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# Forward Action to Stabilize multiple Time-Delays MIMO Systems

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**Abstract**
In this paper, the problem of closed-loop stability of linear time-invariant systems with multiple time-delays is studied. We prove that stability of closed-loop configurations of such systems with unitary feedback can be guaranteed by means of a forward control action. The case of single-input single-output (SISO) systems is first considered, and then, an extension to the more general case of multi-input multi-output (MIMO) systems with multiple time-delays is dealt with. Numerical examples illustrating the theoretical results are also discussed.

**Keywords** Linear systems · Time-delay · Closed-loop stability · Positive-real systems

## 1 Introduction

Time-delay systems are of great interest in several areas of engineering, including man–machine interfaces [1], communication [2] and thermodynamic processes [3], and, for this reason, constitute a topic investigated in many works [4, 5]. More specifically, modeling and control of time-delay systems is an active research field, with particularly interesting problems arising when closed-loop configurations involving time-delay systems are considered [6]. In these configurations, the time-delay plays a fundamental role, for what concerns both the robustness of the closed-loop scheme and the system performance in general [7, 8]. Notably, stability in time-delay systems can be either delay-dependent, when it occurs for specific values of the delay, or delay-independent, when it can be achieved for any value of the delay. Various problems regarding time-delay systems are still open [9–11]. Recent studies include investigations of the more general cases of time-varying delays [12] and nonlinear time-delay systems [8] that also find applications in modeling human-robot interactions [13]. Likewise, the effect of delays in the stability of multi-agent systems is of great importance and subject of intense research [14–17].

In this paper, we show that it is possible to define a control action that ensures the delay-independent stability of time-delay systems in a closed-loop feedback scheme with unitary feedback. The control is based on a forward action, which is commonly used in combination with feedback for the rejection of specific, frequently occurring disturbances [18], but has been shown to be useful also to guarantee in time-delay systems some properties such as positive-realness or negative-imaginaryness [19]. Here, we show that it also represents a viable strategy to achieve the delay-independent stability of linear time-invariant systems with multiple delays.

The rest of the paper is organized as follows: in Sect. 2, the mathematical preliminaries regarding in particular time-delay systems and forward control actions are reported. Our main results for the SISO and MIMO cases are discussed in Sect. 3. Numerical examples are reported in Sect. 4. In Sect. 5, the conclusions are drawn.

## 2 Preliminaries

In this section, some basic concepts and preliminary definitions are briefly presented.

Let us start introducing some notation. We indicate with $\mathcal{C}$ the set of complex numbers, with $\mathcal{C}_+$ the set $\{s \in \mathcal{C} \mid \Re(s) > 0\}$ and with $\bar{\mathcal{C}}_+$ its closure, i.e., $\bar{\mathcal{C}}_+ = \{s \in \bar{\mathcal{C}} \mid \Re(s) \geq 0\}$.

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In addition, let $A \in \mathcal{C}^{m\times m}$, then $A^\dagger$ indicates its Hermitian transpose.

In the following, we will consider linear time-invariant (LTI) time-delay SISO systems described by the transfer function

$$\tilde{G}(s) = G(s)e^{-sT} \tag{1}$$

where $T > 0$ represents the time-delay, and LTI MIMO systems with multiple time-delays described by the following transfer function matrix

$$\tilde{\mathrm{G}}(s) = \mathrm{G}(s) \circ \mathrm{D}(s) \tag{2}$$

that is given by the Hadamard product of a transfer function matrix $\mathrm{G}(s) : \mathcal{C} \to \mathcal{C}^{m\times m}$ and a matrix $\mathrm{D}(s)$ with elements $\mathrm{D}_{ij}(s) = e^{-sT_{i,j}}$ with $T_{i,j} \geq 0$ representing the time-delays of the system [18].

For these systems, we will consider a closed-loop configuration having unitary feedback and including a forward action. To the aim of studying the stability of this configuration, we will use a few definitions and criteria that are here stated in general terms, referring to systems having a characteristic quasipolynomial $a(s, e^{-sT})$ in the case of a single time-delay $T$, or $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ in the case of multiple time-delays $T_h$, $h = 1, \ldots, q$.

We begin with the notion of asymptotic stability [4].

**Definition 1** The characteristic quasipolynomial $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ is said to be asymptotically stable if

$$a(s, e^{-sT_1}, \ldots, e^{-sT_q}) \neq 0 \ \forall s \in \bar{\mathcal{C}}_+ \tag{3}$$

It is said to be asymptotically stable independently of the time-delay if (3) is valid for all $T_h \geq 0$. The system with characteristic quasipolynomial $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ is asymptotically stable if and only if its characteristic quasipolynomial is asymptotically stable. In addition, it is asymptotically stable independently of the time-delay if its characteristic quasipolynomial is such.

Consider now the case of a system with a single time-delay, having characteristic polynomial $a(s, e^{-sT})$, and assume that the system is asymptotically stable in absence of delay, i.e., for $T = 0$. Then, the delay margin $\bar{T}$ can be defined as follows:

$$\bar{T} = \min\{T \geq 0 \mid a(j\omega, e^{-j\omega T}) = 0 \text{ for some } \omega \in \mathbb{R}\} \tag{4}$$

The system is asymptotically stable independently of the delay if $\bar{T} = \infty$. If, on the contrary, $\bar{T}$ is finite, then the system is asymptotically stable for any delay $T \in [0, \bar{T})$.

The following lemma provides a criterion for stability (known as the direct method) for a system with characteristic quasipolynomial $a(s, e^{-sT})$ [4]. For convenience, we introduce the variable $z = e^{-sT}$ and rewrite the characteristic quasipolynomial as $a(s, z)$.

**Lemma 1** *Consider the LTI time-delay system with characteristic quasipolynomial $a(s, z)$ and the following set of the two polynomial equations*

$$\begin{cases} a(s, z) = 0 \\ a(-s, z^{-1}) = 0 \end{cases} \tag{5}$$

*If the quasipolynomial $a(s, z)$ is asymptotically stable for $T = 0$ and no solution $s = j\omega$ of (5) exists, then the system is delay-independent asymptotically stable.*

Let us now consider the more general case of systems with multiple time-delays and rewrite the characteristic quasipolynomial $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ as follows:

$$\begin{aligned} a(s, e^{-sT_1}, \ldots, e^{-sT_q}) &= \\ &= a_0(s) + a_1(s)e^{-T_1 s} + \ldots + a_q(s)e^{-T_q s} \\ &= \sum_{h=0}^{q} a_h(s)e^{-T_h s} \end{aligned} \tag{6}$$

where $T_0 = 0$. The following lemma [4] provides a result for the stability of a system with this characteristic quasipolynomial that is independent of the time-delays $T_h \geq 0$.

**Lemma 2** *Given an LTI system with characteristic quasipolynomial $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ as in (6), then the roots of $a(s, e^{-sT_1}, \ldots, e^{-sT_q})$ are in the open left half plane $\forall T_h \geq 0$ if the following conditions hold:*

1. *$a_0(s)$ has no roots in the closed right half plane $\bar{\mathcal{C}}_+$;*
2. *$\sum_{h=0}^{q} a_h(s)$ has no roots in the closed right half plane $\bar{\mathcal{C}}_+$;*
3. *$\sum_{h=1}^{q} \frac{|a_h(j\omega)|}{|a_0(j\omega)|} < 1, \ \forall \omega > 0$.*

We now recall the definition of positive-real transfer matrices and the hyperstability theorem that will be useful in the following.

**Definition 2** [20] Let $\tilde{\mathrm{G}} : \mathcal{C} \to \mathcal{C}^{m\times m}$ be a transfer matrix. $\tilde{\mathrm{G}}$ is positive-real if:

- all elements of $\tilde{\mathrm{G}}(s)$ are analytic in $\Re(s) > 0$;
- $\tilde{\mathrm{G}}(s)$ is real for real positive $s$;
- the Hermitian matrix $\mathrm{M}_P = \tilde{\mathrm{G}}(s) + \tilde{\mathrm{G}}^\dagger(s)$ is positive semi-definite for $\Re(s) > 0$.

***Remark 1*** If $\tilde{\mathrm{G}}(s)$ is a symmetric matrix, then $\mathrm{M}_P = \tilde{\mathrm{G}}(s) + \tilde{\mathrm{G}}^\dagger(s) = 2\Re\left(\tilde{\mathrm{G}}(s)\right)$.

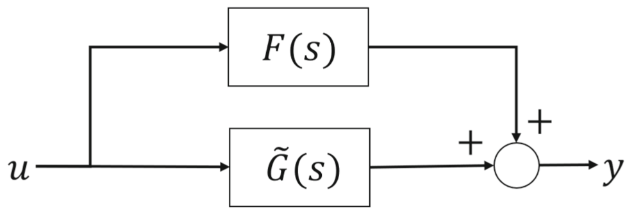


**Fig. 1** Scheme of the forward action F($s$) applied to a system with transfer function matrix $\tilde{\mathrm{G}}(s)$

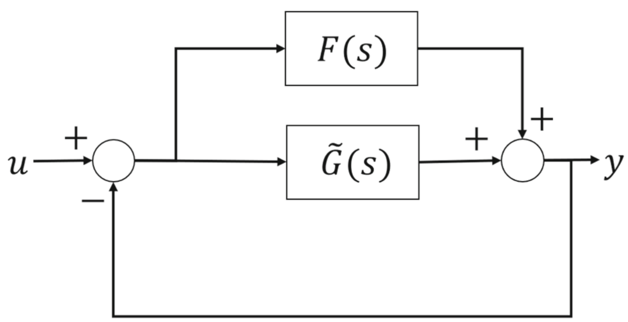


**Fig. 2** Closed-loop configuration with unitary feedback and forward action F($s$) applied to system $\tilde{\mathrm{G}}(s)$

***Remark 2*** If $\mathrm{M}_P$ is positive-definite, then the transfer matrix $\tilde{\mathrm{G}}$ is said to be strictly positive-real.

**Definition 3** [20] An LTI system is said to be (strictly) positive-real if its transfer matrix $\tilde{\mathrm{G}}$ is (strictly) positive-real.

Next, we recall the hyperstability theorem [20].

**Theorem 1** *Consider the feedback configuration of two positive-real systems. Then, the closed-loop system is asymptotically stable if at least one of the two systems is strictly positive-real.*

Finally, we introduce the configuration taken into account in our paper. With the term forward action, we indicate the system with transfer matrix F($s$) interconnected to the system with transfer matrix $\tilde{\mathrm{G}}(s)$ as shown in Fig. 1. In turn, this pair of systems working in parallel is considered connected into a feedback configuration with unitary feedback as shown in Fig. 2.

## 3 Forward action for closed-loop delay-independent stability

This section is divided into two parts. The first one is referred to SISO systems, whereas the second part to square MIMO systems. In both cases, the proofs of the main theorems are organized in two steps. The first step proves that a forward action able to assure delay-independent closed-loop stability exists. The second step is constructive and provides the value of the forward gain $k$ to achieve the delay-independent closed-loop stability.

Let us start considering the SISO case and introduce a proposition useful to prove the stability result.

**Proposition 1** *Consider a SISO time-delay system $\tilde{G}(s) = G(s)e^{-sT}$ with $G(s)$ being asymptotically stable. Then, this system can be made positive-real with a forward action $F(s) = k$, that is, there exists a gain $k > 0$ such that $\bar{G}(s) = \tilde{G}(s) + k$ is positive-real, with the value of $k$ not depending on $T$.*

***Proof*** Let us consider $s = \sigma + j\omega$ and calculate the real part of $\tilde{G}(s)$:

$$\Re(\tilde{G}(s)) = e^{-\sigma T} G_r(\sigma, \omega)\cos(\omega T) + e^{-\sigma T} G_i(\sigma, \omega)\sin(\omega T) \quad (7)$$

where $G_r(\sigma, \omega) = \Re(G(s))$ and $G_i(\sigma, \omega) = \Im(G(s))$.

Now, let $m_{1r} = \min_{\sigma,\omega}(G_r(\sigma, \omega))$ and $m_{2r} = \max_{\sigma,\omega}(G_r(\sigma, \omega))$ and define $M_1 = \max(| m_{1r} |, | m_{2r} |)$. Analogously, let $m_{1i} = \min_{\sigma,\omega}(G_i(\sigma, \omega))$ and $m_{2i} = \max_{\sigma,\omega}(G_i(\sigma, \omega))$ and define $M_2 = \max(| m_{1i} |, | m_{2i} |)$. Then, it immediately follows that $\bar{G}(s) = \tilde{G}(s) + k$ is positive-real if $k \geq M_1 + M_2$. □

Note that the condition on $k$ in the Proposition 1 does not depend on $T$. Also note that the characteristic quasipolynomial for a SISO system with transfer function $\tilde{G}(s) = G(s)e^{-sT} = \frac{N(s)}{D(s)}e^{-sT}$ in a feedback configuration with unitary feedback is given by

$$a(s, e^{-sT}) = D(s) + N(s)e^{-sT} \quad (8)$$

We are now ready to illustrate our main result for SISO systems expressed by the following theorem. □

**Theorem 2** *Consider a SISO time-delay system $\tilde{G}(s) = G(s)e^{-sT}$ with $G(s) = \frac{N(s)}{D(s)}$ being minimum phase and asymptotically stable. In addition, suppose that the characteristic quasipolynomial $p(s) = D(s) + N(s)e^{-sT} = 0$ for $T = 0$ has only roots with negative real part. Then, there exists a forward action $F(s) = k$ such that $\bar{G}(s) = G(s)e^{-sT} + k$ in closed-loop configuration with unitary feedback is delay-independent asymptotically stable.*

***Proof*** First, we prove that with a forward action $F(s) = k$, it is possible to obtain a system $\tilde{G}(s) + F(s)$ that is closed-loop asymptotically stable (always assuming unitary feedback), and then, we illustrate a procedure in order to determine the values of $k$ that guarantee the stability.

To prove the closed-loop stability of $\tilde{G}(s) + F(s)$, we start noticing that, by applying the results of Proposition 1, we can select a forward action $F(s) = k$ such that the system $\tilde{G}(s) + F(s)$ is positive-real, independently from the value of the time-delay $T$. Now, we observe that the closed-loop configuration corresponds to that shown in Fig. 2, where in the direct chain we have a positive-real system, and in the indirect chain another positive-real system (the unitary gain). We can hence apply the hyperstability Theorem 1 to conclude that $\tilde{G}(s) + F(s)$ in closed-loop configuration with unitary feedback is asymptotically stable. This result is independent of the value of the time delay $T$. Also notice, as a confirm, that for the special case of $k \to \infty$, then the closed-loop poles are the roots of $D(s)$ that are all with negative real part.

From these considerations, we conclude that $k$ can be selected such that to obtain closed-loop asymptotic stability in a independent way from the delay $T$. Now, a procedure to find the minimum of such values of $k$ is proposed.

Let us consider the quasipolynomial $\bar{a}(s, e^{-sT})$ associated to system $\bar{G}(s) = \frac{N(s)}{D(s)} e^{-sT} + k$. This is given by

$$\begin{aligned} \bar{a}(s, e^{-sT}) &= N(s)e^{-sT} + kD(s) + D(s) \\ &= N(s)e^{-sT} + D(s)(k+1) \end{aligned} \tag{9}$$

Let us now apply Lemma 2. We have to consider the following two equations

$$\begin{aligned} & N(s)z + \alpha D(s) = 0 \\ & \frac{N(-s)}{z} + \alpha D(-s) = 0 \end{aligned} \tag{10}$$

with $\alpha = k + 1$. We can solve Eqs. (10) for the variable $z$ and obtain:

$$\alpha^2 D(s)D(-s) - N(s)N(-s) = 0 \tag{11}$$

Let us now define $P(s) = \alpha^2 D(s)D(-s) - N(s)N(-s)$. If the polynomial $P(s)$ is different from zero $\forall s = j\omega$, then the system is delay-independent stable. To check when this condition is true, we note that $P(s)$ is the characteristic polynomial of the following Hamiltonian matrix

$$\mathrm{H} = \begin{bmatrix} \mathrm{A} & \frac{\mathrm{BB}^T}{\alpha^2} \\ -\mathrm{CC}^T & -\mathrm{A}^T \end{bmatrix} \tag{12}$$

where A, B, and C are the state matrices of a minimal realization of $G(s) = \frac{N(s)}{D(s)}$. Since the minimum value of $\alpha$ for which $P(j\omega) = 0$ does not admit solution (or, equivalently, the characteristic polynomial of the Hamiltonian matrix H does not have solutions on the imaginary axis) corresponds to the $H_\infty$ norm of $G(s)$, $\alpha_{min}$ can be calculated as $\alpha_{min} = \|G(s)\|_\infty$. We conclude that the asymptotic stability of the closed-loop configuration (with unitary feedback) of the system $\bar{G}(s) = G(s)e^{-sT} + k$ holds, independently of the delay, $\forall k > k_c = \|G(s)\|_\infty - 1$. □

Let us now consider the MIMO case. Also for MIMO systems, we first prove a result showing how it is possible with a forward action to make positive-real a system with multiple time-delays and then we state our main result on the closed-loop stability. □

**Proposition 2** *Consider a $m \times m$ MIMO system with multiple time-delays having transfer function matrix $\tilde{\mathrm{G}}(s) = \mathrm{G}(s) \circ \mathrm{D}(s)$ given by the Hadamard product of an asymptotically stable transfer function matrix $\mathrm{G}(s)$ and a matrix $\mathrm{D}(s)$ with elements $\mathrm{D}_{ij}(s) = e^{-sT_{i,j}}$ with $T_{i,j} \geq 0$. Then, this system can be made positive-real $\forall T_{i,j}$ with a forward action $\mathrm{F}(s) = k\mathrm{I}_m$, that is, there exists a gain $k > 0$ such that $\bar{\mathrm{G}}(s) = \tilde{\mathrm{G}}(s) + k\mathrm{I}_m$ is positive-real.*

***Proof*** Let us consider the Hermitian matrix $\mathrm{M} = \tilde{\mathrm{G}}(s) + \tilde{\mathrm{G}}^\dagger(s)$ with $s = \sigma + j\omega$. The generic element of this matrix can be written as follows:

$$\begin{aligned} \mathrm{M}_{ij} &= \mathrm{G}_{ij}(\sigma + j\omega)e^{-\sigma T_{i,j}}e^{-j\omega T_{i,j}} + \mathrm{G}_{ij}(\sigma - j\omega)e^{-\sigma T_{i,j}}e^{j\omega T_{i,j}} \\ &= \mathrm{G}_{ij}(\sigma + j\omega)e^{-\sigma T_{i,j}}(\cos(\omega T_{i,j}) - j\sin(\omega T_{i,j})) + \\ &\quad \mathrm{G}_{ij}(\sigma + j\omega)e^{-\sigma T_{i,j}}(\cos(\omega T_{i,j}) + j\sin(\omega T_{i,j})) \end{aligned} \tag{13}$$

Since all the poles of $\mathrm{G}_{ij}(s)$ are strictly in the left half plane, each term $\mathrm{M}_{ij}(\sigma, \omega, T_{i,j})$ has both the real and the imaginary part bounded. In addition, the maximum of the upper bound and the minimum of the lower bound are found in correspondence of the value $T_{i,j} = 0$.

Let us now consider the matrix M and, in particular, the $m$ determinants of its leading principal sub-matrices [21]:

$$\begin{aligned} & D_1 = \mathrm{M}_{11} \\ & D_2 = \det \begin{pmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \\ \mathrm{M}_{21} & \mathrm{M}_{22} \end{pmatrix} \\ & \cdots \\ & D_m = \det \mathrm{M} \end{aligned} \tag{14}$$

Since, by definition, determinants are built by summing products of elements of the matrix, these are real quantities derived by operations on bounded functions. Consequently, there exists $m$ positive quantities, namely $K_1$, $K_2$,...,$K_m$ such that:

$$\begin{aligned} & D_1 + K_1 > 0 \\ & D_2 + K_2 > 0 \\ & \ldots \\ & \ldots \\ & D_m + K_m > 0 \end{aligned} \tag{15}$$

By the Sylvester criterion [21], we conclude that the Hermitian matrix $\bar{\mathrm{M}} = \mathrm{M} + K\mathrm{I}_m$ with $K > \max\{K_1, K_2, \ldots, K_m\}$ is positive definite. Hence, the system with transfer function matrix $\bar{\mathrm{G}}(s) = \tilde{\mathrm{G}}(s) + \mathrm{F}(s)$ can be made positive-real by applying a forward action $\mathrm{F}(s) = k\mathrm{I}_m$ with $k = \frac{K}{2} > \frac{\max\{k_1, k_2, \ldots, k_m\}}{2}$. □

Let us now consider the MIMO system with multiple time-delays and transfer function matrix given by $\tilde{\mathrm{G}}(s) = \mathrm{G}(s) \circ \mathrm{D}(s)$. Then, for this system let us consider the unitary feedback closed-loop configuration and calculate $\det(\mathrm{I}_m + \tilde{\mathrm{G}}(s)) = \frac{\tilde{p}(s)}{\tilde{q}(s)}$. The characteristic quasipolynomial of the closed-loop system will be thus given by $\tilde{p}(s)$ and will depend on $s$ and on the time-delays $T_{i,j}$.

We are now ready to illustrate our main result for MIMO systems with multiple time-delays, that is, the possibility of closed-loop stabilization of the system $\tilde{\mathrm{G}}(s)$ $\forall T_{i,j}$ after the application of a forward action of the type $\mathrm{F}(s) = k\mathrm{I}_m$. □

**Theorem 3** *Consider a $m \times m$ MIMO system with multiple time-delays, having transfer function matrix $\tilde{\mathrm{G}}(s) = \mathrm{G}(s) \circ \mathrm{D}(s)$ given by the Hadamard product of an asymptotically stable transfer function matrix $\mathrm{G}(s)$ and a matrix $\mathrm{D}(s)$ with elements $\mathrm{D}_{ij}(s) = e^{-sT_{i,j}}$ with $T_{i,j} \geq 0$. In addition, suppose that the characteristic quasipolynomial $\tilde{p}(s)$ for $T_{i,j} = 0$ $\forall i, j$ has only roots with negative real part. Then, there exists a forward action $\mathrm{F}(s) = k\mathrm{I}_m$ able to guarantee that the unitary feedback closed-loop configuration of system $\bar{\mathrm{G}}(s) = \tilde{\mathrm{G}}(s) + k\mathrm{I}_m$ is asymptotically stable, for any value of the delays $T_{i,j} \geq 0$.*

***Proof*** By applying the results of Proposition 2 and the hyperstability Theorem 1, we conclude that there exists $k$ such that the system $\bar{\mathrm{G}}(s) = \tilde{\mathrm{G}}(s) + k\mathrm{I}_m$ in closed-loop configuration with unitary feedback (Fig. 2) is stable, independently of the values of $T_{i,j}$. To find the values of $k$ such that this property holds, the following procedure may be adopted.

Let us consider the unitary feedback closed-loop configuration of system $\tilde{\mathrm{G}}(s)$ and calculate $\det(\mathrm{I}_m + \bar{\mathrm{G}}(s)) = \frac{p(s)}{q(s)}$. The closed-loop system is stable if and only if the roots of $p(s)$ have negative real part. Using the forward action $\mathrm{F}(s) = k\mathrm{I}_m$, we obtain:

$$\det(\mathrm{I}_m + \tilde{\mathrm{G}}(s) + k\mathrm{I}_m) = \det(\alpha\mathrm{I}_m + \tilde{\mathrm{G}}(s)) = \frac{\bar{a}_0(s) + \sum_{h=1}^{q} \bar{a}_h(\alpha, s)e^{-sT_h}}{q(s)} \tag{16}$$

where $\alpha = 1 + k$ and $T_h$ with $h = 1, \ldots, q$ are linear combinations of the time-delays $T_{i,j}$.

Taking into account Lemma 2, we have to prove that the following transfer function

$$L(s, \alpha) = \frac{\sum_{h=1}^{q} \bar{a}_h(s, \alpha)}{\bar{a}_0(s)} \tag{17}$$

is bounded real. Hence, if $L(s, \alpha)$ is bounded real, then all the roots of $p(s)$ are in the open left half plane.

Let us, therefore, build a minimal realization of $L(s, \alpha)$ and indicate with $\mathrm{A}_L(\alpha)$, $\mathrm{B}_L(\alpha)$, and $\mathrm{C}_L(\alpha)$ the state matrices of this realization. Then, we can find the minimum value of $\alpha$, and so of $k$, such that the system $\bar{\mathrm{G}}(s) = \tilde{\mathrm{G}}(s) + k\mathrm{I}_m$ is closed-loop stable $\forall T_{i,j}$ by considering the Hamiltonian matrix:

$$\mathrm{H}_L = \begin{bmatrix} \mathrm{A}_L(\alpha) & \mathrm{B}_L(\alpha)\mathrm{B}_L(\alpha)^T \\ -\mathrm{C}_L(\alpha)\mathrm{C}_L(\alpha)^T & -\mathrm{A}_L(\alpha)^T \end{bmatrix} \tag{18}$$

and checking the minimum value of $\alpha$ such that $\mathrm{H}_L$ does not have eigenvalues on the imaginary axis. □

## 4 Numerical examples

In this section, some examples illustrating the results of the paper for the SISO and MIMO case are reported.

### 4.1 Example 1

Let us consider the system with the following transfer function:

$$\tilde{G}(s) = \frac{2e^{-sT}}{s+1} \tag{19}$$

To apply Theorem 2, we have to consider a minimal realization of $G(s) = \frac{2}{s+1}$. We can select for instance: $A = -1, B = 1$, and $C = 2$. In this case, the Hamiltonian matrix H in Eq. (12) becomes: $\mathrm{H} = \begin{bmatrix} -1 & \frac{1}{\alpha^2} \\ -4 & 1 \end{bmatrix}$. The characteristic polynomial of this matrix is as follows:

$$\det(\lambda\mathrm{I}_2 - \mathrm{H}) = \det\begin{pmatrix} \lambda + 1 & -\frac{1}{\alpha^2} \\ 4 & \lambda - 1 \end{pmatrix} = \lambda^2 - 1 + \frac{4}{\alpha^2} \tag{20}$$

Solving $\det(\lambda\mathrm{I}_2 - \mathrm{H}) = 0$ for $\lambda = j\omega$, we find $\omega^2 = \frac{4}{\alpha^2} - 1$. Hence, this polynomial has imaginary roots if $\frac{4-\alpha^2}{\alpha^2} > 0$. Therefore, $\alpha_{min} = 2$ and $k_c = 1$, thus a forward action $F(s) = k$, with $k > 1$ ensures the closed-loop stability, independently of the actual value of the time-delay $T$. In Fig. 3, the temporal evolution of the impulse response of the closed-loop system is reported for $T = 10$ s selecting three values of $k$, namely $k = 0.85$, $k = 1$, and $k = 1.2$. The closed-loop stability is observed with a forward action $F(s) = k$ with $k \geq 1$. It is evident that the bound $k_c$ is conservative. In Fig. 4, the largest real part of the closed-loop poles obtained for different values of $k$ and $T$ is reported, considering a fourth-order Padé approximation for the time-delay block. As it can be observed, the theoretical bound $k_c = 1$ is reached when the time-delay $T$ increases.

### 4.2 Example 2

Let us consider the MIMO system with transfer function matrix $\tilde{\mathrm{G}}(s)$ defined as follows:

$$\tilde{\mathrm{G}}(s) = \begin{bmatrix} \frac{2e^{-T_{1,1}s}}{s+1} & \frac{e^{-T_{1,2}s}}{(s+1)(s+2)} \\ \frac{e^{-T_{1,2}s}}{(s+1)(s+2)} & \frac{2e^{-T_{2,2}s}}{s+2} \end{bmatrix} \tag{21}$$

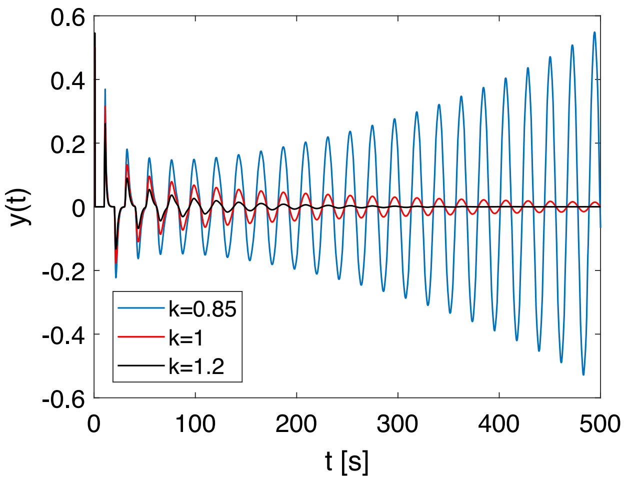


**Fig. 3** Temporal evolution of the impulse response of the closed-loop system in Example 1 considering a forward action $F(s) = k$ with $k = 0.85$ (blue line), $k = 1$ (red line), and $k = 1.2$ (black line)

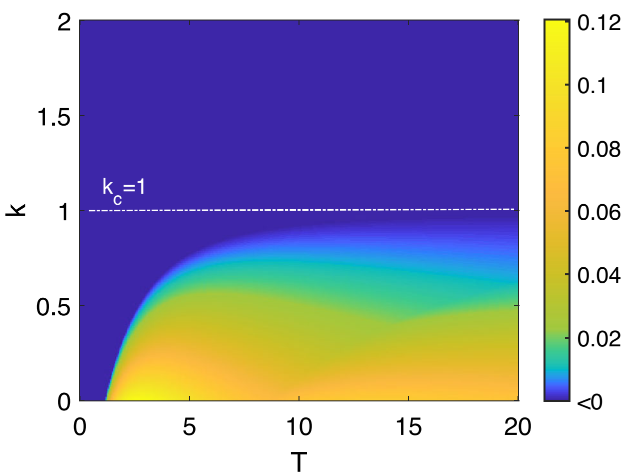


**Fig. 4** Largest real part of the closed-loop poles of the system in Example 1 considering a forward action $F(s) = k$ in the $k - T$ parameter space

This system is in the form $\tilde{\mathrm{G}}(s) = \mathrm{G}(s) \circ \mathrm{D}(s)$ with

$$\mathrm{G}(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{2}{s+2} \end{bmatrix}$$

and

$$\mathrm{D}(s) = \begin{bmatrix} e^{-T_{1,1}s} & e^{-T_{1,2}s} \\ e^{-T_{1,2}s} & e^{-T_{2,2}s} \end{bmatrix}$$

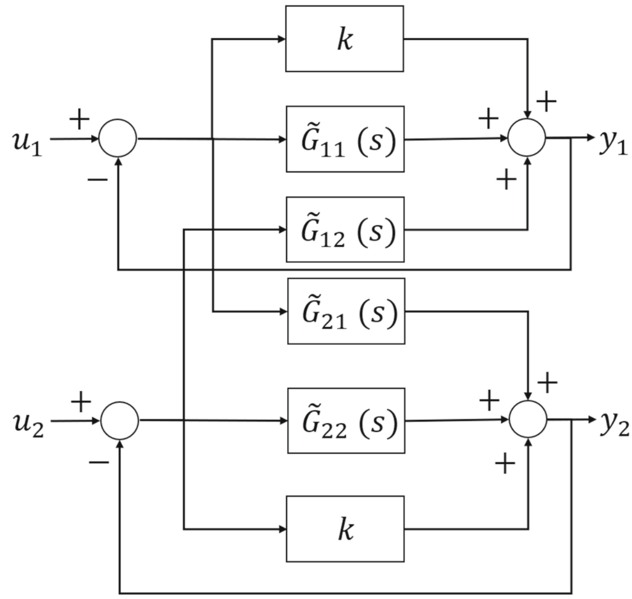


**Fig. 5** Closed-loop configuration with unitary feedback and forward action $\mathrm{F}(s)$ applied to system $\tilde{\mathrm{G}}(s)$ as in Example 2

To apply Theorem 3, the quasi-polynomial $p(s)$ has to be calculated from $\det(\mathrm{I}_2 + k\mathrm{I}_2 + \tilde{G}(s)) = \frac{p(s)}{q(s)}$, or equivalently, from $\det(\alpha \mathrm{I}_2 + \tilde{G}(s)) = \frac{p(s)}{q(s)}$. For this example, we get: $p(s) = p_0(s) + p_1(s)e^{-T_{1,1}s} + p_2(s)e^{-2T_{1,2}s} + p_3(s)e^{-T_{2,2}s} + p_4(s)e^{-(T_{1,1}+T_{2,2})s}$ with:

$$\begin{aligned} p_0(s) &= \alpha^2 s^4 + 6\alpha^2 s^3 + 13\alpha^2 s^2 + 12\alpha^2 s + 4\alpha^2 \\ p_1(s) &= 2\alpha s^3 + 10\alpha s^2 + 16\alpha s + 8\alpha \\ p_2(s) &= -1 \\ p_3(s) &= 2\alpha s^3 + 8\alpha s^2 + 10\alpha s + 4\alpha \\ p_4(s) &= 4s^2 + 12s + 8 \end{aligned}$$

From this, we calculate the transfer function $L(s, \alpha) = \sum_{i=1}^{4} \frac{p_i(s)}{p_0(s)}$, obtaining:

$$L(s, \alpha) = \frac{\frac{4s^3}{\alpha} + \frac{(18a+4)s^2}{\alpha^2} + \frac{(26\alpha+12)s}{\alpha^2} + \frac{12\alpha+7}{\alpha^2}}{s^4 + 6s^3 + 13s^2 + 12s + 4}$$

To find the values of $\alpha$ such that $L(s, \alpha)$ is bounded real, we consider the following realization of $L(s, \alpha)$:

$$\mathrm{A}_L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -12 & -13 & -6 \end{bmatrix}; \quad \mathrm{B}_L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\mathrm{C}_L = \begin{bmatrix} \frac{12\alpha+7}{\alpha^2} & \frac{26\alpha+12}{\alpha^2} & \frac{18\alpha+4}{\alpha^2} & \frac{4}{\alpha} \end{bmatrix}; \quad \mathrm{D}_L = 0$$

that yields:

$$
\mathrm{H}_L = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-4 & -12 & -13 & -6 & 0 & 0 & 0 & 1 \\
-\frac{(12\alpha+7)^2}{\alpha^4} & -\frac{(12\alpha+7)(26\alpha+12)}{\alpha^4} & -\frac{(12\alpha+7)(18\alpha+4)}{\alpha^4} & -\frac{4(12\alpha+7)}{\alpha^3} & 0 & 0 & 0 & 4 \\
-\frac{(12\alpha+7)(26\alpha+12)}{\alpha^4} & -\frac{(26\alpha+12)^2}{\alpha^4} & -\frac{(26\alpha+12)(18\alpha+4)}{\alpha^4} & -\frac{4(26\alpha+12)}{\alpha^3} & -1 & 0 & 0 & 12 \\
-\frac{(12\alpha+7)(18\alpha+4)}{\alpha^4} & -\frac{(26\alpha+12)(18\alpha+4)}{\alpha^4} & -\frac{(18*a+4)^2}{\alpha^4} & -\frac{4(18\alpha+4)}{\alpha^3} & 0 & -1 & 0 & 13 \\
-\frac{4(12\alpha+7)}{\alpha^3} & -\frac{4(26\alpha+12)}{\alpha^3} & -\frac{4(18\alpha+4)}{\alpha^3} & -\frac{16}{\alpha^2} & 0 & 0 & -1 & 6
\end{bmatrix}
$$

The value of $\alpha$ for which $\mathrm{H}_L$ has imaginary eigenvalues is $\alpha_{min} = 3.5$. Therefore, $k_c = \alpha_{min} - 1 = 2.5$. It follows that if $k > k_c = 2.5$, then the system $\bar{\mathrm{G}}(s) = k\mathrm{I}_2 + \tilde{\mathrm{G}}(s)$ (that results from the application to $\tilde{\mathrm{G}}(s)$ of the forward action $\mathrm{F}(s) = k\mathrm{I}_2$) is closed-loop delay-independent stable (Fig. 5).

The temporal evolution of the outputs of the closed-loop configuration with unitary feedback, with $T_{1,1} = 20$ s, $T_{1,2} = 40$ s, and $T_{2,2} = 50$ s, and a forward action $\mathrm{F}(s) = k\mathrm{I}_2$, when the inputs are two identical unit steps, are reported in Fig. 6 for $k = 1$ and for $k = 2.5$. In the case $k \geq k_c$, as predicted by Theorem 3, the closed-loop system is asymptotically stable.

Temporal evolution of the step responses of the closed-loop system in Example 2 considering a forward action $\mathrm{F}(s) = k\mathrm{I}_2$ with $k = 1$ (dash-dot lines), and $k = 2.5$ (continuous lines). Time-delays are fixed as $T_{1,1} = 20$ s, $T_{1,2} = 40$ s, and $T_{2,2} = 50$ s.

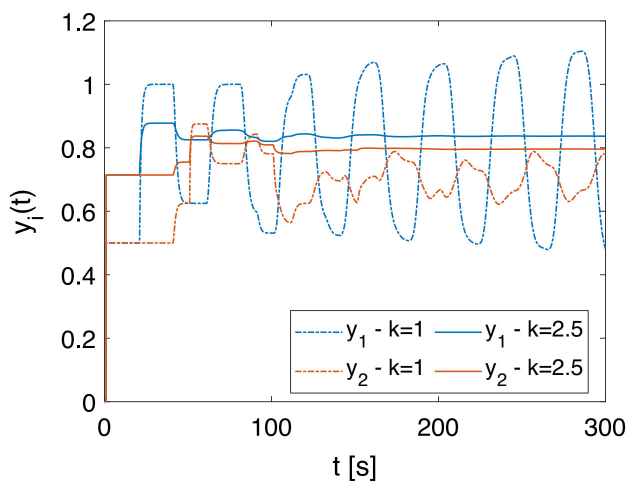


**Fig. 6** Temporal evolution of the step responses of the closed-loop system in Example 2 considering a forward action $\mathrm{F}(s) = k\mathrm{I}_2$ with $k = 1$ (dash-dot lines), and $k = 2.5$ (continuous lines). Time-delays are fixed as $T_{1,1} = 20$ s, $T_{1,2} = 40$ s, and $T_{2,2} = 50$ s

## 4.3 Example 3

Let us consider the not symmetric MIMO system with transfer function $\tilde{\mathrm{G}}(s)$:

$$
\tilde{\mathrm{G}}(s) = \begin{bmatrix} \frac{10e^{-T_1 s}}{s+1} & \frac{e^{-T_2 s}}{(s+2)^2} \\ \frac{4e^{-T_3 s}}{(s+4)} & \frac{10e^{-T_4 s}}{s+4} \end{bmatrix} \tag{22}
$$

This system is in the form $\tilde{\mathrm{G}}(s) = \mathrm{G(s)} \circ \mathrm{D(s)}$ with

$$
\mathrm{G(s)} = \begin{bmatrix} \frac{10}{s+1} & \frac{1}{(s+2)^2} \\ \frac{4}{(s+4)} & \frac{10}{s+4} \end{bmatrix}
$$

and

$$
\mathrm{D(s)} = \begin{bmatrix} e^{-T_1 s} & e^{-T_2 s} \\ e^{-T_3 s} & e^{-T_4 s} \end{bmatrix}
$$

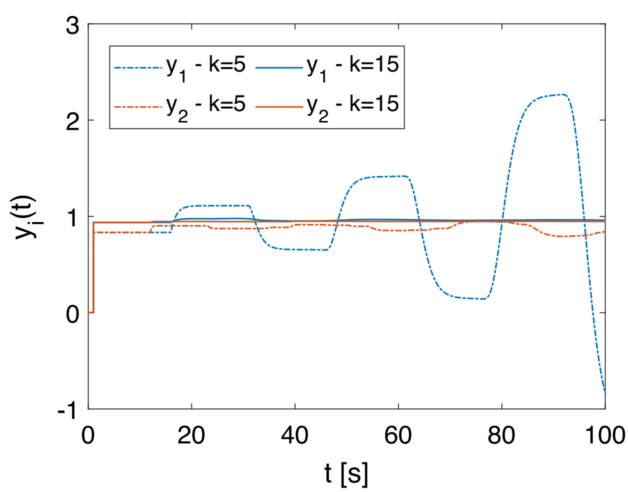


**Fig. 7** Temporal evolution of the step responses of the closed-loop system in Example 3 considering a forward action $\mathrm{F}(s) = k\mathrm{I}_2$ with $k = 5$ (dash-dot lines), and $k = 15$ (continuous lines). $T_{1,1} = 15$ s, $T_{1,2} = 30.5$ s, $T_{2,1} = 38.2$ s, and $T_{2,2} = 11$ s

To apply Theorem 3, we first calculate $p(s)$, obtaining $p(s) = p_0(s)+p_1(s)e^{-T_1 s}+p_2(s)e^{-(T_1+T_4)s}+p_3(s)e^{-T_4 s}+p_4(s)e^{-(T_2+T_3)s}$ with:

$$
\begin{aligned}
p_0(s) &= \alpha^2 s^4 + 9\alpha^2 s^3 + 28\alpha^2 s^2 + 36\alpha^2 s + 16\alpha^2 \\
p_1(s) &= 10\alpha s^3 + 80\alpha s^2 + 200\alpha s + 160\alpha \\
p_2(s) &= 100 s^2 + 400 s + 400 \\
p_3(s) &= 10\alpha s^3 + 50\alpha s^2 + 80\alpha s + 40\alpha \\
p_4(s) &= -4s - 4
\end{aligned}
$$

This yields the following $L(s, \alpha)$:

$$L(s,\alpha) = \frac{\frac{20s^3}{\alpha} + \frac{(100+130\alpha)s^2}{\alpha^2} + \frac{(396+280\alpha)s}{\alpha^2} + \frac{396+200\alpha}{\alpha^2}}{s^4 + 9s^3 + 28s^2 + 36s + 16} \tag{23}$$

A minimal realization of $L(s, \alpha)$ is given by:

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -16 & -36 & -28 & -9 \end{bmatrix}; \qquad \mathrm{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix};$$
$$\mathrm{C} = \begin{bmatrix} \frac{396+200\alpha}{\alpha^2} & \frac{396+280\alpha}{\alpha^2} & \frac{100+130\alpha}{\alpha^2} & \frac{20}{\alpha} \end{bmatrix}; \quad \mathrm{D}_L = 0$$

The corresponding Hamiltonian matrix $\mathrm{H}_L$ is as follows:

$$\mathrm{H}_L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -16 & -36 & -28 & -9 & 0 & 0 & 0 & 1 \\ -\frac{(200\alpha+396)^2}{\alpha^4} & -\frac{(200\alpha+396)(280\alpha+396)}{\alpha^4} & -\frac{(130\alpha+100)(200\alpha+396)}{\alpha^4} & -\frac{20(200\alpha+396)}{\alpha^3} & 0 & 0 & 0 & 16 \\ -\frac{(200\alpha+396)(280\alpha+396)}{\alpha^4} & -\frac{(280\alpha+396)^2}{\alpha^4} & -\frac{(130\alpha+100)(280\alpha+396)}{\alpha^4} & -\frac{20(280\alpha+396)}{\alpha^3} & -1 & 0 & 0 & 36 \\ -\frac{(130\alpha+100)(200\alpha+396)}{\alpha^4} & -\frac{(130\alpha+100)(280\alpha+396)}{\alpha^4} & -\frac{(130a+100)^2}{\alpha^4} & -\frac{20(130\alpha+100)}{\alpha^3} & 0 & -1 & 0 & 28 \\ -\frac{20(200\alpha+396)}{\alpha^3} & -\frac{20(280\alpha+396)}{\alpha^3} & -\frac{20(130\alpha+100)}{\alpha^3} & -\frac{400}{\alpha^2} & 0 & 0 & -1 & 9 \end{bmatrix}$$

The value of $\alpha$ for which $\mathrm{H}_L$ has imaginary eigenvalues is $\alpha_{min} = 14.235$. It follows that $k_c = \alpha_{min} - 1 = 13.235$. Therefore, for $k > k_c = 13.235$, the system $\bar{\mathrm{G}}(s) = k\mathrm{I}_2 + \tilde{\mathrm{G}}(s)$ is closed-loop delay-independent asymptotically stable.

The numerical simulations reported in Fig. 7 show the outputs of the closed-loop system when the inputs are two identical unit steps.The time-delays have been chosen as $T_{1,1} = 15$ s, $T_{1,2} = 30.5$ s, $T_{2,1} = 38.2$ s, and $T_{2,2} = 11$ s. The system is subjected to a forward action $\mathrm{F}(s) = k\mathrm{I}_2$ with $k = 5$ and $k = 15$, confirming the theoretical predictions that for $k > 13.235$ the closed-loop system is asymptotically stable.

## 5 Conclusion

In this paper, the problem of designing a forward action to be added to a time-delay system such that to obtain a closed-loop delay-independent stable system has been dealt with. In particular, the suitability of the approach has been proved for both SISO and (square) MIMO systems with multiple time-delays. The proof considered is constructive, therefore also providing a procedure to determine suitable values of the gain of the forward control action. Some numerical examples to illustrate the proposed strategy have been also reported.

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## Declarations

**Conflict of interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.



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