Research Article

Salvatore Leonardi* and Nikolaos S. Papageorgiou

Positive solutions for a class of singular (p, q)-equations

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Abstract: We consider a nonlinear singular Dirichlet problem driven by the (p, q)-Laplacian and a reaction where the singular term $u^{-\eta}$ is multiplied by a strictly positive Carathéodory function $f(z, u)$. By using a topological approach, based on the Leray-Schauder alternative principle, we show the existence of a smooth positive solution.

Keywords: topological approach, fixed point, regularity theory, purely singular problem, truncation

MSC 2020: 35J20, 35J75

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with a C^2-boundary $\partial\Omega$. In this article, we study the following singular Dirichlet (p, q)-equation:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \dfrac{f(z, u(z))}{u(z)^\eta}, \;\; u > 0 \text{ in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $1 < q < p$ and $0 < \eta < 1$.

By Δ_r, with $r \in (1, +\infty)$, we denote the r-Laplace differential operator defined by $\Delta_r u = \operatorname{div}(|Du|^{r-1}Du)$ for all $u \in W_0^{1,r}(\Omega)$.

In problem (1.1), the equation is driven by the sum of two such operators with different exponents. So, the differential operator is not homogeneous. In the reaction (right-hand side) of (1.1), we have the product of a singular term $u^{-\eta}$ with a Carathéodory function $f(z, u)$ (i.e., for all $x \in \mathbb{R}$, the map $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the map $z \mapsto f(z, x)$ is continuous), which is positive and bounded away from zero.

In the past, singular problems were examined with the singular term and $f(z, u)$ decoupled, i.e., $f(z, u)$ enters in the equation as an additive perturbation of the singular term. We refer to the works of Giacomoni et al. [6], Papageorgiou et al. [20,21], and the references therein.

That formulation allowed the use of the unique solution of the purely singular problem (the Dirichlet problem with reaction only the singular term $u^{-\eta}$), as a lower solution of the problem. This was the crucial step to bypass the singularity, deal with C^1-functionals, and use the results of the critical point theory. This is no longer possible for problem (1.1). Our approach here is topological based on the fixed point theory, and

* **Corresponding author: Salvatore Leonardi,** Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria, 6 – 95125, Catania, Italy, e-mail: leonardi@dmi.unict.it
Nikolaos S. Papageorgiou: Department of Mathematics, Technical University, Zorografou Campus, Athens 15780, Greece, e-mail: npapg@math.ntua.gr

in particular, we use the Leray-Schauder alternative principle (Section 2). This approach is analogous to the one used in problems with convection [22].

In the literature, the only work dealing with a singular problem like (1.1) is that by Dhanya et al. [3] who study the problem

$$\begin{cases} -\Delta u = \dfrac{f(u(z))}{u(z)^{\eta}}, \quad u > 0 \quad \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases}$$

with $f : [0, +\infty) \to [0, +\infty)$ being $C^1([0, +\infty))$, $f(0) > 0$ and $0 < \eta < 1$.

In [3], the authors assume the existence of ordered pairs of upper and lower solutions and, using the order fixed point theory of Amann [1], prove existence and multiplicity results. Our approach here is different, and we do not assume existence of ordered pairs of upper and lower solutions.

Additional literature with problems that are solved using similar techniques can be found in [2,4,9,11–17,23].

2 Mathematical background and hypotheses

Let X, Y be Banach spaces, $C \subseteq X$ a nonempty set, and $\varphi : C \to Y$. We say that φ is compact if φ is continuous and maps bounded sets in C to relatively compact sets of Y. The Leray-Schauder alternative principle says the following (see Papageorgiou and Kyritsi [18], p. 242):

Proposition 2.1. *If X is a Banach space, $C \subseteq X$ is a nonempty convex set with $0 \in C$, $\varphi : C \to C$ is compact and $L = \{u \in C : u = \lambda\varphi(u) \text{ for some } \lambda \in (0, 1)\}$, then either L is unbounded or $\varphi(\cdot)$ has a fixed point.*

In the analysis of problem (1.1), the main spaces are $W_0^{1,p}(\Omega)$ and $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$. By $\|\cdot\|$, we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_{L^p(\Omega)} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\bar{\Omega})$ is ordered, with positive (order) cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : 0 \le u(z) \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0, \right\}$$

with $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ and $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

Let $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega [|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, v \in W_0^{1,p}(\Omega).$$

Proposition 2.2. *The operator $V(\cdot)$ is bounded (i.e., maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too).*

Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define

$$u^{\pm}(z) = \max\{\pm u(z), 0\} \quad \text{for all } z \in \Omega.$$

Evidently $u = u^+ - u^-$, $|u| = u^+ + u^-$, and if $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$.

If $u, v : \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \le v(z)$ for all $z \in \Omega$, then we define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \quad \text{for a.a. } z \in \Omega\}.$$

Our hypotheses on the function $f(z, x)$ are the following:

- (**H**) $f : \Omega \times \mathbb{R} \to (0, +\infty)$ is a Carathéodory function for which there exist a constant $c_0 > 0$ and a positive function $\hat{a} \in L^\infty(\Omega)$ such that

$$c_0 \leq f(z, x) \leq \hat{a}(z)[1 + x^{p-1}] \quad \text{for a.a. } z \in \Omega, \ \text{all } x \geq 0.$$

We will need the following regularity result that complements those by Giacomoni et al. [6,7].
In what follows,

$$\hat{d}(z) = \text{dist}(z, \partial\Omega) \quad \text{for all } z \in \bar{\Omega}.$$

Proposition 2.3. *Let $h : \Omega \to (0, +\infty)$ be such that $f \in L^\infty_{\text{loc}}(\Omega)$, and there exists a constant $\hat{c} > 0$ for which $h(z) \leq \frac{\hat{c}}{\hat{d}(z)^\eta}$ for a.a. $z \in \Omega$. Assume $\hat{u} \in W_0^{1,p}(\Omega)$ is a distributional solution of*

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = h(z) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{2.1}$$

then $\hat{u} \in C_0^{1,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ and $\|\hat{u}\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq M$, with $M = M(p, q, \Omega, \eta, \hat{c})$.

Proof. Hardy's inequality ([19], p. 66) implies that for every $g \in W_0^{1,2}(\Omega)$, $\frac{g}{\hat{d}} \in L^2(\Omega)$. Therefore, for every $g \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} |\langle h, g\rangle| &= \left|\int_\Omega hg \mathrm{d}z\right| \\ &\leq \int_\Omega |h|\ |g| \mathrm{d}z \\ &\leq \int_\Omega \frac{\hat{c}}{\hat{d}^\eta}|g| \mathrm{d}z \\ &= \hat{c}\int_\Omega \hat{d}^{1-\eta}\frac{|g|}{\hat{d}} \mathrm{d}z \\ &\leq c_1 \int_\Omega \frac{|g|}{\hat{d}} \mathrm{d}z \quad \text{for some constant } c_1 > 0 \text{ (since } \hat{d} \in \text{int } C_+) \\ &\leq c_2 \left\|\frac{g}{\hat{d}}\right\|_{L^2(\Omega)} \quad \text{for some constant } c_2 > 0 \\ &\leq c_3\|Dg\|_{L^2(\Omega)} \quad \text{for some constant } c_3 > 0 \text{ (by Hardy's inequality)} \\ &\Rightarrow h \in W^{-1,2}(\Omega). \end{aligned}$$

We take into account the linear Dirichlet problem

$$-\Delta u(z) = h(z) \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0. \tag{2.2}$$

Consider $A \in \mathscr{L}(W_0^{1,2}(\Omega), W^{-1,2}(\Omega))$ defined as

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \quad \text{for all } u, h \in W_0^{1,2}(\Omega).$$

Evidently A is strictly monotone, coercive, thus surjective. So, problem (2.2) has a unique solution $\tilde{u} \in W_0^{1,2}(\Omega)$, $\tilde{u} \geq 0$, $\tilde{u} \neq 0$ (if $h \not\equiv 0$). (see [8])

Also we consider the purely singular problem

$$-\Delta u(z) = \hat{c}u(z)^{-\eta} \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

From Papageorgiou et al. [21], Proposition 3, we know that this problem has a unique solution $\bar{u} \in \operatorname{int} C_+$. Since $\hat{u} \in \operatorname{int} C_+$, using Proposition 4.1.22, p. 274, of Papageorgiou et al. [19], we can find a constant $c_4 > 0$ such that

$$\bar{u} \le c_4 \hat{d} \Rightarrow \hat{d}^{-\eta} \le \frac{c_4^\eta}{\bar{u}^\eta}. \tag{2.3}$$

Let $\bar{u}_* = c_4^\eta \bar{u} \in \operatorname{int} C_+$. We have

$$\begin{aligned} \Delta\bar{u} + h &\le \Delta\bar{u} + \hat{c}\hat{d}^{-\eta} \quad \text{(see (2.3))} \\ &\le \Delta\bar{u}_* + \frac{c_4^\eta \hat{c}}{\bar{u}^\eta} \\ &= c_4^\eta[\Delta\bar{u} + \hat{c}\bar{u}^{-\eta}] \\ &= 0 \quad \text{in } \Omega. \end{aligned} \tag{2.4}$$

Then (2.2), (2.4), and the weak comparison principle (Pucci and Serrin [24], Theorem 3.4.1, p. 61), imply that

$$0 \le \tilde{u} \le \bar{u}_*.$$

Invoking Theorem B1 of Giacomoni et al. [6], we can find $\beta \in (0,1)$ such that $\tilde{u} \in C_0^{1,\beta}(\bar{\Omega})$.

Let $a(y) = |y|^{p-2}y + |y|^{q-2}y$ for all $y \in \mathbb{R}^N$ and rewrite [3] as follows:

$$\operatorname{div}(a(D\hat{u}) + D\tilde{u}) = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

The nonlinear regularity theory of Lieberman [10] implies that

$$\hat{u} \in C_0^{1,\alpha}(\bar{\Omega}) \quad \text{with } 0 < \alpha < 1, \quad \|\hat{u}\|_{C_0^{1,\alpha}(\bar{\Omega})} \le M. \qquad \square$$

3 Existence of positive solution

As we already mentioned in Section 1, our approach to problem (1.1) is topological, based on the fixed point theory and in particular Proposition 2.1 (the Leray-Schauder alternative principle). To reach that point, we employ the method of frozen variable. So, let $w \in C_0^1(\bar{\Omega})$ and consider the following purely singular Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \dfrac{f(z, |w(z)|)}{u(z)^\eta}, & u > 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

From Proposition 3 of Gasinski and Papageorgiou [5] (see also Papageorgiou et al. [20]), we have the following existence and uniqueness result.

Proposition 3.1. *Problem* (3.1) *has a unique solution* $\bar{u} = \bar{u}(w) \in \operatorname{int} C_+$.

Based on this proposition, we can define the solution map $k : C_0^1(\bar{\Omega}) \to C_0^1(\bar{\Omega})$ by $k(w) = \bar{u} \in \operatorname{int} C_+$. Our aim is to apply Proposition 2.1 on this map. To this end, we need to know that $k(\cdot)$ is compact.

Proposition 3.2. *If hypotheses* (H) *hold, then the solution map* $k : C_0^1(\bar{\Omega}) \to C_0^1(\bar{\Omega})$ *is compact.*

Proof. First we show that $k(\cdot)$ is continuous. So, let $w_n \to w$ in $C_0^1(\bar{\Omega})$ and set $u_n = k(w_n)$, $n \in \mathbb{N}$. Let $M_0 > \sup_{n\in\mathbb{N}} \|f(\cdot, |w(\cdot)|)\|_{L^\infty(\Omega)}$ and consider the following purely singular problem:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = \dfrac{M_0}{u(z)^\eta}, \quad u > 0 \quad \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.2}$$

We know that problem (3.2) has a unique solution $\bar{u} \in \operatorname{int} C_+$ [5]. Fix $n \in \mathbb{N}$ and introduce the Carathéodory function $e_n(z, x)$ defined by

$$e_n(z, x) = \begin{cases} M_0 u_n(z)^{-\eta} & \text{if } x \le u_n(z) \\ M_0 x^{-\eta} & \text{if } x > u_n(z). \end{cases} \tag{3.3}$$

We consider the following nonlinear Dirichlet problem:

$$-\Delta_p u(z) - \Delta_q u(z) = e_n(z, u(z)) \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0. \tag{3.4}$$

As in Proposition 10 of Papageorgiou et al. [20], we show that problem (3.4) has a unique positive solution $u_n^* \in \operatorname{int} C_+$. We have

$$\langle V(u_n^*), h\rangle = \int_\Omega e_n(z, u_n^*) h \mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.5}$$

We choose the test function $(u_n - u_n^*)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned} \langle V(u_n^*), (u_n - u_n^*)^+\rangle &= \int_\Omega \frac{M_0}{u_n^\eta}(u_n - u_n^*)^+ \mathrm{d}z \quad \text{(see (3.3))} \\ &\ge \int_\Omega \frac{f(z, |w|)}{u_n^\eta}(u_n - u_n^*)^+ \mathrm{d}z \quad \text{(recall the choice of } M_0) \\ &= \langle V(u_n), (u_n - u_n^*)^+\rangle, \\ &\Rightarrow u_n \le u_n^* \quad \text{(see Proposition 2.2)}. \end{aligned} \tag{3.6}$$

From (3.6) and (3.3), it follows that u_n^* is a positive solution of problem (3.2), and so $u_n^* = \bar{u} \in \operatorname{int} C_+$ (by the uniqueness of the solution). Therefore,

$$u_n \le \bar{u} \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

Next let $c_0 > 0$ as postulated by hypotheses (H). We consider the following purely singular Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) = c_0\, u(z)^{-\eta}, \quad u > 0 \quad \text{in } \Omega \\ u_{|\partial\Omega} = 0. \end{cases} \tag{3.8}$$

As mentioned earlier, problem (3.8) has a unique solution $\tilde{u} \in \operatorname{int} C_+$. By using this solution, we introduce the Carathéodory function $\theta_n(z, x)$ defined by

$$\theta_n(z, x) = \begin{cases} f(z, |w_n(z)|)\tilde{u}(z)^{-\eta} & \text{if } x \le \tilde{u}(z) \\ f(z, |w_n(z)|) x^{-\eta} & \text{if } x > \tilde{u}(z). \end{cases} \tag{3.9}$$

We consider the following Dirichlet problem:

$$-\Delta_p u(z) - \Delta_q u(z) = \theta_n(z, u(z)) \quad \text{in } \Omega, \quad u_{|\partial\Omega} = 0. \tag{3.10}$$

As mentioned earlier (problem (3.4)), problem (3.10) has a unique solution $\tilde{u}_n^* \in \operatorname{int} C_+$. We have

$$\langle V(\tilde{u}_n^*), h\rangle = \int_\Omega \theta_n(z, \tilde{u}_n^*)\, h \mathrm{d}z \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.11}$$

In (3.11), we use the test function $h = (\tilde{u} - \tilde{u}_n^*)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}\langle V(\tilde{u}_n^*), (\tilde{u} - \tilde{u}_n^*)^+\rangle &= \int_\Omega f(z, |w_n|)\tilde{u}^{-\eta}(\tilde{u} - \tilde{u}_n^*)^+ dz \quad (\text{see } (3.9)) \\ &\geq \int_\Omega \frac{c_0}{\tilde{u}^\eta}(\tilde{u} - \tilde{u}_n^*)^+ dz \quad (\text{see hypotheses } (H)) \\ &= \langle V(\tilde{u}), (\tilde{u} - \tilde{u}_n^*)^+\rangle, \\ &\Rightarrow \tilde{u} \leq \tilde{u}_n^*.\end{aligned} \tag{3.12}$$

From (3.12), (3.9), and (3.11), it follows that $\tilde{u}_n^*$ is a solution of problem (3.1); hence, $\tilde{u}_n^* = u_n \in \operatorname{int} C_+$ (by the uniqueness of the solution). So, we have

$$\tilde{u} \leq u_n \quad \text{for all } n \in \mathbb{N} \quad (\text{see } (3.12)). \tag{3.13}$$

From (3.7) and (3.13), it follows that

$$u_n \in [\tilde{u}, \bar{u}] \quad \text{for all } n \in \mathbb{N}. \tag{3.14}$$

Then Proposition 2.3 implies that for some $\alpha \in (0, 1)$, we have

$$u_n \in C_0^{1,\alpha}(\bar{\Omega}), \quad \|u_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_5 \quad \text{for some constant } c_5 > 0, \text{ all } n \in \mathbb{N}. \tag{3.15}$$

From (3.15) and since $C_0^{1,\alpha}(\bar{\Omega}) \hookrightarrow C_0^1(\bar{\Omega})$ compactly, at least for a subsequence, we have

$$u_n \to \hat{u} \quad \text{in } C^1(\bar{\Omega}). \tag{3.16}$$

For every $n \in \mathbb{N}$ and every $g \in W_0^{1,p}(\Omega)$, we have

$$\langle V(u_n), g\rangle = \int_\Omega \frac{f(z, |w_n|)}{u_n^\eta} g\, dz. \tag{3.17}$$

From (3.16) and Proposition 2.2, we have

$$\langle V(u_n), g\rangle \to \langle V(\hat{u}), g\rangle. \tag{3.18}$$

Also we have

$$0 \leq \frac{f(z, |w_n|)}{u_n^\eta}|g| \leq M_0 \frac{|g|}{\tilde{u}^\eta} \quad (\text{see } (3.14)). \tag{3.19}$$

Recall that $\tilde{u} \in \operatorname{int} C_+$. So, we can find a constant $c_6 > 0$ such that

$$c_6\hat{d} \leq \tilde{u} \quad (\text{see [19], p. 274}).$$

Therefore we have

$$\frac{|g|}{\tilde{u}^\eta} \leq c_7 \frac{|g|}{\hat{d}^\eta} \quad \text{for some constant } c_7 > 0. \tag{3.20}$$

Via Hardy's inequality (see [19], p. 66), we see that $\frac{|g|}{\hat{d}^\eta} \in L^p(\Omega) \hookrightarrow L^1(\Omega)$. Moreover, we have

$$\frac{f(z, |w_n(z)|)}{u_n(z)^\eta}|g(z)| \to \frac{f(z, |w(z)|)}{\hat{u}(z)^\eta}|g(z)| \quad \text{a.e. in } \Omega. \tag{3.21}$$

From (3.19)–(3.21) and the Lebesgue dominated convergence theorem, we infer that

$$\int_\Omega \frac{f(z, |w_n(z)|)}{u_n(z)^\eta} g(z)\, dz \to \int_\Omega \frac{f(z, |w(z)|)}{\hat{u}(z)^\eta} g(z)\, dz \quad \text{for all } g \in W_0^{1,p}(\Omega). \tag{3.22}$$

So, if in (3.17) we pass to the limit as $n \to +\infty$ and use (3.18) and (3.22), then

$$\langle V(\hat{u}), g\rangle = \int_\Omega \frac{f(z, |w(z)|)}{\hat{u}(z)^\eta} g(z)\, dz \quad \text{for all } g \in W_0^{1,p}(\Omega), \quad \tilde{u} \leq \hat{u} \leq \bar{u} \quad (\text{see } (3.14)). \tag{3.23}$$

Therefore, $\hat{u} = k(w) \in \operatorname{int} C_+$, and this proves the continuity of the solution map $k(\cdot)$.

Next let $D \subseteq C_0^1(\bar{\Omega})$ be bounded, and let $C = k(D) \subseteq C_0^1(\bar{\Omega})$.

For every $u \in C$, we have $u = k(w)$ for some $w \in D$. We know that

$$\tilde{u} \le u \quad \text{(see (3.23))}. \tag{3.24}$$

Since $\tilde{u} \in \operatorname{int} C_+$, as mentioned earlier, we can find a constant $c_8 > 0$ such that

$$c_8\hat{d} \le \tilde{u}, \quad \Rightarrow u^{-\eta} \le \tilde{u}^{-\eta} \le c_9\hat{d}^{-\eta} \quad \text{for some constant } c_9 > 0 \ \ (3.24).$$

We also have

$$0 \le \frac{f(z, |w|)}{u^\eta} \le \frac{M_D}{\hat{d}^\eta} \quad \text{for some constant } M_D > 0, \ \text{all } w \in D.$$

Then by Proposition 2.3, we have that $C \subseteq C_0^{1,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ is bounded. The compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$ implies that $C \subseteq C_0^1(\bar{\Omega})$ is relatively compact. We conclude that the solution map $k(\cdot)$ is compact. ☐

Let now

$$L = \{u \in C_0^1(\bar{\Omega}) : u = \lambda\, k(u), 0 < \lambda < 1\}.$$

Proposition 3.3. *If hypotheses (H) hold, then $L \subseteq C_0^1(\bar{\Omega})$ is bounded.*

Proof. Let $u \in L$. We have

$$\frac{1}{\lambda}u = k(u) \quad \text{with } 0 < \lambda < 1.$$

This inequality means that

$$\left\langle V\left(\frac{1}{\lambda}u\right), g\right\rangle = \int_\Omega \frac{\lambda^\eta f(z, u)}{u^\eta} g\,dz \quad \text{for all } g \in W_0^{1,p}(\Omega). \tag{3.25}$$

If in (3.25) we choose the test function $g = u \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
\frac{1}{\lambda^{p-1}}\|u\|^p &\le \int_\Omega \lambda^\eta f(z, u)u^{1-\eta}dz,\\
&\Rightarrow \|u\|^p \le \int_\Omega \hat{a}(z)[u^{1-\eta} + u^{p-\eta}]dz \quad (\text{recall } 0 < \lambda < 1)\\
&\le c_{10}[1 + \|u\|^{p-\eta}] \quad \text{for some constant } c_{10} > 0,\\
&\Rightarrow L \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded.}
\end{aligned}$$

Then as in the proof of Lemma A6 of Giacomoni et al. [6] (see also [21], Proposition A1), we infer that

$$L \subseteq L^\infty(\Omega) \quad \text{is bounded.}$$

We have

$$\begin{aligned}
0 \le \frac{f(z, u)}{\left(\frac{1}{\lambda}\,u\right)^\eta} &\le \frac{M_L}{\left(\frac{1}{\lambda}\,u\right)^\eta} \quad \text{with } M_L = \sup_{u\in L}\|f(\cdot,u(\cdot))\|_{L^\infty(\Omega)}\\
&\le \frac{M_L}{\tilde{u}^\eta} \quad \left(\text{since } \tilde{u} \le \frac{1}{\lambda}\,u\right)\\
&\le \frac{c_{11}M_L}{\hat{d}^\eta} \quad \text{for some constant } c_{11} > 0 \ \ (\text{since } \tilde{u} \in \operatorname{int} C_+).
\end{aligned}$$

The Proposition 2.3 implies that $L \subseteq C_0^1(\bar{\Omega})$ is bounded. ☐

Theorem 3.4. *If hypotheses* (H) *hold, then problem* (1.1) *has a solution* $\hat{u} \in \operatorname{int} C_+$ *and if*

$$\frac{f(z,x)}{x^\eta} \quad \textit{is non increasing on } (0,+\infty),$$

then this solution is unique.

Proof. Propositions 3.2 and 3.3 permit the use of Proposition 2.1. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} = k(\hat{u}) \quad \Rightarrow \hat{u} \in \operatorname{int} C_+ \quad \text{solves problem (1.1).}$$

Suppose that the quotient function $x \to \frac{f(z,x)}{x^\eta}$ is nonincreasing on $(0+\infty)$. Let $\hat{v} \in W_0^{1,p}(\Omega)$ be another positive solution of (1.1). We have

$$\begin{aligned}\langle V(\hat{u}), (\hat{u}-\hat{v})^+\rangle &= \int_\Omega \frac{f(z,\hat{u})}{\hat{u}^\eta}(\hat{u}-\hat{v})^+ dz \\ &\le \int_\Omega \frac{f(z,\hat{v})}{\hat{v}^\eta}(\hat{u}-\hat{v})^+ dz \\ &= \langle V(\hat{v}), (\hat{u}-\hat{v})^+\rangle, \\ &\Rightarrow \hat{u} \le \hat{v} \quad \text{(see Proposition (2.2)).}\end{aligned}$$

Reversing the roles of $\hat{u}$ and $\hat{v}$ in the aforementioned argument, we also show that $\hat{v} \le \hat{u}$. So, finally $\hat{u} = \hat{v}$, and this means that $\hat{u} \in \operatorname{int} C_+$ is the unique positive solution of (1.1). □

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