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Pareto efficiency without topology

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Abstract

Given a vector optimization problem over spaces endowed with a topological linear structure, existence results for optima (efficient points) are known. Relying only on the linear structure, the set of properly efficient points from a convex set is proved to be nonempty and the sets of Proper efficient points and Pareto efficient points coincide, provided that the set of internal points picked from the corresponding cone is nonempty. This result is appealing since the scalarization of the vector optimization problem is valid without topological requirements. A As an important consequence, we provide the Second Welfare Theorem in vector lattices and especially in Lebesgue spaces holds without topology.

Keywords Pareto efficient points · Proper efficienct points · Supporting hyperplanes · Welfare theorems

1 Motivation of the paper

In vector optimization problems dealing with the existence of efficient points x_0, one can consider a partially ordered vector space. We assume throughout the paper that any vector space is defined over the real numbers. Henceforth, we consider a vector space L, an ordering cone $K \subseteq L$ and a *constraint set* $A \subseteq L$ such that $x \in (x_0 - K)$ and $x \in A$ would imply $x = x_0$, provided that the cone K is pointed. These are *minimal points* of A, also called *Pareto efficient points*. A similar definition holds for maximal points, it suffices to consider minimal points with respect to the ordering cone $-K$. The general problem in vector optimization is to determine properties for the set of minimal points $\mathsf{E}(A, K)$ of A, with respect to the cone K. The *scalarization*

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of such a problem is nothing but finding a linear functional f such that $f(x) \geqslant f(x_0)$, for some $x_0 \in \mathsf{E}(A,K)$. Thus x_0 is a solution to the *linear programming* problem given by minimizing $f(x)$ subject to $x \in A$. When $x_0 \in \mathsf{E}(A,K)$ is such that x_0 solves the linear programming problem over A and f is a *strictly* positive linear functional, with respect to K, then x_0 is a *proper efficient point*. The linear functional for which scalarization is provided relies on x_0 itself. The definition of a strictly positive linear functional with respect to the cone K is given in the Appendix.

In most of the research devoted to Pareto efficient points, authors use topological properties either of the cone being used for real-valued or vector minimization problems, or topological properties of the vector space itself. Indeed the latter is either a locally convex topological vector space or some normed linear space. Thus, vector optimization problems involve questions about the (topological) interior points of the ordering cone. In this paper we show that algebraic interior points may replace topological interior points in studying vector optimization problems. This approach yields results similar or 'stronger' than 'density results' of the vector optimization problems. Density arguments generalize the Arrow-Barankin-Blackwell Theorem originally developed in finite dimensional spaces and prescribe *how* the set of Pareto efficient points or proper efficient points may be approximated by a suitable subset using the topological structure of the space being considered, see for example Gong [11]. or Ng and Zheng [15] and the references therein. Instead, our algebraic approach tied to convexity enables us to use The Eidelheit's separation theorem as stated in [8] and to show equality between the set of Pareto efficient points and the set of proper efficient points. We apply our results to the special case of Lebesgue spaces. Eventually we consider applications to optimal allocation in exchange economies.

2 Pareto efficient and proper efficient points

Here and throughout the paper we let L be a nonempty vector space. In the following we recall the definition of Pareto efficient and proper efficient points whenever L is endowed with some topology and A, K are its subsets.

Definition 2.1 A *Pareto efficient point* of a non-empty set A, with respect to the cone K, is any $x_0 \in A$ such that $(x_0 - K) \cap A = \{x_0\}$. The set of these points in A is denoted by $\mathsf{E}(A,K)$.

Definition 2.2 A *proper efficient point* of some A, with respect to the cone K, is any $x_0 \in A$ such that $f(x) \geqslant f(x_0)$, for every $x \in A$ and f is a strictly positive functional, with respect to K. Namely, $f(k) > 0$ for any $k \in K \setminus \{0\}$. The set of these points is denoted by $\mathsf{Pos}(A,K)$.

The above definition of proper efficient points is due to Gong [11]. A general version of the Arrow–Barankin–Blackwell Theorem for normed linear spaces is the following:

Theorem 2.3 *If $K \subseteq L$ is a closed cone and $A \subseteq L$ is non-empty convex and closed set, then* $\mathsf{E}(A, K) \subseteq \overline{\mathsf{Pos}(A, K)}$.

The finite dimensional version of the above theorem appears initially in Arrow et al. [6]. A seminal reference to the above result in normed linear spaces is contained in Borwein [7]. The notion of proper efficient points can be found in Geoffrion [10] as applied to the study of *multi-criteria optimization* problems. Positive 'components' akin to nonnegative Lagrange multipliers means that every constraint condition cannot be omitted, otherwise they are *binding* and then may be discarded.

In the rest of the paper we relax the topological requirement for L and use its linear structure. The constrain set $A \subset L$ is nonempty and $K \subseteq L$ is a cone. We additionally need A to be convex. Further, for a vector optimization problem the determination of the optima in $\mathsf{E}(A, K)$ is understood, while its scalarization is achieved if for some $x_0 \in \mathsf{E}(A, K)$ there exists some linear functional $f_{x_0} \in L'$ such that $x_0 \in \mathsf{Pos}(A, K)$, where L' is the algebraic dual of L. A systematic study of vector optimization problems in infinite dimensional spaces began with Borwein [7] as an attempt to determine the solution set $\mathsf{E}(A, K)$. The motivation for using infinite dimensional spaces is the modeling of uncertainty, either for the values of the objective functional or for the constraint set. Other seminal works on the validity of the Arrow–Barankin–Blackwell Theorem are Fu [9], Jahn [12] and Petschke [16]. The assumptions of these papers are almost the same, and rely on the properties of a *base* for the cone K, which induces the partial ordering on K. We remind that a base for the cone K is some convex set B of the cone K, such that for any $k \in K \setminus \{0\}$ there exists a unique real number $t_k > 0$, such that $t_k \cdot k \in B$. $t \cdot x$ denotes the product between any vector x of L and some real number t. In [11] the existence of a norm bounded base of the ordering cone implies the definition of a sequence of *expansion cones*, which are useful for the study of the efficient points of normed spaces. In [1] a similar approach is provided for characterizing proper efficiency by replacing the topological setting by the algebraic one. However, a solid cone should be defined in a suitable way. Since usually a cone K is called *solid* if the norm-interior of the dual wedge K^0 of K is nonempty, the algebraic analogue requires the existence of algebraic interior points in K^0. Indeed, in [17] algebraic notions replace the topological ones in the study of Pareto efficient points with respect to expansion cones.

3 Internal and interior points of a cone

In this section we prove that there exist classes of normed vector lattices in which a set of norm-interior points of the ordering cone (inducing the lattice structure) is empty. On the other hand, the set of (algebraic) *internal* points of the cone is nonempty. If this is the case, internal points of vector lattices allow the use of the Eidelheit's separation theorem. Henceforth, we specialize to Lebesgue spaces

L^p, namely $L^p(\Omega, \mathcal{F}, \mathrm{P})$, where $1 \leqslant p < +\infty$ and $(\Omega, \mathcal{F}, \mathrm{P})$ is an atomless complete probability space. In particular, it supports distributions of random variables whose possible cardinal number is equal to that of the set of real numbers. We make use of the following sets (see Appendix 6):

- $I_x = \cup_{n=1}^{\infty}[-nx, nx]$, the solid subspace generated by $x \in L^1 \setminus \{0\}$ which is L^1-dense then yielding x as a quasi-interior point, where $[-nx, nx] = \{y \in L^1 \mid nx \geqslant y \geqslant -nx\}$ is an order interval for $n \in \mathbb{N}$;
- the *principal ideal* generated by x, $E_x = \{y \in L^1 \mid |y| \leqslant tx \text{ for some } t > 0\}$.

The former definition relies on the norm topology, while the latter is purely algebraic and depends on the lattice structure of L^1.

Proposition 3.1 *The set of norm-interior points of L^1_+ is empty.*

Proof L^1 is an infinite dimensional vector lattice, with respect to the pointwise partial ordering, then the conclusion arises from Jameson [13, Th.4.4.4]. □

Proposition 3.2 *The set of quasi-interior points of L^1_+ is nonempty.*

Proof Any $x \in L^1_+$ such that $f(x) > 0$, for any $f \in L^\infty_+ \setminus \{0\}$, is a quasi-interior point. This arises from Aliprantis and Border [2, Th.8.54], since the closure of a subspace of L^1 under the weak topology and the norm-closure of the same subspace do coincide. Now, it suffices to prove that $I_x = E_x$, where $x \in L^1_+ \setminus \{0\}$. We notice that $I_x \subseteq E_x$. For the opposite inclusion, if $y \in E_x$, and $n = [t] + 1$, then $y \in [-nx, nx]$, where $[t]$ denotes the integer part of t. □

Theorem 3.3 *Any quasi-interior point x* of *L^1_+ is an internal point of L^1_+.*

Proof If $x \in L^1_+ \setminus \{0\}$ is a quasi-interior point of L^1_+, then for any $x_0 \in L^1$, there exists some sequence $(x_n)_{n\in\mathbb{N}}$ lying in I_x, which is norm-convergent to x_0. Hence, for some $\epsilon > 0$, there exists some $n_0(\epsilon)$, such that $\|x_0 - x_n\|_1 < \epsilon$, for any $n \geqslant n_0 = n_0(\epsilon)$, which also depends on x_0. Thus, $-x_{n_0} \in I_x$ and $-x_{n_0} \in [-k_{n_0}x, k_{n_0}x]$ for some $k_{n_0} \in \mathbb{N}$. Now, we get $x_0 - x_{n_0} \in x_0 + [-k_{n_0}x, k_{n_0}x]$. Henceforth, $x_0 + k_{n_0}x \in L^1_+$ and $X + \frac{1}{k_{n_0}}x_0 \in L^1_+$. Since the order interval $[-k_{n_0}x, k_{n_0}x]$ may be enlarged enough, just pick $\delta(x_0) = \frac{1}{k_{n_0}} > 0$ such that $x + tx_0 \in L^1_+$, for any $t \in \mathbb{R}$, where $|t| \leqslant \frac{1}{k_{n_0}}$ and the proof is complete. □

4 Vector optimization using internal points

In this section, we show how internal points of a cone affect scalarization of vector optimization problems. First, we characterize proper efficient points of a convex constraint set A through internal points of the ordering cone K.

Theorem 4.1 *Given a vector space L and a convex subset A, let $K \subseteq L$ be a cone whose algebraic interior is nonempty. If $x_0 \in A$ is such that $A \cap (x_0 - K_0) = \emptyset$, then $x_0 \in \mathsf{Pos}(A, K)$.*

Proof Since $(x_0 - K_0) = (x_0 - K)_0 \cap A = \emptyset$, where K_0 and $(x_0 - K)_0$ denote the internal points of K and $x_0 - K$, respectively By Eidelheit's separation theorem there exists some linear functional $f \neq 0$, lying in the algebraic dual L' of L such that

$$\inf\{f(x_0) - f(k) \mid k \in K\} \geqslant \sup\{f(a) \mid a \in A\},$$

where the supremum is not equal to $-\infty$, since A is nonempty. From the above separation inequality, we obtain that $f(x_0) - f(k) \geqslant f(x_0)$, for any $k \in K$. Hence, $g = -f \in K^0$. If we suppose that there exists some $k_0 \in K \setminus \{0\}$ such that $g(k_0) > 0$ then $nk_0 \in K$ for any $n \in \mathbb{N}$, but this is a contradiction since the separation inequality is violated as far as $n \to +\infty$. Hence, g is a strictly positive functional with respect to K. The separation inequality now implies that $g(a) \geqslant g(x_0)$ for any $a \in A$, namely $x_0 \in \mathsf{Pos}(A, K)$. □

Next, we state our main result in this section: using again convexity and internal points, optima are both properly efficient and Pareto efficient.

Theorem 4.2 *Let L be a vector space and $K \subseteq L$ be a cone whose algebraic interior is nonempty. For a convex constraint set $A \subseteq L$, $\mathsf{Pos}(A, K) = \mathsf{E}(A, K)$.*

Proof Let us suppose that there exists some $x_1 \in \mathsf{E}(A, K)$, such that x_1 is not an element of the nonempty set $\mathsf{Pos}(A, K)$. Therefore, $A \cap (x_1 - K) = \{x_1\}$ and $(x_1 - K)_0 = (x_1 - K_0)$, so we obtain that $A \cap (x_1 - K_0) = \emptyset$. Thus, we repeat the separation argument in Theorem 4.1, which implies that $x_1 \in \mathsf{Pos}(A, K) \subseteq \mathsf{E}(A, K)$. This contradicts $x_1 \notin \mathsf{Pos}(A, K)$, hence $x_1 \in \mathsf{E}(A, K)$ and this yields $\mathsf{Pos}(A, K) = \mathsf{E}(A, K)$. □

Finally, we apply our previous results to Lebesgue spaces.

Proposition 4.3 *If $L = L^p$, where $1 \leqslant p < +\infty$, then for any convex subset A of L, it holds $\mathsf{Pos}(A, K) = \mathsf{E}(A, K)$.*

Proof Any quasi-interior point is an algebraic interior point, if $K = L^p_+$. The result is straightforward. □

5 Implications in mathematical economics

In mathematical economics, a prominent topic is the study of Exchange Economies, having a finite number of consumers $i = 1, \ldots, I$. Infinite dimensional spaces are applied to accommodate uncertainty in economic phenomena. In particular, infinite dimensional vector lattices have a widespread application, because properties of vector lattices may replace the topological requirements used in the proof of existence results in welfare economics. These results rely on the order structure of vector lattices or other partially ordered linear spaces, without using the locally convex topological setting. The result presented in this paragraph is a part of this direction of study. The most important contributions in the same fashion are Aliprantis and Brown [4] and Aliprantis et al. [5]. The last paper is devoted to partially ordered vector spaces, which possess the Riesz Decomposition Property. A previous work, which is important since it is devoted to partially ordered spaces the prices of which lie in the *topological dual* of some vector lattice is [14].

Each consumer's consumption set is L_+, which denotes the positive cone of some vector lattice having quasi-interior points, such as L^p spaces. For this purpose $1 \leqslant p < +\infty$. The endowment of consumers are denoted by $\omega_i \in L_+$, for any $i = 1, \dots, I$. The utility function of any consumer is denoted by $u^i : L_+ \to \mathbb{R}$, for any $i = 1, \dots, I$. The set of allocations is the set $\mathcal{A}_\omega = \{x = (x_1, \dots x_I) \in (L_+)^I | \sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i = \omega\}$. The sets of utility improvement with respect to the allocation x are actually $A_i = \{x \in L_+ | u^i(x) \geqslant u^i(x_i)\}$, for any $i = 1, \dots, I$. We suppose that A_i is convex for any $x \in \mathcal{A}_\omega$ and for any $i = 1, \dots, I$.

Remark 5.1 We do not require additional topological structure of any A_i.

Now we show that every consumer's bundle is indeed a proper efficient point with respect to the problem of maximizing utility in a Pareto optimal way.

Proposition 5.2 $x_i \in \mathsf{Pos}(A_i, L_+)$, *for any* $i = 1, \dots, I$.

Proof Since the set of quasi-interior points of L_+ is non-empty, application of Theorem 4.1, Proposition 4 and Theorem 4.2 of the previous section provides the desired result. □

Finally, we arrive to the following welfare allocation result.

Theorem 5.3 *Every allocation is supported by a strictly positive price p.*

Proof For any $i = 1, \dots, I$, the Proposition 5.2 implies that $p_i \cdot y_i \geqslant p_i \cdot x_i$, for any $y_i \in A_i$ and any $i = 1, \dots, I$. By letting $p = \vee_{i=1}^I p_i$ the conclusion arises.

Remark 5.4 Theorem 5.3 is a 'version' of the second fundamental theorem of welfare economics in infinite dimensional spaces. The standard form for finite dimensional spaces is for example in Aliprantis et al. [3, Th.1.6.10]. We assume that $(\Omega, \mathcal{F}, \mathsf{P})$ is a complete atomless probability space supporting the commodity space $L^p(\Omega, \mathcal{F}, \mathsf{P})$, where $1 \leqslant p < \infty$, which is also partially ordered in the usual pointwise sense and this in turn induces its lattice structure, namely $x \geqslant y$ if and only if $x(\omega) \geqslant y(\omega)$, for P-almost every ω. The lattice structure of L' is given in the following way: If

$h, g \in L'$, then $(h \vee g)(x) = \max\{h(x), g(x)\}$ and $(h \wedge g)(x) = \min\{h(x), g(x)\}$, for any $x \in L_+$.

6 Appendix: Partially ordered vector spaces

Let L be a vector space and let $K \subseteq L$ be nonempty with $K \neq \{0\}$. We call K a *wedge* of L if

(i) $K + K \subseteq K$,
(ii) $\lambda K = \{\lambda k \mid k \in K\} \subseteq K$, for any $\lambda \in \mathbb{R}_+$, where λk denotes the product between the scalar λ and the vector k.

If in addition $K \cap (-K) = \{0\}$, where 0 is the zero vector of L, then K is a *cone* of L. We write $-K = \{x \in L \mid -x \in K\}$ for the *negative* of K. Sometimes a cone is mentioned as *pointed cone* and a wedge is termed as *cone*. Any cone K induces a partial ordering on L as follows:

$$y \geqslant x \quad \text{if and only if} \quad y - x \in K.$$

This can be written $x \leqslant_K y$. In the case $K \neq \{0\}$, the partial ordering is reflexive, antisymmetric, transitive and compatible with the linear structure of L:

(i) $x \geqslant x$ for any $x \in L$;
(ii) If $x \geqslant y$ and $y \geqslant x$, then $x = y$;
(iii) If $x \geqslant y$ and $y \geqslant z$, then $x \geqslant z$;
(iv) If $x \geqslant y$, then $\lambda x \geqslant \lambda y$ for any $\lambda \in \mathbb{R}_+$;
(v) If $x \geqslant y$, then $x + z \geqslant y + z$, for any $z \in L$.

If L' is the algebraic dual of L, that is the vector space of all linear functionals of L, then

$$K^0 = \{f \in L' \mid f(x) \geqslant 0, x \in L\}$$

is called the *polar* wedge of K. For $A \subseteq L$ a vector $a \in A$ is an *internal point* of A if given some $x \in L$, there exists a real number $\delta > 0$ such that $a + \lambda x \in A$, for any $\lambda \in \mathbb{R}$, with $|\lambda| \leqslant \delta$. An internal point is often called *algebraic interior* point. Any $f \in K^0$ is called a *positive* functional with respect to K. If K is a cone, then any $f \in K^0$, such that $f(k) > 0$, if $k \in K \setminus \{0\}$ is called *strictly positive*, with respect to K. The set of all internal points of A is denoted by A_0. We recall the Eidelheit's separation theorem: *Suppose that C, D are convex subsets of L such that $C_0 \neq \emptyset$ and $C_0 \cap D = \emptyset$. Then there is a non-zero functional f of L', such that*

$$\inf_{x \in C} f(x) \geqslant \inf_{z \in D} f(z).$$

An *order interval* of the linear space L with respect to the partial ordering implied by the cone K is the set $[a, b] = \{x \in L \mid b \geqslant x \geqslant a\} = (a + K) \cap (b - K)$. If $(L, \geqslant)$ is a partially ordered vector space where for each pair of vectors there exists a supremum in L, we call it a *vector lattice*. A subset S of the vector lattice L is **solid** if for any $x \in S$, such that $y \in L$ and $|y| \leqslant |x|$, then $y \in S$. The *solid subspace* generated by $x \in L \setminus \{0\}$ is defined as $I_x = \cup_{n=1}^{\infty}[-nx, nx]$, where $n \in \mathbb{N}$. If L is a normed linear space and I_x is dense in L, then x is called *quasi-interior* point.

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