Review Article

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Some recent results on singular *p*-Laplacian equations

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Abstract: A short account of some recent existence, multiplicity, and uniqueness results for singular p-Laplacian problems either in bounded domains or in the whole space is performed, with a special attention to the case of convective reactions. An extensive bibliography is also provided.

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1 Introduction

When studying quasi-linear elliptic systems in the whole space and with singular, possibly convective, reactions, a natural preliminary step is looking for the previous literature on equations of the same type, which we have done in the latest years.

At first, this led us to investigate *singular p-Laplacian Dirichlet problems* as

$$\begin{cases} -\Delta_p u = h(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < \infty$, the symbol Δ_p denotes the p-Laplace operator, namely

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$, and $h \in C^0(\Omega \times \mathbb{R}^+ \times \mathbb{R}^N)$ satisfies

$$\lim_{t\to 0^+} h(x, t, \xi) = \infty.$$

If $p = 2$, then various special (chiefly nonconvective) cases of (1.1) have been thoroughly studied (see Section 3.1). Both surveys [1–3] and a monograph [4], besides many proceeding papers, are already available. The main purpose of Section 3 is to provide a short account on some recent existence, multiplicity, or uniqueness results for $p \neq 2$ and the relevant technical approaches. Let us also point out [5–7]. Saoudi's work [5] treats a singular $p(x)$-Laplacian Robin problem, whereas [6,7] are devoted to singular (p, q)-Laplacian equations with Neumann and Robin boundary conditions, respectively (see also [8]). Section 4 aims at

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performing the same as regards *singular p-Laplacian problems in the whole space*. So, it deals with situations like

$$\begin{cases} -\Delta_p u = h(x, u, \nabla u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.2}$$

To the best of our knowledge, except [4], even when $p = 2$ and h does not depend on ∇u, there are no surveys concerning (1.2). Hence, this probably represents the first contribution.

Both sections are divided into four parts. Sub-section 1 is a historical sketch of the case $p = 2$. Sub-sections 2 and 3 treat existence, multiplicity, and uniqueness in the nonconvective case. Sub-section 4 is devoted to singular problems with convection. Since the literature on (1.1)–(1.2) is by now very wide and our knowledge is limited, significant works may have been overlooked, for which we apologize in advance. Moreover, for the sake of brevity, we did not treat singular parabolic boundary-value problems and instead refer the reader to [2,9–12].

2 Basic notation

Let $X(\Omega)$ be a real-valued function space on a nonempty measurable set $\Omega \subseteq \mathbb{R}^N$. If $u_1, u_2 \in X(\Omega)$, and $u_1(x) < u_2(x)$ a.e. in Ω, then we simply write $u_1 < u_2$. The meaning of $u_1 \le u_2$, etc. is analogous. Put

$$X(\Omega)_+ := \{u \in X(\Omega) : u \ge 0\}.$$

The symbol $u \in X_{\mathrm{loc}}(\Omega)$ means that $u : \Omega \to \mathbb{R}$ and $u\lfloor_K \in X(K)$ for all nonempty compact subset K of Ω. Given $1 < r < N$, define

$$r' := \frac{r}{r-1}, \qquad r^* := \frac{Nr}{N-r}.$$

Let us next recall the notion and some relevant properties of the so-called Beppo Levi space $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, addressing the reader to [13, Chapter II] for a complete treatment. Set

$$\mathcal{D}^{1,r} := \{z \in L^1_{\mathrm{loc}}(\mathbb{R}^N) : |\nabla z| \in L^r(\mathbb{R}^N)\}$$

and denote by $\mathcal{R}$ the equivalence relation that identifies two elements in $\mathcal{D}^{1,r}$ whose difference is a constant. The quotient set $\dot{\mathcal{D}}^{1,r}$, endowed with the norm

$$\|u\|_{1,r} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^r \, \mathrm{d}x \right)^{1/r},$$

turns out complete. Write $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$ for the subspace of $\dot{\mathcal{D}}^{1,r}$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under $\|\cdot\|_{1,r}$, namely

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{1,r}}.$$

$\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, usually called Beppo Levi space, is reflexive and continuously embeds in $L^{r^*}(\mathbb{R}^N)$, i.e.,

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) \hookrightarrow L^{r^*}(\mathbb{R}^N). \tag{2.1}$$

Consequently, if $u \in \mathcal{D}_0^{1,r}(\mathbb{R}^N)$, then u vanishes at infinity, meaning that the set $\{x \in \mathbb{R}^N : |u(x)| \ge \varepsilon\}$ has finite measure for any $\varepsilon > 0$.

3 Problems in bounded domains

3.1 The case $p = 2$

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$; let $a : \Omega \to \mathbb{R}_0^+$ be nontrivial measurable; and let $\gamma > 0$. The simplest singular elliptic Dirichlet problem is written as:

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Since the pioneering papers [14–18], a wealth of existence, uniqueness or multiplicity, and regularity results concerning (3.1) have been published. We refer the reader to the monograph [4] as well as the surveys [1,2] for an exhaustive account. Roughly speaking, four basic questions can be identified:

- Find the right conditions on the datum a. Usually, $a \in L^q(\Omega)$ with $q \geq 1$ is enough for existence. However, starting from the works [19,20], the case when a is a bounded Radon measure took interest.
- Consider nonmonotone singular terms. This is a difficult task, mainly when we want to guarantee the uniqueness of solutions.
- Insert convective terms on the right-hand side. For equations driven by the Laplacian, good references are [4, section 9] and [21]. Otherwise, cf. [22–25].
- Substitute the Laplacian with more general elliptic operators. Obviously, the first attempt might be to consider equations driven by the p-Laplacian, and this section aims to provide a short account of the recent literature. However, further possibly nonhomogeneous operators have been considered; see, e.g., [14,15,20,26–31].

Incidentally, we recall that (3.1) stems from important applied questions, such as the study of heat conduction in electrically conducting materials [32], chemical heterogeneous catalysts [33], and non-Newtonian fluids [34].

3.2 Existence and multiplicity

Consider the model problem

$$\begin{cases} -\Delta_p u = a(x)u^{-\gamma} + \lambda f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $a : \Omega \to \mathbb{R}_0^+$ denotes a nonzero measurable function, $\gamma, \lambda > 0$, while $f : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies Carathéodory's conditions. Let us stress that, here, *the parameter λ multiplies the nonsingular term.*

In 2006, Perera and Silva investigated (3.2) under the following assumptions, where f is allowed to change sign.

(a_1) There exist $\varphi_0 \in C_0^1(\overline{\Omega})_+$ and $\hat{q} > N$ such that $a\varphi_0^{-\gamma} \in L^{\hat{q}}(\Omega)$.

(a_2) With appropriate $\delta, c_1 > 0$, one has

$$f(x, t) \geq -c_1 a(x) \text{ in } \Omega \times [0, \delta].$$

(a_3) To every $M > 0$, there are corresponding $h \in L^1(\Omega)$ and $c_2 > 0$ such that

$$-h(x) \leq f(x, t) \leq c_2 \quad \forall (x, t) \in \Omega \times [0, M].$$

(a_4) With appropriate $q \in]1, p^*[$ and $c_3 > 0$, one has

$$f(x, t) \leq c_3(t^{q-1} + 1) \text{ in } \Omega \times \mathbb{R}_0^+.$$

(a_5) There are $t_0 > 0$ and $\mu > p$ such that

$$0 < \mu \int_0^t f(x,\tau)d\tau \le tf(x,t) \quad \forall (x,t) \in \Omega \times [t_0, +\infty[.$$

They seek distributional solutions to (3.2), i.e., functions $u \in W_0^{1,p}(\Omega)$ such that $u > 0$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx = \int_\Omega au^{-\gamma}\varphi dx + \int_\Omega f(\cdot,u)\varphi dx \quad \forall \varphi \in C_0^\infty(\Omega).$$

Theorem 3.1. ([35], Theorems 1.1 and 1.2) *Let* (a_1)–(a_3) *be satisfied. Then Problem* (3.2) *admits a distributional solution for every* $\lambda > 0$ *small. If, in addition,* (a_4) *and* (a_5) *hold true, then a further distributional solution exists by decreasing* λ *when necessary.*

Proofs employ perturbation arguments and variational methods, which were previously introduced in [36]. An immediate but hopefully useful consequence of Theorem 3.1 is as follows:

Corollary 3.2. *Let* (a_1) *be fulfilled. Suppose f does not depend on x and, moreover,* $f(t) \ge 0$ *in a neighborhood of zero once* $\operatorname{ess\,inf}_\Omega a = 0$. *Then, for every* $\lambda > 0$ *sufficiently small, the problem*

$$-\Delta_p u = a(x)u^{-\gamma} + \lambda f(u) \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{3.3}$$

possesses a distributional solution.

Further results concerning (3.3) can be found in the work of Aranda and Godoy [37], where a continuous nonincreasing function $g(u)$ takes the place of $u^{-\gamma}$ and, from a technical point of view, fixed point theorems for nonlinear eigenvalue problems are exploited.

The case $\lambda = 0$ in (3.3) was well investigated by Canino, Sciunzi, and Trombetta [38], with a special attention to uniqueness (see Section 3.3). Here, given $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$,

$$u = 0 \text{ on } \partial\Omega \overset{\text{def}}{\Leftrightarrow} u \ge 0 \text{ and } (u-\varepsilon)^+ \in W_0^{1,p}(\Omega) \quad \forall \varepsilon > 0.$$

Theorem 3.3. ([38], Theorem 1.3) *Let* $\lambda = 0$. *If* $\gamma \ge 1$ *and* $a \in L^1(\Omega)$, *then* (3.3) *admits a distributional solution* $u \in W_{\mathrm{loc}}^{1,p}(\Omega)$ *such that* $\operatorname{ess\,inf}_K u > 0$ *for any compact set* $K \subseteq \Omega$. *Moreover,* $u^{1+(\gamma-1)/p} \in W_0^{1,p}(\Omega)$. *If* $0 < \gamma < 1$, *then* (3.3) *has a solution* $u \in W_0^{1,p}(\Omega)$ *in each of the following cases*:

- $1 < p < N$ *and* $a \in L^m(\Omega)$, *with* $m := \left(\frac{p^*}{1-\gamma}\right)'$.
- $p = N$ *and* $a \in L^m(\Omega)$ *for some* $m > 1$.
- $p > N$ *and* $a \in L^1(\Omega)$.

The proof of this result relies on a technique that was previously introduced in [29] for the semi-linear case. It employs truncation and regularization arguments. The work [39] contains a version of Theorem 3.3 for the so-called Φ-Laplacian. A more general problem patterned after

$$-\Delta_p u = \mu u^{-\gamma} \text{ in } \Omega, \ u > 0 \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega, \tag{3.4}$$

where μ denotes a nonnegative bounded Radon measure on Ω while $\gamma \ge 0$ is thoroughly studied in [40]; see also [41] and references therein.

Finally, as regards Problem (3.2) again, papers [42–45] do not require Ambrosetti-Rabinowitz's condition (a_5), whereas [46] establishes the existence of at least three weak solutions. Moreover, a possibly nonhomogeneous elliptic operator is considered in [44], but $\lambda = 1$.

The nice paper [47] investigates the problem

$$\begin{cases} -\Delta_p u = \lambda u^{-\gamma} + u^{q-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $0 < \gamma < 1$ and $1 < p < q < p^*$. It should be noted that, here, contrary to above, *the parameter λ multiplies the singular term*. Combining known variational methods with a $C^{1,\alpha}(\overline{\Omega})$-regularity result [47, Theorem 2.2] for solutions to (3.5) and a strong comparison principle [47, Theorem 2.3], the authors obtain the following.

Theorem 3.4. ([47], Theorem 2.1) *Suppose $0 < \gamma < 1$ and $1 < p < q < p^*$. Then, there is $\Lambda > 0$ such that* (3.5) *has*:

- *at least two ordered solutions in $C^1(\overline{\Omega})$ for every $\lambda \in]0, \Lambda[$,*
- *at least one solution in $C^1(\overline{\Omega})$ when $\lambda = \Lambda$, and*
- *no solutions once $\lambda > \Lambda$.*

The case $q = p^*$ is also studied, and it is shown that $\gamma < 1$ is a reasonable sufficient (and likely optimal) condition to obtain $C^1(\overline{\Omega})$-solutions of (3.5).

If $p = 2$ and, roughly speaking, $a \equiv -1$ while f does not depend on u, then Problem (3.2) was fruitfully studied in [48].

We end this section by pointing out two very recent works, namely [49], which deals with possibly nonmonotone singular reactions (see also [50,51], essentially based on sub-super-solution methods), and [31], which is devoted to singular equations driven by the (p, q)-Laplace operator $u \mapsto \Delta_p u + \Delta_q u$.

3.3 Uniqueness

Surprisingly enough, if $p \neq 2$, then the uniqueness of solutions looks a difficult matter, even for the model problem

$$\begin{cases} -\Delta_p u = a(x) u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

As observed in [38], this is mainly caused by the fact that, in general, solutions do not belong to $W_0^{1,p}(\Omega)$ once $\gamma \geq 1$. The paper [38] provides two different results. The first one (Theorem 1.4) holds in star-shaped domains, while the other is as follows.

Theorem 3.5. ([38], Theorem 1.5) *Assume that either $\gamma \leq 1$ and $a \in L^1(\Omega)$ or $\gamma > 1$ and*

- *$a \in L^m(\Omega)$ for some $m > \frac{N}{p}$ if $1 < p < N$,*
- *$a \in L^m(\Omega)$ with $m > 1$ when $p = N$, and*
- *$a \in L^1(\Omega)$ if $p > N$.*

Then (3.6) *possesses a unique distributional solution.*

We next point out that, for $\gamma \leq 1$, Theorem 3.4 of [40] establishes the uniqueness of *renormalized* solutions to (3.4).

The situation becomes quite clear when $p = 2$ and one seeks sufficiently regular solutions. Denote by φ_1 a positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$.

Theorem 3.6. ([17], Theorems 1 and 2) *Let* $p = 2$ *and let* $a \in C^{0,\alpha}(\overline{\Omega})$ *be positive. Then* (3.6) *has a unique solution* $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$. *Moreover,*

- *there exist* $c_1, c_2 > 0$ *such that* $c_1\varphi_1^{2/(1+\gamma)} \le u \le c_2\varphi_1^{2/(1+\gamma)}$ *in* $\overline{\Omega}$,
- $u \in H_0^1(\Omega) \iff \gamma < 3$, *and*
- $\gamma > 1 \implies u \notin C^1(\overline{\Omega})$.

See also the nice paper [52]. As regards weak solutions, one has the following:

Theorem 3.7. ([53], Theorem 3.1) *Suppose* $p = 2$ *and* $a \in L^1(\Omega)$, *then* (3.6) *admits at most one solution belonging to* $H_0^1(\Omega)$.

Another uniqueness case occurs when $\gamma > 1$.

Theorem 3.8. ([53], Theorem 1.3) *If* $p = 2$, $\gamma > 1$, *and* $a \in L^1(\Omega)$, *then* (3.6) *possesses at most one solution* $u \in H^1_{\mathrm{loc}}(\Omega)$ *such that* $u^{(\gamma+1)/2} \in H_0^1(\Omega)$.

3.4 Equations with convective terms

Consider the problem

$$\begin{cases} -\Delta_p u = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where $p < N$ while $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ and $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}_0^+$ satisfy Carathéodory's conditions. In 2019, Liu, Motreanu, and Zheng established the existence of solutions $u \in W_0^{1,p}(\Omega)$ to (3.7) under the following hypotheses, where λ_1 stands for the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$.

(h_1) There exist $c_0, c_1, c_2 > 0$ such that $c_1 + c_2\lambda_1^{1-1/p} < \lambda_1$ and

$$f(x, t, \xi) \le c_0 + c_1 t^{p-1} + c_2|\xi|^{p-1} \ \forall (x, t, \xi) \in \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N.$$

(h_2) $g(x, \cdot)$ is nonincreasing on $(0, 1]$ for all $x \in \Omega$ and $g(\cdot, 1) \not\equiv 0$.

(h_3) With appropriate $\theta \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$, $\hat{q} > \max\{N, p'\}$, and $\varepsilon_0 > 0$, the map $x \mapsto g(x, \varepsilon\theta(x))$ belongs to $L^{\hat{q}}(\Omega)$ for any $\varepsilon \in (0, \varepsilon_0)$.

Condition (h_3) was previously introduced by Faraci and Puglisi [54]. It represents a natural generalization of (a_1) in Section 3.2.

Theorem 3.9. ([55], Theorem 25) *Let* (h_1)–(h_3) *be satisfied. Then* (3.7) *has a solution* $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$.

We think it is worthwhile to sketch the main ideas of the proof. For every fixed $w \in C_0^1(\overline{\Omega})$, an intermediate problem, where ∇w replaces ∇u in $f(x, u, \nabla u)$ and the singular term remains unchanged, is considered. The authors construct a positive sub-solution $\underline{u} \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ independently of w and show the existence of a solution greater than $\underline{u}$. If $\mathcal{S}(w)$ denotes the set of such solutions, then, via suitable properties of the multi-function $w \mapsto \mathcal{S}(w)$, it is proved that the map Γ, which assigns to every w the minimal element of $\mathcal{S}(w)$, is completely continuous. Now, applying Leray-Schauder's alternative principle to Γ yields a solution $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ to (3.7).

The recent paper [24], partially patterned after [55], treats the Robin problem

$$\begin{cases} -\operatorname{div} A(\nabla u) = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_A} + \beta u^{p-1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ denotes a continuous strictly monotone map having suitable properties, which basically stem from Lieberman's nonlinear regularity theory [56] and Pucci-Serrin's maximum principle [57]. By the way, the conditions on A include classical nonhomogeneous operators such as the (p, q)-Laplacian. Moreover, β is a positive constant while $\frac{\partial}{\partial \nu_A}$ indicates the co-normal derivative associated with A. If $p = 2$, then a uniqueness result is also presented; cf. [24, Theorem 4.2].

The special case $A(\xi) := |\xi|^{p-2}\xi$, $g(x, t) := t^{-\gamma}$ for some $0 < \gamma < 1$, and $\beta = 0$ (which reduces (3.8) to a Neumann problem) has been investigated in [8] without imposing any global growth condition on $t \mapsto f(x, t, \xi)$. Instead, a kind of oscillatory behavior near zero is taken on. For such an f, the work [25] establishes the existence of a solution $u \in C^1_0(\overline{\Omega})$ to the parametric problem

$$\begin{cases} -\operatorname{div} A(\nabla u) = f(x, u, \nabla u) + \lambda u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

provided $\lambda > 0$ is small enough.

Finally, the very recent paper [58] treats Φ-Laplacian equations with strongly singular reactions perturbed by gradient terms.

4 Problems on the whole space

4.1 The case $p = 2$

Let $N \geq 3$, let $a : \mathbb{R}^N \to \mathbb{R}^+_0$ be nontrivial measurable, and let $\gamma > 0$. The simplest singular elliptic problem in the whole space is written as:

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

Sometimes it is also required that $u(x) \to 0$ as $|x| \to \infty$. Since the pioneering papers [59–62], some existence and uniqueness results concerning (4.1) have been published. We refer the reader to the monograph [4] for a deep account. Roughly speaking, four basic questions can be identified:

- Find the right hypotheses on a. Usually, $a \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)_+$ and

$$\int_1^\infty r \max_{|x|=r} a(x)\, dr < \infty$$

  (cf. condition (a_8) below) guarantee both existence and uniqueness of solutions $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.
- Replace $u^{-\gamma}$ with a function $f(u)$ such that $\lim_{t\to 0^+} f(t) = \infty$. This was done in [63,64] for decreasing f. Later on, nonmonotone singular reactions were also fruitfully treated [65–67].
- Put convective terms on the right-hand side. For equations driven by the Laplacian, a good reference is [4, section 9.8]; cf. in addition [68,69].
- Generalize the left-hand side of the equation. The case of a second-order uniformly elliptic operator is treated in [27,70], while [71] deals with $u \mapsto -\Delta u + c(x)u$, where $c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)_+$.

The equation of Problem (4.1) arises in the boundary-layer theory of viscous fluids [72–74] and is called *Lane-Emden-Fowler equation*. Its importance in scientific applications has by now been widely recognized; see, e.g., [75].

4.2 Existence and multiplicity

To the best of our knowledge, the first paper treating singular p-Laplacian equations on the whole space is that of Goncalves and Santos [76], published in 2004. The authors consider the problem

$$\begin{cases} -\Delta_p u = a(x)f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{4.2}$$

where $a \in C^0(\mathbb{R}^N)_+$ is radially symmetric while $f \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, and assume that:

(a_6) The function $t \mapsto \frac{f(t)}{t^{p-1}}$ is nonincreasing on $\mathbb{R}^+$.

(a_7) $\liminf_{t\to 0^+} f(t) > 0$ as well as $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0$.

(a_8) If $\Phi(r) := \max_{|x|=r} a(x)$, $r > 0$, then

$$0 < \int_1^\infty [r\Phi(r)]^{\frac{1}{p-1}} \mathrm{d}r < \infty \quad \text{for } 1 < p \le 2,$$

$$0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} \Phi(r)\, \mathrm{d}r < \infty \text{ for } p > 2.$$

Theorem 4.1. ([76], Theorem 1.1) *Under* (a_6)–(a_8), *Problem* (4.2) *admits*:

- *A radially symmetric solution* $u \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ *when* $p < N$.
- *No radially symmetric solution in* $C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ *if* $p \ge N$.

The proof exploits fixed point arguments, the shooting method, and sub-super-solution techniques.

One year later, Covei [77] did not assume a to be radially symmetric but locally Hölder continuous and positive, replaced conditions (a_6) and (a_7) with those below, and obtained similar results. See also [78], where the asymptotic behavior of solutions is described.

($a_{6'}$) The function $t \mapsto \frac{f(t)}{(t+\beta)^{p-1}}$ turns out decreasing on $\mathbb{R}^+$ for some $\beta > 0$.

($a_{7'}$) $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = \infty$ and $f(t) \le c$ for any t that is large enough.

The work [79] treats the parametric problem

$$\begin{cases} -\Delta_p u = a(x)u^{-\gamma} + \lambda b(x)u^{q-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.3}$$

where $1 < p < N$, $0 < \gamma < 1$, $\lambda > 0$, $\max\{p, 2\} < q < p^*$, and the coefficients fulfill

$$a \in L^{\frac{p^*}{p^*-(1-\gamma)}}(\mathbb{R}^N)_+, \;\; a \not\equiv 0, \;\; b \in L^{\frac{p^*}{p^*-q}}(\mathbb{R}^N), \;\; b > 0. \tag{4.4}$$

Theorem 4.2. ([79], Theorem 1.2) *If* (4.4) *holds, then there exists* $\Lambda > 0$ *such that* (4.3) *possesses*

- *at least two solutions in* $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *for every* $\lambda \in\,]0, \Lambda[$,
- *at least one solution belonging to* $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$ *when* $\lambda = \Lambda$, *and*
- *no solutions once* $\lambda > \Lambda$.

It may be of interest to point out that this result is proved by combining sub-super-solution methods with the mountain pass theorem for continuous functionals.

Remark 4.3. If $b \equiv 0$, then Problem (4.3) reduces to a well-known one, that is very important in scientific applications; cf. [80, Remark 2.2].

A meaningful case occurs when $a, b : \mathbb{R}^N \to \mathbb{R}_0^+$ turn out nonzero locally Hölder continuous functions. In fact, define

$$M(x) := \max\{a(x), b(x)\}, \quad x \in \mathbb{R}^N. \tag{4.5}$$

From [81, Remarks 1–2] it follows:

Lemma 4.4. *Suppose that $p < N$, the functions $a, b : \mathbb{R}^N \to \mathbb{R}_0^+$ are nontrivial and locally Hölder continuous, while* (a_8) *holds with M in place of a. Then the problem*

$$\begin{cases} -\Delta_p w = M(x) & \text{in } \mathbb{R}^N, \\ w > 0 & \text{in } \mathbb{R}^N, \\ w(x) \to 0 & \text{as } |x| \to \infty \end{cases} \tag{4.6}$$

admits a solution $w_M \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for suitable $\alpha \in\,]0, 1[$.

Via sub-super-solution techniques, Lemma 4.4 gives rise to the following:

Theorem 4.5. ([81], Theorem 1.1) *Let $\gamma > 0$, let $p < q$, and let M be given by* (4.5). *Under the assumptions of* Lemma 4.4, *there exists $\lambda^* > 0$ such that* (4.3) *has*:
- *At least one solution $u \in C^1(\mathbb{R}^N)$ for every $0 \le \lambda < \lambda^*$. Moreover, $u(x) \to 0$ as $|x| \to \infty$.*
- *No solution once $\lambda > \lambda^*$.*

This result was next generalized under various aspects by the same author and Rezende [82]; cf. also [80].

Finally, infinite semi-positone problems, i.e., $\lim_{t\to 0^+} f(t) = -\infty$, were fruitfully investigated in [83]. Precisely, given $a \in L^\infty(\mathbb{R}^N)$ and $f \in C^0(\mathbb{R}^+)$, consider the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{4.7}$$

where $\lambda > 0$ and $1 < p < N$. The following conditions will be posited.

(a_9) There exists $\gamma \in\,]0, 1[$ such that $\lim_{t\to 0^+} t^\gamma f(t) = c_0 \in \mathbb{R}^-$.

(a_{10}) $\lim_{t\to\infty} f(t) = \infty$ but $\lim_{t\to\infty} \frac{f(t)}{t^{p-1}} = 0$.

(a_{11}) $\inf_{|x|=r} a(x) > 0$ for all $r > 0$ and $0 < a(x) < \frac{C_0}{|x|^\sigma}$ in $\mathbb{R}^N \setminus \{0\}$ with suitable $C_0 > 0$, $\sigma > N + \gamma \frac{N-p}{p-1}$.

Theorem 4.6. ([83], Theorem 1.4) *If* (a_9)–(a_{11}) *hold and λ is sufficiently large, then* (4.7) *has a solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.*

4.3 Uniqueness

As far as we know, uniqueness has been addressed only in [76, Remark 1.2] and [77, Section 2] under the key assumption (a_6') above. The arguments of both papers rely on a famous result by Diaz and Saa [84]. Theorem 1.3 of [85] contains a nice idea to achieve uniqueness for singular problems in exterior domains.

4.4 Equations with convective terms

To the best of our knowledge, there is only one paper concerning singular quasi-linear elliptic equations in the whole space and with convective terms, namely [86]. It treats the problem

$$\begin{cases} -\mathrm{div}\, A(\nabla u) = f(x,u) + g(x,\nabla u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.8}$$

where $N \geq 2$ and $1 < p < N$. The differential operator $u \mapsto \mathrm{div}\, A(\nabla u)$ is as in (3.8), while $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}_0^+$ and $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_0^+$ fulfill Carathéodory's conditions. Moreover,

$$\begin{aligned} &\liminf_{t\to 0^+} f(x,t) > 0 \text{ uniformly with respect to } x \in B_\sigma(x_0), \\ &f(x,t) \leq h(x)t^{-\gamma} \text{ in } \mathbb{R}^N \times \mathbb{R}^+, \text{ where } h \in L^1(\mathbb{R}^N) \cap L^\eta(\mathbb{R}^N), \end{aligned} \tag{4.9}$$

and

$$g(x,\xi) \leq k(x)|\xi|^r \text{ in } \mathbb{R}^N \times \mathbb{R}^N, \text{ with } k \in L^1(\mathbb{R}^N) \cap L^\theta(\mathbb{R}^N). \tag{4.10}$$

Here, $x_0 \in \mathbb{R}^N$, $\sigma \in]0,1[$, $\gamma \geq 1$, $r \in [0, p-1[$, as well as

$$\eta > (p^*)', \quad \text{and} \quad \theta > \left(\frac{1}{(p^*)'} - \frac{r}{p}\right)^{-1}. \tag{4.11}$$

Theorem 4.7. ([86], Theorem 1.2) *Under* (4.9)–(4.11), *there exists a distributional solution* $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ *to* (4.8) *such that* $\operatorname{ess\,inf}_K u > 0$ *for every compact set* $K \subseteq \mathbb{R}^N$.

To prove this result, the authors first use sub-super-solution techniques to solve some auxiliary problems obtained by shifting the singular term and working in balls. A compactness result, jointly with a fine local energy estimate on super-level sets of solutions, then yields the conclusion.

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