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A note on gradient estimates for *p*-Laplacian equations

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Abstract
The aim of this short paper is to show that some assumptions in Guarnotta et al. (Adv Nonlinear Anal 11:741–756, 2022) can be relaxed and even dropped when looking for weak solutions instead of strong ones. This improvement is a consequence of two results concerning gradient terms: an L^∞ estimate, which exploits nonlinear potential theory, and a compactness result, based on the classical Riesz-Fréchet-Kolmogorov theorem.

Keywords A priori estimates · Compactness · Convection terms · Strong solutions

Mathematics Subject Classification 35J15 · 35J47 · 35D30 · 35D35

1 Introduction

In this brief note, whose starting point is [10], we consider the problem

$$\begin{cases} -\Delta_p u = f(x, u, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = g(x, u, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P}$$

where $N \geq 2$, $1 < p, q < N$, $\Delta_r z := \operatorname{div}(|\nabla z|^{r-2}\nabla z)$ denotes the r-Laplacian of z for $1 < r < +\infty$, while $f, g : \mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N} \to (0, +\infty)$ are Carathéodory functions satisfying the following hypotheses.

<u>$H_1(f)$</u> There exist $\alpha_1 \in (-1, 0]$, $\beta_1, \delta_1 \in [0, q-1)$, $\gamma_1 \in [0, p-1)$, $m_1, \hat{m}_1 > 0$, and a measurable $a_1 : \mathbb{R}^N \to (0, +\infty)$ such that

$$m_1 a_1(x) s_1^{\alpha_1} s_2^{\beta_1} \leq f(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \leq \hat{m}_1 a_1(x) \left(s_1^{\alpha_1} s_2^{\beta_1} + |\mathbf{t}_1|^{\gamma_1} + |\mathbf{t}_2|^{\delta_1} \right)$$

in $\mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N}$. Moreover, $\operatorname*{ess\,inf}_{B_\rho} a_1 > 0$ for all $\rho > 0$.

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H$_1$(g) There exist $\beta_2 \in (-1, 0]$, $\alpha_2, \gamma_2 \in [0, p-1)$, $\delta_2 \in [0, q-1)$, $m_2, \hat{m}_2 > 0$, and a measurable $a_2 : \mathbb{R}^N \to (0, +\infty)$ such that

$$m_2 a_2(x) s_1^{\alpha_2} s_2^{\beta_2} \leq g(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \leq \hat{m}_2 a_2(x) \left(s_1^{\alpha_2} s_2^{\beta_2} + |\mathbf{t}_1|^{\gamma_2} + |\mathbf{t}_2|^{\delta_2} \right)$$

in $\mathbb{R}^N \times (0, +\infty)^2 \times \mathbb{R}^{2N}$. Moreover, $\operatorname{ess\,inf}_{B_\rho} a_2 > 0$ for all $\rho > 0$.

H$_1$(a) There exist $\zeta_1, \zeta_2 \in (N, +\infty]$ such that $a_i \in L^1(\mathbb{R}^N) \cap L^{\zeta_i}(\mathbb{R}^N)$, $i = 1, 2$, where

$$\frac{1}{\zeta_1} < 1 - \frac{p}{p^*} - \theta_1, \quad \frac{1}{\zeta_2} < 1 - \frac{q}{q^*} - \theta_2,$$

with

$$\theta_1 := \max\left\{ \frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < 1 - \frac{p}{p^*}, \quad \theta_2 := \max\left\{ \frac{\alpha_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < 1 - \frac{q}{q^*}.$$

H$_2$ If $\eta_1 := \max\{\beta_1, \delta_1\}$ and $\eta_2 := \max\{\alpha_2, \gamma_2\}$ then

$$\eta_1 \eta_2 < (p - 1 - \gamma_1)(q - 1 - \delta_2).$$

In the sequel, by H$_1$ we mean the set of hypotheses H$_1$(f), H$_1$(g), and H$_1$(a).

Unlike [10], we restrict our attention to weak solutions instead of strong ones, which allows us to weaken several conditions. In particular,

- $p, q > 2 - \frac{1}{N}$ is relaxed to $p, q > 1$,
- assumption H$_3$ (cf. [10, p. 743]), ensuring a high local summability of reactions, is dropped, and
- no high local summability for a_1, a_2 is required (cf. H$_1$(f)–H$_1$(g)).

Let us briefly comment these improvements, focusing our attention on the first equation of (P), since arguments do not exploit any system structure. The lower bound concerning p was used to prove [10, Lemma 2.1] and, jointly with H$_3$, to guarantee the strong convergence of $\{|\nabla u_n|^{p-2}\nabla u_n\}$ in $L^{p'}_{\text{loc}}(\mathbb{R}^N)$, being $\{u_n\}$ a sequence of solutions (precisely, their first components) to problems that approximate (P); see [10, formula (4.5)]. On the other hand, in hypothesis $a_1 \in L^{s_p}_{\text{loc}}(\mathbb{R}^N)$ the number s_p was supposed to be greater than $p'N$. Thanks to [5, p. 830], this ensures the local $C^{1,\alpha}$-regularity of each u_n; cf. [10, Lemma 3.1]. However, by [12], the same holds true once $s_p > N$, so that we can take $s_p := \zeta_1$ with no additional conditions, where ζ_1 stems from H$_1$(a). Moreover, exploiting [12] instead of [5] yields that H$_3'$ in [10, Remark 4.4] can be relaxed to

$$\frac{1}{s_p} + \max\left\{ \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < \frac{1}{N}, \quad \frac{1}{s_q} + \max\left\{ \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < \frac{1}{N}.$$

The following example aims to catch the essence of these improvements.

Example 1.1 Let $0 < \varepsilon < \frac{1}{N}$ and let $\sigma > N$. Then the functions

$$f(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) := |x|^{-\frac{N}{N+2\sigma}} (1 + |x|)^{-N} \left[\left(\frac{s_2^\varepsilon}{s_1} \right)^{\frac{1}{2}} + |\mathbf{t}_1|^{\frac{\varepsilon}{2}} + |\mathbf{t}_2|^{\frac{\varepsilon}{2}} \right],$$

$$g(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) := |x|^{-\frac{N}{N+2\sigma}} (1 + |x|)^{-N} \left[\left(\frac{s_1^\varepsilon}{s_2} \right)^{\frac{1}{2}} + |\mathbf{t}_1|^{\frac{\varepsilon}{2}} + |\mathbf{t}_2|^{\frac{\varepsilon}{2}} \right]$$

satisfy hypotheses H_1–H_2 with $p = q := 1 + 2\varepsilon$. In fact, pick $\beta_1 = \alpha_2 = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 := \frac{\varepsilon}{2}$, $m_1 = \hat{m}_1 = m_2 = \hat{m}_2 := 1$, and $\zeta_1 = \zeta_2 := N + \sigma$. To verify H_1(a), observe at first that

$$\frac{1+2\varepsilon}{N} - \frac{\varepsilon/2}{1+2\varepsilon} > \frac{1}{N} - \frac{\varepsilon}{2} > \frac{1}{2N}, \qquad \left(\frac{1+2\varepsilon}{N} - \frac{\varepsilon/2}{1+2\varepsilon}\right)^{-1} - N < 2N - N = N$$

by the choice of ε. So,

$$\frac{1}{\zeta_1} < 1 - \frac{p}{p^*} - \theta_1 \quad \Leftrightarrow \quad \frac{1}{N+\sigma} < \frac{1+2\varepsilon}{N} - \frac{\varepsilon/2}{1+2\varepsilon}$$
$$\Leftrightarrow \quad \sigma > \left(\frac{1+2\varepsilon}{N} - \frac{\varepsilon/2}{1+2\varepsilon}\right)^{-1} - N,$$

which is true because $\sigma > N$. It is worth noticing that $p, q \leq 2 - \frac{1}{N}$, namely $1 + 2\varepsilon \leq 2 - \frac{1}{N}$, whenever $\varepsilon \leq \frac{1}{2N'}$, as well as $\zeta_1, \zeta_2 \leq p'N$, i.e. $N + \sigma \leq \frac{1+2\varepsilon}{2\varepsilon} N$, when $\varepsilon \leq \frac{N}{2\sigma}$. A concrete case can be obtained taking $N := 3$, $\sigma := 4$, and $\varepsilon := \frac{1}{4}$.

Convergence of gradient terms comes into play whenever a second-order differential problem needs to be approximated: this can occur due to the lack of ellipticity (or uniform ellipticity) of the principal part and/or the presence of non-smooth nonlinearities; see, e.g., [11, Theorem 3.3]. An approximation procedure is necessary also in the context of singular problems, that is, problems whose reaction term blows up when the solution approaches to zero, as (P). The very recent papers [8, 9] provide an account on this topic.

Here, we proceed as follows. Lemma 2.1 of [10] is restated in a new, general fashion and its proof is made adapting the one of [10]; vide Lemma 2.4. Next, we establish a compactness result (Lemma 2.5) for gradient terms, which is self-contained (unlike the alternative arguments mentioned in Remark 2.6) and relies on the classical Riesz-Fréchet-Kolmogorov L^p-compactness criterion. Finally, the proof of [10, Lemma 4.1] is modified to get a weak solution of (P) under assumptions H_1–H_2 and the unavailability of [10, Lemma 4.3], pertaining strong solutions, in this context is commented (see Remark 2.7).

Notations

Hereafter, Ω denotes a bounded domain in $\mathbb{R}^N$, $N \geq 2$, while $p \in (1, +\infty)$. We set $p' := \frac{p}{p-1}$ and, provided $p < N$, $p^* := \frac{Np}{N-p}$. If $p \geq N$ then $p^* := \infty$ and $(p^*)' := 1$. Write $\operatorname{dist}(A, B)$ for the distance between the nonempty sets $A, B \subseteq \mathbb{R}^N$. The symbol $B_R(x)$ indicates the (open) ball having center $x \in \mathbb{R}^N$ and radius $R > 0$, while $\overline{B}_R(x)$ stands for the closure of $B_R(x)$. Moreover, $B_R(x) \Subset \Omega$ means that $\overline{B}_R(x) \subseteq \Omega$. Centers of balls will be omitted when they are irrelevant. We denote by $|E|$ the N-dimensional Lebesgue measure of the set $E \subseteq \mathbb{R}^N$.

Let $C_c^\infty(\mathbb{R}^N)$ be the space of compactly supported test functions and let $\|\cdot\|_p$ be the usual norm in $L^p(\mathbb{R}^N)$. The Beppo Levi space $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ is defined as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{1,p} := \|\nabla u\|_p.$$

We know that $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space. Moreover, the Sobolev-type embedding $\mathcal{D}_0^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$ entails

$$\mathcal{D}_0^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : |\nabla u| \in L^p(\mathbb{R}^N)\}.$$

Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, a distributional solution to the equation

$$-\Delta_p u = f(x) \quad \text{in} \quad \mathbb{R}^N \tag{1.1}$$

is a function $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, \mathrm{d}x = \int_\Omega f\phi \, \mathrm{d}x \quad \forall \phi \in C^\infty_c(\mathbb{R}^N). \tag{1.2}$$

If $f \in L^{(p^*)'}(\mathbb{R}^N)$ then by weak solution of (1.1) we mean a function $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$ satisfying (1.2) for all $\phi \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$. Analogous definitions hold when Ω replaces $\mathbb{R}^N$ or f depends also on $u, \nabla u$. Further details can be found in [10, Section 2].

Finally, C and $C(\cdot)$ represent generic positive constants, which may change value at each passage. Possible arguments emphasize their dependence on written variables.

2 Main results

The main result of the paper is the following.

Theorem 2.1 *Let* H_1–H_2 *be satisfied. Then problem* (P) *possesses a weak solution* $(u, v) \in \mathcal{D}^{1,p}_0(\mathbb{R}^N) \times \mathcal{D}^{1,q}_0(\mathbb{R}^N)$.

For every $f \in L^2_{\mathrm{loc}}(\Omega)$, we define the nonlinear potential

$$P_f(x, R) := \int_0^R \left(\frac{|f|^2(B_\rho(x))}{\rho^{N-2}}\right)^{\frac{1}{2}} \frac{\mathrm{d}\rho}{\rho}, \quad \text{where} \quad |f|^2(B_\rho(x)) := \|f\|^2_{L^2(B_\rho(x))}.$$

The following basic result was established in [6].

Proposition 2.2 *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a distributional solution to*

$$-\Delta_p u = f(x) \quad in \quad \Omega, \tag{2.1}$$

with $f \in L^r_{\mathrm{loc}}(\Omega)$, $r := \max\{2, (p^*)'\}$. *Then there exists* $C = C(N, p) > 0$ *such that*

$$\|\nabla u\|_{L^\infty(B_R)} \le C\left[\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^p \, \mathrm{d}x\right)^{\frac{1}{p}} + \|P_f(\cdot, 2R)\|^{\frac{1}{p-1}}_{L^\infty(B_{2R})}\right]$$

for any $B_{2R} \Subset \Omega$.

Remark 2.3 As observed in [6, p. 1363], the condition $r \ge (p^*)'$ is not used to prove the result, but it guarantees that u is a weak solution, and not merely a *very weak solution*. In the latter case, an approximation procedure yields the existence of a very weak solution $u \in W^{1,p-1}(\Omega)$ of (2.1). For a thorough treatment on approximable solutions, see [3].

Lemma 2.4 *Let* $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$ *be a distributional solution to*

$$-\Delta_p u = f(x) \quad in \quad \mathbb{R}^N,$$

with $f \in L^r(\mathbb{R}^N)$, $r > N$. *Then* $\nabla u \in L^\infty(\mathbb{R}^N)$. *More precisely, there exists* $C = C(N, p) > 0$ *such that*

$$\|\nabla u\|^{p-1}_{L^\infty(\mathbb{R}^N)} \le C\left(\|\nabla u\|^{p-1}_{L^p(\mathbb{R}^N)} + \|f\|_{L^r(\mathbb{R}^N)}\right).$$

Proof Pick any $x \in \mathbb{R}^N$. By Proposition 2.2 and Hölder's inequality (with exponents $\frac{r}{2}$ and $\frac{r}{r-2}$), after observing that $r > N \geq \max\{2, (p^*)'\}$, we get

$$
\begin{aligned}
|\nabla u(x)|^{p-1} &\leq \|\nabla u\|^{p-1}_{L^\infty(B_1(x))} \\
&\leq C\left[\left(\frac{1}{|B_2(x)|}\int_{B_2(x)} |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p'}} + \|P_f(\cdot, 2)\|_{L^\infty(B_2(x))}\right] \\
&\leq C\left[\|\nabla u\|^{p-1}_{L^p(\mathbb{R}^N)} + \sup_{y\in B_2(x)} \int_0^2 \rho^{-\frac{N}{2}} \|f\|_{L^2(B_\rho(y))} \,\mathrm{d}\rho\right] \\
&\leq C\left[\|\nabla u\|^{p-1}_{L^p(\mathbb{R}^N)} + \|f\|_{L^r(\mathbb{R}^N)} \int_0^2 \rho^{-\frac{N}{r}} \,\mathrm{d}\rho\right] \\
&\leq C\left(\|\nabla u\|^{p-1}_{L^p(\mathbb{R}^N)} + \|f\|_{L^r(\mathbb{R}^N)}\right).
\end{aligned}
$$

Taking the supremum in $x \in \mathbb{R}^N$ on the left yields the conclusion. □

For every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $x \in B_R \Subset \Omega$, and $h \in \mathbb{R}^N$ such that $|h| < \mathrm{dist}(B_R, \partial\Omega)$, we set

$$u_h(x) := u(x+h), \quad \delta_h u := u_h - u.$$

Analogous definitions hold for vector-valued functions.

Lemma 2.5 *Let $\{u_n\} \subseteq W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\{f_n\} \subseteq L^{r'}_{\mathrm{loc}}(\Omega)$, $r \in (1, p^*)$, be such that u_n is a distributional solution to*

$$-\Delta_p u_n = f_n(x) \quad \text{in } \Omega$$

for all $n \in \mathbb{N}$. Suppose that:

- (K_1) *$\{\nabla u_n\}$ is bounded in $L^p_{\mathrm{loc}}(\Omega)$;*
- (K_2) *$\{f_n\}$ is bounded in $L^{r'}_{\mathrm{loc}}(\Omega)$;*
- (K_3) *$u_n \to u$ in $L^p_{\mathrm{loc}}(\Omega) \cap L^r_{\mathrm{loc}}(\Omega)$.*

Then $\{\nabla u_n\}$ admits a strongly convergent subsequence in $L^p_{\mathrm{loc}}(\Omega)$.

Proof Fix $R > 0$ fulfilling $B_R \Subset \Omega$. A density argument produces

$$\int_{B_R} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\phi \,\mathrm{d}x = \int_{B_R} f_n\phi \,\mathrm{d}x \tag{2.2}$$

for every $n \in \mathbb{N}$ and $\phi \in W^{1,p}_0(B_R)$. Now, pick $t, s > 0$ such that $B_t \Subset B_s \Subset B_R$ and $\eta \in C^\infty_c(B_s)$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_t, and $|\nabla\eta| \leq \frac{C}{s-t}$ for some $C > 0$. If $V_n := |\nabla u_n|^{p-2}\nabla u_n$ then using (2.2) with $\phi := \eta^2\delta_h u_n$, where $|h| < R - s$, gives

$$\int_{B_R} \eta^2\, V_n \cdot \delta_h(\nabla u_n) \,\mathrm{d}x + 2\int_{B_R} \eta\, \delta_h u_n\, V_n \cdot \nabla\eta \,\mathrm{d}x = \int_{B_R} f_n\phi \,\mathrm{d}x. \tag{2.3}$$

Next, exploit (2.2) with ϕ_{-h}, perform the change of variable $x \mapsto x + h$ on the left-hand side, and recall that $B_{s+|h|} \subseteq B_R$, to achieve

$$\int_{B_R} \eta^2\, (V_n)_h \cdot \delta_h(\nabla u_n) \,\mathrm{d}x + 2\int_{B_R} \eta\, \delta_h u_n\, (V_n)_h \cdot \nabla\eta \,\mathrm{d}x = \int_{B_R} f_n\phi_{-h} \,\mathrm{d}x. \tag{2.4}$$

Subtracting (2.3) from (2.4) yields

$$\int_{B_R} \eta^2 \, \delta_h V_n \cdot \delta_h(\nabla u_n) \, dx + 2\int_{B_R} \eta \, \delta_h u_n \, \delta_h V_n \cdot \nabla\eta \, dx = \int_{B_R} f_n \delta_{-h}\phi \, dx.$$

Since $\operatorname{supp}\eta \subseteq B_s$, this entails

$$\begin{aligned}
\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n) \, dx &\le \int_{B_R} \eta^2 \, \delta_h V_n \cdot \delta_h(\nabla u_n) \, dx \\
&\le 2\int_{B_R} |\delta_h u_n||\delta_h V_n||\nabla\eta| \, dx + \int_{B_R} |f_n||\delta_{-h}\phi| \, dx \\
&\le \frac{C}{s-t} \|\delta_h u_n\|_{L^p(B_s)} \|\delta_h V_n\|_{L^{p'}(B_s)} + \|f_n\|_{L^{r'}(B_s)} \|\delta_{-h}\phi\|_{L^r(B_s)} \\
&\le \frac{C}{s-t} \|\delta_h u_n\|_{L^p(B_R)} \left(\|(V_n)_h\|_{L^{p'}(B_s)} + \|V_n\|_{L^{p'}(B_s)} \right) \\
&\quad + \|f_n\|_{L^{r'}(B_R)} \left(\|\phi_{-h}\|_{L^r(B_s)} + \|\phi\|_{L^r(B_s)} \right) \\
&\le \frac{2C}{s-t} \|\delta_h u_n\|_{L^p(B_R)} \|V_n\|_{L^{p'}(B_R)} + 2\|f_n\|_{L^{r'}(B_R)} \|\delta_h u_n\|_{L^r(B_R)} \\
&\le C\left(\|\delta_h u_n\|_{L^p(B_R)} \|\nabla u_n\|^{p-1}_{L^p(B_R)} + \|f_n\|_{L^{r'}(B_R)} \|\delta_h u_n\|_{L^r(B_R)} \right),
\end{aligned} \tag{2.5}$$

where Hölder's inequality has been used twice, while $C = C(N, t, s) > 0$. Notice that, thanks to (K_1)–(K_3) and [2, Exercise 4.34], the last term of (2.5) vanishes as $h \to 0^+$ uniformly in n. Let us now distinguish two cases, namely $p \ge 2$ and $p \in (1, 2)$.

Case 1. If $p \ge 2$ then

$$\begin{aligned}
\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n) \, dx &= \int_{B_t} (|\nabla(u_n)_h|^{p-2}\nabla(u_n)_h - |\nabla u_n|^{p-2}\nabla u_n) \cdot (\nabla(u_n)_h - \nabla u_n) \, dx \\
&\ge C\, \|(\nabla u_n)_h - \nabla u_n\|^p_{L^p(B_t)} = C\, \|\delta_h(\nabla u_n)\|^p_{L^p(B_t)},
\end{aligned} \tag{2.6}$$

with $C > 0$ small enough; cf. [13, Chapter 12, inequality (I)]. By (2.5)–(2.6) we thus obtain $\delta_h(\nabla u_n) \to 0$ in $L^p(B_t)$ as $h \to 0^+$ uniformly in n, and the Riesz-Fréchet-Kolmogorov L^p-compactness criterion yields the conclusion, because $t > 0$ was arbitrary.

Case 2. For $p \in (1, 2)$ one has (see [13, Chapter 12, inequality (VII)])

$$\begin{aligned}
\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n) \, dx &= \int_{B_t} (|\nabla(u_n)_h|^{p-2}\nabla(u_n)_h - |\nabla u_n|^{p-2}\nabla u_n) \cdot (\nabla(u_n)_h - \nabla u_n) \, dx \\
&\ge C\int_{B_t} (1 + |\nabla(u_n)_h|^2 + |\nabla u_n|^2)^{\frac{p-2}{2}} |\nabla(u_n)_h - \nabla u_n|^2 \, dx \\
&= C\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, dx,
\end{aligned} \tag{2.7}$$

where $C > 0$ is sufficiently small while $W_{nh} := (1 + |\nabla(u_n)_h|^2 + |\nabla u_n|^2)^{\frac{p-2}{2}}$. Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, besides (K_1), produce

$$
\begin{aligned}
\|\delta_h(\nabla u_n)\|^p_{L^p(B_t)} &= \int_{B_t} W_{nh}^{\frac{p}{2}} \, |\delta_h(\nabla u_n)|^p \, W_{nh}^{-\frac{p}{2}} \, \mathrm{d}x \\
&\le \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_t} W_{nh}^{\frac{p}{p-2}} \, \mathrm{d}x \right)^{\frac{2-p}{2}} \\
&\le \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(|B_t| + 2 \, \|\nabla u_n\|^p_{L^p(B_R)} \right)^{\frac{2-p}{2}} \\
&\le C \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} .
\end{aligned}
\tag{2.8}
$$

Reasoning as in the case above, the conclusion directly follows from (2.5), (2.7), and (2.8). □

Remark 2.6 Lemma 2.5 can be proved also (in a less direct way) through a result by Boccardo and Murat [1] which, under the hypotheses of Lemma 2.5, ensures that

$$\nabla u_n \to \nabla u \quad \text{in } \; L^q_{\mathrm{loc}}(\Omega) \quad \forall q \in (1, p). \tag{2.9}$$

Evidently, (2.9) implies $\nabla u_n(x) \to \nabla u(x)$ for almost every $x \in \Omega$. A development of this approach, allowing $q = p$, is contained in [7, Lemma 2.5 and Remark 3]. Another way [4, 11] to get convergence of gradient terms relies on a differentiability result for the stress field, i.e., the field whose divergence represents the elliptic operator (as $|\nabla u|^{p-2}\nabla u$ for the p-Laplacian). In fact, by Rellich-Kondrachov's theorem [2, Theorem 9.16], such a differentiability allows to gain compactness.

Proof of Theorem 2.1 The reasoning is patterned after that of [10, Lemma 4.1]. So, here, we only sketch it. Pick $r, s > 1$ such that

$$\frac{1}{\zeta_1} + \theta_1 < \frac{1}{r'} < 1 - \frac{p}{p^*}, \quad \frac{1}{\zeta_2} + \theta_2 < \frac{1}{s'} < 1 - \frac{q}{q^*}, \tag{2.10}$$

which is possible thanks to H_1(a). Fix $\rho > 0$ and define $\varepsilon_n := \frac{1}{n}$, $n \in \mathbb{N}$. By [10, Lemmas 3.5–3.8], for every $n \in \mathbb{N}$ there exists $(u_n, v_n) \in (\mathcal{D}^{1,p}_0(\mathbb{R}^N) \times \mathcal{D}^{1,q}_0(\mathbb{R}^N)) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)^2$ solution to

$$
\begin{cases}
-\Delta_p u = f(x, u + \varepsilon_n, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\
-\Delta_q v = g(x, u, v + \varepsilon_n, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\
u, v > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\tag{P^{ε_n}}
$$

such that the following properties hold true, with appropriate $(u, v) \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N)$ and $M, \sigma_{2\rho} > 0$:

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u, v) && \text{in } \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N); \\ (u_n, v_n) &\to (u, v) && \text{in } W^{1,p}(B_{2\rho}) \times W^{1,q}(B_{2\rho}); \\ (u_n, v_n) &\to (u, v) && \text{in } L^r(B_{2\rho}) \times L^s(B_{2\rho}); \\ (\nabla u_n, \nabla v_n) &\to (\nabla u, \nabla v) && \text{a.e. in } \mathbb{R}^N; \\ \max\left\{\|u_n\|_{L^\infty(\mathbb{R}^N)}, \|v_n\|_{L^\infty(\mathbb{R}^N)}\right\} &\le M && \forall n \in \mathbb{N}; \\ \min\left\{\inf_{B_{2\rho}} u_n, \inf_{B_{2\rho}} v_n\right\} &\ge \sigma_{2\rho} && \forall n \in \mathbb{N}. \end{aligned} \tag{2.11}$$

Hence, $H_1(f)$ and (2.11) yield, for almost every $x \in B_{2\rho}$,

$$\begin{aligned} & f(x, u_n(x) + \varepsilon_n, v_n(x), \nabla u_n(x), \nabla v_n(x)) \\ & \le \hat{m}_1 a_1(x) \left[(u_n(x) + \varepsilon_n)^{\alpha_1} v_n(x)^{\beta_1} + |\nabla u_n(x)|^{\gamma_1} + |\nabla v_n(x)|^{\delta_1}\right] \\ & \le \hat{m}_1 a_1(x) \left(\sigma_{2\rho}^{\alpha_1} M^{\beta_1} + |\nabla u_n(x)|^{\gamma_1} + |\nabla v_n(x)|^{\delta_1}\right). \end{aligned} \tag{2.12}$$

By (2.11) the sequence $\{(\nabla u_n, \nabla v_n)\}$ is bounded in $L^p(B_{2\rho}) \times L^q(B_{2\rho})$. Exploiting $H_1(a)$, (2.10), and (2.12) we thus see that

$$\{f(\cdot, u_n + \varepsilon_n, v_n, \nabla u_n, \nabla v_n)\} \quad \text{is bounded in } \; L^{r'}(B_{2\rho}).$$

Accordingly, Lemma 2.5, with $\Omega := B_{2\rho}$, besides (2.11), produces $\nabla u_n \to \nabla u$ in $L^p(B_\rho)$. Now the proof goes on exactly as in [10, Lemma 4.1], ensuring that (u, v) is a distributional solution to (P). The conclusion is achieved through [10, Lemma 4.2], which shows that any distributional solution to (P) turns out a weak one. □

Remark 2.7 An advantage of using differentiability results for the stress field (see Remark 2.6) in this context is the possibility to obtain strong solutions of (P), as done in [10, Lemma 4.3]. Indeed, otherwise we do not know how to give a pointwise (a.e.) sense to the p-Laplacian operator, seen as the divergence of the stress field $|\nabla u|^{p-2}\nabla u$. This issue is linked to a well-known conjecture for (2.1), which can be stated as

$$f \in L^r_{\text{loc}}(\Omega) \quad \overset{?}{\Leftrightarrow} \quad |\nabla u|^{p-2}\nabla u \in W^{1,r}_{\text{loc}}(\Omega).$$

For a discussion about this conjecture, see [11, Section 1].

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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