

A TWO-FUNCTION EXTENSION OF A MINIMAX THEOREM

ORNELLA NASELLI

In this note we extend a topological minimax theorem due to Ricceri ([2]) to the case of two functions.

1. Introduction and statement of the main result

Let X, Y be two non-empty sets and let φ be a real-valued function on $X \times Y$.
Set

$$\varphi_* = \sup_{y \in Y} \inf_{x \in X} \varphi(x, y)$$

and

$$\varphi^* = \inf_{x \in X} \sup_{y \in Y} \varphi(x, y)$$

It is clear that

$$\varphi_* \leq \varphi^*.$$

This is called the trivial minimax inequality. The opposite inequality

$$\varphi^* \leq \varphi_*$$

Entrato in redazione: 0 0

AMS 2010 Subject Classification: 90C47; 49K35

Keywords: minimax, semicontinuity, global minimum points

is called non-trivial minimax inequality and of course it is equivalent to the minimax equality

$$\varphi_* = \varphi^* \quad (1)$$

Starting from the pioneristic work of von Neumann ([8]), many results ensuring (1) were established. For an introductory bibliography see, for example, the classical survey of Simons ([5]).

Now, let $f, g : X \times Y \rightarrow \mathbb{R}$, with $f(x, y) \leq g(x, y)$ for every $x \in X, y \in Y$. We call non-trivial minimax inequality involving f, g the following

$$f^* \leq g_* \quad (2)$$

So, if $f = g$, (2) is equivalent to (1). For a given minimax theorem for one function, it is an usual fact to see whether it is possible to find a two-function version of it. The most natural way to get this is to split the hypotheses on φ to f and g . For example, the two-function version of the most classical Fan-Sion's theorem ([7]) (Theorem 1.1 below) has been obtained by Simons ([6], Th. 1.4) (Theorem 1.2 below).

Theorem 1.1. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, and let $\Psi : X \times Y \rightarrow \mathbb{R}$ be quasi-convex and lower semicontinuous in X , and quasi-concave and upper semicontinuous in Y . Then, (1) holds.*

Theorem 1.2. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, let $f : X \times Y \rightarrow \mathbb{R}$ be quasi-concave in Y and lower semicontinuous in X , and let $g : X \times Y \rightarrow \mathbb{R}$ be upper semicontinuous in Y and quasi-convex in X , with $f \leq g$ on $X \times Y$. Then, (2) holds.*

In [2], Ricceri proved the following result:

Theorem 1.3. *Let X be a topological space, $I \subseteq \mathbb{R}$ an open interval and $\Psi : X \times I \rightarrow \mathbb{R}$ a function satisfying the following conditions:*

- a) for each $x \in X$, the function $\Psi(x, \cdot)$ is quasi-concave and continuous*
 - b) for each $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ is lower semicontinuous and inf-compact*
 - c) for every $\lambda^* \in I$, the function $\Psi(\cdot, \lambda^*)$ has only one global minimum point*
- Under such hypotheses, (1) holds.*

The aim of the present paper is to establish the following extension of Theorem 1.3 to two functions.

Theorem 1.4. *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval and $f, g : X \times I \rightarrow \mathbb{R}$ two functions satisfying the following conditions:*

H1) for every $(x, \lambda) \in X \times I$ one has $f(x, \lambda) \leq g(x, \lambda)$

H2) the function g is lower semicontinuous in $X \times I$

H3) for every $x \in X$, the function $g(x, \cdot)$ is continuous

H4) for every $\lambda \in I$, the function $f(\cdot, \lambda)$ is lower semicontinuous and inf-compact

H5) for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave

H6) for every $\lambda \in I$, the function $g(\cdot, \lambda)$ has only one global minimum point

Under such hypotheses, (2) holds.

To realize that when $f = g = \Psi$ Theorem 1.4 gives Theorem 1.3, one has to observe that conditions a), b) of Theorem 1.3, by Lemma 4 of [4], imply that the function Ψ is lower semicontinuous in $X \times I$.

Finally, for the reader's convenience, we recall the following result ([1], Th. 2.3) that will be the main tool used to prove Theorem 1.4.

For a generic set $S \subseteq X \times I$, for each $(x, \lambda) \in X \times I$, we set

$$S_x = \{\mu \in I : (x, \mu) \in S\}$$

$$S^\lambda = \{u \in X : (u, \lambda) \in S\}$$

Theorem 1.5. *Let X be a topological space, $I \subseteq \mathbb{R}$ a compact interval and $S, T \subseteq X \times I$. Assume that S is connected and $S^\lambda \neq \emptyset$ for all $\lambda \in I$, while T_x is non-empty and connected for all $x \in X$, and T^λ is open for all $\lambda \in I$.*

Then, one has $S \cap T \neq \emptyset$.

Remark 1.6. In [3], Theorem 1.3 has been extended to the case where I is an arbitrary convex set in a topological vector space. It is an open challenging problem to know whether the same holds for Theorem 1.4.

2. Proof of Theorem 1.4

We argue by contradiction. So, assume that

$$g_* < f^* \quad (3)$$

and fix $r \in \mathbb{R}$ satisfying

$$g_* < r < f^* \quad (4)$$

For each $\lambda \in I$, let x_λ be the only global minimum point of $g(\cdot, \lambda)$. Let us show that the function $\lambda \rightarrow x_\lambda$ is continuous. To this end, it is clearly enough to show that if $\{\lambda_n\}$ is a sequence in I converging to $\bar{\lambda} \in I$, then $x_{\bar{\lambda}}$ is a cluster point of x_{λ_n} . Let $[a, b] \subseteq I$ a compact interval containing the sequence $\{\lambda_n\}$. From H5) it follows that

$$\cup_{\lambda \in [a, b]} \{x \in X : f(x, \lambda) \leq r\} \subseteq \{x \in X : f(x, a) \leq r\} \cup \{x \in X : f(x, b) \leq r\}$$

and so, from H1)

$$\cup_{\lambda \in [a, b]} \{x \in X : g(x, \lambda) \leq r\} \subseteq \{x \in X : f(x, a) \leq r\} \cup \{x \in X : f(x, b) \leq r\} \quad (5)$$

Since, for every $n \in \mathbb{N}$, x_{λ_n} belongs to the left-hand side of (5), from H4) and (5) it follows that the sequence $\{x_{\lambda_n}\}$ is contained in a compact set and so it has a cluster point \bar{x} . Then, $(\bar{x}, \bar{\lambda})$ is a cluster point of $\{(x_{\lambda_n}, \lambda_n)\}$ in $X \times [a, b]$. Let us show that

$$g(\bar{x}, \bar{\lambda}) \leq \limsup_n g(x_{\lambda_n}, \lambda_n) \quad (6)$$

Assume the contrary. Choose η such that

$$\limsup_n g(x_{\lambda_n}, \lambda_n) < \eta < g(\bar{x}, \bar{\lambda})$$

This implies that there exist $\alpha \in \mathbb{N}$ and, by H2), a neighbourhood U of $(\bar{x}, \bar{\lambda})$ such that, for every $(x, \lambda) \in U$ and every $n > \alpha$, one has

$$g(x_{\lambda_n}, \lambda_n) < \eta < g(x, \lambda) \quad (7)$$

Since $(\bar{x}, \bar{\lambda})$ is a cluster point of $\{(x_{\lambda_n}, \lambda_n)\}$, there exists $\bar{n} > \alpha$ such that $(x_{\lambda_{\bar{n}}}, \lambda_{\bar{n}}) \in U$ and so, by (7)

$$g(x_{\lambda_{\bar{n}}}, \lambda_{\bar{n}}) < \eta < g(x_{\lambda_{\bar{n}}}, \lambda_{\bar{n}})$$

that is absurde.

Now, let us fix $x \in X$. Taking (6) and H3) into account, we have

$$g(\bar{x}, \bar{\lambda}) \leq \limsup_n g(x_{\lambda_n}, \lambda_n) \leq \lim_n g(x, \lambda_n) = g(x, \bar{\lambda})$$

Thus, \bar{x} is a global minimum point for $g(x, \bar{\lambda})$, and so $\bar{x} = x_{\bar{\lambda}}$. This prove the claim.

Now, let $\{I_n\}$ be an increasing sequence of compact intervals whose union is I . We claim that there exists $n \in \mathbb{N}$ such that

$$\sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \quad (8)$$

Arguing by contradiction, suppose that, for every $n \in \mathbb{N}$, one has

$$\inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda)$$

For every $n \in \mathbb{N}$, let us put

$$C_n = \left\{ x \in X : \sup_{\lambda \in I_n} f(x, \lambda) \leq r \right\}$$

Each set C_n is non-empty: otherwise, by (4), one would have

$$r \leq \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) \leq g_* < r$$

Since the sequence $\{I_n\}$ is increasing, the sequence $\{C_n\}$ is decreasing. Summarizing, $\{C_n\}$ is a decreasing sequence of non-empty closed and compact sets. So, there exists $x^* \in \bigcap_{n \in \mathbb{N}} C_n$.

From the fact that for every $n \in \mathbb{N}$ and for every $\lambda \in I_n$ one has $f(x^*, \lambda) \leq r$, it follows that $f(x^*, \lambda) \leq r$ for every $\lambda \in I$ and so one can conclude that

$$\inf_{x \in X} \sup_{\lambda \in I} f(x, \lambda) \leq r$$

against (4). Now, fix $n \in \mathbb{N}$ for which (8) is satisfied and choose s such that

$$\sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) < s < \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \quad (9)$$

Let us put

$$S = \{(x_\lambda, \lambda) : \lambda \in I_n\}$$

$$T = \{(x, \lambda) \in X \times I_n : f(x, \lambda) > s\}$$

Thanks to the continuity of the function $\lambda \rightarrow x_\lambda$ we can say that the set S is connected in $X \times I_n$. Observe now that, for every $x \in X$, the set T_x is non-empty from (9) and connected from H5) and that, for every $\lambda \in I_n$, the set T^λ is open from H4). Then, thanks to Theorem 1.5, one has $S \cap T \neq \emptyset$. But, for every $\lambda \in I_n$, one has, from (9) and H1)

$$f(x_\lambda, \lambda) \leq g(x_\lambda, \lambda) = \inf_{x \in X} g(x, \lambda) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) < s$$

and so, $S \cap T = \emptyset$. This contradiction completes the proof.

Acknowledgements

The author has been supported by Università degli Studi di Catania, PIACERI 2024-2026, Linea di intervento 1, Progetto "PAFA".

REFERENCES

- [1] B. Ricceri, *Some topological mini-max theorems via an alternative principle for multifunctions*, Arch. Math. (Basel) 60 (1993), 367-377
- [2] B. Ricceri, *Multiplicity of global minima for parametrized functions*, Rend. Lincei Mat. Appl. 21 (2010), 47-57
- [3] B. Ricceri, *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl. 2017, 99-152
- [4] J. Saint Raymond, *On a minimax theorem*, Arch. Math. (Basel) 74 (2000), 432-437
- [5] S. Simons, *Minimax theorems and their proofs*, in *Minimax and applications*, Nonconvex Optim. Appl., vol. 4, Kluwer Acad. Publ., Dordrecht (1995), 1-23
- [6] S. Simons, *Two-function minimax theorems and variational inequalities for functions on compact and noncompact sets, with some comments on fixed point theorems*, Proceedings of simposia in pure Mathematics, vol. 45 (1986), 377-391
- [7] M. Sion, *On general minimax theorems*, Pacific J. Math. 8 (1958), 171-176
- [8] J. von Neumann, *Zur Theorie der Gesellschaftspiele*, Math. Ann. 100 (1928), 295-320

ORNELLA NASELLI

Dipartimento di Matematica e Informatica

Università di Catania

e-mail: ornella.naselli@unict.it