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A GAME THEORY MODEL OF TARIFF RATE QUOTAS IN INTERNATIONAL TRADE

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Abstract. In this paper, we consider several international companies with production facilities in different countries that aim to sell a homogeneous product to different demand markets in different countries. The importing countries set up a system of two-tiered tariff rate quotas: a lower tariff applies until the quantity of the product to be purchased is below a prescribed threshold, and an upper tariff applies once the threshold is exceeded. We assume that the threshold is a variable controlled by each importing country and that the upper tariff is a function of it. We propose a generalized Nash game model in which the exporting companies compete with each other and the importing countries. The game is investigated in the framework of variational inequalities and a monotonicity result is proved under mild assumptions. Finally, some preliminary numerical experiments illustrate the proposed model.

1. **Introduction.** The growing development of international trade has increased the world-wide circulation of any kind of goods and commodities. Thus, countries where production of certain agricultural or industrial products is insufficient, can import such products from companies whose production sites are located in different countries. Importing countries aim at satisfying their needs at the most favorable market conditions, but national political authorities also wish that the flows of imported goods do not cause damage to their domestic markets. Moreover, they want to avoid that unregulated flows act as a brake on the autonomous development of domestic production sites, when possible. As a consequence, national authorities and trade organizations have put in place various policy instruments to control the quotas of imported goods in an effective manner. For instance, the European Union established in 2021 a new detailed system of rules for the management of autonomous tariff quotas of the Union for certain agricultural and industrial products [3], which has been amended very recently [4]. Moreover, the Consolidated Tariff Schedules (CTS) Database contains the agreed maximum tariffs that world trade organization (WTO) members can impose on imported products from other WTO members [27].

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At the time of revising this paper, in March 2025, the United States upset world trade by imposing a series of extremely high tariffs, thus causing retaliation from major trading partners [1, 22, 23, 26]. These actions have escalated into what can be safely described as the most impressive "tariff war" of the last decades. Among the economic sectors affected by the new protectionist policies we mention: the automobile sector, the steel and aluminum industry, and the industry of the derivatives of metal products. Specifically, on March 27, 2025, President Donald Trump announced a 25% tariff on all imported vehicles, that will take effect on April 2. In retaliation the EU announced a 50% tariff on U.S. bourbon, and President Trump further responded with a 200% tariff on all EU wines and alcoholic products. On March 12, 2025, the U.S. reintroduced and expanded a 25% tariff on steel and aluminum imports. The EU and Canada retaliated immediately: The EU imposed countermeasures on U.S. goods worth approximately \$28 billion. Canada responded with retaliatory tariffs on American goods worth \$20.7 billion, targeting imports like computers and sports equipment. Let us notice that the U.S. protectionist policy includes derivative metal products such as: stainless steel sinks, air conditioner coils and Bulldozer blades. These extended measures affect around \$150 billion worth of goods. Notably, Canada's steel and aluminum products now face a doubled tariff rate of 50%.

A widely used policy instrument aimed at effectively controlling the imports from foreign countries is that of *tariff rate quotas* (TRQs) in which a certain lower tariff is applied to imports until a threshold (*quota*) is reached and then a higher tariff is applied to all subsequent imports. In a stronger (highly protectionist) version, when the quota is overcome the higher tariff can be applied to the entire imports. The ubiquitous presence of tariff rate quotas in international trade has thus called for new mathematical approaches which include this instrument in international supply chain models. In the following, we mention some relatively recent research papers. In the framework of spatial price equilibrium models see, for instance, [2]. A more complex model can be found in [25], where the author considers a linear model describing the oligopolistic behavior of producers, subject to TRQs. In the related papers [14] and [15] the authors further generalize the model in [25], but consider a single production site per country, with separable cost functions and fixed transportation costs. A far more general model has been proposed in [20] where the authors consider the case of multiple companies with multiple production sites and multiple demand markets in each country. Moreover, their transportation costs are flow-dependent and the production and demand price functions can depend on vectors of production and consumption volumes, respectively. They use the variational inequality approach to investigate the theoretical properties of their model and to perform numerical experiments. A similar methodology has been utilized in [19] which deals with a supply chain network equilibrium model with differentiated products, in which firms compete on product quantities and quality, under a strict quota or tariff regime.

While our approach was inspired by the last two papers mentioned above, in relation to them we differ in a number of aspects. First of all, we consider a two-tiered tariff system in which the quota, that is the threshold which separates the two taxation regimes, is not fixed, but is a variable controlled by the importing countries. A fixed under-quota tariff is applied to the imports until the quota is reached, while an over-quota tariff is applied to the exceeding quantity. The over-quota tariff is a function of the quota. We thus provide a new and more flexible trade policy instrument for the importing country. Let us mention that, even in the fixed quota case, our model differs from the one in [20] which entails that once the (fixed) quota has been overcome, the over-quota tariff is applied to the overall imported quantity (i.e. including the under-quota imports), which would correspond to a very strong protectionist policy. On the other hand, we can control the degree of protectionism by adjusting the parameters in the over-quota tariff function, or even varying its functional form. With respect to [19], we mention here that our model encompasses two models put forward therein as special cases, as it will be specified in the subsequent section (the authors of [19] consider, however, an additional decision vector related to the quality of the product). There is another methodological difference in our approach with respect to [20], where the global supply chain is described by a Nash equilibrium problem where the players are the profit-maximizing companies, and their utility functions include, as an exogenous parameter, an equilibrium Lagrange multiplier whose meaning is clarified after the model is built and solved. Instead, we consider a generalized Nash equilibrium problem (GNEP) with shared constraints, where the exporting companies compete with each other and also with the importing countries, which entails that the equilibrium flows, the equilibrium quotas, and the associated over-tariffs, can be explained as the consequence of a (simulated or actual) bargaining procedure.

The paper is organized as follows. In the following Section 2, after introducing some notation, we describe the generalized Nash equilibrium problem and provide its variational inequality formulation. In Section 3 we investigate the monotonicity properties of the operator of the variational inequality, while in Section 4 we illustrate our model by means of some numerical experiments. A small concluding section summarizes our results and outlines possible research developments.

2. **The model.**

Our model deals with a set I of companies which produce or transform a given homogeneous product, with $i \in I$ denoting the index of the generic company and $|I| = n$, and a set J of countries, with $|J| = m$. The product is requested by various demand markets located in different countries. The companies may also have production sites in different countries. We denote with j the generic index of an exporting country and with k the generic index of an importing country, while S_{ij} denotes the set of the production sites of company i in country j, with $n^i_j = |S_{ij}|$, and s being the generic index of a production site. The set of all the production sites of company i is denoted with $S_i = \bigcup_{j \in J} S_{ij}$. The demand markets of each country k are collected in the set D_k, with $n_k = |D_k|$, where the generic demand market is indexed by $l \in D_k$, and subsequently collected in the set $D = \bigcup_{k \in J} D_k$ of all the demand markets. In a two-tiered tariff quotas system, if the quantity of product requested from country k and produced in country j does not exceed a certain quota, then country k will charge the company with production site in country j a fixed, under-quota, tariff τ^u_{jk}. In our model, this quota is not fixed but is a decision variable of the importing country and the over-quota tariff will depend on it.

We now explain the notation used to describe the variables controlled by the companies. Specifically, we denote with q_{ijskl} the quantity that the company i produces in its production site $s \in S_{ij}$ of country $j \in J$ and ships to the demand market $l \in D_k$ of country $k \in J$. We split q_{ijskl} into two parts: x_{ijskl} denotes the part of q_{ijskl} on which the company pays a sub-quota tariff, while y_{ijskl} denotes the part of q_{ijskl} on which the company pays an over-quota tariff. For each company $i \in I$ these variables are grouped into the following vectors:

$$q^i = (q_{ijskl})_{j,s,k,l}, \quad x^i = (x_{ijskl})_{j,s,k,l}, \quad y^i = (y_{ijskl})_{j,s,k,l} \in \mathbb{R}^{(\sum_{j=1}^m n_j^i)(\sum_{k=1}^m n_k)}.$$

These vectors are, in turn, grouped into the overall vectors:

$$q = (q^1, \dots, q^n), \quad x = (x^1, \dots, x^n), \quad y = (y^1, \dots, y^n) \in \mathbb{R}^N,$$

where $N = \sum_{i=1}^n (\sum_{j=1}^m n_j^i)(\sum_{k=1}^m n_k)$.

Country k controls the variables z_{jk} for any $j \in J \setminus \{k\}$, which represent the quotas which have to be overcome by all production sites in country j to be charged an extra tariff. For each country $k \in J$ we group these variables into the vector

$$z^k = (z_{1k}, \dots, z_{k-1,k}, z_{k+1,k}, \dots, z_{mk}).$$

We assume that all the monetary amounts of this model are expressed in a common currency (e.g. US dollars).

The production site s of any company i, located in country j, faces a production cost given by:

$$f_{ijs} = f_{ijs}(q), \qquad \forall j \in J, \ s \in S_{ij}. \tag{1}$$

Furthermore, the transportation cost faced by the company i for shipping the product from its production site s in country j to the demand market l located in country k is denoted with:

$$c_{ijskl} = c_{ijskl}(q), \qquad \forall j, k \in J, \ s \in S_{ij}, \ l \in D_k. \tag{2}$$

For each pair (j, k) of exporting and importing countries, with $j \neq k$, the government of the importing country k establishes the over-quota tariff to apply when importing from country according to a function

$$t_{jk} = t_{jk}(z_{jk}). \tag{3}$$

The demand of each market $l \in D_k$ is denoted with d_l and must satisfy the conservation of flow equation

$$d_l = \sum_{i=1}^n \sum_{j=1}^m \sum_{s \in S_{ij}} q_{ijskl}. \tag{4}$$

The demand markets purchase the good according to the demand price function:

$$\tilde{\rho}_l(d) = \rho_l(q), \qquad \forall l \in D_k, \tag{5}$$

where we have used the conservation law to express the price as a function of the flows.

We now describe the optimization problem faced by each company $i \in I$ which wishes to maximize its net gain:

$$\begin{aligned} \max_{(q^i, x^i, y^i)} \ U_i(q, x, y, z) = & \sum_{j \in J} \sum_{s \in S_{ij}} \sum_{k \in J} \sum_{l \in D_k} \rho_l(q) q_{ijskl} - \sum_{j \in J} \sum_{s \in S_{ij}} f_{ijs}(q) \\ & - \sum_{j \in J} \sum_{s \in S_{ij}} \sum_{k \in J} \sum_{l \in D_k} c_{ijskl}(q) - \sum_{j \in J} \sum_{\substack{k \in J \\ k \neq j}} \tau_{jk}^u \sum_{s \in S_{ij}} \sum_{l \in D_k} x_{ijskl} \\ & - \sum_{j \in J} \sum_{\substack{k \in J \\ k \neq j}} t_{jk}(z_{jk}) \sum_{s \in S_{ij}} \sum_{l \in D_k} y_{ijskl} \end{aligned} \tag{6}$$

subject to

$$\sum_{k\in J}\sum_{l\in D_k} q_{ijskl} \le Q_{ijs}, \qquad j\in J,\ s\in S_{ij}, \tag{7}$$

$$q_{ijskl} = x_{ijskl} + y_{ijskl}, \qquad j\in J,\ s\in S_{ij},\ k\in J,\ l\in D_k, \tag{8}$$

$$\sum_{i'\in I}\sum_{s\in S_{i'j}}\sum_{l\in D_k} x_{i'jskl} \le z_{jk}, \qquad j,k\in J,\ j\neq k, \tag{9}$$

$$x_{ijskl},\ y_{ijskl} \ge 0, \qquad j\in J,\ s\in S_{ij},\ k\in J,\ l\in D_k. \tag{10}$$

The objective function (6) represents the profit of company i, and is given by the difference between the revenue due to selling the product in all demand markets (first term), the production costs (second term), the transportation costs (third term), the under-quota tariffs (fourth term), and the over-quota tariffs (last term). In constraints (7), Q_{ijs} represents the production capacity of production site s located in country j. Notice that (9) is a constraint shared by all the companies which ship the product from country j to country k.

On the other hand, the optimization problem faced by any country $k\in J$ is

$$\begin{aligned}\max_{z^k}\ V_k(q,x,y,z^k) = &\sum_{\substack{j\in J\\ j\neq k}} \tau^u_{jk} \sum_{i\in I}\sum_{s\in S_{ij}}\sum_{l\in D_k} x_{ijskl}\\ &+ \sum_{\substack{j\in J\\ j\neq k}} t_{jk}(z_{jk}) \sum_{i\in I}\sum_{s\in S_{ij}}\sum_{l\in D_k} y_{ijskl},\end{aligned} \tag{11}$$

subject to

$$\sum_{i\in I}\sum_{s\in S_{ij}}\sum_{l\in D_k} x_{ijskl} \le z_{jk}, \qquad j\in J,\ j\neq k, \tag{12}$$

$$z^u_{jk} \le z_{jk} \le z^o_{jk}, \qquad j\in J,\ j\neq k, \tag{13}$$

where constraints (12) are the same as (9), but we have rewritten them again for clarity, since the optimization variable is now z.

Remark 2.1. We remark that our model encompasses the models in subsections 3.2.1 and 3.2.2 of [19] as special cases. Indeed, if we set $t_{jk}(z_{jk}) = \tau^u_{jk}$ for any $j,k\in J$, we obtain the case of a single tariff in 3.2.2, while if $y=0$, z is fixed and $\tau^u_{jk}=0$, we obtain the *strict quota* model in 3.2.1.

In order to formalize the optimization problems above as a noncooperative game, we now specify the strategy set of players, starting with that of company i, for each $i\in I$. Notice that this set depends not only on the vectors controlled by the other companies, but also on the vector z controlled by the importing countries:

$$\begin{aligned} &K_i(q^{-i},x^{-i},y^{-i},z)\\ &= \Bigg\{ (q^i,x^i,y^i):\ x^i,y^i\ge 0, \quad q^i = x^i+y^i,\\ &\quad \sum_{k\in J}\sum_{l\in D_k} q_{ijskl} \le Q_{ijs}, \qquad \forall j\in J,\ \forall s\in S_{ij},\\ &\quad \sum_{i'\neq i}\sum_{s\in S_{i'j}}\sum_{l\in D_k} x_{i'jskl} + \sum_{s\in S_{ij}}\sum_{l\in D_k} x_{ijskl} \le z_{jk}, \quad \forall j,k\in J,\ j\neq k \Bigg\}. \end{aligned} \tag{14}$$

Each country $k \in J$ chooses its strategy z^k in the set

$$C_k(x) = \Big\{ z^k \in \mathbb{R}^{m-1} : \sum_{i \in I} \sum_{s \in S_{ij}} \sum_{l \in D_k} x_{ijskl} \leq z_{jk}, \quad \forall j \in J,\ j \neq k \\ z^u_{jk} \leq z_{jk} \leq z^o_{jk}, \quad \forall j \in J,\ j \neq k \Big\}. \tag{15}$$

We are now in position to formalize our game as a generalized Nash equilibrium problem with shared constraints.

Definition 2.2. A vector $(q^*, x^*, y^*, z^*) \in \mathbb{R}^{3N+m(m-1)}$ is a generalized Nash equilibrium of the considered game if for any company $i \in I$ and country $k \in J$ the conditions

$$U_i(q^{i*}, x^{i*}, y^{i*}, q^{-i*}, x^{-i*}, y^{-i*}, z^*) \geq U_i(q^i, x^i, y^i, q^{-i*}, x^{-i*}, y^{-i*}, z^*), \\ V_k(q^*, x^*, y^*, z^{k*}) \geq V_k(q^*, x^*, y^*, z^k) \tag{16}$$

hold for any $(q^i, x^i, y^i) \in K_i(q^{-i*}, x^{-i*}, y^{-i*}, z^*)$ and $z^k \in C_k(x^*)$.

This kind of game was introduced a long time ago by Rosen in his influential paper [24] and has been investigated in more recent times within the context of quasivariational and variational inequalities (see, e.g., [5, 6, 7, 8, 10, 16, 17]).

In order to find a class of solutions (called variational solutions) of the game above, we will investigate the variational inequality $VI(F, K)$, where the map $F : \mathbb{R}^{3N+m(m-1)} \to \mathbb{R}^{3N+m(m-1)}$ is the *pseudogradient* of the game, i.e.,

$$F(w) = -\Big(\nabla_{q^1} U_1(w), \ldots, \nabla_{q^n} U_n(w), \nabla_{x^1} U_1(w), \ldots, \nabla_{x^n} U_n(w), \\ \nabla_{y^1} U_1(w), \ldots, \nabla_{y^n} U_n(w), \nabla_{z^1} V_1(w), \ldots, \nabla_{z^m} V_m(w) \Big), \tag{17}$$

with $w = (q, x, y, z)$, and the set K is given by

$$K = \Big\{ (q, x, y, z) \in \mathbb{R}^{3N+m(m-1)}_+ : \ q^i = x^i + y^i, \qquad \forall i \in I \\ \sum_{k \in J} \sum_{l \in D_k} q_{ijskl} \leq Q_{ijs}, \qquad \forall i \in I,\ \forall j \in J,\ \forall s \in S_{ij}, \\ \sum_{i \in I} \sum_{s \in S_{ij}} \sum_{l \in D_k} x_{ijskl} \leq z_{jk}, \qquad \forall j, k \in J,\ j \neq k \\ z^u_{jk} \leq z_{jk} \leq z^o_{jk}, \qquad \forall j, k \in J,\ j \neq k \Big\}. \tag{18}$$

The reader interested in the theory of variational inequalities can refer to [11, 12, 18]. Here we only mention that in the case where K is convex, closed and bounded, and F is continuous, the problem $VI(F, K)$ has solutions.

For the sake of clarity, we show below the components of the pseudogradient F. We first compute the derivatives due to the companies:

$$\frac{\partial U_i(q, x, y, z)}{\partial q_{ijskl}} = \rho_l(q) + \sum_{j' \in J} \sum_{s' \in S_{ij'}} \sum_{k' \in J} \sum_{l' \in D_{k'}} \frac{\partial \rho_{l'}(q)}{\partial q_{ijskl}} q_{ij's'k'l'} \\ - \sum_{j' \in J} \sum_{s' \in S_{ij'}} \frac{\partial f_{ij's'}(q)}{\partial q_{ijskl}} - \sum_{j' \in J} \sum_{s' \in S_{ij'}} \sum_{k' \in J} \sum_{l' \in D_{k'}} \frac{\partial c_{ij's'k'l'}(q)}{\partial q_{ijskl}},$$

$$\frac{\partial U_i(q, x, y, z)}{\partial x_{ijskl}} = -\tau^u_{jk},$$

$$\frac{\partial U_i(q,x,y,z)}{\partial y_{ijskl}} = -t_{jk}(z_{jk}).$$

Moreover, for any $h, k \in J$, with $h \neq k$, we have

$$\begin{aligned}\frac{\partial V_k(q,x,y,z^k)}{\partial z_{hk}} &= \frac{\partial}{\partial z_{hk}} \sum_{\substack{j\in J\\ j\neq k}} t_{jk}(z_{jk}) \sum_{i\in I} \sum_{s\in S_{ij}} \sum_{l\in D_k} y_{ijskl} \\ &= t'_{hk}(z_{hk}) \sum_{i\in I} \sum_{s\in S_{ih}} \sum_{l\in D_k} y_{ihskl}.\end{aligned}$$

We are now in position to write down the variational inequality $VI(F,K)$: find $(q^*, x^*, y^*, z^*) \in K$ such that:

$$\begin{aligned}&\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\Big[-\sum_{j'\in J}\sum_{s'\in S_{ij'}}\sum_{k'\in J}\sum_{l'\in D_{k'}} \frac{\partial \rho_l(q^*)}{\partial q_{ijskl}} q^*_{ij's'k'l'} - \rho_l(q^*)\\ &+\sum_{j'\in J}\sum_{s'\in S_{ij'}} \frac{\partial f_{ij's'}(q^*)}{\partial q_{ijskl}} + \sum_{j'\in J}\sum_{s'\in S_{ij'}}\sum_{k'\in J}\sum_{l'\in D_{k'}} \frac{\partial c_{ij's'k'l'}(q^*)}{\partial q_{ijskl}}\Big](q_{ijskl} - q^*_{ijskl})\\ &+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k} \tau^u_{jk}\,(x_{ijskl} - x^*_{ijskl})\\ &+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k} t_{jk}(z^*_{jk})\,(y_{ijskl} - y^*_{ijskl})\\ &-\sum_{k\in J}\sum_{\substack{j\in J\\ k\neq j}}\sum_{i\in I}\sum_{s\in S_{ij}}\sum_{l\in D_k} t'_{jk}(z^*_{jk}) y^*_{ijskl}(z_{jk} - z^*_{jk}) \geq 0, \qquad \forall (q,x,y,z)\in K.\end{aligned} \tag{19}$$

We recall that any solution of $VI(F,K)$ is a generalized Nash equilibrium of the proposed game provided that the utility function U_i of any company $i \in I$ is concave with respect to the variables (q^i, x^i, y^i) and the utility function V_k of any country $k \in J$ is concave with respect to the variables z^k.

3. **Monotonicity properties.** It is well-known that suitable monotonicity properties of the map F are important for the convergence of solution methods (see, e.g., [12, 18]). This section gives some sufficient conditions on the problem data for the map F to be monotone on $\mathbb{R}_+^{3N+m(m-1)}$, that is

$$[F(v) - F(w)]^\top (v - w) \geq 0, \qquad \forall v, w \in \mathbb{R}_+^{3N+m(m-1)}.$$

Theorem 3.1. *Assume that $n \geq 4$, for any $k \in J$ and $l \in D_k$ the demand price function $\tilde{\rho}_l = \tilde{\rho}_l(d_l)$ is twice continuously differentiable, decreasing, concave and the condition*

$$2\,\tilde{\rho}'_l(d_l) \leq (n-3)\, d_l\, \tilde{\rho}''_l(d_l) \tag{20}$$

holds for any $d_l \geq 0$; the production cost function is

$$f_{ijs}(q) = \phi_{ijs}\left(\sum_{k'\in J}\sum_{l'\in D_k} q_{ijsk'l'}\right),$$

where $\phi_{ijs} : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable convex function, for any $i \in I$, $j \in J$ and $s \in S_{ij}$; the transportation cost function is

$$c_{ijskl}(q) = \gamma_{ijskl}(q_{ijskl}),$$

where $\gamma_{ijskl}: \mathbb{R} \to \mathbb{R}$ is a continuously differentiable convex function, for any $i \in I$, $j \in J$, $s \in S_{ij}$, $k \in J$ and $l \in D_k$; the tariff function $t_{jk}: \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable concave function for any $j, k \in J$ with $j \neq k$.

Then, the map F defined in (17) is monotone on $\mathbb{R}^{3N+m(m-1)}_+$.

Proof. First, we prove that the utility functions of all companies and countries are concave with respect to their own variables, so that every solution of $VI(F, K)$ is a generalized Nash equilibrium of the game. For any $i \in I$, $k \in J$ and $l \in D_k$, we denote $p_{ikl} = \sum_{j\in J}\sum_{s\in S_{ij}} q_{ijskl}$ the total quantity that company i ships to the demand market l of country k, and $p_{kl} = (p_{1kl}, \dots, p_{nkl})$. Then the first term of function U_i is

$$\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k} \rho_l(q) q_{ijskl} = \sum_{k\in J}\sum_{l\in D_k} R_{ikl}(p_{kl}),$$

where the function $R_{ikl}(p_{kl}) = \tilde{\rho}_l(d_l) p_{ikl}$ is concave with respect to p_{ikl} since

$$\frac{\partial^2 R_{ikl}}{\partial p_{ikl}^2}(p_{kl}) = \tilde{\rho}_l''(d_l) p_{ikl} + 2\tilde{\rho}_l'(d_l) < 0.$$

Hence, the first term of U_i is concave with respect to q^i. Moreover, the convexity of f_{ijs} and c_{ijskl} guarantees that U_i is concave with respect to the variables (q^i, x^i, y^i) for any $i \in I$. Moreover, the function V_k is concave with respect to z^k since t_{jk} is concave for any $j \neq k$.

It follows from the assumptions that

$$\begin{aligned}
\frac{\partial U_i}{\partial q_{ijskl}}(w) &= \tilde{\rho}_l(d_l) + \tilde{\rho}_l'(d_l) \sum_{j'\in J}\sum_{s'\in S_{ij'}} q_{ij's'kl} - \phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k} q_{ijsk'l'}\right) \\
&\quad - \gamma_{ijskl}'(q_{ijskl}) \\
&= \tilde{\rho}_l(d_l) + \tilde{\rho}_l'(d_l) p_{ikl} - \phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k} q_{ijsk'l'}\right) - \gamma_{ijskl}'(q_{ijskl}),
\end{aligned}$$

$$\frac{\partial U_i}{\partial x_{ijskl}}(w) = \begin{cases} -\tau_{jk}^u & \text{if } j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial U_i}{\partial y_{ijskl}}(w) = \begin{cases} -t_{jk}(z_{jk}) & \text{if } j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial V_k}{\partial z_{jk}}(w) = \begin{cases} t_{jk}'(z_{jk}) \sum\limits_{i\in I}\sum\limits_{s\in S_{ij}}\sum\limits_{l\in D_k} y_{ijskl} & \text{if } j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

hold for any $i \in I$, $j \in J$, $s \in S_{ij}$, $k \in J$, $l \in D_k$ and $w \in \mathbb{R}^{3N+m(m-1)}$.

The map F can be decomposed as $F = F^1 + F^2$, where

$$\begin{aligned}
F^1(w) &= -\left(\nabla_{q^1} U_1(w), \dots, \nabla_{q^n} U_n(w), 0, \dots, 0\right), \\
F^2(w) &= -\big(0, \dots, 0, \nabla_{x^1} U_1(w), \dots, \nabla_{x^n} U_n(w), \nabla_{y^1} U_1(w), \dots, \nabla_{y^n} U_n(w), \\
&\qquad \nabla_{z^1} V_1(w), \dots, \nabla_{z^m} V_m(w)\big).
\end{aligned}$$

We now prove that F^1 is monotone. We consider the map $G_{kl}: \mathbb{R}^n \to \mathbb{R}^n$ defined as follows:

$$G_{kl}(p_{kl}) = \left(-\tilde{\rho}_l'(d_l) p_{1kl}, \dots, -\tilde{\rho}_l'(d_l) p_{nkl}\right).$$

The Jacobian matrix of G_{kl} is

$$\nabla G_{kl}(p_{kl}) = \begin{bmatrix} -p_{1kl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) & -p_{1kl}\tilde{\rho}_l''(d_l) & \dots & -p_{1kl}\tilde{\rho}_l''(d_l) \\ -p_{2kl}\tilde{\rho}_l''(d_l) & -p_{2kl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) & \dots & -p_{2kl}\tilde{\rho}_l''(d_l) \\ \vdots & \vdots & \ddots & \vdots \\ -p_{nkl}\tilde{\rho}_l''(d_l) & -p_{nkl}\tilde{\rho}_l''(d_l) & \dots & -p_{nkl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) \end{bmatrix}$$

and its symmetric part is

$$\frac{\nabla G_{kl}(p_{kl}) + \nabla G_{kl}(p_{kl})^\top}{2} =$$
$$\begin{bmatrix} -p_{1kl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) & -\tilde{\rho}_l''(d_l)(p_{1kl}+p_{2kl})/2 & \dots & -\tilde{\rho}_l''(d_l)(p_{1kl}+p_{nkl})/2 \\ -\tilde{\rho}_l''(d_l)(p_{1kl}+p_{2kl})/2 & -p_{2kl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) & \dots & -\tilde{\rho}_l''(d_l)(p_{2kl}+p_{nkl})/2 \\ \vdots & \vdots & \ddots & \vdots \\ -\tilde{\rho}_l''(d_l)(p_{1kl}+p_{nkl})/2 & -\tilde{\rho}_l''(d_l)(p_{2kl}+p_{nkl})/2 & \dots & -p_{nkl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) \end{bmatrix}.$$

The assumptions on the demand price functions guarantees that $\tilde{\rho}_l'(d_l)$ and $\tilde{\rho}_l''(d_l)$ are non-positive functions. Moreover, condition (20) implies that the latter matrix is diagonally dominant. In fact, for any $i \in I$, the following chain of equalities and inequalities holds:

$$\begin{aligned} -p_{ikl}\tilde{\rho}_l''(d_l) - \tilde{\rho}_l'(d_l) &\geq -p_{ikl}\tilde{\rho}_l''(d_l) - \frac{(n-3)}{2}\, d_l\, \tilde{\rho}_l''(d_l) \\ &= \frac{-\tilde{\rho}_l''(d_l)}{2}\left[2p_{ikl} + (n-3)\, d_l\right] \\ &\geq \frac{-\tilde{\rho}_l''(d_l)}{2}\left[2p_{ikl} + (n-3)\, p_{ikl} + \sum_{i'\neq i} p_{ikl}\right] \\ &= \frac{-\tilde{\rho}_l''(d_l)}{2}\left[(n-1)\, p_{ikl} + \sum_{i'\neq i} p_{ikl}\right]. \end{aligned}$$

Hence, the matrix $\nabla G_{kl}(p_{kl})$ is positive semidefinite for any $p_{kl} \geq 0$ and the map G_{kl} is monotone on $\mathbb{R}^n_+$.

Let us consider two any vectors $w, \tilde{w} \in \mathbb{R}_+^{3N+m(m-1)}$. Then, we have

$$\begin{aligned} &[F^1(w) - F^1(\tilde{w})]^\top (w - \tilde{w}) \\ =&\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[-\frac{\partial U_i}{\partial q_{ijskl}}(w) + \frac{\partial U_i}{\partial q_{ijskl}}(\tilde{w})\right](q_{ijskl} - \tilde{q}_{ijskl}) \\ =&\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[\phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k} q_{ijsk'l'}\right)\right. \\ &\qquad\qquad\left. -\phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k} \tilde{q}_{ijsk'l'}\right)\right](q_{ijskl} - \tilde{q}_{ijskl}) \\ &+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[\gamma_{ijskl}'(q_{ijskl}) - \gamma_{ijskl}'(\tilde{q}_{ijskl})\right](q_{ijskl} - \tilde{q}_{ijskl}) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[\tilde\rho_l(\tilde d_l)-\tilde\rho_l(d_l)\right](q_{ijskl}-\tilde q_{ijskl})\\
&+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[\tilde\rho_l'(\tilde d_l)\tilde p_{ikl}-\tilde\rho_l'(d_l)p_{ikl}\right](q_{ijskl}-\tilde q_{ijskl})\\
=&\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\left[\phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k}q_{ijsk'l'}\right)\right.\\
&\qquad\left.-\phi_{ijs}'\left(\sum_{k'\in J}\sum_{l'\in D_k}\tilde q_{ijsk'l'}\right)\right]\left[\sum_{k\in J}\sum_{l\in D_k}q_{ijskl}-\sum_{k\in J}\sum_{l\in D_k}\tilde q_{ijskl}\right]\\
&+\sum_{i\in I}\sum_{j\in J}\sum_{s\in S_{ij}}\sum_{k\in J}\sum_{l\in D_k}\left[\gamma_{ijskl}'(q_{ijskl})-\gamma_{ijskl}'(\tilde q_{ijskl})\right](q_{ijskl}-\tilde q_{ijskl})\\
&+\sum_{k\in J}\sum_{l\in D_k}\left[\tilde\rho_l(\tilde d_l)-\tilde\rho_l(d_l)\right](d_l-\tilde d_l)\\
&+\sum_{i\in I}\sum_{k\in J}\sum_{l\in D_k}\left[\tilde\rho_l'(\tilde d_l)\tilde p_{ikl}-\tilde\rho_l'(d_l)p_{ikl}\right](p_{ikl}-\tilde p_{ikl})\\
\geq&\sum_{i\in I}\sum_{k\in J}\sum_{l\in D_k}\left[\tilde\rho_l'(\tilde d_l)\tilde p_{ikl}-\tilde\rho_l'(d_l)p_{ikl}\right](p_{ikl}-\tilde p_{ikl})\\
=&\sum_{k\in J}\sum_{l\in D_k}\left[G_{kl}(p_{kl})-G_{kl}(\tilde p_{kl})\right]^\top\left[p_{kl}-\tilde p_{kl}\right]\\
\geq&\ 0,
\end{aligned}
$$

where the first inequality follows from the convexity of ϕ_{ijs}, the convexity of γ_{ijskl} and the assumption that $\tilde\rho_l$ is decreasing, while the second inequality follows from the monotonicity of G_{kl}. Therefore, F^1 is monotone on $\mathbb{R}_+^{3N+m(m-1)}$.

The map F^2 can be rewritten as

$$F^2(w)=\left(F_q^2(w),\ F_x^2(w),\ F_y^2(w),\ F_z^2(w)\right),$$

where

$$
\begin{aligned}
&F_q^2(w)=0, && F_x^2(w)=-\left(\nabla_{x^1}U_1(w),\dots,\nabla_{x^n}U_n(w)\right),\\
&F_y^2(w)=-\left(\nabla_{y^1}U_1(w),\dots,\nabla_{y^n}U_n(w)\right), && F_z^2(w)=-\left(\nabla_{z^1}V_1(w),\dots,\nabla_{z^m}V_m(w)\right).
\end{aligned}
$$

The Jacobian matrix of F^2 is

$$\nabla F^2(w)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\nabla_zF_y^2(w)\\0&0&\nabla_yF_z^2(w)&\nabla_zF_z^2(w)\end{bmatrix},$$

where $\nabla_zF_y^2(w)$ is the Jacobian matrix of F_y^2 with respect to the variables z, i.e.,

$$\left(\nabla_zF_y^2(w)\right)_{ijskl,\ j'k'}=\begin{cases}t_{jk}'(z_{jk}) & \text{if } j=j',k=k',j\neq k,\\ 0 & \text{otherwise,}\end{cases}$$

$\nabla_y F_z^2(w)$ is the Jacobian matrix of F_z^2 with respect to the variables y, i.e.,

$$\left(\nabla_y F_z^2(w)\right)_{jk,\ i'j's'k'l'} = \begin{cases} -t'_{jk}(z_{jk}) & \text{if } j = j', k = k', j \neq k, \\ 0 & \text{otherwise,} \end{cases}$$

and $\nabla_z F_z^2(w)$ is the Jacobian matrix of F_z^2 with respect to the variables z, i.e.,

$$\left(\nabla_z F_z^2(w)\right)_{jk,\ j'k'} = \begin{cases} -t''_{jk}(z_{jk}) \sum\limits_{i\in I} \sum\limits_{s\in S_{ij}} \sum\limits_{l\in D_k} y_{ijskl} & \text{if } j = j', k = k', j \neq k, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\left(\nabla_z F_y^2(w)\right)^\top = -\nabla_y F_z^2(w)$ and $\nabla_z F_z^2(w)$ is a diagonal matrix with non-negative elements since t_{jk} are concave functions. Hence, the symmetric part of $\nabla F^2(w)$ is diagonal and positive semidefinite for any $w \geq 0$, thus F^2 is monotone on $\mathbb{R}_+^{3N+m(m-1)}$.

Finally, the monotonicity of F^1 and F^2 guarantees that F is monotone on $\mathbb{R}_+^{3N+m(m-1)}$. □

Remark 3.2. To provide an economic interpretation of condition (20), we consider, for the sake of clarity, the case of an economy with one demand market. Notice that the elasticity of price with respect to demand can be defined as

$$E_1(d) = \frac{d\,\tilde{\rho}'(d)}{\tilde{\rho}(d)},$$

which measures the percentage change in price relative to a percentage change in demand (see, e.g., [21]). On the other hand, we deal with the quantity

$$E_2(d) = \frac{d\,\tilde{\rho}''(d)}{\tilde{\rho}'(d)},$$

which can be interpreted as a measure of how the marginal effect of demand on price (i.e., $\tilde{\rho}'(d)$) changes relative to the second derivative $\tilde{\rho}''(d)$, scaled by demand. Thus, it can be interpreted as the *elasticity of marginal price sensitivity* or *elasticity of marginal price response*. Hence, condition (20) can be equivalently written as

$$E_2(d) = \frac{d\,\tilde{\rho}''(d)}{\tilde{\rho}'(d)} \leq \frac{2}{n-3}.$$

Notice that the ratio $E_2(d)$ is non-negative since $\tilde{\rho}$ is decreasing ($\tilde{\rho}'(d) < 0$) and concave ($\tilde{\rho}''(d) \leq 0$), i.e., the price decreases as demand rises but at a diminishing rate. Therefore, condition (20) ensures that the rate at which the price response flattens is not too large.

Finally, we remark that the assumptions of Theorem 3.1 on the demand price functions are satisfied by a quite large class of functions. For instance, any decreasing affine function is concave and satisfies condition (20). Moreover, any quadratic function of the form

$$\tilde{\rho}(d) = -\frac{1}{2}ad^2 - bd + c, \tag{21}$$

with $a, b > 0$, is concave, decreasing for any $d \geq 0$, and satisfies (20) provided that $b/a \geq (n-5)d_{\max}/2$, where $d_{\max}$ is an upper bound of the demand d.

4. **Numerical experiments.** We now illustrate the model described above with some numerical experiments. We consider four different scenarios: a first baseline scenario without tariffs, a second scenario with low under-quota tariffs, a third scenario with high under-quota tariffs and a last scenario with high under-quota tariffs and strict quotas.

Example 4.1 (*without tariff rate quotas*)**.** We have four companies ($I = \{1, 2, 3, 4\}$) and three countries ($J = \{A, B, C\}$). Company 1 has two production sites located in country A ($S_{1A} = \{1, 2\}$), company 2 has one production site in country A and one in country B ($S_{2A} = \{3\}$, $S_{2B} = \{4\}$), company 3 has only one production site in country B ($S_{3B} = \{5\}$), and company 4 has only one production site in country C ($S_{4C} = \{6\}$). Moreover, each country has a single demand market, which is denoted for simplicity by the same letter as the country ($D_A = \{A\}$, $D_B = \{B\}$, $D_C = \{C\}$). Figure 1 illustrates the considered supply chain network.

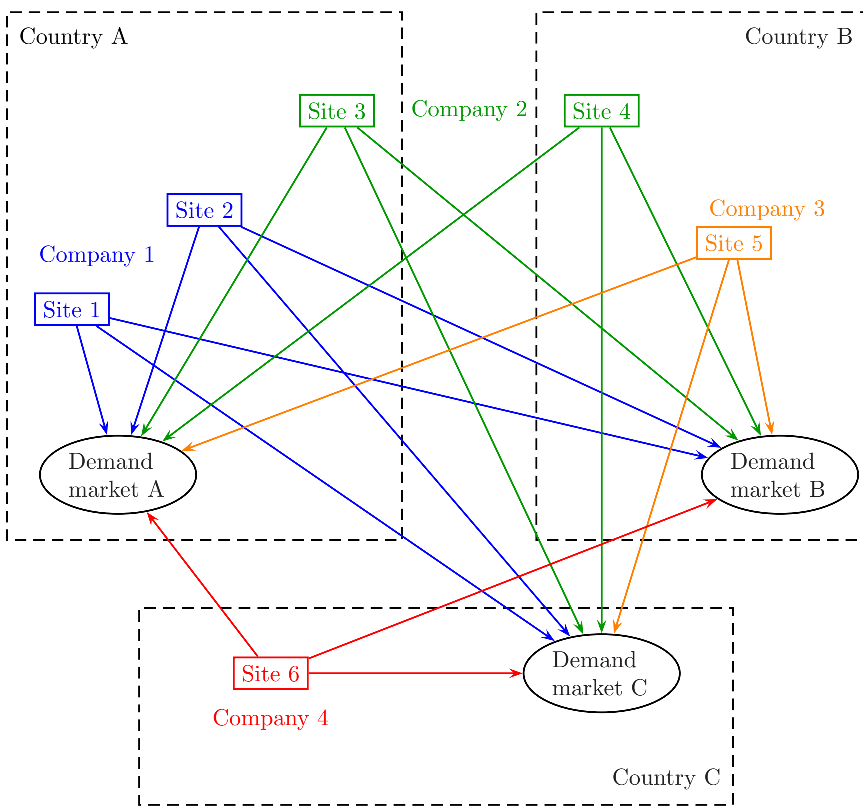

FIGURE 1. Supply chain network of Example 4.1.

We assume that the production capacity $Q_{ijs} = 50$ for any $i \in I$, $j \in J$ and $s \in S_{ij}$, and the production cost faced by each company at any production site is a quadratic function of the total quantity produced at that site:

$$f_{1A1} = 0.01(q_{1A1AA} + q_{1A1BB} + q_{1A1CC})^2 + q_{1A1AA} + q_{1A1BB} + q_{1A1CC},$$
$$f_{1A2} = 0.02(q_{1A2AA} + q_{1A2BB} + q_{1A2CC})^2 + 1.1(q_{1A2AA} + q_{1A2BB} + q_{1A2CC}),$$

$$f_{2A3} = 0.03(q_{2A3AA} + q_{2A3BB} + q_{2A3CC})^2 + 1.2(q_{2A3AA} + q_{2A3BB} + q_{2A3CC}),$$
$$f_{2B4} = 0.01(q_{2B4AA} + q_{2B4BB} + q_{2B4CC})^2 + 0.9(q_{2B4AA} + q_{2B4BB} + q_{2B4CC}),$$
$$f_{3B5} = 0.02(q_{3B5AA} + q_{3B5BB} + q_{3B5CC})^2 + 0.8(q_{3B5AA} + q_{3B5BB} + q_{3B5CC}),$$
$$f_{4C6} = 0.005(q_{4C6AA} + q_{4C6BB} + q_{4C6CC})^2 + 0.1(q_{4C6AA} + q_{4C6BB} + q_{4C6CC}),$$

while the transportation costs are assumed to be quadratic separable functions:

$$\begin{aligned}
c_{1A1AA} &= 0.1q_{1A1AA}^2 + 0.3q_{1A1AA} & c_{1A2AA} &= 0.1q_{1A2AA}^2 + 0.3q_{1A2AA},\\
c_{1A1BB} &= 0.2q_{1A1BB}^2 + 0.5q_{1A1BB} & c_{1A2BB} &= 0.2q_{1A2BB}^2 + 0.5q_{1A2BB},\\
c_{1A1CC} &= 0.2q_{1A1CC}^2 + 0.5q_{1A1CC} & c_{1A2CC} &= 0.2q_{1A2CC}^2 + 0.5q_{1A2CC},\\
c_{2A3AA} &= 0.2q_{2A3AA}^2 + 0.5q_{2A3AA} & c_{2B4AA} &= 0.1q_{2B4AA}^2 + 0.3q_{2B4AA},\\
c_{2A3BB} &= 0.1q_{2A3BB}^2 + 0.3q_{2A3BB} & c_{2B4BB} &= 0.2q_{2B4BB}^2 + 0.5q_{2B4BB},\\
c_{2A3CC} &= 0.2q_{2A3CC}^2 + 0.5q_{2A3CC} & c_{2B4CC} &= 0.2q_{2B4CC}^2 + 0.5q_{2B4CC},\\
c_{3B5AA} &= 0.1q_{3B5AA}^2 + 0.3q_{3B5AA} & c_{4C6AA} &= 0.1q_{4C6AA}^2 + 0.3q_{4C6AA},\\
c_{3B5BB} &= 0.2q_{3B5BB}^2 + 0.4q_{3B5BB} & c_{4C6BB} &= 0.1q_{4C6BB}^2 + 0.3q_{4C6BB},\\
c_{3B5CC} &= 0.1q_{3B5CC}^2 + 0.5q_{3B5CC} & c_{4C6CC} &= 0.05q_{4C6CC}^2 + 0.3q_{4C6CC}.
\end{aligned}$$

The price functions of the demand markets are assumed to be of the form (21), i.e., quadratic, concave and decreasing functions of the demand:

$$\tilde{\rho}_A(d_A) = -0.001d_A^2 - 0.03d_A + 9,$$
$$\tilde{\rho}_B(d_B) = -0.001d_B^2 - 0.02d_B + 10,$$
$$\tilde{\rho}_C(d_C) = -0.001d_C^2 - 0.01d_C + 5,$$

where

$$d_A = q_{1A1AA} + q_{1A2AA} + q_{2A3AA} + q_{2B4AA} + q_{3B5AA} + q_{4C6AA},$$
$$d_B = q_{1A1BB} + q_{1A2BB} + q_{2A3BB} + q_{2B4BB} + q_{3B5BB} + q_{4C6BB},$$
$$d_C = q_{1A1CC} + q_{1A2CC} + q_{2A3CC} + q_{2B4CC} + q_{3B5CC} + q_{4C6CC}.$$

In this first example we assume that there are no tariff rate quotas, that is we set the under-quota tariffs $\tau_{jk}^u = 0$ and the over-quota tariff functions $t_{jk} \equiv 0$ for any $j, k \in J$ with $j \neq k$. The choice of parameters was partially inspired by [20].

Notice that the above functions satisfy the assumptions of Theorem 3.1, thus the map F defined in (17) is monotone. In order to solve the variational inequality $VI(F, K)$, we implemented the extragradient method by Korpelevich [13]. Computations were implemented in MATLAB R2024b and tested on an Apple M1 Max running under macOS 15.3. Table 1 shows the equilibrium solution found together with the equilibrium prices at the demand markets and the utilities of companies and countries.

Example 4.2 (*with low under-quota tariffs*)**.** This example has the same data as Example 4.1 except that the tariff rates are not zero. Specifically, for any pair $j, k \in J$, with $j \neq k$ we assume that the under-quota tariffs are $\tau_{jk}^u = 0.1$ and the over-quota tariff functions are $t_{jk}(z_{jk}) = 0.1 + 0.01z_{jk} - 0.001z_{jk}^2$, where any variable $z_{jk} \in [0, 10]$. Table 2 shows the equilibrium flows and quotas, the equilibrium prices at the demand markets, the profit of each company and tariff payments

TABLE 1. Equilibrium flows, utility of companies, tariff payments collected by countries and equilibrium prices in the scenario without tariff rate quotas (Example 4.1).

		Company 1		Company 2		Company 3	Company 4
	Country	Site 1	Site 2	Site 3	Site 4	Site 5	Site 6
Variables q	A	8.02	6.37	2.53	9.74	9.48	12.98
	B	6.80	5.98	9.51	6.10	7.78	16.40
	C	3.34	2.51	1.77	3.61	5.24	15.82
Variables x	A	8.02	6.37	2.53	9.74	9.48	12.98
	B	6.80	5.98	9.51	6.10	7.78	16.40
	C	3.34	2.51	1.77	3.61	5.24	15.82
Variables y	A	0.00	0.00	0.00	0.00	0.00	0.00
	B	0.00	0.00	0.00	0.00	0.00	0.00
	C	0.00	0.00	0.00	0.00	0.00	0.00

	Utility	Equilibrium price
Country A	0.00	5.11
Country B	0.00	6.18
Country C	0.00	3.63
Company 1	87.64	
Company 2	91.99	
Company 3	55.09	
Company 4	64.88	

collected by countries. Notice that the equilibrium flows between countries decrease slightly compared to the equilibrium solution in Example 4.1, while flows within countries increase. Furthermore, all companies are willing to pay the over-quota tariffs to export the product to countries A and B which are more profitable than C. On the contrary, the quantity exported by companies 1, 2 and 3 to country C is slightly above the threshold because the demand market of C is the least profitable. Finally, we remark that the companies' utilities decrease slightly compared to the equilibrium solution in Example 4.1, while equilibrium prices increase slightly in all demand markets.

Example 4.3 (*with high under-quota tariffs*)**.** This example has the same data as Example 4.1 except that the tariff rates: for any pair $j, k \in J$, with $j \neq k$ we assume that the under-quota tariffs are $\tau^u_{jk} = 0.9$ and the over-quota tariff functions are $t_{jk}(z_{jk}) = 0.9 + 0.01 z_{jk} - 0.001 z^2_{jk}$, where any variable $z_{jk} \in [0, 10]$. Table 3 reports the equilibrium flows and quotas, the equilibrium prices at the demand markets, the profit of each company and tariff payments collected by countries. As tariffs increase, flows between countries decrease compared to both the equilibrium solutions in Examples 4.1 and 4.2, while flows within countries increase. All companies still find it profitable to export to countries A and B, but in smaller quantities than in the solution of Example 4.2, similarly company 4's exports to countries A and

TABLE 2. Equilibrium flows and quotas, utility of companies, tariff payments collected by countries and equilibrium prices in the scenario with low under-quota tariffs (Example 4.2).

		Company 1		Company 2		Company 3	Company 4
	Country	Site 1	Site 2	Site 3	Site 4	Site 5	Site 6
Variables q	A	8.13	6.49	2.71	9.46	9.28	12.75
	B	6.70	5.88	9.27	6.28	7.90	16.20
	C	3.19	2.37	1.65	3.47	4.97	16.21
Variables x	A	8.13	6.49	2.71	2.99	2.91	6.03
	B	1.63	1.21	2.91	6.28	7.90	6.06
	C	2.51	2.10	1.65	2.65	3.41	16.21
Variables y	A	0.00	0.00	0.00	6.47	6.38	6.73
	B	5.07	4.66	6.36	0.00	0.00	10.14
	C	0.68	0.27	0.00	0.81	1.57	0.00

	Country	A	B	C
Variables z	A	–	5.75	6.25
	B	5.90	–	6.06
	C	6.02	6.06	–

	Utility	Equilibrium price
Country A	3.62	5.15
Country B	4.44	6.23
Country C	1.64	3.67
Company 1	86.83	
Company 2	90.16	
Company 3	54.11	
Company 4	63.00	

B still exceed the threshold, but in smaller quantities than in the solution of Example 4.2. The tariffs collected by the countries increase significantly compared to Example 4.2, the profits of the companies decrease, especially for company 4, and the equilibrium prices increase in all demand markets.

Example 4.4 (*with high under-quota tariffs and strict quotas*)**.** This example has the same data as Example 4.1 except that the tariff rates. Here, we assume that there is a regime of strict quotas: we set $z^u_{jk} = z^o_{jk} = 5$ for any pair $j, k \in J$, with $j \neq k$, the under-quota tariffs $\tau^u_{jk} = 0.9$ and the over-quota tariff functions $t_{jk}(z_{jk}) = 0.9 + 100z_{jk} - 0.001z^2_{jk}$, so that companies do not find it profitable to export beyond the set threshold. Table 4 reports the equilibrium flows, the equilibrium prices at the demand markets, the profit of each company and tariff payments collected by countries. The results show that, for each country, imports from other countries reach the threshold set. Compared to the solution in Example 4.3, flows between countries decrease, while flows within countries increase. Notice that the tariffs collected by countries A and B fall sharply because their imports fall sharply,

TABLE 3. Equilibrium flows and quotas, utility of companies, tariff payments collected by countries and equilibrium prices in the scenario with high under-quota tariffs (Example 4.3).

		Company 1		Company 2		Company 3	Company 4
	Country	Site 1	Site 2	Site 3	Site 4	Site 5	Site 6
Variables q	A	8.85	7.27	3.87	7.58	7.98	11.25
	B	6.01	5.22	7.70	7.39	8.69	14.85
	C	2.23	1.46	0.81	2.51	3.29	18.27
Variables x	A	8.85	7.27	3.87	2.90	3.11	5.76
	B	1.81	1.41	2.65	7.39	8.69	6.00
	C	2.23	1.46	0.81	2.33	2.72	18.27
Variables y	A	0.00	0.00	0.00	4.68	4.88	5.50
	B	4.21	3.81	5.05	0.00	0.00	8.85
	C	0.00	0.00	0.00	0.18	0.57	0.00

	Country	A	B	C
Variables z	A	–	5.87	4.50
	B	6.00	–	5.05
	C	5.76	6.00	–

	Utility	Equilibrium price
Country A	24.50	5.40
Country B	30.93	6.51
Country C	9.30	3.90
Company 1	82.30	
Company 2	79.08	
Company 3	48.30	
Company 4	50.99	

while the tariffs collected by country C fall slightly. Finally, companies 1, 2, and 3 increase their profits because the equilibrium prices of market demands A and B have increased, while company 4's profit increases only slightly because its exports to countries A and B decrease and the equilibrium price in market C is essentially unchanged.

5. **Conclusions and further research perspectives.** In this paper, we have proposed a new model of the international supply chain subject to tariff rate quotas. At the time of writing, the tariff debate in the US, China and the EU is evolving, and we expect that new policy instruments will be needed to address the growing complexity of global trade. In this regard, we believe our model is flexible enough to accommodate future scenarios, but can also accommodate further developments. For example, an importing country may decide to tax differently different companies with production facilities in the same exporting country. Another interesting line of research could be a comparative analysis of the performance of different production facilities of the same companies, especially if they are located in different countries.

TABLE 4. Equilibrium flows and quotas, utility of companies, tariff payments collected by countries and equilibrium prices in the scenario with high under-quota tariffs and strict quotas (Example 4.4).

		Company 1		Company 2		Company 3	Company 4
	Country	Site 1	Site 2	Site 3	Site 4	Site 5	Site 6
Variables q	A	11.57	10.05	7.58	1.68	3.32	5.00
	B	2.69	1.92	0.39	12.81	12.45	5.00
	C	2.25	1.50	1.25	2.13	2.87	18.89
Variables x	A	11.57	10.05	7.58	1.68	3.32	5.00
	B	2.69	1.92	0.39	12.81	12.45	5.00
	C	2.25	1.50	1.25	2.13	2.87	18.89
Variables y	A	0.00	0.00	0.00	0.00	0.00	0.00
	B	0.00	0.00	0.00	0.00	0.00	0.00
	C	0.00	0.00	0.00	0.00	0.00	0.00

	Utility	Equilibrium price
Country A	9.00	6.29
Country B	9.00	8.05
Country C	8.97	3.88
Company 1	104.60	
Company 2	83.02	
Company 3	64.48	
Company 4	52.54	

Finally, it would be interesting to extend the analysis to stochastic contexts, e.g., where the demand price and/or transportation cost functions contain uncertain terms. In such a case, the problem under consideration should be reformulated as a stochastic (or random) variational inequality, for which several methodologies exist in the literature (see, e.g., [9]).

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REFERENCES

[1] US imposes 25% steel and aluminum tariffs, sparking global backlash, AP News March 12, 2025. Available from: https://apnews.com/article/e5a6295577275045db3484b71c979bfb/.

[2] P. M. Bishop, C. F. Nicholson, J. E. Pratt and A. M. Novakovic, *Tariff-Rate Quotas: Difficult to Model or Plain Simple? Trade Working Papers 22011*, East Asian Bureau of Economic Research, 2001.

[3] *Council Regulation (EU) 2021/2283 of 20 December 2021 Opening and Providing for the Management of Autonomous Tariff Quotas of the Union for Certain Agricultural and Industrial Products, and Repealing Regulation (EU) No 1388/2013*, 2021. Available from: `https://eur-lex.europa.eu/eli/reg/2021/2283/oj`.

[4] *Council Regulation (EU) 2024/1829 of 25 June 2024 Amending Regulation (EU) 2021/2283 Opening and Providing for the Management of Autonomous Tariff Quotas of the Union for Certain Agricultural and Industrial Products*, Document 32024R1829, 2024. Available from: `https://eur-lex.europa.eu/legal-content/EN/TXT/?uri=celex%3A32024R1829`.

[5] D. Di Lorenzo, M. Passacantando and M. Sciandrone, A convergent inexact solution method for equilibrium problems, *Optim. Methods Soft.*, **29** (2014), 979-991.

[6] F. Facchinei, A. Fischer and V. Piccialli, On generalized Nash games and variational inequalities, *Oper. Res. Lett.*, **35** (2007), 159-164.

[7] F. Faraci and F. Raciti, On generalized Nash equilibrium problems in infinite dimension: The Lagrange multipliers approach, *Optimization*, **64** (2015), 321-338.

[8] D. Gabay and H. Moulin, On the uniqueness and stability of Nash equilibria in noncooperative games, *Applied Stochastic Control in Econometrics and Management Science*, North-Holland, Amsterdam, 1980, 271-294.

[9] J. Gwinner, B. Jadamba, A. A. Khan and F. Raciti, *Uncertainty Quantification in Variational Inequalities: Theory, Numerics, and Applications*, Chapman and Hall/CRC, 2021.

[10] P. T. Harker, Generalized Nash games and quasi-variational inequalities, *Eur. J. Oper. Res.*, **54** (1991), 81-94.

[11] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, 2000.

[12] I. V. Konnov, *Equilibrium Models and Variational Inequalities*, Elsevier, 2007.

[13] G. M. Korpelevich, The extragradient method for finding saddle points and other problems, *Matekon*, **13** (1977), 35-49.

[14] K. Maeda, N. Suzuki and H. M. Kaiser, *An Economic Evaluation of the New, Agricultural Trade Negotiations: A Nonlinear Imperfectly Competitive Spatial Equilibrium Approach*, Working paper, Department of Applied Economics and Management, Cornell University, Ithaca, 2001.

[15] K. Maeda, N. Suzuki and H. M. Kaiser, Evaluating international agricultural policies using a dynamic nonlinear imperfectly competitive spatial equilibrium model, *J. Fac. Agric. Kyushu Univ.*, **50** (2005), 767-790.

[16] G. Mastroeni, M. Pappalardo and F. Raciti, Generalized Nash equilibrium problems and variational inequalities in Lebesgue spaces, *Minimax Theory Appl.*, **5** (2020), 47-64.

[17] K. Nabetani, P. Tseng and M. Fukushima, Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints, *Comput. Optim. Appl.*, **48** (2011), 423-452.

[18] A. Nagurney, *Network Economics. A Variational Inequality Approach*, Springer, 1999.

[19] A. Nagurney, D. Besik and D. Li, Strict quotas or tariffs? Implications for product quality and consumer welfare in differentiated product supply chains, *Transp. Res. E Logist. Transp. Rev.*, **129** (2019), 136-161.

[20] A. Nagurney, D. Besik and L. S. Nagurney, Global supply chain networks and tariff rate quotas: Equilibrium analysis with application to agricultural products, *J. Global Optim.*, **75** (2019), 439-460.

[21] M. Parkin, M. Powell and K. Matthews, *Economics*, Addison-Wesley, 2002.

[22] *Trump Says Larger Tariffs Could be Imposed on Canada, EU if they Cause US 'Economic Harm'*, Reuters March 27, 2025. Available from: `https://www.reuters.com/markets/trump-says-larger-tariffs-could-be-imposed-canada-eu-if-they-cause-us-economic-2025-03-27`.

[23] *Trump's Expanded Metals Tariffs Hit Goods from Horseshoes to Bulldozer Blades*, Reuters March 11, 2025. Available from: `https://www.reuters.com/markets/commodities/trumps-expanded-metals-tariffs-hit-goods-horseshoes-bulldozer-blades-2025-03-11/`.

[24] J. B. Rosen, Existence and uniqueness of equilibrium points for concave n-person games, *Econometrica*, **33** (1965), 520-534.

[25] C. Shono, *WTO to Kokusai-Nuseihin-Boeki (The World Dairy Trade and WTO). Norinlbkci-Kyokai*, Tokyo (2001). (in Japanese)

[26] *Trump Tariffs: Tracking the Economic Impact of the Trump Trade War*, Tax Foundation March 25, 2025. Available from: https://taxfoundation.org/research/all/federal/trump-tariffs-trade-war/.

[27] *Get Tariff Data*, World Trade organization, 2024. Available from: https://www.wto.org/ENGLISH/tratop_e/tariffs_e/tariff_data_e.htm.

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