_{S_\mathcal{M}}) = \{\prod_{j=1}^{n} y_j\} \cup \{\prod_{j\in\mu(i)} z_j \cdot \prod_{h\notin\mu(i)} y_h \mid i = 1, \dots, n\}.$$

In the sequel, we will denote the elements in $\mathcal{G}(I_{S_\mathcal{M}})$ by $f_0 := \prod_{j=1}^{n} y_j$ and, for $i = 1, \dots, n$, by $f_i := \prod_{j\in\mu(i)} z_j \cdot \prod_{h\notin\mu(i)} y_h$.

Remark 2.5.18. $I_{S_\mathcal{M}}$ is a squarefree monomial ideal of $k[\mathcal{N}_\mathcal{M}]$. Moreover, $I_{S_\mathcal{M}}$ is Cohen-Macaulay since it is the determinantal ideal of the Hilbert-Burch matrix (of standard form) $\nu^{-1}(\mathcal{M})$. Therefore, by Remark 2.4.13, $S_\mathcal{M}$ is connected.

In the order to prove that $S_\mathcal{M}$ is a g-towerizable set, in the next step we will characterize the elements in $S_\mathcal{M}$.

Step 3

We need a simple lemma.

Lemma 2.5.19 Let $i, j \in \{1, \dots, n\}$,

(i) if $j \in \mu(i)$ then $\mu(j) \subseteq \mu(i)$;

(ii) if $j, h \in \mu(i)$ and $j \leq h$ then $j \in \mu(h)$;

(iii) there exists $h \in \{1, \dots, n\}$ such that $\mu(i) \cap \mu(j) = \mu(h)$.

Proof. (i) and (ii) follow by definition of μ.

(iii)Let $h := \max(\mu(i) \cap \mu(j))$, by item (i) we have $\mu(h) \subseteq \mu(i) \cap \mu(j)$. Let $k \in \mu(i) \cap \mu(j)$ then $h \geq k$ and so, by item (ii), we get $k \in \mu(h)$. □

The following proposition characterize the elements in $S_\mathcal{M}$.

Proposition 2.5.20 With the notation above, we have

(i) $\{y_a, z_b\} \in S_\mathcal{M}$ iff $b \in \mu(a)$.

(ii) $\{y_a, y_b\} \in S_\mathcal{M}$, iff $a \notin \mu(b)$ and $b \notin \mu(a)$.

Proof. (i)If $\{y_a, z_b\} \in S_{\mathcal{M}}$ then $f_a \in (y_a, z_b)$. But $f_a \notin (y_a)$, hence $f_a \in (z_b)$, i.e. $b \in \mu(a)$. Vice versa, let $b \in \mu(a)$. If $f_k \notin (y_a, z_b)$, we have $b \notin \mu(k)$ and $a \in \mu(k)$. Then, by Lemma 2.5.19 (i), we get a contradiction because

$$b \in \mu(a) \subseteq \mu(k) \text{ and } b \notin \mu(k).$$

(ii) If $\{y_a, y_b\} \in S_{\mathcal{M}}$ then $f_a \in (y_a, y_b)$. But $f_a \notin (y_a)$, hence $f_a \in (y_b)$, i.e. $b \notin \mu(a)$, analogously we get $a \notin \mu(b)$. On the other hand, let $a \notin \mu(b)$ and $b \notin \mu(a)$. If $f_k \notin (y_a, y_b)$, we have $a, b \in \mu(k)$ and then, by Lemma 2.5.19 (ii), or $a \in \mu(b)$ either $b \in \mu(a)$. □

Now we denote by

$$S^0_{\mathcal{M}} := \{\{y_i, z_j\} \in S_{\mathcal{M}} | j \in \mu(n)\} \cup \{\{y_i, y_j\} \in S_{\mathcal{M}}\},$$

and by

$$S^1_{\mathcal{M}} := \{\{y_i, z_j\} \in S_{\mathcal{M}} | j \notin \mu(n)\}.$$

From the previous proposition we have the following partition

$$S_{\mathcal{M}} = S^0_{\mathcal{M}} \cup S^1_{\mathcal{M}}.$$

In the next step we will prove that $S^0_{\mathcal{M}}$ is a towerizable set.

Step 4

In order to prove that $S^0_{\mathcal{M}}$ is a towerizable set we need to introduce some order on $\mathcal{N}_{\mathcal{M}}$. So, we set

$$\psi_n(i) := \max(\mu(i) \cap \mu(n)).$$

Therefore, using ψ_n we can introduce a partial order on the set $\{y_1, \ldots, y_n\}$. We say that

$$y_i > y_j \text{ iff } \psi_n(i) > \psi_n(j).$$

It is easy to check that $\psi_n(i) \leq i$ and $\psi_n(i) = i$ iff $i = \sigma^u(n)$ for some u. Now we describe some other properties of ψ_n.

Lemma 2.5.21. *If $\psi_n(i) > \psi_n(j)$ and $j \in \mu(i)$ then $j \in \mu(n)$*

Proof. If $j \notin \mu(n)$ then since $j, \psi_n(i) \in \mu(i)$ and $j \notin \mu(\psi_n(i))$ we get $\psi_n(i) \in \mu(j)$. Hence, we have $\mu(i) \cap \mu(n) = \mu(j) \cap \mu(n)$ that contradicts $\psi_n(i) > \psi_n(j)$. □

Furthermore we have the following lemma.

Lemma 2.5.22 With the notation above, we have

(i) If $\{y_i, y_j\} \in S_{\mathcal{M}}$ and $\psi_n(i) \geq \psi_n(j)$ then $j \notin \mu(n)$;

(ii) If $j \notin \mu(n)$ and $\psi_n(i) > \psi_n(j)$ then $\{y_i, y_j\} \in S_{\mathcal{M}}$.

Proof. (i) If $j \in \mu(n)$ then $j \in \mu(j) \cap \mu(n) \supseteq \mu(i) \cap \mu(n)$. Hence $j \in \mu(i)$ which contradicts $\{y_i, y_j\} \in S_{\mathcal{M}}$.

(ii) Let $\{y_i, y_j\} \notin S_{\mathcal{M}}$. Then, if $j \in \mu(i)$, by Lemma 2.5.21, we have $j \in \mu(n)$. Otherwise, if $i \in \mu(j)$, we have $\psi_n(i) \leq \psi_n(j)$. □

Remark 2.5.23. Observe that, by Proposition 2.5.20 (i), if $\{y_i, z_j\} \in S^0_{\mathcal{M}}$ then $\psi_n(j) \leq \psi_n(i)$.

Lemma 2.5.24. *Let i, j be such that $\psi(i) = \psi(j)$ and $\{y_i, y_j\} \in S_{\mathcal{M}}$ then for all h such that $\psi(h) > \psi(i)$ we have $\{y_h, y_i\} \in S_{\mathcal{M}}$.*

Proof. By Lemma 2.5.22 (i), $j \notin \mu(n)$. Moreover, by Lemma 2.5.22 (ii), since $\psi_n(h) > \psi_n(i)$ we get $\{y_h, y_i\} \in S_{\mathcal{M}}$. □

Theorem 2.5.25. *$S^0_{\mathcal{M}}$ is a towerizable set.*

Proof. We proceed by induction on $n = \operatorname{rank} \mathcal{M} = \operatorname{rank} \nu^{-1}(\mathcal{M})$. If $n = 1$ then $S^0_{\mathcal{M}} = \{\{y_1, z_1\}\}$ that is trivially a towerizable set. Let us suppose the statement true up to $n - 1$. We set

$$T^0 := \{(y_i, z_j) | \{y_i, z_j\} \in S^0_{\mathcal{M}}\} \cup \{(y_i, y_j) | \{y_i, y_j\} \in S_{\mathcal{M}}, \psi(i) > \psi(j)\}.$$

Note that $\pi_1(T^0) = \{y_1, \ldots, y_n\}$.

Let $i, j \in \{1, \ldots, n\}$ be such that $\psi_n(i) \geq \psi_n(j)$, we want show that $T^0_{(y_i,\bullet)} \supseteq T^0_{(y_j,\bullet)}$. Let $(y_j, z_h) \in T^0$ then by Proposition 2.5.20 (i) we have $h \in \mu(j)$. So

$$h \in \mu(j) \cap \mu(n) \subseteq \mu(i) \cap \mu(n).$$

Hence $h \in \mu(i)$, i.e. $(y_i, z_h) \in T^0$.

Let $(y_j, y_h) \in T^0$ then we have $\psi_n(i) \geq \psi_n(j) > \psi_n(h)$, and so by Lemma 2.5.22 (ii), it is enough to show that $h \notin \mu(n)$. This follows by Lemma 2.5.22 (i) since $(y_j, y_h) \in T^0$ and $\psi_n(j) > \psi_n(h)$. So T^0 is a tower set.

Now for each $n_k \in \mu(n)$, we set

$$B(n_k) := \{\{y_i, y_j\} |\ \psi_n(i) = \psi_n(j) = n_k\}.$$

We remark that by Lemma 2.5.22 (i), if $\{y_i, y_j\} \in B(n_k)$ then $i, j \notin \mu(n)$. We claim that $B(n_k)$ is a towerizable set. If this is the case then there exists a tower set $T(n_k) \in \mathcal{T}_{B(n_k)}$, for each $n_k \in \mu(n)$, and therefore, we see that

$T := T^0 \cup \{T(n_k) \mid n_k \in \mu(n)\} \in \mathcal{T}_{S^0_{\mathcal{M}}}$ is a tower set. This follows from the construction of T^0, by Lemma 2.5.22 (ii) and by Lemma 2.5.24, since we can refine the partial order in $\pi_1(T)$ to a total order just using the ordering of each $\pi_1(T(n_k))$. To get the claim take $n_k \in \mu(n)$ and let σ' be the restriction of σ on

$$\bigcup_{\{i \,:\, \psi_n(i)=n_k\}} \mu(i).$$

In correspondence with σ' we have a sub-matrix $\mathcal{M}'$ of $\mathcal{M}$, that is a Hilbert-Burch matrix of standard form. $\mathcal{M}'$ is obtained by $\mathcal{M}$ by deleting the row and column u if $u \notin \cup_{\{i \,:\, \psi_n(i)=n_k\}}\mu(i)$. Since $\operatorname{rank} \mathcal{M}' < n$, by inductive hypothesis $S_{\mathcal{M}'}$ is a towerizable set, moreover, by Proposition 2.5.20 (i) and (ii),

$$B(n_k) = S_{\mathcal{M}'} \setminus \{\{y_i, z_j\} \mid j \in \mu_{\mathcal{M}'}(i)\}$$

that is easily also a tower set. □

Corollary 2.5.26. *Let $\mathcal{M}$ be a matrix of standard form such that $\sigma(a) = a - 1$, for any $a \in \{1, \ldots, n\}$, then $S_{\mathcal{M}}$ is a towerizable set.*

Proof. It follow by Proposition 2.5.25, after we observe that, by Proposition 2.5.16, $F_n = \prod_{i=1}^n M_i$, hence $S_{\mathcal{M}} = S^0_{\mathcal{M}} = \{\{y_i, z_j\} \mid i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, i\}\}$. □

Let $T \in \mathcal{T}_{S^0_{\mathcal{M}}}$ the tower set constructed in the proof of Theorem 2.5.25 and

$$T' := \{(y_i, z_j) \mid \{y_i, z_j\} \in S^1_{\mathcal{M}}\},$$

in the last step we prove that, with the notation of Definition 2.5.9, $T \cup T' \in \mathcal{T}_{S_{\mathcal{M}}}$ is a g-tower set.

Step 5

In order to prove that $S_{\mathcal{M}}$ is a g-towerizable set we have to verify on $T \cup T'$ the last two conditions in Definition 2.5.9. So, the following theorem conclude the proof of Theorem 2.5.14.

Proposition 2.5.27. *With the above notation, let $j \notin \mu(n)$ and let $T'_{(\bullet, z_j)} = \{y_{i_1}, \ldots, y_{i_t}\}$, with $y_{i_h} < y_{i_{h+1}}$, then*

1. *$z_j \notin \pi_1(T) \cup \pi_2(T)$;*
2. *$F_T(y_{i_h}) \subseteq \{y_{i_{h+1}}, \ldots, y_{a_t}\}$, for every y_{i_h}.*

Proof. The proof of item 1 is immediate.

Moreover, since $\varphi(T \cup T')$ is CM we have, by Lemma 2.2.2, $\varphi(T \cup \{(y_a, z_j) \mid a : j \in m(a)\})$ is also CM. So, by Theorem 2.3.10, $T'_{(\bullet, z_j)} = \{y_a \mid j \in m(a)\} \subseteq \pi_1(T)$ is self-covered in T. By remark 2.5.10 we are done. □

Lemma 2.5.28. *Let S' be a restriction of S, if S' is CM then also S is CM.*

Proof. It follows since I_S and $I_{S'}$ have the same number of minimal generators and so a Hilbert-Burch matrix of $I_{S'}$, via the map ν, is a Hilbert-Burch matrix of I_S. □

Collecting all the results of this section we finally get the following theorem.

Theorem 2.5.29. *$S \subseteq C_{2,\mathcal{N}}$ is CM iff there exists S' a restriction of S which is a g-towerizable set.*

Proof. The necessary part is the Theorem 2.5.14. On the other hand if S' is a g-towerizable which is a restriction of S then S' is CM by Theorem 2.5.12. So, by Lemma 2.5.28 also S is CM. □

Example 2.5.30. Let $\mathcal{N} = \{x_1, \ldots, x_{11}\}$ and

$$S := \big\{\{x_1, x_2\}, \{x_1, x_4\}, \{x_1, x_6\}, \{x_1, x_8\}, \{x_1, x_{10}\},$$
$$\{x_3, x_4\}, \{x_3, x_8\}, \{x_3, x_{10}\}, \{x_4, x_6\}, \{x_5, x_6\}, \{x_5, x_{10}\},$$
$$\{x_6, x_7\}, \{x_6, x_8\}, \{x_6, x_{10}\}, \{x_8, x_9\}, \{x_8, x_{10}\}, \{x_{10}, x_{11}\}\big\}.$$

Note that the following *picture* describes a structure for the elements in S.

	x_1	x_3	x_6	x_8	x_5	x_{11}	x_9	x_5	x_7
x_{10}	X	X	X	X	X	X			
x_8	X	X	X				X		
x_4	X	X	X						
x_6	X							X	X
x_2	X								

Moreover, the following represents a restriction S' of S.

	x_1	x_3	x_6	x_8	x_5	x_{11}	x_9	y
x_{10}	X	X	X	X	X	X		
x_8	X	X	X				X	
x_4	X	X	X					
x_6	X							X
x_2	X							

where $\nu(y) = x_5 x_7$ and ν is the identity map on the set $\{x_1, \ldots, \hat{x}_7, \ldots x_{11}\}$. Note that S' is a g-towerizable set and so S is CM.

Part II

The Betti Weak Lefschetz Property

Chapter 3

Basic facts about the Weak Lefschetz Property

In this chapter we introduce notation, give basic definitions and recall some well-known results about the Weak Lefschetz Property.

The aim of Section 3.1 is to recall some basic facts about the Weak Lefschetz algebras, a more complete description can be found in [MN1] and in [HMMN] where the authors give an overview of the Lefschetz properties from many perspectives. Finally Section 3.2 describes the WLP for the standard graded modules.

3.1 The Weak Lefschetz Property

Let $R := k[x_1, \dots, x_c]$ be the standard polynomial ring over a field of characteristic zero.

Let $\ell \in R_1$ be a linear form and $A = R/I$ be a standard graded algebra, we will use the following terminology:

$$I_{[\ell]} := I + (\ell),\ A_{[\ell]} := R/I_{[\ell]},\ \overline{R}_{[\ell]} := R/(\ell),\ \overline{I}_{[\ell]} := I_{[\ell]}/(\ell) \text{ and } \overline{A}_{[\ell]} := \overline{R}_{[\ell]}/\overline{I}_{[\ell]}$$

(of course, $\overline{A}_{[\ell]} \cong A_{[\ell]}$ as rings), when there is no ambiguity, we will write $\overline{R}$, $\overline{I}$ and $\overline{A}$, instead of $\overline{R}_{[\ell]}$, $\overline{I}_{[\ell]}$ and $\overline{A}_{[\ell]}$.

In this setting A_j will denote the k-vector space consisting of the j-th component of the graded algebra $A = R/I$. Let i be an integer, we will denote by $\varphi_{\ell,i} : A_i \to A_{i+1}$ the linear map (as k-vector spaces) obtained by multiplication by ℓ. An Artinian standard graded algebra is said to have the **Weak Lefschetz Property**, WLP for short, if there is a linear form $\ell \in R_1$ such that, for every integer i, the linear map $\varphi_{\ell,i} : A_i \to A_{i+1}$ has maximal rank (such a linear form will be called a **WL form**). An algebra with the WLP will be call **Weak Lefschetz algebra**.

In the case of one variable, all the algebras have the WLP since all the homogeneous ideals are principal. The case of two variables also has a nice result.

Theorem 3.1.1 ([HMNW], Proposition 4.4.). *Let I be a homogeneous ideal in $k[x,y]$ then R/I has the WLP.*

3.1.1 Hilbert Function and WLP

A O-sequence $(1, h_1, \ldots, h_s)$, with $h_s \neq 0$ is called **unimodal** if there is an integer t such that

$$h_1 \leq \cdots \leq h_t \geq h_{t+1} \geq \cdots \geq h_s.$$

If A has the WLP and ℓ is a Lefschetz form such that the multiplication map $\varphi_{\ell,i}$ is surjective then, for any $r \geq 1$, the multiplication map $\varphi_{\ell,i+r}$ is also surjective. This follows immediately from the fact that A is a standard algebra. As a consequence we see that if A is an Artinian algebra with the WLP then the Hilbert function of A is unimodal.

But the WLP is a strictly stronger condition. An Artinian standard graded algebra can have an unimodal Hilbert Function and fails the WLP.

Example 3.1.2. Let $I = (x^2, xy, xz) + (x, y, z)^4 \subseteq k[x, y, z]$. Observe that R/I has unimodal Hilbert function,

$$HF_{R/I} = (1, 3, 3, 4, 0, \ldots).$$

But, as above observed, an algebra with this h-vector fails the WLP, since the multiplication map from $(R/I)_1$ to $(R/I)_2$ could never be surjective.

We have, for every integer i, the following exact sequence

$$0 \to \operatorname{Ker}\varphi_{\ell,i} \to A_i \xrightarrow{\varphi_{\ell,i}} A_{i+1} \to (A_{[\ell]})_{i+1} \to 0.$$

Therefore

$$H_{A_{[\ell]}}(i+1) = \Delta H_A(i+1) + \dim_k \operatorname{Ker}\varphi_{\ell,i}. \tag{3.1}$$

By this equation, and using the unimodality of H_A, it follows that if ℓ is a WL form for A then

$$H_{A_{[\ell]}} = \Delta H_A^+,$$

where, for any n,

$$\Delta H_A^+(n) = \max\{\Delta H_A(n), 0\}.$$

The previous property can be reversed.

Remark 3.1.3. A has the WLP if and only if there is a linear form $\ell \in R_1$ for which $H_{A_{[\ell]}} = \Delta H_A^+$. Indeed, from this equality and equation (3.1), one gets that, for all i, $\varphi_{\ell,i} : A_i \to A_{i+1}$ has maximal rank.

Remark 3.1.3 allows us to give an equivalent definition of WLP for an algebra just looking at its generic linear quotient.

Proposition 3.1.4. *Let A be an Artinian standard graded algebra. The following are equivalent*

i) A has the WLP;

ii) there is an element $\ell \in R_1$ such that $H_{A_{[\ell]}} = \Delta H_A^+$.

The **Weak Lefschetz O-sequences**, i.e. the Hilbert functions of algebras with the WLP, have been completely classified.

Theorem 3.1.5 (Proposition 3.5, [HMNW]). *Let $H = (1, h_1, h_2, \dots, h_s)$ be a finite sequence of positive integers. Then H is the Hilbert function of a graded Artinian WL algebra A if and only if H is an unimodal O-sequence and ΔH_A^+ is a O-sequence.*

Migliore and Zanello showed that some Hilbert function force the WLP to hold.

Theorem 3.1.6 ([MZ1]). *Let $H = (1, h_1, h_2, \dots, h_s)$ be a O-sequence, and let t be the smallest integer such that $h_t \le t$. Then all the Artinian algebras having Hilbert function H are WL algebras if and only if, for all $i = 1, 2, \dots, t-1$, we have*

$$h_{i-1} = ((h_i)_{(i)})_{-1}^{-1}.$$

3.1.2 Graded Betti numbers and WLP

Let H be a Weak Lefschetz O-sequence, then there is a sharp upper bound on the graded Betti numbers among k-algebras having Hilbert function H and the Weak Lefschetz property.

Let H be a Weak Lefschetz O-sequence and t, t' such that

$$H = (h_0 < h_1 < \cdots < h_t = \cdots = h_{t'} > h_{t'+1} \ge \cdots \ge h_s > 0).$$

We denote by $\{\hat{\beta}_{ij}(\Delta H_A^+)\}$ the maximal Betti numbers for ΔH_A^+, namely the graded Betti numbers of the lex segment ideal $L_{\Delta H_A^+}$ in $\bar{R} = R/\ell$, having Hilbert function ΔH_A^+, see Theorem 1.2.3, i.e.

$$\hat{\beta}_{ij}(\Delta H_A^+) := \hat{\beta}_{ij}(L_{\Delta H_A^+}).$$

With this settings we have the following theorem.

Theorem 3.1.7 (Theorem 3.20, [HMNW]). *With the above notation, we have*

i) Let $A = R/I$ be a k-algebra with the WLP, and let $\ell \in R_1$ be a WL form. Then the graded Betti numbers of A satisfy

$$\beta_{ij}(A) \leq \begin{cases} \hat{\beta}_{ij}(\Delta H_A^+) & \text{if } j-i \leq t' \\ \hat{\beta}_{ij}(\Delta H_A^+) - \Delta h_A(j-i) \cdot \binom{c}{i} & \text{if } t'+1 \leq j-i \leq t+1 \\ -\Delta h_A(j-i) \cdot \binom{c}{i} & \text{if } j-i \geq t+2 \end{cases}$$

ii) Let H be a Weak Lefschetz O-sequence. Then there is an Artinian algebra $A = R/I$ having the WLP and H as Hilbert function such that equality is true in i) for all integers i, j.

3.1.3 Level Algebras and WLP

It is hard to establish if a standard algebra R/I does have the WLP, the following result is helpful for level Algebras.

Theorem 3.1.8 (Proposition 2.1, [MMN]). *Let $A = R/I$ be an Artinian standard graded algebra and let ℓ be a general linear form. $\varphi_i := \varphi_{\ell,\ i} : A_i \to A_{i+1}$ the multiplication map by ℓ. Let $d_0 \geq 0$*

i) If φ_{d_0} is surjective then φ_d is surjective for all $d \geq d_0$.

ii) If φ_{d_0} is injective φ_d is injective for all $d \leq d_0$.

iii) In particular, if A is level and $H_A(d_0) = H_A(d_0+1)$ then A has the WLP if and only if φ_{d_0} is injective (and then is an isomorphism).

Hence to prove the WLP for a level algebra it is enough to look just at two (or occasionally one) critical degrees. However, also for Gorenstein algebras many questions are still open.

3.1.4 Gorenstein Algebras and WLP

It was showeded by Stanley, see [St3], that any monomial complete intersection algebra has the WLP.

Theorem 3.1.9. *Let $I \subseteq R = k[x_1, \ldots, x_c]$ be an Artinian monomial complete intersection algebra, i.e.*

$$I = (x_1^{a_1}, \ldots, x_c^{a_c}).$$

Let ℓ be a general linear form. Then

$$\varphi_{\ell,i} : (R/I)_i \to (R/I)_{i+1}$$

have maximal rank, for any i.

A consequence of this theorem is that a general complete intersection algebra with fixed generator degrees has the WLP. So the question is if all Artinian complete intersections have the WLP. The question is trivial in one and two variables. In more variables we only have the following result.

Theorem 3.1.10 ([HMNW], Theorem 2.3.). *Let $R = k[x, y, z]$. Let $I = (F_1, F_2, F_3)$ be a complete intersection ideal. Then R/I has the WLP.*

Complete intersection algebras are special cases of Gorenstein algebras. Gorenstein algebras can fail the WLP, the first example was given by Stanley in 1978, see [St1]. He showed that $(1, 13, 12, 13, 1)$ is a Gorenstein h-vector which clearly fails the WLP since it is not unimodal. Bernstein-Iarrobino ([BI]), Boij-Laksov ([BL]) and Boij ([Bo]) gave later many other not unimodal Gorenstein h-vectors of codimension 5 or greater.

WLP does not necessarily hold even for a Gorenstein algebra with unimodal Hilbert function. For instance, an example in codimension 4 was given by Ikeda [Ik] in 1996. The question in three variables is still open, but in recent years has some progress been made. In [RZ2] the authors prove that Gorenstein algebras with Hilbert function

$$h_1 < h_2 < \cdots < h_t = h_{t+1} = h_{t+2} \geq h_{t+3} \cdots$$

have the WLP.

In [BMMNZ] the authors reduce the problem to an investigation of the WLP for the compressed Gorenstein algebras of odd socle degree. They answer affirmatively for the case $H = (1, 3, 6, 6, 3, 1)$. We recall that an Artinian level standard algebra $A = \oplus_{i=0}^{d} A_i$ of type t is said to be a **compressed algebra** if its Hilbert function is given by

$$H_A(i) = \min\{\dim_k R_i, t \dim_k R_{d-i}\}.$$

In four variables the results are very limited, the biggest obstruction is that we do not have a characterization theorem for Gorenstein algebras in more then three variables. Ikeda, as mentioned above, showed that WLP not necessarily holds for a Gorenstein algebra. On the other hand in [MNZ] the authors show that if the Hilbert function of a Gorenstein algebra A is $H_A = (1, 4, h_2, h_3, h_4, \ldots)$ and $h_4 \leq 33$ then A has the WLP. More recently Seo and Srinivasan, [SS], extended this result to $h_4 = 34$. Then a Gorenstein algebra $A = R/I$ has the WLP if I has a generator of degree 4.

3.1.5 Almost complete intersection and WLP

We recall in the previous section that if I is a complete intersection ideal of height 3 then R/I has the WLP. So a question we can ask if this property continues to be true increasing the number of minimal generators for the ideal defining the algebra. For instance, for the Artinian standard graded algebras of codimension 3 defined by an **almost complete intersection ideal** i.e. an ideal minimally generated by 4 forms.

In [RZ5] the authors prove that the Hilbert function of an almost complete intersection algebra is a WL O-sequence. However, in [BK] the authors show that $A := k[x, y, z]/(x^3, y^3, z^3, xyz)$ fails the WLP. The question of WLP for almost complete intersection algebras is currently highly studied, especially for monomial level algebras, see for instance [MMN], [CN], [CN1] and [LZ].

3.2 WLP for standard modules over polynomial rings

In the previous section, we studied some general properties about WLP. We described how difficult is to give a full answer for algebras which are supposed to be WL, for example the complete intersection algebras. There are very few results concerning the WLP for graded modules over a polynomial ring, so also the "two variables case" is interesting. In this section we study some useful general properties and in particular we look for some conditions which ensure the WLP for graded modules over a polynomial ring in two variables,most of these results can be found in [FT].

3.2.1 Some useful Lemmas

Let R be the standard graded polynomial ring in c variables over a field k of characteristic zero. Let $M = M_0 \oplus M_1 \oplus \cdots \oplus M_s$ be an Artinian graded R-module with h-vector $H_M = (h_0, \ldots, h_s)$. We recall that the Weak Lefschetz property is an open condition, so a module M has the WLP if, for each i, there exists a linear form $\ell_i \in R_1$ such that the map $\times \ell_i : M_i \to M_{i+1}$ has maximal rank, for each i. An Artinian graded R-module which have the WLP is called **WL module**.

Remark 3.2.1. Let M be a WL module with h-vector $H_M = (h_0, \ldots, h_s)$. If $h_i > h_{i+1}$, by definition, there is a linear form ℓ_i such that the multiplication map given by $\ell_i : M_i \to M_{i+1}$ is surjective. Thus ℓ_i injective for the dual spaces.

Let $M = M_0 \oplus M_1 \oplus \cdots \oplus M_s$ be an Artinian graded R-module with h-vector $H_M = (h_0, \ldots, h_s)$. In this section, we are interested to find a criterion to check the if the WLP holds for M. By definition and by remark 3.2.1, we can assume $M = M_0 \oplus M_1$ and its h-vector to be $H_M = (h_0, h_1)$ with $h_0 \leq h_1$.

We start with the following intuitive observation.

Remark 3.2.2. Let $M = M_0 \oplus M_1$ be a graded R-module with h-vector $H_M = (h_0, h_1)$, let $h_0 \leq h_1$. If N is a submodule of M with a decreasing h-vector then the multiplication map by a generic linear form is not injective on N. Thus M fails the WLP.

The following lemma generalizes the Remark 3.2.2.

Lemma 3.2.3. *Let $M = M_0 \oplus M_1$ be a R-module with h-vector $H_M = (h_0, h_1)$ and $h_0 \leq h_1$. Let N be a submodule of M with h-vector $H_N = (r, r)$. Then*

- *If N has the WLP then M has the WLP if and only if M/N has the WLP;*
- *If N fails the WLP then M fails the WLP.*

Proof. The Hilbert function of M/N has the same behavior of the Hilbert function of M so we need to check the injectivity. Let $\ell \in R_1$ be a generic linear form and assume N has the WLP. Let $m \in M$ such that $\ell \bar{m} = 0_{M/N}$, then $\ell m \in N$. Since ℓ is surjective on N there exists $w \in N_0$ such that $\ell w = \ell m$, therefore, since ℓ is injective on N, $m = w$. If $\ell m = 0$ then $\ell \bar{m} = 0$ so if M/N has the WLP $m \in N$ but the multiplication by ℓ is injective on N and so $m = 0$. □

Example 3.2.4. Let $R = k[x, y, z]$ and $I := (x^3, x^2z, xy^2, xyz) + \mathfrak{m}^4$. Let $M = \langle (R/I)_2 \rangle (-2)$, and let $\ell \in R_1$ be a generic linear form. We easy compute that $H_M = (6, 6)$, so we have to check the injectivity of ℓ. Note that $N = \langle xy + I \rangle$ has Hilbert function $H_N = (1, 1)$ and trivially N has the WLP. So by Lemma 3.2.3 M has the WLP only if M/N has it. But $\ell \overline{x}^2 + I = 0 \in M/N$ and then M fails the WLP.

A module M is said to be **indecomposable** if whenever $M = N \oplus P$, where N and P are submodules of M, then either $N = (0)$ or $P = (0)$.

Indecomposable modules play an important role in the study of WLP, as describe the following remark.

Remark 3.2.5. Let M be a graded R-module and let suppose M can be decomposed as a direct sum of indecomposable submodules

$$M = N_1 \oplus N_2 \oplus \ldots \oplus N_t;$$

then M has the WLP if and only if all the direct summands N_i have the WLP and their Hilbert functions have an uniform behavior, i.e. if, in each spot, the Hilbert function of one summand is strictly increasing (strictly decreasing) then the Hilbert functions of the other summands are also strictly increasing (strictly decreasing).

3.2.2 WLP for standard modules over $K[x, y]$

In this section we study the WLP for standard graded modules over the standard graded polynomial ring $R = k[x, y]$, where k has characteristic 0. Our aim is to find which conditions ensure the WLP for a graded R-module. It is known that cyclic R-modules have the WLP. It is easy to find a non-cyclic R-module who fails the WLP.

Example 3.2.6. Let $I_1 = (x^2, xy, y^2) \subset R$ and $I_2 = (x, y) \subset R$. Let $M = R/I_1 \oplus R/I_2$ be the standard graded module over R. The Hilbert function H_M of M is $H_M = (2, 2, 0, \ldots)$. The multiplication by any generic linear form from M_0 to M_1 can not be injective because it is not injective on the second component.

3.2.3 An algorithm to check the WLP

In order to give a systematic method to check the WLP let

$$\times x : M_0 \to M_1 \quad \text{and} \quad \times y : M_0 \to M_1$$

be the multiplication maps by x and y.

The following algorithm checks the WLP for Artinian R-modules

Step	START(M)		
	$\downarrow$		
1	$\times x$ is injective	$\xrightarrow{yes}$	M has the WLP
	$\downarrow$ *no*		
2	$\times y$ is injective	$\xrightarrow{yes}$	M has the WLP
	$\downarrow$ *no*		
3	$\operatorname{Ker}(\times x) \cap \operatorname{Ker}(\times y) \neq (0)$	$\xrightarrow{yes}$	M does not have the WLP
	$\downarrow$ *no*		
4	$y \operatorname{Ker}(\times x) \cap x \operatorname{Ker}(\times y) \neq (0)$	$\xrightarrow{yes}$	M does not have the WLP
	$\downarrow$ *no*		
5	$M \leftarrow \overline{M}$ and go to start		

The first two steps are clear. Now let be $\dim_k \operatorname{Ker}(\times x) = r > 0$ and $\dim_k \operatorname{Ker}(\times y) = s > 0$. If $m \in \operatorname{Ker}(\times x) \cap \operatorname{Ker}(\times y)$ we have $\ell m = 0$ for a generic linear form $\ell \in R_1$, i.e. the algorithm ends at the third step. Now we can assume that $\operatorname{Ker}(\times x) \cap \operatorname{Ker}(\times y) = (0)$ and go to the next step. If $y \operatorname{Ker}(\times x) \cap x \operatorname{Ker}(\times y) \neq (0)$, then

$$\dim_k(y \operatorname{Ker}(\times x) + x \operatorname{Ker}(\times y)) < r + s.$$

This means that the submodule $\operatorname{Ker}(\times x) + \operatorname{Ker}(\times y) \subseteq M$ have a strictly decreasing Hilbert function so, by Remark 3.2.2, M fails the WLP. Finally we have

$$\dim_k(y \operatorname{Ker}(\times x)) = \dim_k \operatorname{Ker}(\times x) = r$$

and

$$\dim_k(x \operatorname{Ker}(\times y)) = \dim_k \operatorname{Ker}(\times y) = s.$$

Since $y \operatorname{Ker}(\times x) \cap x \operatorname{Ker}(\times y) = (0)$, we get $N := \langle \operatorname{Ker}(\times x) + \operatorname{Ker}(\times y) \rangle \subset M$ has Hilbert function $H_N = (r + s, r + s)$ and $x + y \in R_1$ is a WL element for N. So by Lemma 3.2.3, M has the WLP if and only if $\overline{M} = M/N$ has the WLP. Note that $H_{\overline{M}} = (h_0 - r - s, h_1 - r - s)$ is still not decreasing, then we can back to the $START$ and we apply the algorithm on $\overline{M}$.

This algorithm ends in a finite number of steps because after each cycle the Hilbert function of the module decreases by at least two in each degree.

The algorithm can be used to study the WLP for indecomposable modules over the standard graded polynomial ring $R = k[x, y]$.

3.2.4 Indecomposable modules and WLP

Using the algorithm in Section 3.2.3, we can prove the following result which reverse the implication of the Remark 3.2.2 for modules over $k[x, y]$.

Theorem 3.2.7. *Let M be an Artinian graded R-module such that every its submodule has a non-decreasing Hilbert function, then M has the WLP.*

Proof. It is enough prove the theorem for a module M with $H_M = (h_i, h_{i+1})$, and $h_i \leq h_{i+1}$. Every submodule of M has a non-decreasing Hilbert function hence we can use directly the algorithm in Section 3.2.3 to check the WLP.

Suppose that the first two steps in the algorithm give us negative answers, by hypothesis the third and fourth step also give a negative response, thus we have to prove the WLP for $\overline{M}$.

We conclude the proof observing that each submodule of $\overline{M}$ has a non-decreasing Hilbert function. Let $T = \langle \operatorname{Ker}(\times x) + \operatorname{Ker}(\times y) \rangle$ and let $\overline{P}$ be a submodule of $\overline{M}$. Then $P + T$ is a submodule of M and it has a non-decreasing

Hilbert function, we prove that $\overline{P}$ also has a non-decreasing Hilbert function. In fact, denoted by $Q := P \cap T$, since Q is a submodule of M it has a non-decreasing Hilbert function, so the proof follows from:

$$\begin{aligned}
\dim_k \overline{P_0} &= \dim_k P_0 - \dim_k(\mathrm{Ker}(\times x) + \mathrm{Ker}(\times y)) + \dim_k Q_0 \\
&= \dim_k P_0 + \dim_k Q_0 - (r+s), \\
\dim_k \overline{P_1} &= \dim_k P_1 - \dim_k(y\,\mathrm{Ker}(\times x) + x\,\mathrm{Ker}(\times y)) + \dim_k Q_1 \\
&= \dim_k P_1 + \dim_k Q_1 - (r+s),
\end{aligned}$$

where $r = \dim_k(\mathrm{Ker}(\times x)$ and $s = \dim_k(\mathrm{Ker}(\times y)$. □

To show that if M indecomposable with Hilbert function $H_M = (h_0, h_1)$ has the WLP we need the following lemma.

Lemma 3.2.8. *Let M be a graded indecomposable R-module with a non-decreasing h-vector, $H_M = (h_0, h_1)$. Then every submodule of M has a non-decreasing h-vector.*

Proof. Let N be a submodule of M with $H_N = (r, s)$, we can assume that N has minimal generators only in degree zero. We prove the statement by induction on $h_0 - r$.

If $r = h_0$, the statement holds because all minimal generators of M are in degree zero, then $M = N$. Assume that the statement is true for each submodule minimally generated by $r > t$ elements, we claim that it true for the case $r = t$.

If $r > s$, by the induction hypothesis the statement is true for the submodule $N + \langle e \rangle$, for each $e \in M_0 \backslash N_0$. This means that xe and ye are linearly independent and the submodule generated by e does not intersect with N, i.e. $\langle e \rangle \cap N = (0)$. Hence $s = r - 1$.

We claim that $M = N \oplus \langle M_0 \backslash N_0 \rangle$, which contradicts to the hypothesis on M. In fact, if $m \in N \cap \langle M_0 \backslash N_0 \rangle$, then $m \in M_1$, so m is not a minimal generator. Since $m \in N$, there is $e_A \in N_0$ such that $\ell_A e_A = m$. Similarly, since $m \in \langle M_0 \backslash N_0 \rangle$, there is $e_B \in M_0 \backslash N_0$ such that $\ell_B e_B = m$, for some $\ell_A, \ell_B \in S_1$.By using the same argument for the submodules $N + \langle e_B \rangle$, we get a contradiction. □

Now we are able to prove the main result of this section:

Theorem 3.2.9. *Let M be a graded indecomposable R-module with h-vector $H_M = (h_0, h_1)$. Then M has the WLP.*

Proof. If $h_0 \leq h_1$, it is followed from Lem 3.2.8 and Theorem 3.2.7.

Otherwise if $h_0 > h_1$, the dual module $\mathrm{Hom}_k(M, k)$ of M is an indecomposable module with a non-decreasing Hilbert function, see [Kr]. Therefore, $\mathrm{Hom}_k(M, k)$ has the WLP, hence M has WLP. $\square$

Theorem 3.2.9 does not holds for indecomposable modules with a long enough Hilbert function. In the following example we have an indecomposable module with Hilbert function of length 4 which fails the WLP.

Example 3.2.10. Let $M = ((y, x^4) + I)/I \subset S/I$ where the degree are shifted by 1 and $I = (y^3, x^2y^2) + (x, y)^6$. The Hilbert function of M is

$$H_M = (1, 2, 2, 2, 2).$$

Then M does not have the WLP. In fact, M has a minimal generator of degree 4, so the multiplication map by any linear form from M_3 to M_4 can not be surjective because the minimal generator x^4+I is not an image of any element in M_3. Since the Hilbert function $H_M(3) = H_M(4) = 2$, this multiplication map is not injective. Furthermore, we can prove that M is indecomposable. In fact, suppose that $M = N_1 \oplus N_2$, then the indecomposable submodule generated by $y + I$ must be contained in one of these components, say $\langle y \rangle \subseteq N_1$.

It is clear that $x^4 + I$ is not in N_1, but neither in N_2, otherwise $x^4y + I \in N_1 \cap N_2$. Therefore, $x^4 + I = (n_1 + I) + (n_2 + I) \in N_1 \oplus N_2$.

Since $H_{N_1}(3) = 1$, we get that $n_1 + I = \alpha x^3y + I$, $\alpha \in k$, then $n_2 + I = x^4 - \alpha x^3y + I$. This contradicts to the fact that $yn_2 + I = x^4y + I \in N_1$.

An interesting property arises studying the WLP for modules M with h-vector $H_M = (n, n)$. By Theorem 3.2.9, if M is indecomposable then it has the WLP, otherwise M has the WLP if and only if its indecomposable submodules have such a h-vector.

3.2.5 Determinant condition to ensure the WLP

Let $M = M_0 \oplus M_1$ be a graded R-module with h-vector $H_M = (n, n)$.

If $\beta_{0,1}(M) > 0$ then M fails the WLP, so we can assume M is minimally generated by elements of degree 0.

In [FT] we prove that if M has the WLP then every set of minimal generators of M has a peculiar property.

Lemma 3.2.11 ([FT], Lemma 2.2)**.** *Let M be a graded R-module with a minimal system of generators*

$$\{e_1, \ldots, e_n\}$$

of degree 0 *and the h-vector* $HF_M = (n, n')$, *where* $n \le n'$. *If* M *has the WLP then there exists a linearly independent set in* M_1 *of the form* $\{z_1e_1, \ldots, z_ne_n\}$, *where* $z_i \in \{x, y\}$ *for* $1 \le i \le n$.

Let M be a graded R-module and let $\{e_1, \ldots, e_n\}$ a minimal system of generators of degree 0, such that

$$\{xe_1, \ldots, xe_r, ye_{r+1}, \ldots, ye_n\}$$

is a basis of M_1. Our aim is to give a procedure to verify if M has the WLP. The multiplication maps by the variables,

$$\times x : M_0 \to M_1, \qquad \times y : M_0 \to M_1$$

are morphisms between vector spaces with the same dimension. Let A and B be the matrices associated to the morphisms $\times x$ and $\times y$ respectively. Then we have

$$A = \left(\begin{array}{c|c} I_r & A' \\ \hline 0 & A'' \end{array}\right) \quad \text{and} \quad B = \left(\begin{array}{c|c} B' & 0 \\ \hline B'' & I_{n-r} \end{array}\right)$$

where I_r and I_{n-r} are the identity matrices of the sizes r and $n-r$, respectively, and 0 is the null matrix having the appropriate size.

It is clear that M has the WLP if and only if there exists $\alpha, \beta \in k$ such that

$$|\alpha A + \beta B| \ne 0.$$

Note that if $|A| \ne 0$ we can choose $\alpha = 1, \beta = 0$, similarly if $|B| \ne 0$, in these cases M has the WLP. Thereafter we can assume $|A| = |B| = 0$, so we only need to check the existence of $\alpha \ne 0$ and $\beta \ne 0$ such that $|\alpha A + \beta B| \ne 0$. We have:

$$\begin{aligned} |\alpha A + \beta B| &= \left| \left(\begin{array}{c|c} \alpha I_r + \beta B' & \alpha A' \\ \hline \beta B'' & \alpha A'' + \beta I_{n-r} \end{array}\right) \right| \\ &= \left| \left(\begin{array}{c|c} \frac{\alpha}{\beta} I_r + B' & A' \\ \hline B'' & A'' + \frac{\beta}{\alpha} I_{n-r} \end{array}\right) \right| \alpha^{n-r}\beta^r. \end{aligned}$$

Let $\gamma = \frac{\alpha}{\beta}$, then the determinant $|\alpha A + \beta B|$ is a polynomial of the form $\frac{1}{\gamma^d} p(\gamma)$ in $k[\gamma, \frac{1}{\gamma}]$, where $p(\gamma) \in k[\gamma]$. If $p(\gamma)$ is the zero polynomial then M does not have the WLP, otherwise there always exists $\tau \in k$ such that $p(\tau) \ne 0$. In this case M has the WLP with a Lefschetz element $\ell = \tau x + y$.

Example 3.2.12. Let $m_1 := x^6, m_2 := x^2y^4, m_3 := x^3y^3 \in R$, and let $M = ((m_1, m_2, m_3) + I)/I \subseteq R/I$ be a graded R-module, where $I = (x, y)^8 + (x^2y^5, x^4y^3)$ and the degrees are shifted by 6.

Observe that xm_1 and ym_1 are linearly independent and both do not belong in the k-vector space

$$\mathrm{Span}_k\{xm_2, xm_3, ym_2, ym_3\}.$$

Therefore, by Lemma 3.2.11 M fails the WLP. To show how the above procedure work, we change the basis of M. Let take

$$\begin{aligned} M_0 &= \mathrm{Span}_k\{(x^6 + x^2y^4) + I, (x^6 - x^2y^4) + I, (x^3y^3) + I\}, \\ M_1 &= \mathrm{Span}_k\{(x^7 + x^3y^4) + I, x^6y + I, x^3y^4 + I\}. \end{aligned}$$

Setting $e_1 = x^6 + I, e_2 = x^2y^4 + I, e_3 = x^3y^3 + I$, we get that $\{xe_1, ye_2, ye_3\}$ is a basis of M_1 which is of the form as in Lemma 3.2.11. Moreover, $ye_1 = xe_2$, $xe_2 = xe_1 - 2ye_3$ and $xe_3 = 0$. So, the matrices given by the maps $\times x$ and $\times y$ are:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By computing $\alpha A + \beta B$, and setting $\tau = \frac{\beta}{\alpha}$ we obtain the matrix:

$$\begin{pmatrix} \tau & 1 & 0 \\ 1 & \frac{1}{\tau} & 0 \\ 0 & -2 & \frac{1}{\tau} \end{pmatrix}$$

which has determinant equal to zero for all τ.

Chapter 4

Linear Quotients of WL Algebras

In this chapter, as made in [FRZ1], we study the Hilbert function and the graded Betti numbers for "generic" linear quotients of Artinian standard graded algebras, especially in the case of Weak Lefschetz algebras. Moreover, we investigate a particular property of Weak Lefschetz algebras, the Betti Weak Lefschetz Property, β-WLP for short.

As showed in the Chapter 3, we can study the WLP for an Artinian algebra A just looking at the *good behavior* of its generic quotient with respect to the Hilbert function, so A has the WLP if and only if $H_{A/\ell A} = \Delta H_A^+$ (see Proposition 3.1.4). The aim of this chapter is to extend this *good behavior* with respect to the graded Betti numbers. In this sense we generalize the notion of WLP to the β-WLP. Studying the analogies between the two property we will see that the β-WLP makes possible to completely determine the graded Betti numbers of a generic linear quotient of such algebras.

An Artinian WL algebra A has, in some sense, a "generic" linear quotient $A/\ell A$. So, it seems totally natural to study these algebras which arise as generic linear quotient of WL Artinian graded algebras and try to understand what they inherit from the starting algebra. Such an investigation is similar to what one does in Algebraic Geometry when one studies the generic hyperplane section of a projective variety. It is well known that if one starts from an arithmetically Cohen-Macaulay variety of dimension > 0 then the generic hyperplane section, done by a regular element, has the same graded Betti numbers of the starting variety and consequently its Hilbert function is just the first difference of the Hilbert function of the variety. In the case of the generic linear quotient of a WL Artinian graded algebra A the question is not so simple, since the form ℓ is no more a regular element. So, while it is easy to see that its Hilbert function is just the positive part of the first difference of the Hilbert function of A, the question is more tricky for the graded Betti numbers.

4.1 Linear quotients of Artinian algebras

Let $A = R/I$ be an Artinian standard graded R-algebra. Let $\ell \in R_1$ be a linear form and denote by $I_{[\ell]} = I + (\ell)$ the ideal sum in R. Our aim is to investigate the properties of the Artinian algebra $A_{[\ell]} = R/I_{[\ell]}$ when ℓ varies in R_1. Let us start with studying the Hilbert function $H_{[\ell]}$ of $A_{[\ell]}$. To do that we set

$$\mathcal{H}_A := \{H_{[\ell]} \mid \ell \in R_1\};$$

we will consider $\mathcal{H}_A$ as a poset with the following natural order: if $H, H' \in \mathcal{H}_A$, we say $H \leq H'$ iff $H(i) \leq H'(i)$ for every i.

Since we have a natural surjection $A \to A_{[\ell]}$ we see that $H_{[\ell]} \leq H_A$ for every $\ell \in R_1$. Moreover, since A is Artinian, $\mathcal{H}_A$ only have a finite number of elements. Now we define

$$S_{A,H} := \{[\ell] \in \mathbb{P}_k(R_1) \mid H_{[\ell]} = H\}.$$

So we have that set-theoretically

$$\mathbb{P}_k(R_1) = \bigcup_{H \in \mathcal{H}_A} S_{A,H}.$$

Observe that $\{S_{A,H}\}_H$ is a partition of $\mathbb{P}_k(R_1)$. Since $\mathcal{H}_A$ is a finite set there exists $\overline{\ell} \in R_1$ such that $S_{A,H_{\overline{\ell}}}$ contains a non empty open subset $U \subseteq \mathbb{P}_k(R_1)$ and there is only one element in $\mathcal{H}_A$ with such a property.

Definition 4.1.1. With the above notation we say that $A_{[\ell]}$ has the generic Hilbert function with respect to A iff $[\ell] \in S_{H_{\overline{\ell}}}$. In this case $H_{\overline{\ell}}$ will be called the Hilbert function of the generic linear section of A and will be denoted by H_A^{gen}.

Remark 4.1.2. Note that A has the WLP iff $H_A^{gen} = \Delta H_A^+$. Namely, when A has the WLP, H_A is unimodal and ΔH_A^+ is a O-sequence. Moreover the set of the Weak Lefschetz forms contains a non empty open subset of $\mathbb{P}_k(R_1)$.

More generally we have the following result.

Proposition 4.1.3. *Let $A = R/I$ be an Artinian standard graded R-algebra. The poset $\mathcal{H}_A$ has only one minimal element, precisely H_A^{gen}.*

Proof. Let i be an integer, $\ell \in R_1$ and let $\varphi_{\ell,i} : A_i \to A_{i+1}$ be the multiplication map by ℓ (as k-vector spaces). Let $r_i := \max\{\operatorname{rank} \varphi_{\ell,i} \mid \ell \in R_1\}$. Of course, the set $T_i := \{[\ell] \in \mathbb{P}_k(R_1) \mid \operatorname{rank} \varphi_{\ell,i} = r_i\}$ contains a non empty open subset of $\mathbb{P}_k(R_1)$. Since A is Artinian only a finite number of T_i is different from $\mathbb{P}_k(R_1)$ so the set $\bigcap_i T_i$ contains a non empty open subset of $\mathbb{P}_k(R_1)$. Let H_A^{min} the function defined by $H_A^{min}(i) := H_A(i) - r_{i-1}$. According to (3.1) for every $[\ell] \in \mathbb{P}_k(R_1)$ $H_{A_{[\ell]}} \geq H_A^{min}$. Moreover for every $[\ell] \in \bigcap_i T_i$, $H_{A_{[\ell]}} = H_A^{min}$ and $H_A^{min} = H_A^{gen}$. □

Because of previous discussions we set

$$S^{gen} = \{[\ell] \in \mathbb{P}_k(R_1) \mid H_{A_{[\ell]}} = H_A^{gen}\}.$$

Now, analogously to what we did previously, if $\ell \in R_1$ we define $\beta_\ell := \beta_{A_{[\ell]}}$

$$\mathcal{B}_A = \{\beta_\ell \mid [\ell] \in S^{gen}\}.$$

Note that $\mathcal{B}_A \subseteq \mathcal{B}_{H^{gen}}$.

Since every element in $\mathcal{B}_A$ is associated to the same Hilbert function H_A^{gen}, by a well known result by Bigatti, Hulett and Pardue (see [Bi], Theorem 4.1, [Hu], Theorem 2, [Pa], Theorem 31) the set $\mathcal{B}_A$ is finite. Now we set

$$Z_\beta := \{[\ell] \in S^{gen} \mid \beta_\ell = \beta\}.$$

Obviously $\{Z_\beta\}_{\beta \in \mathcal{B}_A}$ is a finite partition of S^{gen}. Consequently, as before, there exists $\overline{\ell} \in R_1$ such that $Z_{\beta_{\overline{\ell}}}$ contains a nonempty open subset of S^{gen}.

Definition 4.1.4. With the above notation we say that $A_{[\ell]}$ has the generic Betti sequence with respect to A iff $[\ell] \in Z_{\beta_{\overline{\ell}}}$. In this case $\beta_{\overline{\ell}}$ will be called the Betti sequence of the generic linear section of A and will be denoted by β^{gen}.

Proposition 4.1.5. *Let $A = R/I$ be an Artinian standard graded R-algebra. The poset $\mathcal{B}_A$ has only one minimal element, precisely β^{gen}.*

Proof. Let $[\ell] \in S^{gen}$; we want to show that $\beta^{gen} \leq \beta_\ell$. Since $[\ell]$ is in the closure of $Z_{\beta^{gen}}$ and both Betti sequences are associated to the same Hilbert function H^{gen}, we can apply Lemma 1.2 in [RZ1] to get our assertion. □

Example 4.1.6. Let $R = k[x, y, z]$ and

$$I = (x^4 + x^2z(y+z), x^3y - xyz(y+z), x^2y^2 + y^2z^2, z^5, y^5, xy^4) \subset R.$$

Then

$$H_{R/I} = (1, 3, 6, 10, 12, 9, 2, 1, 0)$$

and

$$\beta_{R/I} = ((4^3, 5^3), (6, 7^{10}, 9), (8^6, 10)).$$

The linear forms $\ell_0 = x + y + z$, $\ell_1 = y + z$ and $\ell_2 = z$ are WL forms for R/I; we have that

$$H_{R/I_{\ell_0}} = H_{R/I_{\ell_1}} = H_{R/I_{\ell_2}} = \Delta H^+_{R/I} = H^{gen} = (1, 2, 3, 4, 2, 0)$$

and

$$\beta_{\overline{A}_{[\ell_0]}} = ((4^3), (6^2));$$
$$\beta_{\overline{A}_{[\ell_1]}} = ((4^3, 5), (5, 6^2));$$
$$\beta_{\overline{A}_{[\ell_2]}} = ((4^3, 5^2), (5^2, 6^2)).$$

Consequently $\beta^{gen} = ((4^3), (6^2))$. Moreover one can see that $\mathcal{B}_A$ contains only these three Betti sequences.

4.2 Linear quotients of Artinian Weak Lefschetz algebras

In Section 4.1 we described as H^{gen}_A and β^{gen}_A have a similar behavior. In the same way as H^{gen}_A is the only minimal element in $\mathcal{H}_A$, β^{gen}_A is the only minimal element in $\mathcal{B}_A$. The lowest value that can be reached by $H_{[\ell]}$ is ΔH^+ and Remark 4.1.2 shows that when this happens A is a WL algebra.

In this section we will search which conditions should be required on β^{gen}_A to obtain an analogue situation. In order to do this we will study the Betti sequences of the linear quotients of Artinian standard graded algebras which have the Weak Lefschetz property.

4.2.1 Relationship between generators

We start with this general fact.

Lemma 4.2.1. *Let $A = R/I$ be a graded standard R-algebra. Let $f \in R_r$ such that the multiplication map $\times f : A_i \to A_{i+r}$ is injective. Let $\rho : R_1 \otimes I_{i+r-1} \to I_{i+r}$ be the natural map and V a k-vector subspace of I_{i+r}, such that $V \cap \operatorname{Im} \rho = 0$. If $J := I + (f)$ and if $\overline{\rho} : R_1 \otimes J_{i+r-1} \to J_{i+r}$ is the natural map, then $V \cap \operatorname{Im} \overline{\rho} = 0$.*

Proof. Let $g \in V \cap \operatorname{Im} \overline{\rho}$. Since $g \in \operatorname{Im} \overline{\rho}$, $g = g_1 + af$ where $g_1 \in \operatorname{Im} \rho$ and $a \in R_i$. Since $\times f$ is injective we get that $a \in I$, so $af \in \operatorname{Im} \rho$, consequently $g \in \operatorname{Im} \rho$ and by hypothesis we deduce that $g = 0$. $\square$

From now on, in this section, $A = R/I$ will be a Weak Lefschetz Artinian standard graded R-algebra and $\ell \in R_1$ a WL form for A. We want to study the graded Betti numbers, $\overline{\beta}_{ij}(\overline{A})$, of the algebra

$$\overline{A} := A/\ell A \cong \overline{R}/\overline{I} \cong R/(I + (\ell))$$

as a $\overline{R}$-algebra. Let H_A be the Hilbert function of A; since A has the Weak Lefschetz property, H_A is unimodal and ΔH_A^+ is a O-sequence. We set

$$t := \max\{j \mid \Delta H_A(j) > 0\}.$$

Let $J := I_{\leq t}$ be the ideal generated by the elements of I with degree less than or equal to t. We consider the following commutative diagram:

$$\begin{array}{ccc} \big(R/I\big)_t & \xrightarrow{\psi} & \big(R/J\big)_{t+1} \\ & \overset{\varphi}{\searrow} & \downarrow^{p} \\ & & \big(R/I\big)_{t+1} \end{array}$$

where the maps ψ and φ are both the multiplication by ℓ (hence $\varphi = \varphi_{\ell,t}$ and p is the natural map).

If $\mathcal{G} = \{g_1, \ldots, g_N\}$ is a minimal set of generators of I, we denote by $\mathcal{G}_{\leq j}$ the subset of $\mathcal{G}$ consisting of the elements which have degree less than or equal to j and by $\overline{\mathcal{G}}$ the set $\{\overline{g} \in \overline{I} | g \in \mathcal{G}\}$.

Remark 4.2.2. Since $\varphi_{\ell,i}$ is injective for $i < t$, using Lemma 4.2.1, we see that $\{\ell\} \cup \mathcal{G}_{\leq t}$ is a minimal set of generators for $(I_{[\ell]})_{\leq t}$. Hence $\overline{\mathcal{G}}_{\leq t}$ is a minimal set of generators for $\overline{I}_{\leq t}$. In particular this implies that $\beta_{0j}(A) = \overline{\beta}_{0j}(\overline{A})$ for $j \leq t$.

The Hilbert function of $\overline{A}$ vanishes in degrees $\geq t+1$, therefore every minimal set of generators of $\overline{I}$ is contained in $\overline{I}_{\leq t+1}$. So, to determine $\overline{\beta}_0(\overline{A})$, it is enough to compute $\overline{\beta}_{0\ t+1}(\overline{A})$. In Proposition 4.2.7 we will put into relation $\overline{\beta}_{0\ t+1}(\overline{A})$ with the map ψ.

Whenever an algebra A has a Weak Lefschetz form ℓ, one can study its homological properties just choosing a general change of variables such that $\ell = x_c$. In this case we will consider on $R = k[x_1, \ldots, x_{c-1}, \ell]$ the graded reverse lexicographic monomial order, with $x_1 > x_2 > \cdots > \ell$.

Using the above monomial order, many results holds.

Remark 4.2.3. If $f \in R$ and $\mathrm{lt}(f)$, the leading term of f, is divisible by ℓ, then f is divisible by ℓ too.

Let $\mathrm{LT}(I)$ the ideal generated by the leading terms of the polynomials in I, then, as showed in [Wi], we have

Proposition 4.2.4. *$R/\operatorname{LT}(I)$ is a Weak Lefschetz algebra and ℓ is a Weak Lefschetz form for it.*

Proof. See [Wi], Proposition 2.8. □

Remark 4.2.5. If $m \in \operatorname{LT}(I)_{\leq t}$ is a monomial and m is divisible by ℓ, then $\frac{m}{\ell} \in \operatorname{LT}(I)$. Namely, let $f \in I$ such that $m = \operatorname{lt}(f)$. Since ℓ divides m, by Remark 4.2.3, ℓ divides f; by the injectivity of $\varphi_{\ell,i}$ for $i < t$ we have that $\frac{f}{\ell} \in I$ so $\frac{m}{\ell} \in \operatorname{LT}(I)$.

Remark 4.2.6. $k[x_1, \dots, x_{c-1}]_{t+1} \subseteq \operatorname{LT}(I)$. Indeed, if there is

$$m \in k[x_1, \dots, x_{c-1}]_{t+1} \setminus \operatorname{LT}(I),$$

by the surjectivity of $\varphi_{\ell,t}$, there exists $u \in R_t$ such that $m - \ell u \in I$, so $m = \operatorname{lt}(m - \ell u) \in \operatorname{LT}(I)$ and this leads to a contradiction.

Proposition 4.2.7. *With the above notation $\overline{\beta}_{0\ t+1}(\overline{A}) = \dim_k(\operatorname{Coker}\psi)$.*

Proof. In order to prove our assertion we take a minimal set of generators $\mathcal{G}$ for I contained in a Gröbner basis. Let $\{g_1, \dots, g_r\}$ be the elements of $\mathcal{G}$ of degree $\leq t$ and $\{g_{r+1}, \dots, g_s\}$ the elements of $\mathcal{G}$ of degree $t+1$. Since φ is surjective, we can choose a k-basis of $(R/I)_{t+1}$, $\{f_1 + I, \dots, f_d + I\}$, in such a way that ℓ divides each f_i. On the other hand a k-basis of $(R/J)_{t+1}$ consists of

$$\{g_{r+1} + J, \dots, g_s + J\} \cup \{f_1 + J, \dots, f_d + J\}.$$

Now we can assume (just by re-ordering) that $g_{r+1}, \dots, g_{r'}$ are not divisible by ℓ and $g_{r'+1}, \dots, g_s$ are divisible by ℓ (with $r' \geq r$). To conclude the proof it will be enough to show that the set $\tilde{g}_{r+1}, \dots, \tilde{g}_{r'}$ is a k-basis of $\operatorname{Coker}\psi$ where $\tilde{g}_i = (g_i + J) + \operatorname{Im}\psi$.

Since $\{\tilde{g}_{r+1}, \dots, \tilde{g}_s, \tilde{f}_1, \dots, \tilde{f}_d\}$ generate $\operatorname{Coker}\psi$ and $g_{r'+1} + J, \dots, g_s + J, f_1 + J, \dots, f_d + J \in \operatorname{Im}\psi$, it follows that $\tilde{g}_{r+1}, \dots, \tilde{g}_{r'}$ is a set of generators for $\operatorname{Coker}\psi$. Now let us suppose that $\lambda_{r+1}\tilde{g}_{r+1} + \cdots + \lambda_{r'}\tilde{g}_{r'} = 0$, with each $\lambda_i \in k$. This implies that

$$\lambda_{r+1}(g_{r+1} + J) + \cdots + \lambda_{r'}(g_{r'} + J) \in \operatorname{Im}\psi$$

i.e. $\lambda_{r+1}g_{r+1} + \cdots + \lambda_{r'}g_{r'} + \ell h \in J$ for some $h \in R_t$. If $\lambda_{r+1}g_{r+1} + \cdots + \lambda_{r'}g_{r'} \neq 0$, then $\operatorname{lt}(\lambda_{r+1}g_{r+1} + \cdots + \lambda_{r'}g_{r'} + \ell h) = \operatorname{lt}(g_e) \in \operatorname{LT}(J) = \operatorname{LT}(g_1, \dots, g_r)$, for some e, $r + 1 \leq e \leq r'$. This is a contradiction since $\mathcal{G}$ is contained in a Gröbner basis. □

Theorem 4.2.8. *With the above notation*

$$\beta_{0\ t+1}(A) - \overline{\beta}_{0\ t+1}(\overline{A}) = \dim_k(\operatorname{Ker}\varphi) - \dim_k(\operatorname{Ker}\psi).$$

Proof. From the following exact sequence

$$0 \longrightarrow \operatorname{Ker}\psi \longrightarrow \left(R/I\right)_t \xrightarrow{\psi} \left(R/J\right)_{t+1} \longrightarrow \operatorname{Coker}\psi \longrightarrow 0.$$

we get

$$\dim_k(\operatorname{Ker}\psi) - \dim_k \left(R/I\right)_t + \dim_k \left(R/J\right)_{t+1} - \dim_k(\operatorname{Coker}\psi) = 0.$$

By Proposition 4.2.7 we have

$$\dim_k(\operatorname{Ker}\psi) - H_A(t) + \dim_k \left(R/J\right)_{t+1} - \overline{\beta}_{0\ t+1}(\overline{A}) = 0;$$

since $J = I_{\leq t}$ we get

$$\beta_{0\ t+1}(A) - \overline{\beta}_{0\ t+1}(\overline{A}) = H_A(t) - H_A(t+1) - \dim_k(\operatorname{Ker}\psi) =$$
$$= \dim_k(\operatorname{Ker}\varphi) - \dim_k(\operatorname{Ker}\psi),$$

using the surjectivity of the map φ. □

Corollary 4.2.9. *With the above notation*

i) ψ *is surjective iff* $\overline{\beta}_{0\ t+1}(\overline{A}) = 0$*;*

ii) ψ *is injective iff* $\overline{\beta}_{0\ t+1}(\overline{A}) = \beta_{0\ t+1}(A) + \Delta H_A(t+1)$.

Proof. Just using Proposition 4.2.7 and Theorem 4.2.8. □

The next target in this section will be to determine the graded Betti numbers of $\overline{A}$. It is important to note that, by a numerical point of view, this computation was made by several authors. See for instance Lemma 8.3 in [MN2], where, inter alia, authors determine the value of $\overline{\beta}_{i,j}(\overline{A})$ for $i + j < t$.

Now we assume a qualitative point of view and we study a minimal free resolution of $\overline{A}$. We will find a strong connection with the minimal free resolution of A, as we will see in Theorem 4.2.12.

4.2.2 Relationship between resolutions

Let us consider a graded minimal free resolution of A as a R-module

$$F_\bullet :\ 0 \to F_{c-1} \to \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to R \to A \to 0 \qquad (4.1)$$

and a graded minimal free resolution of $\overline{A}$ as a $\overline{R}$-module

$$G_\bullet :\ 0 \to G_{c-2} \to \cdots \to G_i \xrightarrow{d'_i} G_{i-1} \to \cdots \to G_0 \to \overline{R} \to \overline{A} \to 0. \qquad (4.2)$$

Let $\pi_\bullet : F_\bullet \to G_\bullet$ a lifting of the natural map of R-modules $\pi : A \to \overline{A}$.

In order to show the results in Theorem 4.2.12 we need some preliminary lemmas.

Lemma 4.2.10. *Let $d : F \to M$ be a map of graded R-modules, with F free R-module. Let $\{y_1, \ldots, y_r\}$ be a minimal set of homogeneous generators for* $\operatorname{Im} d$, $\deg y_i \leq \deg y_{i+1}$ *for $1 \leq i \leq r-1$. Let us suppose that there is a free basis $\{e_1, \ldots, e_r\}$ of F such that $de_i = y_i$ for $1 \leq i \leq r$. Let $z_1, \ldots, z_s \in F$ be homogeneous elements such that $dz_i = y_i$ for $1 \leq i \leq s$.*

Then $\{z_1, \ldots, z_s\}$ is a part of a free basis of F.

Proof. Let

$$z_j = \sum_{h=1}^{r} a_h e_h.$$

Notice that $a_h = 0$ when $\deg e_h > \deg z_j$ and $a_h \in k$ when $\deg e_h = \deg z_j$. Applying d we get

$$y_j = \sum_{h=1}^{r} a_h y_h.$$

By the minimality of $\{y_1, \ldots, y_r\}$ we have that $a_j = 1$ and $a_h = 0$ for $h \neq j$ and when $\deg e_h = \deg z_j$. Therefore

$$z_j = \sum_{\deg e_h < \deg z_j} a_h e_h + e_j.$$

Now the set $\{z_1, \ldots, z_s, e_{s+1}, \ldots, e_r\}$ is a free basis of F. Namely it can be obtained by $\{e_1, \ldots, e_r\}$ with a transformation whose matrix is a triangular matrix with 1's in the diagonal.

□

Lemma 4.2.11. *Let R and $\overline{R} = R/(\ell)$ be as above and consider the following commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{\ d\ } & M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tau} \\ G & \xrightarrow{\ d'\ } & N \end{array}$$

where F and M are R-modules, F R-free, G and N are $\overline{R}$-modules, G $\overline{R}$-free, d, π, τ are R-morphisms and d' is a $\overline{R}$-morphism.

Let $\{e_1, \ldots, e_r\}$ be a free basis of F such that $\{d(e_1), \ldots, d(e_r)\}$ is a minimal set of generators for $\operatorname{Im} d$ *and $\{\tau d(e_1), \ldots, \tau d(e_r)\}$ is part of a minimal set of generators for* $\operatorname{Im} d'$.

If $z \in \ker \pi$ then $z = \ell y$ for some $y \in F$.

Proof. First observe that, by the commutativity of the diagram

$$d'\pi(e_i) = \tau d(e_i,)$$

so, by Lemma 4.2.10, $\{\pi(e_1),\dots,\pi(e_r)\}$ are $\overline{R}$-linearly independent elements of G. Now since $z=\sum_{i=1}^r a_ie_i \in \ker\pi$, with $a_i\in R$, we have $\sum_{i=1}^r a_i\pi(e_i)=0$, hence $\sum_{i=1}^r \overline{a}_i\pi(e_i)=0$, where $\overline{a}_i$ is the image of a_i in the natural map $R\to\overline{R}$. This implies $\overline{a}_i=0$ for all i, i.e. $a_i\in(\ell)$, so $z=\ell y$. □

We are now ready to analyze the minimal free resolution of $\overline{A}$

Theorem 4.2.12. *With the above notation, for every $i\geq 0$, let*

$$\{\gamma_{i1},\dots,\gamma_{i\beta_i}\},\quad \deg\gamma_{i1}\leq\ldots\leq\deg\gamma_{i\beta_i},$$

be a minimal set of generators for $\operatorname{Im} d_i$, and $u_i:=|\{j\mid \deg\gamma_{ij}\leq t+i\}|$. If $u_i>0$ then $\{\pi_{i-1}(\gamma_{i1}),\dots,\pi_{i-1}(\gamma_{iu_i})\}$ can be completed to a minimal set of generators for $\operatorname{Im} d_i'$ with elements of degree $\geq t+i$.

Proof. For the case $i=0$ see Remark 4.2.2.

Let $\gamma_{i1},\dots,\gamma_{iu_i}$ be as in the hypotheses and assume that there is γ_{ij}, $1\leq j\leq u_i$ such that $\pi_{i-1}(\gamma_{ij})=\sum_{h\neq j}\overline{\mu_h}\pi_{i-1}(\gamma_{ih})$, $\overline{\mu_h}\in\overline{R}$. Then $\gamma:=\gamma_{ij}-\sum_{h\neq j}\mu_h\gamma_{ih}\in\operatorname{Ker}\pi_{i-1}$. Now let $(e_1,\dots,e_{\beta_{i-1}})$ be a basis of F_{i-1} such that $d_{i-1}(e_h)=\gamma_{i-1\,h}$, for $1\leq h\leq\beta_{i-1}$. Since $\gamma\in F_{i-1}$ it can be written as $\gamma=\sum_{h=1}^{\beta_{i-1}}a_he_h$. Note that since $u_i>0$ then $u_h>0$ for $0\leq h\leq i$. On the other hand $\gamma\in\operatorname{Ker} d_{i-1}$ and $\deg\gamma\leq t+i$, therefore $a_h=0$ when $\deg e_h\geq t+i$, so $\gamma=\sum_{h=1}^{u_{i-1}}a_he_h$, (notice that the number of e_h of degree $\leq t+i-1$ is u_{i-1}). By induction $\{\pi_{i-2}(\gamma_{i-1\,1}),\dots,\pi_{i-2}(\gamma_{i-1u_{i-1}})\}$ can be completed to a minimal set of generators for $\operatorname{Im} d_{i-1}'$ with elements of degree $\geq t+i-1$. Now

$$d_{i-1}'\pi_{i-1}(e_h)=\pi_{i-2}d_{i-1}(e_h)=\pi_{i-2}(\gamma_{i-1\,h}),\ \text{for}\ 1\leq h\leq\beta_{i-1};$$

so we can apply Lemma 4.2.11 to get $\gamma=\ell\gamma'$, $\gamma'\in\operatorname{Ker} d_{i-1}$. Hence $\gamma_{ij}=\sum_{h\neq j}\mu_h\gamma_{ih}+\ell\gamma'$; this contradicts the minimality of the set $\{\gamma_{i1},\dots,\gamma_{i\beta_i}\}$.

In order to conclude the proof it is enough to show that each element in $\operatorname{Im} d_i'$ of degree $\leq t+i-1$ is in the submodule generated by

$$\pi_{i-1}(\gamma_{i1}),\dots,\pi_{i-1}(\gamma_{iu_i}).$$

Now, let $\delta\in\operatorname{Im} d_i'$ be an element of degree $\leq t+i-1$. By Lemma 4.2.10, the elements $\pi_{i-1}(e_1),\dots,\pi_{i-1}(e_{u_{i-1}})$ can be completed to a basis of G_{i-1} with elements of degree $\geq t+i$. Therefore $\delta=\sum_{h=1}^{u_{i-1}}b_h\pi_{i-1}(e_h)$, so $\delta=\pi_{i-1}(\xi)$, for some ξ. Consequently $d_{i-1}(\xi)\in\operatorname{Ker}\pi_{i-2}$; using again Lemma 4.2.11 one finds $d_{i-1}(\xi)=\ell\xi'$, with $\xi'\in\operatorname{Ker} d_{i-2}$ and $\deg\xi'\leq t+i-2$, therefore $d_{i-1}(\xi)=\ell d_{i-1}(\eta)$, for some η, so $\xi-\ell\eta\in\operatorname{Im} d_i$, hence $\xi-\ell\eta=\sum_{h=1}^{u_i}c_h\gamma_{ih}$; applying π_{i-1} to both sides of the previous equality we get $\delta=\sum_{h=1}^{u_i}c_h\pi_{i-1}(\gamma_{ih})$. □

Thanks to Theorem 4.2.12 we can give a partial structure to the minimal free resolution $G_\bullet$ of $\overline{A}$. To do that we decompose each F_i and each G_i in (4.1) and (4.2) in this way

$$F_i = F_i' \oplus F_i'' \text{ and } G_i = G_i' \oplus G_i''$$

where $F_i' := (e_1, \ldots, e_{u_i})$ (therefore F_i'' is generated in degree $> t+i$); $G_i' := (\pi_i(e_1), \ldots, \pi_i(e_{u_i}))$ and G_i'' is generated only in degree $t+i$ and $t+i+1$ (this is due to the fact that the degree of the last syzygy is $t+c-1$ and $G_\bullet$ is a minimal graded free resolution of a Cohen-Macauly ring).

Corollary 4.2.13. *Let us consider the commutative diagram*

$$\begin{array}{ccc} F_i' \oplus F_i'' & \xrightarrow{d_i} & F_{i-1}' \oplus F_{i-1}'' \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_{i-1}} \\ G_i' \oplus G_i'' & \xrightarrow{d_i'} & G_{i-1}' \oplus G_{i-1}''. \end{array}$$

With a suitable choice of the free bases, if

$$\begin{pmatrix} M & N_1 \\ 0 & N_2 \end{pmatrix}$$

is a matrix representing d_i, where $M = (m_{hk})$ is a matrix of size $u_{i-1} \times u_i$, then

$$\begin{pmatrix} \pi(M) & P_1 \\ 0 & P_2 \end{pmatrix}$$

is a matrix associated to d_i', where $\pi(M) = (\pi(m_{hk}))$.

Proof. This is a direct consequence of Theorem 4.2.12. □

Theorem 4.2.12 give a description of the graded Betti numbers of $\overline{A}$.

Corollary 4.2.14.

$$\overline{\beta}_{ih}(\overline{A}) = \beta_{ih}(A), \text{ for } i \geq 0 \text{ and } h \leq t+i-1.$$

Moreover $\overline{\beta}_{i\ t+i}(\overline{A}) \geq \beta_{i\ t+i}(A)$.

Proof. It follows immediately by Theorem 4.2.12. □

By Theorem 4.2.12 we can easily deduce also a property on the last graded Betti numbers of the Weak Lefschetz algebras.

Corollary 4.2.15. $\beta_{c-1\ j}(A) = 0$ *for all* $j \leq t+c-1$.

Proof. Just apply Corollary 4.2.14, to the case $i = c - 1$. □

The result in Corollary 4.2.14 can be clarified in the case $i = 1$.

Corollary 4.2.16. $\beta_{1\ t+1}(A) + \dim_k(\operatorname{Ker}\psi) = \overline{\beta}_{1\ t+1}(\overline{A})$.

Proof. Since $\Delta H_A(i) = H_{\overline{A}}(i)$, for $i \leq t$ we have

$$\Delta^c H_A(t+1) - \Delta^{c-1} H_{\overline{A}}(t+1) = \Delta H_A(t+1).$$

Then

$$\sum_{i=0}^{c-1}(-1)^i \beta_{i\ t+1}(A) = \sum_{i=0}^{c-2}(-1)^i \overline{\beta}_{i\ t+1}(\overline{A}) - \Delta H_A(t+1).$$

By Corollary 4.2.14 we know that $\beta_{i\ t+1}(A) = \overline{\beta}_{i\ t+1}(\overline{A})$ for every $i \geq 2$; now applying Proposition 4.2.8 and using the fact that φ is surjective, we get our conclusion. □

In Corollary 4.2.14 we gave an inequality for the i-th graded Betti numbers of A and $\overline{A}$ in degree $t + i$. Now we will give a characterization when the equality happens.

Proposition 4.2.17. *$\beta_{i\ t+i}(A) = \overline{\beta}_{i\ t+i}(\overline{A})$ for every $i \geq 0$ iff ψ is injective.*

Proof. If $\beta_{i\ t+i}(A) = \overline{\beta}_{i\ t+i}(\overline{A})$ for every i then $\beta_{1\ t+1}(A) = \overline{\beta}_{1\ t+1}(\overline{A})$, hence by Corollary 4.2.16 we get the injectivity of ψ.

Now let us suppose that ψ is injective. For $i = 0$ see Remark 4.2.2. For $i = 1$ see Corollary 4.2.16. Let $i > 1$; it is enough to show that for every $\sigma \in \operatorname{Ker} d'_{i-1}$, with $\deg\sigma \leq t+i$, there exists $\rho \in \operatorname{Ker} d_{i-1}$ such that $\pi_{i-1}(\rho) = \sigma$. Since $\sigma \in \operatorname{Ker} d'_{i-1}$, it belongs to the submodule of G_{i-1} generated by the elements of degree $\leq t+i-1$. So, by the inductive hypothesis and by Lemma 4.2.10, calling $D = \{j \mid \deg e_j < t+i\}$, we can write

$$\sigma = \sum_{j\in D} a_j \pi_{i-1}(e_j).$$

Applying d'_{i-1} we have

$$0 = d'_{i-1}(\sum_{j\in D} a_j \pi_{i-1}(e_j)) = \pi_{i-2} d_{i-1}(\sum_{j\in D} a_j e_j) = \pi_{i-2}(\sum_{j\in D} a_j \gamma_{i-1 j})$$

consequently

$$\sum_{j\in D} a_j \gamma_{i-1 j} \in \ker \pi_{i-2} \cap \ker d_{i-2}$$

so by Lemma 4.2.11

$$\sum_{j \in D} a_j \gamma_{i-1j} = \ell\eta,$$

$\ell\eta \in \ker d_{i-2} = \operatorname{Im} d_{i-1}$. Since $\ell\eta \in \ker d_{i-2}$ we have $\eta \in \ker d_{i-2} = \operatorname{Im} d_{i-1}$ i.e. $\eta = d_{i-1}(\gamma)$, for some γ. Now we set $\rho := \sum_{j \in D} a_j e_j - \ell\gamma \in F_{i-1}$ hence $\pi_{i-1}(\rho) = \sigma$ and $d_{i-1}(\rho) = 0$. □

4.3 The Betti Weak Lefschetz Property

Collecting some results of the previous section we can give a description of the graded Betti numbers of $\overline{A}$.

$$\overline{\beta}_{ij}(\overline{A}) = \begin{cases} \beta_{ij}(A) & \text{if } j \leq t+i-1 \\ \beta_{ij}(A) + m_i & \text{if } j = t+i \\ \sum_{h \geq i+1} (-1)^{h+i+1} \beta_{h\ j}(A) + (-1)^{i+1} \Delta^{c-1} \Delta H^{+}_{A}(j) + m_{i+1} & \text{if } j = t+i+1 \\ 0 & \text{if } j > t+i+1 \end{cases} \tag{4.3}$$

where $m_i \geq 0$ and in particular $m_0 = 0$ and $m_1 = \dim_k \ker \psi$.

Indeed, this follows by Corollary 4.2.14 and by the following computation.

$$-\Delta^{c-1} H_{\overline{A}}(t+i+1) = \sum_{h \geq 0} (-1)^h \overline{\beta}_{h\ t+i+1}(\overline{A})$$

then

$$-\Delta^{c-1} H_{\overline{A}}(t+i+1) =$$

$$= (-1)^i \overline{\beta}_{i\ t+i+1}(\overline{A}) + (-1)^{i+1} \overline{\beta}_{i+1\ t+i+1}(\overline{A}) + \sum_{h \geq i+2} (-1)^h \overline{\beta}_{h\ t+i+1}(\overline{A}) =$$

$$= (-1)^i \overline{\beta}_{i\ t+i+1}(\overline{A}) + (-1)^{i+1} (\beta_{i+1\ t+i+1}(A) + m_{i+1}) + \sum_{h \geq i+2} (-1)^h \beta_{h\ t+i+1}(A),$$

so by multiplying by $(-1)^i$ we get $\overline{\beta}_{i\ t+i+1}(\overline{A}) =$

$$= m_{i+1} + (-1)^{i+1} \Delta^{c-1} H_{\overline{A}}(t+i+1) + \sum_{h \geq i+1} (-1)^{h+i+1} \beta_{h\ t+i+1}(A).$$

Note that the first and the last parts of the description (4.3), can be deduced by Proposition 3.13 in [HMNW] which is in some sense dual to the formula (4.3).

Remark 4.3.1. If $c = 3$ the graded Betti numbers of $\overline{A}$ are determined by $\dim_k \operatorname{Ker} \psi$.

The Weak Lefschetz property for an Artinian standard graded algebra A induces a natural relationship between its Hilbert function and the Hilbert function of the generic linear quotient of A. The previous results suggest to study a property which preserves a *good behavior* also for the graded Betti numbers.

In the sequel, if I is an ideal of R and $\ell \in R_1$ is a Weak Lefschetz form for R/I, we will denote by ψ_ℓ the map $\psi_\ell : \big(R/I\big)_t \to \big(R/(I_{\leq t})\big)_{t+1}$ defined by multiplication by ℓ (t as defined at the beginning of Section 3).

Definition 4.3.2. We say that $A = R/I$ has the *Betti Weak Lefschetz Property*, briefly β-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ is injective.

A linear form ℓ satisfying the conditions as above will be said a β-WL form. In the example 4.1.6 the linear form ℓ_0 is a β-WL form (hence R/I is a β-WL algebra) and the linear forms ℓ_1 and ℓ_2 are WL forms but they are not β-WL forms.

An equivalent version of this definition can be given looking at the graded Betti numbers of the algebra

Proposition 4.3.3. *Let A be a standard graded R-algebra. The following are equivalent*

i) A has the β-WLP and ℓ is a β-WL form;

ii) The graded Betti numbers of $A/\ell A$ are determined by (4.3) with $m_i = 0$ for every i.

Proof. It follow using Proposition 4.2.17 and equations (4.3). □

Note that there are algebras with the WLP having trivially β-WLP. For instance, if $A = R/I$ has the WLP and I is generated in degree $\geq t+1$, then A has the β-WLP. (It is enough to observe that $I_{\leq t} = (0)$).

Next results put into relation the β-WLP with some particular WL algebras.

We recall that a sequence H is said to be a Weak Lefschetz sequence if it occurs as the Hilbert function of some standard graded Artinian algebra with the Weak Lefschetz property.

It is known from [HMNW], Theorem 3.20, that if H is a Weak Lefschetz sequence then the set

$$\mathcal{B}_H^{\mathrm{WL}} = \{\beta_A \mid H_A = H \text{ and } A \text{ has the WLP}\}$$

admits exactly one maximal element, say, β^H.

Proposition 4.3.4. *Let H be a Weak Lefschetz sequence and let $A = R/I$ be an Artinian algebra with $H_A = H$ such that A has the WLP. If $\beta_{0\ t+1}(A) = \beta^H_{0\ t+1}$ then A has the β-WLP.*

Proof. Let ℓ be a WL form for A, from Theorem 3.20 in [HMNW] it follows that

$$\beta^H_{0\ t+1} = \hat{\beta}_{0\ t+1} - \Delta H_A(t+1),$$

where $\hat{\beta}$ is the Betti sequence of the lex-segment ideal $L \subset \overline{R} = R/(\ell)$ such that $H_{\overline{R}/L} = \Delta H^+$. By Theorem 4.2.8 we have

$$\beta_{0\ t+1}(A) = \overline{\beta}_{0\ t+1}(\overline{A}) - \Delta H_A(t+1) - \dim_k \operatorname{Ker} \psi_\ell.$$

So

$$\overline{\beta}_{0\ t+1}(\overline{A}) - \dim_k \operatorname{Ker} \psi_\ell = \hat{\beta}_{0t+1},$$

and, using the maximality of $\hat{\beta}$, we get $\operatorname{Ker} \psi_\ell = 0$. □

Corollary 4.3.5. *Let $A = R/I$ be an Artinian algebra with the WLP such that $\beta_A = \beta^{H_A}$ then A has the β-WLP.*

It is known that if H is the Hilbert function of an Artinian Gorenstein standard graded R-algebra of codimension 3 and $\vartheta - 3$ is its socle degree then the set of the Gorenstein Betti sequences compatible with H

$$\mathcal{G}_H = \{\beta_A \mid H_A = H \text{ and } A \text{ is a Gorenstein Algebra}\}$$

has only one maximal element β^{max} and only one minimal element β^{min} (see [RZ4] Proposition 3.7 and Remark 3.8). According to the paper [RZ2] there exists a Gorenstein Betti sequence $\gamma^H \in \mathcal{G}_H$, such that every Artinian Gorenstein standard graded R-algebra with Betti sequence $\geq \gamma^H$ has the WLP (see Corollary 2.7 in [RZ2]). We recall that

$$\gamma^H_{0i} = \begin{cases} \beta^{\max}_{0i} & \text{for } i = t+1, \vartheta - t - 1 \\ \beta^{\min}_{0i} & \text{otherwise} \end{cases}.$$

Actually in the next proposition we can improve that result.

Proposition 4.3.6. *Let H be the Hilbert function of an Artinian Gorenstein standard graded R-algebra of codimension 3. Then every R-algebra A with Betti sequence $\beta_A \in \mathcal{G}_H$ and $\beta_A \geq \gamma^H$ has the β-WLP.*

Proof. Let $A = R/I$ be an Artinian Gorenstein standard graded R-algebra of codimension 3 such that $\beta_A \geq \gamma^H$. Repeating the same arguments in Theorem 2.5 in citeRZ2 $I_{\leq t} = fI'$ where I' is a perfect ideal of height ≤ 2, and f some form in R. So if we take ℓ to be a linear regular form in R/I', such that ℓ does not divide f, then ℓ is a β-WL form. □

4.4 Failing the β-WLP

In this section we make a tour of some algebras which fail the β-WLP.

Let $A = R/I$ be a complete intersection Artinian standard k-algebra which have the Weak Lefschetz property. Let $I = (g_1, \dots, g_c)$, and $\deg g_i \leq \deg g_{i+1}$ for $1 \leq i \leq c-1$. For such an algebra it is easy to study the behavior with respect to the β-WLP defined in the previous section.

Proposition 4.4.1. *Let A be as above then*

1) If $\deg g_c > t$ then A has the β-WLP.

2) If $\deg g_c \leq t$ and $\Delta H_A(t+1) = 0$ then A has the β-WLP.

3) If $\deg g_c \leq t$ and $\Delta H_A(t+1) \neq 0$ then A has not the β-WLP.

Proof. If $\deg g_c > t$ then $I_{\leq t}$ is generated by a regular sequence of length $< c$, hence $\operatorname{depth} R/(I_{\leq t}) > 0$, so there is a linear form ℓ which is regular for $R/(I_{\leq t})$, therefore ψ_ℓ is injective.

If $\deg g_c \leq t$ then $I_{\leq t} = I$, therefore $\psi_\ell = \varphi_{\ell,t}$ for every $\ell \in R_1$, so the conclusions of items 2 and 3 follow by the Weak Lefschetz property of A. □

The item 3 of the previous proposition in particular says that ψ_ℓ is not injective but still it has maximal rank. This suggests to give a weaker form of the Definition 4.3.2.

4.4.1 The β_0-WLP

Definition 4.4.2. We say that $A = R/I$ has the *generators Weak Lefschetz Property*, briefly β_0-WLP, if there exists $\ell \in R_1$ such that

1. ℓ is a Weak Lefschetz form for A;
2. ψ_ℓ has maximal rank.

Note that for complete intersection algebras the WLP and the β_0-WLP are equivalent.

Of course, not every β_0-WL Artinian algebras are β-WL algebras. Just take a WL complete intersection algebra whose generators have degree $\leq t$ and $\Delta H_A(t+1) \neq 0$ (as in the item 3 in Proposition 4.4.1).

Remark 4.4.3. Note that the graded Betti numbers of a β-WL algebra A determine the graded Betti numbers of $A/(\ell)$, for a generic $\ell \in R_1$. Analogously when A is a β_0-WL algebra, $\beta_0(A)$ determines $\overline{\beta}_0(A/(\ell))$, for a generic $\ell \in R_1$. Precisely $\overline{\beta}_{0j}(A/\ell) = \beta_{0j}(A)$ for $j \leq t$ and $\overline{\beta}_{0j}(A/\ell) = 0$, otherwise (see Corollary 4.2.9).

4.4.2 Examples

Next two examples show that there are Weak Lefschetz Artinian algebras which fail the β_0-WLP.

Example 4.4.4. Let $R = k[x, y, z]$, and

$$I = (x^4, x^2y^2, xy^3, xz^3, y^5, z^5).$$

Then

$$H_{R/I} = (1, 3, 6, 10, 11, 8, 4, 2, 1, 0).$$

In this case $t = 4$ and

$$J := I_{\leq 4} = (x^4, x^2y^2, xy^3, xz^3).$$

One can check that $\ell = x + y + z$ is a WL form for R/I. On the other hand the Hilbert function of R/J is

$$H_{R/J} = (1, 3, 6, 10, 11, 10, 8, 8, 9, \ldots)$$

and $H_{R/J}(n) = n+1$ for $n \geq 7$. If $\psi_\ell : (R/J)_4 \to (R/J)_5$ were surjective then the multiplication by ℓ should be surjective also in the successive degrees. This is clearly impossible in degrees ≥ 7.

Example 4.4.5. Let $R = k[x, y, z]$, and

$$I = (x^5, x^3y^2, x^4z, y^6, z^6, y^3z^3).$$

So we have

$$H_{R/I} = (1, 3, 6, 10, 15, 18, 17, 14, 8, 3, 0, \ldots).$$

In this case $t = 5$ and $J := I_{\leq 5} = (x^5, x^3y^2, x^4z)$. One can check that $\ell = x + y + z$ is a WL form for R/I. On the other hand

$$H_{R/J}(5) = 18 < 20 = H_{R/J}(6).$$

For every linear form ℓ we have $\ell x^4y \in J$ and $x^4y \notin J$. So ψ_ℓ cannot be injective.

In the next example we have a β_0-WL algebra A without the β-WLP for which not all WL forms are β_0-WL forms.

Example 4.4.6. Let $R = k[x, y, z]$, and

$$I = (x^4, x^3y, x^2y^2 - xz^3, xy^3 + yz^3, y^5 + z^5).$$

So the Hilbert function of $A = R/I$ is

$$H_A = (1, 3, 6, 10, 11, 9, 5, 2, 0, \ldots)$$

and the graded Betti numbers are

$$\beta_A = ((4^4, 5), (5, 6, 7^3, 8, 9^2), (8, 9, 10^2)).$$

The linear forms $\ell_0 = z$ and $\ell_1 = x + y + z$ are both WL forms since one can check that

$$H_{\overline{A}_{[\ell_0]}} = H_{\overline{A}_{[\ell_1]}} = (1, 2, 3, 4, 1, 0, \ldots).$$

Computing the Betti sequences, we have

$$\beta_{\overline{A}_{[\ell_0]}} = ((4^4, 5), (5^3, 6)),$$

and

$$\beta_{\overline{A}_{[\ell_1]}} = ((4^4), (5^2, 6)).$$

Thus ℓ_0 is not a β_0-WL form since $\overline{\beta}_{05}(\overline{A}_{[\ell_0]}) \neq 0$. ℓ_1 is a β_0-WL form for A and A is not β-WL algebra since $\dim_k(R/J)_4 = 11 > \dim_k(R/J)_5 = 10$.

Next example shows that the β_0-WLP does not determine the graded Betti numbers of the its generic linear quotient.

Example 4.4.7. $R = k[x, y, z, w]$

$$I = (x, y)^5 + (x, z)^5 + (y, z)^5 + (x^2y^2z + w^5),$$

$$H_{R/I} = (1, 4, 10, 20, 35, 40, 38, 32, 22, 7, 1, 0, \ldots)$$

$\ell_0 = w$ and $\ell_1 = x + y + z + w$ are both β_0-WL forms, in fact

$$H_{\overline{A}_{[\ell_0]}} = H_{\overline{A}_{[\ell_1]}} = (1, 3, 6, 10, 15, 5, 0, \ldots).$$

Computing the Betti sequences, we have that

$$\beta_{\overline{A}_{[\ell_0]}} = ((5^{16}), (6^{20}, 7^4), (7^4, 8^5))$$

whereas

$$\beta_{\overline{A}_{[\ell_1]}} = ((5^{16}), (6^{20}, 7^3), (7^3, 8^5)).$$

Next examples show as we can have a β-WL algebra R/I even if $R/I_{\leq t}$ is already Artinian.

Example 4.4.8. Let $R = k[x, y, z]$, and

$$I = (x^5, y^5, z^5, x^3y^3, y^3z^3).$$

Then

$$H_{R/I} = (1, 3, 6, 10, 15, 18, 17, 12, 7, 3, 1, 0).$$

So $t = 5$, $I_{\leq 5} = J := (x^5, y^5, z^5)$ and $\dim_k(R/J)_5 = 18$, $\dim_k(R/J)_6 = 19$.

Now $x + y + z$ is a WL-form for J so ψ_{x+y+z} is injective.

Example 4.4.9. Let $R = k[x, y, z, w]$, and

$$I = (x^2, y^4, z^4, w^4, xy^2z^2, xy^2w^2, xz^2w^2).$$

Then

$$H_{R/I} = (1, 4, 9, 16, 22, 21, 13, 6, 3, 1, 0)$$

So $t = 4$ e $J = (x^2, y^4, z^4, w^4)$ and $\dim_k(R/J)_6 = 24$. $\psi_{x+y+z+w}$ is injective because $x+y+z+w$ is a Lefschetz Element for J, R/J is an Artinian algebra since J is complete intersection algebra.

Next example shows that although for a β-WL algebra A the graded Betti numbers of its generic linear quotient $\overline{A}$ are determined, we can find in such a sequence ghost terms which were not in β_A.

Example 4.4.10. Let $R = k[x, y, z, w]$, and

$$I = (x^3, x^2y, x^2z, w^3, (x + y + z + w)^4, (x - y + z + w)^4, (x + y - z + w)^4,$$

$$(x - y - z + w)^4, (x + y + 2z + w)^4, (x + 2y + z + w)^4);$$

Then the Hilbert function of $A = R/I$ is

$$H_A = (1, 4, 10, 16, 16, 3, 0, \ldots).$$

Observe that $t = 3$. If we take the weak Lefschetz form $\ell = x + 3y + z + 2w$, we have

$$H_{\overline{A}} = (1, 3, 6, 6, 0, \ldots).$$

Looking at the Betti sequences of both algebras

$$\beta_{R/I} = ((3^4, 4^6), (4^3, 6^{30}), (5, 7^{30}, 8^2), (8^6, 9^3))$$

$$\beta_{\overline{R}/\overline{I}} = ((3^4, 4^6), (4^3, 5^{13}), (5, 6^6))$$

we find a ghost term for $\overline{A}$ which is not so for A. From $\beta_{14}(A) = \overline{\beta}_{14}(\overline{A})$, it follows that ℓ is a β-WL form for A.

In next example we have a β_0-WL algebra A without the β-WLP, and not all WL forms are β_0-WL forms.

Example 4.4.11. Let $R = k[x, y, z]$, and

$$I = (x^4, x^3y, x^2y^2 - xz^3, xy^3 + yz^3, y^5 + z^5).$$

Then

$$H_{R/I} = (1, 3, 6, 10, 11, 9, 5, 2, 0, \ldots),$$

and

$$\beta_A = ((4^4, 5), (5, 6, 7^3, 8, 9^2), (8, 9, 10^2)).$$

$\ell_0 = z$ and $\ell_1 = x + y + z$ are both WL forms, in fact

$$H_{\overline{A}_{[\ell_0]}} = H_{\overline{A}_{[\ell_1]}} = (1, 2, 3, 4, 1, 0, \ldots).$$

Computing the Betti sequences, we have that

$$\beta_{\overline{A}_{[\ell_0]}} = ((4^4, 5), (5^3, 6))$$

and ℓ_1 is a β_0-WL form for A

$$\beta_{\overline{A}_{[\ell_1]}} = ((4^4), (5^2, 6)).$$

Bibliography

[AB] M. Auslander, D. A. Buchsbaum. *Homological dimension in local rings.* Transactions of the American Mathematical Society 85, no. **2** (1957): 390-405.

[BH] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, volume **39**, 1996.

[Bi] A. M. Bigatti, *Upper Bounds For The Betti Numbers Of A Given Hilbert Function*, Comm. Algebra 21 (1993), no. 7, 2317-2334.

[BI] D. Bernstein, A. Iarrobino, *A nonunimodal graded Gorenstein Artin algebra in codimension five*, Comm. in Algebra 20 (1992), No. 8, 2323-2336.

[BK] H. Brenner, A. Kaid, *Syzygy bundles on $\mathbb{P}^2$ and the Weak Lefschetz property*, Ill. J. Math. **51**(4) (2007), 1299-1308.

[BL] M. Boij, D. Laksov, *Nonunimodality of graded Gorenstein Artin algebras*, Proc. Amer. Math. Soc. 120 (1994), 1083-1092.

[BMMNZ] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel, F. Zanello, *On the Weak Lefschetz Property for Artinian Gorenstein algebras of codimension three*, preprint arXiv:1302.5742 [math.AC].

[Bo] M. Boij, *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys*, Comm. in Algebra 23 (1995), No. 1, 97-103.

[BS] D. Bayer, M. Stillman, *A criterion for detecting m-regularity.* Invent. Math., **87**, 1-11 (1987)

[Bu] L. Burch, *On ideals of finite homological dimension in local rings*, Proc. Cambridge Philos. , (1968) Soc. 64: 941948

[Ca] G. Campanella, *Standard bases of perfect homogeneous polynomial ideals of height 2*, J. of Alg. **101**(1) (1986), 47–60.

[CI] Y.H. Cho and A. Iarrobino, *Inverse systems of zero dimensional schemes in P^n*, J. of Algebra, Volume 366, 15 September 2012, Pages 42-77.

[CN] D. Cook II and U. Nagel, *The weak Lefschetz property, monomial ideals, and lozenges*. Illinois J. Math 55.1 (2012): 377-395.

[CN1] D. Cook II and U. Nagel, *Enumerations deciding the weak Lefschetz property*, arXiv preprint arXiv:1105.6062 (2011).

[Di] S. Diesel, *Irreducibility and dimension theorems for families of height 3 Gorenstein algebras*, Pac. J. of Math. **172**(4) (1996), 365-397.

[Ei1] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Berlin, New York: Springer-Verlag.

[Ei2] D. Eisenbud, *The Geometry of Syzygies: A Second Course in Algebraic Geometry and Commutative Algebra*. Vol. **229**. Springer, 2005.

[FRZ1] G. Favacchio, A. Ragusa, G. Zappalà, *Linear quotients of Artinian Weak Lefschetz algebras.* Journal of Pure and Applied Algebra Volume **217**, Issue 10, October 2013, Pages 1955-1966.

[FRZ2] G. Favacchio, A. Ragusa, G. Zappalà, *Tower sets and other general configurations with the Cohen-Macaulay property.* Preprint.

[FT] G. Favacchio, P.D. Thieu, *On the weak Lefschetz property of graded modules over $K[x, y]$* Le Matematiche **67**(1) (2012), 223-235.

[Ga] A. Galligo, *Apropos du theoreme de preparation de Weierstrass.* In: Fonc- tions de plusiers variables complexes. Lect. Notes Math., **409**. Springer (1974).

[GHM] A. V. Geramita, B. Harbourne, and J. Migliore. *Star configurations in Pn.* Journal of Algebra **376** (2013): 279-299.

[GHS] A. V. Geramita, T.Y. Harima,, S. Shin, *Extremal Point Sets and Gorenstein Ideals.* Adv. in Math. **152**(1) (2000), 78-119.

[Ha] R. Hartshorne, *Connectedness of the Hilbert scheme.* Publ. IHES, **29**, 548 (1966).

[HH] J. Herzog, T. Hibi - *Monomial Ideals*, Graduate Texts in Mathematics Volume **260**, 2011, pp 3-22.

[Hi] D. Hilbert, *Ueber die Theorie der algebraischen Formen*, Mathematische Annalen (in German)36 (4): 473-534, (1890).

[HMMN] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe, *Lefschetz properties*. In The Lefschetz Properties, pp. 97-140. Springer Berlin Heidelberg, 2013.

[HMNW] T. Harima, J. Migliore, U. Nagel and J. Watanabe, *The weak and strong Lefschetz properties for Artinian K-algebras*. J. Algebra **262** (2003), no. 1, 99-126.

[Ho] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial com- plexes*. In: McDonald, B.R., Morris, R.A. (eds.) Ring theory II. Lect. Notes in Pure and Appl. Math., **26**. M. Dekker (1977).

[Hu] H. Hulett, *Maximum Betti numbers for a given Hilbert function*, Comm. Algebra 21 (1993), no. 7, 2335-2350.

[Ik] H. Ikeda, *Results on Dilworth and Rees numbers of Artinian local rings*, Japan. J. Math. **22** (1996), 147–158.

[IS] A. Iarrobino and H. Srinivasan, *Some Gorenstein Artinian algebras of embedding dimension four: components of PGor (H) for H=(1, 4, 7,, 1).* Journal of Pure and Applied Algebra 201.1 (2005): 62-96.

[Kr] H. Krause, *An axiomatic description of a duality for modules.* Adv. Math. **130** (1997), no. 2, 280-286.

[LZ] J Li, F Zanello Monomial complete intersections, the weak Lefschetz property and plane partitions. Discrete Mathematics, 2010

[Ma1] F. S. Macaulay. *Some properties of enumeration in the theory of modular systems.* Proc. London Math. Soc., 26:531-555, 1927.

[Ma2] F.H.S. Macaulay. *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press, Cambridge, U.K. (1916). MR1281612 (95i:13001)

[Mi] J. Migliore, *Introduction to Liason theory and deficiency module.* Vol. 165. Birkhauser Boston, 1998.

[MMN] J. Migliore, R. Miró-Roig and U. Nagel, *Monomial ideals, almost complete intersections and the weak Lefschetz property.* Trans. Amer. Math. Soc. **363** (2011), no. 1, 229-257.

[MN1] J. Migliore, U. Nagel, *A tour of the Weak and Strong Lefschetz Properties*, arXiv:1109.5718

[MN2] J. Migliore, U. Nagel, *Reduced Arithmetically Gorenstein Schemes and Simplicial Polytopes with maximal Betti numbers*, Advances in Mathematics 180.1 (2003): 1-63.

[MNZ] Juan C. Migliore, Uwe Nagel, Fabrizio Zanello, *A characterization of Gorenstein Hilbert functions in codimension four with small initial degree*, Math. Res. Lett. 15 (2008), no. 2, 331349.

[Mu] J. R. Munkeres, *Elements of algebraic topology.* Vol. 2. Reading: Addison-Wesley, 1984.

[MZ1] J. Migliore and F. Zanello, *The Hilbert functions which force the weak Lefschetz property.* J. Pure Appl. Algebra **210** (2007), no. 2, 465-471.

[MZ2] J. Migliore, F. Zanello, *The strength of the weak Lefschetz property*, Illinois J. Math. 52 (2008), no. 4, 1417-1433.

[Pa] K. Pardue, *Deformation Classes Of Graded Modules And Maximal Betti Numbers*, Ill. J.Math. **40** (1996), 564-585.

[Pe] I. Peeva, *Consecutive cancellations in Betti numbers.* Proceedings of the American Mathematical Society 132.12 (2004): 3503-3507.

[Re] G.A. Reisner, *CohenMacaulay quotients of polynomial rings.* Adv. in Math., **21**, 30-49 (1976).

[Ri] B. Richert, *Smallest graded Betti numbers*, J. Algebra 244 (2001), 236-259.

[RZ1] A. Ragusa, G. Zappalà, *On the reducibility of the postulation Hilbert scheme*, Rend. Circ. Mat. Palermo (2) 53 (2004), no. 3, 401-406.

[RZ2] A. Ragusa, G. Zappalà, *On the Weak-Lefschetz property for Artinian Gorenstein algebras*, Rendiconti del Circolo Matematico di Palermo (2011): 1-8.

[RZ3] A. Ragusa, G. Zappalà, *Looking For Minimal Graded Betti Numbers*, Illinois Journal Of Mathematics, Volume 49, Number 2, Summer 2005, Pages 453 -473.

[RZ4] A. Ragusa, G. Zappalà, *Properties of 3-codimensional Gorenstein schemes*, Comm. Algebra, 29 (1), 303-318 (2001).

[RZ5] A. Ragusa, G. Zappalà, *On the Weak Lefschetz Property for Hilbert functions of almost complete intersections*. Collectanea Mathematica 64.1 (2013): 73-83.

[RZ6] A. Ragusa, G. Zappalà, *Partial intersections and graded Betti numbers*. Contributions to Algebra and Geometry 44.1 (2003): 285-302.

[SS] S. Seo and H. Srinivasan, *On unimodality of Hilbert functions of Gorenstein Artinian algebras of embedding dimension four*, Comm. Algebra , 40, no. 8 (2012), 2893-2905.

[St1] R. Stanley, *Hilbert functions of graded algebras*, Adv. in Math. **28** (1978), 57-83.

[St2] R. Stanley, *The upper bound conjecture and CohenMacaulay rings.* Stud. Appl. Math., **54**, 135-142 (1975).

[St3] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property,* SIAM J. Algebraic Discrete Methods **1** (1980), 168-184.

[Stu] B. Sturmfels, *Gröbner bases and Stanley decompositions of determinantal rings*. Math. Z., **205**, 137-144 (1990)

[Wi] A. Wiebe, *The Lefschetz property for componentwise linear ideals and Gotzmann ideals*, Comm. Algebra 32 (2004), no. 12, 4601-4611.