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Research article

Oscillation criteria for sublinear and superlinear first-order difference equations of neutral type with several delays

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Abstract: The purpose of this paper is to investigate the oscillatory behaviour of a class of first-order sublinear and superlinear neutral difference equations. Some conditions are established by applying Banach's Contraction mapping principle, Knaster-Tarski fixed point theorem and using several inequalities. We provide some examples to illustrate the outreach of the main results.

Keywords: difference equation; oscillation; neutral; fixed point theorem; sublinear; superlinear
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1. Introduction

In this paper, we consider the following class of first-order nonlinear neutral difference equations

$$\Delta[\mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta - \tau)] + \sum_{j=1}^{s} Q_j(\theta)\mathcal{F}(\mathcal{U}(\theta - \sigma_j)) = 0, \tag{1.1}$$

where $\theta \in \mathbb{N}(\theta_0) = \{\theta_0, \theta_0 + 1, \ldots\}$, $\mathcal{P}(\theta)$, $Q_j(\theta)$, $j = 1, 2, \ldots, s$ are discrete arguments and real valued functions such that $Q_j(\theta) > 0$ for $\theta \in \mathbb{N}(\theta_0)$, $\mathcal{F} \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing function such that $x\mathcal{F}(x) > 0$ for $x \neq 0$ and Δ is defined by $\Delta u(\theta) = u(\theta + 1) - u(\theta)$. Let $\rho = \max\{\tau, \sigma_j\}$, $j = 1, 2, \ldots, s$. A real valued function $\mathcal{U}(\theta)$ defined on $\mathbb{N}(\theta_0 - \rho)$ is said to be a solution of (1.1) if it satisfies (1.1) for $\theta \geq \theta_0$ with the initial conditions $\mathcal{U}(r) = \phi(r)$, where $\phi(r)$, $r = \theta_0 - \rho, \cdots, \theta_0$ are given real constants. A solution

$\mathcal{U}(\theta)$ of (1.1) is said to be oscillatory if for every positive integer $N > 0$ there exists $\theta \geq N$ such that $\mathcal{U}(\theta)\mathcal{U}(\theta + 1) \leq 0$. Otherwise, $\mathcal{U}(\theta)$ is said to be nonoscillatory.

In the continuous-time system [12], Kolmanovski et al. coined the term "neutral" in the mid-twentieth century. Myshkis [17] used the classification of equations as retarded, neutral, or advanced in 1955. In particular, the problem of oscillation of solutions of neutral differential/difference equations arises in a wide variety of real-world phenomena observed in Science and Engineering, like in population dynamics, stability theory, lossless transmission lines in computer networks, vibrating masses attached to an elastic bar, etc., see [1, 5, 9]. As a result, many researchers are interested in the dynamical behaviour of neutral difference equations.

Georgiou et al. [7] in particular introduced the study of oscillatory behaviour of solutions of neutral difference equations (1.1) with $\mathcal{P}(\theta) = \mathcal{P}$, $Q_j(\theta) = Q$ for $j = 1, 2, ..., s$, and $\mathcal{F}(x) = x$ and established four sufficient conditions that ensure the oscillation of (1.1). Since then, many scholars have studied various generalizations of (1.1) and improved the oscillation conditions by using methods like the summation averaging method, comparison method, inequality techniques, etc., see [3, 4, 8, 11, 13–16, 18–29] and references cited therein. In particular, Gao et al. [6] studied (1.1) with $\mathcal{P}(\theta) = -1$, $\mathcal{F}(x) = x$ and $j = 1$.

Theorem 1.1 ([6]). *Assume that $Q(\theta) > 0$ for $\theta \geq 0$. Then all solutions of the Eq* (1.1) *are strongly oscillatory if and only if equation $\Delta^2\mathcal{Z}(\theta - 1) + \frac{1}{\tau}Q(\theta)\mathcal{Z}(\theta) = 0$ is oscillatory.*

Furthermore, Graef et al. [10] established new oscillation criteria for (1.1) with $\mathcal{P}(\theta) > 1$, $j = 1$ and $\mathcal{F}(x)/x \geq M > 0$ by comparing it to the oscillatory behaviour of second order difference equations.

Theorem 1.2 ([10]). *Assume that $m = \sigma_1 - \tau$ and that*

$$Q(\theta) = \frac{M(\mathcal{P} - 1)}{\mathcal{P}^2}Q(\theta) \geq \frac{m^m}{(m + 1)^{m+1}} \quad \textit{for all large } \theta.$$

If every solution of the difference equation

$$\Delta^2\mathcal{Z}(\theta - 1) + \frac{2(m + 1)^{m+1}}{m^{m+1}}\left[Q(\theta) - \frac{m^m}{(m + 1)^{m+1}}\right]\mathcal{Z}(\theta) = 0, \quad \textit{for all large } \theta$$

is oscillatory, then every solution of the nonlinear neutral delay difference equation (1.1) *(with $\mathcal{P}(\theta) > 1$, $j = 1$ and $\mathcal{F}(x)/x \geq M > 0$) is oscillatory.*

However, to the best of our knowledge, there are no results that are sufficient as well as necessary for the oscillation of (1.1). With this motivation, our main aim is to find the necessary and sufficient conditions for the oscillation of solutions of (1.1) when the nonlinear function $\mathcal{F}$ is either sublinear or superlinear. For the general theory of difference equations and a survey of excellent results in the oscillation theory for difference equations, we refer the reader to the monographs by Agarwal [1], Agarwal et al. [2] and Györi et al. [9].

Next, we mention the following fixed point theorems for the completeness of the paper.

Theorem 1.3 (Knaster-Tarski fixed point theorem, [2, 9]). *Let X be a partially ordered Banach space with ordering $\leq$. Let S be a subset of X with the following properties: the infimum of S belongs to S and every nonempty subset of S has a supremum which belongs to S. Let $T : S \to S$ be an increasing mapping, i.e, $x \leq y$ implies that $Tx \leq Ty$. Then T has a fixed point in S.*

Theorem 1.4 (Banach's contraction principle, [2, 9]). *Let X be a complete metric space and T be a contraction mapping on X. Then T has exactly one fixed point on X.*

2. Main results

Let us denote by $X = l_\infty^{\theta^*}$ the Banach space of real valued bounded functions $\mathcal{U}(\theta)$ for $\theta \geq \theta^* > \theta_0$ with the norm

$$\|\mathcal{U}\| = \sup\{|\mathcal{U}(\theta)| : \theta \geq \theta^*\}.$$

Theorem 2.1. *Consider* $-1 < a_1 \leq \mathcal{P}(\theta) \leq 0$ *and* $\tau \geq \sigma_j,\ j = 1, 2, \ldots, s$. *Assume that*

(H_1) $\mathcal{F}(wz) = \mathcal{F}(w)\mathcal{F}(z)$, $\mathcal{F}(-w) = -\mathcal{F}(w)$, $w, z \in \mathbb{R}$.
(H_2) $\mathcal{F}$ *satisfies* $\int_0^c \frac{dx}{\mathcal{F}(x)} < \infty,\ 0 < c < \infty$.

Then, every solution of (1.1) *is oscillatory if and only if*

(H_3) $\displaystyle\sum_{\theta=\theta_0}^{\infty}\sum_{j=1}^{s} Q_j(\theta) = \infty.$

Proof. We argue by contradiction, assuming that $\mathcal{U}(\theta)$ is a nonoscillatory solution of (1.1) such that $\mathcal{U}(\theta) > 0$ or $\mathcal{U}(\theta) < 0$ for $\theta \geq \theta_0$. By symmetry of Eq (1.1) we may assume that $\mathcal{U}(\theta)$, $\mathcal{U}(\theta - \tau)$, $\mathcal{U}(\theta - \sigma_j) > 0$ for $\theta \geq \theta_1 = \theta_0 + \rho$, $j = 1, 2, \ldots, s$. Setting

$$\mathcal{Z}(\theta) = \mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta - \tau) \tag{2.1}$$

in (1.1), we get

$$\Delta\mathcal{Z}(\theta) = -\sum_{j=1}^{s} Q_j(\theta)\mathcal{F}(\mathcal{U}(\theta - \sigma_j)) < 0, \tag{2.2}$$

for $\theta \geq \theta_1$. Hence, $\mathcal{Z}(\theta)$ is nonincreasing for $\theta \geq \theta_2$. So, there exists $\theta_3 > \theta_2$ such that $\mathcal{Z}(\theta) > 0$ or $\mathcal{Z}(\theta) < 0$ for $\theta \geq \theta_3$.

Case 1. Consider $\mathcal{Z}(\theta) > 0$ for $\theta \geq \theta_3$. Here $\mathcal{Z}(\theta) \leq \mathcal{U}(\theta)$ for $\theta \geq \theta_3$. Consequently, (2.2) can be written as

$$\Delta\mathcal{Z}(\theta) \leq -\sum_{j=1}^{s} Q_j(\theta)\mathcal{F}(\mathcal{Z}(\theta - \sigma_j))$$

for $\theta \geq \theta_4 > \theta_3 + \sigma_j$. Using the fact that $\mathcal{Z}(\theta)$ is nonincreasing, the last inequality can be written as

$$\sum_{j=1}^{s} Q_j(\theta) \leq -\frac{\Delta\mathcal{Z}(\theta)}{\mathcal{F}(\mathcal{Z}(\theta))}.$$

If $\mathcal{Z}(\theta + 1) < x < \mathcal{Z}(\theta)$, then $\frac{1}{\mathcal{F}(\mathcal{Z}(\theta))} \leq \frac{1}{\mathcal{F}(x)}$. Therefore, the last inequality implies that

$$\sum_{j=1}^{s} Q_j(\theta) \leq -\int_{\mathcal{Z}(\theta)}^{\mathcal{Z}(\theta+1)} \frac{dx}{\mathcal{F}(x)}.$$

Summing the preceding inequality from θ_4 to $\theta - 1$, we have

$$\sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s} Q_j(k) \leq -\sum_{k=\theta_4}^{\theta-1}\int_{\mathcal{Z}(k)}^{\mathcal{Z}(k+1)} \frac{dx}{\mathcal{F}(x)} = -\int_{\mathcal{Z}(\theta_4)}^{\mathcal{Z}(\theta)} \frac{du}{\mathcal{F}(u)},$$

a contradiction to (H_3) because of letting $\theta \to \infty$ and (H_2).

Case 2. Consider $\mathcal{Z}(\theta) < 0$ for $\theta \geq \theta_3$. Clearly, $\mathcal{Z}(\theta) < 0$ implies that

$$\mathcal{U}(\theta) < -\mathcal{P}(\theta)\mathcal{U}(\theta - \tau) \leq \mathcal{U}(\theta - \tau) \quad \text{for } \theta \geq \theta_3.$$

Proceeding inductively, we get

$$\mathcal{U}(\theta) \leq \max\{\mathcal{U}(\theta_3), \mathcal{U}(\theta_3 + 1), \cdots, \mathcal{U}(\theta_3 + \tau - 1)\}.$$

Consequently, $\mathcal{U}(\theta)$ is bounded for $\theta \geq \theta_3$. As a result, $\mathcal{Z}(\theta)$ is bounded and $\lim_{\theta\to\infty} \mathcal{Z}(\theta)$ exists. We may note that, $\mathcal{Z}(\theta) < 0$ and (2.1) imply $\mathcal{Z}(\theta+\tau-\sigma_j) > a_1\mathcal{U}(\theta-\sigma_j)$, $j = 1, 2, \ldots, s$. Therefore, using (H_1) and (1.1) we obtain

$$\Delta\mathcal{Z}(\theta) + \sum_{j=1}^{s} \frac{\mathcal{Q}_j(\theta)}{\mathcal{F}(a_1)} \mathcal{F}(\mathcal{Z}(\theta + \tau - \sigma_j)) \leq 0. \tag{2.3}$$

Since $\mathcal{Z}(\theta)$ is nonincreasing for $\theta \geq \theta_3$, we can find $\theta_4 > \theta_3$ and $c > 0$ so that $\mathcal{Z}(\theta) \leq -c$ for $\theta \geq \theta_4$. Consequently, (2.3) becomes

$$\Delta\mathcal{Z}(\theta) + \mathcal{F}\left(-\frac{c}{a_1}\right) \sum_{j=1}^{s} \mathcal{Q}_j(\theta) \leq 0. \tag{2.4}$$

Summing from θ_4 to $\theta - 1$, (2.4) gives

$$\mathcal{F}\left(-\frac{c}{a_1}\right) \sum_{k=\theta_4}^{\theta-1} \sum_{j=1}^{s} \mathcal{Q}_j(k) \leq -\mathcal{Z}(\theta) + \mathcal{Z}(\theta_4) < \infty \ \text{ as } \theta \to \infty,$$

a contradiction to (H_3).

To prove the necessary part, we assume that

$$\sum_{\theta=\theta_1}^{\infty} \sum_{j=1}^{s} \mathcal{Q}_j(\theta) < \infty.$$

So, we can choose $\theta_2 > \theta_1$ such that

$$\sum_{k=\theta}^{\infty} \sum_{j=1}^{s} \mathcal{Q}_j(k) < \frac{(1 + a_1)\kappa_2 - \alpha}{\mathcal{F}(\kappa_2)}, \ \theta \geq \theta_2, \tag{2.5}$$

where κ_1 and κ_2 are two positive constants such that

$$\kappa_1 < \alpha < (1 + a_1)\kappa_2.$$

Let $K = \{\mathcal{U} \in X : \mathcal{U}(\theta) \geq 0 \text{ for } \theta \geq \theta_2\}$. Next, we define a partial order on X, that is, for $\mathcal{U}_1, \mathcal{U}_2 \in X$, $\mathcal{U}_1 \leq \mathcal{U}_2$ if and only if $\mathcal{U}_2 - \mathcal{U}_1 \in K$. Thus, X is a partially ordered Banach space. Set

$$\Psi = \{\mathcal{U} \in X : \kappa_1 \leq \mathcal{U}(\theta) \leq \kappa_2, \theta \geq \theta_2\}.$$

Obviously, for every subset Ψ^* of Ψ both $\inf \Psi^*$ and $\sup \Psi^*$ exist in Ψ. Now, for $\mathcal{U} \in \Psi$ we define the following map:

$$(\Gamma\mathcal{U})(\theta) = \begin{cases} \mathcal{U}(\theta_2 + \rho), \ \theta_2 \leq \theta \leq \theta_2 + \rho \\ \alpha - \mathcal{P}(\theta)\mathcal{U}(\theta - \tau) + \sum_{k=\theta}^{\infty} \sum_{j=1}^{s} \mathcal{Q}_j(k)\mathcal{F}(\mathcal{U}(k - \sigma_j)), \ \theta \geq \theta_2 + \rho. \end{cases}$$

For $\mathcal{U} \in \Psi$ and using (2.5), we have

$$\begin{aligned}(\Gamma\mathcal{U})(\theta) &= \alpha - \mathcal{P}(\theta)\mathcal{U}(\theta - \tau) + \sum_{k=\theta}^{\infty} \sum_{j=1}^{s} \mathcal{Q}_j(k)\mathcal{F}(\mathcal{U}(k - \sigma_j)), \ \theta \geq \theta_2 + \rho \\ &\leq \alpha - \mathcal{P}(\theta)\mathcal{U}(\theta - \tau) + \mathcal{F}(\kappa_2) \sum_{k=\theta}^{\infty} \sum_{j=1}^{s} \mathcal{Q}_j(k) \\ &\leq \alpha - a_1\kappa_2 + \mathcal{F}(\kappa_2) \left[\frac{(1 + a_1)\kappa_2 - \alpha}{\mathcal{F}(\kappa_2)} \right] \\ &= \kappa_2\end{aligned}$$

and

$$(\Gamma\mathcal{U})(\theta) \geq \alpha \geq \kappa_1.$$

Therefore, $\Gamma\mathcal{U} \in \Psi$ for all $\mathcal{U} \in \Psi$ and $\theta \geq \theta_2$. Let $\mathcal{U}_1, \mathcal{U}_2 \in \Psi$ such that $\mathcal{U}_1 \leq \mathcal{U}_2$. It is not difficult to verify that $\Gamma\mathcal{U}_1 \leq \Gamma\mathcal{U}_2$. Therefore, by Theorem 1.3, Γ has a $\mathcal{U} \in \Psi$ such that $\Gamma\mathcal{U} = \mathcal{U}$. Thus, $\mathcal{U}(\theta)$ is a nonoscillatory solution of (1.1) with $\liminf_{\theta\to\infty} \mathcal{U}(\theta) \geq \kappa_1 > 0$. Thus, the proof is completed. □

Example 2.2. *For $\theta > 3$, let us consider*

$$\Delta[\mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta - 3)] + \mathcal{Q}_1(\theta)\mathcal{U}^{\frac{1}{3}}(\theta - 1) + \mathcal{Q}_2(\theta)\mathcal{U}^{\frac{1}{3}}(\theta - 2) = 0, \tag{2.6}$$

where $\mathcal{P}(\theta) = -\frac{1}{3}(1 + (-1)^\theta)$, $\mathcal{Q}_1(\theta) = \frac{1}{3}$, $\mathcal{Q}_2(\theta) = 3$, $s = 2$, $\tau = 3$, $\sigma_1 = 1$, $\sigma_2 = 2$ and $\mathcal{F}(x) = x^{\frac{1}{3}}$. Clearly, $-1 < -2/3 \leq \mathcal{P} \leq 0$. Also

$$\sum_{k=\tau}^{\infty} \sum_{j=1}^{2} \mathcal{Q}_j(k) = \infty.$$

Therefore, by Theorem 2.1 every solution of (2.6) *oscillates.*

Theorem 2.3. *Consider $-\infty < a_3 \leq \mathcal{P}(n) \leq a_4 < -1$ and $\tau \geq \sigma_j$, $j = 1, 2, \ldots, s$. Assume that $\mathcal{F}$ is a Lipschitz function on any $[\alpha, \beta]$ where $0 < \alpha < \beta < \infty$, that (H_1) holds and that*

(H_4) $\mathcal{F}$ satisfies $\int_c^\infty \frac{dx}{\mathcal{F}(x)} < \infty$, $c > 0$.

Then, every solution of (1.1) *is oscillatory if and only if* (H_3) *holds.*

Proof. To prove the sufficiency part, we follow the proof of Theorem 2.1 and we can conclude that if $\mathcal{U}(\theta)$ is a positive solution, then $\mathcal{Z}(\theta)$ is nonincreasing for $\theta \geq \theta_2$. So, there exists $\theta_3 > \theta_2$ such that $\mathcal{Z}(\theta) > 0$ or $\mathcal{Z}(\theta) < 0$ for $\theta \geq \theta_3$.

Case 1. Consider $\mathcal{Z}(\theta) > 0$ for $\theta \geq \theta_3$. We have

$$\mathcal{U}(\theta) \geq -\mathcal{P}(\theta)\mathcal{U}(\theta-\tau) \geq \mathcal{U}(\theta-\tau).$$

Proceeding inductively, we get

$$\mathcal{U}(\theta) \geq \min\{\mathcal{U}(\theta_3), \mathcal{U}(\theta_3+1), \cdots, \mathcal{U}(\theta_3+\tau-1)\}$$

which implies that $\mathcal{U}(\theta)$ is bounded below by, say, M for $\theta \geq \theta_4$. Summing (2.2) from θ_4 to $\theta-1$, we get

$$\mathcal{Z}(\theta) - \mathcal{Z}(\theta_4) + \sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s} Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)) = 0$$

implying

$$\mathcal{Z}(\theta) = \mathcal{Z}(\theta_4) - \sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s} Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)),$$

that is,

$$\mathcal{Z}(\theta) \leq \mathcal{Z}(\theta_4) - \mathcal{F}(M)\sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s} Q_j(k) \to -\infty \text{ as } \theta \to \infty,$$

a contradiction to the fact that $\mathcal{Z}(\theta) > 0$ for $\theta \geq \theta_4$.

Case 2 ($\mathcal{Z}(\theta) < 0$). Since $\mathcal{Z}(\theta) < 0$ for $\theta \geq \theta_3$, then we can find $\theta_4 > \theta_3$ such that $\mathcal{Z}(\theta+\tau-\sigma_j) \geq a_3\mathcal{U}(\theta-\sigma_j)$ for $j = 1, 2, \ldots, s$. Hence, (1.1) reduces to

$$\Delta\mathcal{Z}(\theta) + \sum_{j=1}^{s}\frac{Q_j(\theta)}{\mathcal{F}(a_3)}\mathcal{F}(\mathcal{Z}(\theta+\tau-\sigma_j)) \leq 0,$$

that is,

$$\Delta\mathcal{Z}(\theta) + \sum_{j=1}^{s}\frac{Q_j(\theta)}{\mathcal{F}(a_3)}\mathcal{F}(\mathcal{Z}(\theta)) \leq 0.$$

Dividing both sides of the last inequality by $\mathcal{F}(\mathcal{Z}(\theta))$, we get

$$\frac{\Delta\mathcal{Z}(\theta)}{\mathcal{F}(\mathcal{Z}(\theta))} + \sum_{j=1}^{s}\frac{Q_j(\theta)}{\mathcal{F}(a_3)} \geq 0.$$

If $\mathcal{Z}(\theta+1) \leq u \leq \mathcal{Z}(\theta)$, then the above inequality can be viewed as

$$\int_{\mathcal{Z}(\theta)}^{\mathcal{Z}(\theta+1)}\frac{du}{\mathcal{F}(u)} + \sum_{j=1}^{s}\frac{Q_j(\theta)}{\mathcal{F}(a_3)} \geq 0.$$

Therefore,

$$\sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s} Q_j(k) \leq -\mathcal{F}(a_3)\sum_{k=\theta_4}^{\theta-1}\int_{\mathcal{Z}(k)}^{\mathcal{Z}(k+1)}\frac{du}{\mathcal{F}(u)} = -\mathcal{F}(a_3)\int_{\mathcal{Z}(\theta_4)}^{\mathcal{Z}(\theta)}\frac{du}{\mathcal{F}(u)},$$

that is,

$$\sum_{k=\theta_4}^{\infty}\sum_{j=1}^{s}Q_j(k)<\infty \text{ as } \theta\to\infty$$

due to (H_4), a contradiction to (H_3).

To prove the necessary part, we assume that

$$\sum_{\theta=\theta_1}^{\infty}\sum_{j=1}^{s}Q_j(\theta)<\infty.$$

So, we can choose $\theta_2>\theta_1$ such that

$$\sum_{k=\theta}^{\infty}\sum_{j=1}^{s}Q_j(k)<\eta,\ \theta\geq\theta_2, \tag{2.7}$$

where

$$\eta=\min\left\{\frac{a_3\kappa_3+\alpha}{\mathcal{L}},\frac{-(\alpha+(1+a_4)\kappa_4)}{\mathcal{L}},\frac{(-1-a_4)}{2\mathcal{L}}\right\},$$

κ_3 and κ_4 are two positive constants such that

$$-a_3\kappa_3<\alpha<(-1-a_4)\kappa_4,$$

and $\mathcal{L}=\max\{\mathcal{L}_1,\mathcal{F}(\kappa_4)\}$, $\mathcal{L}_1$ is the Lipschitz constant of $\mathcal{F}$ on $[\kappa_3,\kappa_4]$. Let

$$\Psi=\{\mathcal{U}\in X:\kappa_3\leq\mathcal{U}(\theta)\leq\kappa_4,\theta\geq\theta_2\}.$$

Ψ is a complete metric space. For $\mathcal{U}\in\Psi$ let us define the map

$$(\Gamma\mathcal{U})(\theta)=\begin{cases}\mathcal{U}(\theta_2+\rho),\ \theta_2\leq\theta\leq\theta_2+\rho\\ -\dfrac{\alpha}{\mathcal{P}(\theta+\tau)}-\dfrac{\mathcal{U}(\theta+\tau)}{\mathcal{P}(\theta+\tau)}+\dfrac{1}{\mathcal{P}(\theta+\tau)}\displaystyle\sum_{k=\theta+\tau}^{\infty}\sum_{j=1}^{s}Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)),\ \theta\geq\theta_2+\rho.\end{cases}$$

For $\mathcal{U}\in\Psi$ and using (2.7), we have

$$(\Gamma\mathcal{U})(\theta)\leq-\frac{\alpha}{\mathcal{P}(\theta+\tau)}-\frac{\mathcal{U}(\theta+\tau)}{\mathcal{P}(\theta+\tau)}\leq-\frac{1}{a_4}[(-1-a_4)\kappa_4+\kappa_4]=\kappa_4$$

and

$$\begin{aligned}(\Gamma\mathcal{U})(\theta)&\geq-\frac{\alpha}{\mathcal{P}(\theta+\tau)}+\frac{1}{\mathcal{P}(\theta+\tau)}\sum_{k=\theta+\tau}^{\infty}\sum_{j=1}^{s}Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j))\\&\geq-\frac{1}{\mathcal{P}(\theta+\tau)}\left[\alpha-\mathcal{F}(\kappa_4)\sum_{k=\theta+\tau}^{\infty}\sum_{j=1}^{s}Q_j(k)\right]\\&\geq-\frac{1}{a_3}[\alpha-\mathcal{F}(\kappa_4)\eta]\end{aligned}$$

$$\geq -\frac{1}{a_3}[\alpha - \alpha - a_3\kappa_3] = \kappa_3$$

implies that $\Gamma\mathcal{U} \in \Psi$ for every $\theta \geq \theta_2$. For $\mathcal{U}_1, \mathcal{U}_2 \in \Psi$, we have

$$|\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| \leq \frac{1}{|\mathcal{P}(\theta+\tau)|}\,|\mathcal{U}_1(\theta+\tau) - \mathcal{U}_2(\theta+\tau)| + \frac{\mathcal{L}_1}{|\mathcal{P}(\theta+\tau)|}\sum_{k=\theta+\tau}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k)|\mathcal{U}_1(k-\sigma_j) - \mathcal{U}_2(k-\sigma_j)|,$$

that is,

$$\begin{aligned}|\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| &\leq -\frac{1}{a_4}\|\mathcal{U}_1 - \mathcal{U}_2\| - \frac{\mathcal{L}_1}{a_4}\|\mathcal{U}_1 - \mathcal{U}_2\|\sum_{k=\theta+\tau}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k)\\ &\leq -\frac{1}{a_4}\|\mathcal{U}_1 - \mathcal{U}_2\| - \frac{\mathcal{L}_1}{a_4}\eta\|\mathcal{U}_1 - \mathcal{U}_2\|\\ &\leq -\frac{1}{a_4}\left(1 - \frac{1+a_4}{2}\right)\|\mathcal{U}_1 - \mathcal{U}_2\|\end{aligned}$$

which implies

$$\|\Gamma\mathcal{U}_1 - \Gamma\mathcal{U}_2\| \leq \lambda\|\mathcal{U}_1 - \mathcal{U}_2\|,$$

where $\lambda = \frac{a_4-1}{2a_4} < 1$, so, Γ is a contraction. Therefore, by Theorem 1.4, Γ has a point $\mathcal{U} \in \Psi$ such that $\Gamma\mathcal{U} = \mathcal{U}$. Consequently, $\mathcal{U}(\theta)$ is a positive solution of (1.1). Thus, the theorem is proved. □

Example 2.4. *For $\theta > 3$, let us consider*

$$\Delta[\mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta-3)] + \mathcal{Q}_1(\theta)\mathcal{U}^3(\theta-1) + \mathcal{Q}_2(\theta)\mathcal{U}^3(\theta-2) = 0, \tag{2.8}$$

where $\mathcal{P}(\theta) = -(3+(-1)^\theta)$, $\mathcal{Q}_1(\theta) = e^\theta$, $\mathcal{Q}_2(\theta) = 8 + e^\theta$, $s = 2$, $\tau = 3$, $\sigma_1 = 1$, $\sigma_2 = 2$ and $\mathcal{F}(x) = x^3$. Here, $-4 \leq \mathcal{P}(\theta) \leq -2$ and

$$\sum_{k=\tau}^{\infty}\sum_{j=1}^{2}\mathcal{Q}_j(k) = \infty.$$

Therefore, every solution of (2.8) *oscillates by Theorem 2.3.*

In the next result we do not need the assumption (H_4).

Theorem 2.5. *Let us assume that all the conditions of Theorem 2.3 hold, except* (H_4)*. Then, every bounded solution of* (1.1) *is oscillatory if and only if* (H_3) *holds.*

Proof. To prove the sufficiency part, we follow the proof of Theorem 2.3 and we can conclude that if $\mathcal{U}(\theta)$ is a bounded solution, then so is $\mathcal{Z}$ and we get $\mathcal{Z}(\theta) < 0$ for $\theta \geq \theta_3$. So, we can find $\theta_4 > \theta_3$ and $c > 0$ so that $\mathcal{Z}(\theta) \leq -c$ for $\theta \geq \theta_4$. Consequently, (1.1) becomes

$$\Delta\mathcal{Z}(\theta) + \mathcal{F}\left(-\frac{c}{a_3}\right)\sum_{j=1}^{s}\mathcal{Q}_j(\theta) \leq 0 \tag{2.9}$$

for $\theta \geq \theta_4$. Summing (2.9) from θ_4 to $\theta - 1$, we get

$$\mathcal{Z}(\theta) - \mathcal{Z}(\theta_4) + \mathcal{F}\left(-\frac{c}{a_3}\right)\sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s}Q_j(k) \leq 0,$$

that is,

$$\mathcal{F}\left(-\frac{c}{a_3}\right)\sum_{k=\theta_4}^{\theta-1}\sum_{j=1}^{s}Q_j(k) \leq \mathcal{Z}(\theta_4) - \mathcal{Z}(\theta) < \infty \text{ as } \theta \to \infty,$$

a contradiction to (H_3).

The necessary part can be obtained following the proof of Theorem 2.3. So, we omit it here. Thus, the theorem is proved. □

Theorem 2.6. *Let* $0 \leq \mathcal{P}(\theta) \leq a_6 < 1$ *and assume that* (H_1) *holds. Let* $\mathcal{F}$ *be a Lipschitz function on any* $[\alpha, \beta]$ *where* $0 < \alpha < \beta < \infty$. *Then, every nonoscillatory solution* $\mathcal{U}$ *of* (1.1) *satisfies* $\lim_{\theta\to\infty}\mathcal{U}(\theta) = 0$ *if and only if* (H_3) *holds.*

Proof. To prove the sufficiency part, we argue by contradiction and we assume that (H3) holds and $\mathcal{U}$ is an eventually positive solution of (1.1) which does not converge to zero. Then, following the proof of Theorem 2.1, we conclude that $\mathcal{Z}(\theta)$ is nonincreasing for $\theta \geq \theta_2$. Clearly, $\lim_{\theta\to\infty}\mathcal{Z}(\theta)$ exists as $\mathcal{Z}(\theta) > 0$ for $\theta \geq \theta_2$. As a result, $\liminf_{\theta\to\infty}\mathcal{U}(\theta)$ exists, that is, $\liminf_{\theta\to\infty}\mathcal{U}(\theta) = l,\ 0 \leq l < \infty$. We claim that $\liminf_{\theta\to\infty}\mathcal{U}(\theta) = 0$. If not, then for $\gamma > 0$ we have $\mathcal{U}(\theta - \sigma_j) > \gamma$ for $\theta \geq \theta_3 > \theta_2$ and $j = 1, 2, \ldots, s$. Therefore, (2.2), we get

$$\sum_{k=\theta_3}^{\theta-1}\sum_{j=1}^{s}Q_j(k)\mathcal{F}(\mathcal{U}(k - \sigma_j)) \leq -\mathcal{Z}(\theta) + \mathcal{Z}(\theta_3),$$

that is,

$$\mathcal{F}(\gamma)\sum_{k=\theta_3}^{\theta-1}\sum_{j=1}^{s}Q_j(k) \leq \mathcal{Z}(\theta_3) < \infty \text{ as } \theta \to \infty,$$

a contradiction to (H_3). Hence, $\liminf_{\theta\to\infty}\mathcal{U}(\theta) = 0$. Since $\lim_{\theta\to\infty}\mathcal{Z}(\theta)$ exists, then by [22, Lemma 2.1], $\lim_{\theta\to\infty}\mathcal{Z}(\theta) = 0$. Consequently,

$$0 = \lim_{\theta\to\infty}\mathcal{Z}(\theta) = \limsup_{\theta\to\infty}\ [\mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta - \tau)] \geq \limsup_{\theta\to\infty}\mathcal{U}(\theta),$$

that is, $\limsup_{\theta\to\infty}\mathcal{U}(\theta) = 0$. Thus, $\lim_{\theta\to\infty}\mathcal{U}(\theta) = 0$, which is a contradiction to the fact that $\mathcal{U}(\theta)$ does not converge to zero.

To proof the necessary part, we use the contrapositive method, that is, when (H_3) does not hold we find an eventually positive solution that does not converge to zero. Assume that

$$\sum_{\theta=\theta_1}^{\infty}\sum_{j=1}^{s}Q_j(\theta) < \infty.$$

So, we choose $\theta_2 > \theta_1$ such that

$$\sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k) < \eta,\ \ \theta \geq \theta_2, \tag{2.10}$$

where

$$\eta = \min\left\{\frac{\kappa_6-\alpha}{\mathcal{L}}, \frac{\alpha-(\kappa_5+a_6\kappa_6)}{\mathcal{L}}, \frac{1-a_6}{2\mathcal{L}}\right\},$$

κ_5 and κ_6 are two positive constants such that

$$\kappa_5 < (1-a_6)\kappa_6,\ \ \kappa_5 + a_6\kappa_6 < \alpha < \kappa_6,$$

and $\mathcal{L} = \max\{\mathcal{L}_1, \mathcal{F}(\kappa_6)\}$, $\mathcal{L}_1$ is the Lipschitz constant of $\mathcal{F}$ on $[\kappa_5, \kappa_6]$. Let

$$\Psi = \{\mathcal{U} \in X : \kappa_5 \leq \mathcal{U}(\theta) \leq \kappa_6, \theta \geq \theta_2\}.$$

For $\mathcal{U} \in \Psi$ let us define the map

$$(\Gamma\mathcal{U})(\theta) = \begin{cases} \mathcal{U}(\theta_2+\rho),\ \ \theta_2 \leq \theta \leq \theta_2+\rho \\ \alpha - \mathcal{P}(\theta)\mathcal{U}(\theta-\tau) + \sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)),\ \ \theta \geq \theta_2+\rho. \end{cases}$$

For $\mathcal{U} \in \Psi$ and using (2.10), we have

$$\begin{aligned}(\Gamma\mathcal{U})(\theta) &\leq \alpha + \sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)) \\ &\leq \alpha + \mathcal{F}(\kappa_6)\sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k) \\ &\leq \alpha + \mathcal{F}(\kappa_6)\eta \leq \kappa_6\end{aligned}$$

and

$$(\Gamma\mathcal{U})(\theta) \geq \alpha - \mathcal{P}(\theta)\mathcal{U}(\theta-\tau) \geq \kappa_5 + a_6\kappa_6 - a_6\kappa_6 = \kappa_5$$

implies that $\Gamma\mathcal{U}(\theta) \in \Psi$ for every $\theta \geq \theta_2$. For $\mathcal{U}_1, \mathcal{U}_2 \in \Psi$, we have

$$\begin{aligned}|\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| \leq{}& |\mathcal{P}(\theta)|\ |\mathcal{U}_1(\theta-\tau) - \mathcal{U}_2(\theta-\tau)| \\ &+ \mathcal{L}_1\sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k)|\mathcal{U}_1(k-\sigma_j) - \mathcal{U}_2(k-\sigma_j)|,\end{aligned}$$

that is,

$$\begin{aligned}|\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| &\leq a_6\|\mathcal{U}_1 - \mathcal{U}_2\| + \mathcal{L}_1\|\mathcal{U}_1 - \mathcal{U}_2\|\sum_{k=\theta}^{\infty}\sum_{j=1}^{s}\mathcal{Q}_j(k) \\ &\leq (a_6 + \eta\mathcal{L}_1)\|\mathcal{U}_1 - \mathcal{U}_2\|\end{aligned}$$

$$\leq \left(a_6 + \frac{1-a_6}{2}\right) \|\mathcal{U}_1 - \mathcal{U}_2\|$$

and then

$$\|\Gamma\mathcal{U}_1 - \Gamma\mathcal{U}_2\| \leq \lambda \|\mathcal{U}_1 - \mathcal{U}_2\|,$$

where $\lambda = \frac{1+a_6}{2} < 1$. So, Γ is a contraction. Therefore, by Theorem 1.4, Γ has a point $\mathcal{U} \in \Psi$ such that $\Gamma\mathcal{U} = \mathcal{U}$. Clearly, $\mathcal{U}(\theta)$ is a positive solution of (1.1) such that $\liminf_{\theta\to\infty} \mathcal{U}(\theta) \geq \kappa_5$. Thus, the theorem is proved. □

Example 2.7. *For $\theta > 4$, let us consider*

$$\Delta[\mathcal{U}(\theta) + \mathcal{P}(\theta)\mathcal{U}(\theta-4)] + \mathcal{Q}_1(\theta)\mathcal{U}(\theta-1) + \mathcal{Q}_2(\theta)\mathcal{U}(\theta-2) = 0, \tag{2.11}$$

where $\mathcal{P}(\theta) = \frac{1}{e}$, $\mathcal{Q}_1(\theta) = e^2 - e$, $\mathcal{Q}_2(\theta) = \frac{e^2-e}{e^4}$, $s = 2$, $\tau = 4$, $\sigma_1 = 1$, $\sigma_2 = 2$ and $\mathcal{F}(x) = x$. Here, $0 \leq \mathcal{P}(\theta) \leq \frac{1}{e} < 1$ and

$$\sum_{k=\tau}^{\infty}\sum_{j=1}^{2} \mathcal{Q}_j(k) = \infty.$$

Hence, every nonoscillatory solution of (2.11) *converges to zero as $\theta \to \infty$ by Theorem 2.6. In particular, $\mathcal{U}(\theta) = e^{-\theta}$ is such a solution of* (2.11).

Theorem 2.8. *Let $1 < a_7 \leq \mathcal{P}(\theta) \leq a_8 < \infty$ and assume that (H_1) holds. Assume that $\mathcal{F}$ is a Lipschitz function on any $[\alpha,\beta]$ where $0 < \alpha < \beta < \infty$. Then, every nonoscillatory solution of* (1.1) *satisfies $\lim_{\theta\to\infty} \mathcal{U}(\theta) = 0$ if and only if (H_3) holds.*

Proof. The proof of the sufficiency part of this theorem is similar to that of Theorem 2.6. To prove the necessary part, we use the contrapositive method, that is, when (H_3) does not hold we find an eventually positive solution that does not converge to zero. Assume that

$$\sum_{\theta=\theta_1}^{\infty}\sum_{j=1}^{s} \mathcal{Q}_j(\theta) < \infty.$$

So, we choose $\theta_2 > \theta_1$ such that

$$\sum_{k=\theta}^{\infty}\sum_{j=1}^{s} \mathcal{Q}_j(k) < \eta, \ \theta \geq \theta_2, \tag{2.12}$$

where

$$\eta = \min\left\{\frac{a_7\kappa_8 - \alpha}{\mathcal{L}}, \frac{\alpha - (\kappa_8 + a_8\kappa_7)}{\mathcal{L}}, \frac{(a_7 - 1)}{2\mathcal{L}}\right\},$$

κ_7 and κ_8 are two positive constants such that

$$a_8\kappa_7 < (a_7 - 1)\kappa_8, \ \kappa_8 + a_8\kappa_7 < \alpha < a_7\kappa_8,$$

and $\mathcal{L} = \max\{\mathcal{L}_1, \mathcal{F}(\kappa_8)\}$, $\mathcal{L}_1$ is the Lipschitz constant of $\mathcal{F}$ on $[\kappa_7, \kappa_8]$. Let

$$\Psi = \{\mathcal{U} \in X : \kappa_7 \leq \mathcal{U}(\theta) \leq \kappa_8, \theta \geq \theta_2\}.$$

For $\mathcal{U} \in \Psi$ let us define the map

$$(\Gamma\mathcal{U})(\theta) = \begin{cases} \mathcal{U}(\theta_2 + \rho), \ \theta_2 \le \theta \le \theta_2 + \rho \\ \dfrac{\alpha}{\mathcal{P}(\theta+\tau)} - \dfrac{\mathcal{U}(\theta+\tau)}{\mathcal{P}(\theta+\tau)} + \dfrac{1}{\mathcal{P}(\theta+\tau)} \displaystyle\sum_{k=\theta+\tau}^{\infty} \sum_{j=1}^{s} Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma_j)), \ \theta \ge \theta_2 + \rho. \end{cases}$$

For $\mathcal{U} \in \Psi$ and using (2.12), we have

$$\begin{aligned} (\Gamma\mathcal{U})(\theta) &\le \frac{\alpha}{\mathcal{P}(\theta+\tau)} + \frac{1}{\mathcal{P}(\theta+\tau)} \sum_{k=\theta+\tau}^{\infty} \sum_{j=1}^{s} Q_j(k)\mathcal{F}(\mathcal{U}(k-\sigma)) \\ &\le \frac{\alpha}{\mathcal{P}(\theta+\tau)} + \frac{\mathcal{F}(\kappa_8)}{\mathcal{P}(\theta+\tau)} \sum_{k=\theta+\tau}^{\infty} \sum_{j=1}^{s} Q_j(k) \\ &\le \frac{1}{a_7}[\alpha + \mathcal{F}(\kappa_8)\eta] \le \kappa_8 \end{aligned}$$

and

$$(\Gamma\mathcal{U})(\theta) \ge \frac{\alpha}{\mathcal{P}(\theta+\tau)} - \frac{\mathcal{U}(\theta+\tau)}{\mathcal{P}(\theta+\tau)} \ge \frac{1}{a_8}[a_8\kappa_7 + \kappa_8 - \kappa_8] = \kappa_7,$$

which implies that $\Gamma\mathcal{U}(\theta) \in \Psi$ for every $\theta \ge \theta_2$. For $\mathcal{U}_1, \mathcal{U}_2 \in \Psi$, we have

$$\begin{aligned} |\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| \le &\frac{1}{|\mathcal{P}(\theta+\tau)|} \ |\mathcal{U}_1(\theta+\tau) - \mathcal{U}_2(\theta+\tau)| \\ &+ \frac{\mathcal{L}_1}{|\mathcal{P}(\theta+\tau)|} \sum_{k=\theta+\tau}^{\infty} \sum_{j=1}^{s} Q_j(k)|\mathcal{U}_1(k-\sigma_j) - \mathcal{U}_2(k-\sigma_j)|, \end{aligned}$$

that is,

$$\begin{aligned} |\Gamma\mathcal{U}_1(\theta) - \Gamma\mathcal{U}_2(\theta)| &\le \frac{1}{a_7}\|\mathcal{U}_1 - \mathcal{U}_2\| + \frac{\mathcal{L}_1}{a_7}\|\mathcal{U}_1 - \mathcal{U}_2\| \sum_{k=\theta+\tau}^{\infty} \sum_{j=1}^{s} Q_j(k) \\ &\le \frac{1}{a_7}\|\mathcal{U}_1 - \mathcal{U}_2\| + \frac{\mathcal{L}_1}{a_7}\eta\|\mathcal{U}_1 - \mathcal{U}_2\| \\ &\le \frac{1}{a_7}\left(1 + \frac{a_7 - 1}{2}\right)\|\mathcal{U}_1 - \mathcal{U}_2\| \end{aligned}$$

implying

$$\|\Gamma\mathcal{U}_1 - \Gamma\mathcal{U}_2\| \le \lambda\|\mathcal{U}_1 - \mathcal{U}_2\|,$$

where $\lambda = \frac{1+a_7}{2a_7} < 1$. So, Γ is a contraction. Hence, by Theorem 1.4, Γ has a point $\mathcal{U} \in \Psi$ such that $\Gamma\mathcal{U} = \mathcal{U}$. Therefore, $\mathcal{U}(\theta)$ is a positive solution of (1.1) such that $\liminf_{\theta\to\infty} \mathcal{U}(\theta) \ge \kappa_7$. Thus, the theorem is proved. □

3. Conclusions

In this work, we established some necessary and sufficient conditions for oscillation of (1.1), namely when $-\infty < \mathcal{P}(\theta) < -1$ and $-1 < \mathcal{P}(\theta) \leq 0$. It is worthy pointing out that we were able to establish sufficient and necessary conditions for the asymptotic behaviour of nonoscillatory solutions of of (1.1) for $0 \leq \mathcal{P}(\theta) < 1$ and $1 < \mathcal{P}(\theta) < \infty$. However, to the best of the authors' knowledge, the case $0 \leq \mathcal{P}(\theta) < \infty$ is still open. Asymptotic behaviour of the solutions of (1.1) for $0 \leq \mathcal{P}(\theta) < 1$ and $1 < \mathcal{P}(\theta) < \infty$ are studied in Theorems 2.6 and 2.8, respectively. In Theorems 2.5–2.8, $\mathcal{F}$ could be linear, sublinear or superlinear. The results contained in this paper extend those of Graef et al. [10], Lin [14] because in these papers the authors assumed that $\alpha = 1 = \beta$. Moreover, this paper extends the results of Gao and Zhang [6]. Finally, we note that the methods employed in this paper can be applied to the study of oscillatory properties examined by Tang and Lin [27] when $m = 1$ and $-\infty < \mathcal{P}(\theta) < \infty$.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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