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Elliptic differential inclusions and applications to implicit equations

tesi di dottorato

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Contents

Introduction

This thesis is devoted to the study of different types of elliptic differential inclusions and their applications to a wide range of implicit equations.

The theory of differential inclusions, started in 1934-1936, as it can be read with more details in [22], was motivated by those dynamical systems whose velocity is not uniquely determined by their state, but it depends on it. Then, it was developed within the framework of set-valued analysis and, in the last years, attracted many authors because of both theoretical aspects and applications to solve problems arising from other fields, like Physics, Mechanics, Engineering, Social and Biological sciences.

In particular, as observed in [4] and [23], deterministic models are not suitable for explaining the evolution of some systems, when they are quite related to phenomena with a high degree of uncertainty, the absence of controls and the variety of available dynamics. These models appear, for example, in control theory, differential game theory and friction dynamics. Differential inclusions have been intensively studied since eighties and, in a first moment, existence results have been established in the ordinary case. Naturally, mathematicians extended previous results to partial differential inclusions, making use of several methods, such as fixed point theory, Leray-Schauder theory, monotone operators and approximation schemes.

We wish to present what an abstract differential inclusion is, following the ideas contained in [6]. Let V, W, Z be linear normed spaces. Consider an operator A from $domA \subseteq W$ to V, a mapping $j : domj \subseteq W \to Z$, and a multifunction $F : domF \subseteq Z \to V$ with closed valued. We pose the following question:

$$\text{Find } u \in W \text{ such that } Au \in F(ju).$$

This represents a very general problem, that requires appropriate hypotheses on j, A and F to be solved, and it has been largely investigated through different techniques. Now, we want to give more details about the point of view adopted in this thesis.

Let Ω be a bounded domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$ and let $1 < p < +\infty$, except when it is otherwise specified. Along our subject, we deal with different types of elliptic inclusions, that we briefly summarize as follows. Next we point out what tools and features have been studied and used in each context.

1) Find $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu \in F(x, u, Du)$ a.e. in $\Omega \subseteq \mathbb{R}^N$, where $3 \leq N < p < +\infty$, $F\colon \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \to 2^{\mathbb{R}^h}$, and L is a suitable linear second-order elliptic operator in non-divergence form.

2) Find $u \in W_0^{1,p}(\Omega)$ satisfying $-\Delta_p u \in F(x, u, \nabla u)$ a.e. in Ω, where $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ is lower semicontinuous and Δ_p denotes the p-Laplace operator.

3) Find $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in F(x, u)$ a.e. in Ω, with hypotheses of upper semicontinuity on $F\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$, that is a result employed, for example, to study an implicit differential equation in a discontinuous framework.

4) Find $u \in H_0^1(\Omega)$ satisfying $-\Delta u \in \partial J(u)$ a.e. in $\Omega \subseteq \mathbb{R}^N, N \geq 1$, where $\partial J\colon \mathbb{R} \to 2^{\mathbb{R}}$ is the gradient of a convenient locally Lipschitz continuous potential.

5) Find $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in \partial J(x, u)$ a.e. in $\Omega \subseteq \mathbb{R}^N$, $1 \leq N < p < +\infty$, being ∂J as in 4).

The abstract framework where inclusions are placed is that of set-valued analysis. In particular, the notions of lower semicontinuous multifunction with decomposable values and selection are crucial.

In 1) and 2) we make use of the Kuratowski and Ryll-Nardzewski Theorem that produces a measurable one, and the Bressan-Colombo-Fryszkowski Theorem, that furnishes a continuous selection, under suitable hypotheses on the multifunction F. Moreover, fixed-point arguments are exploited, like the classical Schauder Theorem and the Leray-Schauder alternative.

In 3), a solution to the problem is obtained via a general result for inclusions of the type $\Psi(u) \in F(x, \Phi(u))$, adapted there thanks to accurate choices of Φ and Ψ related to the p-Laplacian and its properties.

4) and 5) have in common the non-smooth setting, where locally Lipschitz continuous functions are used as well as the generalized gradient. In 4) we present a different approach to differential inclusions, based on variational methods. Instead, 5) shows how previous techniques from set-valued analysis can be used in this context.

One of the most interesting feature of differential inclusions is the possibility of being used to solve implicit differential equations, so we give just some technical details to specify how it can be possible.

Now, let Y be a suitable subset of $\mathbb{R}^h$, let us consider a real-valued function f defined on $\Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \times Y$ and let $\mathcal{L}$ be a second-order elliptic operator that will be specified later according to the setting. Moreover, we put

$$Lu = (\mathcal{L}u_1, \mathcal{L}u_2, \cdots, \mathcal{L}u_h).$$

In Chapter 3 we study implicit equations of the type

$$f(x, u, Du, Lu) = 0, \tag{1}$$

with homogenuous Dirichlet boundary condition.

Through Theorem 1.10, we reduce (1) to the elliptic differential inclusion

$$Lu \in F(x, u, Du), \tag{2}$$

where F is a lower semicontinuous multiselection of the multifuncion

$$(x, z, w) \to \{y \in Y : f(x, z, w, y) = 0\}.$$

Eventually, we use different results to solve (2) according to the hypotheses on f and on the type of the elliptic operator $\mathcal{L}$, that, according to the framework, may be a linear second-order one, the p-Laplacian, or the classical Laplacian.

The original part of the thesis, based on [35], consists in the study of the following implicit elliptic problem

$$u \in W_0^{1,p}(\Omega), \qquad f(x, u, \nabla u, \Delta_p u) = 0 \quad \text{in } \Omega. \tag{3}$$

We focus on the particular case when the function f can be expressed in the form $f(x, z, w, y) = \varphi(x, z, w) - \psi(y)$, where Y is a nonempty interval of $\mathbb{R}$, φ is a real-valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and ψ is a real-valued function defined on Y, which depends only on the p-Laplacian $\Delta_p u$. We further distinguish among the case where φ is a Carathéodory function and depends on x, u, and ∇u, and the case where φ is allowed to be highly discontinuous in each variable, for which the dependance on the gradient is not allowed.

As before, we reduce (3) to the elliptic differential inclusion problem:

$$\begin{cases} -\Delta_p u \in F(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Both the case when Y coincides with the whole space $\mathbb{R}$ and when Y is a closed interval of $\mathbb{R}$ will be treated. One of the main results is the following, where $\lambda_{1,p}$ indicates the first eigenvalue of the p-Laplacian in the space $W_0^{1,p}(\Omega)$.

Theorem 0.1
Let $\varphi\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function and let $\psi\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that:

(i) for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the set $\{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0\}$ has empty interior;

(ii) for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the function $y \mapsto \varphi(x, z, w) - \psi(y)$ changes sign;

(iii) there exist $a \in L^{p'}(\Omega, \mathbb{R}_0^+), b, c \geq 0$, with $\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{1/p}} < 1$, such that

$$\sup\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + b|z|^{p-1} + c|w|^{p-1},$$

for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\psi(-\Delta_p u) = \varphi(x, u, \nabla u) \qquad in \quad \Omega. \tag{4}$$

When φ is discontinuous, we construct an appropriate upper semicontinuous multi-function F related with ψ^{-1} and φ, and then we solve the elliptic differential inclusion $-\Delta_p u \in F(x, u)$.

The result below is obtained, where we denote by π_0 and π_1 the projections of $\Omega \times \mathbb{R}$ on Ω and $\mathbb{R}$, respectively.

Theorem 0.2
Let $\mathcal{F} = \{A \subseteq \Omega \times \mathbb{R} : A$ is measurable and $m(\pi_i(A)) = 0$ for some $i \in \{0, 1\}\}$, $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval which does not contain 0, ψ a continuous real-valued function defined on (α, β), φ a real-valued function defined on $\Omega \times \mathbb{R}$, and $p > N$. Suppose that

(i) φ is $\mathcal{L}(\Omega \times \mathbb{R})$-measurable and essentially bounded;

(ii) the set $D_\varphi = \{(x, z) \in \Omega \times \mathbb{R} : \varphi$ is discontinuous at $(x, z)\}$ belongs to $\mathcal{F}$;

(iii) $\varphi^{-1}(r) \setminus \operatorname{int}(\varphi^{-1}(r)) \in \mathcal{F}$ for every $r \in \psi((\alpha, \beta))$;

(iv) $\overline{\varphi(S \setminus D_\varphi)} \subseteq \psi((\alpha, \beta))$.

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\psi(-\Delta_p u) = \varphi(x, u) \qquad \text{in} \quad \Omega.$$

We finally present some particular cases and applications of previous results. For example, equations of the type

$$-\Delta_p u = f(x) + \mu(|u| + |\nabla u|)^\gamma - \lambda e^{-\Delta_p u}$$

or

$$-\Delta_p u \cos(-\Delta_p u) = f(x),$$

are solved in $W_0^{1,p}(\Omega)$.

Chapter 1

Preliminaries

In this chapter we gather basic definitions and general results that will be used later. The abstract framework of set-valued analysis is then presented, with a special attention to the crucial notion of selection for a multifunction. Furthermore, we introduce a linear, second-order, elliptic differential operator and then the p-Laplace operator as well as some basic properties.

From now on, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$. The symbol $\mathcal{L}(\Omega)$ will denote the Lebesgue σ-algebra of Ω and $m(\Omega)$ the Lebesgue measure of Ω.
Given a nonnegative integer k, $W^{k,p}(\Omega)$ stands for the Sobolev space of all real-valued functions defined on Ω whose weak partial derivatives up to the order k lie in $L^p(\Omega)$. If

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \text{ and } u \in W^{k,p}(\Omega),$$

we set

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \text{ and } D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}$$

(in the weak sense). A norm on $W^{k,p}(\Omega)$ is introduced by defining

$$\|u\|_{k,p} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_p, \ u \in W^{k,p}(\Omega).$$

Given a positive integer h, we denote by $W^{k,p}(\Omega, \mathbb{R}^h)$ the space of all functions $u : \Omega \to \mathbb{R}^h, u = (u_1, u_2, \dots, u_h)$, such that $u_i \in W^{k,p}(\Omega)$ for every $i = 1, 2 \dots, h$. The norm in this space is defined by

$$\|u\|_{W^{k,p}(\Omega,\mathbb{R}^h)} = \sum_{i=1}^{h} \|u_i\|_{k,p} \ , \quad u = (u_1, u_2, \dots, u_n) \in W^{k,p}(\Omega, \mathbb{R}^h).$$

In this work, we often deal with the case $k = h = 1$, namely the space $W^{1,p}(\Omega)$ that is reflexive and separable for $1 < p < +\infty$. Of fundamental importance the Sobolev space $W_0^{1,p}(\Omega)$ will be, which stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Roughly speaking, $W_0^{1,p}(\Omega)$ is the space of all $u \in W^{1,p}(\Omega)$ such that $u(x) = 0$ for every $x \in \partial\Omega$.
On $W_0^{1,p}(\Omega)$ we introduce the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx \right)^{1/p}, \qquad u \in W_0^{1,p}(\Omega).$$

It is useful to observe that for all $u \in W_0^{1,p}(\Omega)$ we have $\|u\| = \|\nabla u\|_{L^p(\Omega)}$.

Let p^* be the critical exponent for the Sobolev embedding $W_0^{1,p}(\Omega) \subseteq L^r(\Omega)$. Recall that

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{otherwise.} \end{cases}$$

If $p \neq N$, then to each $r \in [1, p^*]$ there corresponds a constant $c_{rp} > 0$ satisfying

$$\|u\|_{L^r(\Omega)} \leq c_{rp}\|u\|, \qquad \forall\, u \in W_0^{1,p}(\Omega),$$

whereas, when $p = N$, for every $r \in [1, +\infty)$ we have

$$\|u\|_{L^r(\Omega)} \leq c_{rN}\|u\|, \qquad \forall\, u \in W_0^{1,N}(\Omega).$$

Finally, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ is compact, provided $1 \leq r < p^*$. When $p > N$, we get $W_0^{1,p}(\Omega) \subseteq L^\infty(\Omega)$ and

$$\|u\|_\infty \leq a\|u\|, \quad u \in W_0^{1,p}(\Omega), \tag{1.1}$$

for suitable $a > 0$; see [12, Ch. IX].

Given $p \in]1, +\infty[$, the symbol p' will denote the conjugate exponent of p while $W^{-1,p'}(\Omega)$ stands for the dual space of $W^{1,p}(\Omega)$. Through [12, Theorem 6.4], we see that $L^{p'}(\Omega)$ compactly embeds in $W^{-1,p'}(\Omega)$. So, there exists $b > 0$ satisfying

$$\|v\|_{W^{-1,p'}(\Omega)} \leq b\|v\|_{L^{p'}(\Omega)}, \quad \forall\, v \in L^{p'}(\Omega). \tag{1.2}$$

1.1 The elliptic setting

We want to continue our subject by presenting the elliptic setting where the first class of problems is placed.
Let $\mathcal{L}$ be the linear, second-order elliptic differential operator defined by

$$\mathcal{L}u = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + \sum_{i,j=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{1.3}$$

where $a_{i,j} \in C^0(\overline{\Omega})$, $a_{ij} = a_{ji}$ for every $i, j = 1, 2, \ldots, N$ and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(\xi_1^2 + \xi_2^2 + \ldots + \xi_N^2)$$

for some $\lambda > 0$, every $x \in \Omega$ and every $(\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N$; $b_i \in L^\infty(\Omega)$ for all $i = 1, 2, \ldots, N$; $c \in L^\infty(\Omega)$ and $c(x) \geq 0$ almost everywhere in Ω.
The operator $\mathcal{L}$ has the following properties; see [21, Theorem 9.15, Lemma 9.17].

1. $\mathcal{L}$ is a one-to-one operator from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$;

2. the inverse operator $\mathcal{L}^{-1} : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is continuous.

We define, for every $u \in (u_1, u_2, \ldots, u_h) \in W^{2,p}(\Omega, \mathbb{R}^h)$,

$$Lu = (\mathcal{L}u_1, \mathcal{L}u_2, \cdots, \mathcal{L}u_h).$$

The linear operator L is one-to-one from $W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ onto $L^p(\Omega, R^h)$ and its inverse L^{-1} is continuous. We denote by $\|L^{-1}\|$ the norm of L^{-1}.

Moreover, for every $(u_1, u_2, \cdots, u_h) \in W^{1,p}(\Omega, \mathbb{R}^h)$, we set

$$Du = (Du_1, Du_2, \cdots, Du_h),$$

where Du_i is the gradient of the function u_i.

A classical example of $\mathcal{L}$ is the Laplace operator $\Delta u = \displaystyle\sum_{i=0}^{N} \frac{\partial^2 u}{\partial x_i^2}$.

1.2 The $p - Laplacian$ operator

One of the most important partial differential equation of the second order is

$$\Delta u = 0,$$

the so-called *Laplace equation*, which represents the prototype for linear elliptic equations. There exists also a non-linear counterpart, the p-Laplace equation or p-harmonic equation, depending on a parameter p. The p-Laplace equation has been much studied during the last fifty years and its theory is by now rather developed, even though some open problems remain. The p-Laplace equation is a degenerate or singular elliptic equation in divergence form:

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

Consequently, the $p - Laplacian$ operator is defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad \forall\, u \in W^{1,p}(\Omega),$$

where $1 < p < +\infty$ and $\Omega \subseteq \mathbb{R}^N$.
It is clear that when $p = 2$ it becomes the classical Laplace operator.
The p-Laplacian is an important example of degenerated/singular quasilinear elliptic operator; for more completeness on the subject see [27].

Now, we want to focus on

$$A_p\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega), \tag{1.4}$$

namely the nonlinear operator stemming from the negative p-Laplacian, i.e.,

$$\langle A_p(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx, \qquad u, v \in W_0^{1,p}(\Omega)$$

and some useful properties concerning the inverse operator

$$A_p^{-1}\colon W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega),$$

that we need to recall; see, e.g., [36, Appendix A].

Lemma 1.1
Let $\Omega \subseteq R^N$ be a bounded domain. The following facts hold true:

(p_1) A_p is bijective and uniformly continuous on bounded sets;

(p_2) A_p^{-1} is $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$-continuous;

(p_3) $\|A_p(u)\|_{W^{-1,p'}(\Omega)} = \|u\|_p^{p-1}$ in $W_0^{1,p}(\Omega)$.

Before stating the next result, we want to focus on $\lambda_{1,p}$, the first eigenvalue of A_p in $W_0^{1,p}(\Omega)$; see [25] for more details. The starting point is the following eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The Liusternik-Schnirelman theory provides a non decreasing sequence $\{\lambda_{n,p}\}$ of nonnegative eigenvalues for (1.5). The first or principal eigenvalue $\lambda_{1,p}$ is isolated and simple. Moreover,

$$\lambda_{1,p} = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx},$$

from which we infer

(p$_4$) $\|u\|_{L^p(\Omega)}^p \leq \dfrac{1}{\lambda_{1,p}}\|u\|_p^p$, for all $u \in W_0^{1,p}(\Omega)$.

Many authors studied problems involving the p-Laplacian and the starting point is the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subseteq R^N$, $f \in W^{-1,p'}(\Omega)$, $p' = p/(p-1)$. We have the following simple result, whose proof relies on the Classical Calculus of Variations and consists on finding a solution that is a critical point of a suitable functional.

Theorem 1.2
Suppose $\Omega \subseteq R^N$ is a bounded domain and $f \in W^{-1,p'}(\Omega)$. Then, problem (1.6) has a solution $u \in W_0^{1,p}(\Omega)$ in the weak sense, that means

$$\int_\Omega (\langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle - fv)dx = 0, \quad \forall v \in W_0^{1,p}(\Omega).$$

1.3 General definitions and properties of set-valued analysis

Let X be a topological space and let $V \subseteq X$. We denote by $\mathrm{int}(V)$ the interior of V and by $\bar{V}$ the closure of V. The symbol $\mathcal{B}(X)$ is used to denote the Borel σ-algebra of X.
If (X, d) is a metric space, we define

$$B(x,r) = \{z \in X : d(x,z) \leq r\}, \quad d(x,V) = \inf_{z\in V} d(x,z) \quad \text{and} \quad d(V,x) = \sup_{z\in V} d(x,z),$$

for every $x \in X, r \geq 0$ and every nonempty set $V \subseteq X$.

Definition 1.3

Let X and Z be two nonempty sets. A multifunction Φ from X into Z is a function from X into the family of all subsets of Z, namely $\Phi\colon X \to 2^Z$.

For every $W \subseteq Z$ we define the set

$$\Phi^-(W) = \{x \in X : \Phi(x) \cap W \neq \emptyset\}$$

and the set

$$\Phi^+(W) = \{x \in X : \Phi(x) \subseteq W\}.$$

Definition 1.4

Let $(X, \mathcal{A})$ be a measurable space and let Z be a topological space. We say that the multifunction $\Phi\colon X \to 2^Z$ is measurable when for every open set $W \subseteq Z$ we have $\Phi^-(W) \in \mathcal{A}$.

Definition 1.5

If X and Z are two topological spaces and, for every open (resp. closed) set $W \subseteq Z$, the set $\Phi^+(W)$ is closed (resp. open) in X, we say that Φ is lower semicontinuous (resp. upper semicontinuous), briefly l.s.c. (resp. u.s.c.).

Definition 1.6

If X and Z are two topological spaces and, for every open (resp. closed) set $W \subseteq Z$, the set $\Phi^-(W)$ is open (resp. closed) in X, we say that Φ is lower semicontinuous (resp. upper semicontinuous).

A useful characterization of the lower semicontinuity is the following; cf. [40, Theorem 1.1].

Proposition 1.7
When (Z, δ) is a metric space, the multifunction Φ is lower semicontinuous if and only if, for every $z \in Z$, the real-valued function $x \mapsto \delta(z, \Phi(x))$, $x \in X$, is upper semicontinuous.

Another condition that ensures this property deals with the weakly convergent subsequences; see [26, Theorem 7.1.7].

Proposition 1.8
If X is a first countable topological space, then the multifunction Φ is lower semicontinuous if and only if, for every $x \in X$, every sequence $\{x_k\}$ in X converging to x and every $z \in \Phi(x)$, there exists a sequence $\{z_k\}$ in Z converging to z and such that $z_k \in \Phi(x_k)$, for all $k \in \mathbb{N}$.

1.4 A fundamental result of lower semicontinuity

Let us first recall a technical lemma, that is [39, Lemma 2.1].

Lemma 1.9
Let S be a connected topological space and let g be a real-valued function defined on S. Let $s_0 \in S$ be a relative maximum point but not absolute for g and suppose that $\mathrm{int}(g^{-1}(g(s_0))) = \emptyset$. Then, there exists $x_0 \in S$ such that $g(x_0) = g(s_0)$ and s_0 is not a relative extremum point for g.

A particular and fruitful multifunction, associated with an equation of the type $f(x, y) = 0$, is presented in the following theorem, that provides some conditions to guarantee its lower semicontinuity; see [38, Theorem 1.1] and [39, Theorem 2.2].

Theorem 1.10
Let C, D be two topological spaces, with D connected and locally connected, and f be a real-valued function defined on $C \times D$. For all $x \in C$ we set

$$\begin{aligned} V(x) &:= \{y \in D : f(x, y) = 0\}, \\ M(x) &:= \{y \in D : y \text{ is a local extremum point for } f(x, \cdot)\}, \\ Q(x) &:= V(x) \setminus M(x). \end{aligned}$$

Suppose that:

(a) for all $x \in C, f(x, \cdot)$ is continuous, and $0 \in \operatorname{int}(f(x, D))$;

(b) for all $x \in C$ and for all A open subset of D, there exists $\bar{y} \in A$ such that $f(x, \bar{y}) \neq 0$;

(c) the set $\{(y', y'') \in D \times D : \{x \in C : f(x, y') < 0 < f(x, y'')\} \text{ is open}\}$ is dense in $D \times D$.

Then, the multifunction Q is lower semicontinuous, with nonempty closed values.

Proof. Fix $x \in C$. Thanks to hypotheses $(a), (b)$ and Lemma 1.9, $Q(x) \neq \emptyset$. Now, we show that $Q(x)$ is closed. Let y_0 be a limit point of $Q(x)$ and note that $y_0 \in V(x)$ because of the continuity of $f(x, \cdot)$. If Ω is an open neighbourhood of y_0, there exists $y_1 \in Q(x) \cap \Omega$ and we can find $y_2, y_3 \in \Omega$ such that

$$f(x, y_2) < f(x, y_1) = 0 < f(x, y_3).$$

So, $y_0 \in Q(x)$ and it follows that $Q(x)$ is closed. Eventually, we prove that Q is lower semicontinuous. Let K be a closed subset of D and x^* a limit point of $Q^+(K)$. By contradiction, suppose that $x^* \notin Q^+(K)$ so that there exists $y^* \in Q(x^*) \setminus K$. We observe that $Y \setminus K$ is open and Y is locally connected in y^* in particular, so there is a connected and open set W^* such that $y^* \in W^* \subseteq D \setminus K$. Moreover, from $y^* \in Q(x^*)$, it follows that there exist $y', y'' \in W^*$ such that:

$$f(x, y') < 0 \quad \text{and} \quad f(x, y'') > 0.$$

Thanks to (c), there exists a neighbourhood U^* of x^* such that

$$f(x,y') < 0 \quad \text{and} \quad f(x,y'') > 0 \quad \forall x \in U^*.$$

Anyway, we can find $x_1 \in Q^+(K) \cap U^*$. Then, in particular, one has:

$$f(x_1,y') < 0 \quad \text{and} \quad f(x_1,y'') > 0. \tag{1.7}$$

Now, we apply Lemma 1.9 by choosing $S = W^*$ and $g = f(x_1,\cdot)_{|W^*}$. Taking into account (1.7), that W^* is open and assumptions $(a),(b)$, we can find $y^{**} \in W^*$ such that $f(x_1,y^{**}) = 0$ and y^{**} is not a relative extremum for $f(x_1,\cdot)_{|W^*}$. It means that $y^{**} \in Q(x_1)$. This is a contradiction because $Q(x_1) \subseteq K$ and $K \cap W^* = \emptyset$. The proof is complete. $\square$

The previous theorem can be applied in different ways, see for examples [38] and [39], where it has been used to establish the existence of implicit functions which are continuous or of first Baire class.

1.5 Selection theorems

We introduce the notion of selection which connects, in a certain sense, the "multi-valued analysis" and the "single-valued analysis". First of all, we recall its definition.

Definition 1.11
A function $\varphi\colon X \to Z$ is said to be a selection of the multifunction $\Phi : X \to 2^Z$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

There are several results which guarantee the existence of selections for l.s.c. or u.s.c. multi-functions with convex values. For example, the famous Michael's selection theorem provides a continuous selection for a l.s.c. multifunction, while Browder's selection theorem produces a continuous selection for an u.s.c. multifunction.
The following is the so-called Kuratowki and Ryll-Nardzewski theorem and it deals with

the existence of a Borel-measurable selection.

Theorem 1.12
Let X be a topological space, Y a complete metric separable space and $\Phi : X \to 2^Y$ a measurable multifunction with closed values. Then, Φ has a Borel-measurable selection.

Moreover, we will further need a selection in the decomposable case, that is, somehow, a notion similar to convexity, but weaker.

Definition 1.13
A nonempty set $K \subseteq L^p(\Omega)$ is said to be decomposable if, for every $w_1, w_2 \in K$ and every measurable set $A \subseteq \Omega$, we have

$$\chi_A \cdot w_1 + (1 - \chi_A) \cdot w_2 \in K,$$

where χ_A is the characteristic function of A.

In several cases the decomposability condition is a good substitute for convexity. So, there exists some results in the field of multi-valued analysis where convexity is replaced by decomposability.
A decomposable set has been considered for the first time in the field of multi-valued analysis by Antosiewicz and Cellina [3] in connection with the problem of the existence of a continuous selection for a continuous multifunction with not necessarily convex values.

A basic connection between decomposable-valued lower semicontinuous multifunctions and continuous selections is established by the following Bressan-Colombo-Fryszkowski's Continuous Selection Theorem; see [5, Proposition 2] and [29, Theorem 2.1].

Theorem 1.14
Let (X, d) be a separable metric space and let Φ be a lower semicontinuous multifunction from X into $L^p(\Omega)$, with decomposable closed values. Then, Φ admits a continuous

selection.

Proof. Let $f_{1,p} : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f_{1,p}(t) = \begin{cases} |t|^{1/p-1}t & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Consider the mapping given by

$$(T_{1,p}g)(y) = f_{1,p}(g(y)) \quad g \in L^1(\Omega),\ y \in \Omega.$$

Then, $T_{1,p} : L^1(\Omega) \to L^p(\Omega)$ is the famous Mazur homeomorphism (cf. [10, p. 139]). For every $x \in X$, we set

$$\Phi_1(x) = \{T_{1,p}^{-1}(w) : w \in \Phi(x)\}.$$

The multifunction $\Phi_1 : X \to L^1(\Omega)$ is lower semicontinuous and with decomposable values thanks to Proposition 1 of [6]. Now, Theorem 3 of [11] furnishes a continuous selection f_1 of Φ_1. Since for all $x \in X$

$$T_{1,p}(f_1(x)) \in T_{1,p}(\Phi_1(x)) = \Phi(x),$$

$f = T_{1,p} \circ f_1$ is the required continuous selection of Φ.

□

In the next example, we want to point out a specific multifunction with decomposable closed values, whose importance will be clearer in the proof of Theorem 2.6.

Example 1.15
Let $G\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ be a closed-valued multifunction such that:

(g_1) G is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable;

(g_2) for almost every $x \in \Omega$, the multifunction $(z, w) \mapsto G(x, z, w)$ turns out to be lower semicontinuous;

(g_3) there exist $a \in L^{p'}(\Omega, \mathbb{R}_0^+), b, c \geq 0$, complying with

$$\sup_{y \in G(x,z,w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Let us consider the multifunction $\mathbb{G} : W_0^{1,p}(\Omega) \to 2^{L^{p'}(\Omega)}$ defined by

$$\mathbb{G}(u) := \{v \in L^{p'}(\Omega) : v(x) \in G(x, u(x), \nabla u(x)) \ a.e. \ in \ \Omega\}. \tag{1.8}$$

Then, $\mathbb{G}$ has nonempty, closed decomposable values.

Proof. We first prove $\mathbb{G}(u) \neq \emptyset$ for every $u \in W_0^{1,p}(\Omega)$. Pick any $u \in W_0^{1,p}(\Omega)$. In view of (g_1) and [15, Theorem III.23] the multifunction $x \mapsto G(x, u(x), \nabla u(x))$ is measurable, because for any open set $B \subseteq \mathbb{R}$ one has

$$\{x \in \Omega : G(x, u(x), \nabla u(x)) \cap B \neq \emptyset\} =$$

$$= \text{proj}_\Omega(G^-(B) \cap \{(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : z = u(x), w = \nabla u(x)\},)$$

being proj_Ω the projection map from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ onto Ω. Hence, Theorem 1.12 gives a measurable function $v : \Omega \to \mathbb{R}$ such that $v(x) \in G(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$. Thanks to (g_3) we obtain

$$\|v\|_{L^{p'}(\Omega)} \leq \|a\|_{L^{p'}(\Omega)} + b\|u\|_{L^p(\Omega)}^{p-1} + c\|\nabla u\|_{L^p(\Omega,\mathbb{R}^N)}^{p-1} < +\infty$$

namely $v \in \mathbb{G}(u)$ and in particular $\mathbb{G}(u) \neq \emptyset$. A standard argument then shows that $\mathbb{G}(u)$ turns out to be a decomposable closed subset of $L^{p'}(\Omega)$. $\square$

Chapter 2

Differential inclusions

Some years ago, in [22], L. Górniewicz writes: *"The theory of differential inclusions was initiated in 1934-1936 with 4 papers. Two of them by the French mathematician A. Marchaud and the remaining two by S. K. Zaremba, a mathematician from Cracow.[...]*
It is noteworthy that A. Marchaud called the equations in question contingent equations. The rapid development of the theory took place at the beginning of the sixties of the previous century, when Cracow Mathematical School headed by Tadeusz Wazewski started working on these issues. The following mathematicians belonging to the group are worth mentioning: Andrzej Lasota, Zdzislaw Opial, Czeslaw Olech, Józef Myjak, Andrzej Pelczar and Andrzej Plis. It is worth adding that Wazewski himself used another term, that is orientor equation. The fundamental role for the theory was played by Wazewski's work titled: 1. On an optimal control problem, Prague, 1964, 692-704.
In that paper, Wazewski demonstrated that each problem of controlling ordinary differential equations of the first order can be articulated with orientor equation terms. That observation served as an essential stimulus to study orientor differential equations and consequently, it contributed to the introduction of the new term, still valid, and that is "differential inclusions". [...]
The theory of differential inclusions is located within the mainstream of non-linear analysis - or to put it more precisely - multi-valued analysis. This theory is intensively developed especially in the countries such as France, Germany, Russia, Italy, Canada and USA. In Poland, there is a large group of mathematicians working on these issues."

2.1 Elliptic differential inclusions

In [5], the abstract differential inclusion introduced at the beginning of the Introduction is presented and analyzed, see [5, Theorem 1]. Then [5, Theorem 2] shows an application to elliptic ones, finding a solution to the problem:

Find $u \in H^2(\Omega, \mathbb{R}^h)$ such that $\Delta u \in F(x, u, Du)$ a.e. in Ω and $u = 0$ on $\partial\Omega$.

Here, we recall Theorem 2 of [5].

Theorem 2.1
Let Ω be a nonempty, open subset of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. Let $F\colon \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \to 2^{\mathbb{R}^h}$ be a closed-valued multifunction. Suppose that:

(b1) F is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$-measurable;

(b2) for almost every $x \in \Omega$, the multifunction $(z, w) \mapsto F(x, z, w)$ turns out to be lower semicontinuous;

(b3) there exist $0 < \gamma < 1$, $M \geq 0$ such that $\|L^{-1}\|\|m\|_p \leq r$ and

$$d(F(x, z, w), 0) \leq m(x) + M(1 + |z| + \|w\|)^\gamma$$

for every $(z, w) \in \mathbb{R}^h \times \mathbb{R}^{Nh}$.

Then, there exists a function $u \in H^2(\Omega, \mathbb{R}^h) \cap H^1_0(\Omega, \mathbb{R}^h)$ such that $\Delta u(x) \in F(x, u(x), Du(x))$.

Its proof is rather similar to that of the following Theorem, which represents an extension to the case $p \neq 2$ and a suitable operator $\mathcal{L}$ instead of Δ, even if we require that p must be strictly greater than N and more regularity for Ω.

Let Ω be a nonempty, bounded, open, connected subset of $\mathbb{R}^N$, $N \geq 3$, with a boundary $\partial\Omega$ of class $C^{1,1}$ and $N < p < +\infty$.

We denote by $C^1(\overline{\Omega})$ the space of all continuously differentiable functions $u : \overline{\Omega} \to \mathbb{R}$, equipped with the norm

$$\|u\|_{C^1(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)| + \sum_{i=1}^{n} \max_{x\in\overline{\Omega}} \left| \frac{\partial u(x)}{\partial x_i} \right|, \quad u \in C^1(\overline{\Omega}).$$

Now, we need to recall that thanks to the Sobolev Imbedding Theorem [1, Theorem 5.4], there exists a constant $\gamma > 0$ such that

$$\|u\|_{C^1(\overline{\Omega},\mathbb{R}^h)} \leq \gamma \|u\|_{W^{2,p}(\Omega,\mathbb{R}^h)} \tag{2.1}$$

for every $u \in W^{2,p}(\Omega, \mathbb{R}^h)$. Moreover, c will denote the smallest constant γ such that (2.1) holds for all $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ $(\star)$.

Theorem 2.2
Let $F\colon \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \to 2^{\mathbb{R}^h}$ be a closed-valued multifunction. Suppose that:

(a1) F is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$-measurable;

(a2) for almost every $x \in \Omega$, the multifunction $(z, w) \mapsto F(x, z, w)$ turns out to be lower semicontinuous;

(a3) there exist $r > 0$ and $m \in L^p(\Omega)$, with $N < p < +\infty$, such that $\|L^{-1}\| \|m\|_p \leq r$ and
$$\sup\{d(F(x, z, w), 0) : |z| \leq cr, \|w\| \leq cr\} \leq m(x)$$
almost everywhere in Ω, where c satisfies $(\star)$.

Then, there exists a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu(x) \in F(x, u(x), Du(x))$ and $|Lu(x)| \leq m(x)$ for almost every $x \in \Omega$.

Proof. We set

$$B_r = \{u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h) : ||u||_{W^{2,p}(\Omega,\mathbb{R}^h)} \leq r\}.$$

Let us consider the embedding operator

$$j : W^{2,p}(\Omega, \mathbb{R}^h) \to W^{1,p}(\Omega, \mathbb{R}^h),$$

that is a compact map, see for istance [18, Theorem 11.2]. Therefore, the set $S_r = \overline{j(B_r)}$ is compact and convex in $W^{1,p}(\Omega, \mathbb{R}^h)$. Let $v \in S_r$ and let $\{u_k\}$ be a sequence in B_r such that $\lim\limits_{k\to+\infty} ||u_k - v||_{W^{1,p}(\Omega,\mathbb{R}^h)}=0$. By (2.1), for every $k \in \mathbb{N}$ and every $x \in \Omega$, one has $|u_k(x)| \leq cr$ and $||Du_k(x)|| \leq cr$. Consequently

$$|v(x)| \leq cr \text{ and } ||Dv(x)|| \leq cr \text{ almost everywhere in } \Omega. \tag{2.2}$$

If $v \in S_r$, we define

$$G(v) = \{w \in L^p(\Omega, \mathbb{R}^h) : w(x) \in F(x, v(x), Dv(x)) \text{ for almost every } x \in \Omega\}. \tag{2.3}$$

We first prove that the set $G(v)$ is nonempty. For any open set $A \subseteq \mathbb{R}^h$ one has

$$\{x \in \Omega : F(x, v(x), Dv(x)) \cap A \neq \emptyset\} =$$

$$\mathrm{pr}_\Omega(F^{-1}(A) \cap \{(x, z, w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} : z = v(x), w = Dv(x)\}),$$

where pr_Ω denotes the projection mapping from $\Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh}$ onto Ω. Owing to assumption (a_1) and [15, Theorem III.23],

$$\{x \in \Omega : F(x, v(x), Dv(x)) \cap A \neq \emptyset\} \in \mathcal{L}(\Omega),$$

so that the multifunction $x \to F(x, v(x), Dv(x))$ is measurable. Therefore, Theorem 1.12 (Kuratowski and Ryll-Nardzewski Selection Theorem) guarantees the existence of a measurable function $w : \Omega \to \mathbb{R}^h$ fulfilling $w(x) \in F(x, v(x), Dv(x))$ for almost every $x \in \Omega$. As v satisfies (2.2), hypothesis $(a3)$ gives $|w(x)| \leq m(x)$ a.e. in Ω, with $m \in L^p(\Omega)$, so $w \in L^p(\Omega, \mathbb{R}^h)$, that is $G(v) \neq \emptyset$.

We want to apply Theorem 1.14 by choosing $X := S_r$ and $\Phi := G$. Let us prove that the multifunction $G : S_r \to 2^{L^p(\Omega,\mathbb{R}^h)}$ defined by (2.3) is lower semicontinuous. Pick $v_0 \in S_r, w_0 \in G(v_0)$ and choose a sequence $\{v_k\}$ in S_r converging to v_0. Since for all $k \in \mathbb{N}$

the function v_k satisfies (2.2), assumption $(a3)$ implies that, for almost every $x \in \Omega$, the set $F(x, v_k(x), Dv_k(x))$ is compact. So, for every $k \in \mathbb{N}$, [15, Theorem III.41] and Theorem 1.12 yield a measurable function $w_k : \Omega \to \mathbb{R}^h$ such that

$$w_k(x) \in F(x, v_k(x), Dv_k(x))$$

and

$$|w_k(x) - w_0(x)| = d(w_0(x), F(x, v_k(x), Dv_k(x)))$$

almost everywhere in Ω. Moreover,

$$w_k \in G(v_k) \text{ for all } k \in \mathbb{N},$$

because of hypothesis $(a3)$. By taking a subsequence if necessary, we may suppose $\lim_{k\to+\infty} v_k = v_0$ and $\lim_{k\to+\infty} Dv_k = Dv_0$ almost everywhere in Ω. Assumption $(a2)$ ensures that, for almost every $x \in \Omega$, the function $(z, w) \to d(v_0(x), F(x, z, w)$ is upper semicontinuous; therefore, for almost every fixed $x \in \Omega$, one has

$$\limsup_{k\to+\infty} |w_k(x) - w_0(x)| = \limsup_{k\to+\infty} d(w_0(x), F(x, v_k(x), Dv_k(x)))$$

$$\leq d(w_0(x), F(x, v_0(x), Dv_0(x))) = 0,$$

so that $\lim_{k\to+\infty} w_k = w_0$ almost everywhere in Ω. Bearing in mind that $|w_k(x)| \leq m(x)$ for all $k \in \mathbb{N}$ and almost $x \in \Omega$, the Lebesgue Dominated Convergence Theorem gives

$$\lim_{k\to+\infty} w_k = w_0 \text{ in } L^p(\Omega, \mathbb{R}^h).$$

A simple argument shows that $G(v)$ is a decomposable closed subset of $L^p(\Omega, \mathbb{R}^h)$. We have now proved that the multifunction G satisfies all tha assumptions of Theorem 1.14. Hence, there is a continuos function $g : S_r \to L^p(\Omega, \mathbb{R}^h)$ such that $g(v) \in G(v)$ for every $v \in S_r$. Obviously, the function $j \circ L^{-1} \circ g : S_r \to W^{1,p}(\Omega, \mathbb{R}^h)$ is continuous. Moreover,

$$j \circ L^{-1} \circ g(S_r) \subseteq S_r$$

since, due to (2.2) and $(a3)$, for every $v \in S_r$, one has

$$||L^{-1}(g(v))||_{W^{2,p}(\Omega,\mathbb{R}^h)} \leq ||L^{-1}|| \; ||g(v)||_{L^p(\Omega,\mathbb{R}^h)} \leq ||L^{-1}|| \; ||m||_p \leq r.$$

Thus, by the Schauder Fixed Point Theorem, there exists $v_0 \in S_r$ such that $j \circ L^{-1} \circ g(v_0) = v_0$. It is immediate to verify that the function $u = L^{-1}(g(v_0))$ satisfies the conclusion. □

Remark 2.3
A simple computation proves that the assumption $(a3)$ of the Theorem 2.2 is verified in the following special case:

$(a3)'$ There exist $\alpha, \beta \in L^p(\Omega)$, with $N < p < +\infty$, and $\gamma > 0$ such that

$$d(F(x,z,w),0) \leq \alpha(x) + \beta(x)(|z| + ||w||)^\gamma$$

for almost every $x \in \Omega$ and every $(z,w) \in \mathbb{R}^h \times \mathbb{R}^{Nh}$ and, if $||\beta||_p > 0$, $2c||L^{-1}|| \; ||\beta||_p < 1$ or $||\alpha||_p \leq (\gamma - 1)[2c\gamma||L^{-1}|| \; ||\beta||_p^{1/\gamma}]^{\gamma/(1-\gamma)}$, according to whether $\gamma = 1$ or $\gamma > 1$.

Because of this Remark, it is evident that, for $p \in]n, +\infty[$ and bounded Ω, the assumptions of Theorem 2.2 are less restictive than those of Theorem 2.1.

2.2 Lower semicontinuous differential inclusions with p-Laplacian

One of the main tool to obtain existence of solutions to an implicit equation of the type $f(x,u,\nabla u,\Delta_p u) = 0$, that we will analyze further, is Theorem 2.6, namely [34, Theorem 3.1], which deals with the existence of solutions for elliptic differential inclusions with lower semicontinuous right-hand side and is based on the selection result for decomposable-valued multifunctions Theorem 1.14.

Let F be a multifunction from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ with nonempty closed values. We want to study the Dirichlet problem:

$$\begin{cases} -\Delta_p u \in F(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

for which we need a notion of solution.

Definition 2.4
A function $u \in W_0^{1,p}(\Omega)$ is called a (weak) solution to (2.4) provided there exists $v \in L^{p'}(\Omega)$, p' being the conjugate exponent of p, such that $v(x) \in F(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla w dx = \int_\Omega v w dx \quad \forall w \in W_0^{1,p}(\Omega).$$

It is noteworty to observe that the right-hand side of (2.4) is neither convex nor upper semi-continuous and, moreover, it depends on the gradient of the solution. This is the reason why you can not use variational methods for possibly non-smooth functionals, that are fundamentals in other contexts.
Other existence results for lower semi-continuous elliptic differential inclusions are contained, for example, in [2, Section 3] and [3, Theorem 2], but they deal with elliptic operators in non-divergence form.

An important tool in the next proof is given by Leray-Schauder Principle, see [41, Theorem 6.A], that we recall.

Theorem 2.5
Let X be a normed linear space. Suppose that:

(i) the operator $T : X \to X$ is compact;

(ii) there exists $r > 0$ such that if the equation $\sigma T(u) = u$ has solutions for some $\sigma \in [0, 1]$, then $\|u\|_X \leq r$.

Then, the equation $T(u) = u$ must have a solution.

The previous result is often called the Leray-Schauder alternative because it states that either the equation $\sigma T(u) = u$ for some $\sigma \in [0,1]$ has solutions with $\|u\|$ arbitrarily large, or else T has a fixed point.

Theorem 2.6
Let $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ be a closed-valued multifunction. Suppose that:

(h_1) F is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable;

(h_2) for almost every $x \in \Omega$, the multifunction $(z,w) \mapsto F(x,z,w)$ turns out to be lower semicontinuous;

(h_3) there exist $a \in L^{p'}(\Omega, \mathbb{R}_0^+), b, c \geq 0$, with $\dfrac{b}{\lambda_{1,p}} + \dfrac{c}{\lambda_{1,p}^{1/p}} < 1$, complying with

$$\inf_{y \in F(x,z,w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Then, (2.4) possesses a solution $u \in W_0^{1,p}(\Omega)$.

Proof. Define, provided $(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,

$$\varphi(x,z,w) := a(x) + b|z|^{p-1} + c|w|^{p-1} \tag{2.5}$$

and

$$G(x,z,w) := \overline{F(x,z,w) \cap B(0, \varphi(x,z,w))}. \tag{2.6}$$

By (h_3) the set $G(x,z,w)$ is nonempty and compact. Thanks to [37, Theorem 0.45 and Theorem 0.48], the multifunction $(x,z,w) \to G(x,z,w)$ turns out to be lower semicontinuous for almost every $x \in \Omega$.
Let us consider the multifunction $\mathbb{G} : W_0^{1,p}(\Omega) \to 2^{L^{p'}(\Omega)}$ defined by

$$\mathbb{G}(u) := \{v \in L^{p'}(\Omega) : v(x) \in G(x, u(x), \nabla u(x)) \ a.e. \ in \ \Omega\}. \tag{2.7}$$

Since one evidently has

$$\sup_{y\in G(x,z,w)} |y| \le a(x) + b|z|^{p-1} + c|w|^{p-1}, \tag{2.8}$$

as seen in Example 1.15, $\mathbb{G}$ takes decomposable closed values and, thanks to [34, Lemma 2.2], it is lower semicontinuous. Thus, Theorem 1.14 yields a continuous selection $g : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ of $\mathbb{G}$, which, by (2.8), is bounded on bounded sets. Through $(p_1)-(p_3)$ we know that A_p^{-1} is a continuous bounded bijecton from $W^{-1,p'}(\Omega)$ into $W_0^{1,p}(\Omega)$. Since the natural embedding $j_{p'} : L^{p'}(\Omega) \to W^{-1,p'}(\Omega)$ is compact, $A_p^{-1} \circ j_{p'}$ enjoys the same property. Consequently, the nonlinear operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$T(u) := A_p^{-1} \circ j_{p'} \circ g(u) \quad \forall u \in W_0^{1,p}(\Omega)$$

turns out to be compact as well and any fixed point $u \in W_0^{1,p}(\Omega)$ of T is a weak solution to the equation

$$A_p(u) = g(u) \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{2.9}$$

On the other hand, if $u \in W_0^{1,p}(\Omega)$ complies with (2.9) then it solves Problem (2.4), because

$$g(u) \in \mathbb{G}(u) \subseteq L^{p'}(\Omega).$$

To get a fixed point of T we shall employ the Leray-Shauder alternative, namely Theorem 2.5. Let us choose $X := W_0^{1,p}(\Omega)$ and suppose $u = \sigma T(u)$ for some $\sigma \in [0,1]$. The choice of T forces

$$\langle A_p(u), v\rangle = \sigma^{p-1} \int_\Omega g(u)(x)v(x)dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{2.10}$$

From $g(u)(x) \in G(x, u(x), \nabla u(x))$, $\sigma \in [0,1]$ and (2.5), it evidently follows

$$|\sigma^{p-1} g(u)(x)| \le \varphi(x, u(x), \nabla u(x)).$$

Letting $v := u$ in (1.4) and exploting (2.5), (2.10) and (p_4), we thus obtain

$$\|u\|_p^p = \int_\Omega |\nabla u(x)|^p dx = \langle A_p(u), u\rangle \le \sigma^{p-1} \int_\Omega |g(u)(x)u(x)|dx \le$$

$$\leq \int_\Omega \varphi(x,u(x),\nabla u(x))|u(x)| = \int_\Omega (a(x)|u(x)| + b|u(x)|^p + c|\nabla u(x)|^{p-1}|u(x)|)dx \leq$$

$$\leq \|a\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} + b\|u\|^p_{L^p(\Omega)} + c\|\nabla u\|^{p/p'}_{L^p(\Omega,\mathbb{R}^N)}\|u\|_{L^p(\Omega)} \leq$$

$$\leq \frac{1}{\lambda_{1,p}^{1/p}}\|a\|_{L_{p'}(\Omega)}\|u\|_p + \left(\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{1/p}}\right)\|u\|_p^p.$$

Therefore, under hypothesis on b and c, any fixed point u of σT is bounded by a constant which does not depend on u and σ, namely

$$\|u\|_p \leq \left(\frac{1}{\lambda_{1,p}^{1/p}}\|a\|_{L_{p'}(\Omega)}\right)^{p'/p}\left(1 - \frac{b}{\lambda_{1,p}} - \frac{c}{\lambda_{1,p}^{1/p}}\right)^{-p'/p}.$$

Now, the Leray-Shauder Fixed Point Theorem 2.5 leads to the conclusion that T has a fixed point. □

The following result represents a priori estimates on $\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)}$, see [34, Proposition 3.3].

Proposition 2.7
Suppose $q > N$. Then, there exists $\hat{C} > 0$, depending on p, q and Ω, such that

$$\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq \hat{C}\|\Delta_p u\|^{1/(p-1)}_{L^q(\Omega)}. \tag{2.11}$$

Theorem 2.8
Let $q > N$ and let $(h_1) - (h_2)$ be satisfied. If, moreover,

(h_3'') for appropriate $a \in L^q(\Omega, \mathbb{R}_0^+)$ and $\psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ nondecreasing with respect to each variable separately one has

$$\inf_{y\in F(x,z,w)} |y| < a(x) + \psi(|z|,|w|) \quad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{2.12}$$

(h_4) there exists $R > 0$ such that

$$\|a\|_{L^q(\Omega)} + m(\Omega)^{1/q}\psi(\delta_\Omega \hat{C} R^{1/(p-1)}, \hat{C} R^{1/(p-1)}) \le R, \tag{2.13}$$

where $\delta_\Omega = \operatorname{diam}(\Omega)$ and $\hat{C}$ is given by Proposition 2.7, then Problem 2.4 possesses at least one solution.

Proof. Since $q > N > p^{*\prime}$, the embedding $j_q : L^q(\Omega) \to W^{-1,p'}(\Omega)$ is compact. To shorten notation, write

$$B_R := \{v \in L^q(\Omega) : \|v\|_{L^q(\Omega)} \le R\}, \tag{2.14}$$

$$A_{pq} := A_p^{-1} \circ j_q, \quad X_R := \overline{co}(A_{pq}(B_R)). \tag{2.15}$$

Obviously, X_R turns out to be a convex compact subset of $W_0^{1,p}(\Omega)$; see [17, Theorem V.2.6]. Inequalities (2.7) and (5) at [12, p. 269] yield, after a standard point-wise convergence argument,

$$\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)} \le \hat{C} R^{1/(p-1)}, \quad \|u\|_{L^\infty(\Omega)} \le \delta_\Omega \hat{C} R^{1/(p-1)} \quad \forall u \in X_R.$$

Now, if $\varphi(x, z, w)$ denotes the right-hand side of (2.12) while $G(x, z, w)$ is as in (2.6) then, by simply adapting the reasoning made to prove [34, Lemma 2.2], we see that the multifunction $\mathbb{G} : X_R \to 2^{L^q(\Omega)}$ defined via (2.7) takes decomposable closed values and is lower semi-continuous. Thus, Theorem 1.14 gives a continuous selection $g : X_R \to L^q(\Omega)$ of $\mathbb{G}$, which turns out to be bounded, because

$$\|g(u)\|_{L^q(\Omega)} \le \|a\|_{L^q(\Omega)} + m(\Omega)^{1/q}\psi(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega,R^N)})$$

$$\le \|a\|_{L^q(\Omega)} + m(\Omega)^{1/q}\psi(\delta_\Omega \hat{C} R^{1/(p-1)}, \hat{C} R^{1/(p-1)})$$

for any $u \in X_R$. Hence, the nonlinear operator $T := A_{pq} \circ g$ is compact and, due to (h_4), complies with $T(X_R) \subseteq X_R$. By the Schauder Fixed Point Theorem [13, Theorem 4.4] there exists $u \in X_R \subseteq W_0^{1,p}(\Omega)$ such that $u = T(u)$, whence $A_p(u) = g(u)$ in $W^{-1,p'}(\Omega)$. This immediately leads to the conclusion. □

2.3 Upper semicontinuous differential inclusions with p-Laplacian

Let us consider a modified version of (2.4):

$$\begin{cases} -\Delta_p u \in F(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where F does not depend on ∇u anymore.
When F is an upper semicontinuous multifunction with convex closed values, we further make use of the following result, [33, Theorem 2.2], to solve the inclusion.

Theorem 2.9
Let U be a nonempty set, $\Phi\colon U \to W_0^{1,p}(\Omega)$, $\Psi\colon U \to L^{p'}(\Omega)$ two operators and $F\colon \Omega\times\mathbb{R} \to 2^{\mathbb{R}}$ a convex closed-valued multifunctions. Suppose that the following conditions hold true:

(i_1) Ψ is bijective and for any $v_h \rightharpoonup v$ in $L^{p'}(\Omega)$ there is a subsequence of $\{\Phi(\Psi^{-1}(v_h))\}$ which converges to $\Phi(\Psi^{-1}(v))$ almost everywhere in Ω. Furthermore, a non-decreasing function $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+ \cup \{+\infty\}$ can be defined in such a way that

$$\|\Phi(u)\|_\infty \le g(\|\Psi(u)\|_{p'}) \qquad \forall\, u \in U;$$

(i_2) $F(\cdot, z)$ is measurable for all $z \in \mathbb{R}$;

(i_3) $F(x, \cdot)$ has a closed graph for almost every $x \in \Omega$;

(i_4) there exists $r > 0$ such that the function

$$\rho(x) := \sup_{|z|\le g(r)} d(0, F(x,z)), \quad x \in \Omega,$$

belongs to $L^{p'}(\Omega)$ and $\|\rho\|_{p'} \le r$.

Then, the problem $\Psi(u) \in F(x, \Phi(u))$ possesses at least one solution $u \in U$ satisfying $|\Psi(u)(x)| \le \rho(x)$ for almost every $x \in \Omega$.

Before dealing with implicit equations in the next chapter, we introduce a significant case when (i_1) of the previous theorem is satisfied, useful in the proof of Theorem 3.13.

Example 2.10
Choose $U := A_p^{-1}(L^{p'}(\Omega))$, where A_p is the nonlinear operator stemming from the negative p-Laplacian, $\Phi(u) := u$ and $\Psi(u) := A_p(u)$, for every $u \in U$. Then, (i_1) of Theorem 2.9 is satisfied.

Proof. Observe that the operator $A_p\colon U \to L^{p'}(\Omega)$ is bijective. Let $v_h \rightharpoonup v$ in $L^{p'}(\Omega)$. Since $\{v_h\}$ is bounded in $L^{p'}(\Omega)$, and $L^{p'}(\Omega)$ compactly embeds in $W^{-1,p'}(\Omega)$, there exists a subsequence, still denoted by $\{v_h\}$, such that $v_h \to v$ in $W^{-1,p'}(\Omega)$. Since, from property (p_2), A_p^{-1} is strongly continuous, it follows that $\{A_p^{-1}(v_h)\}$ converges to $A_p^{-1}(v)$ almost everywhere in Ω.

Let now $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+ \cup \{+\infty\}$ be the function such that

$$g(t) := a(bt)^{1/(p-1)} \qquad \forall\, t \in \mathbb{R}_0^+,$$

where the constants a and b derive from the inequalities (1.1) and (1.2). Clearly, g is monotone increasing in $\mathbb{R}_0^+$. Moreover, taking into account property (p3), if $u \in U$ then

$$\|u\|_\infty \le a\|u\| = a\|A_p(u)\|_{W^{-1,p'}(\Omega)}^{1/(p-1)} \le a(b\|A_p(u)\|_{p'})^{1/(p-1)} = g(\|A_p(u)\|_{p'}).$$

This completes the proof. $\square$

Chapter 3

Implicit elliptic differential equations

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$, let $1 < p < +\infty$ and let Y be a nonempty, closed, connected and locally connected subset of $\mathbb{R}^h$. Let f be a real-valued function defined on $\Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \times Y$ and let $\mathcal{L}$ be a second order elliptic operator that will be specified later according to the setting. The aim of this chapter is to study implicit elliptic equations of the type

$$f(x, u, Du, Lu) = 0, \tag{3.1}$$

under homogeneous Dirichlet boundary condition.
A basic case occurs when $h = 1$ and for every fixed $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ the equation $f(x, z, w, y) = 0$ can be solved respect to y and so (3.1) reduced to a quasilinear equation

$$\mathcal{L}u = \varphi(x, u, Du),$$

for which numerous existence results are available. In the paper [8] the function f is supposed to be of the form

$$f(x, z, w, y) = y - g(x, z, w, y).$$

Our aim is to prove the existence of a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$, for an appropriate p, such that $Lu(x) \in Y$ and $f(x, u(x), Du(x), Lu(x)) = 0$ for almost every $x \in \Omega$. We make use of Theorem 1.10 to reduce our problem to the elliptic differential inclusion

$$Lu \in F(x, u, Du), \tag{3.2}$$

where F is a lower semicontinuous multiselection of the multifuncion

$$(x, z, w) \to \{y \in Y : f(x, z, w, y) = 0\}.$$

Eventually, we can apply various results to solve (3.2) according to the hypotheses on f and on the type of the elliptic operator.
In Section 3.1 operator $\mathcal{L}$ is as in (1.3), while in Section 3.2 we deal with the p-Laplacian.

3.1 Implicit elliptic differential equations involving a linear second-order elliptic operator

Here, we want to apply Theorem 2.2, that- as we have seen- is a modified version of an existence result for elliptic differential inclusions with lower semicontinuous right-hand side, Theorem 2.1, based on a selection theorem for decomposable-valued multifunctions. Hypotheses on Y and f are quite general bacause we do not need that Y is compact or f satisfies a Lipschitz condition for the last variable.

In the following result, that is Theorem 3.2 of [29], we suppose $N \geq 3$.

Theorem 3.1
Let Y be a nonempty, closed, connected and locally connected subset of $\mathbb{R}^h$ and let f be a real-valued function defined on $\Omega \times \mathbb{R}^h \times R^{Nh} \times Y$. Suppose that:

1. for every $(x, z, w) \in \Omega \times \mathbb{R}^h \times R^{Nh}$, the function $y \to f(x, z, w, y)$ is continuous, $0 \in \operatorname{int}(f(x, z, w, Y))$ and the set $\{y \in Y : f(x, z, w, y) = 0\}$ has empty interior in Y;

2. there exists a set $\Lambda \subseteq Y \times Y$, Λ dense in $Y \times Y$, such that, for every $x \in \Omega$ and $(y', y'') \in \Lambda$, the set $\{(z, w) \in \mathbb{R}^h \times \mathbb{R}^{Nh} : f(x, z, w, y') < 0 < f(x, z, w, y'')\}$ is open;

3. there is a countable set $\Lambda^* \subseteq Y \times Y$, Λ^* dense in $Y \times Y$, such that, for every $(y', y'') \in \Lambda^*$, the set $\{(x, z, w) \in \Omega \times \mathbb{R}^h \times R^{Nh} : f(x, z, w, y') < 0 < f(x, z, w, y'')\}$ belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$;

4. there exist $r > 0$ and $m \in L^p(\Omega)$, $N < p < +\infty$, such that $\|L^{-1}\| \|m\|_p \leq r$ and $\{y \in Y : f(x, z, w, y) = 0$, y is not a local extremum point for $f(x, z, w, \cdot)\} \subseteq B(0, m(x))$, for almost every $x \in \Omega$ and every $(z, w) \in \mathbb{R}^h \times \mathbb{R}^{Nh}$ with $|z| \leq cr$, $\|w\| \leq cr$, where c satisfies $(\star)$.

Then, there is a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu(x) \in Y$ and $f(x, u(x), Du(x), Lu(x)) = 0$ for almost every $x \in \Omega$.

Proof. Fix $x \in \Omega$. For every $(z, w) \in \mathbb{R}^h \times \mathbb{R}^{Nh}$, we define

$$F(x, z, w) = \{y \in Y : f(x, z, w, y) = 0, y \text{ is not a local extremum point for } f(x, z, w, \cdot)\}.$$

Owing to the assumptions, Theorem 1.10 holds. Hence, for every $(z, w) \in \mathbb{R}^h \times \mathbb{R}^{Nh}$, the set $F(x, z, w)$ is nonempty and closed, and the multifunction $(z, w) \to F(x, z, w)$ is lower semicontinuous.
Let A be a nonempty, connected, open subset of Y. We prove that

$$F^-(A) = \bigcup_{(y', y'') \in \Lambda^* \cap (A \times A)} \{(x, z, w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} : f(x, z, w, y') < 0 < f(x, z, w, y'')\},$$

so that, by hypothesis 3., it yields $F^-(A) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$.

If $(x, z, w) \in F^-(A)$ then there is $y \in A$ such that $f(x, z, w, y) = 0$ and y is not a local extremum point for $\xi \to f(x, z, w, \xi)$. By hypotesis 1., this implies that there are two open sets $A', A'' \subseteq A$ so that $f(x, z, w, y') < 0$ for every $y' \in A'$ and $f(x, z, w, y'') > 0$ for every $y'' \in A''$. Bearing in mind that $\Lambda^* \cap (A' \times A'') \neq \emptyset$, we get $(y', y'') \in \Lambda^* \cap (A \times A)$ satisfying $f(x, z, w, y') < 0 < f(x, z, w, y'')$.

Conversely, let $(x, z, w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh}$ and let $(y', y'') \in \Lambda^* \cap (A' \times A'')$ be such that $f(x, z, w, y') < 0 < f(x, z, w, y'')$. Since A is connected, there is $y \in A$ fulfilling $f(x, z, w, y) = 0$. If y is not a local extremum point for $\xi \to f(x, z, w, \xi)$, then $(x, z, w) \in$

$F^-(A)$. Otherwise, assumption 1. and Lemma 1.9 give $y^* \in A$ such that $f(x,z,w,y^*) = 0$ and y^* is not a local extremum point for $\xi \to f(x,z,w,\xi)$.

Since $F^-(A) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$ and the space Y has a countable base of connected open sets, we see that the multifunction F is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^h \times \mathbb{R}^{Nh})$-measurable.

Now, observe that assumption 4. implies

$$\sup\{d(F(x,z,w),0) : |z| \leq cr, ||w|| \leq cr\} \leq m(x)$$

for almost every $x \in \Omega$, so that Theorem 2.2 can be applied. Therefore, there is a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ satisfying $Lu(x) \in F(x, u(x), Du(x))$ almost everywhere in Ω. This completes the proof. $\square$

In [19], Frigon and Kaczynski have employed a similar method to study the solvability of some boundary value problems for implicit ordinary differential equations. There, in place of the Ricceri's result, a topological selection theorem by Bielawski and Górniewicz, namely [8, Theorem 2.5], which requires different conditions on f, was applied.

The following result derives directly from Theorem 3.1 and it deals with a particular case of the equation (3.1).

Theorem 3.2
Let Y be as in Theorem 3.1, let ψ be a continuous real-valued function defined on Y and let φ be a continuous real-valued function defined on $\Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh}$. Assume that:

1. for every $\sigma \in \mathrm{int}(\psi(Y))$ the set $\psi^{-1}(\sigma)$ has empty interior in Y,
2. there exist $r > 0$ and $m \in L^p(\Omega)$, $N < p < +\infty$, such that $\|L^{-1}\|\|m\|_p \leq r$ and $\varphi(x,z,w)) \in \psi(Y \cap B(0, m(x)))$ for every $(x,z,w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh}$ with $|z| \leq cr$ and $\|w\| \leq cr$, where c satisfies $(\star)$.

Then, there exists a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ such that $Lu(x) \in Y$ and $\psi(Lu(x)) = \varphi(x, u(x), Du(x))$ for almost every $x \in \Omega$.

Proof. The function ψ satisfies all the assumptions of [39, Theorem 2.4]. Hence, there is a set $Y^* \subseteq Y$ such that $\psi^{-1}(\sigma) \cap Y^*$ is nonempty and closed in $\mathbb{R}^h$ for each $\sigma \in \psi(Y)$ and the multifunction $\sigma \to \psi^{-1}(\sigma) \cap Y^*$, $\sigma \in \psi(Y)$, is lower semicontinuous. For every $(x, z, w) \in \Omega \times \mathbb{R}^h \times \mathbb{R}^{nh}$, we set

$$F(x, z, w) = \begin{cases} \psi^{-1}(\varphi(x, z, w)) \cap Y^*, & \text{if } |z| \leq cr \text{ and } ||w|| \leq cr \\ \mathbb{R}^h, & \text{otherwise.} \end{cases}$$

It is a simple matter to see that the multifunction $F : \Omega \times \mathbb{R}^h \times \mathbb{R}^{Nh} \to 2^{\mathbb{R}^h}$ so defined is lower semicontinuous. Consequently, the hypotheses (h_1) and (h_2) of Theorem 2.2 are verified. Now, observe that assumption 2. implies

$$\sup\{d(F(x, z, w), 0) : |z| \leq cr, ||w|| \leq cr\} \leq m(x) \text{ for all } x \in \Omega.$$

Therefore, Theorem 3.1 can be applied to conclude that there is a function $u \in W^{2,p}(\Omega, \mathbb{R}^h) \cap W_0^{1,p}(\Omega, \mathbb{R}^h)$ satisfying $Lu(x) \in F(x, u(x), Du(x))$ and $|Lu(x)| \leq m(x)$ almost everywhere in Ω. Since for every $x \in \Omega$ one has $|u(x)| \leq cr$ and $||Du(x)|| \leq cr$, the proof is complete. □

3.2 Implicit elliptic differential equations with p-Laplacian

Both this section and the next one are mainly based on [35]. Our aim is to combine techniques and results seen in the previous Chapters to give some new results.

To this aim, we choose $h = 1$ and $L = \Delta_p$ in (3.1), obtaining the following implicit elliptic problem

$$u \in W_0^{1,p}(\Omega), \qquad f(x, u, \nabla u, \Delta_p u) = 0 \quad \text{in } \Omega. \tag{3.3}$$

In particular, we focus on the case when the function f can be expressed in the form

$$f(x, z, w, y) = \varphi(x, z, w) - \psi(y),$$

where Y is a nonempty interval of $\mathbb{R}$, φ is a real-valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N$, and ψ is a real-valued function defined on Y, which depends only on the p-Laplacian $\Delta_p u$.

We further distinguish among the case where φ is a Carathéodory function depending on x, u, and ∇u, and the case where φ is allowed to be highly discontinuous in each variable, for which the dependance on the gradient is not permitted.

In both cases we first reduce problem (3.3) to an elliptic differential inclusion, but methods used are different, depending on the regularity of the function φ and on the structure of the problem.

More precisely, when φ is a Carathéodory function, we make use of a result in [38] to obtain the inclusion

$$-\Delta_p u \in F(x, u, \nabla u), \tag{3.4}$$

where F is a lower semicontinuous selection of the multifunction

$$(x, z, w) \mapsto \{y \in Y : \varphi(x, z, w) - \psi(y) = 0\}.$$

3.2.1 The case of φ as a Carathéodory function

Now, we deal with the existence of solutions to the equation

$$\psi(-\Delta_p u) = \varphi(x, u, \nabla u). \tag{3.5}$$

We first consider the case $Y = \mathbb{R}$. Throughout the section, $p \in]1, +\infty[$ and the following assumptions will be posited:

(i) for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the set $\{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0\}$ has empty interior;

(ii) for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the function $y \mapsto \varphi(x, z, w) - \psi(y)$ changes sign.

Theorem 3.3
Let $\varphi \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function and let $\psi \colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that (i)-(ii) hold true and, moreover,

(iii) there exist $a \in L^{p'}(\Omega, \mathbb{R}_0^+)$, $b, c \geq 0$, with $\dfrac{b}{\lambda_{1,p}} + \dfrac{c}{\lambda_{1,p}^{1/p}} < 1$, such that

$$\sup\{|y| : y \in \psi^{-1}(\varphi(x,z,w))\} < a(x) + b|z|^{p-1} + c|w|^{p-1},$$

for all $(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation (3.5).

Proof. Fix any $x \in \Omega$. In order to apply Theorem 1.10, we have to verify conditions (a), (b) and (c). Choose $C = \mathbb{R} \times \mathbb{R}^N$, $D = \mathbb{R}$, $f(z,w,y) = \varphi(x,z,w) - \psi(y)$, and for every $(z,w) \in \mathbb{R} \times \mathbb{R}^N$ set

$$\begin{aligned} F(x,z,w) := \{y \in \mathbb{R} : \varphi(x,z,&w) - \psi(y) = 0, \\ &y \text{ is not a local extremum point of } \psi(\cdot)\}. \end{aligned}$$

Hypothesis (ii) directly yields (a). Moreover, in our context, (b) is equivalent to say that, for all $(z,w) \in \mathbb{R} \times \mathbb{R}^N$, the set $U := \{y \in \mathbb{R} : \varphi(x,z,w) - \psi(y) \neq 0\}$ is dense in $\mathbb{R}$. Since, by (i), the set $\mathbb{R} \setminus U$ has empty interior, it follows that U is dense in $\mathbb{R}$, as desired.

Let us next analyze the set

$$\begin{aligned} \big\{(y',y'') \in \mathbb{R} \times \mathbb{R} : \{(z,w) \in \mathbb{R} \times \mathbb{R}^N : \varphi(x,&z,w) - \psi(y') < 0 \\ &< \varphi(x,z,w) - \psi(y'')\} \text{ is open}\big\}. \end{aligned} \tag{3.6}$$

If one can find $y', y'' \in \mathbb{R}$ such that

$$\varphi(x,z,w) - \psi(y') < 0 < \varphi(x,z,w) - \psi(y''),$$

then $\varphi(x,z,w) \in]\psi(y''), \psi(y')[$. So, the set

$$\{(z,w) \in \mathbb{R} \times \mathbb{R}^N : \varphi(x,z,w) - \psi(y') < 0 < \varphi(x,z,w) - \psi(y'')\},$$

turns out to be open, because $\varphi(x,\cdot,\cdot)$ is continuous. Otherwise it is empty. So, the set (3.6) is $\mathbb{R} \times \mathbb{R}$, and (c) follows.

Therefore, thanks to Theorem 1.10, the multifunction $F(x, \cdot, \cdot)$ is lower semicontinuous, with nonempty closed values.

Moreover, for all $y', y'' \in \mathbb{R}$ we have

$$
\begin{aligned}
\{(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \varphi(x,z,w) - \psi(y') < 0 < \varphi(x,z,w) - \psi(y'')\} = \\
\{(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \varphi(x,z,w) \in]\psi(y''), \psi(y')[\,\} \\
\in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N),
\end{aligned}
$$

cf. [15, Lemma III.14]. Therefore, condition 3. of Theorem 3.1, with $\Lambda^* = \mathbb{R} \times \mathbb{R}$, is satisfied. Arguing as there we see that if $A \subseteq \mathbb{R}$ is open then

$$
\begin{aligned}
F^-(A) = \bigcup_{(y',y'') \in A \times A} \{(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \\
\varphi(x,z,w) - \psi(y') < 0 < \varphi(x,z,w) - \psi(y'')\}.
\end{aligned}
$$

This actually means that $F^-(A) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$, i.e. F is measurable.

Finally, fix any $y \in F(x,z,w)$. Since $y \in \psi^{-1}(\varphi(x,z,w))$, thanks to hypothesis (iii), we have

$$
\inf_{y \in F(x,z,w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N.
$$

So all the hypotheses of Theorem 2.6 are fulfilled, and we get a solution $u \in W_0^{1,p}(\Omega)$ to equation (2.4). Taking into account the definition of F, we have $\psi(-\Delta_p u) = \varphi(x, u, \nabla u)$, that is the thesis. $\square$

Remark 3.4

A very simple situation when hypothesis (iii) occurs is the following.

Suppose that $\varphi(\Omega \times \mathbb{R} \times \mathbb{R}^N) \subseteq [\alpha, \beta]$ and ψ is such that $\psi^{-1}(B)$ is bounded, for every B bounded subset of $\mathbb{R}$. If $(x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, we get $\varphi(x,z,w) \in [\alpha, \beta]$, and so $\psi^{-1}(\varphi(x,z,w)) \subseteq \psi^{-1}([\alpha, \beta])$. Then, if we choose $a \in L^{p'}(\Omega, \mathbb{R}_0^+)$ such that $a(x) > \sup\{|y| : y \in \psi^{-1}([\alpha, \beta])\}$ for all $x \in \Omega$, we obtain

$$
|\psi^{-1}(\varphi(x,z,w))| < a(x) \leq a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}^N,
$$

that is hypothesis (iii).

As an application of the previous result, we consider the following example.

Example 3.5
Let $g \in L^2(\Omega)$ and $\gamma \in]0,1[$. Then, for every $\lambda \neq 0, \mu \in \mathbb{R}$, there exists a solution $u \in W_0^{1,2}(\Omega)$ to the equation

$$-\Delta u = g(x) + \mu(|u| + |\nabla u|)^\gamma + \lambda \sin(-\Delta u). \tag{3.7}$$

Proof. Fix $\lambda, \mu \in \mathbb{R}$. For every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and every $y \in \mathbb{R}$, set

$$\varphi(x, z, w) := g(x) + \mu(|z| + |w|)^\gamma, \qquad \psi(y) := y - \lambda \sin y.$$

Since $\lim_{y \to \pm\infty} (y - \lambda \sin y) = \pm\infty$, the function $y \mapsto \varphi(x, z, w) - \psi(y)$ surely changes sign. Moreover, since it vanishes only at points of $\mathbb{R}$ and not in intervals, the set

$$\{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0\}$$

has empty interior in $\mathbb{R}$. Hence, hypotheses (i) and (ii) are fulfilled.

Fix now $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. In order to verify hypothesis (iii), we want to find $b, c \geq 0$, with $\frac{b}{\lambda_{1,2}} + \frac{c}{\lambda_{1,2}^{1/2}} < 1$, and $a \in L^2(\Omega, \mathbb{R}_0^+)$ such that

$$\max\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + b|z| + c|w|. \tag{3.8}$$

Notice that we can consider the maximum in (3.8) instead of the supremum, since the set $\psi^{-1}(\varphi(x, z, w))$ is compact. Of course, (3.8) is equivalent to prove that $|y| < a(x) + b|z| + c|w|$, for every y solution of the equation

$$\psi(y) = \varphi(x, z, w). \tag{3.9}$$

Thanks to Young's inequality with exponents $1/\gamma$ and $1/(1-\gamma)$, we have

$$\begin{aligned} |\varphi(x,z,w)| &= |g(x)+\mu(|z|+|w|)^\gamma| \le |g(x)|+|\mu||z|^\gamma+|\mu||w|^\gamma \\ &\le |g(x)|+\varepsilon|z|+\varepsilon|w|+C_{\gamma,\varepsilon,\mu} \\ &\le \tilde{g}(x)+\varepsilon|z|+\varepsilon|w|, \end{aligned} \tag{3.10}$$

where $\tilde{g}(x) := |g(x)|+C_{\gamma,\varepsilon,\mu}$ for every $x\in\Omega$. Then, if $\tilde{y}$ is a solution to (3.9), from (3.10) it follows that

$$|\psi(\tilde{y})| = |\varphi(x,z,w)| \le \tilde{g}(x)+\varepsilon|z|+\varepsilon|w|.$$

On the other hand, by the definition of ψ, we have

$$|\psi(\tilde{y})| = |\tilde{y}-\lambda\sin\tilde{y}| \ge |\tilde{y}|-|\lambda|,$$

which implies that

$$\begin{aligned} |\tilde{y}| &\le |\psi(\tilde{y})|+|\lambda| \le \tilde{g}(x)+|\lambda|+\varepsilon|z|+\varepsilon|w| < \\ &< \bar{g}(x)+\varepsilon|z|+\varepsilon|w|, \end{aligned}$$

where $\bar{g}(x) := \tilde{g}(x)+2|\lambda|$, for every $x\in\Omega$. Observe that $\bar{g}\in L^2(\Omega,\mathbb{R}_0^+)$.
Then, if we choose ε in such a way that

$$\frac{\varepsilon}{\lambda_{1,2}}+\frac{\varepsilon}{\lambda_{1,2}^{1/2}} < 1,$$

we have hypothesis (iii) with $a := \bar{g}$ and $b := c := \varepsilon$. Thanks to Theorem 3.3, there exists a solution $u\in W_0^{1,2}(\Omega)$ to equation (3.7).

$\square$

In the following example the function ψ exhibits a behavior very different from the previous one.

Example 3.6
Let $p\in[2,+\infty[$, $f\in L^{p'}(\Omega)$ and $\gamma\in]0,p-1[$. Then, for every $\mu\in\mathbb{R}$ and $\lambda\in\mathbb{R}^+$, there

exists a solution $u \in W_0^{1,p}(\Omega)$ to the equation

$$-\Delta_p u = f(x) + \mu(|u| + |\nabla u|)^\gamma - \lambda e^{-\Delta_p u}. \tag{3.11}$$

Proof. Fix $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^+$. As before, for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and $y \in \mathbb{R}$, we set

$$\varphi(x, z, w) := f(x) + \mu(|z| + |w|)^\gamma, \qquad \psi(y) := y + \lambda e^y.$$

Since $\lim_{y\to\pm\infty}(y + \lambda e^y) = \pm\infty$, one immediately gets that (i) and (ii) are fulfilled. In order to verify hypothesis (iii), we argue as in Example 3.5. First of all, applying Young's inequality with exponents $\frac{p-1}{\gamma}, \frac{p-1}{p-1-\gamma} > 1$, we have

$$\begin{aligned}
|\varphi(x, z, w)| &= |f(x) + \mu(|z| + |w|)^\gamma| \le |f(x)| + 2^{p-1}(|\mu||z|^\gamma + |\mu||w|^\gamma) \\
&\le |f(x)| + \varepsilon|z|^{p-1} + \varepsilon|w|^{p-1} + C_{\gamma,\varepsilon,\mu} \\
&= \tilde{f}(x) + \varepsilon|z|^{p-1} + \varepsilon|w|^{p-1},
\end{aligned}$$

where $\tilde{f}(x) := |f(x)| + C_{\gamma,\varepsilon,\mu}$ for every $x \in \Omega$. Let now $\tilde{y}$ be a solution to the equation $\varphi(x, z, w) - \psi(y) = 0$. Then, from the previous inequality, we have

$$|\psi(\tilde{y})| = |\varphi(x, z, w)| \le \tilde{f}(x) + \varepsilon|z|^{p-1} + \varepsilon|w|^{p-1}.$$

On the other hand, for every $y \in \mathbb{R}$, and in particular for $\tilde{y}$, we have

$$|\psi(\tilde{y})| = |\tilde{y} + \lambda e^{\tilde{y}}| \ge |\tilde{y}| - |\xi|, \tag{3.12}$$

ξ being the only solution to the equation $y + \lambda e^y = 0$. Indeed, fix $y \in \mathbb{R}$. If $y \ge \xi$, we have

$$\begin{aligned}
|y + \lambda e^y| &= |y + \lambda e^y - \xi - \lambda e^\xi| = |y - \xi + \lambda(e^y - e^\xi)| \\
&\ge |y - \xi| \ge |y| - |\xi|,
\end{aligned}$$

which is (3.12). Suppose now that $y < \xi$, then

$$
\begin{aligned}
|y + \lambda e^y| = |y - \xi + \lambda(e^y - e^\xi)| &= |\xi - y + \lambda(e^\xi - e^y)| \\
&\geq |\xi - y| \geq |y| - |\xi|,
\end{aligned}
$$

which again gives (3.12). This implies that

$$
\begin{aligned}
|\tilde{y}| &\leq |\psi(\tilde{y})| + |\xi| \leq \tilde{f}(x) + \varepsilon|z|^{p-1} + \varepsilon|w|^{p-1} + |\xi| \\
&< \bar{f}(x) + \varepsilon|z|^{p-1} + \varepsilon|w|^{p-1},
\end{aligned}
$$

$\bar{f}(x) := \tilde{f}(x) + 2|\xi|$, for every $x \in \Omega$ (note that one cannot have $\xi = 0$). Observe that $\bar{f} \in L^{p'}(\Omega, \mathbb{R}_0^+)$. Then, if we choose ε in such a way that

$$
\frac{\varepsilon}{\lambda_{1,p}} + \frac{\varepsilon}{\lambda_{1,p}^{1/p}} < 1,
$$

we have hypothesis (iii) with $a := \bar{f}$ and $b := c := \varepsilon$. Thanks to Theorem 3.3, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation (3.11). □

To state the next result, we set $\delta_\Omega := \mathrm{diam}(\Omega)$ and denote by $\hat{C}$ the constant given by Proposition 2.7.

Theorem 3.7

Let φ and ψ as in Theorem 3.3. Suppose that hypotheses (i)-(ii) hold true and, moreover,

(iii)$'$ there exist $a \in L^q(\Omega, \mathbb{R}_0^+)$, $q > N$, $g\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ nondecreasing with respect to each variable separately, such that

$$
\sup\{|y| : y \in \psi^{-1}(\varphi(x, z, w))\} < a(x) + g(|z|, |w|),
$$

for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,

(iv) there exists $R > 0$ such that

$$
\|a\|_{L^q(\Omega)} + m(\Omega)^{1/q} g(\delta_\Omega \hat{C} R^{1/(p-1)}, \hat{C} R^{1/(p-1)}) \leq R.
$$

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation (3.5).

Proof. As before, fix $x \in \Omega$, and for all $(z, w) \in \mathbb{R} \times \mathbb{R}^N$, define

$$F(x, z, w) := \{y \in \mathbb{R} : \varphi(x, z, w) - \psi(y) = 0, \\ y \text{ is not a local extremum point of } \psi(\cdot)\}.$$

Reasoning like in the previous theorem, the multifunction F actually takes nonempty closed values, is lower semicontinuous w.r.t. (z, w) and $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable.

Fix now $y \in F(x, z, w)$, that is $y \in \psi^{-1}(\varphi(x, z, w))$. Then, by hypothesis (iii)$'$, we have

$$\inf_{y \in F(x,z,w)} |y| < a(x) + g(|z|, |w|) \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Taking into account (iv), we see that all the hypotheses of Theorem 2.8 are fulfilled. Therefore, there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in F(x, u\nabla u)$. Exploiting the definition of F, this means that u is a solution to equation (3.5).

□

As an application of the previous result, we consider the following example, which has been inspired by [24, Corollary 1]. Observe that, unlike [24], here we consider a function φ which is not necessarily continuous w.r.t. the variable x, but only in a suitable $L^q(\Omega)$. Moreover, we deal with partial differential equations.

Example 3.8
Let $h \in L^q(\Omega)$, with $q > N$. Then, for every $k \in \mathbb{R}$, there exists a solution $u \in W_0^{1,2}(\Omega)$ to the equation

$$-\Delta u = h(x) + u^3 + |\nabla u|^2 + k \sin(-\Delta u). \tag{3.13}$$

Proof. Fix $k \in \mathbb{R}$ and for all $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ and all $y \in \mathbb{R}$ define

$$\varphi(x, z, w) := h(x) + z^3 + |w|^2, \qquad \psi(y) := y - k \sin y.$$

Reasoning like in Example 3.5, we have that hypotheses (i)-(ii) are fulfilled. In order to verify hypothesis (iii)$'$, let $g(|z|,|w|) := |z|^3 + |w|^2$, for all $(z,w) \in \mathbb{R} \times \mathbb{R}^N$. Of course $g\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is nondecreasing w.r.t. each variable separately. Let $\tilde{y}$ be a solution to equation $\varphi(x,z,w) = \psi(y)$. It follows that

$$\begin{aligned}|\psi(\tilde{y})| = |\tilde{y} - k \sin \tilde{y}| = |\varphi(x,z,w)| \leq \\ \leq |h(x)| + |z|^3 + |w|^2 = |h(x)| + g(|z|,|w|).\end{aligned}$$

On the other hand, we always have

$$|\psi(\tilde{y})| = |\tilde{y} - k \sin \tilde{y}| \geq |\tilde{y}| - |k|,$$

which implies that

$$\begin{aligned}|\tilde{y}| \leq |\psi(\tilde{y})| + |k| \leq |h(x)| + g(|z|,|w|) + |k| < \\ < \bar{h}(x) + g(|z|,|w|),\end{aligned}$$

where $\bar{h}(x) := |h(x)| + 1$, for every $x \in \Omega$. Of course, $\bar{h} \in L^q(\Omega, \mathbb{R}_0^+)$. Hence we get hypothesis (iii).

Moreover, in order to verify hypothesis (iv), we have to check if there exists $R > 0$ such that

$$\|\bar{h}\|_{L^q(\Omega)} + m(\Omega)^{1/q} g(\delta_\Omega \hat{C} R, \hat{C} R) \leq R,$$

that is

$$\|\bar{h}\|_{L^q(\Omega)} + m(\Omega)^{1/q} \delta_\Omega^3 \hat{C}^3 R^3 + m(\Omega)^{1/q} \hat{C}^2 R^2 \leq R. \tag{3.14}$$

If $0 < R << 1$, choosing $\bar{h}$ in such a way that $\|\bar{h}\|_{L^q(\Omega)} < \frac{R}{2}$, we have that (3.14) is immediately satisfied, since the terms containing R^2 and R^3 are negligible with respect to R. So all the hypotheses of Theorem 3.7 are fulfilled, and we obtain a solution $u \in W_0^{1,2}(\Omega)$ to equation (3.13).

□

Next result provides solutions to equation (3.5) when the function ψ is of the form $y \mapsto y - h(y)$, with h continuous and bounded. Note that here we have to require a specific growth condition on φ.

Theorem 3.9
Let $\varphi\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, and let $h \in L^\infty(\mathbb{R})$ be continuous. Suppose that (i)-(ii) hold true and, moreover,

(iii)$''$ there exist $f \in L^{p'}(\Omega, \mathbb{R}_0^+)$, with $f(x) \geq \|h\|_\infty$ for all $x \in \Omega$, $\mu > 0$, $\gamma \in]0, p-1[$ such that

$$\sup_{(x,z,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N} |\varphi(x,z,w)| < f(x) + \mu(|z| + |w|)^\gamma.$$

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation

$$-\Delta_p u - h(-\Delta_p u) = \varphi(x, u, \nabla u). \tag{3.15}$$

Proof. Fix $x \in \Omega$ and define, for all $(z, w) \in \mathbb{R} \times \mathbb{R}^N$,

$$\begin{aligned} F(x,z,w) := \{y \in \mathbb{R} : \varphi(x,z,w) - (y - h(y)) = 0, \\ y \text{ is not a local extremum point of } y \mapsto y - h(y)\}. \end{aligned}$$

Reasoning as in the above proofs ensures that F is lower semicontinuous w.r.t. (z, w), with nonempty closed values, and $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable.

Fix $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. If $y \in F(x, z, w)$, then it solves the equation $\varphi(x, z, w) = y - h(y)$. Two cases occur. First, $\gamma \in [1, p-1[$. Applying Young's inequality with exponents $\frac{p-1}{\gamma}, \frac{p-1}{p-1-\gamma} > 1$, we have

$$\begin{aligned} |y| = |y - h(y) + h(y)| &\leq |y - h(y)| + |h(y)| \leq |\varphi(x,z,w)| + \|h\|_\infty \\ &< f(x) + \mu(|z| + |w|)^\gamma + \|h\|_\infty \\ &\leq 2f(x) + 2^{\gamma-1}\mu(|z|^\gamma + |w|^\gamma) \\ &\leq 2f(x) + 2^{\gamma-1}\mu(\varepsilon|z|^{p-1} + \varepsilon|w|^{p-1} + K_\varepsilon) \\ &\leq 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu\varepsilon(|z|^{p-1} + |w|^{p-1}), \end{aligned}$$

that is $|y| < 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu\varepsilon(|z|^{p-1} + |w|^{p-1})$, where $C_\varepsilon := 2^{\gamma-1}\mu K_\varepsilon$. Hence

$$\inf_{y\in F(x,z,w)} |y| < 2f(x) + C_\varepsilon + 2^{\gamma-1}\mu\varepsilon(|z|^{p-1} + |w|^{p-1}).$$

If we choose ε in such a way that

$$\frac{2^{\gamma-1}\mu\varepsilon}{\lambda_{1,p}} + \frac{2^{\gamma-1}\mu\varepsilon}{\lambda_{1,p}^{1/p}} < 1,$$

hypothesis (h3) of Theorem 2.6 is fulfilled, with $a := 2f + C_\varepsilon \in L^{p'}(\Omega, \mathbb{R}_0^+)$ and $b := c := 2^{\gamma-1}\mu\varepsilon$.

Suppose now $\gamma \in]0,1[$. Since, for every $a, b \geq 0$ we have $(a+b)^\gamma \leq a^\gamma + b^\gamma$, reasoning as before yields

$$|y| < 2f(x) + \tilde{C}_\varepsilon + \mu\varepsilon(|z|^{p-1} + |w|^{p-1}),$$

where $\tilde{C}_\varepsilon := \mu K_\varepsilon$. If we now choose ε in such a way that

$$\frac{\mu\varepsilon}{\lambda_{1,p}} + \frac{\mu\varepsilon}{\lambda_{1,p}^{1/p}} < 1,$$

hypothesis (h3) of Theorem 2.6 is again fulfilled, with $a := 2f + \tilde{C}_\varepsilon \in L^{p'}(\Omega, \mathbb{R}_0^+)$ and $b := c := \mu\varepsilon$.

In both cases, there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in F(x, u, \nabla u)$. Through a familiar argument, this entails that u is a solution to equation (3.15).

□

We conclude this section considering the case when Y is a closed interval of $\mathbb{R}$. Observe that here no growth conditions on φ are required.

Theorem 3.10
Let $\varphi\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, and $\psi\colon [\alpha, \beta] \to \mathbb{R}$ be continuous. Suppose that:

1. for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the set $\{y \in [\alpha, \beta] : \varphi(x, z, w) - \psi(y) = 0\}$ has empty interior;
2. for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, the function $y \mapsto \varphi(x, z, w) - \psi(y)$ changes sign in $[\alpha, \beta]$.

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation (3.5).

Proof. As before, fix $x \in \Omega$, and for all $(z, w) \in \mathbb{R} \times \mathbb{R}^N$ define

$$F(x, z, w) := \{y \in [\alpha, \beta] : \varphi(x, z, w) - \psi(y) = 0,$$
$$y \text{ is not a local extremum point of } \psi(\cdot)\}.$$

Reasoning as in the previous results, we have that F takes nonempty closed values, is lower semicontinuous w.r.t. (z, w) and $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable.

Fix now $y \in F(x, z, w)$. In particular we have that $|y| \leq \max\{|\alpha|, |\beta|\}$. Then, hypothesis (h3) of Theorem 2.6 is immediately satisfied with $a(x) = 2\max\{|\alpha|, |\beta|\}$ for all $x \in \Omega$ and $b = c = 0$. Therefore, there exists $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u \in F(x, u, \nabla u)$, so u is a solution to equation (3.5).

□

As application of the previous theorem, we consider two examples, which differ from the nonlinear behavior of the function ψ. In both cases, the condition which allows us to get a solution is the boundedness of the function φ.

Example 3.11
Let $f \in L^\infty(\Omega)$, $k \in \mathbb{N}$, k even and such that $k\pi > \|f\|_\infty$, and let $\psi\colon [-k\pi, k\pi] \to \mathbb{R}$ be defined by $\psi(y) = y \cos y$. Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to the equation

$$\psi(-\Delta_p u) = f \qquad \text{in } \Omega. \tag{3.16}$$

Proof. Observe that assumption (1) of Theorem 3.10 is clearly satisfied. Moreover, for every $x \in \Omega$, we have

$$\begin{aligned} & f(x) - \psi(k\pi) = f(x) - k\pi \cos(k\pi) = f(x) - k\pi\,(-1)^k = f(x) - k\pi < 0 \\ \text{and} \qquad & f(x) - \psi(-k\pi) = f(x) + k\pi \cos(-k\pi) = f(x) + k\pi > 0. \end{aligned}$$

Therefore, hypothesis (2) is also satisfied. Thanks to Theorem 3.10, there exists at least a solution $u \in W_0^{1,p}(\Omega)$ to equation (3.16).

□

Note that the interval $[\alpha, \beta]$ could not be necessarily bounded, as in the case of the following example.

Example 3.12
Let $p \in]1, +\infty[$, $f \in L^{p'}(\Omega)$, $\gamma > 0$ and $\varphi \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$. Suppose that there exists $\lambda \in \mathbb{R}^+$ such that

$$\sup_{(x,z,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N} |\varphi(x, z, w)| < \lambda. \tag{3.17}$$

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation

$$\varphi(x, u, \nabla u) - \lambda e^{\Delta_p u} - \Delta_p u = 0.$$

Proof. Define $\psi(y) := \lambda e^{-y} - y$ for every $y \in [0, +\infty[$. Observe that hypothesis (1) is immediately satisfied. Moreover, thanks to (3.17), for every $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ we have

$$\varphi(x, z, w) - \psi(0) = \varphi(x, z, w) - \lambda < 0$$
$$\text{and} \quad \lim_{y\to+\infty} (\varphi(x, z, w) - \psi(y)) = +\infty.$$

Therefore, hypothesis (2) holds true too, and the conclusion follows from Theorem 3.10.

□

3.2.2 The discontinuous framework

The interest for a Dirichlet problem for an elliptic equation involving the p-Laplacian and containing discontinuous nonlinearities, was inspired by the paper [32].

Here, the discontinuity is represented by φ that appears in (3.5). Following essentially the reasoning made in Theorem 3.1 of [32], we construct an appropriate upper semicontinuous multifunction F related with ψ^{-1} and φ, and then we solve the elliptic differential inclusion $-\Delta_p u \in F(x, u)$ using Theorem 2.9. So, we obtain Theorem 3.13, that extends [32, Theorem 3.1] to the case $p \neq 2$.

For every $(x, z) \in S := \Omega \times \mathbb{R}$, we set $\pi_0(x, z) = x$ and $\pi_1(x, z) = z$. Moreover, we fix $p > N$ and define

$$\mathcal{F} = \{A \subseteq S : A \text{ is measurable and there exists } i \in \{0, 1\} \text{ such that } m(\pi_i(A)) = 0\},$$

namely π_0 and π_1 denote the projections of $\Omega \times \mathbb{R}$ on Ω and $\mathbb{R}$, respectively.

Theorem 3.13
Let $(\alpha, \beta) \subseteq \mathbb{R}$ be an interval which does not contain 0, ψ a continuous real-valued function defined on (α, β) and φ a real-valued function defined on $\Omega \times \mathbb{R}$. Suppose that the following conditions hold true:

(i) φ is $\mathcal{L}(\Omega \times \mathbb{R})$-measurable and essentially bounded;

(ii) the set $D_\varphi = \{(x, z) \in S : \varphi \text{ is discontinuous at } (x, z)\}$ belongs to $\mathcal{F}$;

(iii) $\varphi^{-1}(r) \setminus \operatorname{int}(\varphi^{-1}(r)) \in \mathcal{F}$ for every $r \in \psi((\alpha, \beta))$;

(iv) $\overline{\varphi(S \backslash D_\varphi)} \subseteq \psi((\alpha, \beta))$.

Then, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\psi(-\Delta_p u) = \varphi(x, u).$$

Proof. The first part of the proof essentially follows [32, Theorem 3.1]. Thanks to assumption (i), there exists a constant $c > 0$ such that

$$S \setminus D_\varphi \subseteq \{(x, z) \in S : |\varphi(x, z)| \leq c\}.$$

Set

$$a = \min \overline{\varphi(S \backslash D_\varphi)} \quad \text{and} \quad b = \max \overline{\varphi(S \backslash D_\varphi)}.$$

Hypothesis (iv) allows us to choose $y', y'' \in (\alpha, \beta)$ in such a way that $\psi(y') = a$ and $\psi(y'') = b$. Pick a continuous function $\lambda\colon [0, 1] \to (\alpha, \beta)$ complying with $\lambda(0) = y'$, $\lambda(1) = y''$, and define $\tilde{\psi}(t) = \psi(\lambda(t))$, for every $t \in [0, 1]$.

If $\tilde{\psi}$ is constant, then $a = b$ and, consequently, $\varphi(S \backslash D_\varphi) = \{a\}$. Let $u \in W_0^{1,p}(\Omega)$ be such that $-\Delta_p u = y'$. Since $\psi(-\Delta_p u) = \psi(y') = a$, then the conclusion will be achieved by showing that the set $\Omega_\varphi = \{x \in \Omega : (x, u(x)) \in D_\varphi\}$ has measure zero.

First of all observe that an elementary computation gives us

$$\Omega_\varphi \subseteq \pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi)) \tag{3.18}$$

and, due to (ii), $m(\pi_i(D_\varphi)) = 0$, for some $i \in \{0, 1\}$.

Suppose that $i = 0$. From (3.18) we obtain

$$m(\Omega_\varphi) \leq m(\pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi))) \leq m(\pi_0(D_\varphi) = 0,$$

and so $m(\Omega_\varphi) = 0$. Suppose now that $i = 1$. From [14, Lemma 1.1], it follows that $\nabla u(x) = 0$ a.e. in $u^{-1}(\pi_1(D_\varphi))$. Moreover, thanks to [28, Theorem 1.1], we have that $y' = 0$ on $\{x \in \Omega : \nabla u(x) = 0\}$, and, in particular, on $\{u^{-1}(\pi_1(D_\varphi)) = 0\}$ (notice that our calculation showed that $u^{-1}(\pi_1(D_\varphi)) \subseteq \{x \in \Omega : \nabla u(x) = 0\}$). Since $y' \in (\alpha, \beta)$, the latter is possible if and only if $m(u^{-1}(\pi_1(D_\varphi))) = 0$.

Again from (3.18) we get

$$m(\Omega_\varphi) \leq m(\pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi))) \leq m(u^{-1}(\pi_1(D_\varphi))),$$

which implies $m(\Omega_\varphi) = 0$.

Suppose now that $\tilde{\psi}$ is non constant and choose $t_1, t_2 \in [0, 1]$ fulfilling

$$\tilde{\psi}(t_1) = \min_{t \in [0,1]} \tilde{\psi}(t), \qquad \tilde{\psi}(t_2) = \max_{t \in [0,1]} \tilde{\psi}(t).$$

Obviously, $t_1 \neq t_2$ and there is no loss of generality in assuming $t_1 < t_2$. Let $h \colon \tilde{\psi}([0, 1]) \to [0, 1]$ be defined by $h(r) = \min(\tilde{\psi}^{-1}(r) \cap [t_1, t_2])$, for every $r \in \tilde{\psi}([0, 1])$.

We claim that h is strictly increasing. Indeed, pick $r_1, r_2 \in \tilde{\psi}([0, 1])$, with $r_1 < r_2$. Then, $h(r_1) \neq h(r_2)$ and $t_1 < h(r_2)$. From $\tilde{\psi}(h(r_2)) = r_2 > r_1$, $\tilde{\psi}(t_1) \leq r_1$, taking into account the continuity of $\tilde{\psi}$, we immediately infer $h(r_1) < h(r_2)$.

Therefore, the family D_k of all discontinuity points of the function $k \colon \mathbb{R} \to (\alpha, \beta)$ given by

$$k(r)=\begin{cases}\lambda(h(\tilde{\psi}(t_1))) & \text{if } r\in]-\infty,\tilde{\psi}(t_1)[\\ \lambda(h(r)) & \text{if } r\in\tilde{\psi}([0,1])\\ \lambda(h(\tilde{\psi}(t_2))) & \text{if } r\in]\tilde{\psi}(t_2),+\infty[\end{cases}$$

is at most countable. Owing to hypotheses (ii) and (iii), this implies that the set

$$D=D_\varphi\cup\{\bigcup_{r\in D_k}[\varphi^{-1}(r)\setminus \operatorname{int}(\varphi^{-1}(r))]\} \tag{3.19}$$

has measure zero.

Define now $f(x,z):=k(\varphi(x,z))$, $(x,z)\in S$. Of course, the function $f\colon S\to\mathbb{R}$ is bounded, since $f(S)\subseteq\lambda([0,1])$. Moreover, arguing like in [32, Theorem 3.1], we have that f is continuous.

Put, for every $(x,z)\in S$,

$$F(x,z)=\overline{\text{co}}\left(\bigcap_{\delta>0}\bigcap_{E\in\mathcal{E}}\overline{f(B_\delta(x,z)\setminus E}\right),$$

where

$$\mathcal{E}=\{E\subseteq S: m(E)=0\}$$
$$\text{and}\qquad B_\delta(x,z)=\{(x',z')\in S: |x-x'|+|z-z'|\le\delta\}.$$

A standard argument (see, e.g., [32, Theorem 3.1]), ensures that F is upper semicontinuous, with nonempty, convex and closed values. Moreover, $F(\cdot,z)$ is measurable for every $z\in\mathbb{R}$, $F(x,\cdot)$ has a closed graph for almost every $x\in\Omega$ and

$$F(x,z)=\{f(x,z)\},\quad\text{for every}\quad (x,z)\in S\setminus D.$$

Consider now the problem

$$-\Delta_p u\in F(x,u),\qquad u\in W_0^{1,p}(\Omega).$$

A solution will be obtained via Theorem 2.9, let us verify its hypotheses. Example 2.10 directly yelds (i_1) by choosing $U:=A_p^{-1}(L^{p'}(\Omega))$, $\Phi(u):=u$ and $\Psi(u):=A_p(u)$, for every $u \in U$. Moreover, it is immediate to observe that hypotheses (i_2) and (i_3) are already satisfied, so we have only to check hypothesis (i_4).

Define, for every $x \in \Omega$, the function $\rho(x) := \sup_{|z| \leq g(r)} d(0, F(x,z))$. We have to verify that $\rho \in L^{p'}(\Omega)$ and $\|\rho\|_{p'} \leq r$. Reasoning as in [31, Theorem 3.1], we see that $\|\rho\|_{p'} \leq r$ once we prove the same property for the function $x \mapsto j(x) := \sup_{|z| \leq g(r)} |f(x,z)|$.

If $z \in \mathbb{R}$ is such that $|z| \leq g(r)$, we have

$$\int_\Omega |f(x,z)|^{p'} dx \leq \int_\Omega \|f\|^{p'}_{L^\infty(S)} dx = \|f\|^{p'}_{L^\infty(S)} m(\Omega),$$

and so

$$\begin{aligned} \int_\Omega |j(x)|^{p'} dx = \int_\Omega | \sup_{|z| \leq g(r)} f(x,z)|^{p'} dx \leq \sup_{|z| \leq g(r)} \int_\Omega |f(x,z)|^{p'} dx \leq \\ \leq \sup_{|z| \leq g(r)} \|f\|^{p'}_{L^\infty(S)} m(\Omega) = \|f\|^{p'}_{L^\infty(S)} m(\Omega). \end{aligned}$$

Therefore, if we choose $r \geq \|f(\cdot, z)\|_\infty m(\Omega)^{1/p'}$, we get $j \in L^{p'}(\Omega)$ and $\|j\|_{p'} \leq r$.

Taking into account the previous observation, we have hypothesis (i_4), and so, thanks to Theorem 2.9, there exists $u \in U \subseteq W_0^{1,p}(\Omega)$ such that

$$-\Delta_p u \in F(x,u) \tag{3.20}$$

and $|\Delta_p u(x)| \leq \rho(x)$, for almost every $x \in \Omega$.

Define $\Omega_f = \{x \in \Omega : (x, u(x)) \in D\}$. From (3.19), it follows that

$$\begin{aligned} \Omega_f \subseteq & \{\pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi))\} \\ & \cup \Big\{ \bigcup_{r \in D_k} \big[\pi_0(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))))\big] \Big\}, \end{aligned}$$

which, in particular, implies

$$\begin{aligned}
m(\Omega_f) &\leq m\big(\pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi))\big) \\
&\quad + m\Big(\bigcup_{r\in D_k} [\pi_0(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))))]\Big) \\
&\leq m\big(\pi_0(D_\varphi) \cap u^{-1}(\pi_1(D_\varphi))\big) \\
&\quad + \bigcup_{r\in D_k} m\big([\pi_0(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))) \cap u^{-1}(\pi_1(\varphi^{-1}(r) \setminus \mathrm{int}(\varphi^{-1}(r))))]\big).
\end{aligned}$$

Assumption (ii) entails $m(\pi_i(D_\varphi)) = 0$ for some $i \in \{0, 1\}$. Likewise, due to (iii), for each $r \in D_k$, there exists $i_r \in \{0, 1\}$ such that $m(\pi_{i_r}(\varphi^{-1}(r)\setminus\mathrm{int}(\varphi^{-1}(r)))) = 0$. Therefore, reasoning as in the case when $\tilde{\psi}$ is constant, and taking into account that the set D_k is at most countable, we obtain $m(\Omega_f) = 0$. This implies $F(x, u(x)) = \{f(x, u(x))\}$ and so, on account (3.20),

$$-\Delta_p u = f(x, u).$$

Eventually,

$$\psi(-\Delta_p u) = \psi(f(x, u)) = \psi(k(\varphi(x, u))) = \varphi(x, u),$$

which completes the proof.

□

Next example provides a simple application of Theorem 3.13.

Example 3.14
Let $(\alpha, \beta) = [1, +\infty[$, $\psi\colon [1, +\infty[\to \mathbb{R}$ be such that $\psi(y) = (2-y)/y$ for every $y \in [1, +\infty[$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(z) = \begin{cases} |z|/(1 + z^2) & \text{if } z \neq 0 \\ 1/2 & \text{if } z = 0. \end{cases}$$

Then, there exists a solution $u \in W_0^{1,p}(\Omega)$ to equation $\psi(-\Delta_p u) = \varphi(u)$.

Proof. Observe that hypotheses (i)-(iii) are immediately satisfied, since φ is bounded,

$D_\varphi = \{0\} \in \mathcal{F}$, and $\varphi^{-1}(r)\setminus\mathrm{int}(\varphi^{-1}(r)) = \varphi^{-1}(r) \in \mathcal{F}$, for every $r \in \psi([1,+\infty[) =]-1,1]$. Moreover, since $\varphi(\mathbb{R}\setminus\{0\}) =]0,1/2]$, we immediately infer that

$$\overline{\varphi(\mathbb{R}\setminus\{0\})} = [0,1/2] \subseteq]-1,1] = \psi([1,+\infty[),$$

that is hypothesis (iv).

Therefore, thanks to Theorem 3.13, we get our thesis.

□

Hypothesis (iv) and the assumption $0 \notin (\alpha,\beta)$ are *essential* to obtain the existence of a solution for equations as in the previous theorem. Below we list two examples, apparently very similar, and such that one admits a solution while the other one doesn't.

Example 3.15

Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(z) = \begin{cases} 0 & \text{if } z \neq 0 \\ 1 & \text{if } z = 0. \end{cases}$$

and let $\psi\colon [1,+\infty[\to \mathbb{R}$ be such that $\psi(y) = y$. Consider the following equation

$$-\Delta_p u = \varphi(u). \tag{3.21}$$

Note that equation (3.21) doesn't have any solution $u \in W_0^{1,p}(\Omega)$. Suppose on the contrary that u is such a solution. Since $\varphi(u) \geq 0$, then from equation (3.21) we have $-\Delta_p u \geq 0$, and so the Strong Maximum Principle implies that $u \equiv 0$ or $u > 0$. Suppose that $u \equiv 0$, then this would imply that $-\Delta_p u \equiv 0$, which is in contrast with (3.21). Suppose now that $u > 0$. Then, taking into account the definition of φ and equation (3.21), we have $-\Delta_p u = 0$. This fact, jointly with the boundary condition $u|_{\partial\Omega} = 0$, implies $u \equiv 0$ which is again impossible.

It is also evident from the definition of φ that hypothesis (iv) and $0 \notin (\alpha,\beta)$ cannot be verified simultaneously.

Fix now $\lambda \in]0,1[$ and consider the function $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{\varphi}(z) = \begin{cases} 1 & \text{if } z \neq 0 \\ \lambda & \text{if } z = 0. \end{cases}$$

In this case $0 \notin [1,+\infty[$ and hypothesis (iv) is now verified, since

$$\{1\} = \overline{\tilde{\varphi}(\mathbb{R} \setminus \{0\})} \subseteq \psi([1,+\infty[) = [1,+\infty[,$$

and so we get a solution to the equation $-\Delta_p u = \tilde{\varphi}(u)$.

Chapter 4

Another point of view

In this chapter, we want to point out another method, different from those used in the previous one, to solve differential inclusions. This important tool is represented by critical point theory and variational methods, which have been used to state existence and multiplicity results, without requiring monotonicity assumption.

4.1 The locally Lipschtiz continuous setting

In [23], the following basic differential inclusion of elliptic type, with homogeneous Dirichlet boundary conditions, has been studied:

$$\begin{cases} -\Delta u \in F(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, is a bounded domain with a C^2-boundary and $F : \mathbb{R} \to 2^{\mathbb{R}}$ is an upper semicontinuous multifunction with compact, convex values satisfying an appropriate growth condition. It is interesting to focus on the method used there, because it hopefully could be applied to other versions of problem (4.1), e.g., with the p-Laplacian operator on the left-hand side, or Neumann boundary conditions.

First of all, recall that $u \in H_0^1(\Omega)$ is a weak solution of (4.1), if there exists $w \in L^2(\Omega)$ such that:

1. $\displaystyle\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega wv dx$ for all $v \in H_0^1(\Omega)$,

2. $w(x) \in F(u(x))$ for a.e. $x \in \Omega$.

Before continuing our subject, we need some basic notions of non-smooth critical point theory.

Definition 4.1
Let X be a Banach space and denote by X^* its dual. A function $J : X \to \mathbb{R}$ is said *locally Lipschitz continuous* if for every $x \in X$ there exist a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|J(z) - J(w)| \leq L_x \|z - w\| \quad \forall z, w \in V_x.$$

Definition 4.2
Given a locally Lipschitz continuous function $J : X \to \mathbb{R}$, we say *generalized directional derivative* at the point $x \in X$ along the direction $z \in X$ the value

$$J^0(x; z) = \limsup_{w \to x,\ t \to 0^+} \frac{J(w + tz) - J(w)}{t} = \inf_{\varepsilon, \delta > 0} \sup_{\|x - w\| < \varepsilon,\ 0 < t < \delta} \frac{J(w + tz) - J(w)}{t}.$$

The set

$$\{x^* \in X^* : \ \langle x^*, z \rangle \leq J^0(x; z) \ \ \forall z \in X\}$$

is called *generalized gradient* or *sub-differential* of J in x and it is usually denoted by $\partial J(x)$. The functionals $x^* \in \partial J(x)$ are said *sub-gradients*.

The generalized gradient has the following properties, that follows from its definition.

Proposition 4.3
The following facts hold true:

i) $\partial(\lambda J)(x) = \lambda \partial J(x)$ for every $\lambda \in \mathbb{R}$ and $x \in X$.

ii) If J and J_1 are two locally Lipschitz continuous functionals, then for every $x \in X$ one has

$$\partial(J + J_1)(x) \subseteq \partial J(x) + \partial J_1(x).$$

Problem (4.1) can be reduced to a variational one, studied in [16], by "shrinking" pointwise the set $F(u)$ to a smaller interval, which happens to be the gradient of a convenient locally Lipschitz continuous potential, namely $J : \mathbb{R} \to \mathbb{R}$ such that

$$\partial J(z) \subseteq F(z) \quad \forall z \in \mathbb{R}. \tag{4.2}$$

Then, we consider the auxiliary problem

$$\begin{cases} -\Delta u \in \partial J(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

If we find a solution to this reduced inclusion problem, we automatically solve (4.1) thanks to (4.2).
Let us suppose that the multifunction F satisfies the following growth hypotheses:

$$\exists a > 0, p \in (1, 2^*) \text{ such that } |y| \leq a(1 + |z|^{p-1}) \quad \forall z \in \mathbb{R},\ y \in F(z). \tag{4.4}$$

Theorem 1.12 produces a Borel measurable selection $f : \mathbb{R} \to \mathbb{R}$ of the multifunction F. By (4.4), we immediately infer that $f \in L^\infty_{loc}(\mathbb{R})$ so we can define

$$J_f(\xi) := \int_0^\xi f(s)ds, \quad \xi \in \mathbb{R}.$$

This function $J_f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and its gradient is given by

$$\partial J_f(z) = [f^-(z), f^+(z)],$$

where f^+ and f^- are defined by

$$f^+(x) := \lim_{\delta \to 0^+} \text{ess sup}\{f(\tau) : |\tau - x| < \delta\},$$

$$f^-(x) := \lim_{\delta \to 0^+} \text{ess inf}\{f(\tau) : |\tau - x| < \delta\}.$$

Let us consider the energy functional related to problem (4.1)

$$\Phi(u) := \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega J_f(u(x))dx \quad \forall u \in H_0^1(\Omega),$$

that is proven to be locally Lipschitz continuous in $H_0^1(\Omega)$.

Before stating Proposition 4.6, we need the notion of critical point and a characterization in this framework.

Let $J : X \to \mathbb{R}_0^+$ be a locally Lipschitz continuous function. We associate to J the function $m_J : X \to \mathbb{R}_0^+$ defined by

$$m_J(x) := \min_{\xi \in \partial J(x)} \|\xi\|_{X^*} \quad \forall x \in X.$$

Definition 4.4
We say that $x_0 \in X$ is a (generalized) critical point of J if $m_J(x_0) = 0$.

Proposition 4.5
The following statements are equivalent:

1. x_0 is a critical point of J;
2. $J^0(x_0; z) \geq 0$ for all $z \in X$;
3. $0 \in \partial J(x_0)$.

Now, we can give a positive answer to problem (4.1), that is [23, Proposition 3.2].

Proposition 4.6
If u is a critical point for the functional Φ, then u is a solution to Problem (4.1).

Proof. Let us consider the functional

$$\psi(u) = \int_\Omega J_f(u(x))dx \quad \forall u \in L^p(\Omega),$$

that is locally Lipschitz continuous, see [23, Proposition 3.2].
Let $u \in L^p(\Omega)$ and $w^* \in \partial\psi(u)$. By [12, Theorem 4.11], there exists $w \in L^{p'}(\Omega)$ such that

$$w^*(v) = \int_\Omega wxdx \quad \forall v \in L^p(\Omega).$$

Moreover, $J_f(\xi)$ satisfies all the hypotheses of [20, Theorem 1.3.17], so

$$w(x) \in \partial J_f(u(x)) \quad \text{a.e. in } \Omega,$$

namely

$$\partial\psi(u) \subseteq \int_\Omega \partial J_f(u(x))dx. \tag{4.5}$$

Since the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and dense with $p \leq 2^*$, the inclusion (4.5) remains true in $H_0^1(\Omega)$.
If we consider the functional Φ, Proposition 4.3 assures that

$$\partial\Phi(u) \subseteq \partial\big(\frac{\|u\|^2}{2}\big) - \partial\psi(u).$$

Since

$$A(u)(v) = \int_\Omega \nabla u \cdot \nabla v dx$$

is the derivative of $u \to \frac{\|u\|^2}{2}$, we obtain

$$\partial\Phi(u) \subseteq A(u) - \int_\Omega \partial J_f(u(x))dx. \tag{4.6}$$

Now, let u be a critical point for the functional Φ, then $0 \in \partial\Phi(u)$. By (4.6), we have

$$A(u) \in \int_\Omega \partial J_f(u(x))dx.$$

This means that there exists $w \in L^{p'}(\Omega)$ such that

$$A(u)(v) = \int_\Omega wvdx$$

for all $v \in H_0^1(\Omega)$, and $w(x) \in \partial J_f(u(x))$ for a.e. in Ω. It follows that

$$f^-(u(x)) \leq w(x) \leq f^+(u(x)).$$

Since the multifunction F takes convex values and for every $s \in \mathbb{R}$

$$\min F(s) \leq f^-(s) \leq f^+(s) \leq \max F(s),$$

we have $w(x) \in F(u(x))$ for a.e. $x \in \Omega$ and so we obtain the thesis. $\square$

The previous result is useful to solve Problem (4.1) by looking for critical points of the functional Φ. Indeed, at this aim it is shown that Φ satisfies a (PS)-type condition in [23].

4.2 A Dirichlet problem with p-Laplacian and set-valued nonlinearity

In this section we want to present a homogeneous Dirichlet problem for a p-Laplacian differential inclusion by using a fixed-point theorem concerning operator inclusions in Banach spaces and exploiting once again Theorem 2.16.

Let $\Omega \subseteq R^N$ and let $J : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the following assumptions:

(J_1) $J(\cdot, z), z \in \mathbb{R}$ is measurable;

(J_2) to every $M > 0$ there corresponds $k(M) > 0$ such that

$$|J(x, z_1) - J(x, z_2)| \leq k|z_1 - z_2|$$

a.e. in Ω and for every $z_1, z_2 \in [-M, M]$;

(J_3) there exist $\varepsilon, r > 0$ such that

$$m(\Omega)^{1-1/p} k(a(br)^{1/(p-1)} + \varepsilon) \leq r.$$

By (J_2) it makes sense to consider the generalized Clarke gradient $\partial J(x, z)$ of $J(x, \cdot)$ at the point $z \in \mathbb{R}$. Let us consider the following problem

$$\begin{cases} -\Delta_p u \in \partial J(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

Theorem 4.7
If $p > N$ and $(J_1) - (J_3)$ hold true, then Problem (4.7) has a solution.

Proof. We want to apply Theorem 2.9, so we have to verify all its hypotheses. Hypothesis (i_1) is guaranteed by Example 2.10 by choosing $U := A_p^{-1}(L^{p'}(\Omega))$, $\Phi(u) := u$ and $\Psi(u) := A_p(u)$, for every $u \in U$.

Now define $F(x, z) := \partial J(x, z),\ (x, z) \in \Omega \times \mathbb{R}$. A simple computation shows that

$$F(x, z) = [-J^0(x, z; -1), J^0(x, z; +1)], \tag{4.8}$$

where, as usual,

$$J^0(x, z; \pm 1) := \limsup_{w \to z, t \to 0^+} \frac{J(x, w \pm t) - J(x, w)}{t}.$$

Thanks to (J_1) the functions $x \to J^0(x, z; \pm 1)$ are measurable in Ω for every $z \in \mathbb{R}$. So, taking account of [30, Proposition 1.1], condition (i_2) of Theorem 2.9 holds.
Let us next verify (i_3). Pick $\{z_h\}, \{y_h\} \subseteq \mathbb{R}$ fulfilling

$$z_h \to z, \quad y_h \to y, \quad y_h \in F(x, z_h) \;\; \forall h \in \mathbb{N}.$$

The upper semicontinuity of $\xi \to J^0(x, \xi; \pm 1)$ combined with (4.8), yield, as $h \to +\infty$,

$$-J^0(x, z; -1) \le y \le J^0(x, z; +1), \qquad \text{namely } y \in F(x, z),$$

which represents the desired conclusion.
Finally, we prove (i_4). First we observe that

$$|J^0(x, z; \pm 1)| \le k(M) \quad \forall M > 0, \; z \in (-M, M).$$

This implies

$$m(x) := \sup_{|z| \le \varphi(r)} \inf\{|y| : y \in F(x, z)\} \le \sup_{|z| < \varphi(r) + \epsilon} \inf\{|y| : y \in F(x, z)\} \le k(\varphi(r) + \epsilon)$$

almost everywhere in Ω. Consequently, by (a_3),

$$||m||_{p'} \le m(\Omega)^{1 - 1/p} k(\varphi(r) + \epsilon) \le r.$$

Now Theorem 2.9 can be applied, and we obtain $u \in U \subseteq W_0^{1,p}(\Omega)$ such that

$$-\Delta_p u(x) = \Psi(u)(x) \in F(x, u(x)) = \partial J(x, u(x))$$

for almost all $x \in \Omega$.

□

Remark 4.8
We want to point out a special case of J. Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function that fulfils the following hypotheses:

(J_4) j turns out to be measurable in each variable separately;

(J_5) to every $M > 0$ there corresponds $k(M) > 0$ such that $|j(x, z)| \leq k(M)$ almost everywhere in Ω and for all $x \in [-M, M]$.

If J is of the form

$$J(x, \xi) := \int_0^\xi j(x, t)dt, \quad (x, \xi) \in \Omega \times \mathbb{R}, \tag{4.9}$$

then it satisfies $(J_1), (J_2)$, and we get

$$\partial J(x, z) = [\underline{j}(x, z), \overline{j}(x, z)],$$

with $\underline{j}, \overline{j}$ being for every $(x, z) \in \Omega \times \mathbb{R}$

$$\underline{j}(x, z) := \lim_{\delta \to 0^+} \text{ess inf}\{j(x, w) : |w - z| < \delta\}$$

$$\overline{j}(x, z) := \lim_{\delta \to 0^+} \text{ess sup}\{j(x, w) : |w - z| < \delta\}$$

Hence, Theorem 4.7 directly leads to the following corollary.

Corollary 4.9
If $(J_3) - (J_5)$ hold true, then there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\underline{j}(x, u(x)) \leq -\Delta_p u(x) \leq \overline{j}(x, u(x)) \tag{4.10}$$

for almost every $x \in \Omega$.

In particular, we observe that (4.10) reduces to the equation $-\Delta_p u = j(x, u)$ at each point where $j(x, \cdot)$ is continuous.

Chapter 5

Possible extensions

Methods used in [35] might be exploited to treat other boundary-value problems. For istance, we would like to study the Neumann problem:

$$\begin{cases} \psi(-\Delta_p u) = \varphi(x, u, \nabla u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\dfrac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\nabla u \cdot n$, with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.
In order to solve (5.1) via the technique adopted in Theorem 3.3, we need a suitable solution of the inclusion:

$$\begin{cases} -\Delta_p u \in F(x, u, \nabla u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega \end{cases}, \tag{5.2}$$

where F is a multifunction arising from φ and ψ.

Another topic of investigation might be the study of the case when the whole space R^N takes the place of Ω. In [9], Theorem 3.7 provides a solution $u \in W^{2,p}(\mathbb{R}^N)$, with

$p > N \geq 3$, to the implicit problem

$$\psi(\mathcal{L}u) = \varphi(x, u),$$

where $\mathcal{L}u := -\Delta u + u$. This result is based on [9, Theorem 3.2], which deals with a semilinear elliptic equation $\mathcal{L}u = f(x, u)$, being f a directionally continuous function. We wonder if a similar result holds for Δ_p instead of $\mathcal{L}$.

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