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Applications of Solvable Lie Algebras to a Class of Third Order Equations

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† M.S. Bruzón sadly passed away prior to the submission of this paper. This is one of her last works.

Abstract: A family of third-order partial differential equations (PDEs) is analyzed. This family broadens out well-known PDEs such as the Korteweg-de Vries equation, the Gardner equation, and the Burgers equation, which model many real-world phenomena. Furthermore, several macroscopic models for semiconductors considering quantum effects—for example, models for the transmission of electrical lines and quantum hydrodynamic models—are governed by third-order PDEs of this family. For this family, all point symmetries have been derived. These symmetries are used to determine group-invariant solutions from three-dimensional solvable subgroups of the complete symmetry group, which allow us to reduce the given PDE to a first-order nonlinear ordinary differential equation (ODE). Finally, exact solutions are obtained by solving the first-order nonlinear ODEs or by taking into account the Type-II hidden symmetries that appear in the reduced second-order ODEs.

Keywords: third-order partial differential equations; lie symmetries; solvable symmetry algebras; group invariant solutions

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1. Introduction

The study of integrable equations that model real-world phenomena has attracted a lot of attention from researchers in the last decades. In [1], Qiao and Liu proposed the following equation

$$u_t = \frac{1}{2}\left(\frac{1}{u^2}\right)_{xxx} - \frac{1}{2}\left(\frac{1}{u^2}\right)_x. \tag{1}$$

They showed that Equation (1) has a bi-Hamiltonian structure and Lax pair, which imply the integrability of the equation, and they stated that although the equation is completely integrable, no smooth solitons have been found.

In [2], Gandarias and Bruzón considered the generalized equation

$$u_t = (g(u))_{xxx} + (f(u))_x, \tag{2}$$

with $f(u)$ and $g(u)$ as arbitrary functions verifying $f'(u) \neq 0$, $g'(u) \neq 0$, and they constructed conservation laws for some subclasses of partial differential equation (PDE) (2).

The purpose of this paper is to analyze the generalized equation

$$u_t = (g(u))_{xxx} + (f(u))_x + h(u)u_{xx}, \tag{3}$$

with $f(u)$, $g(u)$, and $h(u)$ as arbitrary functions verifying $f'(u) \neq 0$, $g'(u) \neq 0$. A special subclass of family (3) was studied in [3].

The family of PDEs (3) that we are going to deal with includes the well-known Korteweg-de Vries equation, the Gardner equation, and the Burgers equation, among others.

Moreover, several macroscopic models for semiconductors considering quantum effects—for instance, models for the transmission of electrical lines and quantum hydrodynamic models, models of aqueous polymer solutions in a bounded domain, or two-dimensional grade-two fluid models—are governed by third-order PDEs. Many of these models are included in family (3). For further detailed examples, the reader is referred, for instance, to [4–9].

Nonlinear evolution equations have attracted the attention of numerous researchers during the last years because they turn out to be more realistic than their linear counterparts in the applications to real-world phenomena which usually include diffusion, convection, or dispersion processes as well as other nonlinear effects.

The main barrier to the systematic analysis of nonlinear PDEs involving arbitrary functions or parameters is that often, there are no tools that can be applied in general for a specific purpose. Indeed, there exist methods that only work for a particular set of equations, but even so, the subsequent mathematics is often quite different from linear PDEs, and they generally present an evident difficulty and complexity. In particular, various direct methods have been developed to deal with the determination of exact solutions of nonlinear PDEs, for instance, the extended simplest equation method [10–12], the tanh-sech method [13–15], the Painlevé analysis [16,17], the variational iteration method [18], the Hirota's method [19], and other special methods.

Symmetries of a PDE are transformations that map the solution space of the PDE into itself. The Lie symmetry method has been proved to be an effective method to analyze PDEs. Symmetry groups have several well-known applications. For instance, invariance solutions can be constructed taking into account the local symmetries admitted. Invariance solutions emerge from solutions of a system of differential equations that involves a smaller number of independent variables. In the case of PDEs with two independent variables, the reduction procedure consists of obtaining a similarity variable that allows us to transform the PDE into an ordinary differential equation (ODE), which is, in general, easier to solve. Thus, symmetry groups can also be combined and applied with other methods to find exact and numerical solutions [20–25]. However, in this paper, we focus only on the application of solvable Lie groups and the reductions obtained from them.

The goal of this paper is to analyze PDE (3) from the viewpoint of Lie symmetries and symmetry reductions. In particular, we focus our attention on deriving group-invariant solutions from admitted three-dimensional solvable symmetry subalgebras of Equation (3), which allow us to reduce the given third-order PDE into a first-order ODE. To the best of our knowledge, the analysis performed in this paper for the PDE family (3) has not been previously carried out. First, in Section 2, we have determined a complete classification of the point symmetries admitted by PDE (3) depending on the arbitrary functions $f(u)$, $g(u)$, and $h(u)$. The results presented for all cases $h(u) \neq 0$ are new. Furthermore, in Section 3, we determine a complete classification of the maximal symmetry groups along with its non-zero commutator structure that PDE (3) admits depending on $f(u)$, $g(u)$, and $h(u)$. In Section 4, we have determined the solvable three- and four-dimensional subgroups of the symmetry group of PDE (3). As far as we know, the analysis set forth in this paper is novel. Although many well-known classes of PDEs, which have been studied over the last years by using point symmetries, are included in family (3), the results obtained in this paper not only include numerous other equations which have not previously studied from the point of view of Lie symmetry reductions but also allow a global analysis of the family considered. In Section 5, we determine group-invariant solutions of Equation (3) from three-dimensional solvable symmetry algebras. Finally, in Section 6, we present the conclusions.

2. Point Symmetries

In order to determine the point symmetries of Equation (3), we consider a one-parameter Lie group of infinitesimal transformations given by

$$\begin{aligned} \hat{t}(t,x,u;\epsilon) &= t+\epsilon\,\tau(t,x,u)+O(\epsilon^2), \\ \hat{x}(t,x,u;\epsilon) &= x+\epsilon\,\xi(t,x,u)+O(\epsilon^2), \\ \hat{u}(t,x,u;\epsilon) &= u+\epsilon\,\eta(t,x,u)+O(\epsilon^2), \end{aligned} \tag{4}$$

where ϵ is the group parameter. We recall that a point transformation group is a Lie point symmetry of Equation (3) if and only if the action of the group (4) leaves the solution space invariant. A general element of the associated Lie algebra of Equation (3) takes the form

$$X=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u. \tag{5}$$

Point symmetries are obtained by applying the symmetry invariance condition

$$pr^{(3)}X(u_t-(g(u))_{xxx}-(f(u))_x-h(u)u_{xx})=0 \quad \text{when} \quad u_t-(g(u))_{xxx}-(f(u))_x-h(u)u_{xx}=0,$$

where $pr^{(3)}X$ is the prolongation of generator X to the space of the derivatives of the dependent variable up to third order. Third-order prolongations are complicated to compute, and they involve a great number of calculations. However, there exists a geometrical way to avoid these prolongations [26,27]. The action of the vector field (5) on the solution space of Equation (3) is similar to the action of the generator

$$\hat{X}=\hat{\eta}\partial_u, \quad \hat{\eta}=\eta-\tau u_t-\xi u_x,$$

which is known as the characteristic form of the point symmetry. Consequently, the set of solutions of Equation (3) is preserved under the transformation (4) provided that

$$pr^{(3)}\hat{X}(u_t-(g(u))_{xxx}-(f(u))_x-h(u)u_{xx})=0, \tag{6}$$

when Equation (3) holds. Here, $pr^{(3)}\hat{X}=\hat{X}+(D_t\hat{\eta})\partial_{u_t}+(D_x\hat{\eta})\partial_{u_x}+\left(D_x^2\hat{\eta}\right)\partial_{u_{xx}}+\left(D_x^3\hat{\eta}\right)\partial_{u_{xxx}}$ is the third prolongation of the vector field $\hat{X}$, and D_t and D_x represent the total derivatives with respect to t and x, respectively.

The symmetry determining Equation (6) leads to a linear system of determining equations. By simplifying this system, we obtain that $\tau=\tau(t)$, $\xi=\xi(t,x)$, $\eta=\eta(t,x,u)$, $f(u)$, $g(u)$, and $h(u)$ must satisfy the following conditions:

$$\begin{aligned} \eta g_{uu}+g_u(\tau_t-3\xi_x)&=0, \\ \eta_{uu}g_u^2+\eta_u g_u g_{uu}+\eta\left(g_u g_{uuu}-g_{uu}^2\right)&=0, \\ \eta_{xxx}g_u+\eta_{xx}h+\eta_x f_u-\eta_t&=0, \\ 3\eta_{ux}g_u^2+3\eta_x g_u g_{uu}+\eta(g_u h_u-g_{uu}h)-3\xi_{xx}g_u^2+\xi_x g_u h&=0, \\ 3\eta_{uux}g_u+\eta_{uu}h+6\eta_{ux}g_{uu}+3\eta_x g_{uuu}-3\xi_{xx}g_{uu}&=0, \\ 3\eta_{uxx}g_u^2+2\eta_{ux}g_u h+3\eta_{xx}g_u g_{uu}+\eta(f_{uu}g_u-f_u g_{uu})-\xi_{xxx}g_u^2-\xi_{xx}g_u h+2\xi_x f_u g_u+\xi_t g_u&=0. \end{aligned} \tag{7}$$

We notice that family (3) is preserved under the equivalence transformation given by

$$\tilde{u}\longrightarrow u+u_0, \quad u_0 \text{ constant}.$$

This allows us to simplify the results achieved on point symmetries. From system (7), if $f(u)$, $g(u)$ and $h(u)$ are arbitrary, we obtain

$$X_1=\partial_x, \quad X_2=\partial_t.$$

Additional generators are admitted by the generalized third-order Equation (3) in the following cases:

1. For arbitrary $g(u)$, $h(u) = 0$, $f(u) = f_1 u + f_2$,

$$X_3 = 3t\partial_t + (x - 2f_1 t)\partial_x.$$

2. For $g(u) = g_0 u^q + g_1$,

2.1. For $h(u) = h_0 u^m$

2.1.1. For $f(u) = f_0 u^{2m-q+2} + f_1 u + f_2$, we obtain

$$X_4 = (3m - 2q + 2)t\partial_t + ((m - q + 1)x - f_1(2m - q + 1)t)\partial_x - u\partial_u.$$

Moreover:

2.1.1.1. If $m = 0$ and $q = 1$ (we suppose $f_0 = 0$ without losing generality), we also obtain

$$\begin{array}{rcl} X_5 & = & 9g_0 t\partial_t + (3g_0 x + 2(h_0^2 - 3f_1 g_0)t)\partial_x - h_0(x + f_1 t)u\partial_u, \\ X_\beta & = & \beta\partial_u, \end{array}$$

where $\beta(t, x)$ verifies $\beta_t - f_1\beta_x - h_0\beta_{xx} - g_0\beta_{xxx} = 0$.

2.1.1.2. If $h_0 = f_0 = 0$, we also obtain X_3.

2.1.1.3. If $h_0 = f_0 = 0$, $q = -\frac{1}{2}$, we also obtain X_3 and

$$X_6 = (x + f_1 t)^2\partial_x - 4(x + f_1 t)u\partial_u.$$

2.1.1.4. If $h_0 = 0$, $q = -\frac{1}{2}$, $m = -\frac{3}{2}$, $\frac{f_0}{g_0} > 0$, we obtain

$$\begin{array}{rcl} X_7 & = & \sin\Big(\sqrt{\frac{f_0}{g_0}}(x + f_1 t)\Big)\partial_x - 2\sqrt{\frac{f_0}{g_0}}\cos\Big(\sqrt{\frac{f_0}{g_0}}(x + f_1 t)\Big)u\,\partial_u, \\ X_8 & = & \cos\Big(\sqrt{\frac{f_0}{g_0}}(x + f_1 t)\Big)\partial_x + 2\sqrt{\frac{f_0}{g_0}}\sin\Big(\sqrt{\frac{f_0}{g_0}}(x + f_1 t)\Big)u\,\partial_u. \end{array}$$

2.1.1.5. If $h_0 = 0$, $q = -\frac{1}{2}$, $m = -\frac{3}{2}$, $\frac{f_0}{g_0} < 0$, we obtain

$$\begin{array}{rcl} X_9 & = & \sinh\Big(\sqrt{-\frac{f_0}{g_0}}(x + f_1 t)\Big)\partial_x - 2\sqrt{-\frac{f_0}{g_0}}\cosh\Big(\sqrt{-\frac{f_0}{g_0}}(x + f_1 t)\Big)u\,\partial_u, \\ X_{10} & = & \cosh\Big(\sqrt{-\frac{f_0}{g_0}}(x + f_1 t)\Big)\partial_x - 2\sqrt{-\frac{f_0}{g_0}}\sinh\Big(\sqrt{-\frac{f_0}{g_0}}(x + f_1 t)\Big)u\,\partial_u. \end{array}$$

2.1.2. For $f(u) = f_0 u^2 + f_1 u + f_2$, $f_0 \neq 0$ and $m = 0$, $q = 1$,

$$X_{11} = 2f_0 t\partial_x - \partial_u.$$

Moreover, if $h_0 = 0$, we also obtain

$$X_{12} = 3t\partial_t + x\partial_x - (2u + \frac{f_1}{f_0})\partial_u.$$

2.1.3. For $f(u) = f_0 u \ln u + f_1 u + f_2$, $f_0 \neq 0$ and $m = \frac{q-1}{2}$, we obtain

$$X_{13} = (q - 1)t\partial_t + ((q - 1)x - 2f_0 t)\partial_x + 2u\partial_u.$$

2.1.4. For $f(u) = f_0 \ln u + f_1 u + f_2$, $f_0 \neq 0$ and $m = \frac{q-2}{2}$, we obtain

$$X_4\big|_{m=\frac{q-2}{2}} \equiv (q + 2)t\partial_t + (qx - 2f_1 t)\partial_x + 2u\partial_u.$$

2.2. For $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{2mu} + f_1 u + f_2$, $m \neq 0$ and $q = 1$, and where f_0 and h_0 are not simultaneously zero, we obtain

$$X_{14} = 3mt\partial_t + m(x - 2f_1 t)\partial_x - \partial_u.$$

3. For $g(u) = g_0 e^{qu} + g_1$

   3.1. For $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{(2m-q)u} + f_1 u + f_2$, we obtain

$$X_{15} = (3m - 2q)t\partial_t + ((m-q)x - f_1(2m-q)t)\partial_x - \partial_u.$$

   Moreover, if $h_0 = f_0 = 0$, we also obtain X_3.

   3.2. If $h(u) = h_0 e^{\frac{q}{2}u}$, $f(u) = f_0 u^2 + f_1 u + f_2$, we obtain

$$X_{16} = qt\partial_t + (qx - 4f_0 t)\partial_x + 2\partial_u.$$

   Moreover, if $h_0 = f_0 = 0$, we also obtain X_3.

4. For $g(u) = g_0 \ln(u) + g_1$

   4.1. For $h(u) = h_0 u^m$, $f(u) = f_0 u^{2m+2} + f_1 u + f_2$, we obtain

$$X_{17} = (3m+2)t\partial_t + ((m+1)x - f_1(2m+1)t)\partial_x - u\partial_u.$$

   Moreover, if $h_0 = f_0 = 0$, we also obtain X_3.

   4.2. If $h(u) = h_0 u^{-\frac{1}{2}}$, $f(u) = f_0 u \ln u + f_1 u + f_2$, $f_0 \neq 0$, we obtain

$$X_{18} = t\partial_t + (x + 2f_0 t)\partial_x - 2u\partial_u.$$

   4.3. If $h(u) = h_0 u^{-1}$, $f(u) = f_0 \ln u + f_1 u + f_2$, $f_0 \neq 0$,

$$X_{19} = t\partial_t - f_1 t\partial_x + u\partial_u.$$

In the above, f_0, f_1, f_2, $g_0 \neq 0$, g_1, h_0, $q \neq 0$, and m represent arbitrary constants. When $h_0 = 0$, without loss of generality, we can set $m = 0$.

3. Maximal Point Symmetry Groups

At this point, it would be very valuable to know the most general symmetry Lie algebra $\mathcal{A}$ that the equation admits depending on the form of the arbitrary functions $f(u)$, $g(u)$, and $h(u)$. We suppose that $\mathcal{A}$ is a r-dimensional Lie algebra with basis $X_1, \ldots, X_r$. Each $X_i \in \mathcal{A}$ defines a linear operator ad $X_i : \mathcal{A} \longrightarrow \mathcal{A}$, ad $X_i(X_j) = [X_i, X_j]$, where $[\ ,\ ]$ represents the Lie bracket. Lie algebras can be represented in tabular form by means of the commutator table for $\mathcal{A}$ which is a $r \times r$ table whose (i,j)-th entry represents the Lie bracket $[X_i, X_j]$. Given X_i, $X_j \in \mathcal{A}$, $[X_i, X_j] = -[X_j, X_i]$, therefore, this table is always skew-symmetric. The commutator table is useful to determine an optimal system of subalgebras or to construct a solvable Lie subalgebra.

As a result of the great number of maximal Lie algebras that Equation (3) admits and in order not to exceed unnecessarily the length of this paper, we do not show the commutator tables. Instead, we will show a basis of generators for each maximal Lie algebra along with the corresponding non-zero Lie brackets. The maximal point symmetry groups for the generalized third-order Equation (3) are generated by:

(i) Two-dimensional

   arbitrary $g(u)$, $h(u)$, $f(u)$
   $\mathcal{A}_1 = span(X_1, X_2)$.

(ii) Three-dimensional

- arbitrary $g(u)$, $h(u) = 0$, $f(u) = f_1 u + f_2$,
  $\mathcal{A}_2 = span(X_1, X_2, X_3)$,
  $[X_1, X_3] = X_1, \quad [X_2, X_3] = -2f_1 X_1 + 3X_2$.
- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^m$, $f(u) = f_0 u^{2m-q+2} + f_1 u + f_2$,
  $\mathcal{A}_3 = span(X_1, X_2, X_4)$,
  $[X_1, X_4] = (m - q + 1)X_1, \quad [X_2, X_4] = -f_1(2m - q + 1)X_1 + (3m - 2q + 2)X_2$.

- $g(u) = g_0 u + g_1$, $h(u) = h_0$, $f(u) = f_0 u^2 + f_1 u + f_2$, $f_0 \neq 0$,
  $\mathcal{A}_4 = span(X_1, X_2, X_{11})$,
  $[X_2, X_{11}] = 2f_0 X_1$.
- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^{\frac{q-1}{2}}$, $f(u) = f_0 u \ln u + f_1 u + f_2$, $f_0 \neq 0$
  $\mathcal{A}_5 = span(X_1, X_2, X_{13})$,
  $[X_1, X_{13}] = (q-1)X_1, \quad [X_2, X_{13}] = -2f_0 X_1 + (q-1)X_2$.
- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^{\frac{q-2}{2}}$, $f(u) = f_0 \ln u + f_1 u + f_2$, $f_0 \neq 0$
  $\mathcal{A}_6 = span(X_1, X_2, X_4|_{m=\frac{q-2}{2}})$,
  $\left[X_1, X_4|_{m=\frac{q-2}{2}}\right] = -\frac{q}{2} X_1, \quad \left[X_2, X_4|_{m=\frac{q-2}{2}}\right] = f_1 X_1 - \frac{q+2}{2} X_2$.
- $g(u) = g_0 u + g_1$, $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{2mu} + f_1 u + f_2$,
  $m \neq 0$, f_0 and h_0 not simultaneously zero,
  $\mathcal{A}_7 = span(X_1, X_2, X_{14})$,
  $[X_1, X_{14}] = mX_1, \quad [X_2, X_{14}] = -2f_1 m X_1 + 3mX_2$.
- $g(u) = g_0 e^{qu} + g_1$, $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{(2m-q)u} + f_1 u + f_2$,
  $\mathcal{A}_8 = span(X_1, X_2, X_{15})$,
  $[X_1, X_{15}] = (m-q)X_1, \quad [X_2, X_{15}] = -f_1(2m-q)X_1 + (3m-2q)X_2$.
- $g(u) = g_0 e^{qu} + g_1$, $h(u) = h_0 e^{\frac{q}{2}u}$, $f(u) = f_0 u^2 + f_1 u + f_2$,
  $\mathcal{A}_9 = span(X_1, X_2, X_{16})$,
  $[X_1, X_{16}] = qX_1, \quad [X_2, X_{16}] = -4f_0 X_1 + qX_2$.
- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^m$, $f(u) = f_0 u^{2m+2} + f_1 u + f_2$,
  $\mathcal{A}_{10} = span(X_1, X_2, X_{17})$,
  $[X_1, X_{17}] = (m+1)X_1, \quad [X_2, X_{17}] = -f_1(2m+1)X_1 + (3m+2)X_2$.
- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^{-\frac{1}{2}}$, $f(u) = f_0 u \ln u + f_1 u + f_2$,
  $\mathcal{A}_{11} = span(X_1, X_2, X_{18})$,
  $[X_1, X_{18}] = X_1, \quad [X_2, X_{18}] = 2f_0 X_1 + X_2$.
- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^{-1}$, $f(u) = f_0 \ln u + f_1 u + f_2$,
  $\mathcal{A}_{12} = span(X_1, X_2, X_{19})$,
  $[X_2, X_{19}] = -f_1 X_1 + X_2$.

(iii) Four-dimensional

- $g(u) = g_0 u^q + g_1$, $q \neq 1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,
  $\mathcal{A}_{13} = span(X_1, X_2, X_3, X_4|_{m=0})$,
  $[X_1, X_3] = X_1, \quad [X_1, X_4|_{m=0}] = (1-q)X_1$,
  $[X_2, X_3] = -2f_1 X_1 + 3X_2, \quad [X_2, X_4|_{m=0}] = (q-1)(f_1 X_1 - 2X_2)$.
- $g(u) = g_0 u + g_1$, $h(u) = 0$, $f(u) = f_0 u^2 + f_1 u + f_2$,
  $\mathcal{A}_{14} = span(X_1, X_2, X_{11}, X_{12})$,
  $[X_1, X_{12}] = X_1, \quad [X_2, X_{11}] = 2f_0 X_1, \quad [X_2, X_{12}] = 3X_2, \quad [X_{11}, X_{12}] = -2X_{11}$.
- $g(u) = g_0 e^{qu} + g_1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,
  $\mathcal{A}_{15} = span(X_1, X_2, X_3, X_{15}|_{m=0})$,
  $[X_1, X_3] = X_1, \quad [X_1, X_{15}|_{m=0}] = -qX_1, \quad [X_2, X_3] = -2f_1 X_1 + 3X_2$,
  $[X_2, X_{15}|_{m=0}] = q(f_1 X_1 - 2X_2)$.
- $g(u) = g_0 \ln u + g_1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,
  $\mathcal{A}_{16} = span(X_1, X_2, X_3, X_{17}|_{m=0})$,
  $[X_1, X_3] = X_1, \quad [X_1, X_{17}|_{m=0}] = X_1, \quad [X_2, X_3] = -2f_1 X_1 + 3X_2$,
  $[X_2, X_{17}|_{m=0}] = -f_1 X_1 + 2X_2$.

(iv) Five-dimensional

- $g(u) = \frac{g_0}{\sqrt{u}} + g_1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,
  $\mathcal{A}_{17} = span(X_1, X_2, X_3, X_4|_{q=-\frac{1}{2}, m=0}, X_6)$,
  $[X_1, X_3] = X_1, \quad \left[X_1, X_4|_{q=-\frac{1}{2}, m=0}\right] = \frac{3}{2} X_1, \quad [X_1, X_6] = 4\left(X_4|_{q=-\frac{1}{2}, m=0} - X_3\right)$,
  $[X_2, X_3] = -2f_1 X_1 + 3X_2, \quad \left[X_2, X_4|_{q=-\frac{1}{2}, m=0}\right] = -\frac{3}{2} f_1 X_1 + 3X_2$,

$$[X_2, X_6] = 4f_1\left(X_4|_{q=-\frac{1}{2},m=0} - X_3\right),$$
$$[X_3, X_6] = X_6, \quad \left[X_4|_{q=-\frac{1}{2},m=0}, X_6\right] = \tfrac{3}{2}X_6.$$

- $g(u) = \frac{g_0}{\sqrt{u}} + g_1,\ h(u) = 0,\ f(u) = \frac{f_0}{\sqrt{u}} + f_1 u + f_2,\ \frac{f_0}{g_0} > 0$
  $\mathcal{A}_{18} = span(X_1, X_2, X_4|_{q=-\frac{1}{2},m=-\frac{3}{2}}, X_7, X_8),$
  $$[X_1, X_7] = \sqrt{\tfrac{f_0}{g_0}}X_8, \quad [X_1, X_8] = -\sqrt{\tfrac{f_0}{g_0}}X_7,$$
  $$\left[X_2, X_4|_{q=-\frac{1}{2},m=-\frac{3}{2}}\right] = \tfrac{3}{2}(f_1X_1 - X_2),$$
  $$[X_2, X_7] = f_1\sqrt{\tfrac{f_0}{g_0}}X_8, \quad [X_2, X_8] = -f_1\sqrt{\tfrac{f_0}{g_0}}X_7, \quad [X_7, X_8] = -\sqrt{\tfrac{f_0}{g_0}}X_1.$$
- $g(u) = \frac{g_0}{\sqrt{u}} + g_1,\ h(u) = 0,\ f(u) = \frac{f_0}{\sqrt{u}} + f_1 u + f_2,\ \frac{f_0}{g_0} < 0$
  $\mathcal{A}_{19} = span(X_1, X_2, X_4|_{q=-\frac{1}{2},m=-\frac{3}{2}}, X_9, X_{10}),$
  $$[X_1, X_9] = \sqrt{-\tfrac{f_0}{g_0}}X_{10}, \quad [X_1, X_{10}] = \sqrt{-\tfrac{f_0}{g_0}}X_9,$$
  $$\left[X_2, X_4|_{q=-\frac{1}{2},m=-\frac{3}{2}}\right] = \tfrac{3}{2}(f_1X_1 - X_2),$$
  $$[X_2, X_9] = f_1\sqrt{-\tfrac{f_0}{g_0}}X_{10}, \quad [X_2, X_{10}] = f_1\sqrt{-\tfrac{f_0}{g_0}}X_9, \quad [X_9, X_{10}] = -\sqrt{-\tfrac{f_0}{g_0}}X_1.$$

(v) ∞-dimensional

$$g(u) = g_0u + g_1,\ h(u) = h_0,\ f(u) = f_1u + f_2,$$
$$\mathcal{A}_{20} = span(X_1, X_2, X_4|_{q=1,m=0}, X_5, X_\beta).$$

4. Solvable Lie Algebras

It is well known that when a PDE with two independent variables admits infinitesimal symmetries, this can be useful to reduce the PDE to an ODE; for details about this reduction procedure, see, e.g., [28]. Nevertheless, it is not always obvious how to solve this ODE. In fact, not all third-order nonlinear ODEs can be solved in explicit form. One alternative is to prove if the third-order nonlinear ODE inherits a three-dimensional solvable Lie group from Equation (3). In this way, Equation (3) can be reduced to quadrature. We recall that the necessary condition to determine a quadrature of Equation (3) is that the starting reduction results from invariance under a point symmetry which belongs to a four-dimensional solvable Lie group. Following reference [27], $\mathcal{A}^k$ is a k-dimensional solvable Lie algebra if there is a chain of subalgebras $\mathcal{A}^{(1)} \subset \mathcal{A}^{(2)} \subset \ldots \subset \mathcal{A}^{(k-1)} \subset \mathcal{A}^{(k)} = \mathcal{A}^k$, with $\mathcal{A}^{(m)}$ an m-dimensional Lie algebra, being $\mathcal{A}^{(m-1)}$ an ideal of $\mathcal{A}^{(m)}$, $m = 1, 2, \ldots, k$. This result can be alternatively formulated as $\mathcal{A} \supset \mathcal{A}^{(1)} \supset \mathcal{A}^{(2)} \supset \ldots \supset \mathcal{A}^{(k)} \supset \mathcal{A}^{(k+1)} = 0$, with $\mathcal{A}^{(m)} = \left[\mathcal{A}^{(m-1)}, \mathcal{A}^{(m-1)}\right]$, $m = 1, 2, \ldots, k \leq \dim \mathcal{A}$.

We focus our attention on three-dimensional solvable symmetry algebras of Equation (3). We denote $\mathcal{A}_\mathcal{S}$ as the possible three-dimensional solvable symmetry algebras and $\mathcal{G}$ as the related solvable symmetry groups. The generators that belong to $\mathcal{A}_\mathcal{S}$ can be taken so that the starting generator leads to an ODE that inherits a two-dimensional symmetry algebra. This condition is similar to the one requiring that the initial one-dimensional symmetry group belongs to a three-dimensional symmetry group spanned by X, Y, Z whose commutator structure is given by

$$[X, Y] = k_1X, \quad [X, Z] = k_2X, \quad [Y, Z] = k_3Y, \tag{8}$$

where k_1, k_2, and k_3 are constants. Let us take a generator X in the abelian subalgebra $\mathcal{A}^{(k)}$; in that case, $\mathcal{A}_X = \mathcal{A}_\mathcal{S}/\text{span}(X) = \text{span}(Y, Z)$ will provide us a two-dimensional symmetry algebra spanned by Y and Z, which will be inherited by the third-order nonlinear ODE obtained for the initial reduction X. Thus, we will be able to transform Equation (3) to a first-order nonlinear ODE. If $\mathcal{A}^{(k)}$ is one-dimensional or three-dimensional, X will be any generator in $\mathcal{A}^{(k)}$. However, in the case that $\mathcal{A}^{(k)}$ is two-dimensional, the form of generator X will be determined by the adjoint action of $\mathcal{G}$ on $\mathcal{A}^{(k)}$. If such an action presents one-dimensional orbits, then X will be either one of the two orbits in $\mathcal{G}$. On the contrary, if the action leads to two-dimensional orbits, X will be any generator belonging to $\mathcal{A}^{(k)}$.

We are interested in those three-dimensional solvable symmetry algebras such that the starting generator X does not belong to $\text{span}(X_1)$ or $\text{span}(X_2)$, since in these cases, we will obtain solutions $u(t,x) = u(t)$ and $u(t,x) = u(x)$, respectively. When this is possible, a convenient choice of X, Y, and Z satisfying condition (8) is shown.

The generalized third-order Equation (3) admits the following three-dimensional solvable symmetry subalgebras:

- Arbitrary $g(u)$, $h(u) = 0$, $f(u) = f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = 3t\partial_t + (x - 2f_1 t)\partial_x, \tag{9}$$

$$\mathcal{A}_2 = \text{span}(X_1, X_2, X_3), \quad \mathcal{A}_2^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_2^{(2)} = 0, \tag{10}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_3. \tag{11}$$

- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^m$, $f(u) = f_0 u^{2m-q+2} + f_1 u + f_2$:

$$X_1 = \partial_x, \quad X_2 = \partial_t, \tag{12}$$

$$X_4 = (3m - 2q + 2)t\partial_t + ((m - q + 1)x - f_1(2m - q + 1)t)\partial_x - u\partial_u,$$

$$\mathcal{A}_3 = \text{span}(X_1, X_2, X_4), \quad \mathcal{A}_3^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_3^{(2)} = 0, \tag{13}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_4. \tag{14}$$

- $g(u) = g_0 u + g_1$, $h(u) = h_0$, $f(u) = f_0 u^2 + f_1 u + f_2$, $f_0 \neq 0$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{11} = 2f_0 t\partial_x - \partial_u, \tag{15}$$

$$\mathcal{A}_4 = \text{span}(X_1, X_2, X_{11}), \quad \mathcal{A}_4^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_4^{(2)} = 0. \tag{16}$$

- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^{\frac{q-1}{2}}$, $f(u) = f_0 u \ln u + f_1 u + f_2$, $f_0 \neq 0$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{13} = (q-1)t\partial_t + ((q-1)x - 2f_0 t)\partial_x + 2u\partial_u, \tag{17}$$

$$\mathcal{A}_5 = \text{span}(X_1, X_2, X_{13}), \quad \mathcal{A}_5^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_5^{(2)} = 0. \tag{18}$$

- $g(u) = g_0 u^q + g_1$, $h(u) = h_0 u^{\frac{q-2}{2}}$, $f(u) = f_0 \ln u + f_1 u + f_2$, $f_0 \neq 0$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_4\big|_{m=\frac{q-2}{2}} \equiv (q+2)t\partial_t + (qx - 2f_1 t)\partial_x + 2u\partial_u, \tag{19}$$

$$\mathcal{A}_6 = \text{span}\Big(X_1, X_2, X_4\big|_{m=\frac{q-2}{2}}\Big), \quad \mathcal{A}_6^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_6^{(2)} = 0, \tag{20}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_4\big|_{m=\frac{q-2}{2}}. \tag{21}$$

- $g(u) = g_0 u + g_1$, $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{2mu} + f_1 u + f_2$, $m \neq 0$,
  f_0 and h_0 not simultaneously zero,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{14} = 3mt\partial_t + m(x - 2f_1 t)\partial_x - \partial_u, \tag{22}$$

$$\mathcal{A}_7 = \text{span}(X_1, X_2, X_{14}), \quad \mathcal{A}_7^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_7^{(2)} = 0, \tag{23}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_{14}. \tag{24}$$

- $g(u) = g_0 e^{qu} + g_1$, $h(u) = h_0 e^{mu}$, $f(u) = f_0 e^{(2m-q)u} + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{15} = (3m - 2q)t\partial_t + ((m - q)x - f_1(2m - q)t)\partial_x - \partial_u, \tag{25}$$

$$\mathcal{A}_8 = \text{span}(X_1, X_2, X_{15}), \quad \mathcal{A}_8^{(1)} = \text{span}(X_1, X_2), \quad \mathcal{A}_8^{(2)} = 0, \tag{26}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_{15}. \tag{27}$$

- $g(u) = g_0 e^{qu} + g_1$, $h(u) = h_0 e^{\frac{q}{2}u}$, $f(u) = f_0 u^2 + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{16} = qt\partial_t + (qx - 4f_0 t)\partial_x + 2\partial_u, \tag{28}$$

$$\mathcal{A}_9 = \mathrm{span}(X_1, X_2, X_{16}), \quad \mathcal{A}_9^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_9^{(2)} = 0. \tag{29}$$

- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^m$, $f(u) = f_0 u^{2m+2} + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{17} = (3m+2)t\partial_t + ((m+1)x - f_1(2m+1)t)\partial_x - u\partial_u, \tag{30}$$

$$\mathcal{A}_{10} = \mathrm{span}(X_1, X_2, X_{17}), \quad \mathcal{A}_{10}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{10}^{(2)} = 0, \tag{31}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_{17}. \tag{32}$$

- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^{-\frac{1}{2}}$, $f(u) = f_0 u \ln u + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{18} = t\partial_t + (x + 2f_0 t)\partial_x - 2u\partial_u, \tag{33}$$

$$\mathcal{A}_{11} = \mathrm{span}(X_1, X_2, X_{18}), \quad \mathcal{A}_{11}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{11}^{(2)} = 0. \tag{34}$$

- $g(u) = g_0 \ln(u) + g_1$, $h(u) = h_0 u^{-1}$, $f(u) = f_0 \ln u + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{19} = t\partial_t - f_1 t\partial_x + u\partial_u, \tag{35}$$

$$\mathcal{A}_{12} = \mathrm{span}(X_1, X_2, X_{19}), \quad \mathcal{A}_{12}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{12}^{(2)} = 0, \tag{36}$$

$$X = -f_1 X_1 + X_2, \quad Y = X_1, \quad Z = X_{19}. \tag{37}$$

Furthermore, the generalized third-order Equation (3) also admits four four-dimensional solvable symmetry algebras:

- $g(u) = g_0 u^q + g_1$, $q \neq 1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = 3t\partial_t + (x - 2f_0 t)\partial_x,$$
$$X_4|_{m=0} = 2(1-q)t\partial_t + (1-q)(x - f_1 t)\partial_x - u\partial_u,$$
$$\mathcal{A}_{13} = \mathrm{span}(X_1, X_2, X_3, X_4|_{m=0}), \quad \mathcal{A}_{13}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{13}^{(2)} = 0.$$

- $g(u) = g_0 u + g_1$, $h(u) = 0$, $f(u) = f_0 u^2 + f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_{11} = 2f_0 t\partial_x - \partial_u, \quad X_{12} = 3t\partial_t + x\partial_x - (2u + \frac{f_1}{f_0}),$$
$$\mathcal{A}_{14} = \mathrm{span}(X_1, X_2, X_{11}, X_{12}), \quad \mathcal{A}_{14}^{(1)} = \mathrm{span}(X_1, X_2, X_{11}), \quad \mathcal{A}_{14}^{(2)} = \mathrm{span}(X_1),$$
$$\mathcal{A}_{14}^{(3)} = 0.$$

- $g(u) = g_0 e^{qu} + g_1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = 3t\partial_t + (x - 2f_0 t)\partial_x,$$
$$X_{15}|_{m=0} = -(2qt\partial_t + q(x - f_1 t)\partial_x + \partial_u),$$
$$\mathcal{A}_{15} = \mathrm{span}(X_1, X_2, X_3, X_{15}|_{m=0}), \quad \mathcal{A}_{15}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{15}^{(2)} = 0.$$

- $g(u) = g_0 \ln u + g_1$, $h(u) = 0$, $f(u) = f_1 u + f_2$,

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = 3t\partial_t + (x - 2f_0 t)\partial_x,$$
$$X_{17}|_{m=0} = 2t\partial_t + (x - f_1 t)\partial_x - u\partial_u,$$
$$\mathcal{A}_{16} = \mathrm{span}(X_1, X_2, X_3, X_{17}|_{m=0}),$$
$$\mathcal{A}_{16}^{(1)} = \mathrm{span}(X_1, X_2), \quad \mathcal{A}_{16}^{(2)} = 0.$$

It should be noted that algebras $\mathcal{A}_{13}$, $\mathcal{A}_{15}$, and $\mathcal{A}_{16}$ include the three-dimensional solvable symmetry algebra $\mathcal{A}_2$ given by (9) and (10), which can be explicitly solved for the initial generator X (11) as will be shown in the next section. Further information about four-dimensional solvable Lie symmetry algebras can be consulted in [26,27,29].

5. Symmetry Reductions and Exact Solutions

In this section, we determine group-invariant solutions of Equation (3) from the three-dimensional solvable symmetry algebras obtained in the previous section such that the starting generator X (8) is not included in span(X_1) or span(X_2).

5.1. Reduction by Using Solvable Lie Algebra $\mathcal{A}_2$

Let us consider the three-dimensional solvable symmetry group given by (9) and (10). Taking into account the symmetry generator X (11), we obtain the invariants

$$z = x + f_1 t, \quad U(z) = u, \tag{38}$$

where $U(z)$ must satisfy the third-order ODE

$$g'(U)U''' + 3g''(U)U'U'' + g'''(U)U'^3 = 0. \tag{39}$$

Equation (39) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z (11) which, in terms of the new variables, are given by

$$Y = \partial_z, \quad Z = z\partial_z, \tag{40}$$

satisfying $[Y, Z] = Y$. Therefore, Equation (39) can be integrated proceeding as follows. Y admits the invariants

$$\omega = U, \quad \chi = U', \tag{41}$$

from which ODE (39) can be transformed into a second-order ODE

$$g'(\omega)\left(\chi'^2 + \chi\chi''\right) + 3g''(\omega)\chi\chi' + g'''(\omega)\chi^2 = 0. \tag{42}$$

Moreover, $V = \mathrm{pr}^{(1)}Z\Big|_{(\omega,\chi)} = -\chi\partial_\chi$ is a symmetry of Equation (42). Invariants of V are given by

$$\phi = \omega, \quad \gamma = \frac{\chi'}{\chi}. \tag{43}$$

By substituting invariants (43) into Equation (42), we obtain a first-order ODE

$$g'(\phi)\left(2\gamma^2 + \gamma'\right) + 3g''(\phi)\gamma + g'''(\phi) = 0, \tag{44}$$

whose general solution is given by

$$\gamma(\phi) = -\frac{2\,g(\phi)g''(\phi) - g'(\phi)^2 + 2\,c_1 g''(\phi)}{2\,g'(\phi)(c_1 + g(\phi))}, \tag{45}$$

where c_1 is an arbitrary constant.

Undoing the change of variables (43), we obtain that

$$\chi(\omega) = \frac{2c_2\sqrt{c_1 + g(\omega)}}{g'(\omega)},$$

where c_2 is a constant of integration, and it is the general solution of Equation (42). Taking into account (41), we determine the solution of Equation (39), which is given by

$$U(z) = g^{-1}\left((c_2 z + c_3)^2 - c_1\right),$$

with c_3 an arbitrary constant.

Finally, by using invariants (38), we obtain the general solution of Equation (3) starting from the symmetry X given by (37)

$$u(x,t) = g^{-1}\Big((c_2(x+f_1t)+c_3)^2 - c_1\Big).$$

5.2. Reduction by Using Solvable Lie algebra $\mathcal{A}_3$

Now, we consider the three-dimensional solvable symmetry group given by (12) and (13). By using the symmetry generator X (14), we obtain the invariants

$$z = x + f_1 t, \quad U(z) = u, \tag{46}$$

where $U(z)$ satisfies

$$\begin{aligned} &g_0 q U^q U''' + h_0 U^{m+1} U'' + 3g_0 q(q-1)U^{q-1}U'U'' \\ &\quad + g_0 q(q-1)(q-2)U^{q-2}U'^3 + f_0(2m-q+2)U^{2m-q+2}U' = 0, \end{aligned} \tag{47}$$

which is a nonlinear third-order ODE. Equation (47) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z (14), which, after being written in the new variables, are given by

$$Y = \partial_z, \quad Z = (m-q+1)z\partial_z - U\partial_U, \tag{48}$$

verifying $[Y,Z] = (m-q+1)Y$. This allows us to integrate Equation (47) as follows. To begin with, Y admits the invariants

$$\omega = U, \quad \chi = U', \tag{49}$$

which implies that ODE (47) can be transformed into a second-order ODE

$$\begin{aligned} &g_0 q\omega^q(\chi'^2 + \chi\chi'') + \left(h_0\omega^{m+1} + 3g_0 q(q-1)\omega^{q-1}\chi\right)\chi' \\ &\quad + g_0 q(q-1)(q-2)\omega^{q-2}\chi^2 + f_0(2m-q+2)\omega^{2m-q+2} = 0. \end{aligned} \tag{50}$$

Furthermore, $V = \mathrm{pr}^{(1)}Z\Big|_{(\omega,\chi)} = -\omega\partial_\omega + (q-m-2)\chi\partial_\chi$ is a symmetry of Equation (50). Symmetry V yields the following invariants

$$\phi = \omega^{q-m-2}\chi, \quad \gamma = \omega^{q-m-1}\chi'. \tag{51}$$

By substituting (51) into Equation (50), we obtain the following first-order ODE

$$\begin{aligned} &g_0 q\phi(\gamma + (q-m-2)\phi)\gamma' + g_0 q\gamma^2 + (h_0 + g_0 q(2q+m-2)\phi)\gamma \\ &\quad + g_0 q(q-1)(q-2)\phi^2 + f_0(2m-q+2) = 0. \end{aligned} \tag{52}$$

5.3. Reduction by Using Solvable Lie Algebra $\mathcal{A}_6$

Equation (3) admits the three-dimensional solvable symmetry group given by (19) and (20). The symmetry generator X (21) yields the invariants

$$z = x + f_1 t, \quad U(z) = u, \tag{53}$$

where $U(z)$ must satisfy the third-order ODE

$$\begin{aligned} &g_0 q U^{q+2}U''' + 3g_0 q(q-1)U^{q+1}U'U'' \\ &\quad + h_0 U^{\frac{q+4}{2}}U'' + g_0 q(q-1)(q-2)U^q U'^3 + f_0 U^2 U' = 0. \end{aligned} \tag{54}$$

Equation (54) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z (21). First, we write the generators Y and Z in terms of the new variables

$$Y = \partial_z, \quad Z = qz\partial_z + 2U\partial_U, \tag{55}$$

which verify $[Y, Z] = qY$. Thus, we can integrate Equation (54) as follows. The generator Y admits the invariants

$$\omega = U, \quad \chi = U', \tag{56}$$

this allows us to transform (54) into a second-order ODE

$$g_0 q \omega^{q+2}\left(\chi\chi'' + \chi'^2\right) + \left(h_0 \omega^{\frac{q+4}{2}} + 3g_0 q(q-1)\omega^{q+1}\chi\right)\chi' + g_0 q(q-1)(q-2)\omega^q \chi^2 + f_0 \omega^2 = 0. \tag{57}$$

Furthermore, Equation (57) admits the generator $V = \mathrm{pr}^{(1)} Z\big|_{(\omega,\chi)} = 2\omega\partial_\omega + (2-q)\chi\partial_\chi$ as a symmetry. Invariants of V are given by

$$\phi = \omega^{\frac{q-2}{2}}\chi, \quad \gamma = \omega^{\frac{q}{2}}\chi'. \tag{58}$$

By substituting (58) into Equation (57), we obtain the following first-order ODE

$$g_0 q \phi(2\gamma + (q-2)\phi)\gamma' + 2g_0 q\gamma^2 + (2h_0 + g_0 q(5q-6)\phi)\gamma + 2g_0 q(q-1)(q-2)\phi^2 + 2f_0 = 0. \tag{59}$$

If $h_0 = 0$, the general solution of Equation (59) can be expressed as

$$\gamma(\phi) = \frac{2-q}{2}\phi - \frac{f_0}{g_0 q^2 \phi}(1 + W^*), \tag{60}$$

where W^* represents the principal value of the Lambert W-function evaluated in $-e^{c_1 - 1 + \frac{g_0 q^3 \phi^2}{2f_0}}$. We recall that the Lambert W-function is the inverse function of

$$f(W) = We^W.$$

5.4. Reduction by using solvable Lie algebra $\mathcal{A}_7$

Equation (3) admits the three-dimensional solvable symmetry group given by (22) and (23). Taking into account X (24), we obtain the invariants

$$z = x + f_1 t, \qquad U(z) = u, \tag{61}$$

where $U(z)$ satisfies the third-order ODE

$$g_0 U''' + h_0 e^{mU} U'' + 2f_0 m e^{2mU} U' = 0. \tag{62}$$

The third-order ODE (62) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z, which in terms of the new variables are given by

$$Y = \partial_z, \quad Z = mz\partial_z - \partial_U,$$

verifying $[Y, Z] = mY$. Therefore, ODE (62) can be integrated as follows. By taking into account generator Y, ODE (62) is reduced to the second-order ODE

$$g_0\left(\chi\chi'' + \chi'^2\right) + h_0 e^{m\omega}\chi' + 2f_0 m e^{2m\omega} = 0, \tag{63}$$

through the use of differential invariants

$$\omega = U, \quad \chi = U'. \tag{64}$$

Moreover, it is not difficult to check that ODE (63) inherits $V = \mathrm{pr}^{(1)} Z\big|_{(\omega,\chi)} = \partial_\omega + m\chi\partial_\chi$. Invariants of V are given by $\phi = e^{-m\omega}\chi$ and $\gamma = e^{-m\omega}\chi'$, from which ODE (63) is reduced to a first-order ODE for $\gamma(\phi)$

$$g_0\phi(\gamma - m\phi)\gamma' + g_0\gamma^2 + (h_0 + g_0 m\phi)\gamma + 2f_0 m = 0. \tag{65}$$

However, when $h_0 = 0$, two new point symmetries, known as Type-II hidden symmetries of ODE (62),

$$V_1 = \frac{1}{\chi}\partial_\chi, \quad V_2 = \frac{\omega}{\chi}\partial_\chi,$$

are admitted by Equation (63). By using V_1, ODE (63) can be reduced to quadrature. We have $\{V, V_1\}$, which constitutes a two-dimensional solvable symmetry algebra of Equation (63), verifying $[V, V_1] = -2mV_1$. Invariants of V_1 are given by

$$\phi = \omega, \quad \gamma = \chi\chi'. \tag{66}$$

Such invariants allow us to reduce ODE (63) to a first-order ODE for $\gamma(\phi)$

$$g_0\gamma' + 2f_0 m e^{2m\phi} = 0. \tag{67}$$

This equation admits the symmetry $\widehat{V} = pr^{(1)}V\Big|_{(\phi,\gamma)} = \partial_\phi + 2m\gamma\partial_\gamma$. The canonical coordinates r, s, s^1, [26–28], where $s^1 = \frac{ds}{dr}$, associated with $\widehat{V}$ are given by

$$r = \frac{e^{2m\phi}}{\gamma}, \quad s = \phi, \quad s^1 = \frac{1}{r\left(2m - \dfrac{\gamma'}{\gamma}\right)}. \tag{68}$$

Hence, the ODE (67) reduces to

$$\frac{ds}{dr} = \frac{g_0}{2mr(g_0 + f_0 r)}. \tag{69}$$

Integrating Equation (69), we obtain

$$s = \frac{1}{2m}\left(\ln\left(\frac{r}{g_0 + f_0 r}\right) - \ln c_1\right), \tag{70}$$

which after undoing change of variable (68) yields the general solution of Equation (67)

$$\gamma(\phi) = \frac{c_1 - f_0 e^{2m\phi}}{g_0}.$$

Reversing the change of variables (66) we find

$$\chi(\omega) = \pm\sqrt{\frac{-f_0 e^{2m\omega} + 2mc_1\omega + c_2}{g_0 m}},$$

which is the general solution of Equation (63). Taking into account (64), we determine the solution of Equation (62), which is given implicitly by

$$z \pm \int^{U(z)} \sqrt{\frac{g_0 m}{-f_0 e^{2my} + 2mc_1 y + c_2}}\, dy + c_3 = 0.$$

Lastly, the general solution of Equation (3) starting from generator X (24) is found by using (61)

$$x + f_1 t \pm \int^{U(x+f_1 t)} \sqrt{\frac{g_0 m}{-f_0 e^{2my} + 2mc_1 y + c_2}}\, dy + c_3 = 0.$$

In the above, c_1, c_2, and c_3 are arbitrary constants.

5.5. Reduction by Using Solvable Lie Algebra $\mathcal{A}_8$

Now, we take into account the three-dimensional solvable symmetry group given by (25) and (26). Taking into account X (27), we obtain the invariants

$$z = x + f_1 t, \quad U(z) = u, \tag{71}$$

where $U(z)$ satisfies the nonlinear third-order ODE

$$g_0 q e^{qU} U''' + 3g_0 q^2 e^{qU} U' U'' + h_0 e^{mU} U'' + g_0 q^3 e^{qU} U'^3 + f_0(2m - q) e^{(2m-q)U} U' = 0. \tag{72}$$

Equation (72) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z which, after being written in the new variables, are given by

$$Y = \partial_z, \quad Z = (m - q) z \partial_z - \partial_U, \tag{73}$$

with the commutator structure $[Y, Z] = (m - q)Y$. Hence, we integrate Equation (72) as follows. From Y, we obtain the invariants

$$\omega = U, \quad \chi = U', \tag{74}$$

from which ODE (72) is transformed into the second-order ODE

$$g_0 q e^{q\omega} \left(\chi \chi'' + \chi'^2 \right) + \left(h_0 e^{m\omega} + 3 g_0 q^2 e^{q\omega} \chi \right) \chi' + g_0 q^3 e^{q\omega} \chi^2 + f_0 (2m - q) e^{(2m-q)\omega} = 0. \tag{75}$$

Moreover, $V = \mathrm{pr}^{(1)} Z \big|_{(\omega,\chi)} \equiv \partial_\omega + (m - q)\chi \partial_\chi$ is a symmetry of Equation (75). Symmetry V yields the following invariants

$$\phi = e^{-(m-q)\omega} \chi, \quad \gamma = e^{-(m-q)\omega} \chi'. \tag{76}$$

By substituting (76) into Equation (75), the following first-order ODE is obtained

$$g_0 q \phi (\gamma - (m - q)\phi) \gamma' + g_0 q \gamma^2 + (h_0 + g_0 q (m + 2q)\phi)\gamma + g_0 q^3 \phi^2 + f_0 (2m - q) = 0. \tag{77}$$

5.6. Reduction by Using Solvable Lie Algebra $\mathcal{A}_{10}$

Now, we consider the three-dimensional solvable symmetry group given by (30) and (31) and consider generator X (32), which yields the invariants

$$z = x + f_1 t, \qquad U(z) = u, \tag{78}$$

where $U(z)$ satisfies the third-order ODE

$$g_0 U^2 U''' - 3 g_0 U U' U'' + h_0 U^{m+3} U'' + 2 g_0 U'^3 + 2 f_0 (m + 1) U^{2m+4} U' = 0. \tag{79}$$

Equation (79) inherits the two-dimensional solvable symmetry algebra spanned by Y and Z, which can be written in the new variables as

$$Y = \partial_z, \quad Z = (m + 1) z \partial_z - U \partial_U,$$

satisfying $[Y, Z] = (m + 1)Y$. This allows us to integrate Equation (79) as follows. From Y, we obtain the invariants

$$\omega = U, \quad \chi = U', \tag{80}$$

therefore, ODE (79) can be transformed into a second-order ODE

$$g_0 \omega^2 (\chi \chi'' + \chi'^2) - 3 g_0 \omega \chi \chi' + g_0 \omega^2 \chi'^2 + h_0 \omega^{m+3} \chi' + 2 g_0 \chi^2 + 2 f_0 (m + 1) \omega^{2m+4} = 0. \tag{81}$$

Furthermore, it can be easily checked that Equation (81) inherits $V = \mathrm{pr}^{(1)}Z\big|_{(\omega,\chi)} \equiv \omega\partial_\omega + (m+2)\chi\partial_\chi$. By using V, whose invariants are given by $\phi = \omega^{-m-2}\chi$ and $\gamma = \omega^{-m-1}\chi'$, Equation (81) can be reduced to a first-order ODE for $\gamma(\phi)$

$$g_0\phi(\gamma-(m+2)\phi)\gamma' + g_0\gamma^2 + (h_0 + g_0(m-2)\phi)\gamma + 2g_0\phi^2 + 2f_0(m+1) = 0. \tag{82}$$

Nevertheless, if $h_0 = 0$, Equation (81) admits two Type-II hidden symmetries

$$V_1 = \frac{\omega^2}{\chi}\partial_\chi, \quad V_2 = \frac{\omega^2\ln\omega}{\chi}\partial_\chi.$$

We have $\{V, V_1\}$, which constitute a two-dimensional solvable symmetry algebra of Equation (81) satisfying $[V, V_1] = -2(m+1)V_1$. From V_1, we obtain the invariants

$$\phi = \omega, \quad \gamma = \chi\chi' - \frac{\chi^2}{\omega}, \tag{83}$$

from which ODE (81) becomes a first-order ODE for $\gamma(\phi)$

$$g_0(\phi\gamma' - \gamma) + 2f_0(m+1)\phi^{2m+3} = 0. \tag{84}$$

This equation admits the symmetry $\widehat{V} = pr^{(1)}V\big|_{(\phi,\gamma)} = \phi\partial_\phi + (2m+3)\gamma\partial_\gamma$. The canonical coordinates r, s, s^1, associated with $\widehat{V}$ are given by

$$r = \phi^{-2m-3}\gamma, \quad s = \ln\phi, \quad s^1 = \frac{\phi^{2m+3}}{\phi\gamma' - (2m+3)\gamma}. \tag{85}$$

Hence, the ODE (84) reduces to

$$\frac{ds}{dr} = -\frac{g_0}{2(m+1)(f_0 + g_0 r)}. \tag{86}$$

Integrating Equation (86), we obtain

$$s = \frac{1}{2(m+1)}\ln\left(\frac{c_1}{f_0 + g_0 r}\right), \tag{87}$$

which after undoing change of variable (85) yields the general solution of Equation (84)

$$\gamma(\phi) = \frac{\phi(c_1 - f_0\phi^{2m+2})}{g_0}.$$

Reversing the change of variables (83), we find

$$\chi(\omega) = \pm\omega\sqrt{\frac{-f_0\omega^{2m+2} + 2(m+1)c_1\ln\omega + c_2}{g_0(m+1)}},$$

which is the general solution of Equation (81). Taking into account (80), we determine the solution of Equation (79), which is given implicitly by

$$z \pm \int^{U(z)} \frac{1}{y}\sqrt{\frac{g_0(m+1)}{-f_0 y^{2m+2} + 2(m+1)c_1\ln y + c_2}}\,dy + c_3 = 0.$$

Lastly, the general solution of Equation (3) starting from generator X (32) is found by using (78)

$$x+f_1t\pm\int^{U(x+f_1t)}\frac{1}{y}\sqrt{\frac{g_0(m+1)}{-f_0y^{2m+2}+2(m+1)c_1\ln y+c_2}}dy+c_3=0.$$

In the above, c_1, c_2, and c_3 are arbitrary constants.

5.7. Reduction by Using Solvable Lie Algebra $\mathcal{A}_{12}$

Finally, we consider the three-dimensional solvable symmetry group given by (35) and (36). Here, by using the symmetry generator X (37), we obtain the invariants

$$z=x+f_1t,\qquad U(z)=u, \tag{88}$$

where $U(z)$ satisfies the third-order ODE

$$g_0U^2U'''-3g_0UU'U''+h_0U^2U''+2g_0U'^3+f_0U^2U'=0. \tag{89}$$

Equation (89) inherits the two-dimensional abelian symmetry algebra spanned by

$$Y=\partial_z,\quad Z=U\partial_U.$$

From Y, we obtain the invariants

$$\omega=U,\quad \chi=U', \tag{90}$$

therefore, ODE (89) can be transformed into a second-order ODE

$$\omega^2(g_0(\chi\chi''+\chi'^2)+h_0\chi'+f_0)-3g_0\omega\chi\chi'+2g_0\chi^2=0. \tag{91}$$

Moreover, Equation (91) inherits $V=\mathrm{pr}^{(1)}Z\Big|_{(\omega,\chi)}\equiv\omega\partial_\omega+\chi\partial_\chi$. By using V, whose invariants are given by $\phi=\frac{\chi}{\omega}$ and $\gamma=\chi'$, Equation (91) can be reduced to a first-order ODE for $\gamma(\phi)$

$$g_0\phi(\gamma-\phi)\gamma'+g_0\gamma^2+(h_0-3g_0\phi)\gamma+2g_0\phi^2+f_0=0. \tag{92}$$

Moreover, if $h_0=0$, Equation (91) admits three Type-II hidden symmetries

$$V_1=\frac{\omega^2}{\chi}\partial_\chi,\quad V_2=\frac{\omega^2\ln\omega}{\chi}\partial_\chi,\quad V_3=\omega\ln\omega\partial_\omega+(\ln\omega+1)\chi\partial_\chi.$$

We have $\{V,V_1\}$, which constitutes a two-dimensional abelian algebra of Equation (91). From V_1, we obtain the invariants

$$\phi=\omega,\quad \gamma=\chi\chi'-\frac{\chi^2}{\omega}, \tag{93}$$

from which ODE (91) becomes a first-order ODE for $\gamma(\phi)$

$$g_0\phi\gamma'-g_0\gamma+f_0\phi=0. \tag{94}$$

which inherits the symmetry $\widehat{V}=pr^{(1)}V\Big|_{(\phi,\gamma)}=\phi\partial_\phi+\gamma\partial_\gamma$. The canonical coordinates r, s, s^1, associated with $\widehat{V}$ are given by

$$r=\frac{\gamma}{\phi},\quad s=\ln\phi,\quad s^1=\frac{\phi}{\phi\gamma'-\gamma}. \tag{95}$$

Hence, the ODE (94) reduces to

$$\frac{ds}{dr} = -\frac{g_0}{f_0}. \tag{96}$$

Integrating Equation (96), we obtain

$$s = \frac{g_0}{f_0}(c_1 - r), \tag{97}$$

which after undoing change of variable (95) yields the general solution of Equation (94)

$$\gamma(\phi) = \frac{\phi(c_1 g_0 - f_0 \ln \phi)}{g_0}.$$

Reversing the change of variables (93), we find

$$\chi(\omega) = \pm\omega\sqrt{\frac{-f_0 \ln^2 \omega + 2c_1 g_0 \ln \omega + c_2}{g_0}},$$

which is the general solution of Equation (91). Taking into account (90), we determine the solution of Equation (89), which is given implicitly by

$$z \pm \int^{U(z)} \frac{1}{y}\sqrt{\frac{g_0}{-f_0 \ln^2 y + 2c_1 g_0 \ln y + c_2}}dy + c_3 = 0.$$

Lastly, the general solution of Equation (3) starting from generator X (37) is found by using (78)

$$x + f_1 t \pm \int^{U(x+f_1 t)} \frac{1}{y}\sqrt{\frac{g_0}{-f_0 \ln^2 y + 2c_1 g_0 \ln y + c_2}}dy + c_3 = 0.$$

In the above, c_1, c_2, and c_3 are arbitrary constants.

6. Conclusions

In this work, a complete classification of the Lie point symmetries admitted by the family of third-order PDEs (3) involving arbitrary functions $f(u)$, $g(u)$, and $h(u)$ have been determined. Additionally, we have derived all the maximal symmetry groups along with its non-zero commutator structure that family (3) admits depending on its arbitrary functions. Furthermore, taking into account the maximal symmetry groups, we have derived the solvable symmetry groups of dimension three or higher admitted by the family (3) for special forms of the functions $f(u)$, $g(u)$, and $h(u)$. Therefore, we apply the symmetry reduction method to determine some exact solutions for family (3) by using the three-dimensional solvable symmetry groups. This allows us to reduce the given PDE (3) into a third-order nonlinear ODE, which inherits a two-dimensional symmetry group. Consequently, the nonlinear PDE is transformed into a first-order nonlinear ODE, even if, unfortunately, it is not always obvious how to solve the first-order nonlinear ODE obtained. Nevertheless, when $h(u) = 0$, the presence of Type-II hidden symmetries in the reduced second-order ODEs yields a reduction of the given nonlinear PDE to a quadrature.

Although there are some PDEs included in family (3) that have been previously studied from the point of view of Lie symmetries and reductions, the point symmetry classification performed in this paper is a novel result itself. This classification not only allows one to analyze the family of PDEs globally but also includes many other equations that have not been previously studied. The same applies to the analysis of solvable Lie algebras and reductions of family (3).

In future work, it is intended to determine a complete classification of local low-order conservation laws for family (3) by using the multiplier approach [30,31]. Moreover, taking into account the conservation laws obtained, we will investigate the potential symmetries that class (3) admits and determine new reductions from them. Finally, we will apply the multi-reduction method proposed in [32] to find all symmetry-invariant conservation laws admitted by PDE (3), which will allow us to reduce the given PDE to first integrals for the ODE, which describes the symmetry-invariant solutions of the PDE.

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