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Locally Balanced G-Designs

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Abstract: Let G be a graph and let K_n be the complete graph of order n. A G-design is a decomposition of the set of edges of K_n in graphs isomorphic to G, which are called *blocks*. It is well-known that a G-design is *balanced* if all the vertices are contained in the number of blocks of G. In this paper, the definition of *locally balanced G-design* is given, generalizing the existing concepts related to balanced designs. Further, locally balanced G-designs are studied in the cases in which $G \cong C_4 + e$ and $G \cong C_4 + P_3$, determining the spectrum.

Keywords: G-designs; balanced; spectrum

MSC: 05C51

1. Introduction

Let $K_v = (X, \mathcal{E})$, the complete graph having a vertex set X of v elements. Given a subgraph G of K_v, it is well-known that a G-design of order v is a pair $\Sigma = (X, \mathcal{B})$, such that the elements of $\mathcal{B}$ are subgraphs of K_v, all isomorphic to G. The elements of G are also called *blocks* and a G-design is also called a G-decomposition of K_v.

Given a G-design $\Sigma = (X, \mathcal{B})$, we denote by $d(x)$ the *degree* of a vertex $x \in X$, which is the number of blocks of $\mathcal{B}$ containing x. A G-design $\Sigma = (X, \mathcal{B})$ is called *balanced* if all the vertices of X have the same degree, i.e., if $d(x)$ is constant for any vertex $x \in X$.

Let $G = (V, \mathcal{E})$ be a graph. An *automorphism class* of G is a subset $A \subseteq V$, such that for every $x, y \in A$, there exists an automorphism φ of G, such that $\varphi(x) = y$. We will denote by $A_1, \ldots, A_s$ the automorphism classes of G.

Given a G-design $\Sigma = (X, \mathcal{B})$ and an automorphism class A_i of G, we denote by $d_{A_i}(x)$ the *degree* of a vertex $x \in X$, which is the number of blocks of Σ containing x as an element of A_i. The degree of an automorphism class A_i is the degree of the vertices of G in A_i.

It is well-known that a G-design $\Sigma = (X, \mathcal{B})$ is *strongly balanced* if, for every $i = 1, \ldots, s$, all the vertices of X are contained in the blocks of Σ the same number of times as element of A_i, i.e., if, for every $i = 1, \ldots, s$, there exists a constant $C_i \in \mathbb{N}$, such that $d_{A_i}(x) = C_i$ for every $x \in X$. In [1], the authors introduced the definition of *strongly balanced G-design*, as a particular *balanced G-design*, determining the spectrum in the case $G \cong P_k$ (P_k path with k vertices). Furthermore, the spectrum of strongly balanced $(C_4 + e)$-designs has been determined in [2]; in [3], the spectrum of strongly balanced G-designs has been determined for all graphs with five non-isolated vertices; in [4], balanced and strongly balanced G-designs are studied with the G tree with six vertices. Note that in [3] (in which the concept of orbit-balanced design is equivalent to strongly balanced design), the concept of degree-balanced G-design has been introduced. This means that the number of times that a vertex appears in a block as an element of degree d is constant for any degree d. The concept of balanced designs has been studied in relation with other designs, see, for example, [5,6], exploring also the case of hypergraph designs. In the case of hypergraphs, the notion of edge-balanced hypergraph designs has been introduced in [7].

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Of course, a strongly balanced G-design is also a balanced G-design. In this paper, we introduce the new concepts of *locally balanced* and *strictly balanced* G-design.

The graphs that will be considered in this paper are the following:

1. the graph $C_4 + e = (V, E)$ is the graph with vertex set $V = \{x, y, z, t, w\}$ and edge set $E = \{\{x, y\}, \{y, z\}, \{z, t\}, \{t, x\}, \{x, w\}\}$; we call it $(4, 1) -$ *kite* graph and it will be denoted by $[(y, z, t, x) - (w)]$;
2. the graph $C_4 + P_3$ is the graph with vertex set $V = \{x, y, z, t, w_1, w_2\}$ and edge set $E = \{\{x, y\}, \{y, z\}, \{z, t\}, \{t, x\}, \{x, w_1\}, \{w_1, w_2\}\}$; we call it $(4, 2) -$ *kite* graph and it will be denoted by $[(y, z, t, x) - (w_1, w_2)]$.

Let us recall that a $(C_4 + e)$-design of order v exists if and only if $v \equiv 0, 1 \mod 5$, $v \geq 10$ (see [8]), and that a $(C_4 + P_3)$-design exists if and only if $v \equiv 0, 1, 4, 9 \mod 12$, $v \geq 9$ (see [9]).

In Section 2, we introduce the definitions of locally balanced and strictly balanced designs. In Section 3, we determine the spectrum of strictly T-balanced $(C_4 + e)$-designs in some cases, precisely when $|T| = 2$ and T contains the automorphism class corresponding to the element of degree 3 in $C_4 + e$. In Section 4, we determine the spectrum of strongly balanced $(C_4 + P_3)$-designs, which is a new result as far as we know, and the spectrum of strictly T-balanced $(C_4 + P_3)$-designs in some cases in which $|T| = 3$ and T contains the automorphism class corresponding to the element of degree 3 in $C_4 + P_3$. All the results contained in the theorems of this paper are obviously original and not proved by anyone before today.

2. Locally Balanced and Strictly Balanced Designs

In this section, we introduce the following new definitions:

Definition 1. *Let $\Sigma = (X, \mathcal{B})$ be a G-design and let $T \subseteq \{A_1, \ldots, A_s\}$, $T \neq \emptyset$. Σ is called:*

- *locally A_i-balanced if there exists a constant C_i such that $d_{A_i}(x) = C_i$, for every $x \in X$;*
- *locally T-balanced if Σ is locally A_i-balanced for any $A_i \in T$;*
- *strictly T-balanced if Σ is locally A_i-balanced for any $A_i \in T$ and if Σ is not locally A_j-balanced for any $A_j \notin T$.*

Note that, if $\Sigma = (X, \mathcal{B})$ is an A_i-balanced G-design of order v, $d_{A_i}(x) = C_i$ for any $x \in X$ and $m = |E(G)|$, then it must be $v \cdot C_i = |A_i| \cdot |B|$. This implies that:

Theorem 1. *If $\Sigma = (X, \mathcal{B})$ is an A_i-balanced G-design of order v and $m = |E(G)|$, then $d_{A_i}(x) = \frac{|A_i|(v-1)}{2m} \in \mathbb{N}$ for any $x \in X$.*

3. Locally Balanced $(C_4 + e)$-Designs

Let $C_4 + e = (V, E)$ be the graph $[(y, z, t, x) - (w)]$. Then, its automorphism classes are $A_1 = \{w\}$, $A_2 = \{x\}$, $A_3 = \{y, t\}$ and $A_4 = \{z\}$. We will focus our attention on the class A_2, which we call the *central class*.

First, we need a few preliminary results.

Proposition 1. *Let X and Y be two disjoint sets, with $|X| = |Y| = 5$. Then, there exists a $(C_4 + e)$-decomposition of $K_{X,Y}$, such that:*

1. *$d_{A_1}(x) = 1$, $d_{A_3}(x) = 2$ for any $x \in X$, $d_{A_1}(y) = d_{A_3}(y) = 0$ for any $y \in Y$*
2. *$d_{A_2}(x) = d_{A_4}(x) = 0$ for any $x \in X$ and $d_{A_2}(y) = d_{A_4}(y) = 1$ for any $y \in Y$.*

Proof. Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{a, b, c, d, e\}$. Then it is sufficient to consider the following blocks:

$$[(1, a, 2, b) - (3)]\ [(4, b, 5, c) - (1)],\ [(2, c, 3, d) - (4)],\ [(5, d, 1, e) - (2)],\ [(3, e, 4, a) - (5)].$$

□

Proposition 2. *There exists a strongly balanced* $(C_4 + e)$*-decomposition of* $K_{10,10}$.

Proof. Let $X = \{x_1, \ldots, x_{10}\}$ and $Y = \{y_1, \ldots, y_{10}\}$. Let us consider $X_1 = \{x_1, \ldots, x_5\}$, $X_2 = \{x_6, \ldots, x_{10}\}$, $Y_1 = \{y_1, \ldots, y_5\}$ and $Y_2 = \{y_6, \ldots, 10\}$. Then it is sufficient to apply Proposition 1 to:

- K_{X_1,Y_1}, in such a way that the vertices of X_1 occupy the positions of A_1 and A_3 and the vertices of Y_1 occupy the positions of A_2 and A_4;
- K_{X_1,Y_2}, in such a way that the vertices of X_1 occupy the positions of A_2 and A_4 and the vertices of Y_2 occupy the positions of A_1 and A_3;
- K_{X_2,Y_1}, in such a way that the vertices of X_2 occupy the positions of A_2 and A_4 and the vertices of Y_1 occupy the positions of A_1 and A_3;
- K_{X_2,Y_2}, in such a way that the vertices of X_2 occupy the positions of A_1 and A_3 and the vertices of Y_2 occupy the positions of A_2 and A_4.

By taking all the blocks of these decompositions we obtain a strongly balanced $(C_4 + e)$-decomposition of $K_{10,10}$. □

Now, we determine the spectrum of the strictly T-balanced $(C_4 + e)$-designs, where T is a proper subset of $\{A_1, A_2, A_3, A_4\}$ and contains the central class A_2. It is not difficult to see that $|T| \leq 2$, otherwise we have a contradiction, because the design would be strongly balanced.

Theorem 2. *There exists a strictly* $\{A_1, A_2\}$*-balanced* $(C_4 + e)$*-design if and only if* $v \equiv 1 \mod 10$, $v \geq 11$.

Proof. By Theorem 1 we obtain the necessary condition.

Let $v = 11$. In such a case, we see that a strictly $\{A_1, A_2\}$-balanced design is $\Sigma = (X, \mathcal{B})$, where $X = \{0, 1, \ldots, 10\}$ and $\mathcal{B}$ is the set of the following blocks:

$$[(3,2,1,0)-(10)],\ [(4,2,10,1)-(9)],\ [(5,7,9,2)-(8)],\ [(6,9,5,3)-(7)],$$
$$[(7,1,3,4)-(6)],\ [(6,2,0,5)-(4)],\ [(8,5,1,6)-(0)],\ [(8,10,6,7)-(2)],$$
$$[(9,4,0,8)-(1)],\ [(10,7,0,9)-(3)],\ [(3,8,4,10)-(5)].$$

Let $v = 10h + 1$, for some $h \geq 2$. Let X_i, for $i = 1, \ldots, j$ be pairwise disjoint sets, such that $|X_i| = 1 =$ for any i and let $\infty \notin X_1 \cup \cdots \cup X_h$. Let $X = \{\infty\} \cup \bigcup_{i=1}^{h} X_i$. Let us consider the following systems:

- for any $i = 1, \ldots, h$ a strictly $\{A_1, A_2\}$-balanced $(C_4 + e)$-design $\Sigma_i = (X_i \cup \{\infty\}, \mathcal{B}_i)$ of order 11;
- for any $i, j = 1, \ldots, h$, $i \neq j$, a strongly balanced $(C_4 + e)$-decomposition $\Sigma_{ij} = (X_i \cup X_j, \mathcal{C}_{ij})$ of K_{X_i,X_j} (by Proposition 2).

Then, it is easy to see that $\Sigma = (X, \bigcup \mathcal{B}_i \cup \bigcup \mathcal{C}_{ij})$ is a strictly $\{A_1, A_2\}$-balanced $(C_4 + e)$-design of order $10h + 1$. □

Theorem 3. *There exists a strictly* $\{A_2, A_3\}$*-balanced* $(C_4 + e)$*-design of order* v *if and only if* $v \equiv 1 \mod 10$, $v \geq 21$.

Proof. The necessary condition is clear by Theorem 1. It is also easy to see that a $(C_4 + e)$ -design of order 11 which is A_2-balanced must be also A_1-balanced. So, it must be $v \geq 21$.

Now, let $v = 21$ and let $X = \{0, 1, \ldots, 20\}$. Consider the system $\Sigma = (X, \mathcal{B})$, having blocks:

- $A_i = [(i+8, i+1, i+6, i) - (i+12)]$ for $i = 0, \ldots, 20$;
- $B_i = [(i+2, i+19, i+1, i) - (i+10)]$ for $i = 0, \ldots, 20$.

The system Σ is cyclic and strongly balanced. Replace the blocks A_2, B_4 and A_{17} with:

$$[(10,3,8,2)-(18)],\ [(6,2,5,4)-(18)] \text{ and } [(4,14,2,17)-(8)].$$

Then, the system Σ' that we obtain is not strongly balanced, but strictly $\{A_2, A_3\}$-balanced.

Let $v = 21 + 10h$, for some $h \geq 1$. Let $X_1, \ldots, X_{h+2}$ pairwise disjoint sets, with $|X_i| = 10$, let $\infty \notin X_1 \cup \cdots \cup X_{h+2}$ and $X = \{\infty\} \cup X_1 \cup \cdots \cup X_{h+2}$. Let us consider:

- a $(C_4 + e)$-design $\Sigma = (X_{h+1} \cup X_{h+2} \cup \{\infty\}, \mathcal{B})$ of order 21 which is strictly $\{A_2, A_3\}$ -balanced;
- a strongly balanced $(C_4 + e)$-design $\Sigma_i = (X_i \cup \{\infty\}, \mathcal{B}_i)$ for $i = 1, \ldots, h$;
- a strongly balanced $(C_4 + e)$-decomposition of K_{X_i,X_j} for $i, j \in \{1, \ldots, h\}$, $i \neq j$, and for $i \in \{1, \ldots, h\}$ and $j = h+1, h+2$, by Proposition 2.

Let $\mathcal{B}'$ be the set of all these blocks and let $\Sigma' = (X, \mathcal{B}')$. Then Σ' is a $(C_4 + e)$-design which is strictly $\{A_2, A_3\}$-balanced. □

Theorem 4. *There exists a strictly $\{A_2, A_4\}$-balanced $(C_4 + e)$-design of order v if and only if $v \equiv 1 \mod 10$, $v \geq 21$.*

Proof. The necessary condition is clear by Theorem 1. As in the previous result, it must be $v \geq 21$. For $v = 21$, take the strongly balanced system Σ considered in Theorem 3 and replace the blocks A_0 and B_2 with the blocks:

$$[(8,1,6,0)-(4)] \text{ and } [(12,0,3,2)-(4)].$$

Then, the system Σ' that we obtain is strictly $\{A_2, A_4\}$-balanced.

Now, for $v > 21$ it is sufficient to repeat the construction of Theorem 3 in order to obtain the statement. □

Theorem 5. *There exists a strictly A_2-balanced $(C_4 + e)$-design of order v if and only if $v \equiv 1 \mod 10$, $v \geq 21$.*

Proof. The necessary condition is clear by Theorem 1. As before, it must be $v \geq 21$, because for $v = 11$, a locally A_2-balanced $(C_4 + e)$-design is also locally A_1-balanced. For $v = 21$, take the strongly balanced system Σ, considered in Theorem 3, and replace the blocks A_1 and B_0 with:

$$[(9,2,19,1)-(13)] \text{ and } [(2,7,1,0)-(10)].$$

In this way, we obtain a strictly $\{A_1, A_2\}$-balanced $(C_4 + e)$-design of order 21. If we also replace the blocks A_0 and B_2, as in Theorem 3, we obtain a strictly A_2-balanced design of order 21.

Now, for $v > 21$, it is sufficient to repeat the construction of Theorem 3 in order to obtain the statement. □

4. Locally Balanced $(C_4 + P_3)$-Designs

Let $C_4 + P_3 = (V, E)$ be the graph $[(y, z, t, x) - (w_1, w_2)]$. Then, its automorphism classes are $A_1 = \{w_2\}$, $A_2 = \{w_1\}$, $A_3 = \{x\}$, $A_4 = \{y, t\}$ and $A_5 = \{z\}$. First, let us determine the spectrum of strongly balanced $(C_4 + P_3)$-designs.

Theorem 6. *There exists a strongly balanced $(C_4 + P_3)$-design if and only if $v \equiv 1 \mod 12$, $v \geq 13$.*

Proof. Let $\Sigma = (X, \mathcal{B})$ be any $(C_4 + P_3)$-design of order v, which is locally A_3-balanced. If $d_{A_3}(x) = C \in \mathbb{N}$ for every $x \in X$, then necessarily:

$$C \cdot v = |\mathcal{B}|,$$

from which $C = \frac{v-1}{12}$ and $v \equiv 1, \mod 12, v \geq 13$.

Now, let $v = 12h + 1$, for some $h \in \mathbb{N}$, $h \geq 1$. Let Σ be the cyclic system on $\{0, 1, \ldots, 12h\}$ having as blocks:

$$A_{i,j} = [(j, i+j, 4h+1+j, h+i+j) - (9h+1+j, 2h+i+j)],$$

for $i = 1, \ldots, h$ and $j = 0, 1, \ldots, 12h$. Then Σ is clearly strongly balanced. □

The determination of the spectrum for a few strictly T-balanced $(C_4 + P_3)$-designs, with T containing the central class A_3, consisting of the only vertex in $(C_4 + P_3)$ having degree three, is given as follows.

Theorem 7. *There exists a strongly balanced $(C_4 + P_3)$-decomposition of $K_{12h,12h}$.*

Proof. First, let us consider two disjoint sets $\{0_1, 1_1, \ldots, 5_1\}$ and $\{0_2, 1_2, \ldots, 5_2\}$. Then, the system $\Sigma = (\{0_1, 1_1, \ldots, 5_1\} \cup \{0_2, 1_2, \ldots, 5_2\}, \mathcal{B})$, having blocks:

$$[((i+1)_2, (i+4)_1, (i+2)_2, i_1) - (i_2, (i+1)_1]$$

for $i = 0, \ldots, 5$ is a $(C_4 + P_3)$-decomposition of $K_{\{0_1,1_1,\ldots,5_1\},\{0_2,1_2,\ldots,5_2\}}$, such that:

- the vertices of A_1, A_3 and A_5 are occupied by $\{0_1, 1_1, \ldots, 5_1\}$, each appearing exactly one time;
- the vertices of A_2 are occupied by $\{0_2, 1_2, \ldots, 5_2\}$, each appearing exactly one time;
- the vertices of A_4 are occupied by $\{0_2, 1_2, \ldots, 5_2\}$, each appearing exactly twice.

Now, let $X = \bigcup_{i=1}^{2h} X_i$ and $Y = \bigcup_{i=1}^{2h} Y_i$, where X_i and Y_i for $i = 1, \ldots, 2h$ are all pairwise disjoint sets, such that $|X_i| = |Y_i| = 6$. Let $D = (\{x_1, \ldots, x_{2h}\} \cup \{y_1, \ldots, y_{2h}\}, E)$ be an oriented complete bipartite graph with partite sets $\{x_1, \ldots, x_{2h}\}$ and $\{y_1, \ldots, y_{2h}\}$ and an Eulerian orientation, which means that $d^+(x_i) = d^-(x_i) = d^+(y_i) = d^-(y_i) = h$ for any $i = 1, \ldots, h$.

If $(x_i, y_j) \in E$ (resp. $(y_j, x_i) \in E$) for some $i, j \in \{1, \ldots, 2h\}$, then consider a $(C_4 + P_3)$-decomposition of K_{X_i,Y_j} such that:

- the vertices of A_1, A_3 and A_5 are occupied by X_i (resp. Y_j), each appearing exactly one time;
- the vertices of A_2 are occupied by Y_j (resp. X_i), each appearing exactly one time;
- the vertices of A_4 are occupied by Y_j (resp. X_i), each appearing exactly twice.

Let Σ' be the system on $X \cup Y$ having all these as blocks. Then it is easy to see that Σ' satisfies the conditions of the statement. □

Theorem 8. *There exists a strictly $\{A_1, A_3, A_4\}$-balanced $(C_4 + P_3)$-design of order* 13.

Proof. Let $X = \mathbb{Z}_{13}$ and let $\mathcal{B}$ be the family of blocks $(C_4 + P_3)$, defined as follows:

$$[(2,8,5,0) - (6,3)],\ [(3,9,6,1) - (7,4)],\ [(4,10,7,2) - (6,5)],$$
$$[(5,11,8,3) - (7,6)],\ [(6,10,9,4) - (8,7)],\ [(7,0,10,5) - (9,8)],$$
$$[(8,1,11,6) - (12,9)],\ [(9,0,12,7) - (11,10)],\ [(10,3,0,8) - (12,11)],$$
$$[(11,4,1,9) - (2,12)],\ [(12,5,2,10) - (1,0)],\ [(0,4,3,11) - (2,1)],$$
$$[(1,5,4,12) - (3,2)].$$

It is possible to verify that $\Sigma = (X, \mathcal{B})$ is a $(C_4 + P_3)$-design of order 13, defined in X. Further, we can control that:

$$d_{A_1}(x) = 1,\, d_{A_3}(x) = 1,\, d_{A_4}(x) = 2$$

for every $x \in X$, and that:

$$d_{A_2}(7) = 2 \text{ and } d_{A_5}(7) = 0.$$

Further, it is:

$$d_{A_2}(0) = 0 \text{ and } d_{A_5}(0) = 2.$$

Therefore, Σ is a strictly $\{A_1, A_3, A_4\}$-balanced system. □

Theorem 9. *There exists a strictly* $\{A_1, A_3, A_4\}$*-balanced* $(C_4 + P_3)$*-design if and only if* $v \equiv 1$, mod 12, $v \geq 13$.

Proof. As before, if a system $\Sigma = (X, \mathcal{B})$ of order v is locally A_3-balanced, then $v \equiv 1$ mod 12, $v \geq 13$.

Construction. $v = 12h + 1 \longrightarrow v + 12$**.** Let $\Sigma_1 = (X_1, \mathcal{B}_1)$, $\Sigma_2 = (X_2, \mathcal{B}_2)$ be two $(C_4 + P_3)$-designs, both strictly $\{A_1, A_3, A_4\}$-balanced, of order, respectively, $v_1 = 12h + 1, h \geq 1$, and $v_2 = 13$, such that $X_1 \cap X_2 = \{\infty\}$. Observe that the vertices of Σ_1 all have degrees h in A_1, h in A_3, $2h$ in A_4 and the vertices of Σ_2 all have degrees 1 in A_1, 1 in A_3 and 2 in A_4.

Let $X = X_1 \cup X_2 = X$. It is $|X| = v = 12(h+1) + 1$. Let Π be a partition on $X_1 - \{\infty\}$ in h classes $C_1, C_2, \ldots, C_h$, all of cardinality 12.

For every $C_i \in \Pi$, consider a strongly balanced system $(C_i \cup X_2 - \{\infty\}, \Gamma_i)$ by Theorem 7. Let $\mathcal{F} = \Gamma_1 \cup \Gamma_2 \cup .... \cup \Gamma_h$. If $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{F}$, we can say that $\Sigma = (X, \mathcal{B})$ is a strictly $\{A_1, A_3, A_4\}$-balanced $(C_4 + P_3)$-design of order $v = 12(h+1) + 1$.

Indeed, in every family Γ_i, $i = 1, 2, ..., h$, all the vertices of $C_i \cup X_2 - \{\infty\}$ occupy the positions of A_1, A_3 and A_4 the same number of times, i.e.:

$$d_{A_1}(x) = 1,\, d_{A_3}(x) = 1,\, d_{A_4}(x) = 2$$

for every $x \in C_i \cup X_2 - \{\infty\}$, in Γ_i. This means that:

$$d_{A_1}(x) = h + 1,\, d_{A_3}(x) = h + 1,\, d_{A_4}(x) = 2h + 2 \quad \forall x \in X_1 \cup X_2,$$

and therefore Σ is a strictly $\{A_1, A_3, A_4\}$-balanced system. □

Theorem 10. *There exists a strictly* $\{A_2, A_3, A_5\}$*-balanced* $(C_4 + P_3)$*-design if and only if* $v \equiv 1$ mod 12, $v \geq 25$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a strictly $\{A_2, A_3, A_5\}$-balanced $(C_4 + P_3)$-design of order v. As in Theorem 9, we see that $v \equiv 1 \mod 12$. Suppose, now, that $v = 13$. In this case, we have:

$$d_{A_2}(x) = 1,\, d_{A_3}(x) = 1,\, d_{A_5}(x) = 1 \quad \forall x \in X.$$

This implies that:

$$d_{A_1}(x) + 2d_{A_4}(x) = 5 \quad \forall x \in X.$$

Clearly, this means that $d_{A_1}(x) \geq 1$ for any $x \in X$ and, since $|\mathcal{B}| = 13$, we obtain $d_{A_1}(x) = 1$ for any $x \in X$, so that Σ is strongly balanced.

Now, let $v = 25$. We want to construct a locally $\{A_2, A_3, A_5\}$-balanced $(C_4 + P_3)$-design which is not locally A_1 and A_4-balanced. So, consider the cyclic system $\Sigma = (\{0, 1, \ldots, 24\}, \mathcal{B})$, having blocks:

$$A_i = [(i+2, i+5, i+1, i) - (i+6, i+11)],$$

and

$$B_i = [(i+8, i+17, i+7, i) - (i+11, i+23)],$$

for $i = 0, 1, \ldots, 24$. Then consider the blocks A_0, A_5, B_0, B_{13} and B_{24} and replace them with the blocks:

$$[(1,5,2,0) - (6,16)],\ [(6,10,7,5) - (11,23)],\ [(7,17,8,0) - (11,6)],$$
$$[(20,5,21,13) - (24,6)],\ [(11,16,7,24) - (10,22)].$$

Then, we obtain a system Σ' of order 25 which is strictly $\{A_2, A_3, A_5\}$-balanced.

Now, let $v = 24h+1$, for some $h \geq 2$. Let $X_1, \ldots, X_h$ be pairwise disjoint sets, such that $|X_i| = 24$ for any $i = 1, \ldots, h$ and let $\infty \notin X_1 \cup \cdots \cup X_h$. Let $X = \bigcup_{i=1}^{h} X_i \cup \{\infty\}$. Let us consider $\Sigma_i = (X_i \cup \{\infty\}, \mathcal{B}_i)$ a $(C_4 + P_3)$-design of order 25 satisfying the conditions of the statement for any $i = 1, \ldots, h$. Moreover, for any $i, j \in \{1, \ldots, h\}$, $i \neq j$, consider a system $\Sigma_{i,j} = (X_i \cup X_j, \mathcal{C}_{ij})$ satisfying the conditions of Theorem 7. Then, clearly $\Sigma = (X, \bigcup \mathcal{B}_i \cup \bigcup \mathcal{C}_{ij})$ is a $(C_4 + P_3)$-design of order $24h+1$ which is strictly $\{A_2, A_3, A_5\}$-balanced.

Let $v = 24h+13$, for some $h \in \mathbb{N}$, $h \geq 1$. Let $X_1, \ldots, X_h$, Y be pairwise disjoint sets, such that $|X_i| = 24$ for any $i = 1, \ldots, h$ and $|Y| = 12$ and let $\infty \notin X_1 \cup \cdots \cup X_h \cup Y$. Let us consider $\Sigma_1 = (\bigcup_{i=1}^{h} X_i \cup \{\infty\}, \mathcal{B}_1)$, a $(C_4 + P_3)$-design of order v which is strictly $\{A_2, A_3, A_5\}$-balanced. Consider also a $(C_4 + P_3)$-design $\Sigma_2 = (Y \cup \{\infty\}, \mathcal{B}_2)$ of order 13 which is strongly balanced by Theorem 6. At last, by Theorem 7 we can consider a $(C_4 + P_3)$-decomposition $\Sigma_3 = (\bigcup X_i \cup Y, \mathcal{B}_3)$ of $K_{Y, \bigcup X_i}$, such that:

- the vertices of A_1, A_2, A_3 and A_5 are occupied by each vertex in Y $2h$ times and by each vertex in X_i, for any $i = 1, \ldots, h$, exactly 1 time;
- the vertices of A_4 are occupied by each vertex in Y exactly $4h$ times and by each vertex in X_i, for any $i = 1, \ldots, h$, exactly 2 times.

Let $\Sigma = (\bigcup X_i \cup Y \cup \{\infty\}, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$. Then clearly Σ is a $(C_4 + P_3)$-design of order v satisfying the conditions of the statement. □

Theorem 11. *There exists a strictly $\{A_1, A_3, A_5\}$-balanced $(C_4 + P_3)$-design if and only if $v \equiv 1 \mod 12$, $v \geq 13$.*

Proof. As before, if a system $\Sigma = (X, \mathcal{B})$ of order v is locally A_3-balanced, then $v \equiv 1 \mod 12$, $v \geq 13$.

Now, let $v = 12h+1$, for some $h \in \mathbb{N}$, $h \geq 1$. Let Σ be the cyclic system on $\{0, 1, \ldots, 12h\}$ having as blocks:

$$A_{i,j} = [(j, i+j, 4h+1+j, h+i+j) - (9h+1+j, 2h+i+j)],$$

for $i = 1, \ldots, h$ and $j = 0, 1, \ldots, 12h$. Then consider the blocks $A_{1,0}$ and $A_{1,11h+1}$ and replace them with the blocks:

$$[(0, 1, 8h+1, h+1) - (9h+1, 2h+1)]$$

and

$$[(11h+1, 11h+2, 3h+1, 1) - (4h+1, h+1)].$$

Then, it is easy to see that the system Σ' that we obtain with such a substitution is locally $\{A_1, A_3, A_5\}$-balanced, but not locally A_2 and A_4-balanced. □

5. Conclusions

In this paper, the notions of *locally balanced G-design* and *strictly T-balanced G-design* have been introduced, with T being a set of automorphism classes of G, following the already well-known concepts of balanced and strongly balanced graph designs. We analyze this problem in the cases that $G = C_4 + e$ and $G = C_4 + P_3$, determining the spectrum of strictly T-balanced G-design for some T.

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