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Nonlocal Controllability of Sobolev-Type Conformable Fractional Stochastic Evolution Inclusions with Clarke Subdifferential

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Abstract
In this paper, Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential and nonlocal conditions are studied. By using fractional calculus, stochastic analysis, properties of Clarke subdifferential and nonsmooth analysis, sufficient conditions for nonlocal controllability for the considered problem are established. Finally, an example is given to illustrate the obtained results.

Keywords Conformable fractional derivative · Stochastic evolution inclusions · Nonlocal controllability · Clarke subdifferential

Mathematics Subject Classification 34K40 · 60H15 · 34A08 · 93B05

1 Introduction

Many real-world problems in science and engineering can be modeled by stochastic differential equations and stochastic differential inclusions (see [1–13]). The integer-order calculus and conventional differential equations are no longer suitable tools for many systems and processes, such as viscoelastic system , dielectric polarization and electromagnetic waves. Hence, in order to avoid this shortcoming of classical derivative, many authors try to replace the classical derivative by a fractional derivative

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because fractional derivatives have been proved that they are a very good way to model many phenomena with memory in various fields of science and engineering [14–18]. Khalil et al. [19] introduced a novel definition named conformable fractional derivative which is an extension of the classical limit definition of the derivative and obeys the classical properties including linearity property, product rule, quotient rule, Rolle's theorem and mean value theorem and coincides with the classical definition of Riemann–Liouville and Caputo on polynomials up to a constant multiple.

In recent years, there have been a lot of results on controllability problems with Clarke subdifferential. Li and Lu [20] discussed the existence and controllability for stochastic evolution inclusions of Clarke's subdifferential type. Zhenhai et al. [21] studied optimal feedback control and controllability for hyperbolic evolution inclusions of Clarke's subdifferential type. Ahmed et al. [22] established sufficient condition for controllability and constrained controllability for nonlocal Hilfer fractional differential systems with Clarke's subdifferential. Raja et al. [23] obtained discussed existence and controllability results for fractional evolution inclusions of order $1 < r < 2$ with Clarke's subdifferential type. Zhenhai and Zeng [24] proved existence and controllability for fractional evolution inclusions of Clarke's subdifferential type.

To the best of our knowledge, nonlocal controllability of nonlocal Sobolev-type conformable fractional stochastic evolution inclusion with Clarke subdifferential has not been studied in this connection and this fact is the motivation of the our work.

Consider the Sobolev-type conformable fractional stochastic evolution inclusion with Clarke subdifferential nonlocal conditions in the following form

$$\begin{cases} D^{\vartheta}\Big(Q[z(t)-m(t,z(t))]\Big) \in A[z(t)-m(t,z(t))] + \sigma(t,z(t)) + By(t) \\ +\varrho(t,z(t))\frac{d\omega(t)}{dt} + \partial G(t,z(t)), \quad t \in I = (0,q], \\ z(0) + \Im(z) = z_0, \end{cases} \tag{1.1}$$

where D^{ϑ} is the conformable fractional derivative, $\frac{1}{2} < \vartheta < 1$, A and Q are linear operators on a Hilbert space Z with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let Γ be another separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\Gamma}$ and norm $\| \cdot \|_{\Gamma}$. Assume $\{\omega(t)\}_{t \geq 0}$ is Γ-valued Wiener process with a finite trace nuclear covariance operator $\Theta \geq 0$. Also, $\| \,.\, \|$ for $L(\Gamma, Z)$, where $L(\Gamma, Z)$ denotes the space of all bounded linear operators from Γ into Z. ∂G is the Clarke's subdifferential of $G(t, z(t))$. The state $z(\cdot)$ takes values in the Hilbert space Z and the control function $y(\cdot)$ is given in $L^2(I, Y)$, the Hilbert space of admissible control functions with Y a Hilbert space and the symbol B stands for a bounded linear operator from Y into Z. The mappings $m : I \times Z \to Z$, $\sigma : I \times Z \to 2^Z$, nonempty, bounded, closed and convex (BCC) multi-valued map, and $\varrho : I \times Z \to L_{\Theta}(\Gamma, Z)$ are nonlinear functions and $\Im : C(I, Z) \to Z$. Here $L_{\Theta}(\Gamma, Z)$ denote the space of all Θ-Hilbert–Schmidt operators from Γ to Z.

The main contributions of this paper are summarized as follows:

- For the first time in the literature, nonlocal controllability of nonlocal Sobolev-type conformable fractional stochastic evolution inclusion with Clarke subdifferential has been investigated.

- By using fractional calculus, stochastic analysis, properties of Clarke subdifferential, nonsmooth analysis and fixed point theorem, new sufficient conditions for nonlocal controllability of the considered system are derived.
- Finally, an example is given to illustrate the theoretical results.

2 Preliminaries

Definition 2.1 (See [19]) Let $0 < \vartheta < 1$. The conformable fractional derivative of order ϑ of a function $f(\cdot)$ for $t > 0$ is defined as follows

$$\frac{d^{\vartheta} f(t)}{dt^{\kappa}} = \lim_{\nu \to 0} \frac{f(t + \nu t^{1-\vartheta}) - f(t)}{\nu}.$$

For $t = 0$, we adopt the following definition:

$$\frac{d^{\vartheta} f(0)}{dt^{\vartheta}} = \lim_{t \to 0^{+}} \frac{d^{\vartheta} f(t)}{dt^{\vartheta}}.$$

The fractional integral $I^{\vartheta}(\cdot)$ associated with the conformable fractional derivative is defined by

$$I^{\vartheta}(f)(t) = \int_0^t \kappa^{\vartheta - 1} f(\kappa)\, \mathrm{d}\kappa.$$

Let $(\Omega, \Upsilon, \{\Upsilon_t\}_{t \geq 0}, P)$ be a complete probability space with a normal filtration $\{\Upsilon_t\}_{t \geq 0}$ satisfying that Υ_0 contains all P-null sets of Υ.

Through this paper, let $\bar{B} := C(I, L^2(\Upsilon, Z))$ be the Banach space of all continuous functions z from I into $L^2(\Upsilon, Z)$, equipped with the supremum norm $\|z\|_{\bar{B}} = \sup_{t \in I}(E\|z(t)\|^2)^{1/2}$, where $L^2(\Upsilon, Z) = L^2(\Omega, \Upsilon, P, Z)$ denotes a Hilbert space of strongly Υ-measurable, H-valued random variables satisfying $E\|z\|^2 < \infty$. $L^2_{\Upsilon}(I, Z)$ will denote the Hilbert space of all random processes Υ_t-adapted measurable defined on J with values in Z with the norm $\|z\|_{L^2_{\Upsilon}(I,Z)} = (\int_0^q E\|z(t)\|_Z^2)^{1/2} < \infty$.

Also, let us introduce the set $D_{\iota} = \{z \in \bar{B} : \|z\|^2_{\bar{B}} \leq \iota\}$, where $\iota > 0$.

The operators $A : D(A) \subset Z \to Z$ and $Q : D(Q) \subset Z \to Z$ satisfy the following hypotheses (see [25]):

$(H1)$ A and Q are closed linear operators,
$(H2)$ $D(Q) \subset D(A)$ and Q is bijective,
$(H3)$ $Q^{-1} : Z \to D(Q)$ is continuous.
Here, $(H1)$ and $(H2)$ together with the closed graph theorem imply the boundedness of the linear operator $AQ^{-1} : Z \to Z$.
$(H4)$ For each $t \in I$ and for $\lambda \in (\rho(AQ^{-1}))$, the resolvent of AQ^{-1}, the resolvent $R(\lambda, AQ^{-1})$ is compact operator.

Lemma 2.1 (See [25]) *Let $S(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $S(t)$ is a compact semigroup.*

From the above fact, AQ^{-1} generates a compact semigroup $\{N(t), t \geq 0\}$ in Z, which means that there exists $T > 1$ such that $\sup_{t \in I} \|N(t)\| \leq T$.

Definition 2.2 (See [26, 27]) Let X be a Banach space with the dual space X^* and $G : X \to R$, be a locally Lipschitz functional on X. The Clarke's generalized directional derivative of G at the point $x \in X$ in the direction $v \in X$, denoted by $G^0(x; v)$ defined by

$$G^0(x; v) = \limsup_{\lambda \to 0^+ \; y \to x} \frac{G(y + \lambda v) - G(y)}{\lambda}.$$

The Clarke's generalized gradient of G at $x \in X$, denoted by $\partial G(x)$, is a subset of X^* given by

$$\partial G(x) = \{x^* \in X^* : G^0(x; v) \geq \langle x^*, v \rangle, \ \forall\, v \in X.\}$$

Definition 2.3 (See [28]) A Υ_t stochastic process $z \in \bar{B}$ is a mild solution of the control system (1.1) if $z(0) = z_0 - \Im(z) \in Z$ and there exist a $\zeta(t) \in L^2_{\Upsilon}(I, Z)$ such that $\zeta(t) \in \partial G(t, z(t))$ for a.e. $t \in I$ and

$$\begin{aligned} z(t) = {} & Q^{-1} N\left(\frac{t^{\vartheta}}{\vartheta}\right) Q[z_0 - \Im(z) - m(0, z(0))] + m(t, z(t)) \\ & + \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^{\vartheta} - \kappa^{\vartheta}}{\vartheta}\right) \sigma(\kappa, z(\kappa))\, \mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^{\vartheta} - \kappa^{\vartheta}}{\vartheta}\right) B y(\kappa)\, \mathrm{d}\kappa + \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^{\vartheta} - \kappa^{\vartheta}}{\vartheta}\right) \zeta(\kappa)\, \mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^{\vartheta} - \kappa^{\vartheta}}{\vartheta}\right) \varrho(\kappa, z(\kappa))\, \mathrm{d}\omega(\kappa), \quad t \in I \end{aligned} \tag{2.1}$$

To establish the results, we need the following hypotheses.

(H5) There exist positive constants $C_1,\ C_2,\ C_3$ such that

$$E\|m(t, z)\|^2 \leq C_1(1 + E\|z\|^2), \quad E\|\sigma(t, z)\|^2 \leq C_2(1 + E\|z\|^2),$$
$$E\|\varrho(t, z)\|^2_{\Theta} \leq C_3(1 + E\|z\|^2),$$

$(H6)$ The function $G : I \times Z \to R$ satisfies the following conditions:

(I) $G(\cdot, z) : I \to R$ is measurable for all $z \in Z$,
(II) $G(t, \cdot) : Z \to R$ is locally Lipschitz continuous for a.e. $t \in I$,
(III) there exist a function $\eta \in L^1(I, R^+)$ and a constant $C_4 > 0$ satisfying

$$E\|\partial G(t, z)\|^2 = \sup\{E\|\zeta(t)\|^2 : \zeta(t) \in \partial G(t, z)\} \leq \eta(t) + C_4 E\|z\|^2,$$

for all $z \in Z$ a.e. $t \in I$ and $z \in Z$.

$(H7)$ The function $\Im : C(I, Z) \to Z$ satisfies the following two conditions:

(i) there exist positive constants C_5 and C_6 such that $E\|\Im(z)\|^2 \leq C_5 E\|z\|^2 + C_6$ for all $z \in Z$
(ii) $\Im$ is completely continuous map.

Now, we define an operator $F : L^2_\Upsilon(I, Z) \to 2^{L^2_\Upsilon(I,Z)}$ as follows
$F(z) = \{\zeta \in L^2_\Upsilon(I, Z) : \zeta(t) \in \partial G(t, z(t)) \text{ a.e. } t \in I \text{ for } z \in L^2_\Upsilon(I, Z)\}$.

Lemma 2.2 (See [29]) *If* $(A6)$ *holds, then for each* $z \in L^2_\Upsilon(I, Z)$, *the set* $F(z)$ *has nonempty, convex and weakly compact values.*

Lemma 2.3 (See [30]) *If* $(A6)$ *holds, the operator* F *satisfies: if* $z_n \to z$ *in* $L^2_\Upsilon(I, Z)$, $\varphi_n \to \varphi$ *weakly in* $L^2_\Upsilon(I, Z)$ *and* $\varphi_n \in F(z_n)$, *then* $\varphi \in F(z)$.

Theorem 2.1 (See [31]) *Let* X *be a locally convex Banach space and* $M_\ell : X \to 2^X$ *be compact convex-valued, upper semicontinuous multi-valued maps such that there exist a closed neighborhood* L *of* 0 *for which* $M_\ell(L)$ *is relatively compact set. If the set* $\Psi = \{z \in X : \alpha z \in M_\ell(z) \text{ for some } \alpha > 1\}$ *is bounded, then* M_ℓ *has a fixed point.*

3 Nonlocal Controllability

Definition 3.1 The system (1.1) is said to be nonlocal controllable on the interval I, if for every initial condition z_0 and $z_1 \in Z$, there exists a stochastic control $y \in L^2(I, Y)$ such that a mild solution $z(\cdot)$ of system (1.1) satisfies $z(q) + \Im(z) = z_1$, where z_1 and q are the preassigned terminal state and time, respectively.

In order to prove the main result, we assume the following hypothesis:

(H8) The linear operator U from $L^2(I, Y)$ into Z defined by

$$Uy = \int_0^q \kappa^{\vartheta-1} Q^{-1} N\left(\frac{q^\vartheta - \kappa^\vartheta}{\vartheta}\right) By(\kappa)\, \mathrm{d}\kappa$$

is invertible with inverse operator U^{-1} taking values in $L^2(I, Y) \setminus \ker U$, and there exist positive constants T_1 and T_2 such that $\|B\|^2 \leq T_1$ and $\|U^{-1}\|^2 \leq T_2$.

Theorem 3.1 *If* (H1)–(H8) *are fulfilled, then* (1.1) *is nonlocal controllable on* I *provided that*

$$\begin{aligned}\wp_2 = &\left\{1 + \frac{T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta - 1)}\right\}\left\{36T^2\|Q^{-1}\|^2\|Q\|^2[C_5(1 + C_1)] + 36C_1\right.\\ &\left.+ \frac{36T^2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta - 1)}[(C_2 + Tr(\Theta)C_3) + C_4 q]\right\}\\ &+ \frac{36T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1} C_5}{(2\vartheta - 1)} < 1. \end{aligned} \tag{3.1}$$

Proof We define the control operator by using the hypothesis $(A8)$

$$y(t) = U^{-1}\{z_1 - \Im(z) - Q^{-1}N\left(\frac{q^{\vartheta}}{\vartheta}\right)Q[z_0 - \Im(z) - m(0, z(0))] - m(q, z(q))$$
$$- \int_0^q \kappa^{\vartheta-1}Q^{-1}N\left(\frac{q^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa, z(\kappa))d\kappa \int_0^q \kappa^{\vartheta-1}Q^{-1}N\left(\frac{q^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa$$
$$- \int_0^q \kappa^{\vartheta-1}Q^{-1}N\left(\frac{q^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa, z(\kappa))\,\mathrm{d}\omega(\kappa)\}(t).$$

Consider the map $M_\ell : \bar{B} \to 2^{\bar{B}}$ as follows

$$M_\ell(z) = \left\{\begin{array}{l} V \in \bar{B} : V(t) = Q^{-1}N\left(\frac{t^{\vartheta}}{\vartheta}\right)Q[z_0 - \Im(z) - m(0, z(0))] + m(t, z(t)) \\ + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa, z(\kappa))\,\mathrm{d}\kappa \\ + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa \\ + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa, z(\kappa))\,\mathrm{d}\omega(\kappa)\;,\; \zeta \in F(z). \end{array}\right\}.$$

We will show the operator M_ℓ has a fixed point.

We subdivide the proof into a sequence of steps.

Step 1: For each $z \in \bar{B}$, $M_\ell(z)$ is nonempty, convex and weakly compact values.

According to Lemma 2.2, it is easy to see that $M_\ell(z)$ has nonempty and weakly compact values. Moreover, as $F(z)$ has convex values, so that if $\rho_1,\ \rho_2 \in F(z)$ then $a\rho_1 + (1-a)\rho_2 \in F(z)$ for all $a \in (0, 1)$, which implies clearly that $M_\ell(z)$ is convex.

Step 2: The operator M_ℓ is bounded on bounded subset of $\bar{B}$.

Obviously, D_ι is a bounded, closed and convex set of $\bar{B}$.

We show that there exists a positive constant τ such that for each $V \in M_\ell(z)$, $z \in D_\iota$, one has $E\|V(t)\|^2 \le \tau$.

If $\chi \in M_\ell(z)$, then there exists a $\zeta \in F(z)$ such that

$$\chi(t) = Q^{-1}N\left(\frac{t^{\vartheta}}{\vartheta}\right)Q[z_0 - \Im(z) - m(0, z(0))] + m(t, z(t))$$
$$+ \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa, z(\kappa))\,\mathrm{d}\kappa$$
$$+ \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa$$
$$+ \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa, z(\kappa))\,\mathrm{d}\omega(\kappa), \quad t \in I. \tag{3.2}$$

Then

$$E\|\chi(t)\|^2 \le 36E\|Q^{-1}N\left(\frac{t^{\vartheta}}{\vartheta}\right)Q[z_0 - \Im(z) - m(0, z(0))]\|^2$$
$$+ 36E\|m(t, z(t))\|^2 + 36E\|\int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa, z(\kappa))\,\mathrm{d}\kappa\|^2$$

$$
\begin{aligned}
&+36E\|\int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta-\kappa^\vartheta}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa\|^2\\
&+36E\|\int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta-\kappa^\vartheta}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\|^2\\
&+36E\|\int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta-\kappa^\vartheta}{\vartheta}\right)\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)\|^2\\
\leq &\Bigg\{36T^2\|Q^{-1}\|^2\|Q\|^2\Big[(E\|z_0\|^2+C_5\iota+C_6)(1+C_1)+C_1\Big]+36C_1(1+\iota)\\
&+\frac{36T^2\|Q^{-1}\|^2q^{2\vartheta-1}}{(2\vartheta-1)}\Big[(C_2+Tr(\Theta)C_3)(1+\iota)+\|\zeta\|_{L^1(I,R^+)}+C_4q\iota\Big]\Bigg\}\\
&\times\left\{1+\frac{T^2T_1T_2\|Q^{-1}\|^2q^{2\vartheta-1}}{(2\vartheta-1)}\right\}\\
&+\frac{36T^2T_1T_2\|Q^{-1}\|^2q^{2\vartheta-1}}{(2\vartheta-1)}\Big[\|z_1\|^2+C_5\iota+C_6\Big]:=\tau.
\end{aligned}
$$

Thus, $M_\ell(D_\iota)$ is bounded in $\bar{B}$.
Step 3: The set $\{M_\ell(z) : z \in D_\iota\}$ is equicontinuous.

For any $z \in D_\iota$, $\chi \in M_\ell(z)$, there exists a $\zeta \in F(z)$ such that (3.2) holds for each $t \in I$.

For $0 < t_1 < t_2 < q$, we can obtain

$$
\begin{aligned}
&E\|\chi(t_2)-\chi(t_1)\|^2\\
&\leq 36E\|Q^{-1}\left(N\left(\frac{t_2^\vartheta}{\vartheta}\right)-N\left(\frac{t_1^\vartheta}{\vartheta}\right)\right)Q[z_0-\Im(z)-m(0,z(0))]\|^2\\
&\quad+36E\|m(t_2,z(t_2))-m(t_1,z(t_1))\|^2\\
&\quad+36E\|\int_0^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^\vartheta-\kappa^\vartheta}{\vartheta}\right)\sigma(\kappa,z(\kappa))\,\mathrm{d}\kappa\\
&\quad-\int_0^{t_1}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_1^\vartheta-\kappa^\vartheta}{\vartheta}\right)\sigma(\kappa,z(\kappa))d\kappa\|^2\\
&\quad+36E\|\int_0^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^\vartheta-\kappa^\vartheta}{\vartheta}\right)\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)\\
&\quad-\int_0^{t_1}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_1^\vartheta-\kappa^\vartheta}{\vartheta}\right)\varrho(\kappa,z(\kappa))d\omega(\kappa)\|^2\\
&\quad+36E\|\int_0^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^\vartheta-\kappa^\vartheta}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\\
&\quad-\int_0^{t_1}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_1^\vartheta-\kappa^\vartheta}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+36E\|\int_0^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa\\
&-\int_0^{t_1}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_1^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(s)\,\mathrm{d}\kappa\|^2\\
=\;&36E\|Q^{-1}\left(N\left(\frac{t_2^{\vartheta}}{\vartheta}\right)-N\left(\frac{t_1^{\vartheta}}{\vartheta}\right)\right)Q[z_0-\Im(z)-m(0,z(0))]\|^2\\
&+36E\|m(t_2,z(t_2))-m(t_1,z(t_1))\|^2\\
&+36E\|\int_{t_1}^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa,z(\kappa))\,\mathrm{d}\kappa\\
&+\int_0^{t_1}\kappa^{\vartheta-1}\left[Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)-Q^{-1}N\left(\frac{t_1^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\right]\sigma(s,x(s))\,\mathrm{d}\kappa\|^2\\
&+36E\|\int_{t_1}^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)\\
&+\int_0^{t_1}\kappa^{\vartheta-1}\left[Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)-Q^{-1}N\left(\frac{t_1^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\right]\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)\|^2\\
&+36E\|\int_{t_1}^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\\
&+\int_0^{t_1}\kappa^{\vartheta-1}\left[Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)-Q^{-1}N\left(\frac{t_1^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\right]\zeta(\kappa)d\kappa\|^2\\
&+36E\|\int_{t_1}^{t_2}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa\\
&+\int_0^{t_1}\left[Q^{-1}N\left(\frac{t_2^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)-Q^{-1}N\left(\frac{t_1^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\right]By(\kappa)d\kappa\|^2.
\end{aligned}
$$

From the compactness of $N(t)(t>0)$, we see that the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Thus, we can conclude that $M_\ell(z)(t)$ is continuous from the right in $(0,q]$. Similarly, for $t_1=0$ and $0<t_2\le q$, we can prove that $E\|\chi(t_2)-\chi(0)\|^2$ tends to zero uniformly with respect to $z\in D_\iota$ as $t_2\to 0$.

Hence, we infer that $\{M_\ell(z)(t):z\in D_\iota\}$ is an equicontinuous family of functions in $\bar{B}$.

Step 4: M_ℓ is completely continuous.

We prove that for all $t\in I,\ \iota>0$, the set $\Pi(t)=\{\chi(t):\chi\in M_\ell(D_\iota)\}$ is relatively compact in Z. Obviously, $\Pi(0)$ is relatively compact in D_ι. Let $0<t\le q$ be fixed, $0<\epsilon<t$, for $z\in D_\iota$, we define

$$
\begin{aligned}
\chi^{\epsilon}(t) &= Q^{-1}N\left(\frac{t^{\vartheta}}{\vartheta}\right)Q[z_0-\Im(z)-m(0,z(0))]+m(t,z(t))\\
&\quad+\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa,z(\kappa))\,\mathrm{d}\kappa\\
&\quad+\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa\\
&\quad+\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\\
&\quad+\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa,z(\kappa))d\omega(\kappa)\\
&= N(\epsilon^{\vartheta})Q^{-1}N\left(\frac{t^{\vartheta}}{\vartheta}-N(\epsilon^{\vartheta})\right)Q[z_0-\Im(z)-m(0,z(0))]+m(t,z(t))\\
&\quad+N(\epsilon^{\vartheta})\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}-N(\epsilon^{\vartheta})\right)\sigma(\kappa,z(\kappa))\,\mathrm{d}\kappa\\
&\quad+N(\epsilon^{\vartheta})\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}-S(\epsilon^{\vartheta})\right)By(\kappa)\,\mathrm{d}\kappa\\
&\quad+N(\epsilon^{\vartheta})\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}-N(\epsilon^{\vartheta})\right)\zeta(\kappa)\,\mathrm{d}\kappa\\
&\quad+N(\epsilon^{\vartheta})\int_0^{t-\epsilon}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}-N(\epsilon^{\vartheta})\right)\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)
\end{aligned}
$$

Since $N(t)$ is compact, then the set $\Pi^{\epsilon}(t)=\{\chi^{\epsilon}(t):\chi^{\epsilon}\in M_{\ell}(D_{\iota})\}$ is relatively compact in Z. Moreover, we have

$$
\begin{aligned}
E\|\chi(t)-\chi^{\epsilon}(t)\|^2 &\le 36E\left\|\int_{t-\epsilon}^{t}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\sigma(\kappa,z(\kappa))\,\mathrm{d}\kappa\right\|^2\\
&\quad+36E\left\|\int_{t-\epsilon}^{t}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa\right\|^2\\
&\quad+36E\left\|\int_{t-\epsilon}^{t}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\zeta(\kappa)\,\mathrm{d}\kappa\right\|^2\\
&\quad+36E\left\|\int_{t-\epsilon}^{t}\kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^{\vartheta}-\kappa^{\vartheta}}{\vartheta}\right)\varrho(\kappa,z(\kappa))\,\mathrm{d}\omega(\kappa)\right\|^2
\end{aligned}
$$

We see that, when $\epsilon\to 0^+$, the inequality above tends to zero. Therefore, the set $\Pi(t)$ is relatively compact in Z. Thus, from Step 3 and the Arzela–Ascoli theorem, M_{ℓ} is completely continuous.
Step 5: M_{ℓ} has a closed graph.

Consider $z_n \to z_*$ in $\bar{B}$, $\chi_n \in M_\ell(z_n)$ and $\chi_n \to \chi_*$ in $\bar{B}$. We will prove that $\chi_* \in M_\ell(z_*)$. Actually, $\chi_n \in M_\ell(z_n)$ implies that there exists a $\zeta_n \in F(z_n)$ such that

$$\begin{aligned}\chi_n(t) = {} & Q^{-1}N\left(\frac{t^\vartheta}{\vartheta}\right)Q[z_0 - \Im(z_n) - m(0, z(0))] + m(t, z_n(t)) \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\sigma(\kappa, z_n(\kappa))\,\mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)By(\kappa)d\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\zeta_n(\kappa)\,\mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\varrho(\kappa, z_n(\kappa))\mathrm{d}\omega(\kappa). \end{aligned} \tag{3.3}$$

From (H5)–(H7), it is easy to show that $\{\Im(z_n), m(\cdot, z_n), \sigma(\cdot, z_n), \zeta_n, \varrho(\cdot, z_n)\}_{n\geq 1} \subseteq Z \times Z \times Z \times L_F^2(J, Z) \times L_\Theta$ is bounded. Hence, moving to a subsequence if necessary, we get

$$(\Im(z_n), m(\cdot, z_n), \sigma(\cdot, z_n), \zeta_n, \varrho(\cdot, z_n)) \to (\Im(z_*), m(\cdot, z_*), \sigma(\cdot, z_*), \zeta_*, \varrho(\cdot, z_*)) \tag{3.4}$$

weakly in $Z \times Z \times Z \times L_F^2(J, Z) \times L_\Theta$.

From (3.3), (3.4) and the compactness of the operator $N(t)$, we have that

$$\begin{aligned}\chi_n(t) \to {} & Q^{-1}N\left(\frac{t^\vartheta}{\vartheta}\right)Q[z_0 - \Im(z_*) - m(0, z(0))] + m(t, z_*(t)) \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\sigma(s, z_*(s))\,\mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)By(\kappa)\,\mathrm{d}\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\zeta_*(s)d\kappa \\ & + \int_0^t \kappa^{\vartheta-1}Q^{-1}N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right)\varrho(s, z_*(s))\,\mathrm{d}\omega(\kappa). \end{aligned} \tag{3.5}$$

Concentrating that $\chi_n \to \chi_*$ in $\bar{B}$ and $\zeta_n \in F(z_n)$. From Lemma 2.3 and (3.5), we can obtain $\zeta_* \in F(z_*)$. Therefore, it can show that $\chi_* \in M_\ell(z_*)$, which implies that M_ℓ has a closed graph. and M_ℓ is a completely continuous multi-valued map with compact value. Thus, from Liu et al. [21], M_ℓ is upper semicontinuous.

Step 6: A priori estimate.

From steps 1–5, we found that M_ℓ is compact convex-valued and upper semicontinuous and $M_\ell(D_\iota)$ is relatively compact. By Theorem 2.1, it remains to show that the set $\Psi = \{z \in \bar{B} : \alpha z \in M_\ell, \ \ \alpha > 1\}$ is bounded. For all $z \in \Psi$, there exists a $\zeta \in F(z)$ such that

$$\begin{aligned} z(t) &= \alpha^{-1} Q^{-1} N\left(\frac{t^\vartheta}{\vartheta}\right) Q[z_0 - \Im(z) - m(0, z(0))] + \alpha^{-1} m(t, z(t)) \\ &\quad + \alpha^{-1} \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \sigma(\kappa, z(\kappa))\, \mathrm{d}\kappa \\ &\quad + \alpha^{-1} \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) B y(\kappa)\, \mathrm{d}\kappa \\ &\quad + \alpha^{-1} \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \zeta(\kappa)\, \mathrm{d}\kappa \\ &\quad + \alpha^{-1} \int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \varrho(\kappa, z(\kappa))\, \mathrm{d}\omega(\kappa). \end{aligned} \tag{3.6}$$

By using the assumptions (A5)–(A7), we can obtain

$$\begin{aligned} E\|z(t)\|^2 &\le 36E\|Q^{-1} N\left(\frac{t^\vartheta}{\vartheta}\right) Q[z_0 - \Im(z) - m(0, z(0))]\|^2 \\ &\quad + 36E\|m(t, z(t))\|^2 + 36E\|\int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \sigma(\kappa, z(\kappa))\, \mathrm{d}\kappa\|^2 \\ &\quad + 36E\|\int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) B y(\kappa) d\kappa\|^2 \\ &\quad + 36E\|\int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \zeta(\kappa)\, \mathrm{d}\kappa\|^2 \\ &\quad + 36E\|\int_0^t \kappa^{\vartheta-1} Q^{-1} N\left(\frac{t^\vartheta - \kappa^\vartheta}{\vartheta}\right) \varrho(\kappa, z(\kappa)) d\omega(\kappa)\|^2 \\ &\le \Bigg\{ 36T^2\|Q^{-1}\|^2\|Q\|^2\Big[(E\|z_0\|^2 + C_5 E\|z(t)\|^2 + C_6)(1 + C_1) + C_1\Big] \\ &\quad + 36C_1(1 + E\|z(t)\|^2) \\ &\quad + \frac{36T^2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta - 1)}\Big[(C_2 + Tr(\Theta)C_3)(1 + E\|z(t)\|^2) \\ &\quad + \|\zeta\|_{L^1(I,R^+)} + C_4 T E\|z(t)\|^2\Big]\Bigg\} \\ &\quad \times \left\{1 + \frac{T^2 T_1 T_2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta - 1)}\right\} \\ &\quad + \frac{36T^2 T_1 T_2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta - 1)}\Big[\|z_1\|^2 + C_5 E\|z(t)\|^2 + C_6\Big] \\ &\le \wp_1 + \wp_2 E\|z(t)\|^2, \end{aligned} \tag{3.7}$$

where

$$\wp_1 = \left\{1 + \frac{T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta-1)}\right\}$$
$$\left\{36T^2\|Q^{-1}\|^2\|Q\|^2[(E\|z_0\|^2 + C_6)(1+C_1) + C_1] + 36C_1\right.$$
$$\left. + \frac{36T^2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta-1)}[(C_2 + Tr(\Theta)C_3) + \|\zeta\|_{L^1(I,R^+)}]\right\}$$
$$+ \frac{36T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1}[\|z_1\|^2 + C_6]}{(2\vartheta-1)}.$$
$$\wp_2 = \left\{1 + \frac{T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta-1)}\right\}\left\{36T^2\|Q^{-1}\|^2\|Q\|^2[C_5(1+C_1)] + 36C_1\right.$$
$$\left. + \frac{36T^2\|Q^{-1}\|^2 q^{2\vartheta-1}}{(2\vartheta-1)}[(C_2 + Tr(\Theta)C_3) + C_4 q]\right\}$$
$$+ \frac{36T^2 T_1 T_2 \|Q^{-1}\|^2 q^{2\vartheta-1} C_5}{(2\vartheta-1)}.$$

Since $\wp_2 < 1$, from (3.7), we obtain

$$\|z\|^2_{\bar{B}} = \sup_{t\in J} E\|z(t)\|^2 \le \wp_1 + \wp_2\|z\|^2_{\bar{B}}.$$

Then, $\|z\|^2_{\bar{B}} \le \frac{\wp_1}{1-\wp_2}$ implies that the set Ψ is bounded. By theorem 2.1, M_ℓ has a fixed point. Therefore, the system (1.1) is nonlocal controllable on I, and the proof is completed. □

4 Example

In this section, we present an example to illustrate the applicability of our results.

Let us consider the control system described by Sobolev-type conformable fractional stochastic evolution inclusions with Clarke generalized gradient:

$$\begin{cases} {}^cD^{\frac{3}{5}}[(1 - \frac{\partial^2}{\partial\mu^2})(z(t,\mu) - m(t,z(t,\mu)))] \in \frac{\partial^2}{\partial\mu^2}[(z(t,\mu) - m(t,z(t,\mu)))] \\ +\frac{1}{20}sin(z(t,\mu)) + \gamma(t,\mu) + \frac{1}{5}cos(z(t,\mu))\frac{d\omega(t)}{dt} + \partial G(t,\mu,z(t,\mu)),\ t\in I = (0,1], \\ z(t,0) = z(t,2) = 0, \quad t\in I, \\ z(0,x) + \sum_{i=1}^{p} a_i\, z(t_i,\mu) = z_0(\mu),\ 0\le\mu\le 2, \end{cases} \tag{4.1}$$

where $0 < t_0 < t_1 < \cdots < t_p < 1$, $z_0(\mu) \in Z = L^2([0,2])$ and ω is a Wiener process. The functions $z(t)(\mu) = z(t,\mu)$, $m(t,z(t))(\mu) = m(t,z(t,\mu))$, $\frac{1}{20}sin(z(t,\mu)) = \sigma(t,z(t,\mu))$, $\frac{1}{5}cos(z(t,\mu)) = \varrho(t,z(t,\mu))$, $G(t,z(t))(\mu) =$

$G(t, \mu, z(t, \mu))$ and $y(t)(\mu) = \gamma(t, \mu)$. The bounded linear operator B is defined by $By = \vartheta(t, \mu),\ t \in I,\ 0 \leq \mu \leq 2,\ y \in Y$.

To study this system, let $Z = \Gamma = Y = L^2([0, 2])$ and the operators $A : D(\Delta) \subset Z \to Z$ and $Q : D(Q) \subset Z \to Z$ are given by $A = \frac{\partial^2}{\partial \mu^2}$ and $Q = 1 - A$ with $D(A) = D(Q) = \{z \in Z, z, \frac{\partial z}{\partial \mu}$ are absolutely continuous, $\frac{\partial^2 z}{\partial \mu^2} \in Z,\ z(0) = z(2) = 0\}$.

Then A and Q can be written as

$$Az = \sum_{n=1}^{\infty} (-n^2)(z, z_n)z_n, \quad z \in D(\Delta), \qquad Qz = \sum_{n=1}^{\infty} (1 + n^2)(z, z_n)z_n, \quad z \in D(B),$$

where $z_n(s) = \sqrt{\frac{2}{\pi}} \sin ns,\ n = 1, 2, \ldots$ is the orthogonal set of eigenvectors of Δ.

Furthermore, for $z \in Z$ we have

$$Q^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1 + n^2}(z, z_n)z_n, \qquad AQ^{-1}z = \sum_{n=1}^{\infty} \frac{-n^2}{1 + n^2}(z, z_n)z_n,$$

$$N(t)z = \sum_{n=1}^{\infty} e^{\frac{-n^2 t}{1+n^2}}(z, z_n)z_n.$$

It is well known that AQ^{-1} generates a strongly continuous semigroup $\{N(t)\}_{t \geq 0}$ which is compact, analytic and self-adjoint in Z. Therefore, with the above choice, the system (4.1) can be written to the abstract (1.1) and all conditions of Theorem 3.1 are satisfied. Thus by Theorem 3.1, the system (4.1) is nonlocal controllable on (0, 1].

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