J. Math. Anal. Appl. 542 (2025) 128807

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Blow-up and global existence for a new class of parabolic $p(\mathrm{x},\cdot)$-Kirchhoff equation involving double phase operator

Ahmed Aberqi [a], Phuong Duc Nguyen [b], Abdesslam Ouaziz [c], Maria Alessandra Ragusa [d,b,*]

[a] *Laboratory LAMA, National School of Applied Sciences, Sidi Mohamed Ben Abdellah University, Fez, Morocco*
[b] *Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Viet Nam*
[c] *Laboratory LAMA, Faculty of Sciences Dhar El Mahraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco*
[d] *Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy*

ARTICLE INFO

Article history:
Received 23 May 2024
Available online 28 August 2024
Submitted by A. Lunardi

Keywords:
p-Kirchhoff
Potential well method
Blow-up existence
Parabolic equation
Global existence
Stabilization

ABSTRACT

In this paper, we are concerned with study solutions for double phase problems involving a Kirchhoff function, and nonlinearity with subcritical growth:

$$\begin{cases} \mathfrak{u}_t + M\left([\mathfrak{u}]^s_{p,q,a}\right)\mathscr{L}^s_{p,q,a}\mathfrak{u} = |\mathfrak{u}|^{r-2}\mathfrak{u}, & \text{in} \quad \mathcal{U}\times[0,T),\\ \mathfrak{u}(\mathrm{x},t)=0 & \text{in} \quad \partial\mathcal{U}\times[0,T),\\ \mathfrak{u}(\mathrm{x},0)=\mathfrak{u}_0(\mathrm{x}), & \text{in} \quad \mathcal{U}, \end{cases}$$

where, $\mathscr{L}^s_{p,q,s}$ is double phase operator, and M is a Kirchhoff function. Combining the Faedo-Galerking approximation with Gronwall's inequality, the existence of a unique weak solution is established. More interestingly, we study the global existence and finite time blow of solution when the initial energy is sub-critical and supercritical. Finally, the stabilization of the solution with positive initial energy is established based on Komornik's integral inequality.

1. Introduction

In this paper, we study the existence and uniqueness results for some new double-phase parabolic equations of Kirchhoff type. More precisely, using potential well theory, Galerkin's method, and Komornik's integral inequality we treat the following problem:

* Corresponding author at: Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy.
E-mail addresses: ahmed.aberqi@usmba.ac.ma (A. Aberqi), nguyenducphuong@iuh.edu.vn (P.D. Nguyen), abdesslam.ouaziz@usmba.ac.ma (A. Ouaziz), maragusa@dmi.unict.it (M.A. Ragusa).

https://doi.org/10.1016/j.jmaa.2024.128807

$$\begin{cases} \mathfrak{u}_t + M\left([\mathfrak{u}]^s_{p,q,a}\right)\mathscr{L}^s_{p,q,a}\mathfrak{u} = |\mathfrak{u}|^{r-2}\mathfrak{u}, & \text{in} \quad \boldsymbol{\mathcal{U}} \times [0,T), \\ \mathfrak{u}(\mathrm{x},t) = 0 & \text{in} \quad \partial\boldsymbol{\mathcal{U}} \times [0,T), \\ \mathfrak{u}(\mathrm{x},0) = \mathfrak{u}_0(\mathrm{x}), & \text{in} \quad \boldsymbol{\mathcal{U}}. \end{cases} \tag{1}$$

Along this work, and without further mentioning $\boldsymbol{\mathcal{U}} \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\boldsymbol{\mathcal{U}}$, $s \in (0,1)$, $a : \boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{U}} \to [0,+\infty)$ is in $L^\infty(\boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{U}})$, and $1 < p < q < q(\beta+1) < r < p^*_s = \frac{Np}{N-sp}$, with p^*_s is the critical Sobolev exponent. The main operator $\mathscr{L}^s_{p,q,a}$ and $[\mathfrak{u}(t)]^s_{p,q,a}$ are called double phase operator and semi-norm given by

$$\mathscr{L}^s_{p,q,a}\mathfrak{u} = 2\lim_{\varepsilon\to0^+} \int\limits_{\boldsymbol{\mathcal{U}}\setminus\mathfrak{B}_\varepsilon(\mathrm{x})} \left[\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^{p-2}}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^{q-2}}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right](\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y}))\mathrm{dy}$$

and

$$[\mathfrak{u}(t)]^s_{p,q,a} = \int\limits_{\boldsymbol{\mathcal{U}}\times\boldsymbol{\mathcal{U}}} \left(\frac{1}{p}\frac{|\mathfrak{u}(\mathrm{x},t)-\mathfrak{u}(\mathrm{y},t)|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + \frac{a(\mathrm{x},\mathrm{y})}{q}\frac{|\mathfrak{u}(\mathrm{x},t)-\mathfrak{u}(\mathrm{y},t)|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}.$$

We assume that $M : [0,\infty) \to [0,\infty)$ is a continuous function satisfying the following assumptions:

($\mathcal{H}_1$) $M \neq 0$ is continuous and non-decreasing on $[0,\infty)$.
($\mathcal{H}_2$) The map $t \mapsto \frac{M(t)}{t^\beta}$ is non-increasing on $[0,\infty)$, where $\beta \geq 0$ is given a constant.

A classical models for M verifying ($\mathcal{H}_1$) and ($\mathcal{H}_2$) due to Kirchhoff is given $M(t) = \ln(1+t)$ and $M(t) = a+bt^{\theta-1}$ where $a, b \geq 0$, $\theta > 1$ and $a+b > 0$. Problem (1) is said double phase type because of the presence of two different elliptic growths p and q. The study of double-phase problems and related functional originates from the seminal paper by Zhikov [35] where he introduced for the first time in literature the related energy functional to (1) defined by

$$\mathrm{u} \mapsto \int\limits_{\boldsymbol{\mathcal{U}}} \left(|\nabla \mathrm{u}|^p + \mu(\mathrm{x})|\nabla \mathrm{u}|^q\right)\mathrm{dx}. \tag{2}$$

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Certainly, the geometry of composites consisting of two different materials with varying power-hardening exponents p and q is determined by the weight coefficient $a(.)$. The functional (2) is a mathematical prototype of a functional whose integrands alter their ellipticity in accordance with the locations where $a(.)$ vanishes or does not. In this direction, the functional (2) has several mathematical applications in the study of duality theory and Lavrentiev gap phenomenon, see [25,26,35–38] for more details. On the other hand, Mingione et al. provide famous results in the regularity theory of local minimizers of functional (2), see, for example, [7,8,11,12] for more details.

A second interesting phenomenon is the appearance of Kirchhoff's function, which was first introduced as a model that describes the small-amplitude vibration of an elastic string

$$\rho h \mathfrak{u}_{tt} + \delta\mathfrak{u}_t = \left(p_0 + \frac{Eh}{L}M(\mathfrak{u})\right)u_{\mathrm{xx}} + f,$$

by Kirchhoff, see [20]. The Kirchhoff function M is a so-called non-degenerate or degenerate case in accordance with $M(0) > 0$ or $M(0) = 0$, respectively. For the non-degenerate case, in [17] the authors studied the following parabolic equation

$$\mathrm{u}_t + M\left(\|\nabla \mathrm{u}\|_p^p\right)\triangle_p \mathrm{u} = |\mathrm{u}|^{m-2}\mathrm{u},$$

when $M(t) = a + bt$ with $2 = p < \frac{m}{2}$ which is a special case of $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. Moreover, He used potential well theory. In addition, He obtained global existence and blow-up in finite time of the weak solutions. For the degenerate case, in the reference [27], the authors studied the following parabolic equation involving the fractional p-Laplacian

$$\mathrm{u}_t + M\left(\|\mathrm{u}\|_{r,p}^p\right)(\triangle_p^r)\mathrm{u} = |\mathrm{u}|^{m-2}\mathrm{u}, \tag{3}$$

where $M(t) = t^s$ with $s > 0$. The authors obtained the existence of global solutions and blow-up time in finite, by using potential well theory, Galerkin method, and some integral inequality. Xiang et al. in the reference [33] continued to study the equation (3), they supposed a more general condition on M, that is

$$M(z) \geq m_0 z^s \quad \text{and } \mu\int_0^z M(\sigma)d\sigma \geq zM(z).$$

Moreover, in [19], the authors studied a Kirchhoff-type parabolic equation

$$\left(1 + |\mathrm{u}|^{m(\mathrm{x})-2}\right)\mathrm{u}_t + M\left(\int_{\mathcal{U}} |\nabla \mathrm{u}|^2 d\mathrm{x}\right)\triangle \mathrm{u} = |\mathrm{u}|^{m(\mathrm{x})-2}\mathrm{u},$$

where $M(z) = a + bz^s$. Readers may refer to [1,4,5,2,3,6,16,15,22,25,31,32,34] and the references therein for more ideas and techniques developed to guarantee the existence of weak solution for the class of nonlocal fractional parabolic and elliptic problems. Motivated by the above works, the objective of this paper is to discuss the effect of a new class of operator and Kirchhoff function verifying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ on the non-global and global existence of a weak solution to the problem (1). For this reason, we use the potential well theory introduced by Payne and Sattinger [28,29]. In our paper, we restrict to subcritical case where

$$\varphi(\mathrm{u}_0(\mathrm{x})) < d_* := d\frac{\left(\frac{1}{p(\beta+1)} - \frac{1}{p}\right)}{\frac{1}{p} - \frac{1}{r}} \text{ and } 2 < r,$$

where d is potential well depth and φ is the energy functional are given in Corollary 3.1 and (4) respectively. To the best of our knowledge, this is the first result in the literature to investigate the global existence and blow-up of solution in the study of parabolic equations involving double-phase operator and Kirchhoff's function.

This paper is organized as follows: Sect 2, provides an overview of fractional Sobolev space and some algebraic inequalities, referencing papers [10,13,16,21,23] for further information. In sect 3.1, using the potential well theory and some inequalities, we will prove the depth of well theory d is positive. In sect 3.2, we obtain uniqueness and existence of a weak solution of problem (1) by Faedo-Galerking method and Gronwall's inequality. In sect 3.3, under suitable conditions, we prove that the weak solution of problem (1) blows up in finite time. In addition, we give the estimate of maximal existence time. In sect 3.4, we prove that the weak solutions of problem (1) exist globally. Moreover, we give a decay estimate of global solutions, by using Komornik's integral inequality. This paper ends with an abstract conclusion and an open question.

2. Preliminaries

2.1. Fractional Sobolev spaces

In this subsection, we recall some necessary properties of fractional Sobolev space. See [10,13] for more details. We start by fixing the fractional exponent s in $(0, 1)$, $\mathcal{U} \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\mathcal{U}$, and $\mathrm{u} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$.

- Let $p \in [1; +\infty[$, the norm in the space $L^p(\mathcal{U})$ is denoted by

$$\|\mathrm{u}\|_{L^p(\mathcal{U})} := \left(\int_{\mathcal{U}} |\mathrm{u}|^p \mathrm{dx} \right)^{\frac{1}{p}}.$$

- For any $p \in [1, \infty)$, we define $W^{s,p}(\mathcal{U})$ as follows:

$$W^{s,p}(\mathcal{U}) = \left\{ \mathrm{u} \in L^p(\mathcal{U}) : \frac{|\mathrm{u(x)} - \mathrm{u(y)}|}{|\mathrm{x} - \mathrm{x}|^{\frac{N}{p}+s}} \in L^p(\mathcal{U} \times \mathcal{U}) \right\},$$

i.e., an intermediary Banach space between $L^p(\mathcal{U})$ and $W^{s,p}(\mathcal{U})$, endowed with the natural norm

$$\|\mathrm{u}\|_{W^{s,p}(\mathcal{U})} = \left(\int_{\mathcal{U}} |\mathrm{u(x)}|^p \mathrm{dx} + [\mathrm{u}]^p_{W^{s,p}(\mathcal{U})} \right)^{\frac{1}{p}},$$

where the term

$$[\mathrm{u}]_{W^{s,p}(\mathcal{U})} = \left(\iint_{\mathcal{U}\times\mathcal{U}} \frac{|\mathrm{u(x)} - \mathrm{u(y)}|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} \mathrm{dxdy} \right)^{\frac{1}{p}}$$

is the so-called Gagliardo semi-norm of u. It is easy to prove that $\|\cdot\|_{W^{s,p}(\mathcal{U})}$ is a norm on $W^{s,p}(\mathcal{U})$.
- The space $\mathrm{W}^{s,p}_0(\mathcal{U})$ is set of the functions defined as:

$$\mathrm{W}^{s,p}_0(\mathcal{U}) := \left\{ \mathrm{u} \in \mathrm{W}^{s,p}\left(\mathbb{R}^{\mathrm{N}}\right), \ \ \mathrm{u} = 0 \text{ a.e. in } \mathbb{R}^{\mathrm{N}} \backslash \mathcal{U} \right\}$$

and the Banach norm in the space $\mathrm{W}^{s,p}_0(\mathcal{U})$ is the Gagliardo semi-norm:

$$\|\mathrm{u}\|_{\mathrm{W}^{s,p}_0(\mathcal{U})} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathrm{u(x)} - \mathrm{u}(y)|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} \mathrm{dxdy} \right)^{\frac{1}{p}}.$$

We recall that by the fractional Poincaré inequality (e.g., in [[13], Theorem 6.5]), there exists a positive constant $c > 0$ such that

$$c^{-1}\|\mathrm{u}\|_{\mathrm{W}^{s,p}(\mathbb{R}^N)} \leq \|\mathrm{u}\|_{\mathrm{W}^{s,p}_0(\mathcal{U})} \leq c\|\mathrm{u}\|_{\mathrm{W}^{s,p}(\mathbb{R}^{\mathrm{N}})}$$

for all $\mathrm{u} \in \mathrm{W}^{s,p}_0(\mathcal{U})$.
- Assuming $p > 1$ the spaces $W^{s,p}(\mathcal{U})$ and $W^{s,p}_0(\mathcal{U})$ are separable and reflexive Banach spaces.

Theorem 2.1. *(See [13,16].) Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Then there exists a positive constant $C = C(N,p,s)$ such that, for any measurable and compactly supported function $f : \mathcal{U} \to \mathbb{R}$ and $1 \leq q < p_s^\star$ we have*

$$\|f\|_{L^q(\mathcal{U})}^p \leq C \int_{\mathcal{U}} \int_{\mathcal{U}} \frac{|f(\mathrm{x}) - f(\mathrm{x})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} \mathrm{d}\mathrm{x}\,\mathrm{d}\mathrm{y},$$

where $p_s^\star = p^\star(N,s)$ is the so-called "fractional critical exponent" and it is equal to $\frac{Np}{N-sp}$. Consequently, the space $W^{s,p}(\mathcal{U})$ is continuously embedded in $L^q(\mathcal{U})$. Moreover, this embedding is compact.

Corollary 2.1. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ such that $sp < N$. Then, the space $\mathrm{W}_0^{s,p}(\mathcal{U})$ is continuously embedded in $L^r(\mathcal{U})$ when $1 \leq q \leq p_s^*$. Moreover, this embedding is compact for any $1 \leq q < p_s^*$.*

Lemma 2.1. *(See [9].) If $1 < p_0 < p\theta < p_1 < \infty$. Then, we have*

$$\|\mathrm{u}\|_{p\theta} \leq \|\mathrm{u}\|_{p_0}^{1-\theta} \|\mathrm{u}\|_{p_1}^{\theta},$$

for any $\mathrm{u} \in L^{p_1}(\mathcal{U})$ with $\theta \in (0,1)$ and $\frac{1}{\theta p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Definition 2.1. Let $\mathrm{u} = \mathrm{u}(\mathrm{x},t)$ be a weak solution of problem (1). We define the maximal existence time $T_{\max}$ of u as follows:

1. If u exists for all $0 \leq t < +\infty$, then $T_{\max} = +\infty$.
2. If there exists a $t_0 \in (0,+\infty)$ such that u exits for $0 \leq t < t_0$, but does not exist at $t = t_0$, then $T_{\max} = t_0$.

2.2. Inequalities algebraic

Lemma 2.2. *(See [23].) Suppose that $0 < T \leq \infty$ and suppose a non-negative function $L(t) \in C^2([0,T),\mathbb{R})$ satisfies*

$$L''(t)L(t) - \alpha\,(L'(t))^2 \geq 0,$$

for some constant $\alpha > 1$. If $L(0) > 0, L'(0) > 0$, then

$$T \leq \frac{L(0)}{(\alpha-1)L'(0)} < \infty \text{ and } L(t) \to \infty \text{ as } t \to T.$$

Lemma 2.3. *(See [18].) Let $G : [0,+\infty) \to [0,+\infty)$ be a non-increasing function. Assume that there exist constants $\sigma \geq 0$ and $C > 0$ such that*

$$\int_t^\infty G^{1+\sigma}(\tau)\mathrm{d}\tau \leq \frac{1}{C} G^\sigma(0) G(t), \quad \text{for all} \quad t \geq 0.$$

Then G has the following decay property:

1) *If $\sigma = 0$, then $G(t) \leq G(0)e^{1-Ct}$ for all $t \geq 0$.*
2) *If $\sigma > 0$, then $G(t) \leq G(0)\left(\frac{1+\sigma}{1+\sigma Ct}\right)^{1/\sigma}$ for all $t \geq 0$.*

3. Main results

3.1. Potential theory

We start this section with the same lemmas, which are useful in proving our main results.

Lemma 3.1. *If Kirchhoff's function satisfies the conditions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*. Then, we have the following results:*

(1) $\theta^{\beta} M(t) \leq M(\theta t)$, *for all* $0 \leq \theta \leq 1$, $t \geq 0$.
(2) $M(\theta t) \leq \theta^{\beta} M(t)$, *for all* $\theta \geq 1$, $t \geq 0$.
(3) $\dfrac{M(\theta)}{\theta^{\beta}} \min\{\theta^{\beta}, t^{\beta}\} \leq M(t) \leq \dfrac{M(\theta)}{\theta^{\beta}} \max\{\theta^{\beta}, t^{\beta}\}$.
(4) $M(t) > 0$, *for all* $t > 0$.
(5) $\dfrac{1}{\beta+1} t M(t) \leq \widehat{M}(t) \leq t M(t)$, *for all* $t \geq 0$.
(6) $\widehat{M}(\theta t) \geq \theta^{\beta+1} \widehat{M}(t)$, *for all* $0 \leq \theta \leq 1, t \geq 0$.
(7) $\widehat{M}(\theta t) \leq \theta^{\beta+1} \widehat{M}(t)$, *for all* $\theta \geq 1, t \geq 0$.

Proof. Let $\theta \in (0,1)$ and $t > 0$. We use the condition $(\mathcal{H}_2)$. Then, we have

$$\frac{M(t)}{t^{\beta}} \leq \frac{M(\theta t)}{(\theta t)^{\beta}}.$$

Obviously, we have (1). Identically, we show (2). We combine (1), (2) with condition $(\mathcal{H}_1)$, we obtain (3). By (3) and $M \neq 0$, we get (4). Doe to M is non-decreasing, we get that

$$\widehat{M}(t) = \int_0^t M(z) \mathrm{d}z \leq \int_0^t M(t) \mathrm{d}z = t M(t).$$

Using the condition $(\mathcal{H}_2)$, we obtain that

$$\widehat{M}(t) = \int_0^t \frac{M(z)}{z^{\beta}} z^{\beta} \mathrm{d}z \geq \int_0^t \frac{M(t)}{t^{\beta}} z^{\beta} \mathrm{d}z = \frac{1}{\beta+1} t M(t).$$

Let $\theta \in [0,1]$, we have

$$\widehat{M}(\theta t) = \int_0^{\theta t} M(z) \mathrm{d}z = \int_0^t \theta M(\theta z) \mathrm{d}z \geq \theta^{\beta+1} \int_0^t M(z) \mathrm{d}z = \theta^{\beta+1} \widehat{M}(t). \quad \square$$

We now define the energy functional $\varphi : W^{s,p}(\boldsymbol{\mathcal{U}}) \to \mathbb{R}$ and Nehari functional $\psi : W^{s,p}(\boldsymbol{\mathcal{U}}) \to \mathbb{R}$ as follows:

$$\varphi(\mathfrak{u}) := \widehat{M}\left([\mathfrak{u}]^s_{p,q,a}\right) - \frac{1}{r} \int_{\mathcal{U}} |\mathfrak{u}|^r \mathrm{dx}, \tag{4}$$

and

$$\psi(\mathfrak{u}) := M\left([\mathfrak{u}]^s_{p,q,a}\right) \int_{\mathcal{U} \times \mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x}, \mathrm{y}) \frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}} \right) \mathrm{dxdy} - \int_{\mathcal{U}} |\mathfrak{u}|^r \mathrm{dx}.$$

Lemma 3.2. *Let $\mathfrak{u} \in W^{s,p}(\mathcal{U})$. Suppose that the conditions $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ are satisfied. Then, we have*

1) $\psi(\varepsilon\mathfrak{u}) > 0$ *for all* $0 < \varepsilon < \varepsilon_*$, $\psi(\varepsilon_*\mathfrak{u}) = 0$, *and* $\psi(\varepsilon\mathfrak{u}) < 0$ *for all* $\varepsilon_* < \varepsilon$.
2) *The map* $\varepsilon \mapsto \varphi(\varepsilon\mathfrak{u})$ *is strictly increasing on* $0 < \varepsilon < \varepsilon_*$, *strictly decreasing on* $\varepsilon_* < \varepsilon < +\infty$, *and* $\sup_{\varepsilon>0} \varphi(\varepsilon\mathfrak{u}) = \varphi(\varepsilon_*\mathfrak{u})$.
3) $\lim_{\varepsilon\to 0} \varphi(\varepsilon\mathfrak{u}) = 0$, *and* $\lim_{\varepsilon\to\infty} \varphi(\varepsilon\mathfrak{u}) = -\infty$.

Proof. From (2) in Lemma 3.1, we get that, for any $\varepsilon > 1$

$$\begin{aligned}
&\widehat{M}\left[\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{\varepsilon^p}{p}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + \frac{a(\mathrm{x},\mathrm{y})\varepsilon^q}{q}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right] - \frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx} \\
&\le \widehat{M}\left[\frac{\varepsilon^q}{p}\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right] - \frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx} \qquad (5) \\
&\le \varepsilon^{q(\beta+1)}\widehat{M}\left[\frac{1}{p}\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right] - \frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx}.
\end{aligned}$$

From (5) and (5) in Lemma 3.1, we get

$$\lim_{\varepsilon\to\infty} \psi(\varepsilon\mathfrak{u}) = -\infty. \qquad (6)$$

Since $q(\beta + 1) < r$. Similarly, we have for any $\varepsilon \in (0, 1)$

$$\begin{aligned}
\psi(\varepsilon\mathfrak{u}) &= M\left[\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{\varepsilon^p}{p}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + \frac{a(\mathrm{x},\mathrm{y})\varepsilon^q}{q}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right] \\
&\quad\times \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\varepsilon^p\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\varepsilon^q\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} - \varepsilon^r\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx} \\
&\ge M\left[\int\limits_{\mathcal{U}\times\mathcal{U}} \varepsilon^q\left(\frac{1}{p}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + \frac{a(\mathrm{x},\mathrm{y})}{q}\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right] \qquad (7) \\
&\quad\times \int\limits_{\mathcal{U}\times\mathcal{U}} \varepsilon^q\left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} - \varepsilon^r\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx} \\
&\ge \varepsilon^{q(\beta+1)}M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} - \varepsilon^r\int\limits_{\mathcal{U}} |\mathfrak{u}|^r\mathrm{dx},
\end{aligned}$$

which yields that

$$\psi(\varepsilon\mathfrak{u}) > 0 \text{ for } \varepsilon > 0 \text{ small enough.} \qquad (8)$$

From (6), (8), and the intermediate value theorem there exist $\varepsilon_* > 0$ such that $\psi(\varepsilon_*\mathfrak{u}) = 0$. For all $\varepsilon > \varepsilon_*$, we get that

$$\psi(\varepsilon\mathfrak{u}) = M\left[\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon^p}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon^q}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]$$
$$\times\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\varepsilon^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\varepsilon^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\varepsilon^r\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$
$$= M\left[\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon_*^p}{p}\left(\frac{\varepsilon}{\varepsilon_*}\right)^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon_*^q}{q}\left(\frac{\varepsilon}{\varepsilon_*}\right)^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]$$
$$\times\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\varepsilon_*^p\left(\frac{\varepsilon}{\varepsilon_*}\right)^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\varepsilon_*^q\left(\frac{\varepsilon}{\varepsilon_*}\right)^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}$$
$$\leq M\left[\left(\frac{\varepsilon}{\varepsilon_*}\right)^q\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon_*^p}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon_*^q}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]$$
$$\times\left(\frac{\varepsilon}{\varepsilon_*}\right)^q\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\varepsilon_*^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\varepsilon_*^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}$$
$$\leq\left(\frac{\varepsilon}{\varepsilon_*}\right)^{q(\beta+1)}M\left([\varepsilon_*\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\varepsilon_*^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\varepsilon_*^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}$$
$$\leq\left[\left(\frac{\varepsilon}{\varepsilon_*}\right)^{q(\beta+1)}-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\right]\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}<0.$$

Similarly, we prove

$$\psi(\varepsilon\mathfrak{u})\geq\left(\frac{\varepsilon}{\varepsilon_*}\right)^{q(\beta+1)}M\left([\varepsilon_*\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\varepsilon_*^p\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\varepsilon_*^q\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}$$
$$-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}$$
$$\geq\left[\left(\frac{\varepsilon}{\varepsilon_*}\right)^{q(\beta+1)}-\left(\frac{\varepsilon}{\varepsilon_*}\right)^r\right]\int\limits_{\mathcal{U}}|\varepsilon_*\mathfrak{u}|^r\mathrm{dx}>0,$$

for any $\varepsilon\in(0,\varepsilon_*)$. Now, we prove (2). Using a direct computation, we have

$$\frac{d\varphi(\varepsilon\mathfrak{u})}{d\varepsilon}=\frac{1}{\varepsilon}\psi(\varepsilon\mathfrak{u}). \tag{9}$$

Combining (9) with (1) in Lemma 3.2, we have the results. For (3). Since $\widehat{M}$ is a continuous function, we can see that

$$\lim_{\varepsilon\to0}\widehat{M}\left[\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon^p}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon^q}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]-\frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$
$$=\widehat{M}(0)=0.$$

From (2) in Lemma 3.1, we get that, for any $\varepsilon>1$

$$\widehat{M}\left[\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon^p}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon^q}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]-\frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$
$$\leq \widehat{M}\left[\frac{\varepsilon^q}{p}\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]-\frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$
$$\leq \varepsilon^{q(\beta+1)}\widehat{M}\left[\frac{1}{q}\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\right]-\frac{\varepsilon^r}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx},$$

which implies that

$$\lim_{\varepsilon\to\infty}\varphi(\varepsilon\mathfrak{u})=-\infty. \quad \square \tag{10}$$

Corollary 3.1. *Let us define the Nehari manifold set*

$$\mathcal{N}=\{\mathfrak{u}\in W:\psi(\mathfrak{u})=0\}\neq\varnothing,$$

and we can talk about $d=\inf_{\mathfrak{u}\in\mathcal{N}}\varphi(\mathfrak{u})$*. The number* d *is called the depth of potential well.*

Considering the following elliptic problem:

$$\begin{cases} M\left([\mathfrak{u}]^s_{p,q,a}\right)\mathscr{L}^s_{p,q,a}\mathfrak{u}=|\mathfrak{u}|^{r-2}\mathfrak{u}, & \text{in} \quad \mathcal{U},\\ \mathfrak{u}(\mathrm{x})=0 & \text{in} \quad \partial\mathcal{U}.\end{cases} \tag{11}$$

From (1) in Lemma 3.2, we prove that the problem (11) has a unique solution in $W^{s,p}(\mathcal{U})$.

Lemma 3.3. *Suppose that the conditions* $(\mathcal{H}_1)$*-*$(\mathcal{H}_2)$ *are satisfied. Then the following assertions hold:*

1) $\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})\leq\left(\frac{1}{q}-\frac{1}{r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)dx\,dy,$
2) $\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})\geq\left(\frac{1}{p(\beta+1)}-\frac{1}{r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)dx\,dy,$
3) *If* $\psi(\mathfrak{u})<0$*, then* $\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})>d_*$.
4) *If* $\psi(\mathfrak{u})\geq 0$*, then* $\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})\geq d_*\epsilon_*^{-p(\beta+1)}$*, and*

$$\psi(\mathfrak{u})\geq\left(1-\epsilon_*^{p(\beta+1)-r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)dx\,dy,$$

for any $\mathfrak{u}\in W^{s,q(\mathrm{x}),p}(\mathcal{U})\setminus\{0\}$.

Proof. For (1). We have

$$\varphi(\mathfrak{u})=\widehat{M}\left([\mathfrak{u}]^s_{p,q,a}\right)-\frac{1}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$
$$\leq M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{1}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{1}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\frac{1}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx}$$

$$\leq \frac{1}{q}M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}-\frac{1}{r}\int\limits_{\mathcal{U}}|\mathfrak{u}|^r\mathrm{dx} \tag{12}$$

$$\leq \left(\frac{1}{q}-\frac{1}{r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}+\frac{1}{r}\psi(\mathfrak{u})$$

From (12), we have

$$\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})\leq\left(\frac{1}{p}-\frac{1}{r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}.$$

Similarly, we prove (2). Suppose that $\psi(\mathfrak{u})<0$. From Lemma 3.1, there exists $\varepsilon_*\in(0,1)$ such that $\psi(\varepsilon_*\mathfrak{u})=0$. Combining definition of d with (1) in Lemma 3.3, we deduce that

$$\begin{aligned}
d&\leq\varphi(\varepsilon_*\mathfrak{u})=\varphi(\varepsilon_*\mathfrak{u})-\frac{1}{r}\psi(\varepsilon_*\mathfrak{u})\\
&\leq\left(\frac{1}{p}-\frac{1}{r}\right)M\left([\varepsilon_*\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon_*^p|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon_*^q|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\\
&\leq\left(\frac{1}{p}-\frac{1}{r}\right)\frac{\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})}{\frac{1}{p(\beta+1)}-\frac{1}{r}},
\end{aligned}$$

which implies

$$d_*:=d\frac{\left(\frac{1}{p(\beta+1)}-\frac{1}{p}\right)}{\frac{1}{p}-\frac{1}{r}}\leq\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u}). \tag{13}$$

Now, we prove (3). We suppose that $\psi(\mathfrak{u})\geq 0$. From Lemma 3.2, there exists $\varepsilon_*>1$ such that $\psi(\varepsilon_*\mathfrak{u})=0$. Therefore, we have

$$\begin{aligned}
d&=\varphi(\varepsilon_*\mathfrak{u})=\varphi(\varepsilon_*\mathfrak{u})-\frac{1}{r}\psi(\varepsilon_*\mathfrak{u})\\
&\leq\left(\frac{1}{q}-\frac{1}{r}\right)M\left([\varepsilon_*\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{\varepsilon_*^p|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+\frac{a(\mathrm{x},\mathrm{y})\varepsilon_*^q|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\\
&\leq\varepsilon_*^{p(\beta+1)}M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}\\
&\leq\varepsilon_*^{p(\beta+1)}\frac{\frac{1}{q}-\frac{1}{r}}{\frac{1}{p(\beta+1)}-\frac{1}{r}}\left(\varphi(\mathfrak{u})-\frac{1}{r}\psi(\mathfrak{u})\right).
\end{aligned}$$

Similarly, we prove that

$$\psi(\mathfrak{u})\geq\left(1-\varepsilon_*^{q(\beta+1)-r}\right)M\left([\mathfrak{u}]^s_{p,q,a}\right)\int\limits_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}+a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}.\qquad\square$$

Lemma 3.4. *Suppose that the conditions ($\mathcal{H}_1$)-($\mathcal{H}_2$) are satisfied. The potential well depth d is positive.*

Proof. Let $\mathfrak{u} \in \mathcal{N}$. It follows that

$$M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^q}{|x-y|^{N+sq}}\right)\mathrm{dxdy} = \int\limits_{\mathcal{U}} |\mathfrak{u}|^r \mathrm{dx} \leq C\|\mathfrak{u}\|^r_{W^{s,p}(\mathcal{U})}, \tag{14}$$

where C is embedding constant of $W^{s,p}(\mathcal{U}) \hookrightarrow L^r(\mathcal{U})$. On the other hand, since $M \neq 0$. Using (1) and (3) in Lemma 3.1, we get

$$\begin{aligned} &M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^q}{|x-y|^{N+sq}}\right)\mathrm{dxdy} \\ &\geq \frac{1}{p^\beta} M\left[\int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^q}{|x-y|^{N+sq}}\right)\mathrm{dxdy}\right] \\ &\quad \times \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^q}{|x-y|^{N+sq}}\right)\mathrm{dxdy} \\ &\geq \frac{M(1)}{p^\beta}\min\left\{\|\mathfrak{u}\|^p_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}\|^q_{W^{s,p}(\mathcal{U})}, \|\mathfrak{u}\|^{p(\beta+1)}_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}\|^{q(\beta+1)}_{W^{s,p}(\mathcal{U})}\right\}, \end{aligned} \tag{15}$$

where $a^- = \inf_{(x,y)\in\mathcal{U}\times\mathcal{U}} a(x,y)$.

If $\|\mathfrak{u}\|^p_{W^{s,p}(\mathcal{U})} \leq 1$. From (14) and (15), we have

$$\frac{M(1)}{p^\beta}\|\mathfrak{u}\|^{p(\beta+1)}_{W^{s,p}(\mathcal{U})} \leq C\|\mathfrak{u}\|^r_{W^{s,p}(\mathcal{U})}, \tag{16}$$

which yield that

$$\|\mathfrak{u}\|_{W^{s,p}(\mathcal{U})} \geq \left(\frac{M(1)}{Cp^\beta}\right)^{\frac{1}{r-p(\beta+1)}}. \tag{17}$$

Due to $\|\mathfrak{u}\|_{W^{s,p}(\mathcal{U})} > 0$ and $p\,(\beta+1)$. Hence, we have for all $\mathfrak{u} \in \mathcal{N}$,

$$\|\mathfrak{u}\|_{W^{s,p}(\mathcal{U})} \geq \min\left\{1, \left(\frac{M(1)}{Cp^\beta}\right)^{\frac{1}{r-p(\beta+1)}}\right\} := \zeta \in (0,1]. \tag{18}$$

From Lemma 3.3, we obtain, for any $\mathfrak{u} \in \mathcal{N}$.

$$\begin{aligned} \varphi(\mathfrak{u}) &= \varphi(\mathfrak{u}) - \frac{1}{r}\psi(\mathfrak{u}) \\ &\geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right) M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|\mathfrak{u}(x)-\mathfrak{u}(y)|^q}{|x-y|^{N+sq}}\right)\mathrm{dxdy} \\ &\geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right)\frac{M(1)}{p^\beta}\min\left\{\|\mathfrak{u}\|^p_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}\|^q_{W^{s,p}(\mathcal{U})}, \|\mathfrak{u}\|^{p(\beta+1)}_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}\|^{q(\beta+1)}_{W^{s,p}(\mathcal{U})}\right\} \\ &\geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right)\frac{M(1)}{p^\beta}\zeta^{p(\beta+1)}. \end{aligned} \tag{19}$$

Using the definition of d, we have

$$d = \inf_{\mathfrak{u}\in\mathcal{N}} \varphi(\mathfrak{u}) \geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right)\frac{M(1)}{p^{\beta}}\zeta^{p(\beta+1)} > 0. \quad \square \tag{20}$$

3.2. Existence and uniqueness of a weak solution

This section is devoted to proving the existence and uniqueness of solutions of (1). To this aim, we use Galerkin's method together with Gronwall–Bellman–Bihari inequality.

Definition 3.1. We say that $\mathfrak{u} = \mathfrak{u}(\mathrm{x}, t) \in L^{\infty}(0, T; W^{s,p}(\boldsymbol{\mathcal{U}}))$ is a weak solution to problem (1) if:

1) $\mathfrak{u}_t \in L^2(0, T; L^2(\boldsymbol{\mathcal{U}}))$,
2) $\int_{\mathcal{U}} \mathfrak{u}_t \mathfrak{v} dx + \langle \mathfrak{u}, \mathfrak{v}\rangle_{W^{s,p}(\mathcal{U})} = \int_{\mathcal{U}} |\mathfrak{u}|^{r-2}\mathfrak{u}\mathfrak{v}\mathrm{dx}$, for all $t \in [0, T]$, and $\mathfrak{v} \in W^{s,p}(\boldsymbol{\mathcal{U}})$,
3) $\mathfrak{u}(\mathrm{x}, 0) = \mathfrak{u}_0(\mathrm{x})$,
4) $\int_0^T \|\mathfrak{u}_\sigma\|_2^2 d\sigma + \varphi(\mathfrak{u}) \leq \varphi(\mathfrak{u}_0(\mathrm{x}))$, for all $t \in [0, T]$,

with $\langle \mathfrak{u}, \mathfrak{v}\rangle_{W^{s,p}(\mathcal{U})} = M\left([\mathfrak{u}]^s_{p,q,a}\right)\int_{\mathcal{U}\times\mathcal{U}}\left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^{p-2}(\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^{q-2}(\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y}))}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\big(\mathfrak{v}(\mathrm{x}) - \mathfrak{v}(\mathrm{y})\big)\mathrm{dx}d\mathrm{y}$.

The first result of our paper is the following:

Theorem 3.1. *Under assumptions* $(\mathcal{H}_1)$ *and* $(\mathcal{H}_2)$*. We suppose* $\mathfrak{u}_0(\mathrm{x}) \in W^{s,p}(\boldsymbol{\mathcal{U}})$ *and following condition holds*

$$p < q < r < p^s_*.$$

Then there exists $T > 0$ *such that problem* (1) *has a solution. In addition, the weak solution is unique if it is bounded.*

Proof. We shall employ the Galerkin's method. The proof will be divided into 5 steps.

Step 1: Galerkin Approximation. In the space $W^{s,p}(\boldsymbol{\mathcal{U}})$, we take a Galerkin basis $\{\mathfrak{w}_j\}_{j=1}^{\infty}$ and define the finite dimensional space

$$V_m := \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_m\}$$

such that $\overline{\cup_{m\in\mathbb{N}} V_m}^{\|\cdot\|_{W^{s,p}(\mathcal{U})}} = W^{s,p}(\boldsymbol{\mathcal{U}})$ and $(\mathfrak{w}_i, \mathfrak{w}_j) = \delta_{ij}$, for all $i, j = 1, 2, \ldots, m$, with δ_{ij} denotes the Kronecker delta. Let $\mathfrak{u}_m(\mathrm{x}, 0)$ be an element of V_m such that

$$\mathfrak{u}_m(\mathrm{x}, 0) = \sum_{j=1}^{m} a_{mj}(0)\mathfrak{w}_j(\mathrm{x}) \to \mathfrak{u}_0(\mathrm{x}) \text{ in } W^{s,p}(\boldsymbol{\mathcal{U}}) \text{ as } m \to \infty. \tag{21}$$

Constructing the approximate solutions $\mathfrak{u}_m(\mathrm{x}, t)$ of the problem (1) by the form

$$\mathfrak{u}_m(\mathrm{x}, t) = \sum_{j=1}^{m} a_{mj}(t)\mathfrak{w}_j(\mathrm{x}),$$

where the coefficient a_{mj} satisfy the system ordinary differential equation

$$\int_{\mathcal{U}} \mathfrak{u}_{mt} w_i \mathrm{dx} + \langle \mathfrak{u}_m, w_i\rangle_{W^{s,p}(\mathcal{U})} = \int_{\mathcal{U}} |\mathfrak{u}_m|^{r-2}\mathfrak{u} w_i \mathrm{dx}, \text{ for all } i = 1, \ldots, n. \tag{22}$$

Since $\mathfrak{u}_0(\mathrm{x}) \in W^{s,p}(\mathcal{U})$, there exists $\{\xi_{mj}, \ \ j = 1, 2, \ldots, m\}$ such that

$$\mathfrak{u}_m(\mathrm{x}, 0) = \sum_{j=1}^{m} \xi_{mj}(0)\mathfrak{w}_j(\mathrm{x}) \to \mathfrak{u}_m(\mathrm{x}) \text{ in } W^{s,p}(\mathcal{U}). \tag{23}$$

Then (22) and (23) are reduced to the following initial value problem for a system of nonlinear ordinary differential equations on a_{mi}:

$$\begin{cases} a'_{mi}(t) = G_i(a_{mi}(t)) \quad t \in [0, t_0] \quad i = 1, \ldots, m, \\ a_{mi}(0) = \xi_{mj}(0) \quad i = 1, \ldots, m, \end{cases} \tag{24}$$

where

$$\begin{aligned} G_i(a_{mi}(t)) = &-M\left(\sum_{j=1}^{m} [a_{mi}(t)\mathfrak{w}_j(\mathrm{x})]^s_{p,q,a}\right) \\ &\times \sum_{j=1}^{m} \int_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^{p-2}(\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y}))}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x}, \mathrm{y}) \frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^{q-2}(\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y}))}{|\mathrm{x} - \mathrm{y}|^{N+sq}} \right) \\ &\times (\mathfrak{w}_i(\mathrm{x}) - \mathfrak{w}_i(\mathrm{y}))\,\mathrm{dx}dy \\ &+ \sum_{j=1}^{m} \int_{\mathcal{U}} |\sum_{j=1}^{m} a_{mi}(t)\mathfrak{w}_j(\mathrm{x})|^{r-2} a_{mi}(t)\mathfrak{w}_j(\mathrm{x})\mathfrak{w}_i(\mathrm{x})\mathrm{dx}. \end{aligned} \tag{25}$$

By the Picard iteration method, there is $t_{0,m} > 0$ depending on $|\xi_{m,i}|$ such that problem (24) admits a unique local solution $a_{mi} \in C^1([0; t_{0;m}])$.

Step 2: The priory estimate. Multiplying the i^{th} equations in (22) by $a_{mi}(t)$ and summing over, i from 1 to m, afterwards, integrating over $(0, t)$ yields

$$\frac{1}{2}\frac{d}{dt}\|\mathfrak{u}_m(t)\|_2^2 + M\left([\mathfrak{u}_m]^s_{p,q,a}\right) \int_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x} - \mathrm{y}|^{N+sp}} + a(\mathrm{x}, \mathrm{y}) \frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x} - \mathrm{y}|^{N+sq}} \right) \mathrm{dxdy} = \int_{\mathcal{U}} |\mathfrak{u}_m|^r \mathrm{dx}. \tag{26}$$

Since $r < p_s^*$ and using Lemma 2.1 with continuous embedding $W^{s,p}(\mathcal{U}) \hookrightarrow L^{p_s^*}(\mathcal{U})$, we deduce that

$$\begin{aligned} \int_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dx} &\leq C\|\mathfrak{u}_m(t)\|_{p_s^*}^{\theta r} \|\mathfrak{u}_m(t)\|_2^{(1-\theta)r} \\ &\leq C\|\mathfrak{u}_m(t)\|_{W^{s,p}(\mathcal{U})}^{\theta r} \|\mathfrak{u}_m(t)\|_2^{(1-\theta)r}, \quad \text{for all } t \in [0, t_{0,m}], \end{aligned} \tag{27}$$

where $\theta \in (0, 1)$ satisfies $\frac{1}{r} = \frac{\theta}{p_s^*} + \frac{1-\theta}{2}$. For any $\varepsilon \in (0, 1)$, the Young inequality yields

$$\int_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dx} \leq C\|\mathfrak{u}_m(t)\|_{W^{s,p}(\mathcal{U})}^{p} + C_\varepsilon \left(\|\mathfrak{u}_m(t)\|_2^2\right)^\gamma, \quad \text{for all } t \in [0, t_{0,m}], \tag{28}$$

where $\gamma = \frac{rp(1-\theta)}{2(p-r\theta)} > 1$. Using (28) and (3) in Lemma 3.1, we get that

$$\begin{aligned}\int\limits_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dx} &\leq \varepsilon \left(1 + \min\left\{\|\mathfrak{u}_m(t)\|^p_{W^{s,p}(\mathcal{U})}, \|\mathfrak{u}_m(t)\|^{p(\beta+1)}_{W^{s,p}(\mathcal{U})}, a^+\|\mathfrak{u}_m(t)\|^q_{W^{s,p}(\mathcal{U})}, \|\mathfrak{u}_m(t)\|^{q(\beta+1)}_{W^{s,p}(\mathcal{U})}\right\}\right) \\ &\quad + C_\varepsilon \left(\|\mathfrak{u}_m(t)\|_2^2\right)^\gamma \\ &\leq \frac{\varepsilon p^\beta}{M(1)} M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy} \\ &\quad + C_\varepsilon \left(\|\mathfrak{u}_m(t)\|_2^2\right)^\gamma + 1 + 2\varepsilon.\end{aligned} \tag{29}$$

It follows from (26) and (29) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathfrak{u}_m(t)\|_2^2 &= -M\left([\mathfrak{u}_m]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy} + \int\limits_{\mathcal{U}} |\mathfrak{u}_m|^r \mathrm{dx} \\ &\leq \left(\frac{\varepsilon p^\beta}{M(1)} - 1\right) M\left([\mathfrak{u}_m]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x}) - \mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy} \\ &\quad + C_\varepsilon \left(\|\mathfrak{u}_m(t)\|_2^2\right)^\gamma + 1 + 2\varepsilon.\end{aligned} \tag{30}$$

Choosing $\varepsilon < \frac{M(1)}{p^\beta}$. So, we deduce that

$$\frac{1}{2}\frac{d}{dt}\|\mathfrak{u}_m(t)\|_2^2 \leq C_\varepsilon \left(\|\mathfrak{u}_m(t)\|_2^2\right)^\gamma + 1 + 2\varepsilon. \tag{31}$$

Integrating (30) with respect to time from 0 to t, we get that

$$\|\mathfrak{u}_m(t)\|_2^2 \leq \|\mathfrak{u}_m(0)\|_2^2 + 2C_4 \int\limits_0^t \|\mathfrak{u}_m(\sigma)\|_2^2 d\sigma + 2(1+2\varepsilon)t. \tag{32}$$

Combining (32) with Gronwall-Bellinan-Bihari inequality, we deduce, there exists $T_0 > 0$ such that

$$\|\mathfrak{u}_m(t)\|_2^2 \leq C(T_0). \tag{33}$$

We next multiply (22) by $a'_m(t)$, sum over j from 1 to m and integrate with respect to time from 0 to t, we have

$$\int\limits_0^t \|\mathfrak{u}_{\sigma m}(\sigma)\|_2^2 d\sigma + \varphi(\mathfrak{u}_m(t)) = \varphi(\mathfrak{u}_m(0)), \quad \text{for all } t \in [0, T_0). \tag{34}$$

So, we deduce that

$$\varphi(\mathfrak{u}_m(0)) \geq \varphi(\mathfrak{u}_m(t)). \tag{35}$$

Using (30), (33), and (35), we get that

$$\begin{aligned}
\varphi(\mathfrak{u}_m(0)) &\geq \varphi(\mathfrak{u}_m(t)) \\
&= \widehat{M}\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right) - \frac{1}{r}\int\limits_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dx} \\
&\geq \frac{1}{p(\beta+1)} M\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} \\
&\quad - \frac{1}{r}\int\limits_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dx} \\
&\geq \left(\frac{1}{p(\beta+1)} - \frac{\varepsilon p^\beta}{M(1)r}\right) M\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right) \\
&\quad \times \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} - \frac{C_4}{r}\|\mathfrak{u}_m(t)\|_2^\gamma - \frac{1+2\varepsilon}{r} \\
&\geq \left(\frac{1}{p(\beta+1)} - \frac{\varepsilon p^\beta}{M(1)r}\right) \frac{M(1)}{p^\beta} \\
&\quad \times \min\left\{\|\mathfrak{u}_m(t)\|^p_{W^{s,p}(\mathcal{U})}, \|\mathfrak{u}_m(t)\|^{p(\beta+1)}_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}_m(t)\|^q_{W^{s,p}(\mathcal{U})}, a^-\|\mathfrak{u}_m(t)\|^{q(\beta+1)}_{W^{s,p}(\mathcal{U})}\right\} \\
&\quad - \frac{C_4 C_6^\gamma + 1 + 2\varepsilon}{r} \\
&\geq -\left(\frac{1}{p(\beta+1)} - \frac{\varepsilon p^\beta}{M(1)r}\right)\frac{M(1)}{p^\beta} - C''(r,\gamma,\varepsilon).
\end{aligned} \tag{36}$$

Using the continuity of φ and (21), we deduce that

$$\lim_{m\to\infty} \varphi(\mathfrak{u}_m(0)) = \varphi(\mathfrak{u}_0(\mathrm{x})). \tag{37}$$

So, we have

$$\varphi(\mathfrak{u}_m(0)) \leq C_7. \tag{38}$$

From (36) and (38) it follows that

$$\int\limits_{\mathcal{U}\times\mathcal{U}} \frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}}\mathrm{dx}d\mathrm{y} \leq C_8. \tag{39}$$

Similarly, we show that

$$\int\limits_{\mathcal{U}\times\mathcal{U}} a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\mathrm{dx}d\mathrm{y} \leq C_8'. \tag{40}$$

From (34), (36), and (38), we get that

$$\int\limits_0^t \|\mathfrak{u}_{m\sigma}(\sigma)\|_2^2 d\sigma \leq C_9. \tag{41}$$

Since M is increasing function, we have

$$\begin{aligned}
&\int\limits_{\mathcal{U}\times\mathcal{U}} |M\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \Bigg(\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^{p-2}\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{N+sp}} \\
&\quad + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^{q-2}\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\Bigg)|^{\frac{q}{q-1}}\mathrm{dx}d\mathrm{y} \\
&\leq \left[M\left(\frac{C_8}{p}+a^+\frac{C_8'}{q}+1\right)\right]^{\frac{q}{q-1}} \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} \\
&\leq \left[M\left(\frac{C_8}{p}+a^+\frac{C_8'}{q}+1\right)\right]^{\frac{q}{q-1}} \left(C_8+a^+C_8'\right).
\end{aligned} \tag{42}$$

Using (39) and Sobolev embedding, we deduce that

$$\begin{aligned}
\int\limits_{\mathcal{U}} ||\mathfrak{u}_m(t)|^{r-2}\mathfrak{u}_m(t)|^{\frac{r}{r-1}}\mathrm{dx} &\leq \int\limits_{\mathcal{U}} ||\mathfrak{u}_m(t)|^r\mathrm{dx} \\
&\leq C_{10}\|\mathfrak{u}_m(t)\|^r_{W^{s,p}(\mathcal{U})} \\
&\leq C_{10}C_8^r.
\end{aligned} \tag{43}$$

Step 3: Pass to the limit. By means of (39), (41), and (42) there exists a function $\mathfrak{u}(t) \in L^\infty(0, T_0; W^{s,p}(\mathcal{U}))$ and a sub-sequence of $\{\mathfrak{u}_m(t)\}_{m\in\mathbb{N}}$ still denote by $\{\mathfrak{u}_m\}_{m\in\mathbb{N}}$ such that

$$\begin{cases}
\mathfrak{u}_m(t) \to \mathfrak{u}(t), & \text{weakly star in} \quad L^\infty(0,T_0;W^{s,p}(\mathcal{U})), \\
\mathfrak{u}_{mt} \to \mathfrak{u}_t, & \text{weakly in} \quad L^\infty\left(0,T_0;L^2(\mathcal{U})\right), \\
M\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right)\int_{\mathcal{U}\times\mathcal{U}}\Big(\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} & \\
\quad + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}_m(\mathrm{x})-\mathfrak{u}_m(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\Big)\mathrm{dxdy} \to \Gamma, & \text{weakly star in} \quad L^\infty\left(0,T_0;L^{\frac{p}{p-1}}(\mathcal{U})\right).
\end{cases} \tag{44}$$

Thanks to Aubin-Lions Simon Lemma (see [30], Corollary 4), it follows from (44) that

$$\mathfrak{u}_m(t) \to \mathfrak{u}(t) \quad \text{strongly in } C\left([0,T_0];L^\vartheta(\mathcal{U})\right) \quad \text{for any } 2 \leq \vartheta \leq r. \tag{45}$$

Therefore,

$$\mathfrak{u}_m(t) \to \mathfrak{u}(t) \quad \text{a.e in } \mathcal{U}\times(0,T_0). \tag{46}$$

So, we get that

$$|\mathfrak{u}_m(t)|^{r-2}\mathfrak{u}_m(t) \to |\mathfrak{u}(t)|^{r-2}\mathfrak{u}(t) \quad \text{a.e in } \mathcal{U}\times(0,T_0) \quad \text{as } m\to\infty. \tag{47}$$

Using Lemma 1.3 in [24] and (43), we get that

$$|\mathfrak{u}_m(t)|^{r-2}\mathfrak{u}_m(t) \to |\mathfrak{u}(t)|^{r-2}\mathfrak{u}(t) \quad \text{a.e in } L^{\frac{r}{r-1}}\mathcal{U}\times(0,T_0) \quad \text{as } m\to\infty. \tag{48}$$

From (44), (48), and letting $m\to\infty$, we get that

$$\int\limits_{\mathcal{U}} \mathfrak{u}_t(t)w(t)d\mathrm{x} + \langle\Gamma, w(t)\rangle = \int\limits_{\mathcal{U}} |\mathfrak{u}(t)|^{r-2}\mathfrak{u}(t)w(t)\mathrm{dx}, \quad \text{for all } (t,w)\in[0,T_0]\times W^{s,p}(\mathcal{U}). \tag{49}$$

All that's left to do is prove that

$$\Gamma = M\left([\mathfrak{u}(t)]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy}.$$

Putting the functional $\mathcal{L} : W^{s,p}(\mathcal{U}) \to \mathbb{R}$ defined by

$$\mathcal{L}\mathfrak{u} = \widehat{M}\left(\frac{1}{p}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{1}{q}\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy}.$$

Similarly, using the same technical appear in proposition 1.1 in the reference [24], we prove that

$$\mathfrak{u} \mapsto \mathcal{L}'\mathfrak{u} = M\left([\mathfrak{u}(t)]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy}$$

and $\mathcal{L}'$ is monotone and hemicontinuous operator. So, we get $\int_0^{T_0} \langle \mathcal{L}'\mathfrak{u}_m(t) - \mathcal{L}'\mathfrak{v}(t), \mathfrak{u}_m(t) - \mathfrak{v}(t)\rangle \,\mathrm{d}t \geq 0$, for all $\mathfrak{v} \in W^{s,p}(\mathcal{U})$. Deducing that

$$\int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{u}_m(t), \mathfrak{u}_m(t)\rangle \,\mathrm{d}t \geq \int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{u}_m(t), \mathfrak{v}(t)\rangle \,\mathrm{d}t - \int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{v}(t), \mathfrak{u}_m(t) - \mathfrak{v}(t)\rangle \,\mathrm{d}t. \tag{50}$$

Integrating (26) over $(0, T_0)$, we get

$$\int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{u}_m(t), \mathfrak{u}_m(t)\rangle = \frac{1}{2}\left(\|\mathfrak{u}_m(0)\|_2^2 - \|\mathfrak{u}_m(T)\|_2^2\right) + \int\limits_{[0,T_0]\times\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{dxd}t. \tag{51}$$

Integrating (49) and taking $\mathfrak{u} = \mathfrak{v}$, we have that

$$\int\limits_0^{T_0} \langle \Gamma, \mathfrak{u}(t)\rangle = \frac{1}{2}\left(\|\mathfrak{u}(0)\|_2^2 - \|\mathfrak{u}(T)\|_2^2\right) + \int\limits_{[0,T_0]\times\mathcal{U}} |\mathfrak{u}(t)|^r \mathrm{dxd}t. \tag{52}$$

From (50), (51), and (52) we have that

$$\begin{aligned} \int\limits_0^{T_0} \langle \Gamma, \mathfrak{u}(t)\rangle &= \lim_{m\to\infty} \int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{u}_m(t), \mathfrak{u}_m(t)\rangle \,\mathrm{d}t \\ &\geq \int\limits_0^{T_0} \langle \Gamma, \mathfrak{v}(t)\rangle - \int\limits_0^{T_0} \langle \mathcal{L}'\mathfrak{v}(t), \mathfrak{u}(t) - \mathfrak{v}(t)\rangle \,\mathrm{d}t. \end{aligned} \tag{53}$$

Therefore, we deduce that $\int_0^{T_0} \langle \Gamma - \mathcal{L}'\mathfrak{u}(t), \mathfrak{u}(t) - \mathfrak{v}(t)\rangle \,\mathrm{d}t \geq 0$. Let $w \in W^{s,p}(\mathcal{U})$ and λ be a real number. Putting $\mathfrak{v} = \mathfrak{u} - \lambda w$, we have

$$\lambda \int\limits_0^{T_0} \langle \Gamma - \mathcal{L}'(\mathfrak{u}(t) - \lambda w(t)), w(t)\rangle \,\mathrm{d}t. \tag{54}$$

Letting $\lambda \to 0^+$ and hemicontinuous property of $\mathcal{L}'$, we get that

$$\int_0^{T_0} \langle \Gamma - \mathcal{L}'(\mathfrak{u}(t), w(t)\rangle \, \mathrm{d}t \geq 0. \tag{55}$$

Letting $\lambda \to 0^-$ in (54), we have

$$\int_0^{T_0} \langle \Gamma - \mathcal{L}'(\mathfrak{u}(t), w(t)\rangle \, \mathrm{d}t \leq 0. \tag{56}$$

Combining (55) with (56), we deduce that $\Gamma = \mathcal{L}'(\mathfrak{u}(t)$.

Step 4: Energy inequality. Fixing $t \in [0, T_0)$. Using (44) and compact embedding $W^{s,p}(\mathcal{U}) \hookrightarrow L^r(\mathcal{U})$, we have that

$$\begin{cases} \lim_{m\to\infty} \int_0^{T_0} \|\mathfrak{u}_{m\sigma}(\sigma)\|_2^2 d\sigma = \int_0^{T_0} \|\mathfrak{u}_\sigma(\sigma)\|_2^2 d\sigma, \\ \mathfrak{u}_m(t) \to \mathfrak{u}(t) \ \text{ weakly in } W^{s,p}(\mathcal{U}) \ \text{ as } m \to \infty, \\ \mathfrak{u}_m(t) \to \mathfrak{u}(t) \ \text{ strongly in } L^r(\mathcal{U}) \ \text{ as } m \to \infty. \end{cases} \tag{57}$$

Using (57), the continuity, and convexity of $\mathcal{L}'$, we deduce that

$$\widehat{M}\left([\mathfrak{u}(t)]^s_{p,q,a}\right) \leq \liminf_{m\to\infty} \widehat{M}\left([\mathfrak{u}_m(t)]^s_{p,q,a}\right). \tag{58}$$

From (57), it follows that

$$\lim_{m\to\infty} \frac{1}{r} \int_{\mathcal{U}} |\mathfrak{u}_m(t)|^r \mathrm{d}\mathrm{x} = \frac{1}{r} \int_{\mathcal{U}} |\mathfrak{u}(t)|^r \mathrm{d}\mathrm{x}. \tag{59}$$

Combining (58) with (59), we have

$$\varphi(\mathfrak{u}(t)) \leq \liminf_{m\to\infty} \varphi(\mathfrak{u}_m(t)). \tag{60}$$

Using (34), (37), (57), and (60), we deduce that

$$\int_0^{T} \|\mathfrak{u}_\sigma\|_2^2 d\sigma + \varphi(\mathfrak{u}) \leq \varphi(\mathfrak{u}_0(\mathrm{x})), \quad \text{for all } t \in [0, T_0). \tag{61}$$

Step 5: Uniqueness of bounded solution. Supposing that the problem (1) admits two bounded weak solutions $\mathfrak{u}_1$ and $\mathfrak{u}_2$. It follows that

$$\langle \partial \mathfrak{u}_1, \mathfrak{v}\rangle + \langle L'\mathfrak{u}_1, \mathfrak{v}\rangle = \int_{\mathcal{U}} |\mathfrak{u}_1|^{r-2}\mathfrak{u}_1\mathfrak{v}\mathrm{d}\mathrm{x}, \tag{62}$$

and

$$\langle \partial \mathfrak{u}_2, \mathfrak{v}\rangle + \langle L'\mathfrak{u}_2, \mathfrak{v}\rangle = \int_{\mathcal{U}} |\mathfrak{u}_2|^{r-2}\mathfrak{u}_2\mathfrak{v}\mathrm{d}\mathrm{x}, \tag{63}$$

for any $\mathfrak{v} \in W^{s,p}(\mathcal{U})$. From (62) and (63), we have

$$\langle \partial(\mathfrak{u}_1 - \mathfrak{u}_2), \mathfrak{v}\rangle + \langle L'(\mathfrak{u}_1 - \mathfrak{u}_2), \mathfrak{v}\rangle = \int_{\mathcal{U}} \left(|\mathfrak{u}_1|^{r-2}\mathfrak{u}_1 - |\mathfrak{u}_2|^{r-2}\mathfrak{u}_2\right) \mathfrak{v} dx. \tag{64}$$

Testing by $\mathfrak{u}_1 - \mathfrak{u}_2$ in (64), we get that

$$\langle \partial\mathfrak{w}, \mathfrak{w}\rangle + \langle L'(\mathfrak{u}_1 - \mathfrak{u}_2), \mathfrak{w}\rangle = \int_{\mathcal{U}} \left(|\mathfrak{u}_1|^{r-2}\mathfrak{u}_1 - |\mathfrak{u}_2|^{r-2}\mathfrak{u}_2\right) \mathfrak{w} dx. \tag{65}$$

Since L' is an operator monotone, we obtain

$$\langle L'(\mathfrak{u}_1 - \mathfrak{u}_2), \mathfrak{w}\rangle \geq 0. \tag{66}$$

Combining (66) with (64), we get that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathfrak{w}\|_2^2 = \langle \partial\mathfrak{w}, \mathfrak{w}\rangle &\leq \int_{\mathcal{U}} \left(|\mathfrak{u}_1|^{r-2}\mathfrak{u}_1 - |\mathfrak{u}_2|^{r-2}\mathfrak{u}_2\right) \mathfrak{w} dx \\ &\leq C_{11}\|\mathfrak{w}\|_2^2, \end{aligned} \tag{67}$$

with C_{11} is a positive constant. Integrating (67) with respect to time from 0 to t, we have

$$\|\mathfrak{w}\|_2^2 \leq 2C_{11} \int_0^t \|\mathfrak{w}(\sigma)\|_2^2 d\sigma.$$

From to Gronwall's inequality, and $\mathfrak{w}(0) = 0$, we have $\mathfrak{w} = 0$. □

3.3. Blow-up existence of solutions

In this section, by means of a differentiable inequality, we prove that the local solutions of problem (1) blow up in finite time.

Considering the following set:

$$\mathcal{Z} := \{\mathfrak{u}_0(x) \in W^{s,p}(\mathcal{U}) : \varphi(\mathfrak{u}_0) < d_*, \psi(\mathfrak{u}_0) < 0\}. \tag{68}$$

Theorem 3.2. *Let $\mathfrak{u}_0 \in \mathcal{Z}$. Assume the $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ are satisfied. Then, we have the following results:*

1. $T_{\max} \leq \frac{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}}{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}(\frac{r}{2}-1) - \|\mathfrak{u}_0\|_2^2} < \infty$,
2. $\int_0^T \|\mathfrak{u}_\sigma\|_2^2 d\sigma + \varphi(\mathfrak{u}) \leq \varphi(\mathfrak{u}_0(x))$, *for all* $t \in [0, T_{\max}]$.

Proof. We prove $\mathfrak{u} \in \mathcal{Z}$ for all $t \in [0, T_{\max})$. By contradiction, there exists $t_0 \in [0, T_{\max})$ such that $\psi(\mathfrak{u}(t)) < 0$ for all $t \in [0, t_0)$ and $\psi(\mathfrak{u}(t_0)) = 0$. From theorem 3 in [14], we have

$$\frac{1}{2}\frac{d}{dt}\|\mathfrak{u}\|_{L^2(\mathcal{U})}^2 = (\mathfrak{u}_t, \mathfrak{u}). \tag{69}$$

Testing by $\mathfrak{u}$ in (2) of Definition 3.1, we get

$$\frac{d}{dt}\|\mathfrak{u}\|^2_{L^2(\mathcal{U})} = (\mathfrak{u}_t, \mathfrak{u}) = -\psi(\mathfrak{u}(t)), \quad \text{for all } t \in [0, t_0). \tag{70}$$

So, the map $t \mapsto \|\mathfrak{u}(t)\|_2^2$ is strictly increasing on $[0, t_0]$. Therefore, $\|\mathfrak{u}(t_0)\| > 0$ which implies that $\mathfrak{u}(t_0) \in \mathcal{N}$. Using the definition of d, we have

$$\varphi(\mathfrak{u}(t_0)) > 0. \tag{71}$$

On the other hand, using 4) in Definition 3.1, we have

$$\varphi(\mathfrak{u}(t_0)) \leq \varphi(\mathfrak{u}_0(\mathrm{x})) < d_* \leq d. \tag{72}$$

From (71) and (72), we arrive a contradiction. Then, we have

$$\mathfrak{u}(t) \in \mathcal{Z}, \quad \text{for all } t \in [0, T_{\max}). \tag{73}$$

Now, considering the following functional

$$L(t) = \int_0^t \|\mathfrak{u}(\sigma)\|_2^2 d\sigma + (T - t)\|\mathfrak{u}_0(\mathrm{x})\|_2^2 + 2at + b, \quad \text{for all } t \in [0, T_{\max}),$$

where a and b are positive constants such that $0 < a < \sqrt{d_* - \varphi(\mathfrak{u}_0)}$ and $b \leq 2a$. By differentiating L and theorem 3 in [14], we have

$$\begin{aligned} L'(t) &= \|\mathfrak{u}(t)\|_2^2 - \|\mathfrak{u}_0(\mathrm{x})\|_2^2 + 2a = \int_0^t \frac{d}{d\sigma}\left(\|\mathfrak{u}(\sigma)\|_2^2\right) d\sigma + 2a \\ &= 2\int_{[0,t]\times\mathcal{U}} \mathfrak{u}_\sigma(\sigma)\mathfrak{u}(\sigma)\mathrm{dx}d\sigma + 2a \end{aligned} \tag{74}$$

and

$$L''(t) = 2\int_{\mathcal{U}} \mathfrak{u}_t(t)\mathfrak{u}(t)\mathrm{dx} = -2\psi(\mathfrak{u}(t)) > 0, \quad \text{for all } t \in [0, T_{\max}). \tag{75}$$

It follows from (73) that

$$L''(t) > 0, \quad \text{for all } t \in [0, T_{\max}). \tag{76}$$

Thanks to Cauchy-Schwarz inequality, we estimate $L'(t)$

$$\begin{aligned} (L'(t))^2 &\leq 4\left[\sqrt{\int_0^t \|\mathfrak{u}_\sigma(\sigma)\|_2^2 d\sigma}\sqrt{\int_0^t \|\mathfrak{u}(\sigma)\|_2^2 d\sigma} + a\right]^2 \\ &\leq 4L(t)\left(\int_0^t \|\mathfrak{u}_\sigma(\sigma)\|_2^2 + a^2\right). \end{aligned} \tag{77}$$

On the other hand, using (3) of Lemma 3.3, we get

$$\varphi(\mathfrak{u}) - \frac{1}{r}\psi(\mathfrak{u}) > d_*.$$

Combining (77) with (4) in Definition 3.1, we have

$$\begin{aligned} L''(t) &= -2\psi(\mathfrak{u}(t)) \\ &\geq 2r\left(d_* - \varphi(\mathfrak{u}(t))\right) \\ &\geq 2r\left[\int_0^t \|\mathfrak{u}_\sigma(\sigma)\|_2^2 d\sigma + \left(d_* - \varphi(\mathfrak{u}(t))\right)\right] \\ &\geq 2r\left[\frac{\left(L'(t)\right)^2}{4L(t)} - a^2 + d_* - \varphi(\mathfrak{u}(t))\right] \end{aligned} \tag{78}$$

It is equivalent to

$$\begin{aligned} L''(t)L(t) - \frac{r}{2}\left(L'(t)\right)^2 &\geq 2L(t)\left(r(d_* - \varphi(\mathfrak{u}(t))) - ra^2\right) \\ &> 0. \end{aligned} \tag{79}$$

A direct computation, we have $L(0) > 0$ and $L'(0) = a > 0$. From Lemma 2.2, we obtain that

$$T \leq \frac{L(0)}{\left(\frac{r}{2}-1\right)L'(0)} = \frac{T\|\mathfrak{u}_0\|_2^2 + b}{2a\left(\frac{r}{2}-1\right) - \|\mathfrak{u}_0\|_2^2}.$$

Therefore, we have

$$T \leq \frac{b}{2a\left(\frac{r}{2}-1\right)} \leq \frac{2a}{2a\left(\frac{r}{2}-1\right) - \|\mathfrak{u}_0\|_2^2} := g(a).$$

Minimizing the function g for all $0 < a < \sqrt{d_* - \varphi(\mathfrak{u}_0)}$, we obtain that

$$T \leq \frac{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}}{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}\left(\frac{r}{2}-1\right) - \|\mathfrak{u}_0\|_2^2}.$$

Letting $T \to T_{\max}$, we have

$$T_{\max} \leq \frac{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}}{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}\left(\frac{r}{2}-1\right) - \|\mathfrak{u}_0\|_2^2} < \infty. \quad \square$$

3.4. Global existence of solutions

Considering the following set:

$$\mathcal{W} := \left\{\mathfrak{u}_0(\mathrm{x}) \in W^{s,p}(\boldsymbol{\mathcal{U}}) : \varphi(\mathfrak{u}_0) < d_*, \psi(\mathfrak{u}_0) > 0\right\}. \tag{80}$$

Theorem 3.3. *Let $\mathfrak{u}_0 \in \mathcal{W}$. Assume the $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ are satisfied. Then, we have following results:*

1. $T_{\max} = \infty$.

2. *The energy functional of a weak solution of problem* (1) *satisfies the following inequality:*

$$\varphi(\mathfrak{u}(t)) \leq \begin{cases} \varphi(\mathfrak{u}_0)\left(\frac{p(\beta+1)}{2+(p(\beta+1)-2)\alpha t}\right)^{\frac{p(\beta+1)}{p(\beta+1)-2}}, & \text{if } p > \frac{2}{(\beta+1)}, \\ \varphi(\mathfrak{u}_0)\, e^{1-\gamma t}, & \text{if } p \leq \frac{2}{(\beta+1)}, \end{cases}$$

where α and γ are some positive constants depending only on p, m, β, M and $\mathfrak{u}_0$.

Proof. Firstly, we prove $\mathfrak{u} \in \mathcal{W}$ for all $t \in [0, T_{\max})$. By contradiction, we show that $\mathfrak{u} \in \mathcal{W}$. Supposing that there exists a $t_0 \in (0, T_{\max})$ such that $\mathfrak{u}(t_0) \in \partial\mathcal{W}$. Noticing that 0 is an interior point of $\mathcal{W}$. So, we have $\varphi(\mathfrak{u}(t_0)) = d$ or $\psi(\mathfrak{u}(t_0)) = 0$, $\mathfrak{u}(t_0) \neq 0$. It follows from 4) in Definition 3.1 that

$$\varphi(\mathfrak{u}(t_0)) \leq \varphi(\mathfrak{u}_0) < d \text{ and } \psi(\mathfrak{u}(t_0)) = 0, \ \ \mathfrak{u}(t_0)) \neq 0. \tag{81}$$

Then $\mathfrak{u}(t_0)) \in \mathcal{N}$. Using the definition of d, we have

$$d \leq \varphi(\mathfrak{u}(t_0)). \tag{82}$$

From (81) and (82), we arrive a contradiction. Finally, we have $\mathfrak{u}(t) \in \mathcal{N}$.

Now, we prove that $T_{\max} = \infty$. From Lemma 3.3, we obtain that

$$\begin{aligned} d_* &> \varphi(\mathfrak{u}_0) \geq \varphi(\mathfrak{u}(t)) \\ &\geq \varphi(\mathfrak{u}(t)) - \frac{1}{r}\psi(\mathfrak{u}(t)) \\ &\geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right) M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x}) - \mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right) \mathrm{dxdy} \\ &\geq \left(\frac{1}{p(\beta+1)} - \frac{1}{r}\right) \frac{M(1)}{p^\beta} \min\left\{\|\mathfrak{u}\|^p_W, \|\mathfrak{u}\|^{p(\beta+1)}_W, a^-\|\mathfrak{u}\|^q_W, a^-\|\mathfrak{u}\|^{q(\beta+1)}_W\right\}. \end{aligned} \tag{83}$$

From (83), we have

$$\|\mathfrak{u}\|_W \leq \max\left\{1, \left[\frac{d_* r(\beta+1)p^{\beta+1}}{M(1)\,(r - p(\beta+1))}\right]^{\frac{1}{p}}\right\} := C_1 > 1. \tag{84}$$

So, $\mathfrak{u}(t)$ is uniformly bounded in time in W. Finally $T_{\max} = +\infty$. Now, we show (2). From (4) in Lemma 3.3, we get

$$\varphi(\mathfrak{u}(t)) - \frac{1}{r}\psi(\mathfrak{u}(t)) \geq d_*\varepsilon_*^{-p(\beta+1)}. \tag{85}$$

However, we have also

$$\varphi(\mathfrak{u}(t)) - \frac{1}{r}\psi(\mathfrak{u}(t)) \leq \varphi(\mathfrak{u}(t)) \leq \varphi(\mathfrak{u}_0(\mathrm{x})). \tag{86}$$

Combining (85) with (86), we have

$$1 < \left(\frac{d_*}{\varphi(\mathfrak{u}_0(\mathrm{x}))}\right)^{\frac{1}{p(\beta+1)}} \leq \varepsilon_*. \tag{87}$$

Thanks to (4), in Lemma and (87), we have

$$\begin{aligned}\psi(\mathfrak{u}(t)) &\geq \left(1-\varepsilon_*^{p(\beta+1)-r}\right) M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} \\ &\geq \left(1-\left(\frac{d_*}{\varphi(\mathfrak{u}_0(\mathrm{x}))}\right)^{\frac{p(\beta+1)-r}{p(\beta+1)}}\right) M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy}.\end{aligned} \tag{88}$$

Using (86) and (2) in Lemma 3.3, we obtain

$$\begin{aligned}\varphi(\mathfrak{u}(t)) - \frac{1}{r}\psi(\mathfrak{u}(t)) &\leq \left(\frac{1}{q}-\frac{1}{r}\right) M\left([\mathfrak{u}]^s_{p,q,a}\right) \int\limits_{\mathcal{U}\times\mathcal{U}} \left(\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^p}{|\mathrm{x}-\mathrm{y}|^{N+sp}} + a(\mathrm{x},\mathrm{y})\frac{|\mathfrak{u}(\mathrm{x})-\mathfrak{u}(\mathrm{y})|^q}{|\mathrm{x}-\mathrm{y}|^{N+sq}}\right)\mathrm{dxdy} \\ &\leq \left(\frac{1}{q}-\frac{1}{r}\right)\left(1-\left(\frac{d_*}{\varphi(\mathfrak{u}_0(\mathrm{x}))}\right)^{\frac{p(\beta+1)-r}{p(\beta+1)}}\right)^{-1} \psi(\mathfrak{u}(t)).\end{aligned}$$

So, we have

$$\varphi(\mathfrak{u}(t)) \leq C_2\psi(\mathfrak{u}(t)), \tag{89}$$

with $C_2 = \left(\frac{1}{q}-\frac{1}{r}\right)\left(1-\left(\frac{d_*}{\varphi(\mathfrak{u}_0(\mathrm{x}))}\right)^{\frac{p(\beta+1)-r}{p(\beta+1)}}\right)^{-1} > 0$. On the other hand, we have

$$\begin{aligned}\varphi(\mathfrak{u}(t)) &\geq \left(\frac{1}{p(\beta+1)}-\frac{1}{r}\right)\frac{M(1)}{p^\beta}\min\left\{\|\mathfrak{u}\|^p_W, \|\mathfrak{u}\|^{p(\beta+1)}_W, a^-\|\mathfrak{u}\|^q_W, a^-\|\mathfrak{u}\|^{q(\beta+1)}_W\right\} \\ &\geq C_3\|\mathfrak{u}\|^{p(\beta+1)}_W,\end{aligned} \tag{90}$$

where $C_3 = \left(\frac{1}{p(\beta+1)}-\frac{1}{r}\right)\frac{M(1)}{p^\beta}C_1^{p-p(\beta+1)} > 0$. Using the embedding $W \hookrightarrow L^2(\mathcal{U})$, we have

$$\|\mathfrak{u}\|_2 \leq C_4\|\mathfrak{u}\|_W.$$

From (90), we have

$$\|\mathfrak{u}\|_2 \leq C_5\left[\varphi(\mathfrak{u}(t))\right]^{\frac{2}{p(\beta+1)}}, \tag{91}$$

with $C_5 := C_4^2 C_3^{\frac{-2}{p(\beta+1)}} > 0$. Let h be an integer such that $t \leq h$. From (91) and (89), we get that

$$\begin{aligned}\int_0^h \varphi(\mathfrak{u}(\sigma))d\sigma &\leq C_2\int_0^h \psi(\mathfrak{u}(\sigma))d\sigma \\ &= \frac{C_2}{2}\left(\|\mathfrak{u}(t)\|_2^2 - \|\mathfrak{u}(h)\|_2^2\right) \\ &\leq \frac{C_2C_5}{2}\left(\varphi(\mathfrak{u}(t))\right)^{\frac{2}{p(\beta+1)}}.\end{aligned}$$

Letting $h \to \infty$, we have

$$\int_0^\infty \varphi(\mathfrak{u}(\sigma))d\sigma \leq \frac{C_2C_5}{2}\left(\varphi(\mathfrak{u}(t))\right)^{\frac{2}{p(\beta+1)}}. \tag{92}$$

We prove that the map $t \mapsto \varphi(\mathfrak{u}(t))$ is decreasing with respect to t.

Case $p > \frac{2}{\beta+1}$. Considering the function $t \mapsto G(t) := (\varphi(\mathfrak{u}(t)))^{\frac{2}{p(\beta+1)}}$. From (92), we can see

$$\int_0^\infty (G(\sigma))^{\frac{p(\beta+1)}{2}} d\sigma \leq \frac{1}{\eta}(G(0))^{\frac{p(\beta+1)}{2}-1} G(t), \tag{93}$$

with $\eta = \frac{2(\varphi(\mathfrak{u}_0(\mathrm{x})))^{1-\frac{p(\beta+1)}{2}}}{C_2C_5}$. From Lemma 2.3, we have

$$G(t) \leq G(0)\left(\frac{p(\beta+1)}{2+(p(\beta+1)-2)\eta t}\right)^{\frac{p(\beta+1)}{p(\beta+1)-2}}.$$

Finally, we have $\varphi(\mathfrak{u}(t) \leq \varphi(\mathfrak{u}_0(\mathrm{x})\left(\frac{p(\beta+1)}{2+(p(\beta+1)-2)\eta t}\right)^{\frac{p(\beta+1)}{p(\beta+1)-2}}$.

Case $p < \frac{2}{\beta+1}$. Since $\varphi(\mathfrak{u}(t)\leq\varphi(\mathfrak{u}_0(\mathrm{x})$, it follows from (92) that $\int_0^\infty \varphi(\mathfrak{u}(\sigma))d\sigma\leq C_5\left(\varphi(\mathfrak{u}_0(\mathrm{x}))\right)\varphi(\mathfrak{u}(t))^{\frac{2}{p(\beta+1)}-1} = \frac{1}{\mu}\varphi(\mathfrak{u}(t))$. From (1) in Lemma 2.3, we have

$$\varphi(\mathfrak{u}(t)) \leq \varphi(\mathfrak{u}_0(\mathrm{x}))\exp(1-\mu t). \quad \square$$

4. Conclusion

In this article, we treat a new general Kirchhoff function satisfying certain conditions $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ involving double phase operator on the existence of the global and blow-up of a weak solution to (1) under the restriction

$$\varphi(\mathfrak{u}_0(\mathrm{x})) < d_* := \frac{\frac{1}{p(\beta+1)}-\frac{1}{r}}{\frac{1}{p}-\frac{1}{r}} d \text{ and } 2 \leq r.$$

We have two the following results:

♣ If $\varphi(\mathfrak{u}_0) \in \mathcal{N}$, we have obtained a blow-up existence and

$$T_{\max} \leq \frac{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}}{2\sqrt{d_* - \varphi(\mathfrak{u}_0)}\left(\frac{r}{2}-1\right) - \|\mathfrak{u}_0\|_2^2} < \infty.$$

♣ If $\varphi(\mathfrak{u}_0) \in \mathcal{W}$, we have obtained a global existence. Moreover, the energy functional φ satisfies the following inequality:

$$\varphi(\mathfrak{u}(t)) \leq \begin{cases} \varphi(\mathfrak{u}_0)\left(\frac{p(\beta+1)}{2+(p(\beta+1)-2)\alpha t}\right)^{\frac{p(\beta+1)}{p(\beta+1)-2}}, & \text{if } p > \frac{2}{(\beta+1)}, \\ \varphi(\mathfrak{u}_0)e^{1-\gamma t}, & \text{if } p \leq \frac{2}{(\beta+1)}. \end{cases}$$

The fast diffusion equation case with $\varphi(\mathfrak{u}_0) \geq d_*$ and $1 < r \leq 2$, is still open.

CRediT authorship contribution statement

The authors declare that their contributions are equal.

Declaration of competing interest

The authors declare to have no conflict of interest.

Data availability

This manuscript contains no associated data.

Acknowledgments

The second and the last authors like to thank Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam, for the opportunity to work in it.

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