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Constrained Optimization Problems in Network Models

PhD Thesis

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*"Nothing happens in the world
that cannot be traced to a
maximization or minimization problem"*

Leonh. Euler

Contents

Introduction

Operations Research is the field of mathematics that deals with solving various application problems, the development and application of quantitative methods for the solution of decision problems that arise in the management of companies and organizations (production planning, optimal allocation of resources), in logistics (scheduling difficulties, the optimization of warehouse management), in the planning of service distribution, in the economic and financial field (investment selection, portfolio choice, price determination of financial derivatives), etc.

Operations Research is a relatively new discipline. The term Operations Research was coined towards the end of the 1930s to describe a new branch of applied sciences deriving from the term Operational Research, and is linked to the first applications of the OR to increase the efficiency of the Second World War military operations.
However, there are important examples of OR methods implemented before this time. The most famous dates back to F. Taylor who in 1885 developed a study on production methods; even earlier, in 1776, G. Monge had studied a transport problem. However, the birth of the OR is linked to studies that in the years immediately preceding the Second World War were conducted in Britain to solve strategic and tactical problems in military operations.
It provides automatic decision-making tools which solve enormous problems by means of mathematical modelling. The OR, therefore, is characterized by the use of mathematical models defined and resolved in order to provide

guidance to decision-makers in the act of choice. It is no coincidence that OR is also known as management science.

In this thesis, we focus our attention on some mathematical models that are decision problems and which are all based on networks and applied to different real situations.

The network framework is usually associated with transportation problems, electrical power transmission, telecommunications, etc., that is with physical networks, where nodes and links are tangible (see [26]). However, networks can be applied also to a very large class of problems where these concepts have no physical characteristics, such as problems arising from economics, finance, information technologies, Internet, etc.
As stated in the book by Nagurney and Dong (see [102]), *supernetworks* are networks that are above and beyond existing networks, which consist of nodes, links, and flows, with nodes corresponding to the locations in space, links to connections in the form of roads, cables, etc., and flows to vehicles, data, etc.

Supply chain networks, consisting of different tiers of decision-makers provide an effective framework for the production of goods, their distribution, and their consumption in today's globalized economies and societies. The representatives of each level may have various objectives: they may aim at maximizing their own profit, or minimizing the risk, or the environmental emissions, or the total costs.

Constrained optimization problems are one of the most important and useful fields of mathematics, particularly in operations research. In this thesis we analyze different thematic areas such as Cloud Computing, Financial Market, Business Management and Cybersecurity and for each of them we formulate the associated linear or nonlinear constrained problems which allows us to solve the decision problems related to the specific applications.

Cloud Computing is a type of Internet-based computing, much used in recent years, that relies on sharing computer processing resources and data to computers and other devices on demand, from any location and at any time rather than having local servers or personal devices to handle applications. This shared IT infrastructure contains large pools of systems that are linked together. Often, virtualization techniques are used to maximize the power of cloud computing. In Chapter 2 we describe the global network of a cloud computing environment with five different layers, represented by hardware/datacenter, infrastructure, platform, application and end-users. Then, we present the mathematical model of the network and study the behavior of the typical IaaS provider in order to find the global optimization problem. A computational procedure for the calculus of the global optimal solutions is proposed, is applied to a numerical example and is compared with a linearization.

In Chapter 3 we present two financial mathematical models, based on networks, which firstly allows us to formulate a new multi-period portfolio selection problem as a Markowitz mean-variance optimization problem with intermediaries and the addition of transaction costs and taxes (on the capital gain). Moreover, by means of the proposed Integer Nonlinear Programming (INLP) Problem, it is possible to establish when it is suitable to buy and to sell financial securities, not only while maximizing the profits but also while minimizing the risk which is weighted by an aversion degree or risk inclination value. In addition, we propose another model which is characterized by short selling, which consists in the sale of non-owned financial instruments with subsequent repurchase, and transfer of securities. We study some numerical examples, whose solutions give us the optimal distributions of securities to be purchased and sold.

In Chapter 4 we first present a supply chain network model with four different tiers of decision makers (suppliers of raw materials, manufacturers, retailers, demand markets), we derive the optimality conditions and the associated variational inequality problem for the representatives of each level and for the total supernetwork and focusing our attention on the behavior of the manufacturers. Then, in a more detailed model, we introduce the distinction by brand of the products of manufacturers and we add the e-commerce to the traditional physical links for the shipments from manufacturers to demand markets. Moreover, to the forward chain we add a reverse chain model where manufacturers, using the unsold product given back from retailers, after reworking, produce a new commodity which will be sold to new retailers. Also in this case we study the behavior of manufacturers obtaining their optimality conditions and the governing variational formulation. Finally, we apply our model to a concrete company (Valle del Dittaino, Italy), obtaining, after introducing additional constraints, the optimal amount of raw material, the optimal shipment of new product as well as the optimal production periods.

In Chapter 5 we propose a new cybersecurity investment supply chain game theory model, assuming that the demands for the product are known and fixed and, hence, the conservation law of each demand market is fulfilled. The model is a Generalized Nash equilibrium problem with nonlinear budget constraints for which we define the variational equilibrium, which provides us with a variational inequality formulation. We construct an equivalent formulation, enabling the analysis of the influence of the conservation laws and the importance of the associated Lagrange multipliers. We find that the marginal expected transaction utility of each retailer depends on this Lagrange multiplier and its sign. Finally, numerical examples with reported equilibrium product flows, cybersecurity investment levels, and Lagrange multipliers, along with individual firm vulnerability and network vulnerability, illustrate the obtained results.

Finally, in Chapter 6 we present the conclusions and the ideas for a future work.

Chapter 1

Theory and Fundamentals

Mathematical models are often applied to real phenomena or situations and they are used in many fields, such as the natural sciences (physics, biology, earth science, meteorology), the engineering disciplines (artificial intelligence, mechanics, computer science), the social sciences (economics, psychology, sociology, management science and political science) and other important areas. This is because a model can be useful to explain what we are describing and to make predictions about the future, allowing us to adopt the best strategy and make the correct decision. Therefore, mathematical modelling is one of the main approaches that mathematicians use to describe real situations.

Thus, the central contribution of Operations Research consists of the introduction of the so-called *model-optimizing approach* for the solution of a decision problem. In this approach, the analysis of a real problem is organized into two phases:

- representation of the problem by means of a mathematical model that should extract the essential aspects and outline the interrelations existing between the different aspects of the phenomenon under examination;
- development of efficient mathematical methods to determine an optimal solution of the problem.

To build a mathematical-optimizing model which represents a particular phenomenon, the significant control parameters must be identified. Having determined the correct model, the OR is responsible for providing an explicit procedure to determine a solution to a problem. This procedure can be represented by analytical mathematical methods or numerical methods which determine the solution of the problem through specific calculation algorithms.

The modeling approach is achieved through different phases:

- Problem analysis,
- Model construction,
- Model analysis,
- Numerical solution,
- Validation of the model.

The first phase consists in *analyzing the structure of the problem* to identify the logical-functional links and objectives, to collect the data. In the subsequent construction phase of the model, also called *formulation*, the main characteristics of the problem are described in mathematical terms. Then follows the *analysis of the model* which provides the analytical deduction of some important properties, such as the existence and uniqueness of the optimal solution, the optimality conditions, that is: the analytical characterization of the optimal solution and the stability of the solutions when the data or any parameters change.

The next phase of *numerical solution* takes place by means of suitable calculation algorithms and the numerical solution thus obtained must then be interpreted from an applicative point of view. This *model validation* can be carried out by experimental verification or simulation methods.

1.1 Optimization models

Many real problems, studied in different disciplines, consist in finding the maximum or the minimum value of a determined function. The branch of study dealing with such problems takes the name of Optimization Theory. Among the various optimization problems there are those where it is necessary to determine the optimal values of a function, whose decision-making variables are subject to constraints expressed by equalities and/or inequalities. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ and the set $K \subseteq \mathbb{R}^n$, an *optimization problem*, called also *mathematical programming problem* (term introduced by Robert Dorfman in 1949), can be formulated as follows:

$$\min_{x \in K} f(x).$$

Therefore, an optimization problem consists in determining, if it exists, a minimum point of the function f among the points of the set K.
The function f is called *objective function* and K is the *feasible set*, that is the set of all possible solutions to the problem. A point $x \in K$ is called feasible solution or candidate solution.
The feasible set K is a subset of $\mathbb{R}^n$ and therefore $x = (x_1, \ldots, x_n)^T$ is an n-dimensional vector and the objective function f is a function of n real variables.
We underline that it is possible to speak indifferently about problems of maximum or minimum because the following relation is valid:

$$\min_{x \in K} f(x) = -\max_{x \in K}(-f(x)).$$

Definition 1. *An optimization problem is said to be unfeasible if $K = \varnothing$.*

Definition 2. *It is said that the optimization problem admits an optimal solution if there exists a point $x^* \in K$ such that it results to be $f(x^*) \leq f(x)$ for all $x \in K$. Point x^* is called optimal solution or global minimum and the corresponding value $f(x^*)$ is called optimal value.*

A first classification of the optimization problems is based on the structure of the feasible set K. If $K = \mathbb{R}^n$, then the problem is said to be *unconstrained.*

If the set K is described by a finite number of inequalities and/or equalities, $K = \{x \in \mathbb{R}^n \;:\; g(x) \leq 0, h(x) = 0\}$, then the problem is said to be *constrained* and has the following formulation:

$$\begin{cases} \min f(x) \\ g_i(x) \leq 0 \quad \forall i = 1, \ldots, m \\ h_j(x) = 0 \quad \forall j = 1, \ldots, p \\ x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Moreover, the Mathematical Programming problems can be classified also according to the nature of the functions that define them:

- *Linear* Programming Problem (LP) if the objective function f and all the functions that define the constraints are linear;

- *Nonlinear* programming problem (NLP) if at least one of the functions defining the problem is not linear.

Finally, depending upon the values permitted for the variables, optimization problems can be classified as *integer* (if all the variables can only take integer values), *mixed integer* (if only some of the variables are constrained to take integer values) or *real valued*, and deterministic or stochastic.

Thanks to the generality of the models, an outsize number of real problems can be represented by programming models. Indeed in this thesis, we present different thematic areas, such as Cloud Computing, Financial Markets, Business Management and Cybersecurity and for each of them we formulate the associated linear or non linear constrained problems which allows us to solve the decision-making problems related to the specific applications.

The kind of model that must be studied and the most suitable approach shape the choice of the mathematical tools to be used, such as dynamic systems, variational inequalities, game theory and many others.
Furthermore, the main role in the formulation of a mathematical model and in decision-making processes is played by the definition of the network on which the whole model is based. Indeed, the network allows the definition

of the different layers of decision-makers involved in the whole process. The related flows and the methods of analysis are applicable not only to physical networks, such as transport and energy networks, production and logistics, but also to complex networks such as supply chains, financial, social and economic networks.

1.2 A brief recall to Lagrange Theory and Variational Inequalities

Modern optimization problems originated towards the end of the last century, but the history of mathematical programming dates back the end of the 1700s, although limited to the case of equality constraints. Indeed, in the second half of the 18th century, G.L. Lagrange studied mathematical programming problems which consisted in minimizing (or maximizing) a given function, subject to a system of constraints expressed by equality.
The well known multiplier method was introduced by Lagrange in 1788, in the first part of his book titled *Mècanique Analytique*, as a tool to determine the stable equilibrium configuration in a specific issue of Mechanics. The Lagrange multipliers were presented for general optimization problems, not referring to any mechanical system, in *Thèorie de fonctions analytiques* (1797).

Given the optimization problem (1.1), the function

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

is called the *Lagrangian function* of the problem.

Assume $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ minimizes $f(x)$ subject to the constraints $g_i(x) \leq 0$, for $i = 1, 2, \ldots, m$ and $h_j(x) = 0$, for $j = 1, 2, \ldots, p$.
Then either:

(i) the vectors $\nabla g_1(x^*), \ldots, \nabla g_m(x^*), \nabla h_1(x^*), \ldots, \nabla h_p(x^*)$ are linearly independent, or

(ii) there exists a vector $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$ and a vector $\mu^* = (\mu_1^*, \dots, \mu_p^*)$ such that

$$\begin{aligned}
&\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0,\\
&g_i(x^*) \leq 0 \quad \forall i = 1, 2, \dots, m,\\
&h_j(x^*) = 0 \quad \forall j = 1, 2, \dots, p,\\
&\lambda_i^* g_i(x^*) = 0 \;\; \forall i = 1, 2, \dots, m, \quad \text{(Complementarity)},\\
&\lambda_i^* \geq 0 \quad \forall i = 1, 2, \dots, m.
\end{aligned}$$

The above conditions are called the **Karush-Kuhn-Tucker conditions** (see [64] and [69]) and are the necessary conditions for the solution of a non-linear programming problem.
This is a generalization of the *Lagrange multiplier method*, applied to problems in which there are also inequality constraints. The first condition is that of the cancellation of the gradient of the Lagrangian function associated with the problem. The second and third conditions are the constraints of the admissibility of point x^*, while the fourth condition is called a complementarity condition or a "complementary deviation", since the multiplier of an inactive constraint must be null. Finally, the last condition is the non-negativity condition of the multiplier associated with the inequality constraints.

A great variety of problems in the real world can be traced back to variational models that are much closer to reality, when their equilibrium condition is expressed as a solution to a system of Variational Inequalities (VI).
The scientific life of the Variational Inequalities Theory has immediately proved to be eventful and surprising. This theory was developed in the seventies as an innovative and effective method to solve a series of equilibrium problems. It was advanced by mathematical physicists to solve, for example, the problem of Signorini (1959), the problem of the obstacle and that of elasto-plastic torsion.
Therefore, the first variational inequality problem was the problem known

as the Signorini problem (see [129]). His student, Gaetano Fichera, dedicated the name to him and resolved it in 1963. In 1964, Guido Stampacchia, a 20th-century Italian mathematician, known for his work on the theory of variational inequalities, generalized the Lax-Milgram theorem (see [132]) in order to study the regularity problem for partial differential equations and the name "variational inequality" was coined by him for all the problems involving inequalities of this kind. Further in-depth analyses of previous studies date back to 1966 by Hartman and Stampacchia (see [57]) and to 1967 by Stampacchia and Jacques-Louis Lions (see [74]).

An explanation of infinite-dimensional variational inequalities and numerous references can be found in the text by Kinderlehrer and Stampacchia (see [66]).

After an intense period of successes and fundamental results obtained with the Variational Inequalities theory, the interest waned, perhaps because of the early death of Stampacchia in 1979, and it seemed that this theory had nothing more to communicate.

On the contrary, in 1980, the breakthrough in finite-dimensional theory occurred when S. Dafermos recognized that the problem of traffic equilibrium, as stated by M.J. Smith (1979), could be formulated in terms of a finite dimensional inequality and, moreover, in this way it is possible to study the existence, uniqueness and stability of the traffic equilibrium problem and calculate the solutions. So began the use of this methodology for the study of problems in economics, management science/operations research, and also in engineering, with a focus on transportation.

At the end of the nineties, researchers started to investigate optimization problems, through a variational approach, by considering also time-dependence. Daniele, Maugeri and Oettli, in [36] and [35] (see also [46]), first studied and analyzed the traffic network equilibrium problem with feasible path flows which have to satisfy capacity constraints dependent from time and traffic demands.

As a result of this, the last decades have witnessed an exceptional interest in Variational Inequalities both in the development of VI theory and its application to equilibrium problems arising in many different contexts and an enormous amount of papers and books have been dedicated to this topic.

For the analysis of economic phenomena, equilibrium is a central concept, therefore, various problems from the world of economics, such as those of spatially distributed economic and oligopolistic markets, migration, pollution and many other problems, have been formulated in terms of a finite dimensional variational inequality and, by means of this theory, they have been solved.

Recently, a lot of problems coming from fields of applied sciences such Operation Reasearch, Physics, Engineering, Biology and Economics are studied as optimization problems with a variational approach.

Now, we introduce the definition of Variational Inequality:

Definition 3 (Variational Inequality Problem)**.** *The finite-dimensional variational inequality problem, $VI(F,K)$, is the problem to find a vector $x^* \in K \subset \mathbb{R}^n$, such that*

$$\left\langle F(x^*)^T, x - x^* \right\rangle \geq 0, \quad \forall x \in K, \tag{1.2}$$

where F is a given continuous function from K to $\mathbb{R}^n$, K is a given closed convex nonempty set and $\langle\ .\ ,\ .\ \rangle$ denotes the inner product in the n-dimensional Euclidean space.

In geometric terms (see Figure 1.1), variational inequality (1.2) states that $F(x^*)^T$ is "orthogonal" to the feasible set K at the point x^*.
We recall that, for two vectors $u, v \in \mathbb{R}^n$, the inner product $\left\langle u^T, v \right\rangle = \|u\| \, \|v\| \, cos\theta$, where θ is the angle between the vectors u and v. Hence, for θ in the range: $0 \leq \theta \leq 90°$, we have that $\left\langle u^T, v \right\rangle \geq 0$. Thus, one can see from Figure 1.1 that x^* is a solution of $VI(F,K)$ if and only if the angle between the vectors $F(x^*)^T$ and $x - x^*$, with x and x^* both in K, is a non-obtuse angle, that is: less than or equal to 90°.
This formulation is particularly convenient because it permits a unified treatment of equilibrium problems and optimization problems.

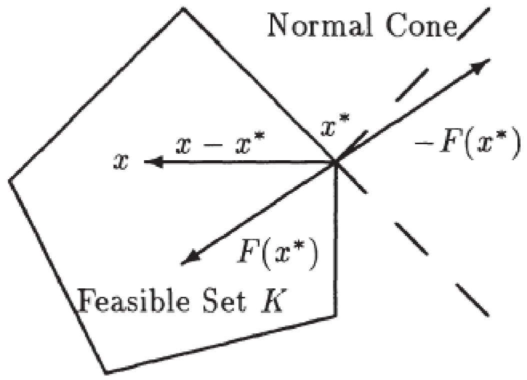

Figure 1.1: Geometric interpretation of $VI(F, K)$

We may formalize this observation using the concept of the normal cone. Specifically, associated with the set K and any vector x' belonging to K, we define the *normal cone* to K at x' as the following:

$$\mathcal{N}(x', K) = \{d \in \mathbb{R}^n : d^T(x - x') \leq 0, \ \forall x \in K\}.$$

Therefore, Variational Inequality (1.2) affirms that a vector $x^* \in K$ solves the $VI(F, K)$ if and only if $-F(x^*)$ is a normal vector to K at x^*.

It is interesting to outline the relationships between variational inequalities and optimization problems. Indeed, a variational inequality is related to an optimization problem when the objective function is a primitive of the operator involved in the inequality itself and optimization problems, constrained and not, can be formulated as VIPs.
The connections between minimum problems and the variational inequalities have been widely studied in the case in which K is a convex set and the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, defined and differentiable on a open set containing K, is a primitive of F, that is $f'(x) = F(x)$.
The precise connection between the $VI(F, K)$ and the Optimization Problem in described in the following results.

Proposition 1. *Let x^* be a solution to the optimization problem:*

$$\begin{aligned} &\min f(x) \\ &\textit{subject to: } x \in K, \end{aligned} \tag{1.3}$$

where f is continuously differentiable and K is closed and convex. Then x^* is a solution to the variational inequality problem:

$$\left\langle \nabla f(x^*)^T, x - x^* \right\rangle \geq 0, \quad \forall x \in K. \tag{1.4}$$

Proof. Let $\phi(t) = f(x^* + t(x - x^*))$, for $t \in [0, 1]$. Since $\phi(t)$ achieves its minimum at $t = 0, 0 \leq \phi'(0) = \nabla f(x^*)^T \cdot (x - x^*)$, that is, x^* is a solution to (1.4). □

Under the assumption of convexity for f, then also the viceversa holds.

Proposition 2. *If $f(x)$ is a convex function and x^* is a solution to $VI(\nabla f, K)$, then x^* is a solution to the optimization problem (1.3).*

Proof. Since $f(x)$ is convex,

$$f(x) \geq f(x^*) + \nabla f(x^*)^T \cdot (x - x^*), \quad \forall x \in K. \tag{1.5}$$

But $\nabla f(x^*)^T \cdot (x - x^*) \geq 0$, since x^* is a solution to $VI(\nabla f, K)$. Therefore, from (1.5) one concludes that

$$f(x) \geq f(x^*), \quad \forall x \in K,$$

that is, x^* is a minimum point of the mathematical programming problem (1.3). □

1.3 Traffic Network

The study of networks and their applications has had a long tradition in engineering, operational research / management and computer science. Indeed,

network analysis is not only associated with problems of transport, transmission of electricity, telecommunications, etc. as is usually done, because its methods are applied not only to physical networks, where nodes and links have tangible embodiments, but also to a much wider class of problems where these concepts do not need physical counterparts.
Recently, the computer science, finance and economy sectors have become rich and fascinating sources of network-based problems and applications.
Global communication networks such as the Internet and cloud computing, often do not have a central authority that regulates and monitors network traffic. Most cooperation among network users is not possible, therefore, network users may behave selfishly based on their private interests regardless of the overall performance of the system. Therefore, while the optimization of the traditional network has a central authority controlling the network, the former mentioned are built and governed by a huge number of entities interacting in an uncoordinated way and distributed according to their individual interests.
Following this, it was interesting to study the result of the combination of classical methods from traditional network optimization and concepts provided by game theory techniques. For this reason, the network traffic in the framework of non-cooperative game theory has been analyzed. The main aspect of this theory is the notion of equilibrium that describes stable results of a non-cooperative game.

The problem of sharing resources has a long history in transport science and economy. Already in the middle of the nineteenth century Kohl, a German geographer, analyzed the problem of spending time and money to move people and goods between different places in the context of urban planning. For users of a transport network, determining their travel routes from their origins to their destinations at the lowest cost is a classic problem of network balance.
The effects of congestion have been taken into account explicitly by Pigou in

[118] (1920), where the author considers a two-node transport network, two links (or routes) and was further developed by Knight in 1924 (see [68]), who qualitatively described egoistic routing in transport networks and observed that a selfish behavior does not necessarily maximize overall performance.

In 1952, Wardrop (see [142]) introduced a game-theoretic model for describing resource sharing problems in the context of road traffic systems. He stated two principles that formalize different concepts of equilibrium.
First Principle: the travel times of all the routes actually used are the same and less than those that would be experienced by a single vehicle on any unused itinerary.
Second Principle: Average travel time is minimal.
Wardrop's traffic model has attracted a lot of interest and inspired a great deal of research, especially after the emergence of enormous non-cooperative systems such as the Internet. It has been utilized to analyze many problems in transportation and communication networks.
A rigorous mathematical formulation of Wardrop's equilibria was provided by Beckmann el al. in [7], in which the authors formulated the network equilibrium problem as a convex optimization problem with a single objective function. Therefore, they established the equivalence between an equilibrium problem and a mathematical programming problem. In this optimization problem a potential function has to be minimized subject to natural flow constraints.
The terms "user-optimized" and "system optimized" transportation networks were invented by Dafermos and Sparrow in 1969 (see [25]) to differentiate between the case where users act in a unilateral way, in their personal interest and that where users choose optimal social paths, as the total costs for the system are reduced to a minimum.
Now we will describe Wardrop's traffic model formally.

Let $N = \{P_1, \ldots, P_p\}$ be the nodes set and $A = \{a_i,\ i = 1, \ldots, n\} \subset N \times N$ the links set, where $a_i = (P_l, P_m)$ is an unidirectional link from P_l to

P_m.
Assume that $W = \{w_j, j = 1, \ldots, l\} \subseteq N \times N$ is the set of all Origin/Destination (O/D) pairs.
Therefore, a Traffic Network is identified by the tern (N, A, W).
Given an O/D pair, $w_j = (P_h, P_k)$, we define $\mathcal{R}_j$ the set of all the paths R_r which join P_h with P_k and $\mathcal{R} = \cup_{j=1}^{l} \mathcal{R}_j$ the set consisting of all the network paths.
We consider the following hypotheses:
1. $\mathcal{R}_j \neq \varnothing,\ j = 1, \ldots, l$ and
2. $m > l$.

The flow on a link a_i is denoted by $f_i \geq 0$. We group the link flows into a vector $f \equiv (f_1, \ldots, f_n) \in \mathbb{R}^n_+$. Similarly, the flow on a path R_r is denoted by $F_r \geq 0$ and we group the path flows into a vector $F \equiv (F_1, \ldots, F_m) \in \mathbb{R}^m_+$.
Introducing Δ, the link-path incidence matrix, of components:

$$\delta_{ir} = \begin{cases} 1 & \text{if } a_i \in R_r \\ 0 & \text{if } a_i \notin R_r, \end{cases}$$

we obtain the following relation:

$$f_i = \sum_{r=1}^{m} \delta_{ir} F_r,\ i = 1, \ldots, n \ \Leftrightarrow\ f = \Delta F.$$

Now we consider the user cost associated with traveling on link a_i that is denoted by $c_i(f) \geq 0$. We group the link costs into a vector $c(f) \in \mathbb{R}^n_+$. Similarly, we have the user cost associated with traveling on path R_r: $C_r(F) \geq 0$ and the vector $C(F) \in \mathbb{R}^m_+$.
Therefore, the correlation between $c(f)$ and $C(F)$ is given by:

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} c_i(f),\ r = 1, \ldots, m \ \Leftrightarrow\ C(F) = \Delta^T c(\Delta F).$$

The travel demand of potential users traveling between O/D pair w_j is denoted by $\rho_j \geq 0$. Group the demands into a vector $\rho \in \mathbb{R}^l_+$ The conservation of flow equations are as follows. The demand for a O/D pair must be equal to the sum of the flows on the paths joining the O/D pair, that is,

$$\rho_j = \sum_{R_r \in \mathcal{R}_j} F_r, \quad \forall j = 1, \dots, l. \tag{1.6}$$

Introducing Φ the pair-path incidence matrix, of components:

$$\phi_{jr} = \begin{cases} 1 & \text{if } R_r \in \mathcal{R}_j \\ 0 & \text{otherwise,} \end{cases}$$

the traffic conservation law become:

$$\sum_{r=1}^{m} \phi_{jr} F_r = \rho_j, \; j = 1, \dots, l \;\Leftrightarrow\; \Phi F = \rho.$$

We define

$$\begin{aligned} \mathbb{K} &= \{F \in \mathbb{R}^m : F \geq 0 \text{ and } \Phi F = \rho\} \\ &= \{F \in \mathbb{R}^m : F_r \geq 0, \; r = 1, \dots, m \text{ and } \sum_{r=1}^{m} \phi_{jr} F_r = \rho_j, \; j = 1, \dots, l\} \end{aligned}$$

the feasible flows set.

Definition 4 (Wardrop, 1952)**.** *$H \in \mathbb{K}$ is an equilibrium distribution if:*
$\forall w_j \in W$ and $\forall R_q, R_s \in \mathcal{R}_j$ if $C_q(H) < C_s(H)$,
then $H_s = 0$.

It is easy to verify (see Smith [130] and Dafermos [24]) that such a definition is characterized by a VI through the subsequent theorem:

Theorem 1. *$H \in \mathbb{K}$ is a Wardrop's equilibrium distribution if and only if H is a solution of the following VI:*
Find $H \in \mathbb{K}$ such that

$$\langle C(H), F - H \rangle \geq 0, \quad \forall F \in K,$$

that is: $\displaystyle\sum_{r=1}^{m} C_r(H)(F_r - H_r) \geq 0, \quad \forall F \in K.$

1.4 A brief introduction to supply chain networks

A supply chain system consists of organizations, companies, people, and resources involved in moving a product or a service from a supplier (who provides products, services, and information that add value for customers and other stakeholders) to customers (the end-users).
The term *supply-chain management* (SCM) was coined in 1982 to refer to the process of planning and controlling the business processes of the supply chain with the scope of satisfying customer requirements as efficiently as possible (see [111]).
Supply chain operations usually involve raw materials, and components that are transformed into a finished good that is bought by end customers.

Because there was a rapid technological progress, organizations with a basic supply chain needed to develop a chain with a more complex structure implying a higher level of interdependence and connectivity between multiple organizations. For this reason, evolution has led to the *supply chain network* (SCN) that can be used to highlight the interactions between the diverse organizations or companies that are part of the same supply chain network. Indeed a SCN shows the links between organizations and how information and materials flow between these links.

Supply chain networks are usually structured on five levels: external suppliers, production centers, distribution centers (DC), demand areas and transport activities.

The Supply chain network is sometimes also called *Network Modeling* because, as will be shown in the next chapters, a mathematical model can be created to design the network strategically in order to optimize it, thereby reducing the cost of the supply chain ([143]).
We underline that it is possible to investigate the impacts deriving from

the addition or cancellation of various nodes in the network (or decision makers) or links (or different transaction methods) and also different production methods. This is possible thanks to the study of the supply chain utilizing a network formalism.
In the next chapters of this thesis, we do this in the context of cloud service supply chains, financial chains with intermediation, supply chain management, including reverse supply chains in the context of recycling and e-commerce networks; and even networks related to cybersecurity. Since the publication of the book *Studies in the Economics of Transportation* by Beckmann, McGuire, and Winsten in 1956, transportation networks have been intensively studied by economists, engineers, as well as operations researchers and management experts.
For more information on the supply chain study, refer to the book by A. Nagurney [99].

Caused by the environmental impact of end-of-life goods, the necessity to create a new kind of network has arisen, the *reverse supply chain network*. This particular network design addresses logistical issues such as collection, processing and recycling of end-of-life goods. Companies that have had the greatest success are those where forward and reverse supply chain processes have been designed, also taking into account the recycling and the disposal of said goods.
Through the reverse supply chain network, organizations can support products from production to disposal creating a *closed-loop system*. This topic is discussed in detail in Chapter 4, where we also analyze a supply chain network in the context of the Information Age with the innovations brought about by electronic commerce.

1.5 Cloud Computing Overview

Cloud computing is rapidly becoming a widely used service in Internet computing. It is a relatively new concept in information technology; indeed, for the first time, cloud computing was marketed in 2006 by Amazon, with the EC2 service, followed by Cloud providers such as Microsoft, Google and numerous other organizations. Figure 1.2, obtained through the Google Trends tool, shows the interest gathered by the keywords *Cloud Compunting* over time, from 2004 to today, by users all over the world. Clearly, interest in cloud computing has grown over the last few years and cloud computing services have grown in popularity, which brings great benefits to all types of computing, including business support.

Cloud Computing offers a way to share distributed resources and services

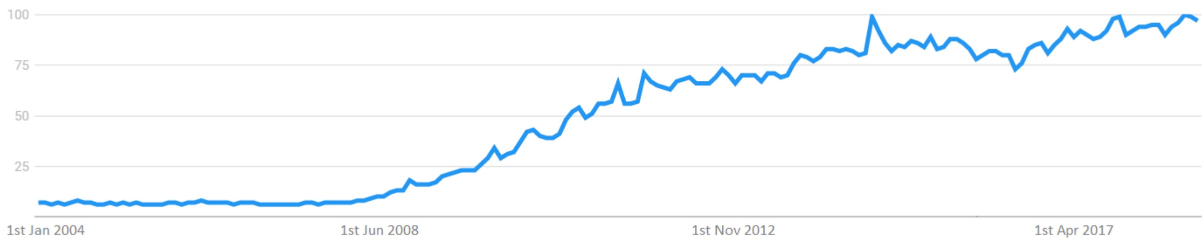

Figure 1.2: Relative popularity of the search query: *Cloud Computing*

that belong to different organizations or sites. It has been used for webmail services, blogs, storage and web hosting and offers various services in the forms of infrastructure, platform and software to meet consumer needs (examples include Google Apps, Amazon Web Services, Microsoft Windows Azure, IBM Smart Cloud, and Salesforce.com). Thanks to this, the way in which the subtree and the IT services are provided has been modified, offering unprecedented computing power and flexibility in the distributed computing environment, providing more efficient computational resources for running websites and web applications.

With the rapid developments in networking technologies and Cloud computing, many companies have adopted a wide variety of Cloud. Therefore, a significant impact on the performance of the whole information infrastructure is the role played by Cloud providers who need to fully utilize their infrastructures (available resources) while satisfying users' performance requirements.

Cloud Computing provides scalable, dynamic, shared, and flexible resources over the Internet from remote data centers to the users and has been extensively studied in literature from different standpoints.
The Cloud infrastructure supports three types of service delivery models which are studied: Software as a Service (SaaS) (see for examples [49], [40]), Platform as a Service (PaaS) ([3]) and Infrastructure as a Service (IaaS) ([81], [48]).
The most efficient way to manage the Cloud infrastructure and the development of effective methods for evaluating the performance of Cloud services have become very important research problems. Therefore, they have attracted the attention of both industry and academia.
Researchers have widely studied the problem from different perspectives, adopting many types of approaches. For example, they analyzed system modeling (see for instance [39] where Network calculus was first applied to develop a profile-based model for Cloud service performance analysis and [58]), system design (see [112]) and network protocol (see [79]).
Recently, among the work on the assessment of the performance of the cloud, the approaches from the point of view of the systems modeling have constituted an active area with much progress because they are beneficial for both service providers and consumers (see for example [147] where authors analyzed Cloud Computing by formulating a mathematical model).
Cloud computing is also an infrastructure where users can have, on demand, the availability of the pool of computing resources as well as the computing power of their own in a network environment ([5], [146]).

Virtualization is one of the key technologies of Cloud computing services which makes it different from Grid computing. Virtualization allows hosting heterogeneous services on shared abstract infrastructures and for this reason it is considered a fundamental element of the cloud network (in [145], authors studied a resource management system with a power saving method for virtual machines). The practical aspects of virtualization related to configuration, networking, and sizing of cloud systems are faced with challenges. When a cloud provider accepts a request from a customer, it must create the appropriate number of virtual machines (VMs) and allocate resources to support them ([138]). Virtualization hides the complexity of managing the physical computing platform and simplifies scalability of computing resources.

In Chapter 2, we analyze a Cloud Computing framework and focus our attention on resource management that is one of the most important issues in Cloud Computing for IaaS, taking into account resource utilization, the cost of turning a server on or off, resource wastage and energy consumption as optimization objectives at the same time. To the best of our knowledge, despite the importance of these matters in cloud environments, there does not exist any work which holds these fields jointly, while most of the researchers have dealt with them separately.

Over the years, numerous researchers have studied and surveyed the issues of security and privacy (see for example [116], [120], [123], [124] and [144]). Indeed in a cloud infrastructure, sensitive information for a customer is kept on geographically dispersed cloud platforms. Cloud resources are vulnerable to abuse, theft, unlawful distribution, harm, and/or compromise, therefore, recurring users' data in a cloud computing environment is one of the most challenging tasks. In this thesis, we propose a new cybersecurity model (see Chapter 5) that could be applied in a general case, indeed, Jensen et al. presented the technical security issues in Cloud Computing (see [61]) which are related more to the problems of web services and web browsers rather than Cloud Computing.

Chapter 2

A Cloud Computing Network and an Optimization Algorithm for IaaS Providers

2.1 Introduction

According to the National Institute of Standards and Technology (see [92]): "Cloud computing is a model for enabling ubiquitous, convenient, on-demand network access to a shared pool of configurable computing resources (e.g., networks, servers, storage, applications, and services) that can be rapidly provisioned and released with minimal management effort or service provider interaction. This cloud model is composed of five essential characteristics, three service models, and four deployment models."
Therefore, cloud computing is a business model in which the user does not buy the product, but purchases the possibility to use such a product, while he does not hold it phisically. Essentially, cloud computing consists in obtaining the services hosted on cloud, that is the storage and processing of data due to hardware resources and localized software on the Internet.
In a cloud computing environment there are some essential characteristics that can be elaborated as follows:

- *On-demand self-service*: Users are able to provision cloud computing resources without requiring human interaction, mostly done through an online control panel or directly with a cloud host provider. The payments vary with each software provider and typically, it is used a pay-for-what-you-use scenario.

- *Broad network access*: Cloud computing resources are accessible over the network, supporting heterogeneous client platforms. Users can access using their smartphones, tablets, laptops, and office computers.

- *Resource pooling*: Cloud computing gives services to multiple customers from the same physical resources, by securely separating the resources on logical level. Physical and virtual resources are dynamically assigned and reassigned according to consumer demand from any location, and at any time.

- *Rapid elasticity*: Resources are provisioned and released on-demand and/or automated based on triggers or parameters. Therefore, companies sometimes can require additional resources in a small period of time and this is where cloud computing comes in to play.

- *Measured service*: Resource usage are monitored, measured, and reported (billed) transparently based on utilization. In short, pay for use, therefore, users and cloud provider can measure storage levels, processing, bandwidth, and the number of user accounts and then users are billed appropriately only paying for what they use.

Cloud Computing offers all the advantages of a cost-effective system, in terms of convenience, flexibility, and proven delivery platform for providing business or consumer IT services over the Internet. However, Cloud Computing presents an added level of risk because essential services are often outsourced to a third party so there are still some challenges to be solved, not least of which are: privacy and cybersecurity (see [16], [97], [101], [115]).
According to NIST there are three Service Models (see [13], [77], [92], [146]):

- *Software as a Service (SaaS)*: a software distribution model in which applications (for example e-mail, DropBox, Google Drive, ...) are not distributed physically, but they are hosted by a vendor or a service provider and made instantaneously available to customers over a network, typically the Internet. Users gain the access to application software and databases but they do not manage or control the underlying cloud infrastructure and platform where the application runs such as the network, the servers, the operating systems, the storage. This eliminates the expenses related to hardware acquisition, provisioning and maintenance, as well as software licensing, installation and support.

- *Platform as a Service (PaaS)*: it can be defined as a computing platform in which developers can build and deploy web applications quickly and easily on a hosted infrastructure using programming languages, libraries, services, and tools supported by the provider but without the complexity of buying and maintaining the software and infrastructure underneath it. In other words, PaaS allows them to leverage the seemingly infinite compute resources of a cloud infrastructure.

- *Infrastructure as a Service (IaaS)*: it is a way of delivering Cloud Computing infrastructure such as servers, storage, network and operating systems (usually in terms of virtual machines) that provides virtualized computing resources over the Internet as an on-demand service. Rather than purchasing servers, software, datacenter space or network equipment, clients, on the contrary, buy those resources as a fully outsourced service on demand. Customers pay on a per-use basis, typically by the hour, week or month. Some providers also charge customers based on the amount of virtual machine space they use (see [140] for a virtualization architecture). Therefore, IaaS refers to online services that abstract the user from the details of infrastructure like physical computing resources, location, data partitioning, scaling, security, backup etc.

Virtualization is the essential technological characteristic of clouds, therefore, in this chapter, we focus our attention on resource management that is one of the most important issues in Cloud Computing for IaaS (see [82]), including allocation, provisioning, mapping and adaptation in a multi-tenancy environment, where users share the same resource.
Analyzing the Cloud Computing network, we take into account resources utilization, monetary cost and energy consumption as optimization objectives at the same time, while most of the researchers have dealt with them separately (see [117]).
Virtualization consists of sharing computer hardware by partitioning the computational resources; often many services need not the total available resources but only a small portion of them. Energy consumption is playing an increasing role in the Cloud Network because of its costs and, for this reason, many researchers use the Server Consolidation (an approach according to which it is better mapping Virtual Machines on fewest possible physical servers) to prevent the wastage of resources (see [136]). In this thesis, taking into account also the physical resources heterogeneity, we aim at optimizing cost of running servers (cost of turning on or off a server, power consumption) and resource wastage.
In [80], the author underlines that in the single-data centre problem, the usual formulation used by researchers is about mapping Virtual Machines to Physical Machines, while, in the multi-IaaS problem, it is more common to investigate the mapping of tasks (platform) to Virtual Machines. In the model we present, the provider can accept or reject a platform execution request, can establish the revenue and can make the decision about the allocation, ensuring the quality of service (determined in a particular agreement).

The chapter is organized as follows. In Section 2.2 we present the mathematical model of cloud computing and we describe the role of the different layers. Then, we analyze the behavior of the typical IaaS provider and derive its optimality conditions given by the desire to maximize its profit while minimizing its operational costs. In Section 2.3 we describe the computational procedure for the calculus of the nonlinear model. We also present a linearization of the constraints and of the objective function, in Section 2.4, in order to compare the global optimal solutions as shown in Section 2.5 with a numerical example. Section 2.6 is dedicated to the conclusions.

2.2 The Matematical Model

In this section we first introduce the global network of a cloud computing environment with different layers (see [14], [18], [19], [26], [27] and [102] for the study of other supply chain networks applied to different problems) and then we present the mathematical model of the network from the perspective of a typical IaaS provider in order to find the optimization problem.
The architecture of a cloud computing environment can be divided into 5 layers: the hardware/datacenter layer, the infrastructure layer, the platform layer, the application layer and the end-users layer (see [13], [146]). We now describe each of them in detail.

The hardware layer: At the highest level of the hierarchy, the hardware layer is sometimes referred to as the server layer because it represents the physical hardware that provides actual resources that make up the cloud. It is responsible for managing the physical resources of the cloud, such as physical servers, routers, switches, power and cooling systems.

The infrastructure layer: Often referred to as the virtualization layer and built below the hardware layer, the infrastructure layer represents the result of various operating systems being installed as Virtual Machines (VMs). Much of the scalability and flexibility of the cloud computing model is derived by the inherent ability of virtual machines to be created and deleted at will. Partitioning the physical resources using virtualization technologies, this layer offers the virtual machines as a service to users. The decision-makers of hardware and infrastructure tiers are *IaaS providers*: $1, \cdots, i, \cdots, I$; they provide virtualized computing services such as storage, CPU and memory packaged as VMs of different sizes, each one with a price per time, over the Internet as an on-demand service (billing subprocess is incorporated into the general information flow of the supply chain, see [75]).

The platform layer: Built below the infrastructure layer, the platform layer consists of operating systems and application frameworks. The purpose of the platform layer is to minimize the burden of deploying applications directly into VM containers. Platforms provide programming and runtime environments to deploy cloud computing applications. The decision-makers of this layer are *PaaS providers*: $1, \cdots, f, \cdots, F$. They offer platform services, such as web, application and database servers and an associated programming model for dynamic and flexible application provisioning (see [2] for a dynamic performance optimization framework). Users (/programmers) can use this environment to develop, test and deploy applications. PaaS provider consumes the services of IaaS providers by requesting VMs with performance characteristics: memory, storage, processing capacity. We make the simplifying assumption that each VM hosts a single platform. Multiple VMs implementing the same platform can also run in parallel.

The application layer: This layer is the most used layer of cloud computing and the most common from a public perspective. The vast majority of consumers utilizes this layer of Cloud Computing because it is accessible via a computer, tablet or smartphone through web-portals. Services at the application level consist of complete applications that do not require development. Such applications can be email, customer relationship management, and other office productivity applications. Different from traditional applications, cloud applications can leverage the automatic-scaling feature to achieve a better performance, availability and a lower operating cost. The decision-makers of application layer are *SaaS providers*: $1, \cdots, a, \cdots, A$. They offer applications to the end-users. Those applications are deployed in PaaS's platforms on topologies which are specific to each application. There can be many applications sharing the same platform.

The end-users layer: At the lowest level of the hierarchy we found a fundamental part of the model, namely the end-users layer, which is made up by an active entity that utilizes the SaaS applications over the Internet (see [62] for an information collaboration system; see [71] and [87] for some descriptions of the roles in the cloud computing value network). While this layer is not a cloud computing service, it is an essential part of the model.

Each layer is loosely coupled with the layers above and below, allowing each layer to evolve separately.
Each decision-maker will have different optimization goals.

Each provider needs to comply with providers of lower tiers (or with end-users) a Service Level Agreement (SLA) contract, and, at the same time, to maximize its own revenue, while minimizing the cost associated with the use of resources supplied by above providers.
Therefore, each provider signs with its customers a SLA contract that determines the minimum throughput and penalties incurred on the basis of the level of performance achieved (see [1]).
When we look at the security of data in the cloud computing, the vendor has to provide some assurance in the SLA contract, so the SLA has to describe different levels of security and their complexity based on the services to make the customer understand the security policies that are being implemented.
Furthermore, each provider (or end-user) behaves selfishly and competes with other providers of the same tier for the use of resources supplied by above providers and for maximizing the revenues obtained providing their resources.
We highlight that the aggregated demands for different resources or services are dynamic over a time horizon. Therefore we present a multi-time period optimization model. We divide the time horizon into T equal time slots. Let

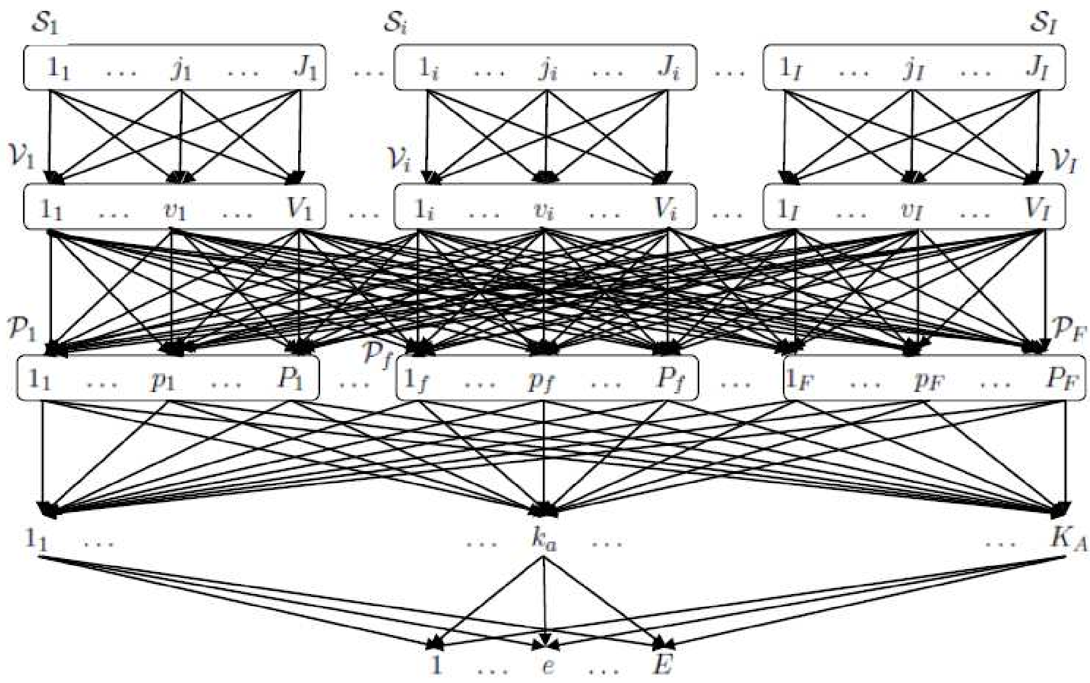

Figure 2.1: The Cloud Computing Network

$\mathcal{T} = \{1, \cdots, t, \cdots, T\}$ be the set of these slots.

We consider a global supply chain network with four tiers of decision-makers; in Figure 2.1, the underlying network structure of the optimization problem is depicted.

Let:

- $\mathcal{I} = \{1, \ldots, i, \ldots, I\}$ be the set of IaaS providers,
  $\mathcal{S} = \{1, \ldots, j, \ldots, J\}$ be the set of all physical servers
  and we consider a partition of $\mathcal{S}$ in $\mathcal{S}_1, \ldots, \mathcal{S}_i, \ldots, \mathcal{S}_I$ where $\mathcal{S}_i$ indicates the set of the i provider's servers,
  $\mathcal{V} = \{1, \ldots, v, \ldots, V\}$ be the set of all VMs
  and we consider a partition of $\mathcal{V}$ in $\mathcal{V}_1, \ldots, \mathcal{V}_i, \ldots, \mathcal{V}_I$ where $\mathcal{V}_i$ indicates the set of the i provider's VMs;

- $\mathcal{F} = \{1, \ldots, f, \ldots, F\}$ be the set of PaaS providers,
  $\mathcal{P} = \{1, \ldots, p, \ldots, P\}$ be the set of all platforms
  and we consider a partition of $\mathcal{P}$ in $\mathcal{P}_1, \ldots, \mathcal{P}_f, \ldots, \mathcal{P}_F$ where $\mathcal{P}_f$ indicates the set of the f provider's platforms;

- $\mathcal{A} = \{1, \ldots, a, \ldots, A\}$ be the set of SaaS providers,
  $\mathcal{K} = \{1, \ldots, k, \ldots, K\}$ be the set of all applications
  and we consider a partition of $\mathcal{K}$ in $\mathcal{K}_1, \ldots, \mathcal{K}_a, \ldots, \mathcal{K}_A$ where $\mathcal{K}_a$ indicates the set of the a provider's applications;

- $\mathcal{E} = \{1, \ldots, e, \ldots, E\}$ be the set of the end-users.

2.2.1 The behavior of the IaaS provider

Now we analyze the behavior of the typical IaaS provider in order to find the global optimization problem.
Let:

- C_i^+, C_i^- be the wear-and-tear cost of turning a server on and off, respectively (see [119]);
- ω^i be the unit power cost (see [50]);
- $\frac{1}{\alpha^i} \in [0, 1]$ be the fraction such that $\dfrac{\omega^i}{\alpha^i}$ is the unit cost of wasted resources;
- $\bar{C}$ be the upper bound of the capacity for each platform;
- $\bar{C}_j$ be the threshold of capacity (that can be CPU, memory, storage and so on) utilization associated with each server;
- C_{vt} be the capacity of VM v hosted by a physical server at time t;
- D_{pt}^i be the capacity demand of each platform p by PaaS provider f to IaaS provider i at time t;

- q^i_{pt} be the number of platforms of the kind $p \in \mathcal{P}_f$ requested by f in the VMs of i at time t;
- $\bar{\rho}_{pt}$ be the upper bound of the price for platform $p \in \mathcal{P}_f$ that f is willing to pay at time t;
- δ^i be the penalty for rejecting a single platform.

Let also ρ^i_{vt} be the revenue variable for the IaaS provider i for a single request execution in VM v at time t.
We use three binary variables x^j_{vt}, y_{jt} and z^v_{pt}, defined as follows:

$$x^j_{vt} = \begin{cases} 1 & \text{if VM } v \text{ is assigned to server } j \text{ at time } t \\ 0 & \text{otherwise;} \end{cases} \tag{2.1}$$

$$y_{jt} = \begin{cases} 1 & \text{if server } j \text{ is in use at time } t \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

and

$$z^v_{pt} = \begin{cases} 1 & \text{if platform } p \text{ is assigned to VM } v \text{ at time } t \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Let $\bar{Z}^i_{f\,t}$ be the prediction demand of VMs made by PaaS provider f at time t.
We assume that $\bar{Z}^i_{f\,t} = \sum_{p \in \mathcal{P}_f} q^i_{pt}\ \forall f \in \mathcal{F},\ \forall t \in T$.
In this context, IaaS provider i can make the decision of accepting or rejecting a platform execution request in order to maximize its own revenue.
The resulting throughput (or acceptance rate) is denoted by:

$$Z^i_{ft} = \sum_{v \in \mathcal{V}_i} \sum_{p \in \mathcal{P}_f} z^v_{pt} \leq \bar{Z}^i_{f\,t} \quad \forall f \in \mathcal{F},\ \forall t \in T.$$

IaaS providers may possibly incur in penalties (see [3], [114]) upon rejection of request executions of $p \in \mathcal{P}_f$: $\delta^i \cdot (\bar{Z}^i_{f\,t} - Z^i_{ft})\ \ \forall f \in \mathcal{F},\ \forall t \in T$.
In order to fix the rejection rate over a fixed threshold and to ensure a minimum availability, i may decide to guarantee a minimum throughput $\underline{Z}^i_{ft}$ at

time t established in SLA between i and f.
IaaS provider's objective is to simultaneously maximize its profit and minimize the total operational cost of running servers (wear-and-tear cost and power consumption) and the resource wastage over the entire time horizon. The problem at time t can therefore be formulated as a global optimization problem as follows (see [3], [4] and [114] for game formulation):

$$\begin{aligned}
&\max_{\rho^i_{vt}, z^v_{pt}, x^j_{vt}, y_{jt}} \Bigg\{ \sum_{p\in\mathcal{P}} \sum_{v\in\mathcal{V}_i} \rho^i_{vt} z^v_{pt} - \sum_{f=1}^{F} \delta^i (\bar{Z}^i_{f\,t} - Z^i_{ft}) \\
&- \sum_{j\in\mathcal{S}_i} \big[C^+ \cdot y_{jt} \cdot (y_{jt} - y_{j(t-1)}) + (C^- \cdot y_{j(t-1)} \cdot (y_{j(t-1)} - y_{jt})) \big] \\
&- \sum_{j\in\mathcal{S}_i} y_{jt} \big[\omega^i_j \cdot (\sum_{v\in\mathcal{V}_i} x^j_{vt} C_{vt}) + \frac{\omega^i_j}{\alpha^i} \cdot (\bar{C}_j - \sum_{v\in\mathcal{V}_i} x^j_{vt} C_{vt}) \big] \Bigg\}
\end{aligned} \tag{2.4}$$

subject to:

$$D^i_{pt} \leq C_{vt} z^v_{pt} + \overline{C} \cdot (1 - z^v_{pt}) \forall p \in \mathcal{P},\ \forall v \in \mathcal{V}_i \tag{2.5}$$

$$\sum_{j\in\mathcal{S}_i} x^j_{vt} \leq 1, \quad \forall v \in \mathcal{V}_i \tag{2.6}$$

$$\sum_{p\in\mathcal{P}} z^v_{pt} = \sum_{j\in\mathcal{S}_i} x^j_{vt}, \quad \forall v \in \mathcal{V}_i \tag{2.7}$$

$$\sum_{v\in\mathcal{V}_i} z^v_{pt} \leq q^i_{pt}, \quad \forall p \in \mathcal{P} \tag{2.8}$$

$$\sum_{v\in\mathcal{V}_i} C_{vt} \cdot x^j_{vt} \leq \bar{C}_j \cdot y_{jt}, \quad \forall j \in \mathcal{S}_i \tag{2.9}$$

$$\underline{Z}^i_{ft} \leq Z^i_{ft} \quad \forall f = 1, \ldots, F \tag{2.10}$$

$$\rho^i_{vt} z^v_{pt} \leq \bar{\rho}_{pt}\, \forall v \in \mathcal{V}_i, \forall p \in \mathcal{P} \tag{2.11}$$

$$\begin{aligned}
&\rho^i_{vt} \geq 0,\ \ y_{jt}, x^j_{vt}, z^v_{pt} \in \{0, 1\} \\
&\qquad \forall j \in \mathcal{S}_i, \forall v \in \mathcal{V}_i, \forall p \in \mathcal{P}.
\end{aligned} \tag{2.12}$$

The first constraint affirms that the capacity requested for platform p cannot exceed the capacity of the VM which is assigned to (otherwise it results to be $z = 0$).

Constraint (2.6) assigns a VM v at most to only one of the servers. Constraint (2.7) necessarily assigns a used VM to only one of the platforms. Constraint (2.8) establishes that the number of platforms of the kind p which are accepted by i for the execution (in its own VMs) is less than or equal to the requested amount.

Constraint (2.9) models the capacity constraint of the servers and it also establishes that:

- $y_{jt} = 0 \;\Rightarrow\; \sum_{v \in \mathcal{V}_i} x^j_{vt} = 0 \quad \forall j \in \mathcal{S}_i$; namely, if a server j is not in use at time t, then there are not VMs assigned to such a server;
- if $\sum_{v \in \mathcal{V}_i} x^j_{vt} > 0 \;\Rightarrow\; y_{jt} = 1 \quad \forall j \in \mathcal{S}_i$; namely, if some VMs are assigned to server j at time t, then such a server must be in use.

Constraint family (2.10) establishes that Z^i_{ft} cannot be smaller than the minimum throughput $\underline{Z}^i_{ft}$.

Constraint (2.11) states that the revenue ρ^i_{vt} for the IaaS provider i for a single request execution in VM v at time t cannot exceed the maximum price that f is willing to pay for the platform p which is hosted by v.

The latest constraint family defines the domain of the variables of the problem.

2.3 A Computational Procedure

The following algorithm describes a computational procedure which allows us to calculate the solutions of the previous nonlinear model (2.4) subject to (2.5)-(2.12) (see [56] for a particle swarm optimization algorithm).

The algorithm starts initializing the values of IaaS variables (step 1).

In step 2, for each PaaS provider, we order all platforms in increasing order with respect to efficiency.

In steps 3-6 we assign the platforms to VMs in order to have the maximum revenue and to satisfy the minimum throughput; similarly, in steps 7-10, we assign the remaining platforms to VMs in order to have the maximum revenue.

Then, we allocate the used VMs into servers minimizing the power costs and the resource wastage, by paying also attention to the state of the servers (on or off) at previous time and to the respective wear-and-tear costs (step 12). In step 13 we analyze the costs of running servers in order to establish if it is convenient to turn on/off the servers minimizing the total operational costs. Furthermore, in step 14, we evaluate if deleting a platform is suitable, by taking into account the penalty for rejecting platform.

Finally, we estimate the total profit (step 15).

1. **Initialization**:
   $\rho^i_{v0} = 0,\ y_{j0} = x^j_{v0} = z^v_{p0} = 0\ \ \forall j \in \mathcal{S}_i, \forall v \in \mathcal{V}_i, \forall p \in \mathcal{P};$
   $\rho^i_{vt} = 0,\ y_{jt} = y_{j(t-1)},\ x^j_{vt} = z^v_{pt} = 0\ \ \forall j \in \mathcal{S}_i, \forall v \in \mathcal{V}_i, \forall p \in \mathcal{P},\ \forall t \in \mathcal{T}.$

2. **Sorting of platforms** and **partition of the set** $\mathcal{P}_f$:
   $\forall f \in \mathcal{F}$ we order platforms $p \in \mathcal{P}_f$ in increasing order with respect to efficiency:
   $$\frac{\bar{\rho}^i_{1_f t}}{D^i_{1_f t}} \geq \dots \geq \frac{\bar{\rho}^i_{p_f t}}{D^i_{p_f t}} \geq \dots \geq \frac{\bar{\rho}^i_{P_f t}}{D^i_{P_f t}}$$
   and we denote by $\mathcal{P}^Z_f$ the set of the first platforms $\underline{Z}^i_{ft}$ of f, by $\mathcal{P}^L_f = \mathcal{P}_f \setminus \mathcal{P}^Z_f$ the set of the remaining platforms and $\tilde{\mathcal{P}}^L_f = \mathcal{P}^L_f$.
   Remark: If $q^i_{pt} > 1$, then we repeat platform p q^i_{pt} times.

3. We denote by $\mathcal{P}^Z = \bigcup_{f \in \mathcal{F}} \mathcal{P}^Z_f$, $\mathcal{P}^L = \bigcup_{f \in \mathcal{F}} \mathcal{P}^L_f$ and $\mathcal{V}^L_i = \mathcal{V}_i$ the set of VMs which have to be assigned.
   Let $\tilde{\mathcal{P}}^Z = \mathcal{P}^Z$.

4-6. **Allocation of P^Z_f's platforms to the VM**:
   Solve `AllocationPZ` (which will be detailed later).
   $\forall \bar{p} \in \mathcal{P}^Z_f$, we chose the VM $\bar{v}$ with smaller capacity such that it contains

the platform; if $D^i_{\bar{p}t} = C_{\bar{v}}$, then we assign $\bar{p}$ to $\bar{v}$; otherwise, we search for a platform, if any, with requested capacity smaller than that of VM $\bar{v}$ and with maximum $\bar{\rho}_p$; in both cases, we choose the maximal revenue: $\rho^i_{vt} = \bar{\rho}_p$.
If all the platforms $p \in \mathcal{P}^Z_f$ require a capacity which is greater than the ones of the available VMs, then it is necessary to make a search in $\mathcal{P}^L_f$ in order to let the minimum throughput $\underline{Z}^i_{ft}$ be satisfied $\forall f \in \mathcal{F}$.
If all the VMs are engaged or all the $\mathcal{P}^Z_f$'s platforms have been assigned, then go to step 7.

7. Let $\tilde{\mathcal{P}}^L = \mathcal{P}^L$.

8-10. **Allocation of P^L_f's platforms to the VMs**:
Solve: `AllocationPL` (which will be detailed later).

11. Let $\mathcal{P}^- = \mathcal{P}^S = \mathcal{P} \setminus \tilde{\mathcal{P}}^L \setminus P^Z$.

12. **Allocation of the VMs**:
We order servers $j \in \mathcal{S}_i$ such that: $\omega_1 \leq \ldots \leq \omega_j \leq \ldots \leq \omega_{|\mathcal{S}_i|}$.
For $j \in \mathcal{S}_i$:
we consider $\mathcal{V}^S_i = \mathcal{V}_i \setminus \mathcal{V}^L_i$; we assign $x^j_{vt} \in \{0, 1\}$ with the method of Branch and Bound (for a linear integer programming 0-1 problem) in such a way as to most saturate the server with lower cost.

Here we pay attention to the status of the servers at time $t-1$:
consider every server $\hat{j}$ which is used at time t and switched off at time $t-1$ such that its VMs can be allocated (according to the capacity constraint) in the server $\bar{j}$, if any, which is not in use at time t and was switched on at time $t-1$. We evaluate whether the sum of turning on, energy and waste costs of $\hat{j}$ plus the mimimum between the turning off cost of the server $\bar{j}$ and its waste cost is greater than the cost associated with the use of $\bar{j}$, then, in this case, it is convenient to allocate the VMs to the server $\bar{j}$ which is already turned on

12.1 If $\exists \bar{j} \in \mathcal{S}_i \, : \, \sum_{v \in \mathcal{V}_i} x_{vt}^{\bar{j}} = 0 \; \&\& \; y_{\bar{j}(t-1)} = 1 \; \Rightarrow$

For $\hat{j} \in \mathcal{S}_i$ such that $y_{\hat{j}(t-1)} = 0 \; \&\& \; \sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} > 0 \; \&\& \; \sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} C_v \leq \bar{C}_{\bar{j}}$:

- if $\left[C^+ + \omega_{\hat{j}}^i \cdot (\sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} C_v) + \frac{\omega_{\hat{j}}^i}{\alpha^i} \cdot (\bar{C}_{\hat{j}} - (\sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} C_v))\right] + \min\{C^-, \frac{\omega_{\bar{j}}^i}{\alpha^i} \cdot \bar{C}_{\bar{j}}\} > \omega_{\bar{j}}^i \cdot (\sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} C_v) + \frac{\omega_{\bar{j}}^i}{\alpha^i} \cdot (\bar{C}_{\bar{j}} - (\sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} C_v)) \; \Rightarrow$
  $y_{\bar{j}t} = 1, \; x_{vt}^{\bar{j}} = x_{vt}^{\hat{j}} \; \forall v \in \mathcal{V}_i,$
  $y_{\hat{j}t} = 0, \; \sum_{v \in \mathcal{V}_i} x_{vt}^{\hat{j}} = 0;$
- otherwise go to step 13.

12.2 Otherwise (if all servers are used: $\forall \bar{j} \in \mathcal{S}_i \; \sum_{v \in \mathcal{V}_i} x_{vt}^{\bar{j}} > 0$ OR if they are not used but they are off: $\sum_{v \in \mathcal{V}_i} x_{vt}^{\bar{j}} = 0 \&\& y_{\bar{j}(t-1)} = 0$) go to step 13.

13. **Evaluation of the server**:
For every server (which is not required for satisfying the minimum throughput), we evaluate the turning on/off cost; specifically,
if, at time t, the server is not in use and it was turned on at time $t-1$, then we evaluate whether it is convenient to turn it off or to pay the waste (namely, the inaction costs),
on the contrary, if, at time t, the server is in use and it was turned off at time $t-1$, then we evaluate whether the penalty cost is greater than the profit. In this case, it is convenient to switch the server on, otherwise it is better to keep it off and to pay the penalty.

For $j \in \mathcal{S}_i$:

– If $\sum_{v\in\mathcal{V}_i} x^j_{vt} = 0$ && $y_{j(t-1)} = 1 \Rightarrow$

$$\frac{\omega^i_j}{\alpha^i}\bar{C}_j - C^- \begin{cases} > 0 & \Rightarrow y_{jt} = 0 \\ \leq 0 & \Rightarrow y_{jt} = 1. \end{cases}$$

– If $\sum_{v\in\mathcal{V}_i} x^j_{vt} > 0$ && $y_{j(t-1)} = 0$ &&

If $\forall \bar{v} :\ x^j_{\bar{v}t} > 0$ and $\forall \bar{p} :\ z^{\bar{v}}_{\bar{p}t} > 0$ we have that $\bar{p} \notin \mathcal{P}^Z \ \Rightarrow$

$$P - C^+ = \Big[\sum_{v\in\mathcal{V}_i}\sum_{p\in\mathcal{P}} x^j_{vt}\rho^i_{vt}$$

$$-[\omega^i_j \cdot (\sum_{v\in\mathcal{V}_i} x^j_{vt}C_v)$$

$$+\frac{\omega^i_j}{\alpha^i}\cdot(\bar{C}_j - (\sum_{v\in\mathcal{V}_i} x^j_{vt}C_v))]\Big] - C^+,$$

$$P - C^+ \begin{cases} \geq 0 & \Rightarrow y_{jt} = 1 \\ < 0 & \Rightarrow \end{cases}$$

$$\Rightarrow C^+ - P - \delta^i(\sum_{p\in\mathcal{P}}\sum_{v\in\mathcal{V}_i} x^j_{vt}z^p_{vt}) \begin{cases} \leq 0 & \Rightarrow y_{jt} = 1 \\ > 0 & \Rightarrow \end{cases}$$

$$\Rightarrow y_{jt} = 0, \sum_{p\in\mathcal{P}} z^{\bar{v}}_{pt} = 0\ \forall \bar{v} \in \mathcal{V}_i : x^j_{\bar{v}t} > 0, \sum_{v\in V_i} x^j_{vt} = 0.$$

Else $y_{jt} = 1$.

14. **Evaluation of the platforms**:
For every platform which has been assigned and which is not required for the minimum throughput, we evaluate whether it is convenient to delete it:
if the profit P, given by the revenue minus the energy and the waste costs, is negative, and if the penalty cost δ^i is less than such a loss, then such a platform will be deleted and the VMs will be allocated to

the servers.

$\forall p^- \in \mathcal{P}^-$ let $v^- \in \mathcal{V}_i$ such that $z^{v^-}_{p^-t} = 1$ and
we denote by $P = \rho^i_{v^-t} z^{v^-}_{p^-t} - \sum_{j \in \mathcal{S}_i} y_{jt}(\omega^i_j x^j_{v^-t} C_{v^-} + \frac{\omega^i_j}{\alpha^i}(\bar{C}_j - x^j_{v^-t} C_{v^-}))$.

- If $P \geq 0$ we denote by $\mathcal{P}^- = \mathcal{P}^- \setminus \{p^-\}$ and
  - if $\mathcal{P}^- = \emptyset \Rightarrow$ go to step 15;
  - otherwise go to step 14.
- If $P < 0$ we estimate $R = \delta^i - |P|$,
  - ▷ if $R \geq 0$ go to step 15;
  - ▷ if $R < 0$: $z^{v-}_{p^-t} = 0$, $\sum_{j \in \mathcal{S}_i} x^j_{v^-t} = 0$, $\rho^i_{v^-t} = 0$ and return to step 12 with $\mathcal{P}^- = \mathcal{P}^- \setminus \{p^-\}$.

15. We estimate:
$$\mathcal{R} = \sum_{p \in \mathcal{P}} \sum_{v \in \mathcal{V}_i} \rho^i_{vt} z^v_{pt} - \sum_{f=1}^{F} \delta^i (\bar{Z}^i_{ft} - Z^i_{ft})$$
$$- \sum_{j \in \mathcal{S}_i} \left[C^+ \cdot y_{jt} \cdot (y_{jt} - y_{j(t-1)}) + (C^- \cdot y_{j(t-1)} \cdot (y_{j(t-1)} - y_{jt}))\right]$$
$$- \sum_{j \in \mathcal{S}_i} y_{jt} \left[\omega^i_j \cdot (\sum_{v \in \mathcal{V}_i} x^i_{vt} C_v) + \frac{\omega^i_j}{\alpha^i} \cdot (\bar{C}_j - \sum_{v \in \mathcal{V}_i} x^i_{vt} C_v)\right].$$

Now we show in detail `AllocationPZ` and `AllocationPL`.

`AllocationPZ:`

4. Let $\bar{p} \in \tilde{\mathcal{P}}^Z$ the first platform which has to be assigned such that
$$\frac{\bar{\rho}^i_{\bar{p}t}}{D^i_{\bar{p}t}} = \max_{p \in \tilde{\mathcal{P}}^Z} \frac{\bar{\rho}^i_{pt}}{D^i_{pt}}.$$

5. - If $\nexists \bar{v} \in \mathcal{V}^L_i$ such that $D^i_{\bar{p}t} \leq C_{\bar{v}} \Rightarrow \tilde{p} = \bar{p}$, where $\bar{p} \in \mathcal{P}_{\bar{f}}$ go to step 6.1.

- Otherwise: let $\bar{v} \in \mathcal{V}_i^L$ such that

$$C_{\bar{v}} = \min_{v \in \mathcal{V}_i^L, D^i_{\bar{p}t} \leq C_{\bar{v}}} C_v.$$

⊗ If $D^i_{\bar{p}t} = C_{\bar{v}} \Rightarrow$ we assign $\bar{p}$ to VM $\bar{v}$: we define $z^{\bar{v}}_{\bar{p}t} = 1$, $\rho^i_{\bar{v}t} = \bar{\rho}_{\bar{p}}$, $\mathcal{V}_i^L = \mathcal{V}_i^L \setminus \{\bar{v}\}$, $\tilde{\mathcal{P}}^Z = \tilde{\mathcal{P}}^Z \setminus \{\bar{p}\}$.

⊗ Otherwise $\Rightarrow$ we select $\bar{\bar{p}} \in \tilde{\mathcal{P}}^Z \cup \mathcal{P}^L_{\bar{f}}$ (where $\bar{f}\,:\,\bar{p} \in \mathcal{P}_{\bar{f}}$) such that:

$$\bar{\rho}_{\bar{\bar{p}}} = \max_{D^i_{\bar{p}t} < D^i_{pt} \leq C_{\bar{v}}} \bar{\rho}_p$$

and we define $z^{\bar{v}}_{\bar{\bar{p}}t} = 1$, $\rho^i_{\bar{v}t} = \bar{\rho}_{\bar{\bar{p}}}$, $\mathcal{V}_i^L = \mathcal{V}_i^L \setminus \{\bar{v}\}$,

- if $\bar{\bar{p}} \in \tilde{\mathcal{P}}^Z \Rightarrow \tilde{\mathcal{P}}^Z = \tilde{\mathcal{P}}^Z \setminus \{\bar{\bar{p}}\}$;
- if $\bar{\bar{p}} \in \mathcal{P}^L_{\bar{f}} \Rightarrow \tilde{\mathcal{P}}^Z = \tilde{\mathcal{P}}^Z \setminus \{\bar{p}\}$, $\mathcal{P}^Z = \mathcal{P}^Z \setminus \{\bar{p}\} \cup \{\bar{\bar{p}}\}$, $\mathcal{P}^L_{\bar{f}} = \mathcal{P}^L_{\bar{f}} \cup \{\bar{p}\} \setminus \{\bar{\bar{p}}\}$, $\mathcal{P}^L = \mathcal{P}^L \cup \{\bar{p}\} \setminus \{\bar{\bar{p}}\}$.

6. – if $\mathcal{V}_i^L = \emptyset$ or $\tilde{\mathcal{P}}^Z = \emptyset$: go to step 7;

   – if $D^i_{pt} > C_v$, $\forall p \in \tilde{\mathcal{P}}^Z$, $\forall v \in \mathcal{V}_i^L$:
   for $\tilde{p} \in \tilde{\mathcal{P}}^Z \cap \mathcal{P}_{\bar{f}}$:

   6.1 we select $\bar{p} \in \tilde{\mathcal{P}}^L_{\bar{f}}$ such that

$$\frac{\bar{\rho}^i_{\bar{p}t}}{D^i_{\bar{p}t}} = \max_{p \in \tilde{\mathcal{P}}^L_{\bar{f}}} \frac{\bar{\rho}^i_{pt}}{D^i_{pt}},$$

   $\tilde{\mathcal{P}}^Z = \tilde{\mathcal{P}}^Z \cup \{\bar{p}\} \setminus \{\tilde{p}\}$, $\mathcal{P}^Z = \mathcal{P}^Z \cup \{\bar{p}\} \setminus \{\tilde{p}\}$, $\mathcal{P}^L_{\bar{f}} = \mathcal{P}^L_{\bar{f}} \cup \{\tilde{p}\} \setminus \{\bar{p}\}$, $\tilde{\mathcal{P}}^L_{\bar{f}} = \tilde{\mathcal{P}}^L_{\bar{f}} \setminus \{\bar{p}\}$, $\mathcal{P}^L = \mathcal{P}^L \cup \{\tilde{p}\} \setminus \{\bar{p}\}$ and go to step 5;

   – otherwise: go to step 4.

`AllocationPL`:

8. Let $\bar{p} \in \tilde{\mathcal{P}}^L$ the first platform which has to be assigned such that

$$\frac{\bar{\rho}^i_{\bar{p}t}}{D^i_{\bar{p}t}} = \max_{p \in \tilde{\mathcal{P}}^L} \frac{\bar{\rho}^i_{pt}}{D^i_{pt}}.$$

9. - If $\nexists \bar{v} \in \mathcal{V}_i^L$ such that $D^i_{\bar{p}t} \leq C_{\bar{v}} \Rightarrow \tilde{P}^L = \tilde{P}^L \setminus \{\bar{p}\}$ and go to step 8.

- Otherwise: let $\bar{v} \in \mathcal{V}_i^L$ such that:

$$C_{\bar{v}} = \min_{v \in \mathcal{V}_i^L, D^i_{\bar{p}t} \leq C_{\bar{v}}} C_v.$$

⊗ If $D^i_{\bar{p}t} = C_{\bar{v}} \Rightarrow$ we assign $\bar{p}$ to VM $\bar{v}$: we define $z^{\bar{v}}_{\bar{p}t} = 1$, $\rho^i_{\bar{v}t} = \bar{\rho}_{\bar{p}}$, $\mathcal{V}_i^L = \mathcal{V}_i^L \setminus \{\bar{v}\}$, $\tilde{\mathcal{P}}^L = \tilde{\mathcal{P}}^L \setminus \{\bar{p}\}$.

Let $\bar{p} \in \tilde{\mathcal{P}}^L_{\bar{f}} \Rightarrow \forall p^- \in \mathcal{P}^Z_{\bar{f}} : \bar{\rho}_{p^-} = \min_{p \in \mathcal{P}^Z_{\bar{f}}} \bar{\rho}_p$ and let $v^- \in \mathcal{V}$ t.c. $z^{v^-}_{p^-t} = 1$
we select $\tilde{p} \in \tilde{\mathcal{P}}^L \cup \{p^-\}$ such that $\rho_{\tilde{p}} = \max\{\max_{p \in \tilde{\mathcal{P}}^L,\, C_p \leq C_{v^-}} \bar{\rho}_p; \bar{\rho}_{p^-}\}$.
If $\bar{\rho}_{\tilde{p}} = \bar{\rho}_{p^-}$ go to step 10,
Else $z^{v^-}_{p^-t} = 0$, $z^{v^-}_{\tilde{p}t} = 1$, $\rho_{v^-} = \bar{\rho}_{\tilde{p}}$, $\mathcal{P}^Z = \mathcal{P}^Z \setminus \{p^-\} \cup \{\bar{p}\}$.

⊗ Otherwise: $\Rightarrow$ we select $\bar{\bar{p}} \in \tilde{\mathcal{P}}^L$ such that:

$$\bar{\rho}_{\bar{\bar{p}}} = \max_{D^i_{\bar{p}t} \leq D^i_{pt} \leq C_{\bar{v}}} \bar{\rho}_p$$

and we define $z^{\bar{v}}_{\bar{\bar{p}}t} = 1$, $\rho^i_{\bar{v}t} = \bar{\rho}_{\bar{\bar{p}}}$, $\mathcal{V}_i^L = \mathcal{V}_i^L \setminus \{\bar{v}\}$, $\tilde{\mathcal{P}}^L = \tilde{\mathcal{P}}^L \setminus \{\bar{\bar{p}}\}$.
Let $\bar{\bar{p}} \in \tilde{\mathcal{P}}^L_{\bar{f}} \Rightarrow \forall p^- \in \mathcal{P}^Z_{\bar{f}} : \bar{\rho}_{p^-} = \min_{p \in \mathcal{P}^Z_{\bar{f}}} \bar{\rho}_p$ and let $v^- \in \mathcal{V}$ s.t. $z^{v^-}_{p^-t} = 1$
we choose $\tilde{p} \in \tilde{\mathcal{P}}^L \cup \{p^-\}$ such that $\rho_{\tilde{p}} = \max\{\max_{p \in \tilde{\mathcal{P}}^L,\, C_p \leq C_{v^-}} \bar{\rho}_p; \bar{\rho}_{p^-}\}$.
If $\bar{\rho}_{\tilde{p}} = \bar{\rho}_{p^-}$ go to step 10,
Else $z^{v^-}_{p^-t} = 0$, $z^{v^-}_{\tilde{p}t} = 1$, $\rho_{v^-} = \bar{\rho}_{\tilde{p}}$, $\mathcal{P}^Z = \mathcal{P}^Z \setminus \{p^-\} \cup \{\bar{\bar{p}}\}$.

10. – If $\mathcal{V}_i^L = \emptyset$ or $\tilde{\mathcal{P}}^L = \emptyset$ or $D^i_{pt} > C_v, \forall v \in \mathcal{V}_i^L$, $\forall p \in \tilde{\mathcal{P}}^L$ go to step 11;

– otherwise go to step 8.

2.4 Linearization

In section 2.2 we obtained a *Mixed-Integer Nonlinear Programming Problem* (see [19] for another Integer Nonlinear programming problem, the related

optimality conditions and the variational inequality formulation) which can be solved by the above algorithm.
In this section we propose a mixed-integer linear programming problem that is equivalent to the previous one in order to compare the solutions coming from our algorithm with those obtained by standard methods.

Theorem 2. *Problem (2.4) under constraints (2.5)-(2.12) is equivalent to the following mixed-integer linear programming problem:*

$$\max_{\rho^i_{vt}, z^v_{pt}, x^j_{vt}, y_{jt}} \Bigg\{ \sum_{v\in\mathcal{V}_i} \rho^i_{vt} - \sum_{f=1}^{F} \delta^i (\bar{Z}^i_{f_t} - Z^i_{ft}) \\ - \sum_{j\in\mathcal{S}_i} \left[C^+ \cdot y^+_{jt} + (C^- \cdot y^-_{jt})\right] \\ - \sum_{j\in\mathcal{S}_i} \left[\omega^i_j \cdot (\sum_{v\in\mathcal{V}_i} x^j_{vt} C_{vt}) + \frac{\omega^i_j}{\alpha^i} \cdot (\bar{C}_j \cdot y_{jt} - \sum_{v\in\mathcal{V}_i} x^j_{vt} C_{vt})\right] \Bigg\} \tag{2.13}$$

subject to:

$$D^i_{pt} \le C_{vt} z^v_{pt} + \overline{C} \cdot (1 - z^v_{pt}) \forall p \in \mathcal{P},\ \forall v \in \mathcal{V}_i \tag{2.14}$$

$$\sum_{j\in\mathcal{S}_i} x^j_{vt} \le 1, \quad \forall v \in \mathcal{V}_i \tag{2.15}$$

$$\sum_{p\in\mathcal{P}} z^v_{pt} = \sum_{j\in\mathcal{S}_i} x^j_{vt}, \quad \forall v \in \mathcal{V}_i \tag{2.16}$$

$$\sum_{v\in\mathcal{V}_i} z^v_{pt} \le q^i_{pt}, \quad \forall p \in \mathcal{P} \tag{2.17}$$

$$\sum_{v\in\mathcal{V}_i} C_{vt} \cdot x^j_{vt} \le \bar{C}_j \cdot y_{jt}, \quad \forall j \in \mathcal{S}_i \tag{2.18}$$

$$\underline{Z}^i_{ft} \le Z^i_{ft} \quad \forall f = 1, \dots, F \tag{2.19}$$

$$\rho^i_{vt} \le \sum_{p\in\mathcal{P}} \bar{\rho}_{pt} z^v_{pt} \ \forall v \in \mathcal{V}_i \tag{2.20}$$

$$\rho^i_{vt} \ge 0, \ \ y_{jt}, x^j_{vt}, z^v_{pt} \in \{0,1\} \\ \forall j \in \mathcal{S}_i, \forall v \in \mathcal{V}_i, \forall p \in \mathcal{P} \tag{2.21}$$

$$y^+_{jt} + y^-_{jt} \le 1 \quad \forall j \in \mathcal{S}_i \tag{2.22}$$

$$y_{jt} - y_{j(t-1)} - y^+_{jt} + y^-_{jt} = 0 \quad \forall j \in \mathcal{S}_i. \tag{2.23}$$

Proof. Problem (2.4) under constraints (2.5)-(2.12) can be linearized by making use of the following procedure.
Constraint (2.11) can be replaced by:

$$\rho^i_{vt} \leq \sum_{p \in \mathcal{P}} \bar{\rho}_{pt} z^v_{pt} \; \forall v \in \mathcal{V}_i;$$

therefore, the first term in the objective function becomes:

$$\sum_{v \in \mathcal{V}_i} \rho^i_{vt};$$

indeed, if $\sum_{p \in \mathcal{P}} z^v_{pt} = 0$ no transactions occur (no platform is assigned to VM v), so the revenue is zero: $\rho^i_{vt} = 0$, otherwise: $\rho^i_{vt} \leq \bar{\rho}_{pt}$ where p s.t. $z^v_{pt} = 1$. Taking into account constraint (2.9) which establishes that, if $y_{jt} = 0$, then $\sum_{v \in \mathcal{V}_i} x^j_{vt}$, we have that the term associated with the energy costs and with the waste can be linearized as follows:

$$\sum_{j \in \mathcal{S}_i} \big[\omega^i_j \cdot (\sum_{v \in \mathcal{V}_i} x^j_{vt} C_{vt}) + \frac{\omega^i_j}{\alpha^i} \cdot (\bar{C}_j y_{jt} - \sum_{v \in \mathcal{V}_i} x^j_{vt} C_{vt})\big].$$

Finally, in order to linearize the term in the objective function which is connected to turning on and off, we define two binary variables for every server j:

$$y^+_{jt} = \begin{cases} 1 & \text{if we turn server } j \text{ on at time } t \\ 0 & \text{otherwise} \end{cases} \tag{2.24}$$

and

$$y^-_{jt} = \begin{cases} 1 & \text{if we turn server } j \text{ off at time } t \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

As a consequence, in the objective function the term becomes:

$$\sum_{j \in \mathcal{S}_i} [C^+ y^+_{jt} + C^- y^-_{jt}]$$

and, further, it is necessary to add the following constraints:

- to avoid that the new variables are simultaneously equal to one:

$$y^+_{jt} + y^-_{jt} \leq 1 \quad \forall j \in \mathcal{S}_i;$$

- to connect the new variables with y_{jt}:

$$y_{jt} - y_{j(t-1)} - y_{jt}^{+} + y_{jt}^{-} = 0 \quad \forall j \in \mathcal{S}_i,$$

  indeed:

  - if we turn server j on, then we have: $y_{jt} = 1$, $y_{j(t-1)} = 0 \Rightarrow y_{jt} - y_{j(t-1)} = 1 = y_{jt}^{+}$;
  - if we turn server j off, then we have: $y_{jt} = 0$, $y_{j(t-1)} = 1 \Rightarrow y_{jt} - y_{j(t-1)} = -1 = -y_{jt}^{-}$;
  - otherwise, if we do not change the state of server j, then we have: $y_{jt} = 0$, $y_{j(t-1)} = 0$ or $y_{jt} = 1$, $y_{j(t-1)} = 1$ and in both cases $\Rightarrow y_{jt} - y_{j(t-1)} = 0 = y_{jt}^{+} - y_{jt}^{-}$ from which we get: $y_{jt}^{+} = y_{jt}^{-} = 0$ (since they are binary variables).

□

Hence, we have proved that the Mixed-Integer Nonlinear Programming Problem (2.4) subject to (2.5)-(2.12) can be reformulated as mixed-integer linear programming problem (2.13) subject to (2.14)-(2.23).

2.5 A Numerical Example

In this section we apply the model to a numerical example.
Since we want to report all the results for transparency purposes, we select the size of problems as reported. The numerical data are inspired by realistic values (see Amazon Web Services that offers different VMs types) and are constructed for easy interpretation purposes.

We consider the network at time t as depicted in Figure 2.2, consisting of:

- 5 servers $\mathcal{S}_i = \{1, 2, 3, 4, 5\}$,
  where $y_{1(t-1)} = 1$, $y_{2(t-1)} = 0$, $y_{3(t-1)} = 1$, $y_{4(t-1)} = 0$, $y_{5(t-1)} = 1$;
  $\bar{C}_1 = 35$, $\bar{C}_2 = 10$, $\bar{C}_3 = 30$, $\bar{C}_4 = 40$, $\bar{C}_5 = 20$;
  $\omega_1 = 0.5$, $\omega_2 = 1$, $\omega_3 = 1.2$, $\omega_4 = 1.5$, $\omega_5 = 3$;

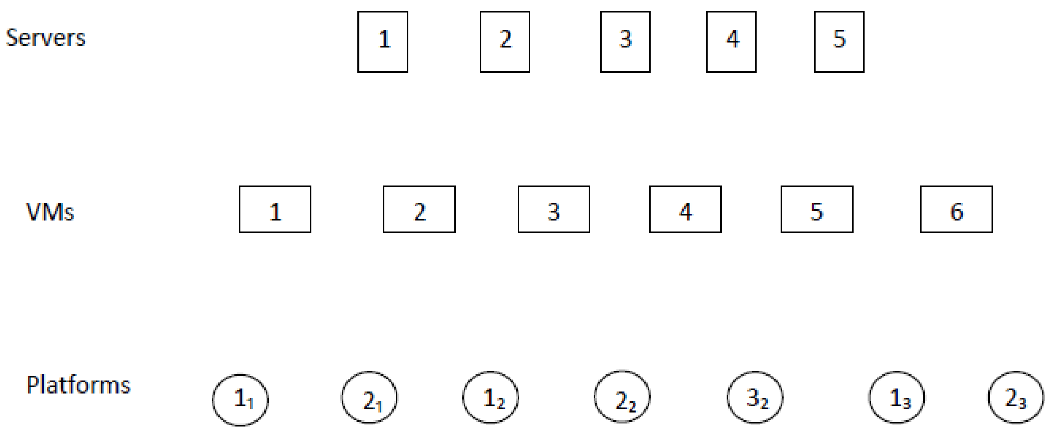

Figure 2.2: The Network of the numerical example

- 6 VMs $\mathcal{V}_i = \{1, 2, 3, 4, 5, 6\}$,
  where $C_1 = 5$, $C_2 = 8$, $C_3 = 13$, $C_4 = 20$, $C_5 = 20$, $C_6 = 20$;

- 3 PaaS providers: $\mathcal{P}_1 = \{1_1, 1_2\}$, $\mathcal{P}_2 = \{1_2, 2_2, 3_2\}$ e $\mathcal{P}_3 = \{1_3, 2_3\}$
  where $D_{1_1} = 5$, $D_{1_2} = 10$, $D_{2_2} = 13$, $D_{3_2} = 7$, $D_{1_3} = 18$, $D_{2_3} = 19$;
  $\bar{\rho}_{1_1} = 20$, $\bar{\rho}_{1_2} = 35$, $\bar{\rho}_{2_2} = 40$, $\bar{\rho}_{3_2} = 21$, $\bar{\rho}_{1_3} = 18$, $\bar{\rho}_{2_3} = 18$;
  $q_{1_1} = 2$, $q_{1_2} = 1$, $q_{2_2} = 1$, $q_{3_2} = 1$, $q_{1_3} = 1$, $q_{2_3} = 1$;
  $\underline{Z}_1 = 1$, $\underline{Z}_2 = 2$, $\underline{Z}_3 = 1$;

and let: $\alpha = 4$, $\delta = 15$, $C^+ = 15$, $C^- = 5$.

Figure 2.3 shows the global optimal solutions of the model solved by applying the algorithm presented in Section 4.
The optimal variables are the following ones:

$y_1 = 1$, $y_2 = 1$,$y_3 = 1$,$y_4 = 0$,$y_5 = 0$;

$x_1^1 = 0$, $x_2^1 = 0$, $x_3^1 = 1$, $x_4^1 = 1$, $x_5^1 = 0$, $x_6^1 = 0$;
$x_1^2 = 0$, $x_2^2 = 1$, $x_3^2 = 0$, $x_4^2 = 0$, $x_5^2 = 0$, $x_6^2 = 0$;
$x_1^3 = 1$, $x_2^3 = 0$, $x_3^3 = 0$, $x_4^3 = 0$, $x_5^3 = 1$, $x_6^3 = 0$;
$x_1^4 = 0$, $x_2^4 = 0$, $x_3^4 = 0$, $x_4^4 = 0$, $x_5^4 = 0$, $x_6^4 = 0$;

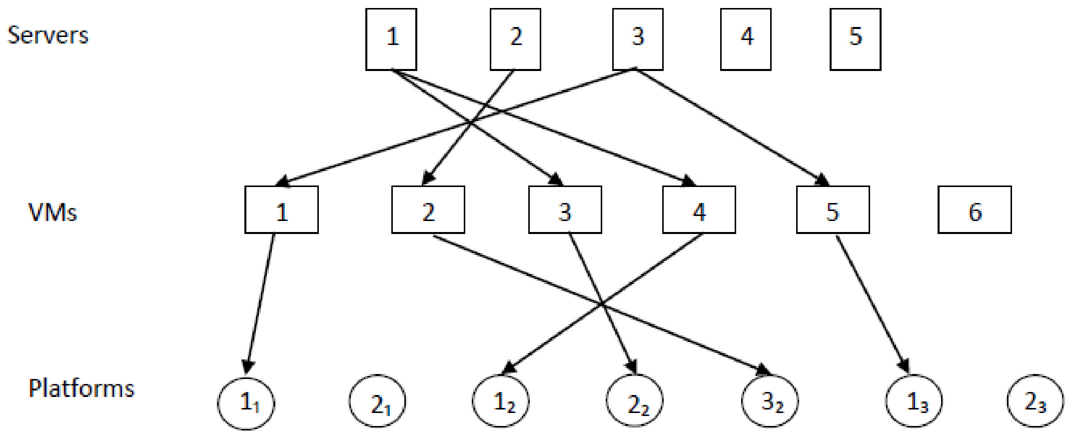

Figure 2.3: Optimal solution of the network

$x_1^5 = 0,\ x_2^5 = 0,\ x_3^5 = 0,\ x_4^5 = 0,\ x_5^5 = 0,\ x_6^5 = 0;$

$z_1^1 = 1,\ z_2^1 = 0,\ z_3^1 = 0,\ z_4^1 = 0,\ z_5^1 = 0,\ z_6^1 = 0,\ z_7^1 = 0;$
$z_1^2 = 0,\ z_2^2 = 0,\ z_3^2 = 0,\ z_4^2 = 0,\ z_5^2 = 1,\ z_6^2 = 0,\ z_7^2 = 0;$
$z_1^3 = 0,\ z_2^3 = 0,\ z_3^3 = 0,\ z_4^3 = 1,\ z_5^3 = 0,\ z_6^3 = 0,\ z_7^3 = 0;$
$z_1^4 = 0,\ z_2^4 = 0,\ z_3^4 = 1,\ z_4^4 = 0,\ z_5^4 = 0,\ z_6^4 = 0,\ z_7^4 = 0;$
$z_1^5 = 0,\ z_2^5 = 0,\ z_3^5 = 0,\ z_4^5 = 0,\ z_5^5 = 0,\ z_5^4 = 1,\ z_7^5 = 0;$
$z_1^6 = 0,\ z_2^6 = 0,\ z_3^6 = 0,\ z_4^6 = 0,\ z_5^6 = 0,\ z_6^6 = 0,\ z_7^6 = 0;$

$\rho_1 = 20€,\ \rho_2 = 21€,\ \rho_3 = 40€,\ \rho_4 = 35€,\ \rho_5 = 18€,\ \rho_6 = 0€;$

The maximum profit obtained by applying the algorithm shown in the previous section is:

$\mathcal{R} = 27.25€.$

We underline that, at step 9, $z_2^6 = 1$ and $\rho_6 = 20€$, and, at step 12, $x_6^4 = 1$ and $y_4 = 1$, but by evaluating, in step 13, the profit, we observe that it is suitable to delete platform 2_1, in fact the algorithm assigns $z_2^6 = 0$, $\rho_6 = 0€$ and $x_6^4 = 0$.

On the contrary, it is not suitable to delete platform 3_2 because its partial profit is positive $P > 0$.
Then, in step 14, by analyzing the costs of running servers, the algorithm assigns $y_4 = 0$ and, furthermore, $y_5 = 0$ (you note that in the previous time server 5 was turned on), whereas, nevertheless in the previous instant server 2 was turned off, the algorithm assigns $y_2 = 1$ in order to maximize the total profit.

We wish to highlight that these global optimal solutions are the same as the optimal solutions of the mixed-integer linear programming problem, which has been solved by using Matlab and Lingo.

2.6 Conclusions

In recent years, interest in Cloud Computing increased mainly because there is no doubt that businesses and people can reap huge benefits from it. Perhaps, the most significant cloud computing benefit is in terms of IT cost savings. Businesses, no matter what type or size is, and people can save substantial capital costs with zero in-house server storage and application requirements. Furthermore, the costs of cloud computing are much more flexible than traditional methods, in fact customers pay for what they use.
Cloud computing is much more reliable and consistent than in-house IT infrastructure and it provides enhanced and simplified IT management and maintenance capabilities through central administration of resources, vendor managed infrastructure and SLA backed agreements. IT infrastructure updates and maintenance are eliminated, as all resources are maintained by the service provider. The SLA ensures the timely and guaranteed delivery, management and maintenance of your IT services.
The cloud computing services are available on-demand, anywhere and in any time; therefore, the providers of cloud services receive a lot of requests. In this chapter we analyze the behavior of the IaaS provider who can make

the decision of accepting or rejecting a platform execution request, can establish the revenue for a request execution in a VM, can make the decision about the allocation of VMs in their servers and can decide if they need to turn their servers on or off in order to maximize their own revenue. We present a nonlinear mathematical model, based on networks, which allows us to simultaneously maximize IaaS provider's profit and minimize their total operational cost of running servers (wear-and-tear cost and power consumption) and the resource wastage.

Furthermore we propose a solution algorithm which describes a computational procedure which allows us to calculate the solutions of the mathematical model.

The theoretical framework is then further illustrated through a numerical example for which the optimal IaaS provider's variables are computed. Such solutions are the same as the optimal global solutions of the linear model.

Chapter 3

Financial Models for a Multi-Period Portfolio Optimization Problem

3.1 Introduction

In financial literature, a portfolio is considered to be a set of financial assets or investments which are owned by an individual (an investor) or a financial institution and consist of various financial instruments such as shares of a company (often referred as equities), government bonds, and so on.

Given a financial portfolio, it is possible to obtain different combinations of expected returns and risks depending on the choices related to the placement of their investments. So it is important to find the combination that allows us to get the best possible strategy (i.e. the best performance for a given level of risk). At this aim the principle of Dominance is introduced.

Let us assume to have two portfolios A and B, and denote by $\mathbb{E}[u_A]$ and $\mathbb{E}[u_B]$ their expected yields, respectively and by r_A^2 and r_B^2 their risks. Portfolio A is said to be efficient and dominant on B ($A \succ B$) if it satisfies the following properties:

- $\mathbb{E}[u_A] \geq \mathbb{E}[u_B]$;

- $r_A^2 \leq r_B^2$;

where at least one of the two inequalities must be strictly satisfied.
If both properties are satisfied as equalities, then the two portfolio are equivalent.

The foundation of portfolio optimization and asset allocation problem is always attributed to the Markowitz's Modern Portfolio Theory (MPT) (see [85] and [86]) based on the mean-variance analysis.

The underlying principle behind Markowitz's theory is that in order to build an efficient portfolio, a combination of securities should be identified to maximize the performance and minimize the total risk by choosing as few correlated securities as possible. The fundamental assumptions of the portfolio theory are as follows:

- investors intend to maximize their ultimate wealth and are at risk;
- the investment period is unique (for Markowitz model, time is not a significant variable);
- transaction costs and taxes are zero and the assets are perfectly divisible;
- expected value and standard deviation are the only parameters that guide the choice;
- the market is perfectly competitive.

Ever since then, in numerous research papers, modifications, extensions and alternatives to MPT have been introduced in order to simplify and reduce the limitations of Markowitz's model.

In [134] and [38] the authors present the inclusion of a risk-free asset in the traditional Markowitz formulation and the optimal risk portfolio can be obtained without any knowledge of the investor's preferences (this is also known as Separation theorem), whereas, in [126] the author, taking into account the risk-free asset and the mean-variance analysis, develops the Capital Asset Pricing Model (CAPM), studied also by Lintner in [76] and Mossin in [95], in which he shows that not all the risk of an asset is rewarded by the market in the form of a higher return, but only the part which can not be reduced by diversification and also he describes how expected portfolios can be calculated by summing the pure (risk-free) rate of interest and the multiplication between the price of risk reduction for efficient portfolios and the portfolio standard deviation, known as Capital Market Line (see [127].

CAPM limits are:

- the investment horizon is one-period;
- you can negotiate any amount of securities (which is almost unrealistic);
- absence of taxes and transaction costs;
- all investors analyze securities in the same way with the same probability estimates;
- regular performance distribution.

It is precisely the existence of this risk-free title in the CAPM that is the main and most significant difference with the Markowitz portfolio selection model because the utility curves are eliminated and thus the strong subjective component in the efficient portfolio selection; indeed, all individuals invest in the same portfolio of tangency, while the weights inside it to the various titles, and in particular to the risk-free title, change.

One of the newest models is the one for calculating optimal portfolio weights developed by Black and Litterman (see [11] and [12]).
The innovative aspect of the Black-Litterman model lies in the fact that,

thanks to the Bayes theorem, it is able to put together two types of information different from each other, namely the market equilibrium and the investor's views on the future trend of the market. The obtained results are then used by the classic media-variance optimization approach to calculate the mean, variance and consequently also the excellent portfolio composition.

Other extensions of the Markowitz model are studied, in which the variance has been replaced by the Value-at-Risk (with threshold) (see [8]) or with the Conditional Value-at-Risk (CVaR) (see [121]).

In an Optimization Portfolio Problem, the multiperiod theory must be taken into account and becomes crucial. A formulation neglecting this feature can easily become misleading.
Therefore, in this work Markovitz's portfolio theory is reviewed for investors with long-term horizons.

In 1969, Samuelson (see [125]) and Merton (see [93]), taking inspiration from Mossin's work (see [96]), formulate and solve a many-period generalization, corresponding to lifetime planning of consumption and investment decisions.
Samuelson and Merton were therefore the first authors to study the problem with long-term horizons, but in their case the investment horizon was irrelevant and the choice of portfolio was considered short-sighted because investors ignored what was going to happen the next period and continued to choose the same portfolio, as opposed to what is studied in this chapter, in which we consider the predictable and variable returns (or profits) over time.

In [133], multiperiod mean-variance models are analyzed and the final goal consists in constructing an approximate downside risk minimization through appropriate constraints.

In financial markets buying and selling securities entail brokerage fees and sometimes lump sum taxes are imposed on the investors.

In [10] the authors used Mixed Integer Programming (MIP) methods to construct portfolios, reducing the number of different stocks and assuming that it is desirable to have a portfolio with a small number of transactions (considered as the processes of rebalancing the portfolio).

Mao (see [83]), Jacob (see [59]) and Levy (see [73]) have examined the fixed transaction costs problem by placing restrictions on the number of securities in the optimal portfolio.

In 2000 Kellerer, Mansini, and Speranza (see [65]) introduced some Mixed Integer Linear Programming (MILP) problems with the presence of transaction costs and studied the problem of portfolio selection with fixed and proportional costs and possibly with minimum transaction lots, but they only allow linear objective function and linear and integer constraints on transaction amounts, so nonlinear constraints cannot be managed.

In 2013, Greco, Matarazzo and Slowinski (see [55]), considering the quantiles as evaluation criteria of the portfolios, solved a multiobjective optimization problem by using a Multiple Criteria Decision Aiding method.

In some models, investors can negotiate any amount of securities, but this hypothesis is unrealistic, as each investor has a maximum budget limit available to invest.

In this work, however, we impose that the resources used are not greater than the available ones, making the model more realistic.

The objective of this chapter is to formulate the multi-period portfolio

selection problem as a Markowitz mean-variance optimization problem by adding not only transaction costs and taxes (on the capital gain)and time-length for some financial securities, but also the *short selling* and the *transfer of financial assets*.
Short selling is the sale of a security that is not owned by the seller or that the seller has borrowed. Short selling is motivated by the belief that a security's price will decline, enabling it to be bought back at a lower price to make a profit. Short selling may be prompted by speculation, or by the desire to hedge the downside risk of a long position in the same security or a related one. Since the risk of loss on a short sale is theoretically infinite, short selling should only be used by experienced traders, who are familiar with the risks. While short selling is frequently vilified and short sellers viewed as ruthless operators out to destroy companies, the reality is that short selling provides liquidity to markets and prevents stocks from being bid up to ridiculously high levels. Although abusive short-selling practices, such as rumor-mongering to drive a stock lower, are illegal, short selling, when done properly, can be a good tool for portfolio risk management. The transfer of financial securities consists in relocating one or more assets from a bank, or a financial intermediary, to another one; in such a way, the investor can seize the opportunities offered by the commercial initiatives of the various financial institutions.

Moreover, by means of the proposed Integer Nonlinear Programming (INLP) Problems, it is possible to establish when it is convenient to buy and to sell financial securities, while maximizing the profits and minimizing the risk.

This chapter is organized as follows. In Section 3.2 we present the financial model consisting of financial securities, issuers, investors, and intermediaries. We derive the optimization problem of each investor based on the maximization of his expected gain and the minimization of his risk portfolio.

In Section 3.3, taking into account also the short selling and the transfer of securities, we introduce another model. In Section 3.4 we apply the model to some numerical examples consisting of a financial network with two issuers, two financial securities and an investor. Section 3.5 summarizes the obtained results.

3.2 A Financial Model with transaction costs, taxes and time-length

We consider a financial network consisting of: n financial securities, and the typical one is denoted by i; S issuers of financial securities, such as companies, banks, countries, etc., and the typical one is denoted by s; K investors (security purchasers) and the typical one is denoted by k; B financial intermediaries, and the typical one is denoted by b. In addition, we consider a partition of the financial securities by means of the sets $\mathcal{A}_1, \ldots, \mathcal{A}_s, \ldots, \mathcal{A}_S$, where $\mathcal{A}_s$ represents the set of financial securities made available by issuer s. A representation of the financial network is depicted in Figure 1.

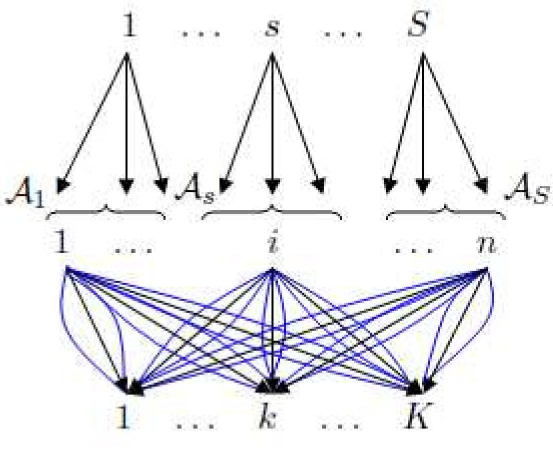

Figure 3.1: Financial Network

We can remark that in the network the financial intermediaries are denoted by parallel edges, since they are not decision makers. We analyze the model in a discrete time horizon: $1, \dots, j, \dots, t$.

Every investor k aims at determining which securities he has to buy and sell, which financial intermediary he has to choose and at what time it is more convenient to buy and sell a security in order to maximize his own profit and minimize his own risk.

For every security i, there is a purchase cost, $C_{i,j}$, which varies over time; moreover, it is necessary to pay a commission to the chosen financial intermediary (often the banks), which consists of a percentage of the purchase cost, $\gamma_k^b \cdot C_{i,j}$, and a flat fee C_k^b.

During the ownership time of the security, it is possible (not necessary) to obtain funds (such as dividends in the case of shares, interests in the case of bonds) $D_{i,j}$ or pay money (for example in the case of an increase in the corporate capital) $P_{i,j}$. Obviously, in the event that one does not get or does not have to pay anything until the expiration or sale of the security, these quantities vanish.

Each investor has the opportunity to sell his own securities and, in this case, he will receive the sum $R_{i,j}$, but he will have to pay a charge to the chosen financial intermediary $\beta_k^b \cdot R_{i,j} + F_k^b$ (similar to purchase) and a taxation on the capital gain or a percentage on the gain obtained from the title. In case of loss, no taxation will be carried out, whereas, on the contrary, generally you have the compensation, but such a situation is not examined in this work. So, we have:

$$\alpha_i^k \left(\frac{|\mathbb{E}[u_{i,j}]| + \mathbb{E}[u_{i,j}]}{2} \right),$$

where $|\mathbb{E}[u_{i,j}]|$ denotes the absolute value of the expected gain, which coincides with the capital gain.

We note that the tax treatment of the capital gain in Italy varies according to the subject k making the gain (individual or individual company or company) and the type of financial security i.

In this work, we will refer to the declarative regime and, moreover, we will assume that, for each security, the financial intermediary of the sale coincides with that of the purchase.

Therefore, we introduce the following binary variables:

$$x_{i,j}^k = \begin{cases} 1 & \text{if security } i \text{ is purchased by } k \text{ at time } j \\ 0 & \text{otherwise} \end{cases}$$

$$\forall i = 1, \dots, n, \quad \forall j = 1, \dots, t, \quad \forall k = 1, \dots, K;$$

$$y_{i,j}^k = \begin{cases} 1 & \text{if security } i \text{ is sold by } k \text{ at time } j \\ 0 & \text{otherwise} \end{cases}$$

$$\forall i = 1, \dots, n, \quad \forall j = 1, \dots, t, \quad \forall k = 1, \dots, K;$$

$$z_i^b = \begin{cases} 1 & \text{if security } i \text{ is purchased and sold by} b \\ 0 & \text{otherwise} \end{cases}$$

$$\forall i = 1, \dots, n, \quad \forall b = 1, \dots, B.$$

Since at the initial time the values $P_{i,j}$, $D_{i,j}$ and $R_{i,j}$ of the subsequent times are unknown, we will use their expected values: $\mathbb{E}[P_{i,j}]$, $\mathbb{E}[D_{i,j}]$ and $\mathbb{E}[R_{i,j}]$.

Thus, the capital gain of a security i which has been purchased or sold, $\mathbb{E}[u_{i,j}]$, if positive, will be given by the difference between the selling price $\mathbb{E}[R_{i,j}]$ and the purchasing price $C_{i,\bar{j}}$ plus all the dividends (interests) and minus all the paid fees (if any) $\mathbb{E}[D_{i,j}] - \mathbb{E}[P_{i,j}]$ while holding the title, that is:

$$\mathbb{E}[u_{i,j}] = \mathbb{E}[R_{i,j}] - C_{i,\bar{j}} + \sum_{\hat{j}=\bar{j}+1}^{j} \big(\mathbb{E}[D_{i,\hat{j}}] - \mathbb{E}[P_{i,\hat{j}}]\big),$$

where $\bar{j}$ and j indicate the purchase and selling time respectively, with $1 \le \bar{j} < j \le t$.

Some financial securities have a length which we denote by τ_i.
So $S = \bar{j} + \tau_i$, which means the time given by the purchasing period plus the length of the title, represents its expiration time.

In order to determine the optimal portfolio of securities, it is necessary to establish the time interval such that $t > \tau_i \quad \forall i = 1, \dots, n$. Therefore, if a title has not a pre-established length, we impose $\tau_i = t - 1$; in such a way, the time of their fictitious expiration coincides with t or will be greater than t.

If a financial security i has not been sold before it expires, the investor who owns this security, when it expires, will receive an amount equal to its expected nominal value $\mathbb{E}[N_{i,\bar{j}+\tau_i}]$ and will have to pay the tax on capital gain, $\mathbb{E}[g_{i,\bar{j}+\tau_i}]$, if positive, as in the case of sale:

$$\alpha_i^k \left(\frac{\left|\mathbb{E}[g_{i,\bar{j}+\tau_i}]\right| + \mathbb{E}[g_{i,\bar{j}+\tau_i}]}{2} \right),$$

where, in this case, $\mathbb{E}[g_{i,\bar{j}+\tau_i}] = \mathbb{E}[N_{i,\bar{j}+\tau_i}] - C_{i,\bar{j}} + \sum_{\hat{j}=\bar{j}+1}^{\bar{j}+\tau_i} \big(\mathbb{E}[D_{i,\hat{j}}] - \mathbb{E}[P_{i,\hat{j}}]\big)$.

If the expiration time of the unsold security exceeds t, or in the case of non-expiration securities, the investor at time t will own the security whose expected value is $\mathbb{E}[N_{i,t}]$.

Every investor k aims at determining the decision variables $x_{i,j}^k, y_{i,j}^k, z_i^b \in \{0,1\} \quad \forall i = 1, \dots, n, \quad \forall j = 1, \dots, t, \quad \forall b = 1, \dots, B$, which means deciding, at every time, which securities is convenient to buy and to sell and through which financial intermediary, in order to maximize the profit of each security, which is obtained by taking into account:

- the purchase cost and the commission to be given to the chosen financial intermediary (given by a percentage on the purchasing cost plus a fixed fee)

$$-C_{i,\bar{j}} - \sum_{b=1}^{B} z_i^b \cdot (\gamma_k^b C_{i,\bar{j}} + C_k^b),$$

if security i was purchased at time $\bar{j}$;

- if security i was purchased at time $\bar{j}$, the investor has the possibility to sell it at time j (with $\bar{j}+1 \leq j \leq \min\{\bar{j}+\tau_i, t\}$, where τ_i is the length of the title);
  if investor k sells his security, he will receive the selling price, but he will have to pay the tax on the capital gain and the commission to the chosen financial intermediary:

$$\mathbb{E}[R_{i,j}] - \alpha_i^k \left(\frac{|\mathbb{E}[u_{i,j}]| + \mathbb{E}[u_{i,j}]}{2} \right) - \sum_{b=1}^{B} z_i^b \cdot (\beta_k^b \mathbb{E}[R_{i,j}] + F_k^b);$$

- during the period of ownership of his security, investor k may receive dividends (or interests) and pay some amounts of money:

$$\sum_{j=\bar{j}+1}^{\min\{\bar{j}+\tau_i,t\}} \left(\mathbb{E}[-P_{i,j} + D_{i,j}] - y_{i,j}^k \sum_{\hat{j}=j+1}^{\min\{\bar{j}+\tau_i,t\}} \left(\mathbb{E}[-P_{i,\hat{j}} + D_{i,\hat{j}}] \right) \right);$$

- if financial security i has not been sold and
  - if the security expires before the final term t,
    then investor k, at time $\bar{j}+\tau_i$ receives the nominal value of the security and pays the tax α_i^k in the event that there is a positive capital gain

$$\sum_{\bar{j}=1}^{t-\tau_i-1} x_{i,\bar{j}}^k \Bigg[(1 - \sum_{j=\bar{j}+1}^{\bar{j}+\tau_i} y_{i,j}^k) \Bigg(\mathbb{E}[N_{i,\bar{j}+\tau_i}] - \alpha_i^k \left(\frac{\left|\mathbb{E}[g_{i,\bar{j}+\tau_i}]\right| + \mathbb{E}[g_{i,\bar{j}+\tau_i}]}{2} \right) \Bigg) \Bigg];$$

– if the expiration of this security is such that $S_i = \bar{j} + \tau_i \geq t$ or the title does not expire (we set $\tau_i = t-1 \Rightarrow \bar{j} + \tau_i = \bar{j} + t - 1 \geq t$), then investor k, at time t, holds a security that has a certain nominal value

$$\sum_{\bar{j}=t-\tau_i}^{t-1} x_{i,\bar{j}}^k \left[(1 - \sum_{j=\bar{j}+1}^{t} y_{i,j}^k) \mathbb{E}[N_{i,t}] \right].$$

Therefore, we are dealing with maximization of the expected gain of the portfolio:

$$\mathbb{E}[e_p^k] = \sum_{i=1}^{n} \sum_{\bar{j}=1}^{t-1} x_{i,\bar{j}}^k \mathbb{E}[e_{i,\bar{j}}^k] = \sum_{i=1}^{n} x_i^k \mathbb{E}[e_i^k],$$

namely:

$$\max \mathbb{E}[e_p^k] = \max \sum_{i=1}^{n} \sum_{\bar{j}=1}^{t-1} x_{i,\bar{j}}^k \mathbb{E}[e_{i,\bar{j}}^k]$$

$$= \max \sum_{i=1}^{n} \Bigg\{ \sum_{\bar{j}=1}^{t-1} x_{i,\bar{j}}^k \Bigg[- C_{i,\bar{j}} - \sum_{b=1}^{B} z_i^b \cdot (\gamma_k^b C_{i,\bar{j}} + C_k^b)$$

$$+ \sum_{j=\bar{j}+1}^{\min\{\bar{j}+\tau_i, t\}} \Bigg(\mathbb{E}[-P_{i,j} + D_{i,j}] + y_{i,j}^k \Bigg(\mathbb{E}[R_{i,j}] - \alpha_i^k \left(\frac{|\mathbb{E}[u_{i,j}]| + \mathbb{E}[u_{i,j}]}{2} \right)$$

$$- \sum_{b=1}^{B} z_i^b \cdot (\beta_k^b \mathbb{E}[R_{i,j}] + F_k^b) - \sum_{\hat{j}=j+1}^{\min\{\bar{j}+\tau_i, t\}} \big(\mathbb{E}[-P_{i,\hat{j}} + D_{i,\hat{j}}] \big) \Bigg) \Bigg) \Bigg]$$

$$+ \sum_{\bar{j}=1}^{t-\tau_i-1} x_{i,\bar{j}}^k \left[(1 - \sum_{j=\bar{j}+1}^{\bar{j}+\tau_i} y_{i,j}^k) \left(\mathbb{E}[N_{i,\bar{j}+\tau_i}] - \alpha_i^k \left(\frac{|\mathbb{E}[g_{i,\bar{j}+\tau_i}]| + \mathbb{E}[g_{i,\bar{j}+\tau_i}]}{2} \right) \right) \right]$$

$$+ \sum_{\bar{j}=t-\tau_i}^{t-1} x_{i,\bar{j}}^k \left[(1 - \sum_{j=\bar{j}+1}^{t} y_{i,j}^k) \mathbb{E}[N_{i,t}] \right] \Bigg\}.$$

Another objective of investor k is to minimize his risk portfolio. In [103] also Nagurney and Ke assumed that the decision-makers seek not only to increase their net revenues but also to minimize risk with the risk being considered as the possibility of suffering losses compared to the expected profit. It can be measured through the use of statistical indices such as the

variance or standard deviation of the asset's earnings distribution.
Given the aleatory gain on security i, e_i^k, the risk of the security, as a variance, will be given by:

$$(\sigma_i^k)^2 = \frac{\sum_{m=1}^{M}(e_m^k - \mathbb{E}[e_i^k])^2}{M-1}.$$

The risk on the portfolio is given by:

$$(\sigma_p^k)^2 = \sum_{i=1}^{n}(x_i^k)^2(\sigma_i^k)^2 + 2\sum_{i=1}^{n-1}\sum_{h>i}^{n} x_i^k x_h^k \sigma_{ih}^k,$$

where σ_{ih}^k is the covariance between securities i and h.

As it is well known, covariance lies in $]-\infty, +\infty[$, hence, it is often more useful to take into account correlation $\rho_{ih}^k = \frac{\sigma_{ih}^k}{\sigma_i^k \sigma_h^k}$ since it lies in $[-1, 1]$ and it measures the correlation or discrepancy between the gains of the securities i and h.

As a consequence, the minimization of the portfolio risk can be expressed as:

$$\min(\sigma_p^k)^2 = \min\left[\sum_{i=1}^{n}(x_i^k)^2(\sigma_i^k)^2 + 2\sum_{i=1}^{n-1}\sum_{h>i}^{n} x_i^k x_h^k \rho_{ih}^k \sigma_i^k \sigma_h^k\right].$$

The overall objective of investor k is to maximize his profit and, at the same time, to minimize his portfolio risk; therefore, we introduce the aversion degree or risk inclination, η_k, which depends on subjective evaluations of the single investor k and on the influences of the external environment that surrounds it (see [63]), and add the term

$$-\eta_k(\sigma_p^k)^2$$

to the objective function to be maximized, obtaining:

$$\max \sum_{i=1}^{n}\left\{\sum_{\bar{j}=1}^{t-1} x_{i,\bar{j}}^k \left[- C_{i,\bar{j}} - \sum_{b=1}^{B} z_i^b \cdot (\gamma_k^b C_{i,\bar{j}} + C_k^b)\right.\right.$$

$$
+ \sum_{j=\bar{j}+1}^{\min\{\bar{j}+\tau_i,t\}} \Bigg(\mathbb{E}[-P_{i,j} + D_{i,j}] + y_{i,j}^k \Big(\mathbb{E}[R_{i,j}] - \alpha_i^k \left(\frac{|\mathbb{E}[u_{i,j}]| + \mathbb{E}[u_{i,j}]}{2} \right)
$$

$$
- \sum_{b=1}^{B} z_i^b \cdot (\beta_k^b \mathbb{E}[R_{i,j}] + F_k^b) - \sum_{\hat{j}=j+1}^{\min\{\bar{j}+\tau_i,t\}} \left(\mathbb{E}[-P_{i,\hat{j}} + D_{i,\hat{j}}] \right) \Big) \Bigg) \Bigg]
$$

$$
+ \sum_{\bar{j}=1}^{t-\tau_i-1} x_{i,\bar{j}}^k \left[(1 - \sum_{j=\bar{j}+1}^{\bar{j}+\tau_i} y_{i,j}^k) \left(\mathbb{E}[N_{i,\bar{j}+\tau_i}] - \alpha_i^k \left(\frac{|\mathbb{E}[g_{i,\bar{j}+\tau_i}]| + \mathbb{E}[g_{i,\bar{j}+\tau_i}]}{2} \right) \right) \right]
$$

$$
+ \sum_{\bar{j}=t-\tau_i}^{t-1} x_{i,\bar{j}}^k \left[(1 - \sum_{j=\bar{j}+1}^{t} y_{i,j}^k) \mathbb{E}[N_{i,t}] \right] \Bigg\}
$$

$$
-\eta_k(\sigma_p^k)^2.
$$

The problem formulation is as follows:

$$
\begin{cases}
\max \mathbb{E}[e_p^k] - \eta_k(\sigma_p^k)^2 & (3.1) \\
\sum_{k=1}^{n} \sum_{j=1}^{t-1} x_{i,j}^k \leq 1 \quad \forall i = 1, \ldots, n & (3.2) \\
y_{i,j}^k \leq \sum_{\bar{j}=j-\tau_{i+1}}^{j-1} x_{i,\bar{j}}^k \quad \forall i = 1, \ldots, n, \quad \forall j = 2, \ldots, t & (3.3) \\
y_{i,j}^k \leq \dfrac{\sum_{\bar{j}=2}^{j-1} (1 - y_{i,\bar{j}}^k)}{j-2} \quad \forall i = 1, \ldots, n, \quad \forall j = 3, \ldots, t & (3.4) \\
(\sigma_{p_k})^2 \leq \bar{R}_k & (3.5) \\
\sum_{i=1}^{n} \sum_{j=1}^{t-1} x_{i,j}^k C_{i,j} \leq \bar{B}_k & (3.6) \\
\sum_{b=1}^{B} z_i^b = \sum_{j=1}^{t-1} x_{i,j}^k \quad \forall i = 1, \ldots, n & (3.7) \\
\sum_{\substack{\max\{j<\bar{j}:\\ D_{i,j}>0\}}}^{\bar{j}} \sum_{z\in\mathcal{A}_s} \sum_{k=1}^{K} (x_{z,j}^k - y_{z,j}^k) \geq 1 \quad \forall s \in \mathcal{S}, D_{i,\bar{j}} > 0 & (3.8) \\
x_{i,j}^k, y_{i,j}^k, z_i^b \in \{0,1\} & \\
\quad \forall i = 1, \ldots, n, \quad \forall j = 1, \ldots, t, \quad \forall b = 1, \ldots, B. & (3.9)
\end{cases}
$$

It is interesting to note that:
(3.2) means that it is possible to buy the same security only once and it can be purchased by a single investor (but there are numerous coincident securities);
(3.3) means that it is possible to sell a security only if it has been purchased previously and has not yet expired;
(3.4) means that you can sell a stock only if it has not yet been sold;
(3.5) means that there is a risk limit, $\bar{R}_k$, which represents the maximum risk limit that the investor is willing to accept;
(3.6) means that there is a budget limit, $\bar{B}_k$, which represents the maximum available budget for an investor;
(3.7) means that for each security, only one financial intermediary can be chosen for purchasing and selling activities;
(3.8) means that each issuer must sell at least one security during the dividend distribution periods, where the dividend $D_{i,\bar{j}}$ at time $\bar{j}$ of security $i \in \mathcal{A}_s$ is given by:

$$D_{i,\bar{j}} = \frac{U^s_{\bar{j}} - R^s_{\bar{j}}}{\displaystyle\sum_{\substack{\max\{j<\bar{j}:\\ D_{i,j}>0\}}}^{\bar{j}} \sum_{z\in\mathcal{A}_s} \sum_{k=1}^{K} (x^k_{z,j} - y^k_{z,j})}.$$

In some particular cases, additional constraints could be included in the model, such as for example:

- $\displaystyle\sum_{j=1}^{t} x^k_{i,j} = 1$, if security i must be purchased;
- $\displaystyle\sum_{j=1}^{t} (x^k_{i,j} + x^k_{h,j} + x^k_{w,j}) \leq 1$, if only one security among i, h and w can be purchased;
- $\displaystyle\sum_{j=1}^{t} (x^k_{i,j} + x^k_{h,j} + x^k_{w,j}) \geq 1$, if only one security among i, h and w must be purchased;

- $\sum_{j=1}^{t} x_{i,j}^{k} \leq \sum_{j=1}^{t} x_{h,j}^{k}$, if security i can be purchased only whether security h is purchased too;
- $x_{i,j}^{k} \leq \sum_{\bar{j}=1}^{j} x_{h,\bar{j}}^{k}, \quad \forall j = 1, \ldots, t$, if security i can be purchased only whether h has already been purchased too;
- $x_{i,j}^{k} \leq \prod_{\bar{j}=1}^{j} (1 - x_{h,\bar{j}}^{k}), \quad \forall j = 1, \ldots, t$, if security i can be purchased only whether security h has not been yet purchased;
- $\sum_{j=1}^{t} x_{i,j}^{k} \leq \frac{1}{2} \sum_{j=1}^{t} \left(x_{h,j}^{k} + x_{w,j}^{k} \right)$ if security i can be purchased only if h and w are purchased too.

Now, we consider the continuous relaxation of problem (3.1)-(3.9) related to the binary variables which can be obtained with constraints (3.17) together with (3.18):

$$
\begin{cases}
\max \mathbb{E}[e_p^k] - \eta_k(\sigma_p^k)^2 \\
\sum_{k=1}^{n}\sum_{j=1}^{t-1} x_{i,j}^k \leq 1 \quad \forall i = 1,\ldots,n & (3.10) \\
y_{i,j}^k \leq \sum_{\bar{j}=j-\tau_i+1}^{j-1} x_{i,\bar{j}}^k \quad \forall i = 1,\ldots,n, \quad \forall j = 2,\ldots,t & (3.11) \\
y_{i,j}^k \leq \frac{\sum_{\bar{j}=2}^{j-1}(1-y_{i,\bar{j}}^k)}{j-2} \quad \forall i = 1,\ldots,n, \quad \forall j = 3,\ldots,t & (3.12) \\
(\sigma_{p_k})^2 \leq \bar{R}_k & (3.13) \\
\sum_{i=1}^{n}\sum_{j=1}^{t-1} x_{i,j}^k C_{i,j} \leq \bar{B}_k & (3.14) \\
\sum_{b=1}^{B} z_i^b = \sum_{j=1}^{t-1} x_{i,j}^k \quad \forall i = 1,\ldots,n & (3.15) \\
\sum_{\substack{\max\{j<\bar{j}:\\ D_{i,j}>0\}}}^{\bar{j}} \sum_{z\in\mathcal{A}_s}\sum_{k=1}^{K}(x_{z,j}^k - y_{z,j}^k) \geq 1 \quad \forall s \in \mathcal{S}, D_{i,\bar{j}} > 0 & (3.16) \\
\sum_{i=1}^{n}\sum_{j=1}^{t-1} x_{i,j}^k(1-x_{i,j}^k) + \sum_{i=1}^{n}\sum_{j=2}^{t} y_{i,j}^k(1-y_{i,j}^k) + \sum_{i=1}^{n}\sum_{b=1}^{B} z_i^b(1-z_i^b) \leq 0 & (3.17) \\
x_{i,j}^k, y_{i,j}^k, z_i^b \in [0,1] \\
\qquad \forall i = 1,\ldots,n, \quad \forall j = 1,\ldots,t, \quad \forall b = 1,\ldots,B. & (3.18)
\end{cases}
$$

Now, we group the variables $x_{i,\bar{j}}^k, \quad i = 1, \ldots, n, \ \bar{j} = 1,\ldots,t-1, \ k = 1,\ldots,K$ into the vector $\mathbf{x} \in [0,1]^{n(t-1)K}$, the variables $y_{i,j}^k, \quad i = 1, \ldots, n, j = 2,\ldots,t, \ k = 1,\ldots,K$ into the vector $\mathbf{y} \in [0,1]^{n(t-1)K}$ and the variables $z_i^b, \quad i = 1, \ldots, n, \ b = 1,\ldots,B$ into the vector $\mathbf{z} \in [0,1]^{nB}$.

3.3 Financial Model with short selling and transfer of securities

In this section we consider the previous financial network and we make the previous multi-period model (3.1-3.9) more realistic by adding not only transaction costs, taxes (on the capital gain) and time-length for some financial securities, but also the short selling and the transfer of financial assets. We take into account the previous binary variables $x^k_{i,j}$ and $y^k_{i,j}$, but we introduce the new following ones:

$$z^b_{1i} = \begin{cases} 1 & \text{if security } i \text{ is purchased by } b \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \dots, n, \quad \forall b = 1, \dots, B;$$

$$z^b_{2i} = \begin{cases} 1 & \text{if security } i \text{ is sold by } b \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \dots, n, \quad \forall b = 1, \dots, B.$$

Further, in order to take into account the short selling, we introduce the following binary variables:

$$w^k_{i,j} = \begin{cases} 1 & \text{if security } i \text{ is purchased by } k \text{ at time } j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \dots, n, \quad \forall j = 2, \dots, t, \quad \forall k = 1, \dots, K;$$

$$h^k_{i,j} = \begin{cases} 1 & \text{if security } i \text{ is sold by } k \text{ at time } j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \dots, n, \quad \forall j = 1, \dots, t-1, \quad \forall k = 1, \dots, K.$$

Let also T^b_i denote the financial title transfer fee, M_b the maximum time limit, fixed by financial intermediary b, within which investor k is obliged to short covering (namely, he has to buy the not owned securities), $\mathbb{E}[p_{i,\bar{j}}] = \mathbb{E}[R_{i,j}] - \mathbb{E}[C_{i,\bar{j}}]$ the new capital gain, $I^b_i(\bar{j} - j)$ the interest, which is a function of time, to be paid to the broker who lends the security which has to be sold in the short selling.

Then, the objective function to maximize is as follows:

$$\mathbb{E}[e^k_p] - \eta_k(\sigma^k_p)^2 = \sum_{i=1}^{n} \Bigg\{ \sum_{\bar{j}=1}^{t-1} x^k_{i,\bar{j}} \Bigg[- C_{i,\bar{j}} - \sum_{b=1}^{B} z^b_{1i} \cdot (\gamma^b_k C_{i,\bar{j}} + C^b_k)$$

$$
\begin{aligned}
&+ \sum_{j=\bar{j}+1}^{\min\{\bar{j}+\tau_i,t\}} \Bigg(\mathbb{E}[-P_{i,j} + D_{i,j}] + y_{i,j}^k \Big(\mathbb{E}[R_{i,j}] - \alpha_i^k \left(\frac{|\mathbb{E}[u_{i,j}]| + \mathbb{E}[u_{i,j}]}{2} \right) \\
&\quad - \sum_{b=1}^{B} z_{2i}^b \cdot (\beta_k^b \mathbb{E}[R_{i,j}] + F_k^b) - \sum_{\hat{j}=j+1}^{\min\{\bar{j}+\tau_i,t\}} \big(\mathbb{E}[-P_{i,\hat{j}} + D_{i,\hat{j}}] \big) \Big) \Bigg) \Bigg] \\
&+ \sum_{\bar{j}=1}^{t-\tau_i-1} x_{i,\bar{j}}^k \Bigg[(1 - \sum_{j=\bar{j}+1}^{\bar{j}+\tau_i} y_{i,j}^k) \left(\mathbb{E}[N_{i,\bar{j}+\tau_i}] - \alpha_i^k \left(\frac{|\mathbb{E}[g_{i,\bar{j}+\tau_i}]| + \mathbb{E}[g_{i,\bar{j}+\tau_i}]}{2} \right) \right) \Bigg] \\
&\quad + \sum_{\bar{j}=t-\tau_i}^{t-1} x_{i,\bar{j}}^k \Bigg[(1 - \sum_{j=\bar{j}+1}^{t} y_{i,j}^k) \mathbb{E}[N_{i,t}] \Bigg] - \sum_{b=1}^{B} \Big(z_{1i}^b \cdot \sum_{\bar{b} \neq b} z_{2i}^{\bar{b}} \cdot T_i^b \Big) \\
&+ \sum_{j=1}^{t-1} h_{i,j}^k \Bigg[\mathbb{E}[R_{i,j}] + \sum_{b=1}^{B} z_{1i}^b \Bigg[- (\beta_k^b \mathbb{E}[R_{i,j}] + F_k^b) - \sum_{\bar{j}=j+1}^{\min\{j+M_b,t\}} w_{i,\bar{j}}^k \Big(\mathbb{E}[C_{i,\bar{j}}] \\
&+ (\gamma_k^b \mathbb{E}[C_{i,\bar{j}}] + C_k^b) + I_b(\bar{j} - j) + \alpha_i^k \left(\frac{|\mathbb{E}[p_{i,\bar{j}}]| + \mathbb{E}[p_{i,\bar{j}}]}{2} \right) \Big) \Bigg] \Bigg] \Bigg\} - \eta_k (\sigma_p^k)^2.
\end{aligned}
$$

The problem formulation is as follows:

$$\max \left\{\mathbb{E}[e_p^k] - \eta_k(\sigma_p^k)^2\right\} \tag{3.19}$$

$$\sum_{k=1}^{n}\sum_{j=1}^{t-1} x_{i,j}^k \le 1 \quad \forall i = 1,\dots,n \tag{3.20}$$

$$y_{i,j}^k \le \sum_{\bar{j}=j-\tau_i+1}^{j-1} x_{i,\bar{j}}^k \quad \forall i = 1,\dots,n, \quad \forall j = 2,\dots,t \tag{3.21}$$

$$y_{i,j}^k \le \frac{\sum_{\bar{j}=2}^{j-1}(1 - y_{i,\bar{j}}^k)}{j-2} \quad \forall i = 1,\dots,n, \quad \forall j = 3,\dots,t \tag{3.22}$$

$$(\sigma_{p_k})^2 \le \bar{R}_k \tag{3.23}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{t-1} x_{i,j}^k C_{i,j} \le \bar{B}_k \tag{3.24}$$

$$\sum_{b=1}^{B} z_{1i}^b = \sum_{j=1}^{t-1} x_{i,j}^k \quad \forall i = 1,\dots,n \tag{3.25}$$

$$\sum_{b=1}^{B} z_{2i}^b = \sum_{\bar{j}=2}^{t} y_{i,\bar{j}}^k \quad \forall i = 1,\dots,n \tag{3.26}$$

$$\sum_{\substack{\max\{j<\bar{j}:\\ D_{i,j}>0\}}}^{\bar{j}} \sum_{z\in\mathcal{A}_s}\sum_{k=1}^{K}(x_{z,j}^k - y_{z,j}^k) \ge 1 \quad \forall s \in \mathcal{S}, D_{i,\bar{j}} > 0 \tag{3.27}$$

$$\sum_{\bar{j}=j+1}^{\min\{j+M_b,t\}} w_{i,\bar{j}}^k = h_{i,j}^k \quad \forall i = 1,\dots,n \quad \forall j = 1,\dots,t-1 \tag{3.28}$$

$$\sum_{k=1}^{K}\sum_{j=1}^{t-1} h_{i,j}^k \le 1 \quad \forall i = 1,\dots,n \tag{3.29}$$

$$\sum_{\bar{j}=1}^{j} x_{i,\bar{j}}^k \Big(\sum_{\hat{j}=\bar{j}+1}^{j} \Big(1 - y_{i,\hat{j}}^k\Big)\Big) \le 1 - h_{i,j}^k \quad \forall i = 1,\dots,n \quad \forall j = 1,\dots,t-1 \tag{3.30}$$

$$x_{i,j}^k, y_{i,j}^k, z_{1i}^b, z_{2i}^b, h_{i,j}^k, w_{i,j}^k \in \{0,1\}$$
$$\forall i = 1,\dots,n, \quad \forall j = 1,\dots,t, \quad \forall b = 1,\dots,B. \tag{3.31}$$

We specify that the meaning of constraints (3.25) and (3.26) is that for each security, only one financial intermediary can be chosen for purchasing and selling activities. Constraint (3.28) means that if investor k sells the title i in the short selling market, he is obliged to buy back the same security within the time established by the financial intermediary. Constraint (3.29) states that it is possible to sell in the short selling market the same security more than once. Finally, constraint (3.30) affirms that investor k cannot sell security i in the short selling market, if he owns security i.

3.4 Numerical Examples

In this section we apply the model to some numerical examples that consist of a financial network with two issuers, two financial securities and an investor, as depicted in Figure 2.
We consider also two financial intermediaries and we analyze the model in the following time horizon: $1, \ldots, 5$.
Since we want to report all the results for transparency purposes, we select

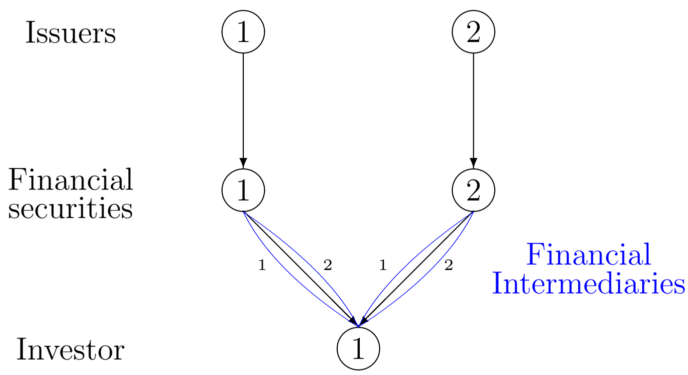

Figure 3.2: Network Topology for the Numerical Example

the size of problems as reported. The numerical data are inspired by realistic values and are constructed for easy interpretation purposes.
To solve the examples we used Matlab on a laptop with an Intel Core2 Duo processor and 4 GB RAM.

3.4.1 Examples: basic model

We assume the following data are given:

C_{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	5	6	6	7
$i=2$	7	7	8	9

$\mathbb{E}[-P_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0	0	-2	0
$i=2$	0	0	0	0

$\mathbb{E}[U_{ij}-R_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0	3	0	15
$i=2$	0	0	0	0

$\mathbb{E}[R_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	7	7	8	1
$i=2$	20	20	20	1

We also assume that $\tau_1 = 2$ and that the second financial security does not expire, so we require $\tau_2 = 4$. Further, we assume that the nominal value of each security at maturity or at time $j = 5$ coincides with its current value (cost) $\mathbb{E}[N_{i,\bar{j}+\tau_i}] = C_{i,\bar{j}+\tau_i}$, $\mathbb{E}[N_{i,5}] = C_{i,4}$, that the maximum budget and risk values are $\bar{B} = 25$ and $\bar{R} = 15$ respectively, that the percentages of taxation are $\alpha_1 = 15\%$ and $\alpha_2 = 10\%$, that commission costs are given by $\beta^1 = \gamma^1 = 5\%$, $\beta^2 = \gamma^2 = 15\%$, $C^1 = F^1 = 0.5$ and $C^2 = F^2 = 2$, that $\eta = 0.2$ is the risk aversion index, $(\sigma_{1j}) = (2,2,2,2,2)$, $(\sigma_{2j}) = (1,1,1,1,1)$ the variances of the titles and $\rho_{12j} = 0\ \forall j = 1,\dots,5$ meaning that the two titles are completely unrelated.

The optimal solutions are calculated by solving the optimization problem, the calculations are performed using the Matlab program.
We get the following optimal solutions:

$$x^*_{14} = x^*_{21} = 1; \qquad x^*_{1j} = 0\ \forall j = 1,2,3;\ x^*_{2j} = 0\ \forall j = 2,3,4;$$

$$y^*_{1j} = 0\ \forall j = 2,3,4,5; \quad y^*_{22} = 1,\ y^*_{2j} = 0\ \forall j = 3,4,5;$$

$$z_1^{1*} = z_2^{1*} = 1, \quad z_1^{2*} = z_2^{2*} = 0.$$

These optimal solutions clearly show that the most convenient choice for the investor is to buy security 1 at time 4 and security 2 at time 1, to sell security 2 at time 2 but never sell security 1.
For both securities, it is better to choose the financial intermediary 1, reaching 22.5 as the total gain.

Now we consider a second example where the investor has a greater degree of risk aversion than the previous one and the variance of the securities is greater. We suppose, in this case, that $\eta = 0.9$ and $(\sigma_{1j}) = (4,4,3,3,2)$, $(\sigma_{2j}) = (2,2,1,1,1)$.
Then, we get the following optimal solutions:

$$x^*_{1j} = 0\ \forall j = 1,2,3,4; \qquad y^*_{1j} = 0\ \forall j = 2,3,4,5;$$

$$z_1^{b*} = 0\ \forall b = 1,2;$$

$$x^*_{21} = 1; \quad x^*_{2j} = 0\ \forall j = 2,3,4;$$

$$y^*_{23} = 1, \quad y^*_{2j} = 0\ \forall j = 2,4,5;$$

$$z_2^{1*} = 1, \quad z_2^{2*} = 0.$$

Therefore, in this case, it is never convenient for the investor to buy security 1, but to buy security 2 at time 1 and sell it at time 3, through the financial intermediary 1, thus obtaining a profit of 5.75.

A third example refers to the case when the degree of risk aversion of the investor is $\eta = 0.2$, as in the first example, but the maximum risk is smaller, that is 4.
In this case we get the following optimal solutions:

$$x^*_{1j} = 0 \ \forall j = 1, 2, 3, 4; \qquad y^*_{1j} = 0 \ \forall j = 2, 3, 4, 5;$$

$$z^{b*}_1 = 0 \ \forall b = 1, 2;$$

$$x^*_{21} = 1; \quad x^*_{2j} = 0 \ \forall j = 2, 3, 4;$$

$$y^*_{23} = 1, \quad y^*_{2j} = 0 \ \forall j = 2, 4, 5;$$

$$z^{1*}_2 = 1, \quad z^{2*}_2 = 0.$$

We remark that such solutions are the same as the ones of the second example, but the total profit is now 8.55.
It is worth observing that not all the total budget is used.

3.4.2 Example: model with short selling and transfer of securities

We assume the following data are given:

C_{ij}	$j=1$	$j=2$	$j=3$	$j=4$
$i=1$	5	6	6	7
$i=2$	9	9	8	9

$\mathbb{E}[R_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	7	7	8	1
$i=2$	20	15	10	1

$\mathbb{E}[-P_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0	0	-2	0
$i=2$	0	0	0	0

$\mathbb{E}[U_{ij}-R_{ij}]$	$j=2$	$j=3$	$j=4$	$j=5$
$i=1$	0	3	0	15
$i=2$	0	0	0	0

We also assume that $\tau_1 = 2$ and that the second financial security does not expire, so we require $\tau_2 = 4$. Further, we assume that the nominal value of each security at maturity or at time $j = 5$ coincides with its current value (cost) $\mathbb{E}[N_{i,\bar{j}+\tau_i}] = C_{i,\bar{j}+\tau_i}$, $\mathbb{E}[N_{i,5}] = C_{i,4}$, that the maximum budget and risk values are $\bar{B} = 25$ and $\bar{R} = 15$ respectively, that the percentages of taxation are $\alpha_1 = 15\%$ and $\alpha_2 = 10\%$, that commission costs are given by $\beta^1 = \gamma^1 = 5\%$, $\beta^2 = \gamma^2 = 15\%$, $C^1 = F^1 = 0.5$ and $C^2 = F^2 = 2$, that $\eta = 0.2$ is the risk aversion index, $(\sigma_{1j}) = (2, 2, 2, 2, 2)$, $(\sigma_{2j}) = (1, 1, 1, 1, 1)$ the variances of the titles and $\rho_{12j} = 0\ \forall j = 1, \dots, 5$ meaning that the two titles are completely unrelated.

We get the following optimal solutions:

$$x^*_{14} = 1; \qquad x^*_{1j} = 0\ \forall j = 1, 2, 3;\ x^*_{2j} = 0\ \forall j = 1, 2, 3, 4;$$

$$y^*_{1j} = y^*_{2j} = 0 \quad \forall j = 2, 3, 4, 5;$$

$$h^*_{1j} = 0\ \forall j = 1, 2, 3, 4; \quad h^*_{22} = 1, \quad h^*_{2j} = 0\ \forall j = 1, 3, 4;$$

$$w^*_{1j} = 0\ \forall j = 2, 3, 4, 5; \quad w^*_{24} = 1, \quad w^*_{2j} = 0\ \forall j = 2, 3, 5;$$

$$z^{1*}_{11} = z^{1*}_{12} = z^{1*}_{22} = 1, \quad z^{2*}_{11} = z^{2*}_{12} = z^{1*}_{21} = z^{2*}_{21} = z^{2*}_{22} = 0.$$

These optimal solutions clearly show that the most convenient choice for the investor is to buy security 1 at time 4 and never sell it; using the short selling, to sell security 2 at time 2 and to buy it at time 4, through the financial intermediary 1.
In this case the total gain is: 21.20.

If, now, we assume that the commission costs are given by $\beta^1 = \gamma^2 = 5\%$, $\beta^2 = \gamma^1 = 15\%$, $C^1 = F^2 = 0.5$ and $C^2 = F^1 = 2$, we see that it is more convenient to choose financial intermediary 1 for the selling and, after a transfer of security 2, financial intermediary 2 for the buying. In this case the total gain is: 19.7.

3.5 Conclusions

In this chapter, we focused our attention on an important problem which is studied by many researchers, namely the Portfolio Optimization problem. Specifically, the Markovitz's portfolio theory is reviewed for investors with long-term horizons.

We presented a financial model, taking into account that in financial markets buying and selling securities entail brokerage fees and sometimes lump sum taxes are imposed on the investors.
Furthermore, in this work, we imposed that the used resources are not greater than the available ones, making the model more realistic.

Therefore, the objective of this chapter was to formulate the multi-period portfolio selection problem as a Markowitz mean-variance optimization problem firstly with the addition of transaction costs and taxes (on the capital gain) and then also taking into account the short selling and the transfer of financial assets.
The presented financial models determine which securities every investor has to buy and sell, which financial intermediary he has to choose and at what time it is more convenient to buy and sell a security in order to maximize his

own profit and minimize his own risk.
For every security, we assumed that there is a purchase cost and it is also necessary to pay a commission to the chosen financial intermediary (often the banks), which consists of a percentage of the purchase cost, and a flat fee. We also assumed that during the ownership time of the security, it is possible to obtain funds (such as dividends in the case of shares, interests in the case of bonds) or pay money (for example in the case of an increase in the corporate capital).
Furthermore, each investor has the opportunity to sell his own securities and, in this case, he will receive a sum, but he will have to pay a charge to the chosen financial intermediary and a taxation on the capital gain or a percentage on the gain obtained from the title.

In this chapter we also took into account that some financial securities have a length or a deadline.
Therefore, we supposed that if the security expires before the final term, then the investor receives the nominal value of the security and pays the tax in the event that there is a positive capital gain.

In the second part of this chapter, inspired by reality, we introduced short selling or financial transactions that consist in the sale of non-owned financial instruments with subsequent repurchase. In addition, we examined the case of transfer of financial assets and, finally, we studied some numerical examples.
In a future work we intend to continue the study of this topic and, in particular, we could analyze the behavior of investors in the presence of the secondary market.
The results in this work add to the growing literature of operations research techniques for portfolio optimization modeling and analysis.

Chapter 4

A Convex Optimization Model for Business Management

4.1 Introduction

A supply chain is a set of activities that includes purchasing, manufacturing, logistics, distribution, marketing (see [113] and [27]). Since the interest in sustainable economics has been growing in the last two decades, the theory of closed-loop supply chains has been deeply developed. Closed-loop supply chains are designed and managed to explicitly consider the reverse and forward supply chain activities over the entire life cycle of the product. A traditional supply chain aims to lower the cost and to maximize the economic benefits. A closed-loop supply chain also seeks to maximize economic benefits, to decrease the consumption of resources and energy and to reduce the emissions of pollutants, all in an effort to create a socially responsible environment, and to balance the economic, social and environmental effects. The traditional supply chain starts with suppliers and ends with users. In the closed-loop supply chain thinking, product flow is circular and reversible and all products must be managed throughout the entire life cycle, and beyond so that waste finds a second life or becomes raw material available for new production or other purposes (see [131]). Closed-loop supply chains have been deeply studied in the case of electronic waste (see [107] and the references therein, [135], [113]), but we shall adapt them to an agribusiness company.

In this chapter, we present a general supply chain network model with four different tiers of decision makers, represented by suppliers of raw materials, manufacturers, retailers, demand markets. In particular, we study a more comprehensive and extensive model than the one in [102] and [107], since we complete the forward chain by adding a reverse chain model where manufacturers, using the unsold product given back from retailers, after reworking, produce a new commodity which will be sold to new retailers (some of them could be the same as the retailers in the forward chain). Moreover, in this new structure we also include electronic commerce. E-commerce has had an enormous effect on the manner in which businesses as well as consumers order goods and have them transported. The primary benefit of the Internet for business is its open access to potential suppliers and customers both within a particular country and beyond national boundaries. Consumers, on the other hand, may obtain goods which they physically could not locate otherwise. In order to derive the optimality conditions for the typical manufacturer and the subsequent variational inequality problem governing the equilibrium of all manufacturers simultaneously, we need to guarantee the convexity of the constraint set and the convexity of the cost functions. An existence result, based on the classical theory due to Stampacchia, is stated.

Finally, we apply our model to a concrete company (Valle del Dittaino, Italy), obtaining, after introducing additional suitable constraints, the optimal amount of raw material, the optimal shipment of new product as well as the optimal production periods.

The purpose of this chapter is to analyze the multiple features of business management by improving existing networks models, where only a few aspects have been taken under consideration.

The chapter is organized as follows. In Section 4.2 we present a model where we allow for a distinction (by brand, small changes, ...) between a product of a manufacturer and the same good produced by other manufacturers. We introduce the reverse logistics and e-commerce and study the optimality conditions of the manufacturer. In Section 4.3 we apply our model to a well-known agribusiness company, the Cooperativa Agricola Valle del Dittaino (www.pandittaino.it), which is located in the heart of Sicily, Italy. Taking into account the company reality, we add some suitable constraints related to the capacity of a single tray and their number, as well as the production time. After analyzing the optimization problem and determining the sales forecast and the production periods, while minimizing the production and the storing costs, we obtain the optimal amount of product that all retailers must sell to all demand markets. We also establish, for every day, the amount of raw materials needed for the products. Section 4.4 is dedicated to the conclusions.

4.2 The Model

We consider a supply chain network with four tiers of decision-makers:

- H suppliers of raw materials;
- G manufacturers, who produce a new product using raw materials;
- S retailers;
- K demand markets.

We assume there are also V carriers, who are not decision-makers in our network since they do not decide on the transactions, but only make the shipment to retailers. Their role will be represented by parallel links connecting the nodes of the second tier with the ones of the third. Moreover, we shall add a new arc representing the possibility for retailers to buy directly the products, using no carriers. Further, it is also allowed to consumers at demand markets to buy the products directly from manufacturers.

Let L be the number of raw materials, I the number of new products. We assume suppliers are spatially distributed in different parts of the world and can be of different types (private suppliers, companies, institutions, local governments, ...).

The framework of the network is depicted in Fig. 4.1: the first tier represents the combination between suppliers and the kind of raw material (see [88]), the second one is the combination between manufacturers and products, in the third tier we represent retailers and, finally, in the last level we depict the demand markets.

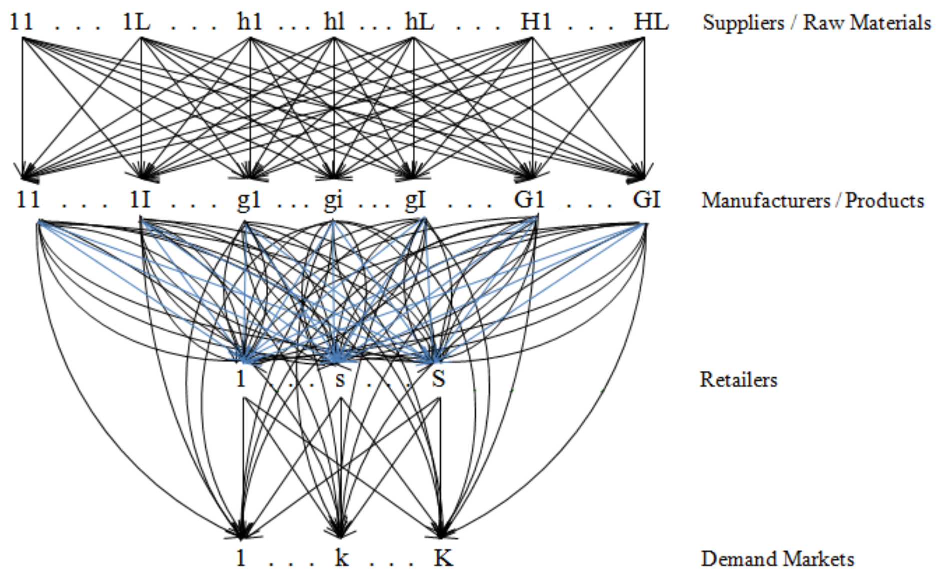

Figure 4.1: Network topology

The aim of suppliers, manufacturers and retailers is to maximize their own profit, while consumers at demand markets aim at minimizing their expenses.

In particular, in this section we shall focus only on the behavior of manufacturers. A detailed presentation of the optimality conditions and the characterization by means of variational inequality problems can be found in [27], [88], [102], [107].

4.2.1 Supply chain network with e-commerce and excesses

We allow there can be a distinction (by brand, small changes, ...) between the product i of manufacturer g and the same good produced by other manufacturers. We shall find the optimality conditions of manufacturer g, who purchases raw materials from suppliers, and, after the production process, sells the new product to retailers (with or without carriers) both through traditional links or via Internet and to demand markets just via Internet (*forward chain*). Moreover, the same manufacturer receives the products unsold by all retailers and, after reworking, sells the new products (which generally differ from the previous ones) to new buyers (*reverse chain*).

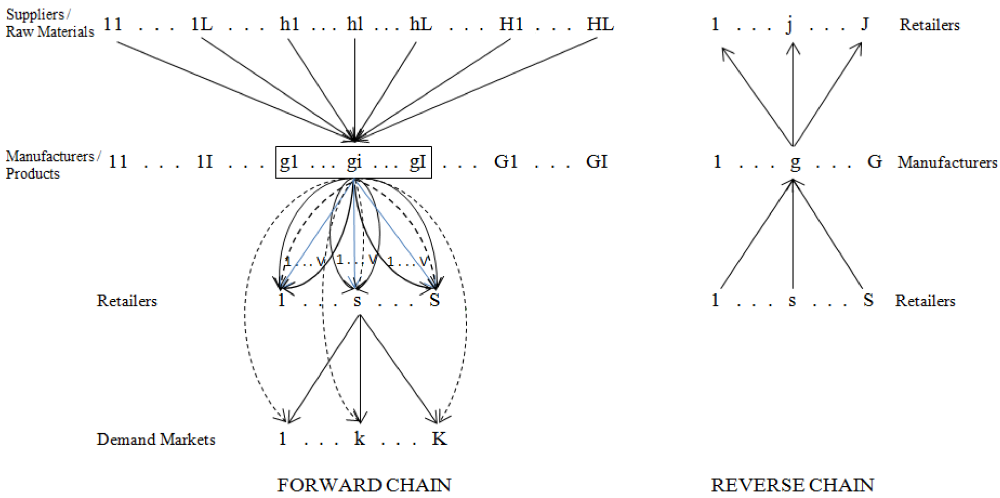

Figure 4.2: Network with e-commerce and reverse logistics

The topology of the network is depicted in Fig. 4.2. We use the following variables:

- $q_{hlgi} \geq 0$ be the amount of raw material l shipped from supplier h to manufacturer g for the product i and let us group all these quantities into the vector $Q_0 = (q_{hlgi})_{\substack{h=1,\dots,H,\, l=1,\dots,L \\ g=1,\dots,G\, i=1,\dots,I}} \in \mathbb{R}_+^{HLGI}$;
- $q^v_{gise} \geq 0$ be the amount of product i sold by manufacturer g to retailer

s directly or throught the carrier v in e mode, where:

- $e = 1$ means physical link,
- $e = 2$ means Internet link

and let us group such quantities into the vector

$$Q_1 = (q^v_{gise})_{\substack{g=1,\dots,G\; i=1,\dots,I \\ s=1,\dots,S\; e=1,2\; v=1,\dots,V+1}} \in \mathbb{R}^{GIS2(V+1)}_+;$$

- $q_{gik} \geq 0$ be the amount of product i sold by manufacturer g to demand market k and let us group such quantities into the vector

$$Q_2 = (q_{gik})_{\substack{g=1,\dots,G, \\ i=1,\dots,I,\; k=1,\dots,K}} \in \mathbb{R}^{GIK}_+;$$

- q^{gi}_{sk} be the amount of product i, produced by manufacturer g, shipped from retailer s to demand market k and let us group such quantities into the vector

$$Q_4 = (q^{gi}_{sk})_{\substack{s=1,\dots,S,\; k=1,\dots,K \\ g=1,\dots,G,\; i=1,\dots,I}} \in \mathbb{R}^{SKGI}_+;$$

When we are in the presence of supply excesses, we remark that the amount of product i bought by retailer s is greater than or equal to the sold amount, that is:

$$\sum_{v=1}^{V+1} \sum_{e=1}^{2} q^v_{gise} \geq \sum_{k=1}^{K} q^{gi}_{sk} \qquad \forall i = 1, \dots, I, \forall s = 1, \dots, S.$$

Now, we present the parameters. Let:

- c_{hlgi} be the transaction cost of the materials from supplier h to manufacturer g for the product i (such costs could also be the same for every i), and let us assume c_{hlgi} is a function of $q_{hg} = (q_{hlgi})_{\substack{l=1,\dots,L \\ i=1,\dots,I}} \in \mathbb{R}^{LI}_+$:

$$c_{hlgi} = c_{hlgi}(q_{hg}) \quad \forall h = 1, \dots, H;\ \forall l = 1, \dots, L;$$
$$\forall g = 1, \dots, G;\ \forall i = 1, \dots, I;$$

- ρ^*_{0hlgi} be the price of raw material l in the transaction between supplier h and manufacturer g;

- c^v_{gise} be the transaction costs of product i from manufacturer g to retailer s directly or through the carrier v in e mode and let us assume c^v_{gise} is a function of q^v_{gise}:

$$c^v_{gise} = c^v_{gise}(q^v_{gise}),\ \forall g = 1, \ldots, G,\ \forall i = 1, \ldots, I,\ \forall s = 1, \ldots, S,$$
$$\forall e = 1, 2,\ \forall v = 1, \ldots, V + 1;$$

- ρ^{v*}_{1gise} be the price of product i charged by g in the transaction with s in e mode, directly or through v;

- c_{gik} be the transaction costs of product i from manufacturer g to demand market k and let us assume c_{gik} is a function of q_{gik}:

$$c_{gik} = c_{gik}(q_{gik})\ \forall g = 1, \ldots, G,\ \forall i = 1, \ldots, I,\ \forall k = 1, \ldots, K;$$

- ρ^*_{1gik} be the price of product i charged by g in the transaction with k;

- c^i_g be the handling cost, associated to product i, of manufacturer g and let us assume:

$$c^i_g = c^i_g(Q_0, Q_1, Q_2) \quad \forall g = 1, \ldots, G,\ \forall i = 1, \ldots, I;$$

- C^i be the fix costs, associated to product i, of manufacturer g, which do not depend on the production, $\forall i = 1, \ldots, I$;

- p_{gi} be the production cost of i and let us assume:

$$p_{gi} = p_{gi}(Q_1, Q_2) \quad \forall g = 1, \ldots, G,\ \forall i = 1, \ldots, I;$$

- α^{gi}_{hl} be the transformation rate of raw material l bought by g from supplier h to produce one unit of i;

- β_{li} be the portion of raw material l which is transformed in one unit of raw material needed to produce i;

- τ be the unit tax charged to manufacturer g by the authorities in order to produce a certain amount of product; so, the total taxes are given by:
$$\tau\Big[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2} q_{gise}^{v} + \sum_{k=1}^{k} q_{gik}\Big];$$

- $r_{isg}^{(v)}$ be the amount of unsold product i that retailer s gives back to manufacturer g, and let us assume $r_{isg}^{(v)}$ is a function of Q_1 and Q_4:
$$r_{isg}^{(v)} = r_{isg}^{(v)}(Q_1, Q_4);$$
specifically the following estimate holds:
$$r_{isg}^{(v)} = \sum_{v=1}^{V+1}\sum_{e=1}^{2} q_{gise}^{v} - \sum_{k=1}^{K} q_{sk}^{gi}$$
$\forall i = 1, \ldots, I,\ \forall s = 1, \ldots, S,\ \forall g = 1, \ldots, G,$
which means that $r_{isg}^{(v)}$ is given by the difference between the shipment that retailers purchase from manufacturers and the shipment from retailers to all demand markets;

- $\hat{c}_{isg}^{(v)}$ be the transaction cost of product i from retailer s to manufacturer g, via the chosen carrier v, and let us assume $\hat{c}_{isg}^{(v)}$ is a function of $r_{isg}^{(v)}$:
$$\hat{c}_{isg}^{(v)} = \hat{c}_{isg}^{(v)}(r_{isg}^{(v)}) = \hat{c}_{isg}^{(v)}(Q_1, Q_4)$$
$$\forall i = 1, \ldots, I,\ \forall s = 1, \ldots, S,\ \forall g = 1, \ldots, G;$$

- $\hat{p}_{gi}$ be the unit reworking cost of the unsold product i of manufacturer g in order to get a new product;

- γ_i be the transformation rate of unsold product i needed to produce one unit of a new product;

- β_i be the portion of unsold product i which can be used to produce a new product;

- $\hat{q}_{gj}$ be the amount of new product sold by manufacturer g to retailer j and let us assume that the shipment of new products sold by g to all retailers is the same as the quantity of new product obtained by the goods given back by all retailers:
$$\sum_{j=1}^{J} \hat{q}_{gj} = \sum_{i=1}^{I} \gamma_i \beta_i \sum_{s=1}^{S} r_{isg}^{(v)} = \sum_{i=1}^{I} \gamma_i \beta_i \Big(\sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} - \sum_{s=1}^{S} \sum_{k=1}^{K} q_{sk}^{gi} \Big);$$
hence, $\hat{q}_{gj}$ is a combination of Q_1 and Q_4.

  We remark that, in order to obtain the actual amount of new product, we have to sum the return and the scrap (caused by a wrong cut, a wrong leavening, a wrong cooking, ...). Since, in our case, the scrap is in a small amount, it will be neglected in this discussion.

- $\hat{\rho}_{1gj}^{*}$ be the price charged to the new product by g in the transaction with j;

- $\hat{c}_{gj}$ be the transaction cost of the new product from manufacturer g to retailer j and let us assume $\hat{c}_{gj}$ is a function of $\hat{q}_{gj}$, that is Q_1 and Q_4:
$$\hat{c}_{gj} = \hat{c}_{gj}(Q_1, Q_4) \; \forall g = 1, \ldots, G, \; \forall j = 1, \ldots, J;$$

- ρ be the unit waste cost, so the total waste cost is given by:
$$\rho \left\{ \sum_{i=1}^{I} \sum_{l=1}^{L} \sum_{h=1}^{H} [(1-\beta_{li}) q_{hlgi}] + \sum_{i=1}^{I} \sum_{s=1}^{S} \Big[(1-\beta_i) r_{isg}^{(v)} \Big] \right\}.$$

We assume there are no demand excesses or deficiencies, therefore the demand at demand market k related to the product i of g is:
$$\delta_k^{gi} = \sum_{s=1}^{S} q_{sk}^{gi} + q_{gik} \qquad \forall k = 1, \ldots, K, \; \forall i = 1, \ldots, I.$$
Moreover, the total demand of product i of g is less than or equal to the produced amount:
$$\sum_{k=1}^{K} \delta_k^{gi} = \sum_{k=1}^{K} \Big[\sum_{s=1}^{S} q_{sk}^{gi} + q_{gik} \Big] = \sum_{k=1}^{K} \sum_{s=1}^{S} q_{sk}^{gi} + \sum_{k=1}^{K} q_{gik}$$

$$\leq \sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2} q^v_{gise} + \sum_{k=1}^{K} q_{gik}, \quad \forall i = 1,\dots,I.$$

Since the total demand is not actually determined, we denote by $[\lambda^{gi}, \mu^{gi}]$ $\forall i = 1,\dots,I$, the range of demand forecasting.

The aim of manufacturer g is to maximize his own profits. Therefore, g's optimality conditions are given by the following problem:

$$\begin{cases}
\max\Big\{ \sum_{i=1}^{I}\Big[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}\rho^{v*}_{1gise}q^v_{gise} + \sum_{k=1}^{K}\rho^{*}_{1gik}q_{gik} - C^i - c^i_g(Q_0,Q_1,Q_2) \\
\quad -\sum_{h=1}^{H}\sum_{l=1}^{L}\rho^{*}_{0hlgi}q_{hlgi} - \sum_{h=1}^{H}\sum_{l=1}^{L}c_{hlgi}(q_{hg}) - p_{gi}(Q_1,Q_2) \\
\quad -\tau\Big[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}q^v_{gise} + \sum_{k=1}^{K}q_{gik}\Big] - \rho\sum_{l=1}^{L}\sum_{h=1}^{H}(1-\beta_{li})q_{hlgi} \\
\quad -\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}c^v_{gise}(q^v_{gise}) - \sum_{k=1}^{K}c_{gik}(q_{gik})\Big] \\
\quad +\sum_{j=1}^{J}\hat{\rho}^{*}_{1gj}\hat{q}_{gj}(Q_1,Q_4) - \sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}\rho^{v*}_{1gise}r^{(v)}_{isg}(Q_1,Q_4) - \sum_{j=1}^{J}\hat{c}_{gj}(Q_1,Q_4) \\
\quad -\sum_{i=1}^{i}\hat{p}_{gi}r^{(v)}_{isg}(Q_1,Q_4) - \sum_{i=1}^{I}\sum_{s=1}^{S}\hat{c}^{(v)}_{isg}(Q_1,Q_4) \\
\quad -\rho\sum_{i=1}^{I}\sum_{s=1}^{S}(1-\beta_i)r^{(v)}_{isg}(Q_1,Q_4)\Big\} \\
\sum_{h=1}^{H}\frac{\beta_{li}}{\alpha^{gi}_{hl}}q_{hlgi} = \sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}q^v_{gise} + \sum_{k=1}^{K}q_{gik}, \ \forall l=1,\dots,L;\ \forall i=1,\dots,I \\
\lambda^{gi} \leq \sum_{k=1}^{K}q_{gik} + \sum_{s=1}^{S}\sum_{k=1}^{K}q^{gi}_{sk} \leq \mu^{gi} \qquad \forall i=1,\dots,I \\
\sum_{v=1}^{V+1}\sum_{e=1}^{2}q^v_{gise} \geq \sum_{k=1}^{K}q^{gi}_{sk} \qquad \forall i=1,\dots,I, \forall s=1,\dots,S \\
q_{hlgi},\ q^v_{gise},\ q_{gik},\ q^{gi}_{sk} \geq 0 \qquad \forall h=1,\dots,H,\ \forall l=1,\dots,L, \forall s=1,\dots,S, \\
\qquad \forall v=1,\dots,V+1,\ \forall e=1,2,\ \forall k=1,\dots,K, \forall i=1,\dots,I.
\end{cases} \tag{4.1}$$

We assume that the objective function is continuously differentiable and concave and this is guaranteed assuming that all the cost functions

$$c_g^i(Q_0, Q_1, Q_2),\ c_{hlgi}(q_{hg}),\ p_{gi}(Q_1, Q_2),\ c_{gise}^v(q_{gise}^v) \text{ and } c_{gik}(q_{gik})$$

are convex functions. Therefore, since the objective function is concave and the feasible set is convex, it is easy to verify that the optimality conditions for all manufacturers simultaneously are characterized by the following variational inequality (for the proof, see [6], Theorem 5):

$$\begin{aligned}
&\text{“Find } (Q_0^*, Q_1^*, Q_2^*, Q_4^*) \in \mathbb{K}: \\
&\sum_{g=1}^{G}\sum_{h=1}^{H}\sum_{l=1}^{L}\sum_{i=1}^{I}\left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{hlgi}} + \rho_{0hlg}^* + \frac{\partial c_{hlgi}(q_{hg}^*)}{\partial q_{hlgi}} + \rho(1-\beta_{li})\right] \\
&\qquad\qquad\left(q_{hlgi} - q_{hlgi}^*\right) \\
&+\sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\sum_{e=1}^{2}\left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{gise}^v} - \rho_{1gise}^{v*} + \frac{\partial p_{gi}(Q_1^*, Q_2^*)}{\partial q_{gise}^v} + \tau\right. \\
&\qquad +\frac{\partial c_{gise}^v(q_{gise}^{v*})}{\partial q_{gise}^v} - \sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gise}^v} + \rho_{1gise}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^v} \\
&\qquad +\sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gise}^v} + \hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^v} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^v} \\
&\qquad \left.+\rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gise}^v}\right]\left(q_{gise}^v - q_{gise}^{v*}\right) \\
&+\sum_{i=1}^{I}\sum_{k=1}^{K}\left[\frac{\partial c_g^i(Q_0^*, Q_1^*, Q_2^*)}{\partial q_{gik}} - \rho_{1gik}^* + \frac{\partial p_{gi}(Q_1^*, Q_2^*)}{\partial q_{gik}} + \tau\right. \\
&\qquad \left.+\frac{\partial c_{gik}(q_{gik}^*)}{\partial q_{gik}}\right]\left(q_{gik} - q_{gik}^*\right) \\
&+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{i=1}^{I}\left[-\sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \sum_{v=1}^{V+1}\rho_{1gis}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}}\right. \\
&\qquad +\sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} \\
&\qquad \left.+\rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}}\right]\left(q_{sk}^{gi} - q_{sk}^{gi*}\right) \geq 0 \\
&\qquad\qquad\qquad \forall (Q_0, Q_1, Q_2, Q_4) \in \mathbb{K}\text{”}
\end{aligned} \tag{4.2}$$

where:

$$\mathbb{K} = \Bigg\{ (Q_0, Q_1, Q_2, Q_4) \in \mathbb{R}_+^{HLGI+GIS2(V+1)+GIK+SGKI} :$$

$$\sum_{h=1}^{H} \frac{\beta_{li}}{\alpha_{hl}^{gi}} q_{hlgi} = \sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} + \sum_{k=1}^{K} q_{gik}, \ \forall l = 1, \dots, L; \ \forall i = 1, \dots, I,$$

$$\lambda^{gi} \leq \sum_{k=1}^{K} q_{gik} + \sum_{s=1}^{S} \sum_{k=1}^{K} q_{sk}^{gi} \leq \mu^{gi} \qquad \forall i = 1, \dots, I,$$

$$\sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} \geq \sum_{k=1}^{K} q_{sk}^{gi} \qquad \forall i = 1, \dots, I, \forall s = 1, \dots, S \Bigg\}.$$

Variational inequality (4.2) represents the optimality conditions of manufacturer g for all products simultaneously.
The solution of (4.2) gives the optimal amount of raw materials manufacturer g has to purchase from all suppliers (Q_0^*), the optimal amount of final product g has to sell to all retailers (Q_1^*) and to all demand markets (Q_2^*) and the optimal amount of products sold by all retailers to all demand markets, in equilibrium.

We now provide an existence result for a solution to variational inequality (4.2).

Theorem 3. *A solution* $(Q_0^*, Q_1^*, Q_2^*, Q_4^*) \in \mathbb{K}$ *to variational inequality (4.2) is guaranteed to exist.*

Proof. The result follows from the classical theory of variational inequalities (see [66]), since the feasible set is compact and the function that enters the variational inequality is continuous. □

4.3 A case study

4.3.1 The Company

We have applied our model to a well-known agribusiness company, the Cooperativa Agricola "*Valle del Dittaino*", which is located in the heart of Sicily among huge expanses of wheat called Valle del Dittaino (Enna, Italy). It is a cooperative of wheat producers founded in 1976 with the aim of enhancing the precious raw material and of verticalizing the entire production process, from the storage up to the grinding and bakery. The company is widely distributed in the Sicilian (and not only) market with more than 900 stores and thirteen types of bread, bread crumbs (plain and flavored), pastries and snacks branded ***Pandittaino*** (see www.pandittaino.it). The cooperative offers a highly competitive product from the point of view of quality and food safety and these two powers are at the base of their marketing strategy. The plant contains the storage silos as well as the mill for the production of flour which is directly connected to the bakery.

4.3.2 The model

The Cooperativa Agricola "*Valle del Dittaino*" has the topology of the network as shown in the previous sections, with four tiers of decision-makers: suppliers of raw materials, the company, retailers, demand markets (and also carriers).

We consider the production of the newest and developing line: the soft wheat bread. From this production line you can get seven different types of goods, which will be distinguished by shape (round or elongated) or duration (fresh or long-life). It is our goal to determine, with the help of the model presented in Section 4.2.1, the optimal amount of raw material that the manufacturer has to buy from all suppliers (Q_0^*), the optimal amount of product that the manufacturer must sell to all retailers (Q_1^*) and to all demand markets (Q_2^*) and the optimal amount of goods that all retailers sell to all demand markets (Q_4^*), in equilibrium, in order to maximize the profits (minimizing the amount of unsold product i that retailer s gives back to manufacturer g).
Please note that, since the e-commerce has not yet been introduced in the company, we will only consider the physical network.

The topology of the network is shown in Fig. 4.3.

Since the production takes place daily, in a continuous fashion, it is neces-

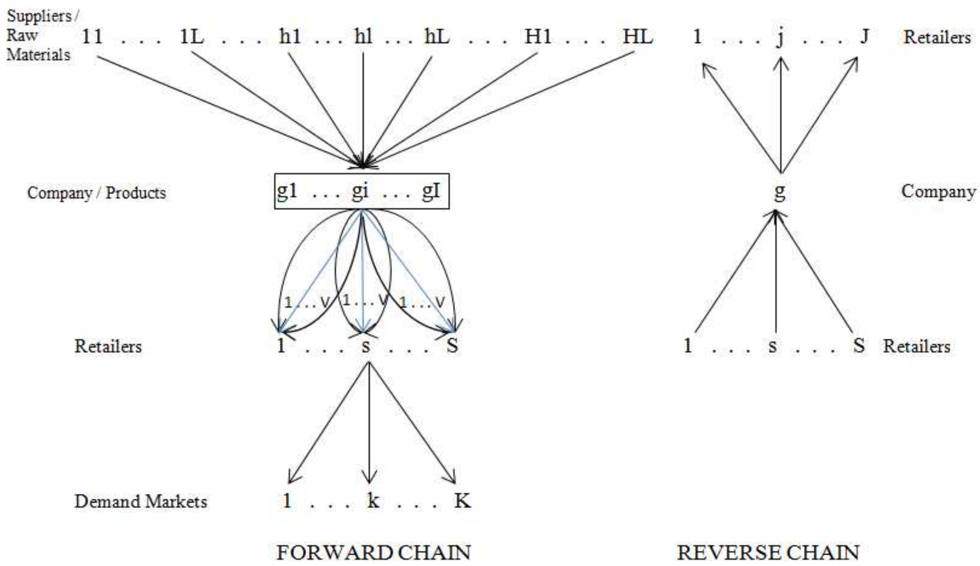

Figure 4.3: Pandittaino's network

sary to estimate the optimal amounts in advance and, therefore, we consider a seven-day interval.

In Appendix A we shall determine the demand forecast of every good.

After obtaining the estimates, for the long-life goods, we shall make use of the well-known (see [141]) Wagner-Whitin algorithm shown in Appendix B for inventory management, in order to determine the production periods (days), which satisfy the expected demand.

Later, we shall apply our model so to determine the optimal amounts we are looking for.

4.3.3 Additional constraints

4.3.3.1 Worsening of the objective function

In our study case, an additional constraint is given by the maximum capacity of every single tray and by the finite number of trays which can be used

following one another.
We can distinguish two different types of sandwiches (type I and type II), according to their format (round or elongated). Let:

- I be the number of products;
- x_i be the amount of the packages to produce, $\forall i = 1, \ldots, j$ containing type I sandwiches and $\forall i = j+1, \ldots, I$ type II sandwiches; we have:

$$x_i = \sum_{s=1}^{S} \sum_{v=1}^{V+1} q_{gis}^{v};$$

  in the general case, we have: $x_i = \sum_{s=1}^{S} \sum_{v=1}^{V+1} \sum_{e=1}^{2} q_{gise}^{v} + \sum_{k=1}^{K} q_{gik};$
- n_i be the number of sandwiches in package i;
- t_I be the number of trays for type I sandwiches;
- t_{II} be the number of trays for type II sandwiches;
- p_I the maximum number of type I sandwiches in every tray;
- p_{II} the maximum number of type II sandwiches in every tray.

Once the trays used for each type of product are over, we need $\frac{h}{4}$ minutes to experience the *format change*, during which there is a worsening of the objective function given by the consumption of energy, labor and other components, waiting for the time it takes to restart the production process.
The number of format changes turns out to be:

$$C = \left\lceil \frac{\sum_{i=1}^{j} n_i x_i}{p_I \cdot t_I} \right\rceil + \left\lceil \frac{\sum_{i=j+1}^{I} n_i x_i}{p_{II} \cdot t_{II}} \right\rceil - 1.$$

We remark that the number of format changes is a function of x_i and, hence, of Q_1.

Therefore, we get: $C = C(Q_1)$. As a consequence, we have to add a new term to the objective function, namely:

$$-C(Q_1) \cdot E,$$

where E is the price of all components of every format change.

4.3.3.2 Time

In the real situation, an important role is played by the production time, which provides an additional constraint, since it is bounded.
Let:

- N_i be the number of sandwiches i produced in one hour;
- T be the total available hours (including also the lead-time).

Therefore, the new constraint is:

$$\sum_{i=1}^{I} \frac{n_i x_i}{N_i} + C \cdot \frac{h}{4} \leq T;$$

namely the time needed to produce all sandwiches plus the time for format changes must be less than or equal to the total available time (in hours).

4.3.4 Optimality conditions and Variational formulation

The company aims at maximizing its profit. Hence, according with the model in Section 4.2.1 and the additional constraints, the optimality conditions for the company are given by the following problem:

$$\left\{\begin{array}{l}
\max\Bigg\{\sum_{i=1}^{I}\Bigg[\sum_{s=1}^{S}\sum_{v=1}^{V+1}\rho_{1gis}^{v*}q_{gis}^{v}-C^{i}-c_{g}^{i}(Q_0,Q_1)-\sum_{h=1}^{H}\sum_{l=1}^{L}\rho_{0hlg}^{*}q_{hlgi}\\
\qquad-\sum_{h=1}^{H}\sum_{l=1}^{L}c_{hlgi}(q_{hg})-p_{gi}(Q_1)-\tau\left[\sum_{s=1}^{S}\sum_{v=1}^{V+1}q_{gis}^{v}\right]\\
\qquad-\rho\sum_{l=1}^{L}\sum_{h=1}^{H}[(1-\beta_{li})q_{hlgi}]-\sum_{s=1}^{S}\sum_{v=1}^{V+1}c_{gis}^{v}(q_{gis}^{v})\Bigg]\\
\qquad+\sum_{j=1}^{J}\hat{\rho}_{1gj}^{*}\hat{q}_{gj}(Q_1,Q_4)-\sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\rho_{1gis}^{v*}r_{isg}^{(v)}(Q_1,Q_4)\\
\qquad-\sum_{j=1}^{J}\hat{c}_{gj}(Q_1,Q_4)-\sum_{i=1}^{I}\hat{p}_{gi}r_{isg}^{(v)}(Q_1,Q_4)-\sum_{i=1}^{I}\sum_{s=1}^{S}\hat{c}_{isg}^{(v)}(Q_1,Q_4)\\
\qquad-\rho\sum_{i=1}^{I}\sum_{s=1}^{S}\left[(1-\beta_i)r_{isg}^{(v)}(Q_1,Q_4)\right]-C(Q_1)E\Bigg\}\\
\sum_{h=1}^{H}\frac{\beta_{li}}{\alpha_{hl}^{gi}}q_{hlgi}=\sum_{s=1}^{S}\sum_{v=1}^{V+1}q_{gis}^{v}+\sum_{k=1}^{K}q_{gik},\qquad\forall l=1,\dots,L,\ \forall i=1,\dots,I\\
\sum_{i=1}^{I}\frac{n_i x_i(Q_1)}{N_i}+C(Q_1)\frac{h}{4}\le T\\
\lambda^{gi}\le\sum_{s=1}^{S}\sum_{k=1}^{K}q_{sk}^{gi}\le\mu^{gi}\qquad\forall i=1,\dots,I\\
\sum_{v=1}^{V+1}q_{gis}^{v}\ge\sum_{k=1}^{K}q_{sk}^{gi}\qquad\forall i=1,\dots,I,\ \forall s=1,\dots,S\\
q_{hlgi},\ q_{gis}^{v},\ q_{gik},\ q_{sk}^{gi}\ge 0\ \forall h=1,\dots,H,\ \forall l=1,\dots,L,\ \forall s=1,\dots,S,\\
\qquad\qquad\forall v=1,\dots,V+1,\ \forall k=1,\dots,K,\ \forall i=1,\dots,I.
\end{array}\right.$$

We assume that the objective function is continuously differentiable and concave and this is guaranteed assuming that all the cost functions

$$c_g^i(Q_0,Q_1),\ c_{hlgi}(q_{hg}),\ p_{gi}(Q_1),\ c_{gis}^{v}(q_{gis}^{v})\ \text{and}\ C(Q_1)E$$

are convex functions. Therefore, since the objective function is concave and the feasible set is convex, it is easy to verify that the optimality conditions for the manufacturer are characterized by the following variational inequality (where δ^{gi} does not appear):

$$\text{"Find } (Q_0^*, Q_1^*, Q_4^*) \in \mathbb{K} \text{ such that:}$$

$$\sum_{h=1}^{H}\sum_{l=1}^{L}\sum_{i=1}^{I}\left[\frac{\partial c_g^i(Q_0^*, Q_1^*)}{\partial q_{hlgi}} + \rho_{0hlg}^* + \frac{\partial c_{hlgi}(q_{hg}^*)}{\partial q_{hlgi}} + \rho(1-\beta_{li})\right]\left(q_{hlgi} - q_{hlgi}^*\right)$$
$$+\sum_{i=1}^{I}\sum_{s=1}^{S}\sum_{v=1}^{V+1}\left[\frac{\partial c_g^i(Q_0^*, Q_1^*)}{\partial q_{gis}^v} - \rho_{1gis}^{v*} + \frac{\partial p_{gi}(Q_1^*)}{\partial q_{gis}^v} + \tau + \frac{\partial c_{gis}^v(q_{gis}^{v*})}{\partial q_{gis}^v}\right.$$
$$-\sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gis}^v} + \rho_{1gis}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gis}^v} + \sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*, Q_4^*)}{\partial q_{gis}^v}$$
$$+\hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gis}^v} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gis}^v} + \rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{gis}^v}$$
$$\left.+E\frac{\partial C(Q_1^*)}{\partial q_{gis}^v}\right]\left(q_{gis}^v - q_{gis}^{v*}\right)$$
$$+\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{i=1}^{I}\left[-\sum_{j=1}^{J}\hat{\rho}_{1gj}^*\frac{\partial \hat{q}_{gj}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \sum_{v=1}^{V+1}\rho_{1gis}^{v*}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}}\right.$$
$$+\sum_{j=1}^{J}\frac{\partial \hat{c}_{gj}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \hat{p}_{gi}\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}} + \frac{\partial \hat{c}_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}}$$
$$\left.+\rho(1-\beta_i)\frac{\partial r_{isg}^{(v)}(Q_1^*, Q_4^*)}{\partial q_{sk}^{gi}}\right]\left(q_{sk}^{gi} - q_{sk}^{gi*}\right) \geq 0,$$
$$\forall (Q_0, Q_1, Q_4) \in \mathbb{K}\text{"}, \tag{4.3}$$

where:

$$\mathbb{K} = \left\{(Q_0, Q_1, Q_4) \in \mathbb{R}_+^{HLGI+GIS2(V+1)+SKGI} :\right.$$
$$\sum_{h=1}^{H}\frac{\beta_{li}}{\alpha_{hl}^{gi}}q_{hlgi} = \sum_{s=1}^{S}\sum_{v=1}^{V+1}q_{gis}^v + \sum_{k=1}^{K}q_{gik},\ \forall l = 1,\ldots,L;\ \forall i = 1,\ldots,I,$$
$$\sum_{i=1}^{I}\frac{n_i x_i(Q_1)}{N_i} + C(Q_1)\frac{h}{4} \leq T,$$
$$\lambda^{gi} \leq \sum_{s=1}^{S}\sum_{k=1}^{K}q_{sk}^{gi} \leq \mu^{gi} \qquad \forall i = 1,\ldots,I,$$

$$\text{and} \sum_{v=1}^{V+1} q_{gis}^{v} \geq \sum_{k=1}^{K} q_{sk}^{gi} \qquad \forall i = 1, \ldots, I, \ \forall s = 1, \ldots, S \Bigg\}.$$

Variational inequality (4.3) represents the optimality conditions for the company g related to all commodities simultaneously.
The solution to (4.3) gives the optimal amount of raw material that g has to buy from all suppliers (Q_0^*), the optimal amount of product sold by g to all retailers (Q_1^*) and the optimal amount of products sold by all retailers to all demand markets (Q_4^*), in equilibrium.

4.3.5 Results

In this section we show the results of the optimization model. We will consider as the discrete time interval the first 7 next days. For each of the long-life products, we get the $x_1, \ldots, x_7$ quantities to be produced and the $s_1, \ldots, s_7$ quantities to be stored in each period, so that the expected demands $d_1, \ldots, d_7$ (obtained in Appendix A) are met. We use the algorithm shown in Appendix B, developed in `C++` code, to determine the periods of production, while minimizing the production and storing costs. By analyzing product 807, as shown in Fig. 4.4, it turns out that the optimal production periods are the days 1, 3, 5 and 6; in the remaining days, however, the demands have to be met from stocks obtained from previous productive days prior. By applying the same algorithm to products 808, 809 and 810, it turns out that only the demand in day 7 must be satisfied by using stocks (Fig. 4.5).

Finally, using the optimal production periods previously obtained, we calculate, for every day, the optimal amounts Q_1^*, by solving our model, that is equivalent to the Variational inequality (4.3), through the mathematical software `Maple` (and also the software `Lingo`).

The following table shows the results:

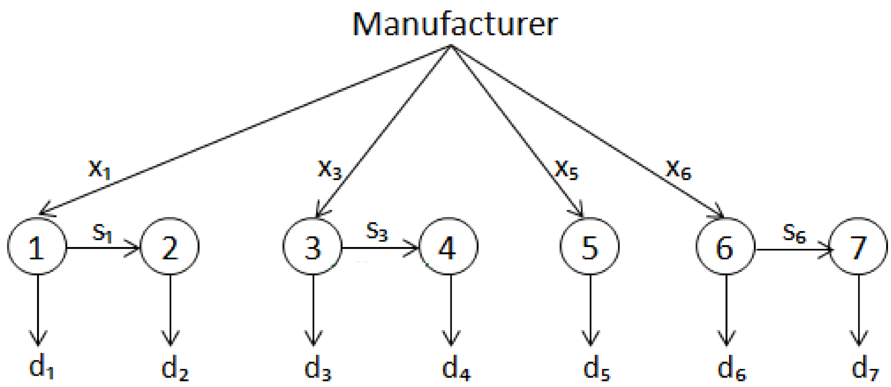

Figure 4.4: Inventory management: product 807

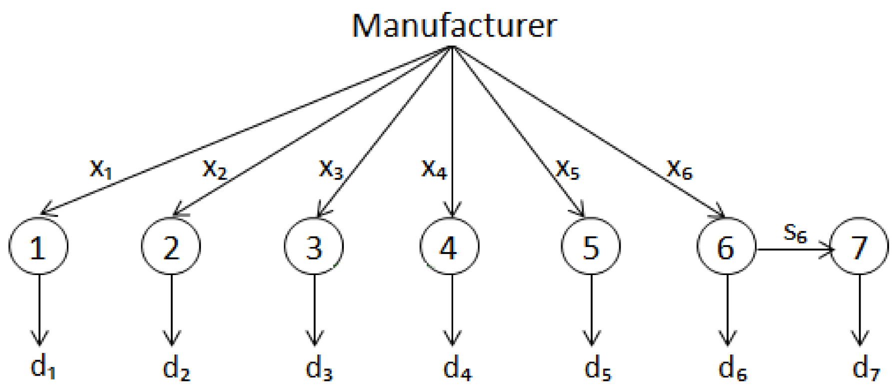

Figure 4.5: Inventory management: products 808-809-810

	Products						
	302	**303**	**807**	**808**	**809**	**810**	**811**
Day 1	*1073*	*526*	*0*	*0*	*0*	*0*	*0*
Day 2	*2211*	*502*	*0*	*140*	*592*	*580*	*406*
Day 3	*2012*	*752*	*2664*	*137*	*494*	*6341*	*1008*
Day 4	*2080*	*672*	*0*	*255*	*672*	*663*	*1069*
Day 5	*2017*	*632*	*3970*	*580*	*290*	*1783*	*750*
Day 6	*3000*	*825*	*2384*	*260*	*568*	*5892*	*95*
Day 7	*926*	*335*	*0*	*0*	*0*	*0*	*150*

We remark that the "*0*" in **Day 1** are due to the demand forecast of the market because that day coincides with Sunday; the others, namely those related to product **807** and **Day 7**, are determined by the inventory management.

The model also gives us the optimal amount of product that all retailers must sell to all demand markets (Q_4^*) and, as expected, they coincide with the produced amount, so that there is the maximization of the objective function.
Therefore, we note that the amount of unsold product i that retailer s gives back to manufacturer g is equal to zero:

$$r_{isg}^{(v)} = \sum_{v=1}^{V+1}\sum_{e=1}^{2} q_{gise}^{v} - \sum_{k=1}^{K} q_{sk}^{gi} = 0,$$

$\forall i = 1, \ldots, I, \ \forall s = 1, \ldots, S, \ \forall g = 1, \ldots, G.$

Once we have the quantities to be produced, by means of the constraint:

$$\sum_{h=1}^{H} \frac{\beta_{li}}{\alpha_{hl}^{gi}} q_{hlgi} = \sum_{s=1}^{S}\sum_{v=1}^{V+1} q_{gis}^{v} + \sum_{k=1}^{K} q_{gik}, \ \forall l = 1, \ldots, L; \forall i = 1, \ldots, I,$$

we can establish, for every day, the amount of raw materials needed for the products (Q_0^*).
For example, in the table below, we show a portion of the bill of material of Day 3:

Day 3	**Products**							**Total**
Raw material	**302**	**303**	**807**	**808**	**809**	**810**	**811**	
Label 66105	*2012*	*752*	*0*	*0*	*0*	*0*	*0*	**2764**
Envelope 66287	*2012*	*0*	*0*	*0*	*0*	*0*	*0*	**2012**
Farina FAR3	*577.14*	*104.72*	*484.37*	*15.41*	*89.92*	*1033.9*	*113.40*	**2418.86 kg**
Salt 90101	*13.84*	*2.512*	*10.656*	*0.338*	*1.976*	*22.764*	*2.490*	**54.576 kg**
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Water 64110	*311.659*	*54.452*	*251.881*	*8.015*	*46.708*	*537.59*	*58.968*	**1269.273 lt**

4.4 Conclusions

In this chapter we initially studied a multilevel supply chain network to which the variation of the different products of the company was added with respect to those produced by competing companies. We also analyzed the chain in a context where both e-commerce and production excesses are present, thus introducing the reverse chain and presenting the optimality conditions and the variational formulation of the single manufacturer. Therefore, we have built the supernetwork structure for a business management with electronic commerce in which suppliers of raw materials provide their commodities to manufacturers who produce i different products and sell their goods to retailers through V different carriers and directly to demand markets. Moreover, we have assumed that the products unsold by retailers can be given back to manufacturers who, after reworking such products, are now able to produce new commodities which will be shipped to new retailers (some of them could coincide with the previous ones). So we have extended the model including a forward and a reverse chain network. We have obtained the optimality conditions of the manufacturer which have been characterized by a variational inequality. Existence results for solutions follow from the classical theory.

Then, we improved the problem by applying it to a company, the Valle del Dittaino, considering a more complete model in which we not only the production excess and reverse logistics were examined, but we also added additional constraints arising from concrete and production limits.

Finally, we have applied such models to a real company, Valle del Dittaino, obtaining the optimal production and storage in a 7 days period. In the case study, we have included some additional constraints related to production and time.

Since every company aspiring to become a leader in the market has to rely on an internal organization capable of managing its work, trying to reduce costs and to get maximum profits, in order to study the best production planning it is necessary to analyze all of the organizational, managerial and strategic activities that govern the company's flow of materials and related information. Therefore, through the historical business series, we have also dealt with the demand forecast analysis needed to solve the maximization problem and a stock management model by implementing an algorithm that enables us to obtain excellent production times, in addition to the optimal amount of goods to be manufactured and sold and raw materials to be purchased.

Given the importance and relevance of the optimization and network models nowadays and the need to continuously create new and innovative ways to exploit for achieving business goals, the most important result of this chapter is the complete and concrete model, which can be applied to many companies in order to improve their production and minimize their waste or can be used in other fields that maintain a multilevel structure such as the described one.

Chapter 5

Cybersecurity Investments with Nonlinear Budget Constraints and Conservation Laws

5.1 Introduction

Supply chains have become increasingly complex as well as global and are now highly dependent on information technology to enhance effectiveness as well as efficiency and to support communications and coordination among the network of suppliers, manufacturers, distributors, and even freight service providers. At the same time, information technology, if not properly secured, can increase the vulnerability of supply chains to cyberattacks. Many examples exist of cyber attacks infiltrating supply chains with a vivid example consisting of the major US retailer Target cyber breach in which attackers entered the system via a third party vendor, an HVAC subcontractor, with an estimated 40 million payment cards stolen in late 2013 and upwards of 70 million other personal records compromised (see [67]). Not only did Target incur financial damages but also reputational costs. Other highly publicized examples have included breaches at the retailer Home Depot, the Sony media company, and the financial services firm JP Morgan Chase. Energy companies as well as healthcare organizations as well as defense companies have also been subject to cyberattacks (cf. [105] and [106]). In addition, the Internet of Things (IoT) has expanded the possible entry points for cyberattacks ([23]).

Of course, cyberattacks are not exclusively a US phenomenon. According to Verizon's 2016 Data Breach Investigations Report, there were 2,260 confirmed data breaches in the previous year at organizations in 82 countries. Numerous other breaches, affecting small and medium-size businesses, have gone unreported and unanalyzed (cf. [137]). In order to illustrate the scope of the negative impacts associated with cybercrime, it has been estimated that the world economy sustained $445 billion in losses from cyberattacks in 2014 (see [15]).

Numerous companies and organizations have now realized that investing in cybersecurity is an imperative. Furthermore, because of the interconnectivity through supply chains and even financial networks, the decisions of an organization in terms of cybersecurity investments can affect the cybersecurity of others. For example, according to Kaspersky Lab, a multinational gang of cybercriminals, known as "Carbanak," infiltrated more than 100 banks across 30 countries and extracted as much as one billion dollars over a period of roughly two years ([72]). Gartner ([94]) and Market Research ([84]) report that organizations in the US are spending $15 billion for security for communications and information systems. Hence, research in cybersecurity investment is garnering increased attention with one of the first research studies on the topic being that of Gordon and Loeb (see [54]).

In this chapter we consider a recently studied cybersecurity investment supply chain game theory model consisting of retailers and consumers at demand markets with each retailer being faced with a nonlinear budget constraint on his security investments (see [101] and [34]). We present an alternative to this model in which the demand for the product at each demand market is known and fixed and, hence, the conservation law of each demand market must be fulfilled. The reason for introducing such a satisfaction of the demands at the demand markets is because there are numerous products in which demand is inelastic as in the case, for example, of infant formula, certain medicines, etc.

The supply chain game theory model with cybersecurity investments in the case of fixed, that is, inelastic, demands, unlike the models of [101] and [34], is characterized by a feasible set such that the strategy of a given retailer is affected by the strategies of the other retailers since the product can come from any (or all) of them. Hence, the governing concept is no longer a Nash equilibrium (cf. [109], [110]) but, rather, is a Generalized Nash equilibrium (see, e.g., [139] and [45]). Recall that, in classical Nash equilibrium problems, the strategies of the players, that is, the decision-makers in the noncooperative game, affect the utility functions of the other players, but the feasible set of each player depends only on his/her strategies. It is worth mentioning that it was Rosen ([122]) who, in his seminal paper, studied a class of GNE problems. In [41] the authors show that the Rosen's class of GNE problems can be solved by finding a solution of a variational inequality. Moreover, the variational solution of a GNE problem with shared constraints has been derived in a general Hilbert space in [44].

In this chapter, we make use of a *variational equilibrium* (cf. [42] and [70]), which is a special kind of GNE. The variational equilibrium allows for a variational inequality formulation of the Generalized Nash equilibrium model. Notably, according to [78] and the references therein, the Lagrange multipliers associated with the shared (that is, the common) constraints are the same for all players in the game, which allows for an elegant economic interpretation. In our model, the demand constraints faced by the retailers are the shared ones, and we then fully investigate these and other relevant Lagrange multipliers in this chapter.

We note that in the papers [101] and [34] the governing Nash equilibrium conditions are formulated in terms of a variational inequality and an analysis of the dual problem and its associated Lagrange multipliers is performed. In particular, in this chapter, the influence of the conservation laws is analyzed and the importance of the associated Lagrange multipliers highlighted. The marginal expected transaction utility for each retailer depends on this Lagrange multiplier and its sign. For other papers on cybersecurity models see also [37], [105], [106], [128], whereas for other studies on the Lagrange theory and its application to variational models we refer to [32], [31], [33], [51], [53], [52], and [135]. For recent research on Generalized Nash equilibrium models in disaster relief supply chains and in commercial supply chains, respectively, see [100] and [108].

In the paper [34] an analysis of the marginal expected cybersecurity investment utilities and their stability is performed and, hence, this work adds to the literature on the study of marginal expected utilities, with a focus on both supply chains and cybersecurity investments, but in the more challenging setting of Generalized Nash equilibrium.

This chapter is organized as follows. In Section 5.2 we present the model, along with such concepts and firm and network vulnerability, define the variational equilibrium, and provide the variational inequality formulation. In Section 5.3 we construct an equivalent formulation by means of the Lagrange multipliers associated with the constraints and the conservation law which define the feasible set. Then we prove the existence of the Lagrange multipliers associated with the equality and inequality constraints by applying the Karush-Kuhn-Tucker conditions (see Theorem 4). In Section 5.4 we analyze the marginal expected transaction utilities and we find that they depend on the Lagrange multipliers and their signs. In Section 5.5 we present detailed numerical examples which emphasize the importance of the Lagrange multipliers and of the inelastic demands in order to maximize the expected utilities. Finally, in Section 5.6, we present the conclusions and the projects for future research.

5.2 The Model

The supply chain network, consisting of retailers and consumers at demand markets, is depicted in Figure 5.1. Each retailer i; $i = 1, \ldots, m$, can transact with demand market j; $j = 1, \ldots, n$, with Q_{ij} denoting the product transaction from i to j. We intend to study the cybersecurity by introducing for each retailer i; $i = 1, \ldots, m$, his cybersecurity or, simply, security, level s_i; $i = 1, \ldots, m$. We group the product transactions for retailer i; $i = 1, \ldots, m$, into the n-dimensional vector Q_i and then we group all such retailer transaction vectors into the mn-dimensional vector Q. The security levels of the retailers are grouped into the m-dimensional vector s.

Then, the cybersecurity level in the supply chain network is the average security and is denoted by $\bar{s}$, where $\bar{s} = \sum_{i=1}^{m} \frac{s_i}{m}$. Also, as in ([105]), a retailer's vulnerability $v_i = 1 - s_i$; $i = 1, \ldots, m$, and the network vulnerability $\bar{v} = 1 - \bar{s}$.

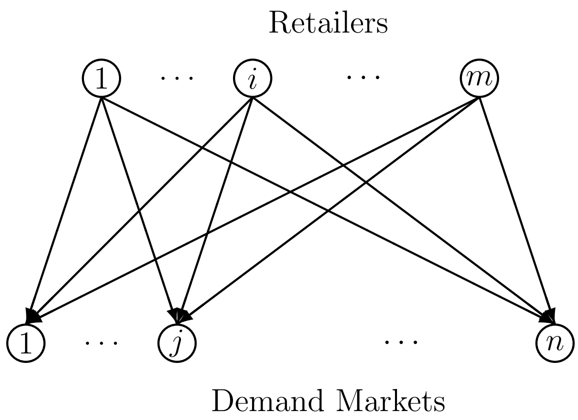

Figure 5.1: The Bipartite Structure of the Supply Chain Network Game Theory Model

The retailers seek to maximize their individual expected utilities, consisting of expected profits, and compete in a noncooperative game in terms of strategies consisting of their respective product transactions and security levels.

The demand at each demand market j, d_j, is assumed to be fixed and known, in contrast to the models in [34], [101], and [105]. The demand d_j must satisfy the following conservation law:

$$d_j = \sum_{i=1}^{m} Q_{ij}, \quad j = 1, \ldots, n. \tag{5.1}$$

The product transactions have to satisfy capacity constraints and must be nonnegative, so that we have the following conditions:

$$0 \leq Q_{ij} \leq \bar{Q}_{ij}, \text{ with } \sum_{i=1}^{m} \overline{Q}_{ij} > d_j \quad i = 1, \ldots, m; j = 1, \ldots, n. \tag{5.2}$$

The cybersecurity level of each retailer i must satisfy the following constraint:

$$0 \leq s_i \leq u_{s_i}, \quad i = 1, \ldots, m, \tag{5.3}$$

where $u_{s_i} < 1$ for all i; $i = 1, \ldots, m$. The larger the value of s_i, the higher the security level, with perfect security reflected in a value of 1. However, since, as noted in [101], we do not expect perfect security to be attainable, we have $u_{s_i} < 1$; $i = 1, \ldots, m$. If $s_i = 0$ this means that retailer i has no security.

The demand price of the product at demand market j, $\rho_j(d, s)$; $j = 1, \ldots, n$, is a function of the vector of demands and the network security. We can expect consumers to be willing to pay more for higher network security. In view of the conservation of flow equations above, we can define $\hat{\rho}_j(Q, s) \equiv \rho_j(d, s)$; $j = 1, \ldots, n$. We assume that the demand price functions are continuously differentiable, monotone **and concave**.

There is an investment cost function h_i; $i = 1, \ldots, m$, associated with achieving a security level s_i with the function assumed to be increasing, continuously differentiable and convex. For a given retailer i, $h_i(0) = 0$ denotes an entirely insecure retailer and $h_i(1) = \infty$ is the investment cost associated with complete security for the retailer. An example of an $h_i(s_i)$ function that satisfies these properties and that is utilized here (see also [101]) is

$$h_i(s_i) = \alpha_i \left(\frac{1}{\sqrt{(1 - s_i)}} - 1 \right) \text{ with } \alpha_i > 0.$$

The term α_i enables distinct retailers to have different investment cost functions based on their size and needs. Such functions have been introduced by [128] and also utilized by [105]. However, in those models, there are no cybersecurity budget constraints and the cybersecurity investment cost functions only appear in the objective functions of the decision-makers.

In the model with nonlinear budget constraints as in [101] each retailer is faced with a limited budget for cybersecurity investment. Hence, the following nonlinear budget constraints must be satisfied:

$$\alpha_i \left(\frac{1}{\sqrt{(1 - s_i)}} - 1 \right) \leq B_i; \quad i = 1, \dots, m, \tag{5.4}$$

that is, each retailer can't exceed his allocated cybersecurity budget.

The profit f_i of retailer i; $i = 1, \dots, m$ (in the absence of a cyberattack and cybersecurity investment), is the difference between his revenue $\sum_{j=1}^{n} \hat{\rho}_j(Q, s) Q_{ij}$ and his costs associated, respectively, with production and transportation: $c_i \sum_{j=1}^{n} Q_{ij} + \sum_{j=1}^{n} c_{ij}(Q_{ij})$, that is,

$$f_i(Q, s) = \sum_{j=1}^{n} \hat{\rho}_j(Q, s) Q_{ij} - c_i \sum_{j=1}^{n} Q_{ij} - \sum_{j=1}^{n} c_{ij}(Q_{ij}), \tag{5.5}$$

where $c_{ij}(Q_{ij})$ are convex functions.

If there is a successful cyberattack on a retailer i; $i = 1, \dots, m$, retailer i incurs an expected financial damage given by

$$D_i p_i,$$

where D_i, the damage incurred by retailer i, takes on a positive value, and p_i is the probability of a successful cyberattack on retailer i, where:

$$p_i = (1 - s_i)(1 - \bar{s}), \quad i = 1, \dots, m, \tag{5.6}$$

with the term $(1 - \bar{s})$ denoting the probability of a cyberattack on the supply chain network and the term $(1 - s_i)$ denoting the probability of success of such an attack on retailer i. We assume that such a probability is a given data on the basis of statistical observations.

Each retailer i; $i = 1, \ldots, m$, hence, seeks to maximize his expected utility, $E(U_i)$, corresponding to his expected profit given by:

$$E(U_i) = (1-p_i)f_i(Q,s) + p_i(f_i(Q,s) - D_i) - h_i(s_i) = f_i(Q,s) - p_iD_i - h_i(s_i). \tag{5.7}$$

Let us remark that, because of the assumptions, $-E(U_i)$ is a convex function (see [89]).

Let $\mathbb{K}^i$ denote the feasible set corresponding to retailer i, where

$$\mathbb{K}^i \equiv \{(Q_i, s_i)|0 \leq Q_{ij} \leq \bar{Q}_{ij}, \forall j,\ 0 \leq s_i \leq u_{s_i},$$

$$\text{and the budget constraint}\, h_i(s_i) - B_i \leq 0, \text{ holds for}\, i\}.$$

We also define

$$\mathbb{K} \equiv \Big\{(Q,s) \in \mathbb{R}^{mn+m} : -Q_{ij} \leq 0,\ Q_{ij} - \overline{Q}_{ij} \leq 0,\ -s_i \leq 0,$$

$$s_i - u_{s_i} \leq 0,\ h(s_i) - B_i \leq 0,\ i = 1, \ldots, m,\ j = 1, \ldots, n\Big\}.$$

In addition, we define the set of shared constraints $\mathcal{S}$ as follows:

$$\mathcal{S} \equiv \{Q|(4.1)\, \text{holds}\}.$$

We now state the following definition.

Definition 5. (A Supply Chain Generalized Nash Equilibrium in Product Transactions and Security Levels) *A product transaction and security level pattern $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, is said to constitute a supply chain Generalized Nash equilibrium if for each retailer i; $i = 1, \ldots, m$,*

$$E(U_i(Q_i^*, s_i^*, \hat{Q}_i^*, \hat{s}_i^*)) \geq E(U_i(Q_i, s_i, \hat{Q}_i^*, \hat{s}_i^*)), \quad \forall (Q_i, s_i) \in \mathbb{K}^i, \forall Q \in \mathcal{S}, \tag{5.8}$$

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_m^*); \quad \textit{and} \quad \hat{s}_i^* \equiv (s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_m^*).$$

Hence, according to the above definition, a supply chain Generalized Nash equilibrium is established if no retailer can unilaterally improve upon his expected utility (expected profit) by choosing an alternative vector of product transactions and security level, given the product flow and security level decisions of the other retailers and the demand constraints.

We now provide the linkage that allows us to analyze and determine the equilibrium solution via a variational inequality through a variational equilibrium ([70] and [78]).

Definition 6. (Variational Equilibrium) *A product transaction and security level pattern* (Q^*, s^*) *is said to be a variational equilibrium of the above Generalized Nash equilibrium if* $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, *is a solution of the variational inequality*

$$-\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}} \times \left(Q_{ij}-Q^*_{ij}\right) - \sum_{i=1}^{m}\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i} \times (s_i - s^*_i) \geq 0,$$
$$\forall (Q,s) \in \mathbb{K}, \forall Q \in \mathcal{S}; \qquad (5.9)$$

namely, $(Q^*, s^*) \in \mathbb{K}$, $Q^* \in \mathcal{S}$, *is a supply chain Generalized Nash equilibrium product transaction and security level pattern if and only if it satisfies the variational inequality*

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[c_i + \frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}} - \hat{\rho}_j(Q^*,s^*) - \sum_{k=1}^{n}\frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial Q_{ij}} \times Q^*_{ik}\right] \times (Q_{ij} - Q^*_{ij})$$
$$+\sum_{i=1}^{m}\left[\frac{\partial h_i(s^*_i)}{\partial s_i} - \left(1 - \sum_{k=1}^{m}\frac{s^*_k}{m} + \frac{1-s^*_i}{m}\right) D_i - \sum_{k=1}^{n}\frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} \times Q^*_{ik}\right]$$
$$\times (s_i - s^*_i) \geq 0, \quad \forall (Q,s) \in \mathbb{K}, \forall Q \in \mathcal{S}. \qquad (5.10)$$

For convenience, we define now the feasible set $\mathcal{K}$ where

$$\mathcal{K} \equiv \Big\{(Q,s) \in \mathbb{R}^{mn+m} : -Q_{ij} \leq 0,\ Q_{ij} - \overline{Q}_{ij} \leq 0,\ -s_i \leq 0,$$

$$s_i - u_{s_i} \leq 0,\ h(s_i) - B_i \leq 0,\ i = 1,\dots,m,\ j = 1,\dots,n,\ \text{and } Q|(4.1)\,\text{holds}\Big\}.$$

Problem (5.10) admits a solution since the classical existence theorem, which requires that the set $\mathcal{K}$ is closed, convex, and bounded and the function entering the variational inequality is continuous, is satisfied (see also [90]).

5.3 Equivalent Formulation of the Variational Inequality

The aim of this section is to find an alternative formulation of the variational inequality (5.9) for the cybersecurity supply chain game theory model with nonlinear budget constraints and conservation laws by means of the Lagrange multipliers associated with the constraints defining the feasible set $\mathcal{K}$. To this end, we remark that $\mathcal{K}$ can be rewritten in the following way:

$$\mathcal{K} = \Big\{(Q,s) \in \mathbb{R}^{mn+m} : -Q_{ij} \le 0,\ Q_{ij} - \overline{Q}_{ij} \le 0,\ -s_i \le 0,\ s_i - u_{s_i} \le 0,$$
$$h_i(s_i) - B_i \le 0,\ \sum_{i=1}^{m} Q_{ij} = d_j,\ i = 1, \dots, m,\ j = 1, \dots, n\Big\}, \tag{5.11}$$

and that variational inequality (5.9) can be equivalently rewritten as a minimization problem. Indeed, by setting:

$$V(Q,s) = -\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} \left(Q_{ij} - Q^*_{ij}\right) - \sum_{i=1}^{m} \frac{\partial E(U_i(Q^*, s^*))}{\partial s_i} \left(s_i - s^*_i\right),$$

we have:

$$V(Q,s) \ge 0 \text{ in } \mathcal{K} \text{ and } \min_{\mathcal{K}} V(Q,s) = V(Q^*, s^*) = 0. \tag{5.12}$$

Then, we can consider the following Lagrange function:

$$\begin{aligned}\mathcal{L}(Q,s,\lambda^1,\lambda^2,\mu^1,\mu^2,\lambda,\gamma) &= V(Q,s)+\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^1_{ij}(-Q_{ij})\\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda^2_{ij}(Q_{ij}-\overline{Q}_{ij})+\sum_{i=1}^{m}\mu^1_i(-s_i)\\ &+ \sum_{i=1}^{m}\mu^2_i(s_i-u_{s_i})+\sum_{i=1}^{m}\lambda_i(h_i(s_i)-B_i)\\ &+ \sum_{j=1}^{n}\gamma_j\left(\sum_{i=1}^{m}Q_{ij}-d_j\right), \end{aligned} \tag{5.13}$$

where $(Q,s)\in\mathbb{R}^{mn+m}$, λ^1, $\lambda^2\in\mathbb{R}^{mn}_+$, $\mu^1,\mu^2\in\mathbb{R}^m_+$, $\lambda\in\mathbb{R}^m_+$, $\gamma\in\mathbb{R}^n$.

It is worth mentioning that Lagrange function (5.13) is different from the one considered in [43].

Hence, we are able to prove the following result, which is interesting in itself, namely, using the Mangasarian Fromowitz constraint qualification condition, if (Q^*,s^*) is a solution of variational inequality (5.9), we are able to prove that KKT conditions (5.14) hold and vice versa from KKT conditions (5.14) variational inequality (5.9) follows. Moreover, for the first time, to the best of our knowledge, we show that strong duality (5.17) holds.

Theorem 4. *The Lagrange multipliers which appear in the Lagrange function* (5.13) *exist and, for all* $i=1,\dots,m$, *and* $j=1,\dots,n$, *the following conditions hold:*

$$\begin{gathered}\overline{\lambda}^1_{ij}(-Q^*_{ij})=0,\quad \overline{\lambda}^2_{ij}(Q^*_{ij}-\overline{Q}_{ij})=0,\\ \overline{\mu}^1_i(-s^*_i)=0,\quad \overline{\mu}^2_i(s^*_i-u_{s_i})=0,\quad \overline{\lambda}_i(h_i(s^*_i)-B_i)=0,\end{gathered} \tag{5.14}$$

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial Q_{ij}}-\overline{\lambda}^1_{ij}+\overline{\lambda}^2_{ij}+\overline{\gamma}_j=0, \tag{5.15}$$

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i}-\overline{\mu}^1_i+\overline{\mu}^2_i+\overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}=0. \tag{5.16}$$

Moreover, also the strong duality holds true; namely:

$$V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s) \tag{5.17}$$

$$= \max_{\substack{\lambda^1,\, \lambda^2 \in \mathbb{R}^{mn}_+,\, \mu^1, \mu^2 \in \mathbb{R}^m_+ \\ \lambda \in \mathbb{R}^m_+,\, \gamma \in \mathbb{R}^n}} \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma).$$

Proof Since the existence of the solution to problem (5.10) has been guaranteed, by virtue of the presence of equality constraints, we must apply the KKT theorem (see [60], Theorem 5.8) in order to obtain the existence of the Lagrange multipliers.

Let us denote by (Q^*, s^*) the solution to (5.10) and let us set:

$$\begin{aligned}
&g_i'(Q) = (-Q_{ij})_{j=1,\dots,n} \le 0,\ i = 1, \dots, m;\\
&I_i'(Q^*) = \{j \in \{1, \dots, n\} : Q^*_{ij} = 0\},\ i = 1, \dots, m;\\
&g_i''(Q) = \left(Q_{ij} - \overline{Q}_{ij}\right)_{j=1,\dots,n} \le 0,\ i = 1, \dots, m;\\
&I_i''(Q^*) = \{j \in \{1, \dots, n\} : Q^*_{ij} - \overline{Q}_{ij} = 0\},\ i = 1, \dots, m;\\
&s_i' = -s_i \le 0,\ i = 1, \dots, m \text{ and } J'_{s_i} = \{i \in \{1, \dots, m\} : s_i^* = 0\},\\
&s_i'' = s_i - u_{s_i} \le 0,\ i = 1, \dots, m \text{ and } J''_{s_i} = \{i \in \{1, \dots, m\} : s_i^* = u_{s_i}\},\\
&s_i''' = h(s_i) - B_i \le 0,\ i = 1, \dots, m \text{ and } J'''_{s_i} = \{i \in \{1, \dots, m\} : h(s_i^*) = B_i\},\\
&h_j(Q) = \sum_{i=1}^{m} Q_{ij} - d_j = 0,\ j = 1, \dots, n.
\end{aligned}$$

We remark that: $I_i'(Q^*) \cap I_i''(Q^*) = \emptyset$. Define also the matrix:

$$Q = \begin{pmatrix} Q_{11} & \dots & Q_{1j} & \dots & Q_{1n} \\ \dots & & & & \\ Q_{i1} & \dots & Q_{ij} & \dots & Q_{in} \\ \dots & & & & \\ Q_{m1} & \dots & Q_{mj} & \dots & Q_{mn} \end{pmatrix}.$$

For the Karush-Kuhn-Tucker theorem under the Mangasarian Fromowitz constraint qualification condition, we must prove that, taking into account that $\nabla g_i'^T(Q^*) = (-1, \dots, -1)$, there exists $Q \in \mathbb{R}^{mn}$ such that $-Q_{ij} < 0$, $i = 1, \dots, m$ and $j \in I_i'(Q^*)$.

Analogously, since $\nabla g_i''^T(Q^*) = (1, \dots, 1)$, we must also prove that there exists $Q \in \mathbb{R}^{mn}$ such that $Q_{ij} < 0$, $i = 1, \dots, m$ and $j \in I_i''(Q^*)$.

Such a Q does exist, because it is enough to choose $Q_{ij} > 0$ when $j \in I_i'(Q^*)$ and $Q_{ij} < 0$ when $j \in I_i''(Q^*)$.

For what concerns the equality constraints $\sum_{i=1}^{m} Q_{ij} - d_j = 0$; $j = 1, \dots, n$, we must prove that the matrix $\nabla h_j(Q^*)$, $j = 1, \dots, n$ is linearly independent and for some vector $Q \in \mathbb{R}^{mn}$ it must be : $\nabla^T h_j(Q^*)Q < 0$, $j = 1, \dots, n$.

We remark that:

$$\nabla h_j^T(Q^*) = \left(\frac{\partial h_j(Q^*)}{\partial Q_{11}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{1n}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{m1}}, \dots, \frac{\partial h_j(Q^*)}{\partial Q_{mn}} \right).$$

Hence:

$$\begin{aligned} \nabla h_1^T(Q^*) &= (1, 1, \dots, 1, 0, 0, \dots, 0, \dots, 0, 0, \dots, 0) \\ \dots \\ \nabla h_n^T(Q^*) &= (0, 0, \dots, 0, 0, 0, \dots, 0, \dots, 1, 1, \dots, 1) \end{aligned}$$

and their linear combination with constants $c_1, \dots, c_n$ is given by:

$$\sum_{j=1}^{n} c_j \nabla h_j^T(Q^*) = (c_1, \dots, c_1, c_2 \dots, c_2, \dots, c_n, \dots, c_n).$$

Such a linear combination is equal to zero if and only if all the coefficients c_j; $j = 1, \dots, n$, are zero.

As a consequence, $\nabla h_j^T(Q^*)$; $j = 1, \dots, n$, are linearly independent. Now we have to prove that for a vector Q of the same type as before, we get:

$$\begin{aligned} \nabla h_1^T(Q^*)Q &= \sum_{j=1}^{n} Q_{1j} = 0, \\ \dots \\ \nabla h_n^T(Q^*)Q &= \sum_{j=1}^{n} Q_{mj} = 0. \end{aligned} \tag{5.18}$$

We note that $Q_{1j} > 0$ if $j \in I_{1j}'(Q^*)$ and $Q_{1j} < 0$ if $j \in I_{1j}''(Q^*)$. Moreover, all the components Q_{ij} cannot be simultaneously equal to zero; otherwise, the equality constraint $\sum_{i=1}^{m} Q_{ij} = d_j$ would be unsatisfied. At the same time, it cannot be that $Q_{ij} = \overline{Q}_{ij}$, since $\sum_{i=1}^{m} \overline{Q}_{ij} > d_j$. Therefore, some Q_{ij} are arbitrarily positive and some Q_{ij} are arbitrarily negative and we can choose them so that (5.18) is verified.

Now, we can proceed with s_i; $i = 1, \dots, m$. We need to find $s^* \in \mathbb{R}^m$ such that:

$$\begin{array}{ll} \nabla s_i'(s_i^*) s_i < 0 & i \in J_s'(s^*) \\ \nabla s_i''(s_i^*) s_i < 0 & i \in J_s''(s^*), \end{array} \quad \text{namely,} \quad \begin{array}{ll} s_i > 0 & i \in J_s'(s^*) \\ s_i < 0 & i \in J_s''(s^*). \end{array} \tag{5.19}$$

Moreover, we need: $\nabla \left(h_j(s_i^*) - B_i\right) s_i < 0$, $i \in J_s'''(s^*)$. Since $h_j(s_i)$ is an increasing function, then $\max h_j(s_i) = h_j(u_{s_i})$. Hence, it must be that:

$$\frac{\partial h_j(u_{s_i})}{\partial s_i} s_i < 0, \quad i \in J_s'''(s^*).$$

We recall that $h_j(s_i) = (1 - s_i)^{-\frac{1}{2}}$, which implies $\frac{\partial h_j(u_{s_i})}{\partial s_i} = \frac{1}{2}(1 - u_{s_i})^{-\frac{3}{2}} > 0$ and that $s_i^* = u_{s_i}$ implies $s_i < 0$, then

$$\frac{\partial h_j(u_{s_i})}{\partial s_i} s_i < 0, \quad i \in J_s'''(s^*).$$

Then, the Lagrange multipliers $\overline{\lambda}^1$, $\overline{\lambda}^2 \in \mathbb{R}^{mn}_+$, $\overline{\mu}^1$, $\overline{\mu}^2$, $\overline{\lambda} \in \mathbb{R}^m_+$, $\overline{\gamma} \in \mathbb{R}^n$, do exist and conditions (5.14), (5.15), and (5.16) hold true (see Th. 5.8 in [60]). Since the inequality constraints are linear or convex and the equality constraints are affine linear, the Lagrange function results to be convex on the whole space $\mathbb{R}^{3mn+2n+3m}$. Then, by virtue of Theorem 3.8, part b in [60], the point (Q^*, s^*) is the minimal solution of the Lagrange function $\mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$ in the whole space $\mathbb{R}^{mn+n}$.

As a consequence, taking into account (5.14), we obtain:

$$\min_{(Q,s)\in\mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma}) = \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$$

$$= V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s),$$

see also Theorem 5.17 in [60] for similar remarks.

Now, we want to prove the strong duality; namely:

$$V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s) =$$

$$= \max_{\substack{\lambda^1,\, \lambda^2 \in \mathbb{R}^{mn}_+,\, \mu^1, \mu^2 \in \mathbb{R}^m_+ \\ \lambda \in \mathbb{R}^m_+,\, \gamma \in \mathbb{R}^n}} \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma).$$

Indeed, for every λ^1, $\lambda^2 \in \mathbb{R}^{mn}_+$, μ^1, $\mu^2 \in \mathbb{R}^m_+$, $\lambda \in \mathbb{R}^m_+$, $\gamma \in \mathbb{R}^n$, we have:

$$\min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq \mathcal{L}(Q^*, s^*, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

and

$$\mathcal{L}(Q^*, s^*, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq \underbrace{V(Q^*, s^*)}_{=0},$$

since in the Lagrange function all the terms except $V(Q^*, s^*)$ are less than or equal to zero.

Moreover,

$$V(Q^*, s^*) = \min_{\mathcal{K}} V(Q, s) = \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma}).$$

Further, we also have:

$$\max_{\substack{\lambda^1,\, \lambda^2 \in \mathbb{R}^{mn}_+,\, \mu^1, \mu^2 \in \mathbb{R}^m_+ \\ \lambda \in \mathbb{R}^m_+,\, \gamma \in \mathbb{R}^n}} \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma) \leq V(Q^*, s^*)$$

$$\leq \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})$$

$$\leq \max_{\substack{\lambda^1,\, \lambda^2 \in \mathbb{R}^{mn}_+,\, \mu^1, \mu^2 \in \mathbb{R}^m_+ \\ \lambda \in \mathbb{R}^m_+,\, \gamma \in \mathbb{R}^n}} \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

which yields:

$$V(Q^*, s^*) = \max_{\substack{\lambda^1,\, \lambda^2 \in \mathbb{R}^{mn}_+,\, \mu^1, \mu^2 \in \mathbb{R}^m_+ \\ \lambda \in \mathbb{R}^m_+,\, \gamma \in \mathbb{R}^n}} \min_{(Q,s) \in \mathbb{R}^{mn+m}} \mathcal{L}(Q, s, \lambda^1, \lambda^2, \mu^1, \mu^2, \lambda, \gamma),$$

and the assertion is proved.

Conditions (5.14)–(5.16) represent an equivalent formulation of variational inequality (5.9) and it is easy to see that from (5.15) and (5.16) the variational inequality (5.9) follows. Indeed, multiplying (5.15) by $(Q_{ij} - Q^*_{ij})$ we obtain:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}}(Q_{ij}-Q^*_{ij})-\overline{\lambda}^1_{ij}(Q_{ij}-Q^*_{ij})+\overline{\lambda}^2_{ij}(Q_{ij}-Q^*_{ij})-\overline{\gamma}_j(Q_{ij}-Q^*_{ij}) = 0$$

and, taking into account (5.14), we have:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}}(Q_{ij} - Q^*_{ij}) = \overline{\lambda}^1_{ij}Q_{ij} - \overline{\lambda}^2_{ij}(Q_{ij} - \overline{Q}_{ij}) + \overline{\gamma}_j(Q_{ij} - Q^*_{ij}) \geq 0.$$

Analogously, multiplying (5.16) by $(s_i - s^*_i)$, we get:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i}(s_i - s^*_i) - \overline{\mu}^1_i(s_i - s^*_i) + \overline{\mu}^2_i(s_i - s^*_i) + \overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}(s_i - s^*_i) = 0.$$

From (5.14), we have:

$$\overline{\mu}^1_i(-s^*_i) = 0, \quad \overline{\mu}^2_i s^*_i = \overline{\mu}^2_i u_{s_i}.$$

Moreover, if $\overline{\lambda}_i > 0$, then $h_i(s^*_i) = B_i = \max h_i(s_i)$, but $h_i(s_i)$ is a nondecreasing function; hence, it attains its maximum value at $s^*_i = u_{s_i}$. Therefore, we get:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i}(s_i - s^*_i) = \overline{\mu}^1_i s_i - \overline{\mu}^2_i(s_i - u_{s_i}) - \overline{\lambda}_i\frac{\partial h_i(s^*_i)}{\partial s_i}(s_i - u_{s_i}) \geq 0$$

because $h_i(s_i)$ is a nonnegative convex function such that $h_i(0) = 0$. Then $h_i(s_i)$ attains the minimum value at 0. Hence, $\frac{\partial h_i(0)}{\partial s_i} \geq 0$ and, since $\frac{\partial h_i(s_i)}{\partial s_i}$ is increasing, it results in:

$$0 \leq \frac{\partial h_i(0)}{\partial s_i} \leq \frac{\partial h_i(s_i)}{\partial s_i}, \quad \forall 0 \leq s_i \leq u_{s_i}.$$

For the above calculations variational inequality (5.9) easily follows. □

The term $\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}}$ is called the *marginal expected transaction utility*, $i = 1, \dots, m$; $j = 1, \dots, n$, and the term $\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i}$ is called the *marginal expected cybersecurity investment utility*, $i = 1, \dots, m$. Our aim is to study such marginal expected utilities by means of (5.14)–(5.16).

5.4 Analysis of Marginal Expected Transaction Utilities and of Marginal Expected Cybersecurity Investment Utilities

From (5.15) we get

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} - \overline{\lambda}^1_{ij} + \overline{\lambda}^2_{ij} + \overline{\gamma}_j = 0, \quad i = 1, \ldots, m; \; j = 1, \ldots, n.$$

So, if $0 < Q^*_{ij} < \overline{Q}_{ij}$, then we get (see also (5.10))

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q^*_{ik} + \overline{\gamma}_j = 0, \tag{5.20}$$

$$i = 1, \ldots, m; \; j = 1, \ldots, n,$$

whereas if $\overline{\lambda}^1_{ij} > 0$, and, hence, $Q^*_{ij} = 0$, and $\overline{\lambda}^2_{ij} = 0$, we get

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q^*_{ik} = \overline{\lambda}^1_{ij} + \overline{\gamma}_j, \tag{5.21}$$

$$i = 1, \ldots, m; \; j = 1, \ldots, n,$$

and if $\overline{\lambda}^2_{ij} > 0$, and, hence, $Q^*_{ij} = \overline{Q}_{ij}$, and $\overline{\lambda}^1_{ij} = 0$, we have

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial Q_{ij}} = c_i + \frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}} - \hat{\rho}_j(Q^*, s^*) - \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q^*_{ik} = -\overline{\lambda}^2_{ij} + \overline{\gamma}_j, \tag{5.22}$$

$$i = 1, \ldots, m; \; j = 1, \ldots, n.$$

Now let us analyze the meaning of equalities (5.20)–(5.22). From equality (5.20), which holds when $0 < Q^*_{ij} < \overline{Q}_{ij}$, we see that for retailer i, who transfers the product Q^*_{ij} to the demand market j, the marginal expected transaction utility is $-\overline{\gamma}_j$. We remark that $-\overline{\gamma}_j \in \mathbb{R}$, but its sign depends on the difference between the marginal expected transaction cost $c_i + \frac{\partial c_{ij}(Q^*_{ij})}{\partial Q_{ij}}$

and the marginal expected transaction revenue $\hat{\rho}_j(Q^*, s^*) + \sum_{\substack{k=1 \\ k \neq i}}^{m} \frac{\partial \hat{\rho}_k}{\partial Q_{ij}} \times Q^*_{ik}$.
Then the positive situation is the one when $\overline{\gamma}_j > 0$ so that the marginal expected transaction revenues exceed the costs.

Equality (5.21) shows that, when there is no trade between retailer i and demand market j; namely, $\overline{\lambda}^1_{ij} > 0$ and equality (5.21) holds, then the marginal expected transaction utility decreases, whereas if $\overline{\lambda}^2_{ij} > 0$; namely, $Q^*_{ij} = \overline{Q}_{ij}$, then the marginal expected transaction utility increases.

In conclusion, we remark that the Lagrange variables $\overline{\gamma}_j$, $\overline{\lambda}^1_{ij}$, $\overline{\lambda}^2_{ij}$, $i = 1, \ldots, m$; $j = 1, \ldots, n$, give a precise evaluation of the behavior of the market with respect to the supply chain product transactions.

The analysis of marginal expected cybersecurity investment utilities is the same as the one performed in subsection 3.2 in [34] as well as the stability of the marginal expected cybersecurity investment utilities is the same as the one performed in subsection 3.3 in [34], but we report them here for the reader's convenience. From (5.16) we have:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i} - \overline{\mu}^1_i + \overline{\mu}^2_i + \overline{\lambda}_i \frac{\partial h_i(s^*)}{\partial s_i} = 0, \quad i = 1, \ldots, m. \tag{5.23}$$

If $0 < s^*_i < u_{s_i}$, then $\overline{\mu}^1_i = \overline{\mu}^2_i = 0$ and we have (see also (5.10))

$$\begin{aligned} &\frac{\partial h_i(s^*_i)}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(s^*_i)}{\partial s_i} \\ &= \left(1 - \sum_{k=1}^{m} \frac{s^*_k}{m} + \frac{1 - s^*_i}{m}\right) D_i + \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} \times Q^*_{ik}. \end{aligned} \tag{5.24}$$

Since $0 < s^*_i < u_{s_i}$, $h(s^*_i)$ cannot be the upper bound B_i; hence, $\overline{\lambda}_i$ is zero and (5.24) becomes:

$$\frac{\partial h_i(s^*_i)}{\partial s_i} = \left(1 - \sum_{k=1}^{m} \frac{s^*_k}{m} + \frac{1 - s^*_i}{m}\right) D_i + \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} \times Q^*_{ik}. \tag{5.25}$$

Equality (5.25) shows that the marginal expected cybersecurity cost is equal to the marginal expected cybersecurity investment revenue plus the term

$\left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$; namely, the marginal expected cybersecurity investment revenue is equal to $\frac{\partial h_i(s_i^*)}{\partial s_i} - \left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$. This is reasonable because $\left(1 - \sum_{k=1}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i$ is the marginal expected damage expense.

If $\overline{\mu}_i^1 > 0$ and, hence, $s_i^* = 0$, and $\overline{\mu}_i^2 = 0$, we get:

$$-\frac{\partial E(U_i(Q^*, s^*))}{\partial s_i} = \frac{\partial h_i(0)}{\partial s_i} - \left(1 - \sum_{\substack{k=1 \\ k\neq i}}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} Q_{ik}^* = \overline{\mu}_i^1. \tag{5.26}$$

In (5.26) minus the marginal expected cybersecurity investment utility is equal to $\overline{\mu}_i^1$; hence, the marginal expected cybersecurity cost is greater than the marginal expected cybersecurity investment revenue plus the marginal damage expense. Then the marginal expected cybersecurity investment revenue is less than the marginal expected cybersecurity cost minus the marginal damage expense. We note that case (5.26) can occur if $\frac{\partial h_i(0)}{\partial s_i}$ is strictly positive.

In contrast, if $\overline{\mu}_i^2 > 0$ and, hence, $s_i^* = u_{s_i}$, retailer j has a marginal gain given by $\overline{\mu}_i^2$, because

$$\begin{aligned} -\frac{\partial E(U_i(Q^*, u_{s_i}))}{\partial s_i} &= -\left(1 - \sum_{\substack{k=1 \\ k\neq i}}^{m} \frac{u_{s_k}}{m} + \frac{1-u_{s_i}}{m}\right) D_i - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*, s^*)}{\partial s_i} \times Q_{ik}^* \\ &+ \frac{\partial h_i(u_{s_i})}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(u_{s_i})}{\partial s_i} = -\overline{\mu}_i^2. \end{aligned} \tag{5.27}$$

We note that $\overline{\lambda}_i$ could also be positive, since, with $s_i^* = u_{s_i}$, $h_i(s_i)$ could reach the upper bound B_i. In (5.27) minus the marginal expected cybersecurity investment utility is equal to $-\overline{\mu}_i^2$. Hence, the marginal expected cybersecurity cost is less than the marginal expected cybersecurity investment revenue plus the marginal damage expense. Then the marginal expected cybersecurity investment revenue is greater than the marginal expected cybersecurity cost minus the marginal damage expense.

From (5.27) we see the importance of the Lagrange variables $\overline{\mu}_i^1$, $\overline{\mu}_i^2$ which describe the effects of the marginal expected cybersecurity investment utilities.

Now let us consider the three cases related to the studied marginal expected cybersecurity investment utilities. Each of these cases holds for certain values of the damage D_i. Let us consider the value D_i for which the first case (5.25) occurs. We see that in this case there is a unique value of D_i for which (5.25) holds and if we vary such a value, also the value s_i^* in (5.25) varies. Now let us consider the value D_i for which (5.26) holds and let us call D_i^* the value of D_i for which we have

$$-\frac{\partial E(U_i(Q^*,s^*))}{\partial s_i} = \frac{\partial h_i(0)}{\partial s_i} - \left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{s_k^*}{m} + \frac{1-s_i^*}{m}\right) D_i^* - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} Q_{ik}^* = 0.$$

Then for $0 < D_i < D_i^*$ the solution (Q^*, s^*) to variational inequality (5.9) remains unchanged because (5.26) still holds for these new values of D_i and the marginal expected cybersecurity investment utility remains negative, but it is increasing with respect to D_i. Analogously, if we consider the value D_i for which (5.27) holds and call D_i^* the value such that

$$\begin{aligned}-\frac{\partial E(U_i(Q^*,u_{s_i}))}{\partial s_i} = &- \left(1 - \sum_{\substack{k=1\\k\neq i}}^{m} \frac{u_{s_k}}{m} + \frac{1-u_{s_i}}{m}\right) D_i^* - \sum_{k=1}^{m} \frac{\partial \hat{\rho}_k(Q^*,s^*)}{\partial s_i} \times Q_{ik}^* \\ &+ \frac{\partial h_i(u_{s_i})}{\partial s_i} + \overline{\lambda}_i \frac{\partial h_i(u_{s_i})}{\partial s_i} = 0,\end{aligned}$$

we see that for $D_i > D_i^*$ the solution (Q^*, s^*) to (5.9) remains unchanged because (5.27) still holds and the marginal expected cybersecurity investment utility remains positive and is increasing with respect to D_i.

5.5 Numerical Examples

The numerical examples consist of a supply chain network with two retailers and two demand markets as depicted in Fig. 5.2.

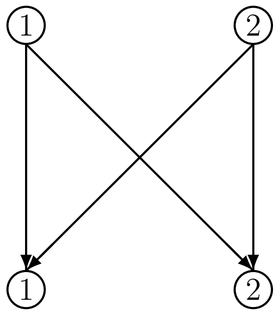

Figure 5.2: Network Topology for the Numerical Examples

The examples are inspired by related examples as in [101] and in [34]. Since we want to report all the results for transparency purposes, we have selected the size of problems as reported.

The cost function data are:

$$\begin{aligned} c_1 &= 5, & c_2 &= 10, \\ c_{11}(Q_{11}) &= .5Q_{11}^2 + Q_{11}, & c_{12}(Q_{12}) &= .25Q_{12}^2 + Q_{12}, \\ c_{21}(Q_{21}) &= .5Q_{21}^2 + Q_{21}, & c_{22}(Q_{22}) &= .25Q_{22}^2 + Q_{22}. \end{aligned}$$

The demand price functions are:

$$\rho_1(d, \overline{s}) = -d_1 + .1\frac{s_1 + s_2}{2} + 100, \quad \rho_2(d, \overline{s}) = -.5d_2 + .2\frac{s_1 + s_2}{2} + 200.$$

The damage parameters are: $D_1 = 200$ and $D_2 = 210$ with the investment functions taking the form:

$$h_1(s_1) = \frac{1}{\sqrt{1 - s_1}} - 1, \quad h_2(s_2) = \frac{1}{\sqrt{1 - s_2}} - 1.$$

The damage parameters are in millions of \$US, the expected profits (and revenues) and the costs are also in millions of \$US. The prices are in thousands of dollars and the product transactions are in thousands. The budgets for the two retailers are identical with $B_1 = B_2 = 2.5$ (in millions of \$US). In this case the bounds on the security levels are $u_{s_1} = u_{s_2} = .91$ and the capacities $\overline{Q}_{ij}$ are set to 100 for all i, j.

Keeping the same structure of the network, we have considered five cases with different values of demands:

Case 1: $d_1 = Q_{11} + Q_{21} = 20$ and $d_2 = Q_{12} + Q_{22} = 80$;

Case 2: $d_1 = Q_{11} + Q_{21} = 40$ and $d_2 = Q_{12} + Q_{22} = 190$;

Case 3: no fixed demands;

Case 4: $d_1 = Q_{11} + Q_{21} = 60$ and $d_2 = Q_{12} + Q_{22} = 280$;

Case 5: $d_1 = Q_{11} + Q_{21} = 80$ and $d_2 = Q_{12} + Q_{22} = 380$.

We remark that Case 3 gives the same results as in the example in [34] which is a Nash equilibrium.
For $i = 1, 2$ we obtain:

$$\begin{aligned}
-\frac{\partial E(U_i(Q,s))}{\partial Q_{i1}} &= 2Q_{i1} + Q_{11} + Q_{21} - .1\frac{s_1+s_2}{2} + c_i - 99, \\
-\frac{\partial E(U_i(Q,s))}{\partial Q_{i2}} &= Q_{i2} + .5Q_{12} + .5Q_{22} - .2\frac{s_1+s_2}{2} + c_i - 199, \\
-\frac{\partial E(U_i(Q,s))}{\partial s_i} &= -\frac{1}{20}Q_{i1} - \frac{1}{10}Q_{i2} - \left(1 - \frac{s_1+s_2}{2} + \frac{1-s_i}{2}\right) D_i \\
&\quad + \frac{1}{2\sqrt{(1-s_i)^3}}.
\end{aligned}$$

Now, we wish to determine the equilibrium solution, taking into account the different values assumed by λ^1, λ^2, μ^1, μ^2, and λ, and searching, among them, the feasible ones. After some algebraic calculations, we realize that for $i = 1, 2$ and $j = 1, 2$ we get the solution when $\overline{\lambda}^1_{ij} = \overline{\lambda}^2_{ij} = \overline{\mu}^1_i = \overline{\lambda}_i = 0$, and $\overline{\mu}^2_i > 0$. Hence, $s^*_1 = s^*_2 = 0.91$ (which is the maximum value).

In this case, the marginal expected transaction utilities are zero, whereas the marginal expected cybersecurity investment utilities are positive; namely, there is a marginal gain, given by $\overline{\mu}_i^2$, $i = 1, 2$. Solving the system:

$$\begin{cases} \dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial Q_{i1}} &= 0 \\ \\ \dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial Q_{i2}} &= 0 \qquad i = 1, 2; \\ \\ \dfrac{\partial \mathcal{L}(Q^*, s^*, \overline{\lambda}^1, \overline{\lambda}^2, \overline{\mu}^1, \overline{\mu}^2, \overline{\lambda}, \overline{\gamma})}{\partial s_i} &= 0 \end{cases}$$

namely:

$$\begin{cases} 3Q_{11}^* + Q_{21}^* - 0.1\dfrac{s_1^* + s_2^*}{2} + c_1 - 99 - \overline{\lambda}_{11}^1 + \overline{\lambda}_{11}^2 + \overline{\gamma}_1 &= 0 \\ \\ Q_{11}^* + 3Q_{21}^* - 0.1\dfrac{s_1^* + s_2^*}{2} + c_2 - 99 - \overline{\lambda}_{21}^1 + \overline{\lambda}_{21}^2 + \overline{\gamma}_1 &= 0 \\ \\ 1.5Q_{12}^* + .5Q_{22}^* - 0.2\dfrac{s_1^* + s_2^*}{2} + c_1 - 199 - \overline{\lambda}_{12}^1 + \overline{\lambda}_{12}^2 + \overline{\gamma}_2 &= 0 \\ .5Q_{12}^* + 1.5Q_{22}^* - 0.2\dfrac{s_1^* + s_2^*}{2} + c_2 - 199 - \overline{\lambda}_{22}^1 + \overline{\lambda}_{22}^2 + \overline{\gamma}_2 &= 0 \\ \\ -\dfrac{1}{20}Q_{11}^* - \dfrac{1}{10}Q_{12}^* - \dfrac{3 - 2s_1^* - s_2^*}{2}D_1 + \dfrac{1 + \overline{\lambda}_1}{2\sqrt{(1 - s_1^*)^3}} - \overline{\mu}_1^1 + \overline{\mu}_1^2 &= 0 \\ \\ -\dfrac{1}{20}Q_{21}^* - \dfrac{1}{10}Q_{22}^* - \dfrac{3 - s_1^* - 2s_2^*}{2}D_2 + \dfrac{1 + \overline{\lambda}_2}{2\sqrt{(1 - s_2^*)^3}} - \overline{\mu}_2^1 + \overline{\mu}_2^2 &= 0, \end{cases}$$

and, therefore, assuming for $i = 1, 2$, $j = 1, 2$, $\overline{\lambda}_{ij}^1 = \overline{\lambda}_{ij}^2 = \overline{\mu}_i^1 = \overline{\lambda}_i = 0$, and

$\overline{\mu}_i^2 > 0$; hence, $s_1^* = s_2^* = 0.91$, and $D_1 = 200$ and $D_2 = 210$, we have:

$$\begin{cases} Q_{11}^* + Q_{21}^* & = & d_1 \\ 3Q_{11}^* + Q_{21}^* & = & 94.091 - \overline{\gamma}_1 \\ Q_{11}^* + 3Q_{21}^* & = & 89.091 - \overline{\gamma}_1 \\ Q_{12}^* + Q_{22}^* & = & d_2 \\ 1.5Q_{12}^* + .5Q_{22}^* & = & 194.182 - \overline{\gamma}_2 \\ .5Q_{12}^* + 1.5Q_{22}^* & = & 189.182 - \overline{\gamma}_2 \\ \overline{\mu}_1^2 & = & \frac{1}{20}Q_{11}^* + \frac{1}{10}Q_{12}^* + \frac{3 - 3 \times .91}{2}200 - \frac{1}{2\sqrt{(1-.91)^3}} \\ \overline{\mu}_2^2 & = & \frac{1}{20}Q_{21}^* + \frac{1}{10}Q_{22}^* + \frac{3 - 3 \times .91}{2}210 - \frac{1}{2\sqrt{(1-.91)^3}}. \end{cases}$$

The previous system, in the five examined cases, has been solved using Wolfram Alpha and the solutions are summarized in Table 5.1. In particular, we have reported the flows, the cybersecurity levels, the retailers' vulnerability, the network vulnerability, the Lagrange multipliers associated to the conservation laws and to the constraints on cybersecurity levels, in equilibrium.

We remark that, since the retailers invest at the upper bound levels of security, both the individual retailers' vulnerability, v_1 and v_2, and that of the network, $\bar{v}$, are low.

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
Q^*_{11}	11.25	21.25	24.148	31.25	41.25
Q^*_{21}	8.75	18.75	21.648	28.75	38.75
Q^*_{12}	42.5	97.5	98.341	142.5	192.5
Q^*_{22}	37.5	92.5	93.341	137.5	187.5
s^*_1	.91	.91	.91	.91	.91
s^*_2	.91	.91	.91	.91	.91
v_1	.09	.09	.09	.09	.09
v_2	.09	.09	.09	.09	.09
$\bar{v}$	.09	.09	.09	.09	.09
$\overline{\gamma}_1 = \frac{\partial E(U_1)}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{12}}$	51.591	11.591	0	-28.409	- 68.409
$\overline{\gamma}_2 = \frac{\partial E(U_1)}{\partial Q_{12}} = \frac{\partial E(U_1)}{\partial Q_{22}}$	111.682	1.682	0	-88.318	-188.318
$\overline{\mu}^2_1 = \frac{\partial E(U_1)}{\partial s_1}$	13.294	19.294	19.523	24.294	29.794
$\overline{\mu}^2_2 = \frac{\partial E(U_2)}{\partial s_2}$	14.019	20.019	20.248	25.019	30.019

Table 5.1: Equilibrium solutions

Moreover, the demand prices charged by the retailers and the expected utilities of each retailer, in the five cases, are summarized in Table 5.2.

Comparing the different results, we see that, for some values of the demands, the marginal expected transaction utilities, $\overline{\gamma}_1$ and $\overline{\gamma}_2$, have a positive value; for other values of the demands, $\overline{\gamma}_1$ and $\overline{\gamma}_2$ have negative values, and, when the demand is not fixed at the values above, $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are zero. On the contrary, the marginal cybersecurity investment utilities $\overline{\mu}^2_1$ and $\overline{\mu}^2_2$ are always increasing when the fixed demands increase too and have a small value when the demands are not fixed. Further, for the corresponding values of the demands, the expected utilities, $E(U_i)$; $i = 1, 2$, achieve the maximum value; for the other values of the demands, $E(U_i)$ decrease and, when the demands are not fixed, then $E(U_i)$ assumes a value which is less than the maximum obtained with certain fixed demands. As a conclusion, we can deduce that

	$\rho_1(d^*, s^*)$	$\rho_2(d^*, s^*)$	$E(U_1)$	$E(U_2)$
Case 1	80.091	160.182	2,798.087	5,804.9935
Case 2	60.091	105.182	8,213.3825	7,313.4232
Case 3	54.2954	104.341	8,123.9298	7,156.6968
Case 4	40.091	60.182	3,217.4817	2,455.0132
Case 5	20.091	10.182	-8,732.5083	-9,344.9765

Table 5.2: Demand prices and expected utilities

the problem has an optimal demand which yields optimal expected utilities and a good value of marginal cybersecurity expected utilities, whereas, when the demands are not fixed, we get a value of cybersecurity expected utilities which is not necessarily the optimal one.

Keeping the same structure as the one depicted in Fig. 5.2, now we study the cybersecurity by introducing the possibility, for each retailer $i = 1, 2$, to have different investment cost functions based on their different sizes and needs.

We assume that the cost functions, the demand price functions, the damage parameters, the budgets for the two retailers, the bounds on the security level, the product transactions capacities are given and are the same as in the previous example, but we suppose now that the investment cost functions are the following:

$$h_1(s_1) = 2\left(\frac{1}{\sqrt{1-s_1}} - 1\right), \quad h_2(s_2) = 3\left(\frac{1}{\sqrt{1-s_2}} - 1\right).$$

Therefore, we are setting $\alpha_1 = 2$ and $\alpha_2 = 3$.

In Table 5.3 we present the solutions, for the five examined cases, computed using the MatLab program.

In particular we remark that, since the cybersecurity levels are not equal to their upper bounds ($s_i < u_{s_i}$) and the budget constraints are satisfied with equality signs, namely, both retailers use the whole budget, we have: $\bar{\mu}_i^2 = 0$ and $\bar{\lambda}_i \neq 0$.

Comparing the different results, we notice that the product transactions Q^*_{ij}

in equilibrium are very similar to the previous ones (when $\alpha_1 = \alpha_2 = 1$) but now the cybersecurity levels are lower, specially when α_i is higher; obviously, in this case, the vulnerability values are bigger.
From Table 5.4 we also see that the marginal expected cybersecurity investment utilities over the marginal expected cybersecurity costs, the Lagrange multipliers $\bar{\lambda}_1$ and $\bar{\lambda}_2$, are always increasing when the demands increase too.

Furthermore, since $\bar{\lambda}_i > 0$ and $\dfrac{\partial h_i(s^*)}{\partial s_i} > 0$, $i = 1, 2$, for every case, we have that $\dfrac{\partial E(U_i)}{\partial s_i} > 0$.
Moreover, the demand prices charged by the retailers and the expected utilities of each retailer, in the five cases with $\alpha_1 = 2$ and $\alpha_2 = 3$, are summarized in Table 5.4.

5.6 Conclusions

In this chapter, we introduced a cybersecurity investment supply chain game theory model consisting of retailers and consumers at demand markets assuming that the demands for the product at the demand markets are known and fixed and, hence, the conservation law of each demand market is fulfilled. The model also has nonlinear budget constraints. This model is a Generalized Nash equilibrium model since not only are the retailers' expected utility functions dependent on one another's strategies but their feasible sets are as well. We proposed a variational equilibrium which allows us to formulate the governing equilibrium conditions as a variational inequality problem, rather than a quasi-variational inequality. We also studied the dual problem and, specifically, we analyzed the Lagrange multipliers associated with the conservation laws and the expected utilities when the demands change. In particular, we have seen that, for certain values of the fixed demand, we can attain the best expected utilities with respect to the demand. In the future we

	Case 1	**Case 2**	**Case 3**	**Case 4**	**Case 5**
Q^*_{11}	11.25	21.25	24.1438	31.25	41.25
Q^*_{21}	8.75	18.75	21.6438	28.75	38.75
Q^*_{12}	42.5	97.5	98.3252	142.5	192.5
Q^*_{22}	37.5	92.5	93.3252	137.5	187.5
s^*_1	.8025	.8025	.8025	.8025	.8025
s^*_2	.7025	.7025	.7025	.7025	.7025
v_1	.1975	.1975	.1975	.1975	.1975
v_2	.2975	.2975	.2975	.2975	.2975
$\bar{v}$	.2475	.2475	.2475	.2475	.2475
$\overline{\gamma}_1 = \frac{\partial E(U_1)}{\partial Q_{11}} = \frac{\partial E(U_1)}{\partial Q_{12}}$	51.5752	11.5752	0	-28.4248	- 68.4248
$\overline{\gamma}_2 = \frac{\partial E(U_1)}{\partial Q_{12}} = \frac{\partial E(U_1)}{\partial Q_{22}}$	111.6505	1.6505	0	-88.3495	-188.35
$\overline{\lambda}_1 = \frac{\frac{\partial E(U_1)}{\partial s_1}}{\frac{\partial h_1(s^*)}{\partial s_1}}$	5.5028	6.0295	6.0495	6.4685	6.9514
$\overline{\lambda}_2 = \frac{\frac{\partial E(U_2)}{\partial s_2}}{\frac{\partial h_2(s^*)}{\partial s_2}}$	8.4566	9.1057	9.1303	9.6466	10.2417

Table 5.3: Equilibrium solutions with $\alpha_1 = 2$ and $\alpha_2 = 3$

would like to continue the study of this topic and, in particular, we will take into account uncertainty on the data which leads to a random formulation of the model (see also [30] for an application to the traffic network models).

The results in this work add to the growing literature of operations research and game theory techniques for cybersecurity modeling and analysis.

	$\rho_1(d^*, s^*)$	$\rho_2(d^*, s^*)$	$E(U_1)$	$E(U_2)$
Case 1	80.0772	160.1545	6,857.8143	6,028.8489
Case 2	60.0772	105.1545	8,202.0839	7,858.6184
Case 3	54.2954	104.341	8,114.7336	7,785.6368
Case 4	40.0772	60.1545	3,204.8084	3,273.8429
Case 5	20.0772	10.1545	-8,746.6946	-8,227.6601

Table 5.4: Demand prices and expected utilities with $\alpha_1 = 2$ and $\alpha_2 = 3$

Chapter 6

Conclusions

The aim of this thesis is to analyze different thematic areas and applications to real situations by using models based on networks.

In particular, in this thesis we investigate initially a Mixed-Integer non-linear programming problem that we solve with a computational procedure and we compare the solutions with those of the correspondent linearization. Later, we analyze some non-linear programming problems (some of which have mixed-integer variables that we solved by obtaining the relaxed problems) and by applying the classical Lagrange theory we get the optimality conditions for all decision makers simultaneously.

The purpose of our mathematical model, in Chapter 2, is to represent a cloud environment. This structured and simplified formulation has the advantage of perceiving some properties of reality which otherwise could be unseen.

This mathematical model could also allow to analytically deduce other properties of the problem which are not yet known and to develop methods for monitoring and diagnosis. But, above all, it allows us to identify a rational strategy for reaching a final goal, which is to maximize the Iaas provider's profit. Further, the model also provides detailed quantitative information on the decisions to be taken.

By using this model, the IaaS provider can make a simulation of reality. Such

a simulation allows him to study the effects of a decision without having to necessarily take it in the reality.
Hence, the studied model is also effective, since it plays an important predictive function by reducing the risks in the choices of the decision makers.
We get a mixed-Integer nonlinear programming problem, which can be solved through the proposed computational algorithm. Such an algorithm can be also used to solve problems of a different nature but with the same framework. A second step is the linearization of the problem. The effectiveness of the model and of the algorithm is tested, by comparing the final data with the results obtained by solving the linearized problem through an existing software.
In the future we aim at studying the behavior of the decision makers at all levels of the network, so as to obtain the optimality and equilibrium conditions, and, as a consequence, the global solution for the entire network.
Another topic we have dealt with in depth in Chapter 3 of this thesis is the financial market. We studied some optimization models based on networks which allow us to formulate two new multi-period portfolio selection problems as Markowitz mean-variance optimization problems with intermediaries, and therefore with transaction costs, the addition of capital gains tax, but also with short selling and transfer of securities. We proposed two constrained Integer nonlinear programming problems with which it is possible to establish if and when it is suitable to buy and to sell financial securities, not only while maximizing the profits, but also while minimizing the risk (through the use of a weight).We applied the Lagrange theory and analyzed the variational inequality also in Chapter 4 and Chapter 5 where we studied an optimization model for business management and cybersecurity investments, respectively.

Appendix A

Achieved goals

In this section, we determine the demand forecast of every good, because it is necessary to estimate the amounts in advance (in the problem there is the total demand of product i and the range of demand forecasting) and, since the production takes place daily in a continuous fashion, we consider a seven-day interval.

Therefore, in this section, we discuss about the time series analysis. Its main aim is to identify an appropriate model that has trajectories that fit to the data, to be able to predict and determine the ranges.
The company software also allowed us to achieve the daily data of sales, starting from the production till today, required for forecasting and production planning.
In this work we make use of the statistical software **R**, with a plurality of commands and functions very useful in the time series analysis.
Before the analysis, we examined the raw data and we made some adjustments to purify the data by discontinuities or effects of different interval duration or time periods considered; in particular we have eliminated cases of *additives outlier*, observations that lie an abnormal distance from other values, by replacing them with appropriate values.
Therefore, we show the graph of the series (Figure A.1) which describes an

increase in the sales in some weeks of the year and especially a regularity at periodic intervals.

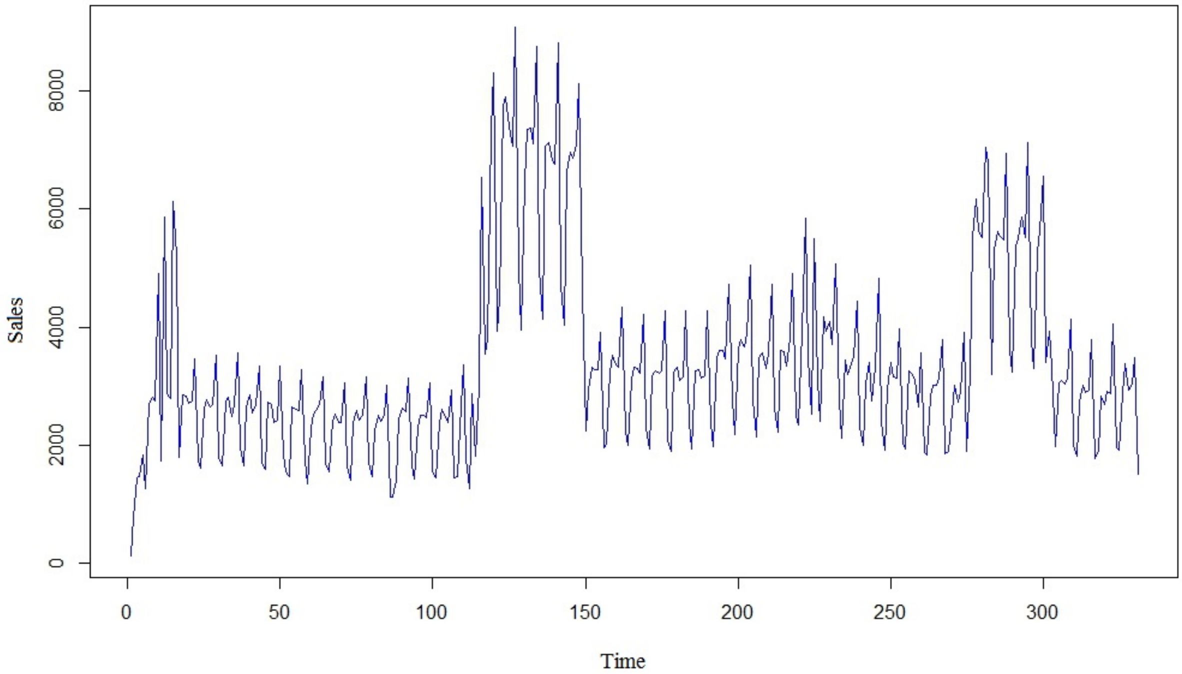

Figure A.1: Time series

This periodicity is confirmed by the correlogram (Figure A.2) which represents the autocorrelation of the series as a function of the lag.

The result, shown in Figure A.3, clearly proves how in all weeks the trend remains similar: we notice a maximum on the first day which decreases, until a minimum on the third day, and after that it has a slight increase.

In order to analyze the series, we have to make it stationary and therefore we have implemented the detrending through the moving average, although we loose some initial and final values (the latter being very useful for the prediction). So we used the *Holt-Winters model* (exponential smoothing), which is denoted in red in Figure A.4, thus resulting in an initial sales forecast in the next week:

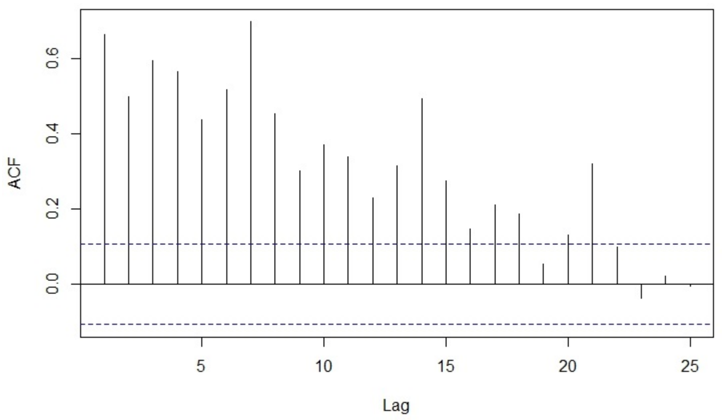

Figure A.2: Correlogram

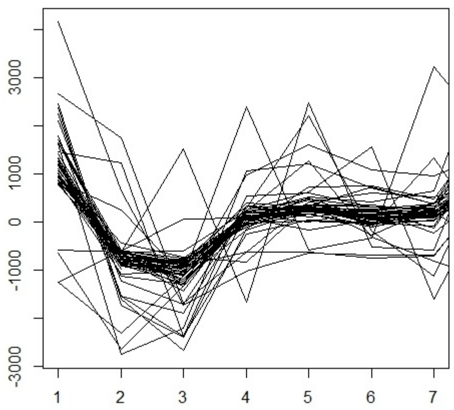

Figure A.3: Profile

```
> serief<-ts(serie1,frequency=7)
> serie.hw<-HoltWinters(serief,seasonal="additive")
> serie.hw
Holt-Winters exponential smoothing with trend and additive seasonal component.
Call:
HoltWinters(x = serief, seasonal = "additive")
Smoothing parameters:
alpha:  0.4344439
beta :  0.04848514
gamma:  0.4232419
```

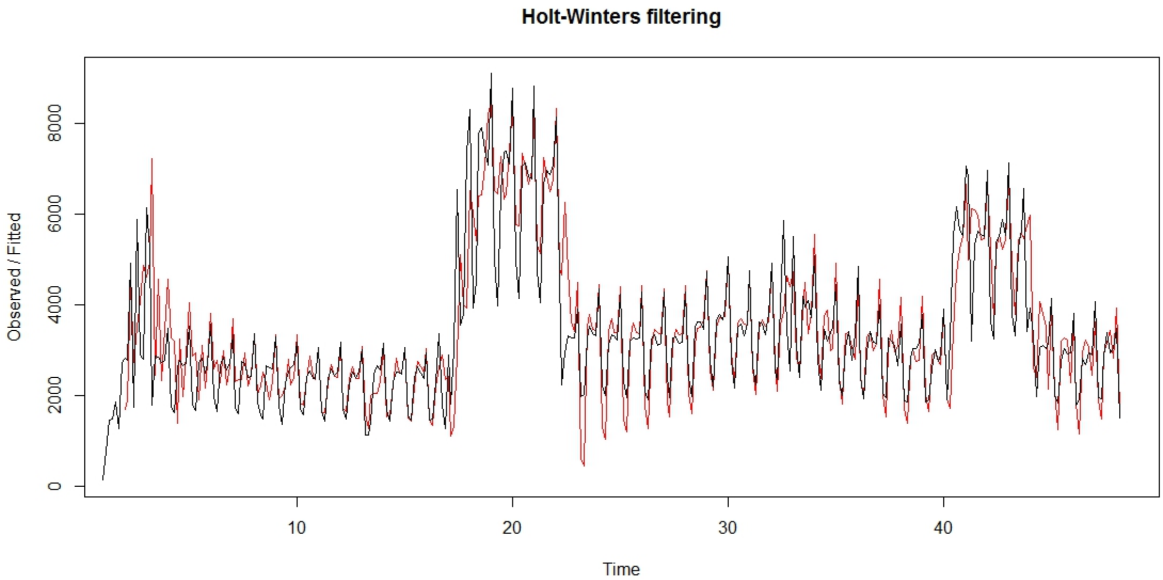

Figure A.4: Holt-Winters

```
Coefficients:
                [,1]
a       2989.25514
b       -36.94766
s1       -1703.41649
s2       -532.79118
s3       -199.48552
s4       -169.15905
s5       -364.33260
s6       514.17507
s7       -1425.35193
> plot(serie.hw)
> prev<-predict(serie.hw,n.ahead=7)
> prev
Time Series:
Start = c(48, 3)
```

```
End = c(49, 2)
Frequency = 7
                fit
[1,]       1248.891
[2,]       2382.569
[3,]       2678.927
[4,]       2672.305
[5,]       2440.184
[6,]       3281.744
[7,]       1305.270
```

A second prediction is obtained by the *decomposition and estimation of the components* with the method *loess* which consists in estimating a locally weighted polynomial regression at the point t_k (with k fixed) using points of its neighborhood.

We show the graph of this decomposition into seasonal, trend and irregular components (Figure A.5).

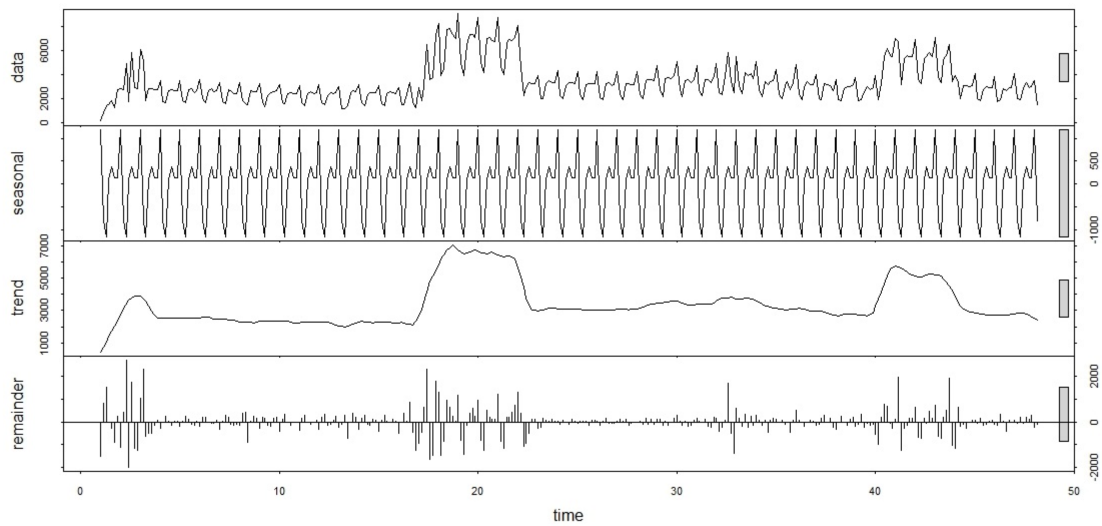

Figure A.5: Decomposition in components

Now we calculate forecasts and ranges of the three components:

```
> prev.stag<-predict(stag.stl,7)
> prev.stag
         Point Forecast       Lo 80       Hi 80       Lo 95       Hi 95
48.28571    -1142.75224 -1142.75224 -1142.75224 -1142.75224 -1142.75224
48.42857       89.56213    89.56213    89.56213    89.56213    89.56213
48.57143      378.56325   378.56325   378.56325   378.56325   378.56325
48.71429      131.13198   131.13198   131.13198   131.13198   131.13198
48.85714      147.16995   147.16995   147.16995   147.16995   147.16995
49.00000     1193.92431  1193.92431  1193.92431  1193.92431  1193.92431
49.14286     -797.59933  -797.59933  -797.59933  -797.59933  -797.59933
> prev.trend<-predict(trend.stl,7)
> prev.trend
         Point Forecast    Lo 80    Hi 80     Lo 95    Hi 95
48.28571       2313.735 2255.400 2372.070 2224.5195 2402.951
48.42857       2220.072 2046.462 2393.683 1954.5575 2485.587
48.57143       2131.398 1859.787 2403.009 1716.0049 2546.791
48.71429       2047.446 1677.720 2417.172 1481.9992 2612.893
48.85714       1967.966 1498.356 2437.575 1249.7603 2686.171
49.00000       1892.718 1321.512 2463.924 1019.1340 2766.302
49.14286       1821.478 1147.356 2495.600  790.4974 2852.459
> prev.res<-predict(res.stl,7)
> prev.res
         Point Forecast     Lo 80    Hi 80     Lo 95    Hi 95
48.28571       2.236474 -712.1711 716.6441 -1090.356 1094.829
48.42857       2.236474 -712.1712 716.6441 -1090.356 1094.829
48.57143       2.236474 -712.1712 716.6441 -1090.356 1094.829
48.71429       2.236474 -712.1712 716.6441 -1090.356 1094.829
48.85714       2.236474 -712.1712 716.6441 -1090.356 1094.829
49.00000       2.236474 -712.1712 716.6441 -1090.356 1094.829
49.14286       2.236474 -712.1712 716.6441 -1090.356 1094.829
```

Therefore, we get the following values of prediction:

```
          Point Forecast
48.28571    1173,219234
48.42857    2311,870604
48.57143    2512,197724
48.71429    2180,814454
48.85714    2117,372424
49.00000    3088,878784
49.14286    1026,115144
```

After estimating the seasonal component, it is necessary to prove that the seasonality over the months shows no trend; in this case we call it trend-season. A simple and immediate way that we use is to refer to the function `seaplot()`.

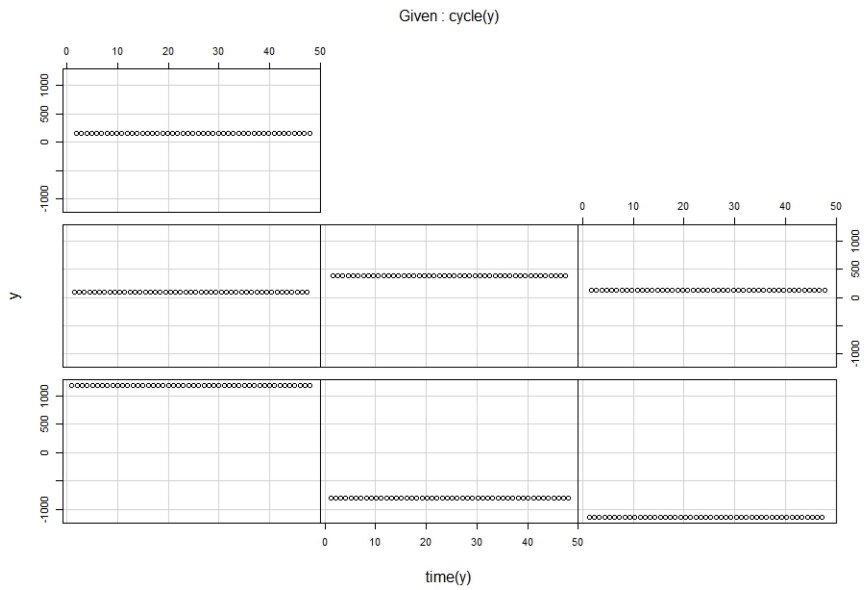

Figure A.6: Seaplot.stl

In our case, the seasonality is constant in every week, in fact, Figure A.6 shows the weekly sub-series of seasonality; the order is from left to right and from bottom to top (thus the first diagram at the bottom left is referred to the seasonality observed on the first day of the week, and so on).

A final prediction is obtained by the *ARIMA model*.
We show the graph (Figure A.7) and the results of the forecast:

```
> plot(previsione)
> previsione
```

	Point Forecast	Lo 80	Hi 80	Lo 95	Hi 95
48.28571	1423.474	572.2470	2274.700	121.6348	2725.313
48.42857	2568.165	1497.7902	3638.539	931.1682	4205.162
48.57143	2819.999	1679.1242	3960.875	1075.1814	4564.818
48.71429	2653.697	1414.4843	3892.909	758.4850	4548.908
48.85714	2535.439	1187.3477	3883.531	473.7111	4597.167
49.00000	3433.762	1978.2263	4889.297	1207.7124	5659.811
49.14286	1421.351	-101.5853	2944.287	-907.7789	3750.480
49.28571	1128.699	-545.8531	2803.251	-1432.3073	3689.705
49.42857	2399.166	581.1618	4217.169	-381.2312	5179.562
49.57143	2591.403	677.6506	4505.156	-335.4290	5518.236
49.71429	2486.855	486.8052	4486.905	-571.9573	5545.668
49.85714	2304.635	203.0068	4406.264	-909.5281	5518.799
50.00000	3264.413	1059.9267	5468.900	-107.0581	6635.885
50.14286	1282.890	-999.7158	3565.496	-2208.0542	4773.834

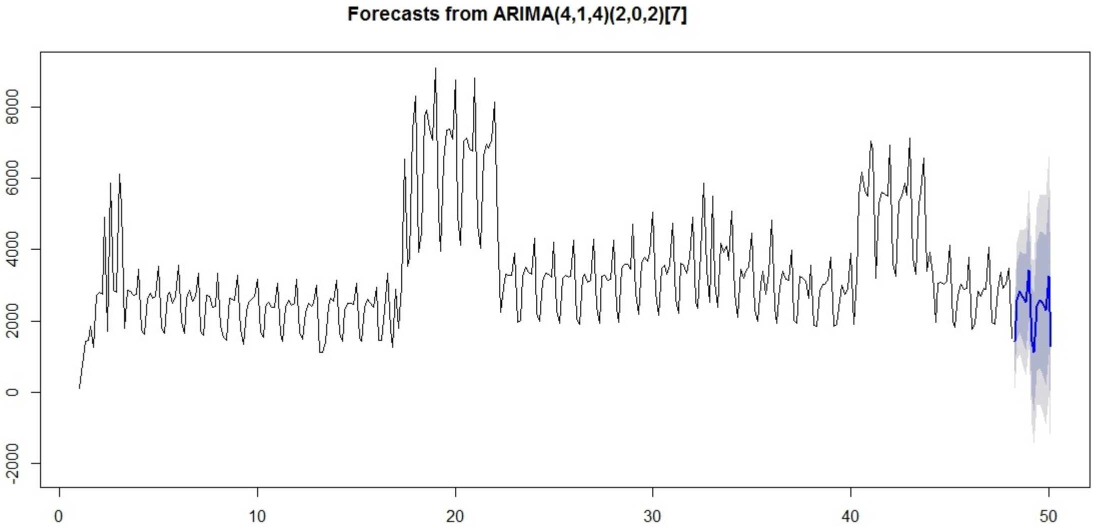

Figure A.7: ARIMA

After estimating the model, we must verify how well the data fit a statistical model. An important element for such a verification is the *coefficient of determination* R^2; it measures the rate of variability explained by the model compared to the variability of Y:

$$R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS}$$

where:

- $ESS = \sum_{t=1}^{T}(\hat{Y}_t - \overline{Y})^2$ is the regression sum of squares, also called the explained sum of squares;
- $TSS = \sum_{t=1}^{T}(Y_t - \overline{Y})^2$ is the total sum of squares (proportional to the variance of the data);
- $RSS = \sum_{t=1}^{T} R_t^2 = \sum_{t=1}^{T}(Y_t - \hat{Y}_y)^2$ is the sum of squares of residuals, also called the residual sum of squares.

When the coefficient R^2 is equal to 1, it means that the regression line perfectly fits the data, while when R^2 is equal to 0, it means that the line does

not fit the data at all.
Therefore, we calculate the coefficient for the Holt-Winters, the decomposition and the ARIMA models getting the following values: `0.8368674`, `0.8781064` and `0.7328934`, respectively. Since we obtain results very close to 1, we can conclude that the models could usefully be used for predictive purposes.

For the three obtained forecasts, we verified the validity of the assumptions (test specification):

- average waste equal to zero (t-test);
- errors normality (Shapiro-Wilk and Jarque-Bera);
- homoscedasticity (Breusch-Pagan);
- correlation between residues (Box-Pierce and Ljung-Box).

Only models passing all tests and having a good determination coefficient R^2 can be accepted. So, since for the examined product, in the *Holt-Winters model* and in the *ARIMA model* residuals are not normally distributed (and therefore the second assumption is not verified), only the *decomposition model* is acceptable and, hence, it is the one we will use for our optimization problem.
With a similar procedure we obtained estimates of all 7 goods produced by the considered production line, but, as a matter of corporate privacy, will not present them.

Appendix B

Inventory management

In this section, we study a model of inventory management in order to determine the production periods (days), which satisfy the demand forecast obtained in the previous section, while minimizing the production and the storing costs. Later, we shall apply our model so to determine the optimal amounts we are looking for.

The Wagner-Whitin model of inventory management (1958) is part of the Lot sizing problems, characterized by the fact that the demand and the production and storage costs may vary over time.
The aim is to determine the production $x_1, x_2, \ldots, x_N$ and the storage $s_1, s_2, \ldots, s_N$ in each period in such a way that the demands $d_1, d_2, \ldots, d_N$ (obtained in the previous section), have, in stock, an initial value $s_0 = 0$. Of course we should try to minimize production and storage costs. Graphically we have a situation shown in Figure B.1.

Since the production and the storage costs may vary over time, we have the following estimates:

- total production cost at time t:

$$C_t(x_t) = A_t\eta(x_t) + c_t(x_t);$$

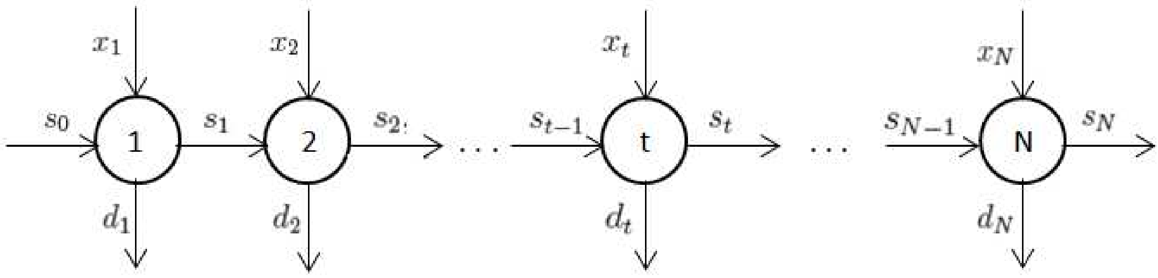

Figure B.1: Inventory management

- total storage cost at time t:

$$H_t(s_t) = \pi_t \eta(s_t) + h_t(s_t);$$

where: A_t and π_t are the fixed production cost and the fixed storage cost respectively;

$$\eta(x_t) = \begin{cases} 1 & \text{if } x_t > 0 \\ 0 & \text{if } x_t \leq 0; \end{cases}$$

c_t and h_t denote the production and the storage costs.
We have to solve the following optimization problem:

$$\begin{cases} \min f(x,s) = \sum_{t=1}^{N} \big(C_t(x_t) + H_t(s_t) \big) \\ x_t + s_{t-1} - s_t = d_t \qquad t = 1, \dots, N \\ x_t, s_t \geq 0 \qquad t = 1, \dots, N. \end{cases}$$

If (x, s) is a solution to the previous problem, then in any period $t \in \{1, \dots, N\}$ one of the following cases holds:

- if $x_t > 0$ and $s_{t-1} = 0$, then we have a positive production and no inventory;
- if $x_t = 0$ and $s_{t-1} > 0$, then we have no production and a positive inventory.

Definition 7. *If $x_t > 0$, then time $t \in \{1, \dots, N\}$ is called* ***productive time****.*

The demand in a nonproductive time $h \in \{1, \dots, N\}$ is satisfied by the production in the last productive time $k \in \{1, \dots, N\}$, where $k < h$.

Definition 8. *The set of nonproductive times when the demand is satisfied in a special productive time is called* ***production interval****.*

Let j be the productive time satisfying the demand of nonproductive times $\{j+1, \dots, k\}$, then these times are subsequent.
After establishing the production interval $\{j, j+1, \dots, k\}$, it is possible to evaluate the production and the storage:

$$\bar{x}_j = \sum_{r=j}^{k} d_r \text{ and } \bar{s}_t = \sum_{r=t+1}^{k} d_r, \quad t \in \{j, j+1, \dots, k\}.$$

Further, we have:

- $\bar{x}_t = 0$ for $t = j+1, \dots, k$;

- $\bar{s}_t > 0$ for $t = j, \dots, k-1$;

- $\bar{s}_{j-1} = \bar{s}_k = 0$.

A solution $(\bar{x}, \bar{s})$ depends on the set of productive times $\bar{J} = \{j_1, \dots, j_q\}$. Indeed, from this set we can deduce the production and the storage at each time.
In any productive interval $\{j, \dots, k\}$, the total cost, given by the sum of production and storage costs, is as follows:

$$M(j,k) = C_j(\bar{x}_j) + \sum_{t=j}^{k-1} H_t(\bar{s}_t) = C_j\Big(\sum_{r=j}^{k} d_r\Big) + \sum_{t=j}^{k-1} H_t\Big(\sum_{r=t+1}^{k} d_r\Big).$$

Now we introduce F_k, namely the minimum production and storage cost required in order to satisfy the demand in the interval $\{1, \dots, k\}$. Hence, F_N represents the optimal value.
We assume that the productive times form the set $J = \{j_1, \dots, j_q\}$; the total production and storage cost is given by:

$$Z(J) = \sum_{s=1}^{q-1} M(j_s, j_{s+1} - 1) + M(j_q, N).$$

As a consequence, we have to find the set $J^* \subseteq \{1, \ldots, N\}$ such that $Z(J^*) = F_N$.

B.1 Computational procedure

We find F_k recursively.
Assume we have to satisfy the demand in the time horizon $\{1, \ldots, k\}$ and we know the optimal solution $J_k = \{j_1, \ldots, j_q\}$. Then, the minimal cost in such a time interval is:

$$F_k = M(j_1, j_2 - 1) + M(j_2, j_3 - 1) + \ldots + M(j_{q-1}, j_q - 1) + M(j_q, k).$$

But, if the horizon is $\{j_1, \ldots, j_q - 1\}$, then we have:

$$F_{j_q - 1} = M(j_1, j_2 - 1) + M(j_2, j_3 - 1) + \ldots + M(j_{q-1}, j_q - 1),$$

therefore:

$$F_k = F_{j_q - 1} + M(j_q, k).$$

Time j_q represents the last productive time of the optimal solution associated with the time interval $\{1, \ldots, k\}$. If $j \in \{1, \ldots, k\}$ is the last productive time, then the minimal production and storage cost is given by $F_{j-1} + M(j, k)$.
As a consequence, $F_k \leq F_{j-1} + M(j, k)\ \forall j \in \{1, \ldots, k\}$.
Hence, we can conclude:

$$F_k = \min_{1 \leq j \leq k} \{F_{j-1} + M(j, k)\}.$$

Such a result is a recursive estimate of F_k depending on the values of F_j con $j = 1, \ldots, k - 1$. If we set $F_0 = 0$, we can calculate all the values $F_1, F_2, \ldots, F_N$, and the last one allows us to solve the problem.
We implemented the algorithm in `C++` code.

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