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Helmholtz, Schrödinger, and the First Non-Euclidean Model of Perceptual Color Space

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This paper explores the groundbreaking contributions of Hermann von Helmholtz and Erwin Schrödinger to the geometry of color space -a 3D space that correlates color distances with perceptual differences. Drawing upon his expertise in non-Euclidean geometry, physics, and psychophysics, Helmholtz introduced the first Riemannian line element in color space between 1891 and 1892, inaugurating a new line of research known as *higher color metric*, a term coined by Schrödinger in 1920. During his tenure at the University of Vienna, Schrödinger extensively worked on color theory and rediscovered Helmholtz's forgotten line element. In his 1920 papers titled "Grundlinien einer Theorie der Farbmetrik im Tagessehen," published in the *Annalen der Physik*, Schrödinger elucidated certain shortcomings in Helmholtz's model and proposed his refined version of the Riemannian line element. This study delves into this captivating chapter in the history of color science, emphasizing the profound impact of Helmholtz's and Schrödinger's work on subsequent research in color metrics up to the present day.

1. Introduction

A skilled experimenter and a profound expert in the latest developments of physiological optics, psychophysics, and non-Euclidean geometry, the polymath Hermann von Helmholtz leveraged his transdisciplinary knowledge to define the first Riemannian line element in color space between 1891 and 1892. This paper explores the path that led to the establishment of the field known as *higher color metric*, a term coined by one of the chief architects of quantum mechanics, Erwin Schrödinger. In his 1920 papers on color theory, Schrödinger rescued Helmholtz's work on color metrics from obscurity. Schrödinger noted, a computational error by Helmholtz, the discovery of which further weakens the connection to experience, is probably the reason why his intellectually interesting idea of a Riemann geometry of color has not been appreciated, as far as I know. The relevant sections have even been deleted from the latest edition of the Helmholtz Handbook of Physiological Optics.[1] On one hand, Schrödinger acknowledged Helmholtz's idea of a Riemannian nature of perceptual color space. On the other, he highlighted certain critical aspects that had eluded Helmholtz's attention in defining the line element. Schrödinger argued that the identification of these shortcomings by Helmholtz's pupils might have led to the omission of crucial excerpts related to the line element in the third edition of Helmholtz's "Handbuch der Physiologischen Optik".[2] Since Schrödinger's rediscovery of Helmholtz's 1892 paper, Helmholtz's treatment of the line element has formed the foundation for all subsequent studies on the geometry of color space, extending to the present day. This work builds upon the author's two more extensive prior contributions,[3] which can be consulted by readers interested in the subject for a more in-depth exploration.

2. Helmholtz's Color Theory: The Beginning

The foundation of modern color science was laid in the mid-19th century when James Clerk Maxwell and Hermann von Helmholtz provided the first comprehensive clarification of long-debated topics within the field. This involved distinguishing between additive and subtractive color mixing, elucidating the correct mechanism underlying human color vision, and determining the number of primary colors. They were well aware of the inherent connection between the science of color and subjective experience, which prompted them to perform experiments to obtain quantitative expressions for color mixing. During the same period, Hermann Günther Grassmann introduced the formal principles of affine geometry to address the characterization of perceptual color space, laying the basis for modern colorimetry. In doing so, these three scientists refined and redefined the color theories of Isaac Newton and Thomas Young, which served as the starting point for their groundbreaking research.

Helmholtz began systematically his research on color in 1849 when he was appointed professor of physiology in Königsberg (now Kaliningrad). Years before, from 1838 to 1842, as a student at the Royal Friedrich-Wilhelm Institute of Medicine and Surgery in Berlin, he had encountered Johannes Peter Müller, who was at that time his professor of physiology. Inspired by the work of Müller, Helmholtz started to find connections between physiology and physics, an aim that he pursued throughout

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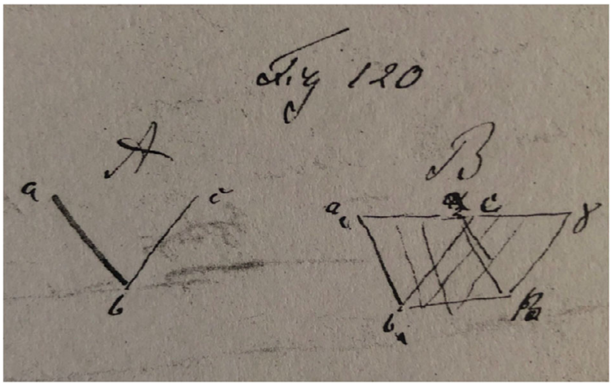

Figure 1. Helmholtz's original sketch of the V-shaped slit experiment.[8] In (A), the V-shaped slit is depicted with its two arms, ab and bc, inclined at a 45° angle on both sides of the vertical. Image (B) illustrates the partial superimposition of the two spectra: a, b, α_2 β represents the spectrum from the slit ab, while c b,γ β refers to the spectrum from bc. Within the central triangular field shared by both spectra, all color bands of one spectrum intersect with those of the other. Reproduced with permission. Copyright Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften, NL Helmholtz Nr. 573, p. 162.

his life and constituted the leitmotiv in his investigation in the field of color. Drawing upon his expertise in physical optics, Helmholtz designed refined instruments for conducting novel observations that significantly contributed to the advancement of color science. The first two papers he published on color appeared in the *Annalen der Physik* in 1852, under the titles "Über Hrn. D. Brewster's neue Analyse des Sonnenlichts" ("On Sir David Brewster's New Analysis of Solar Light")[4] and "Über die Theorie der zusammengesetzten Farben" ("On the Theory of Compound Colors").[5] In the former paper, Helmholtz expressed his critique of the color theory proposed by another prominent natural philosopher, David Brewster. Similar to Goethe, Brewster postulated the existence of three primary colors: red, yellow, and blue.[6] Helmholtz replicated Brewster's experiments and demonstrated that the Scottish physicist had succumbed to several subjective illusions regarding color. Moreover, he clarified that the confusion surrounding color mixing, and thus Brewster's theory, arose from considering two distinct processes as equivalent: additive and subtractive color mixing. In his second contribution, Helmholtz provided a coherent and comprehensive explanation of the differentiation between these two phenomena. According to him, these processes belong to separate domains: pigment mixing pertains to the physical domain, whereas light composition falls within the physiological domain (a conclusion shared by Maxwell). In this paper, Helmholtz also offered a detailed account of an experimental device he designed to observe binary combinations of spectral colors at various intensities. He allowed sunlight to pass through a V-shaped slit inserted in a dark screen before falling into a flint prism borrowed from his colleague Franz Ernst Neumann in Königsberg. This uniquely shaped aperture produced two partially overlapping spectra, as depicted in **Figure 1**. By observing the spectra produced through the slit, Helmholtz could achieve all the combinations resulting from the optical mixing of two spectral colors. The intensity of the two colors could be adjusted by altering the prism's angle from its original position, making it more or less oblique. A telescope, positioned at an appropriate distance from the prism, was used to examine these combinations. Upon individually inspecting each pair of superimposed spectral colors, Helmholtz identified only one pair of complementary colors, namely, yellow and indigo-blue. This finding contradicted Newton's color mixing theory, which envisaged the necessity of an infinite number of pairs of complementary colors. Furthermore, when analyzing the results, Helmholtz discovered that by mixing red and green, as well as green and violet lights, he could not obtain colors as saturated as spectral yellow and blue, respectively. Consequently, he proposed that a minimum of five primary colors (red, yellow, green, blue, and violet) was necessary to produce all the spectral hues. However, this evidence was in opposition to Young's trichromatic theory, which postulated the existence of three types of photoreceptors in the human retina, each sensitive to one of the three primary colors: red, violet, and green. Within a few years, Helmholtz reversed his position on Young's theory, which assumed a central role in all his subsequent investigations. He officially embraced Young's trichromatic theory after Maxwell's contributions between 1855 and 1860.[7]

During the summer of 1854, Helmholtz submitted a third essay on color to the physicist and editor of *Annalen der Physik und Chemie*, Johann Christian Poggendorff, entitled "Über die Zusammensetzung von Spectralfarben" ("On the Composition of Spectral Colors").[9] Here, he proposed a new method that allowed him to simultaneously cover only a small area with two spectral colors, enabling the study of their mixture. The V-shaped slit experiment posed challenges in achieving accurate judgments of the mixtures, particularly the paler ones. This difficulty arose because the individual colors occupied a considerable area, even when observed through a telescope. Furthermore, numerous other brilliant colors appeared in the field of view due to contrast effects. To overcome these limitations, Helmholtz designed the apparatus sketched in **Figure 2**.

He employed a refined color mixer with two adjustable slits, manipulable both in position (for selecting specific wavelength ranges within the spectrum) and in width (for controlling intensities). Helmholtz identified seven pairs of complementary colors, which mixed together produced white, and measured the intensity, i.e., the slit width, of the light for each complementary color.

His aim was to obtain a 2D representation of the space of perceived colors. In 1853, Grassmann[10] demonstrated that human color vision depends on three variables: *hue* (wavelength), *shade*, and *tint*. Shortly afterward, Maxwell confirmed the three-dimensional nature of color space, proposing *red*, *green*, and *blue* as independent variables, in line with Young's suggestions. This marked the birth of the concept of color space as a three-dimensional structure describing colors. Within the color space, each color can be specified using three coordinates or parameters that define its position. The two-dimensional representation of the color space, with *hue* and *tint* as the two independent variables while maintaining constant brightness, is commonly kwon as a *color diagram*.[11] **Figure 3**, illustrating Helmholtz's 2D color space (or color diagram), reveals that, during the mixing of colored lights, distinct amounts (i.e., intensities) of the two colors were necessary to produce white for each combination of complementary colors.[12] Helmholtz thus realized that the shape of the color diagram was neither a circle, as proposed by Newton in his "Opticks",[13] nor an equilateral triangle, as

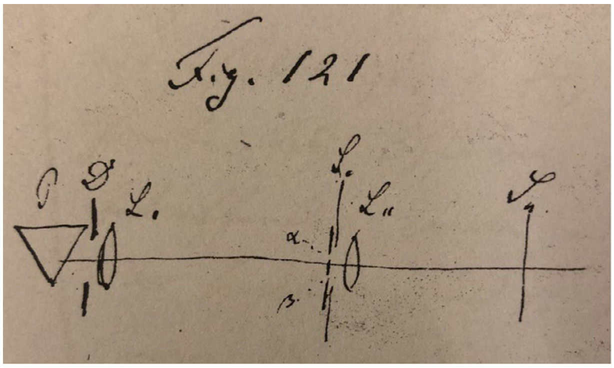

Figure 2. Original drawing of Helmholtz's color mixer. A heliostat allowed sunlight to be reflected through a vertical slit into a dark room. The beam of light passed through a prism and a lens, denoted by P and L_I, respectively, with a rectangular diaphragm, D, positioned between them. In the focal plane of the lens L_I, Helmholtz placed a screen S_I, which contained two vertical slits adjustable both in position and in width. Any two beams of spectral colors, indicated by α and β, could thus be combined by another achromatic lens, L_{II}, projecting an image on a second screen, S_{II}. The area on the second screen appeared colored by a uniform mixture of the two lights. Reproduced with permission. Copyright Archiv der Berlin-Brandenburgischen Akademie der Wissenschaften, NL Helmholtz Nr. 573, p. 163.

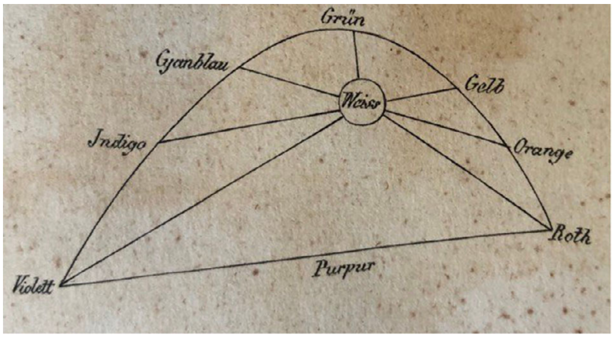

Figure 3. Helmholtz's experimentally derived color diagram illustrating color mixing at constant brightness.[15]

envisaged by Young,[14] but rather a distinctly asymmetric curve: a truncated hyperbola. Observing that equal distances did not correspond to equal perceptual distances in his color diagram, Helmholtz stated that the color space could not be uniform. This was the first experimental evidence that prompted him to explore non-Euclidean geometry for defining a line element in color space.

3. Helmholtz's Line Element

Precisely in physical optics, two examples were available to me of other manifolds which can be portrayed spatially and are variable in several respects. Namely, the color system, which Riemann also cites, and the measuring out of the visual field by visual estimation. Both show certain fundamental differences from the metrical system of geometry, and stimulated comparison.[16]

In 1868, Helmholtz published his memoir titled "Über die Tatsachen, die der Geometrie zugrunde liegen" ("On the Facts Underlying Geometry"),[17] delving into the foundation of geometry – a topic highly debated during that period. In this context, Helmholtz identified the color space as one of the few candidates capable of providing empirical evidence for the existence of geometries beyond the Euclidean framework, something already discussed by the eminent nineteenth-century mathematician Bernhard Riemann. Riemann, indeed, introduced the idea of multidimensional manifolds in his seminal 1854 lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" ("On the Hypotheses that Lie at the Basis of Geometry"). In this work, he pinpointed the space of color as the only example of a continuous manifold with several dimensions. Riemann's lecture was only published posthumously by Richard Dedekind in 1867, and it appeared in the 13th volume of the *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*.[18] It is noteworthy to mention how Helmholtz became aware of Riemann's work. Only a few months before the publication of his own memoir, Helmholtz corresponded with Ernst Schering, the author of "Bernhard Riemann Gedächtnis" in 1867.[19] On April 21, 1868, Helmholtz wrote to Schering: In thanking you for sending me the two little notes about Riemann, there is one question I should like to ask. In your notice of his life, I find it stated that he gave a *Habilitationsvorlesung* on the Hypotheses of Geometry. I have myself been occupied with this subject for the last two years in connection with my work in physiological optics but have not yet completed or published the work because I hoped to make certain points more general. [...] I venture to ask you to let me know if Riemann's essay is already in print, or if there is any prospect of its being published shortly, as it seems to me most desirable; in the event of Riemann having taken the same point of departure, my own work would become useless, and I need not go on expending as much time and headache as it has already cost me.[20] Shortly after, Helmholtz acknowledged Schering for sending a copy of Riemann's work: I am much obliged for the copy of Riemann's *Habilitationsschrift*. Herewith I send you a short account of the part of my own studies of this subject which is not covered by Riemann's work, begging you to lay it before the Royal Society to be published in the Gottinger Anzeigen (Proceedings of the Society).[21] The request was promptly accepted. Helmholtz's "On the Facts" was published in June 1868.

While Riemann's insight remained a conjecture, Helmholtz took a step further by integrating his knowledge in non-Euclidean geometry and experimental psychology to derive the line element for the space of perceived colors. As the foundation for his line-element derivation, Helmholtz considered the Weber-Fechner law from the burgeoning field of psychophysics. The Weber-Fechner law was presented in 1860 by the German physicist and experimental psychologist Gustav Theodor Fechner,[22] who extended the work carried out by his mentor and collaborator, the German physiologist and anatomist Ernst Heinrich Weber. Fechner conducted experiments in which subjects were asked to compare the intensity of sensations caused by two physical stimuli. One stimulus was kept constant, while one parameter of the second stimulus was systematically varied. Fechner investigated various physical stimuli, such as light intensity, weight, and temperature. Through these experiments, Fechner identified the smallest amount of physical difference between stimuli that

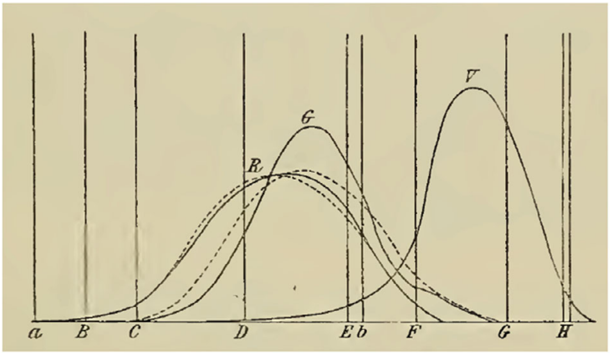

Figure 4. König' and Dieterici's elementary curves for individuals with trichromatic vision (normal eye: R, G, and V; eye affected by deficiencies: dotted curves, and V). The abscissa displays Fraunhofer lines as reference points.[27]

produced a *just noticeable difference* in sensations. This relationship is given by the following expression:

$$dp = k \cdot \frac{dS}{S} \tag{1}$$

where dp represents the perceived difference in sensation intensity, dS indicates the variation in stimulus intensity, S refers to the intensity of the physical stimulus, and k denotes a constant. Through integration, the following expression for sensation intensity, p, can be obtained:

$$p = kln\left(\frac{S}{S_0}\right) \tag{2}$$

where S_0 stands for the minimum value of the perceived stimulus.

Thanks to the work conducted by his collaborators in Berlin, Arthur König, Conrad Dieterici, and Eugen Brodhun, Helmholtz succeeded in extending the one-dimensional Weber-Fechner law to the three-dimensional color space. Drawing on the experimental findings collected and analyzed by his collaborators from 1883 onward, Helmholtz presented his line element in his 1892 paper "Kürzeste Linien im Farbensystem" ("The Shortest Lines in the Color System"),[23] where it assumed the following form:

$$dE^2 = \left(\frac{dx}{x+a}\right)^2 + \left(\frac{dy}{y+b}\right)^2 + \left(\frac{dz}{z+c}\right)^2 \tag{3}$$

In this equation, dE represents the *Empfindungsunterschied* (difference in sensation), while x, y, and z denote the *physiologische Urfarben* (physiological primary colors), and a, b, and c indicate self-light constants. The physiological primary colors were expressed as linear homogeneous equations of R (red), G (green), and V (violet), known as *Elementarfarben* or elementary colors, whose sensitivity curves were experimentally derived by Helmholtz's assistants, Arthur König and Conrad Dieterici.[24] These were essentially the sensitivity curves of the three receptors in the human retina,[25] as depicted in **Figure 4**. The new set of primary colors, the physiological primary colors (x, y, and z), was constrained to ensure that R, G, and V values for spectral colors did not yield negative values.

The three quadratic terms in Equation (3) differ only in the magnitudes of the self-light constants, a difference that becomes increasingly unimportant as the brightness increases. At such high brightness levels, the line element predicts, for pure brightness discrimination, that the Weber-Fechner fraction will have the same value for all colors, according to König' and Brodhun's experimental findings:[26]

$$\frac{\Delta x}{x} = \frac{\Delta y}{y} = \frac{\Delta z}{z} \tag{4}$$

4. Schrödinger's "Grundlinien"

Prominent space was dedicated to Helmholtz's line-element treatment in the second revised edition of the "Handbuch der physiologischen Optik".[28] However, both the mathematical derivation and the underlying idea were omitted by Helmholtz's successors in the third edition, published in 1911, seventeen years after Helmholtz's death. This omission probably contributed to making his outstanding work fall into oblivion for at least a quarter century, until Erwin Schrödinger rediscovered its precious value.

Before writing his well-known papers on wave mechanics, Schrödinger extensively researched color theory. After completing his doctorate in 1910 at the University of Vienna, he became an assistant to Franz Serafin Exner. Exner was exploring, at the time, the implications of the Young-Maxwell-Helmholtz color theory, a pursuit he continued until his death. Motivated by Exner's research, Schrödinger began delving into color metrics. The outcomes of Schrödinger's research in the field were submitted in 1920 to the *Annalen der Physik*, appearing as three separate contributions in the comprehensive work titled "Grundlinien einer Theorie der Farbmetrik im Tagessehen" ("Outline of a Theory of Color Measurement for Daylight Vision").[29] Schrödinger's "Grundlinien" represented his major accomplishment in color theory, earning him the prestigious Haitinger Prize from the Austrian Academy of Sciences for his foundational contributions to color theory. Schrödinger identified two categories into which the efforts to quantify color fall: Their distinction lies in the criterion used for the adjustment of two adjoining color fields on an experimental device. Either a criterion of complete identity is applied exclusively (indistinguishability), or other criteria are applied (such as just-noticeable difference, maximum similarity, or eventual maximum contrast). Measurement results of the first kind do form a unified, internally consistent system whose simple axiomatic rules have been formalized by Grassmann, which rules have been experimentally verified by König. I believe that this system of rules – frequently known as the laws of light mixture – may reasonably be called lower color metric, to distinguish it from higher color metric, whose laws are much more complicated and much less well understood.[30] According to Schrödinger's definition, the pioneering color-matching experiments carried out by Maxwell in the 1850s fell within the domain of lower color metric. Between 1855 and 1860, Maxwell obtained the first color-matching equations using instruments that enabled direct comparison, in both color and intensity, between white light and a combination of three spectral colors. This process laid the foundation for his color diagram, an equilateral triangle in the Euclidean plane, with the primary colors—red, green, and blue—positioned at the vertices (a precursor to the RGB color diagram). In Maxwell's diagram, distances between colors were expressed in Euclidean terms. Lower color metric, as noted by J. J. Vos, precisely describes the laws of color mixing evident in these diagrams and their 3D extension. Helmholtz's work on the line element, on the other hand, dealt with the smallest perceptual differences between colors. Helmholtz examined two initially indistinguishable adjacent color fields, adjusting the color of one field slightly (in terms of hue, shade, or tint) until a *just noticeable difference* was identified. In line with Schrödinger's clarification, these experiments belong to the domain of higher color metric, "higher" in the sense that it builds on the fundamentals of lower color metrics.[31] In this case, a Riemannian space turned out to be essential for translating color differences into distances within the three-dimensional color space. This geometry, as noted by Moore, is of the same kind as that used by Einstein in his general theory of relativity, albeit in a three-dimensional color space rather than the four-dimensional space-time.[32] It is noteworthy that Schrödinger had previously acquainted himself with Riemann geometry while working on issues related to general relativity.[33] Undoubtedly, his comprehensive understanding of the Riemannian formalism enabled him to envision its application in the context of color theory.

In this work, we only focus on Part III of the "Grundlinien" which is devoted to higher color metrics. Here, Schrödinger responded directly to Helmholtz's hypothesis of a Riemannian line element for color space. After critiquing the editors of the third edition of the "Handbuch" for omitting the entire treatment of Helmholtz's line element, Schrödinger pointed out certain shortcomings in Helmholtz's model. It is noteworthy to mention some of them. Schrödinger first observed that the primaries, x, y, and z, chosen by Helmholtz, had to be computed *ad hoc* to maintain the line element for just distinguishable color pairs approximately constant. In Helmholtz's treatment, moreover, brightness was not considered an additive property of color, contradicting the latest results on photometry, particularly the experimental findings of William De Wiveleslie Abney and Edward Robert Festing.[34] Schrödinger also emphasized another significant aspect related to the nature of color space. Although Helmholtz's system was Riemannian, the color space demonstrated to be isometric to the Euclidean space. Indeed, neglecting the self-lights constant in Equation (3), Helmholtz's color space is Euclidean on a logarithmic basis. These deficiencies prompted Schrödinger to undertake a comprehensive review of the entire metrical theory of color perception, and his proposal for a new line element (ds), compatible with both the additivity law and the Weber-Fechner law, took shape:

$$ds^2 = \frac{1}{a_1 x_1 + a_2 x_2 + a_3 x_3} \cdot \left(\frac{a_1 dx_1^2}{x_1} + \frac{a_2 dx_2^2}{x_2} + \frac{a_3 dx_3^2}{x_3} \right) \tag{5}$$

In Equation (5), a_1, a_2, and a_3 indicate constants, experimentally derived by Exner, which correspond approximately to the brightness of the three fundamental processes defined as x_1, x_2, and x_3, as experimentally determined by Helmholtz's collaborators König and Dieterici in 1892.[35] Contrary to Helmholtz's metric, in Schrödinger's formulation, the color space does not exhibit isometry with Euclidean one. A highly significant application of Schrödinger's line element was the derivation of the lines of constant hue (wavelength) inside Young's triangular color diagram, as shown in **Figure 5**. These lines, as it turned out, were curved: colors with constant hues lay on the geodesics between fully saturated color and white. The derivation of the geodesic lines of constant hue, now called *isochromes*, provided a crucial clue for the non-Euclidean interpretation of the geometry of color space.

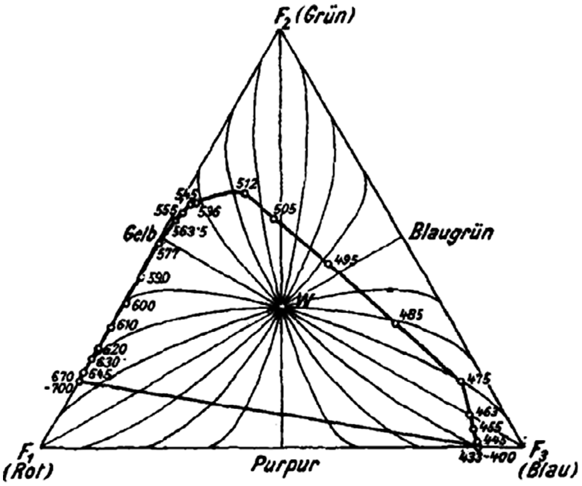

Figure 5. Schrödinger's lines of constant hue inside Young's triangle (Red-Blue-Green) showing the spectrum locus determined by König and Dieterici (thicker black line with indication of wavelength in nm).[36]

5. Conclusion

Following Helmholtz's and Schrödinger's pioneering contributions, numerous endeavors have been made to formulate line elements in color space[37] so that higher color metric started to contribute a proper branch of color theory, as Turner pointed out.[38] A variety of formulas for both line elements and color differences were presented by the leading figures in the field of color science, such as Johannes J. Vos and Pieter L. Walraven from the Institute for Perception TNO of Soesterberg, during the *Memorial Symposium on Color Metrics*, dedicated to Helmholtz in commemoration of the 150th anniversary of his birth.[39] The systematic exploration of color space metrics remains at the forefront of current scientific research, gaining significance due to its pivotal role in various technological applications, notably in computer graphics, color printing processes, and medical investigations. At present, there is no universally accepted color space, and different models are employed based on specific applications. Helmholtz's pioneering 1892 work on the line element has paved the way for contemporary inquiries into the Riemannian nature of color space. However, it is worth emphasizing that the fundamental nature of color space, whether Euclidean or non-Euclidean, remains an open problem and continues to be one of the most debated topics in colorimetry.[40]

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Conflict of Interest

The authors declare no conflict of interest.

Keywords

history of color theory, history of physics

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[11] a) Before Helmholtz's work, one of the most widely accepted representations of the color diagram was an equilateral triangle, influenced by the painter's tradition, with the three primary colors at the vertices (see J. H. Lambert, Beschreibung einer mit dem Caulauischen Wachse ausgemalten Farbenpyramide, Berlin, Haude und Spener **1772**; b) T. Mayer, *Opera inedita Tobiae Mayeri*, G. C. Lichtenberg, Göttingen **1775**.

[12] To predict the outcome of a mixture of colors, Helmholtz applied Newton's barycentric rule as outlined in the volume "Opticks", Book I, Part III, Proposition VI, grounding it on empirical foundations (see Ed: D. Cahan, Hermann von Helmholtz and the foundations of nineteenth-century science, University of California, Berkely, Los Angeles, London 1993, p. 235). Newton established a theoretical method for predicting the compound color resulting from a composition of seven primary colors. These colors were arranged along the arches of a circumference, and at the center of gravity of these sections, he depicted a small circle. The area of each circle was proportional to the number of rays of the colors entering a given mixture, allowing the common center of gravity of these circles to be easily determined. In Helmholtz's treatment, the amount of each color was defined by the slit width, precisely measurable using two pairs of Gravesande slits.

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[36] See Ref. [29c], p. 515.

[37] The first significant implementations of the line element theory were carried out by the American physicist David MacAdam, who experimentally derived ellipses for color stimuli in 1942 (see D. L. MacAdam, Visual sensitivities to color differences in daylight, Journal of the Optical Society of America 1942, 32, 5, 247. https://doi.org/10.1364/JOSA.32.000247), and the English physicist Walter Stiles, who proposed a modified version of Helmholtz's line element in 1946 (see W. S. Stiles, A modified Helmholtz line-element in brightness-colour space), Proceedings of the Physical Society **1946**, *58*, 41.

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