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Symmetries, Equivalence and Decoupling of First Order PDE's

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Declaration of Authorship

I, Matteo GORGONE, declare that this thesis titled, "Symmetries, Equivalence and Decoupling of First Order PDE's" and the work presented in it are my own. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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Messina, January 2017

The Author

"The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should still take simplicity into consideration in a subordinate way to beauty. It often happens that the requirements of simplicity and of beauty are the same, but where they clash the latter must take precedence."

— Paul Adrien Maurice Dirac

"Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now, as in the past, are inspired by the art of mathematics rather than by any prospect of ultimate usefulness."

— Eric Temple Bell

"I am interested in mathematics only as a creative art."

— Godfrey Harold Hardy

UNIVERSITY OF CATANIA

Abstract

University of Messina (Partner Institution)
Department of Mathematics and Computer Science,
Physics and Earth Sciences

Doctor of Philosophy

Symmetries, Equivalence and Decoupling of First Order PDE's

by Matteo GORGONE

The present Ph.D. Thesis is concerned with first order PDE's and to the structural conditions allowing for their transformation into an equivalent, and somehow simpler, form. Most of the results are framed in the context of the classical theory of the Lie symmetries of differential equations, and on the analysis of some invariant quantities. The thesis is organized in 5 main sections. The first two Chapters present the basic elements of the Lie theory and some introductory facts about first order PDE's, with special emphasis on quasilinear ones. Chapter 3 is devoted to investigate equivalence transformations, *i.e.*, point transformations suitable to deal with classes of differential equations involving arbitrary elements. The general framework of equivalence transformations is then applied to a class of systems of first order PDE's, consisting of a linear conservation law and four general balance laws involving some arbitrary continuously differentiable functions, in order to identify the elements of the class that can be mapped to a system of autonomous conservation laws. Chapter 4 is concerned with the transformation of nonlinear first order systems of differential equations to a simpler form. At first, the reduction to an equivalent first order autonomous and homogeneous quasilinear form is considered. A theorem providing necessary conditions is given, and the reduction to quasilinear form is performed by constructing the canonical variables associated to the Lie point symmetries admitted by the nonlinear system. Then, a general nonlinear system of first order PDE's involving the derivatives of the unknown variables in polynomial form is considered, and a theorem giving necessary and sufficient conditions in order to map it to an autonomous system polynomially homogeneous in the derivatives is established. Several classes of first order Monge–Ampère systems, either with constant coefficients or with coefficients depending on the field variables, provided that the coefficients entering their equations satisfy some constraints, are reduced to quasilinear (or linear) form. Chapter 5 faces the decoupling problem of general quasilinear first order systems. Starting from the direct decoupling problem of hyperbolic quasilinear first order systems in two independent variables and two or three dependent variables, we observe that the decoupling conditions can be written in terms of the eigenvalues and eigenvectors of the coefficient matrix. This allows to obtain a completely general result. At first, general autonomous and homogeneous quasilinear first order systems (either hyperbolic or not) are discussed, and the necessary and sufficient conditions for the decoupling in two or more subsystems proved. Then, the analysis is extended to the case of nonhomogeneous and/or nonautonomous systems. The conditions, as one expects, involve just the properties of the eigenvalues and the eigenvectors (together with the generalized eigenvectors, if needed) of the coefficient matrix; in particular, the conditions for the full decoupling of a hyperbolic system in non–interacting subsystems have a physical interpretation since require the vanishing both of the change of characteristic speeds of a subsystem across a wave of the other subsystems, and of the interaction coefficients between waves of different subsystems. Moreover, when the required decoupling conditions are satisfied, we have also the differential constraints whose integration provides the variable transformation leading to the (partially or fully) decoupled system. All the results are extended to the decoupling of nonhomogeneous and/or nonautonomous quasilinear first order systems.

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Thanks to all of them, life becomes exciting.

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Matteo Gorgone

Contents

Quod tu fuisti ego sum, quod tu es et ego ero.

To whom knows it

1 Introduction

THE results presented in this Ph.D. Thesis are concerned with first order systems of partial differential equations and to the structural conditions allowing for their transformation into an equivalent, and somehow simpler, form. Most of the results are framed in the context of the classical theory (and of some of its recent generalizations and extensions) of the Lie symmetries of differential equations, and on the analysis of some invariant quantities.

The concept of *symmetry*, in everyday language, refers to a sense of harmonious and beautiful proportion and balance. In mathematics, a symmetry is just a transformation which does not change a mathematical object! The set of symmetries of an object is a group.

In mathematical physics, symmetry, with the meaning of invariance under suitable transformations, has become one of the most powerful, elegant and useful tools for the formulation of the *laws of nature*. For instance, the reproducibility of experiments in different places at different times relies on the invariance of the laws of nature under space translation and rotation (homogeneity and isotropy of space), as well as time translation (homogeneity of time) [66]. Without such regularities, physical events probably would remain out of our knowledge, and their formulation would be impossible. An important implication of symmetries is the existence of conservation laws. This connection has been proved by Emmy Noether [64] in her famous theorem, which states that for every suitable continuous symmetry there is a corresponding conserved quantity.

Towards the end of the nineteenth century, Sophus Lie realized that many of the available integration techniques for solving differential equations could be unified and extended using group theory. He introduced the notion, known now as Lie group, in order to study the solutions of ordinary differential equations [54, 55], and showed the following main property: the order of an ordinary differential equation can be reduced by one if it is invariant under a one–parameter Lie group of point transformations. The Lie symmetry methods are central in the modern approach to nonlinear ordinary differential equations. Lie devoted the remainder of his mathematical career to investigate these continuous groups of transformations leaving differential equations invariant, creating what is now called the *symmetry analysis of differential equations* that had an impact on many areas of mathematically based sciences.

In Chapter 2, the basic elements of the theory of Lie groups of transformations of differential equations are presented to keep this thesis self–contained. Lie's theory enables one to derive solutions of differential equations in a completely algorithmic way without appealing to special lucky guesses, and introduces the notion of *Lie point symmetry* of a system as a local group of transformations that maps every solution of the system to another solution of the same system. Elementary examples of Lie groups are translations, rotations and scalings. For any Lie group of point transformations there exists a set of privileged variables (the *canonical variables*) in terms of which the Lie group expresses in its simplest form; these play a crucial role in the proof of some theorems presented in this thesis. The application of Lie's theory to differential equations is completely algorithmic; however, it usually involves a lot of cumbersome and tedious calculations. For instance, in looking for symmetries of a system of partial differential equations, it is not uncommon to have to handle hundreds of equations to find a single solution. In this thesis, most of the calculations were made by using powerful Computer Algebra Systems (CAS) like Mathematica® [94] (commercial) and Reduce [37] (open source), also using specific packages for the necessary algebraic manipulations [3, 68].

The key idea of Lie's theory of symmetry analysis of differential equations relies on the invariance of the equation under a transformation of independent and dependent variables. This transformation forms a local group of point transformations establishing a diffeomorphism on the space of independent and dependent variables, mapping solutions of the equations to other solutions. Any transformation of the independent and dependent variables in turn induces a transformation of the derivatives. Lie groups are intimately connected to Lie algebras [28, 40], and a brief sketch about Lie algebras realized in terms of generators of Lie groups of transformations is provided. The algebraic structure of the admitted Lie symmetries is crucial in some theorems presented in the subsequent chapters.

In Chapter 3, some introductory facts about first order systems of partial differential equations, with special emphasis on quasilinear ones, are recalled. In view of the results presented in Chapter 5, we also recall some known results concerned with the construction of mappings from a given (*source*) system of differential equations to another equivalent (*target*) one [6, 19, 22, 23, 25, 26, 39, 65, 66, 80]. By considering general nonautonomous and/or nonhomogeneous first order quasilinear systems of partial differential equations it has been shown that their reduction to autonomous and homogeneous quasilinear form is possible if and only if a suitable algebra of point symmetries is admitted.

Since in many applications we have differential equations involving arbitrary elements (constants or functions), so that one often needs to deal with *classes of differential equations*, it may be convenient to use equivalence transformations. For instance, if one is interested in identifying the systems of balance laws (possibly nonautonomous) that can be transformed by an invertible point transformation to systems of autonomous conservation laws, equivalence transformations, *i.e.*, point transformations that preserve the differential structure of the equations in the class but may change the form of the constitutive functions and/or parameters, provide useful.

Equivalence transformations live in an augmented space of independent, dependent and additional variables representing values taken by the arbitrary elements. The algorithm for the determination of such transformations consists of the following steps:

- consider the augmented space where the independent variables, the dependent variables and the arbitrary functions live;
- assume the arbitrary parameters determining the class of differential equations as dependent variables;

- impose, by using Lie infinitesimal criterion [76], the invariance of the class in the augmented space;
- project the admitted equivalence transformations into the space of independent and dependent variables and integrate the corresponding Lie's equations, so determining some finite transformations mapping the system to an equivalent one with the same differential structure but involving different arbitrary elements.

In Chapter 4, this general procedure has been applied to a class of $(3+1)$–dimensional systems of first order partial differential equations consisting of a linear conservation law and four general balance laws involving some arbitrary continuously differentiable functions. The aim is to identify, for a given equivalence transformation, the elements of the class that can be mapped to a system of autonomous conservation laws. The equivalence transformations are determined, and the finite transformations corresponding to the admitted generators are given. By constructing the finite transformations corresponding to a suitable linear combination of the admitted Lie point symmetries, under particular assumptions, a model of physical interest has been considered: an ideal gas in a non–inertial frame rotating with constant angular velocity around the vertical axis and subject to gravity and Coriolis forces, that can be mapped to an equivalent system where gravity and Coriolis forces disappear.

In Chapter 5, we consider nonlinear first order systems of differential equations, and investigate their reduction to a simpler form. At first, the reduction to an equivalent first order autonomous and homogeneous quasilinear form is considered. A theorem providing necessary conditions is given, and the reduction to quasilinear form is performed by constructing the canonical variables associated to the Lie point symmetries admitted by the nonlinear system. The fact that the conditions are only necessary is proved by exhibiting a nonlinear system satisfying the hypotheses of the theorem that is not reducible to quasilinear form. Several examples of first order systems of nonlinear partial differential equations that can be transformed, under suitable conditions, to quasilinear autonomous and homogeneous systems are given. The nonlinear first order systems are obtained from second order $(1+1)$–, $(2+1)$– and $(3+1)$–dimensional Monge–Ampère equations. In the second part of Chapter 5, a general nonlinear system of first order partial differential equations involving the derivatives of the unknown variables in polynomial form is considered, and a theorem giving necessary and sufficient conditions in order to map it to an autonomous system polynomially homogeneous in the derivatives is established. The theorem involves the Lie point symmetries admitted by the nonlinear source system, and the proof is constructive, in the sense that it leads to the algorithmic construction of the invertible mapping performing the task. First order nonlinear systems polynomial in the derivatives where the theorem applies are considered; several classes of first order Monge–Ampère systems, either with constant coefficients or with coefficients depending on the field variables, provided that the coefficients entering their equations satisfy some constraints, are reduced to quasilinear (or linear) form.

Chapter 6 faces the decoupling problem of general quasilinear first order systems. In fact, for quasilinear first order systems, it may be interesting, from a computational point of view, to look for the conditions (if any) leading to their possible decoupling into smaller non–interacting subsystems (full decoupling), or their reduction to a set of smaller subsystems that can be solved separately in hierarchy (partial decoupling). In literature, the problem has been analyzed by Nijenhuis [63] who, in the case of strictly hyperbolic systems, provided necessary and sufficient conditions for the decoupling into non–interacting one–dimensional subsystems requiring the vanishing of the corresponding Nijenhuis tensor. Later, other results have been given by Bogoyavlenskij [9, 10], who provided necessary and sufficient conditions, with a geometric formalism, by using Nijenhuis [63] and Haantjes [36] tensors. In the beginning of Chapter 6, we considered the decoupling problem of hyperbolic quasilinear first order systems in two independent variables and two or three dependent variables that can be in principle nonautonomous and/or nonhomogeneous. By means of a direct approach, we derived the conditions on the source system and the transformation allowing us to obtain a system that results partially or fully decoupled in some subsystems. Such conditions are written in terms of the eigenvalues and eigenvectors of the coefficient matrix. The results can be applied to the class of Galilean first order systems in two and three dependent variables. From a physical point of view, in the family of 2×2 Galilean first order systems the one–dimensional Euler equations of barotropic fluids which, with a suitable constitutive law, are mapped to the partially decoupled form, have been characterized.

In the rest of Chapter 6, a generalization of the results found by the direct approach for the decoupling problem in the case of quasilinear first order systems involving two or three dependent variables is presented. At first, we discuss general autonomous and homogeneous quasilinear first order systems (either hyperbolic or not), and prove the necessary and sufficient conditions for the decoupling in two or more subsystems. Then, we extend the analysis also to the case of nonhomogeneous and/or nonautonomous systems. The conditions, as one expects, involve just the properties of the eigenvalues and the eigenvectors (together with the generalized eigenvectors, if needed) of the coefficient matrix; in particular, the conditions for the full decoupling of a hyperbolic system in non–interacting subsystems have a physical interpretation since require the vanishing both of the change of characteristic speeds of a subsystem across a wave of the other subsystems, and of the interaction coefficients between waves of different subsystems. Moreover, when the required decoupling conditions are satisfied, we have also the differential constraints whose integration provides the variable transformation leading to the (partially or fully) decoupled system. All the results are extended to the decoupling of nonhomogeneous and/or nonautonomous quasilinear first order systems. Some examples of physical interest where the procedure works are also given. The theorem was applied to the one–dimensional Euler equations of an ideal gas with the special value of the adiabatic index $\Gamma = 3$ [17, p. 88], and to a nonlinear model describing the motion of a moving threadline with a particular constitutive law for the tension.

Most of the original results presented in this thesis are contained in the papers [31, 32, 33, 34, 35].

2 Lie groups of transformations

In this Chapter, also to fix the notation that we will use throughout this thesis, we give a brief account of the basic concepts of Lie groups theory of differential equations; the interested reader may find a more detailed exposition in several well known monographies [3, 5, 6, 7, 8, 14, 16, 42, 43, 44, 45, 62, 73, 74, 75, 76, 87].

2.1 Basic theory of Lie groups of transformations

Definition 2.1.1 (Group of transformations)**.** *Let us consider a domain $D \subseteq \mathbb{R}^N$ and a subset $S \subseteq \mathbb{R}$. The set of transformations*

$$\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}; a), \qquad \mathbf{Z} : D \times S \rightarrow D, \tag{2.1}$$

depending on the parameter a, forms a one–parameter group of transformations *on D if:*

1. *For each value of the parameter $a \in S$ the transformations are one–to–one onto D;*
2. *S with the law of composition $\mu : D \times D \rightarrow D$ is a group with identity e;*
3. *$\mathbf{Z}(\mathbf{z}; e) = \mathbf{z}$, $\forall\, \mathbf{z} \in D$;*
4. *$\mathbf{Z}(\mathbf{Z}(\mathbf{z}; a); b) = \mathbf{Z}(\mathbf{z}; \mu(a, b))$, $\forall\, \mathbf{z} \in D$, $\forall\, a, b \in S$.*

Definition 2.1.2 (Lie group of transformations)**.** *The group of transformations (2.1) defines a* one–parameter Lie group of transformations *if, in addition to satisfying the axioms of the previous definition,*

1. *a is a continuous parameter, i.e., S is an interval in $\mathbb{R}$;*
2. *$\mathbf{Z}$ is C^∞ with respect to $\mathbf{z}$ in D and an analytic function of a in S;*
3. *the group operation $\mu(a, b)$ and the inversion a^{-1} are analytic functions $\forall\, a, b \in S$.*

Due to the analyticity of the group operation μ, it is always possible to reparametrize the Lie group in such a way the group operation becomes the ordinary sum in $\mathbb{R}$ so that $e = 0$.

Expanding (2.1) in powers of a around $a = 0$, we get (in a neighbourhood of $a = 0$):

$$\begin{aligned} \mathbf{z}^\star &= \mathbf{z} + a \left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0} + \frac{a^2}{2} \left.\frac{\partial^2 \mathbf{Z}(\mathbf{z}; a)}{\partial a^2}\right|_{a=0} + \cdots = \\ &= \mathbf{z} + a \left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0} + O(a^2). \end{aligned} \tag{2.2}$$

By setting

$$\boldsymbol{\zeta}(\mathbf{z}) = \left.\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a}\right|_{a=0}, \tag{2.3}$$

the relation

$$\mathbf{z}^\star = \mathbf{z} + a\boldsymbol{\zeta}(\mathbf{z}) \tag{2.4}$$

defines the *Lie infinitesimal transformation*; the components of $\boldsymbol{\zeta}(\mathbf{z})$ are called the *infinitesimals* of (2.1).

Lie's First Fundamental Theorem ensures that the infinitesimal transformations contain the essential information for characterizing a one–parameter Lie group of transformations.

Theorem 2.1.1 (First Fundamental Theorem of Lie)**.** *The Lie group of transformations (2.1) can be found as solution of the initial value problem for the system of first order differential equations*

$$\frac{d\mathbf{z}^\star}{da} = \boldsymbol{\zeta}(\mathbf{z}^\star), \qquad \mathbf{z}^\star(0) = \mathbf{z}. \tag{2.5}$$

Proof. If $\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}; a)$, taking into account that

$$\mathbf{Z}(\mathbf{z}; a + \epsilon) = \mathbf{Z}(\mathbf{z}^\star; \epsilon), \tag{2.6}$$

and expanding both sides in powers of ϵ around $\epsilon = 0$ we get:

$$\begin{aligned} \mathbf{Z}(\mathbf{z}; a+\epsilon) &= \mathbf{Z}(\mathbf{z}; a) + \epsilon\frac{\partial \mathbf{Z}(\mathbf{z}; a)}{\partial a} + O(\epsilon^2) = \\ &= \mathbf{z}^\star + \epsilon\frac{d\mathbf{z}^\star}{da} + O(\epsilon^2), \\ \mathbf{Z}(\mathbf{z}^\star; \epsilon) &= \mathbf{Z}(\mathbf{z}^\star; 0) + \epsilon\left.\frac{\partial \mathbf{Z}(\mathbf{z}^\star; \epsilon)}{\partial \epsilon}\right|_{\epsilon=0} + O(\epsilon^2) = \\ &= \mathbf{z}^\star + \epsilon\boldsymbol{\zeta}(\mathbf{z}^\star) + O(\epsilon^2), \end{aligned} \tag{2.7}$$

where $\mathbf{z}^\star$ is given by (2.1).

By comparing these expressions it follows that $\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}; a)$ satisfies the initial value problem (2.5).

Conversely, since the infinitesimals $\boldsymbol{\zeta}$ and their first order partial derivatives are continuous, the Cauchy existence and uniqueness theorem for the initial value problem (2.5) implies that the solution of (2.5) exists and is unique. This solution has to be (2.1), and this completes the proof. □

Theorem 2.1.1 means that the infinitesimal transformation uniquely characterizes the Lie group of point transformations, and so it is justified the term *infinitesimal generator of the group* given to $\boldsymbol{\zeta}(\mathbf{z})$.

To the infinitesimal generator $\boldsymbol{\zeta}(\mathbf{z})$ (which is a vector field) of the one–parameter Lie group of transformations (2.1) it is associated the first order differential operator (the *symbol* in Lie's notation)

$$\Xi = \boldsymbol{\zeta}(\mathbf{z}) \cdot \boldsymbol{\nabla} = \zeta^1(\mathbf{z})\frac{\partial}{\partial z_1} + \cdots + \zeta^N(\mathbf{z})\frac{\partial}{\partial z_N}. \tag{2.8}$$

For any differentiable function $F(\mathbf{z})$ it is

$$\Xi(F) = \boldsymbol{\zeta}(\mathbf{z}) \cdot \boldsymbol{\nabla} F = \zeta^1(\mathbf{z})\frac{\partial F}{\partial z_1} + \cdots + \zeta^N(\mathbf{z})\frac{\partial F}{\partial z_N}, \tag{2.9}$$

and, in particular,

$$\Xi(\mathbf{z}) = \boldsymbol{\zeta}(\mathbf{z}). \tag{2.10}$$

A one–parameter Lie group of transformations, which by Theorem 2.1.1 is *equivalent* to its infinitesimal transformation, is also *equivalent* to its infinitesimal operator.

The following theorem shows that the use of the infinitesimal operator leads to an algorithm for finding the explicit solution of the initial value problem (2.5).

Theorem 2.1.2. *The one–parameter Lie group of transformations (2.1) is equivalent to*

$$\mathbf{z}^\star = \exp(a\Xi)(\mathbf{z}) = \mathbf{z} + a\Xi(\mathbf{z}) + \frac{a^2}{2}\Xi^2(\mathbf{z}) + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}\Xi^k(\mathbf{z}), \tag{2.11}$$

where the operator Ξ *is defined by (2.8), and* $\Xi^k(\mathbf{z}) = \Xi\left(\Xi^{k-1}(\mathbf{z})\right)$*; in particular, it is* $\Xi^0(\mathbf{z}) = \mathbf{z}$*.*

Proof. Let us have

$$\Xi = \zeta^1(\mathbf{z})\frac{\partial}{\partial z_1} + \cdots + \zeta^N(\mathbf{z})\frac{\partial}{\partial z_N}, \tag{2.12}$$

and

$$\Xi^\star = \zeta^1(\mathbf{z}^\star)\frac{\partial}{\partial z_1^\star} + \cdots + \zeta^N(\mathbf{z}^\star)\frac{\partial}{\partial z_N^\star}, \tag{2.13}$$

where $\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}; a)$. By expanding (2.1) in Taylor series around $a = 0$, one obtains:

$$\mathbf{z}^\star = \sum_{k=0}^{\infty} \frac{a^k}{k!} \left.\frac{\partial^k \mathbf{Z}(\mathbf{z}; a)}{\partial a^k}\right|_{a=0} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \left.\frac{d^k \mathbf{z}^\star}{da^k}\right|_{a=0}. \tag{2.14}$$

Since for any differentiable function $F(\mathbf{z})$ it is

$$\frac{dF(\mathbf{z}^\star)}{da} = \sum_{i=1}^{N} \frac{\partial F(\mathbf{z}^\star)}{\partial z_i^\star}\frac{dz_i^\star}{da} = \sum_{i=1}^{N} \zeta^i(\mathbf{z}^\star)\frac{\partial F(\mathbf{z}^\star)}{\partial z_i^\star} = \Xi^\star\left(F(\mathbf{z}^\star)\right), \tag{2.15}$$

in particular, it follows that

$$\begin{aligned} \frac{d\mathbf{z}^\star}{da} &= \Xi^\star(\mathbf{z}^\star), \\ \frac{d^2\mathbf{z}^\star}{da^2} &= \frac{d}{da}\left(\frac{d\mathbf{z}^\star}{da}\right) = \Xi^\star\left(\Xi^\star(\mathbf{z}^\star)\right) = \Xi^{\star 2}\left(\mathbf{z}^\star\right), \end{aligned} \tag{2.16}$$

and, more in general:

$$\frac{d^k\mathbf{z}^\star}{da^k} = \Xi^{\star k}\left(\mathbf{z}^\star\right), \qquad k \in \mathbb{N}. \tag{2.17}$$

From (2.17) it follows:

$$\left.\frac{d^k\mathbf{z}^\star}{da^k}\right|_{a=0} = \Xi^k(\mathbf{z}), \qquad k \in \mathbb{N}, \tag{2.18}$$

which gives the (2.11). □

Henceforth, the Taylor series expansion about $a = 0$ of a function $\mathbf{Z}(\mathbf{z}; a)$, which defines the Lie group of transformations (2.1), is determined by the coefficient of its $O(a)$ term, *i.e.*, by the infinitesimals $\boldsymbol{\zeta}(\mathbf{z})$.

Thus, one can find explicitly a one–parameter Lie group of transformations from its infinitesimal transformation, by expressing the group in terms of a power series (2.11), called *Lie series*, or by solving the initial value problem (2.5).

Now, we can introduce the concept of invariance of a function with respect to a Lie group of transformations, and prove the related invariance criterion.

Definition 2.1.3. *An infinitely differentiable function $F(\mathbf{z})$ is said to be an* invariant function *(or, simply, an* invariant*) of the Lie group of transformations (2.1) if and only if for any group of transformations (2.1) the condition*

$$F(\mathbf{z}^\star) \equiv F(\mathbf{z}) \tag{2.19}$$

holds true.

If $F(\mathbf{z})$ is an invariant function of (2.1), then it is simply called *an invariant* of (2.1). The invariance of a function is characterized in a very simple way by means of the infinitesimal generator of the group, as the following theorem shows.

Theorem 2.1.3. *$F(\mathbf{z})$ is invariant under (2.1) if and only if*

$$\Xi(F(\mathbf{z})) = 0. \tag{2.20}$$

It can be also defined the invariance of a surface of $\mathbb{R}^N$ with respect to a Lie group. A surface $F(\mathbf{z}) = 0$ is said to be an *invariant surface* with respect to the one–parameter Lie group (2.1) if $F(\mathbf{z}^\star) = 0$ when $F(\mathbf{z}) = 0$.

As a consequence of the Theorem 2.1.3, the following theorem immediately follows.

Theorem 2.1.4. *A surface $F(\mathbf{z}) = 0$ is invariant with respect to the group (2.1) if and only if*

$$\Xi(F(\mathbf{z})) = 0 \qquad \text{when} \qquad F(\mathbf{z}) = 0. \tag{2.21}$$

2.2 Canonical variables

For any Lie group there exists a set of privileged variables (*canonical variables*) in terms of which the Lie group (and the associated infinitesimal generator) expresses in its simplest form.

Given in $\mathbb{R}^N$ the one–parameter Lie group of transformations (2.1), let us suppose to make the change of variables defined by the one–to–one and C^1 transformation:

$$\mathbf{y} = \mathbf{Y}(\mathbf{z}), \qquad \mathbf{y} = (y_1, \ldots, y_N). \tag{2.22}$$

If

$$\Xi = \sum_{i=1}^{N} \zeta^i(\mathbf{z}) \frac{\partial}{\partial z_i} \tag{2.23}$$

is the infinitesimal generator in terms of the coordinates $\mathbf{z}$, the corresponding generator in terms of $\mathbf{y}$ is

$$\widetilde{\Xi} = \sum_{i=1}^{N} \widetilde{\zeta}^i(\mathbf{y}) \frac{\partial}{\partial y_i}, \tag{2.24}$$

where, in order to have the same group action, it is $\Xi = \widetilde{\Xi}$ with $\widetilde{\zeta}^i(\mathbf{y}) = \Xi\,(y_i)$ $(i = 1, \dots, N)$. In fact, by using the chain rule, we have:

$$\Xi = \sum_{i=1}^{N} \zeta^i(\mathbf{z}) \frac{\partial}{\partial z_i} = \sum_{i,j=1}^{N} \zeta^i(\mathbf{z}) \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} = \sum_{j=1}^{N} \widetilde{\zeta}^j(\mathbf{y}) \frac{\partial}{\partial y_j} = \widetilde{\Xi}, \tag{2.25}$$

where

$$\widetilde{\zeta}^j(\mathbf{y}) = \sum_{i=1}^{N} \zeta^i(\mathbf{z}) \frac{\partial y_j}{\partial z_i} = \Xi\,(y_j)\,, \qquad j = 1, \dots, N. \tag{2.26}$$

If we choose the function $\mathbf{Y}(\mathbf{z})$ such that the conditions

$$\begin{aligned} &\Xi(Y_i(\mathbf{z})) = 0, \qquad i = 1, \dots, N-1, \\ &\Xi(Y_N(\mathbf{z})) = 1 \end{aligned} \tag{2.27}$$

hold true, then the infinitesimal generator expresses as

$$\Xi = \frac{\partial}{\partial y_N}, \tag{2.28}$$

and the Lie group writes as

$$\begin{aligned} &y_i^\star = y_i, \qquad i = 1, \dots, N-1, \\ &y_N^\star = y_N + a. \end{aligned} \tag{2.29}$$

The canonical variables of Lie groups of point transformations play a crucial role in the proof of some theorems presented in this thesis.

2.3 Multi parameter Lie groups

A Lie group of transformations may depend as well on many parameters,

$$\mathbf{z}^\star = \mathbf{Z}(\mathbf{z}; \mathbf{a}), \tag{2.30}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_r) \in S \subseteq \mathbb{R}^r$. The $r \times N$ *infinitesimal matrix* $\boldsymbol{\chi}(\mathbf{z})$ with entries

$$\chi_\alpha^j(\mathbf{z}) = \left.\frac{\partial z_j^\star}{\partial a_\alpha}\right|_{\mathbf{a}=\mathbf{0}} = \left.\frac{\partial Z_j(\mathbf{z}, \mathbf{a})}{\partial a_\alpha}\right|_{\mathbf{a}=\mathbf{0}} \tag{2.31}$$

$(\alpha = 1, \dots, r;\ j = 1, \dots, N)$ may be constructed, and, for each parameter a_α of the r–parameter Lie group of transformations (2.30), the *infinitesimal* *generator* Ξ_α,

$$\Xi_\alpha = \sum_{j=1}^{N} \chi_\alpha^j(\mathbf{z})\frac{\partial}{\partial z_j}, \qquad \alpha = 1,\dots,r, \tag{2.32}$$

is defined. The infinitesimal generator

$$\Xi = \sum_{\alpha=1}^{r} \sigma_\alpha \Xi_\alpha = \sum_{j=1}^{N} \zeta^j(\mathbf{z})\frac{\partial}{\partial z_j}, \qquad \zeta^j(\mathbf{z}) = \sum_{\alpha=1}^{r} \sigma_\alpha \chi_\alpha^j(\mathbf{z}), \tag{2.33}$$

where $\sigma_1,\dots,\sigma_r$ are fixed real constants, in turn defines a one–parameter subgroup of an r–parameter Lie group of transformations.

2.4 Lie groups of differential equations

In considering Lie groups of point transformations associated to a given differential equation $\mathcal{E}$ involving n independent variables $\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n$, and m dependent variables $\mathbf{u} = (u_1,\dots,u_m) \in \mathbb{R}^m$, it is convenient to distinguish the independent variables from the dependent ones, so that we can write such a group of transformations in the form

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x},\mathbf{u};a), \qquad \mathbf{u}^\star = \mathbf{U}(\mathbf{x},\mathbf{u};a), \tag{2.34}$$

acting on the space $\mathbb{R}^{n+m}$ of the variables $(\mathbf{x},\mathbf{u})$. Also, let

$$\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x}) \equiv (\Theta_1(\mathbf{x}),\dots,\Theta_m(\mathbf{x})) \tag{2.35}$$

be a solution of the equation $\mathcal{E}$.

A Lie group of transformations of the form (2.34) admitted by $\mathcal{E}$ has the two equivalent properties:

- a transformation of the group maps any solution of $\mathcal{E}$ into another solution of $\mathcal{E}$;
- a transformation of the group leaves $\mathcal{E}$ invariant, say, $\mathcal{E}$ reads the same in terms of the variables $(\mathbf{x},\mathbf{u})$ and in terms of the transformed variables $(\mathbf{x}^\star,\mathbf{u}^\star)$.

Since a differential equation involves, in addition to $\mathbf{x}$ and $\mathbf{u}$, the derivatives up to some finite order of the latter with respect to the former ones, we have to determine the transformation of such derivatives. This is accomplished by *prolonging* the action of the group. The transformations (2.34) determine suitable transformations for the derivatives of the dependent variables $\mathbf{u}$ with respect to the independent variables $\mathbf{x}$.

Let $\mathbf{u}^{(1)}$ denote the vector whose $m \cdot n$ components are all first order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$,

$$\mathbf{u}^{(1)} \equiv \left(\frac{\partial u_1}{\partial x_1},\dots,\frac{\partial u_1}{\partial x_n},\dots,\frac{\partial u_m}{\partial x_1},\dots,\frac{\partial u_m}{\partial x_n}\right), \tag{2.36}$$

and, in general, let $\mathbf{u}^{(k)}$ denote the vector whose components are the $m \cdot \binom{n+k-1}{k}$ k–th order partial derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$.

The transformations of the derivatives of the dependent variables (obtained by requiring that the transformations preserve the *contact conditions*) lead to natural extensions (*prolongations*) of the one–parameter Lie group of transformations (2.34). While the one–parameter Lie group of transformations (2.34) acts on the space $(\mathbf{x}, \mathbf{u})$, the extended group acts on the space $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)})$, and, more in general, on the *jet space* $(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})$. Since all the information about a Lie group of transformations is contained in its infinitesimal generator, we need to compute its prolongations:

- the first order prolongation

$$\Xi^{(1)} = \Xi + \sum_{A=1}^{m}\sum_{i=1}^{n} \eta^A_{[i]}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)})\frac{\partial}{\partial u_{A,i}}, \qquad u_{A,i} = \frac{\partial u_A}{\partial x_i}, \tag{2.37}$$

  with

$$\eta^A_{[i]}(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}) = \frac{D\eta^A}{Dx_i} - \frac{D\xi^j}{Dx_i}\frac{\partial u_A}{\partial x_j}; \tag{2.38}$$

- the general k–th order prolongation recursively defined by

$$\Xi^{(k)} = \Xi^{(k-1)} + \sum_{A=1}^{m}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n} \eta^A_{[i_1\ldots i_k]}\frac{\partial}{\partial u_{A,i_1\ldots i_k}}, \tag{2.39}$$

  where $u_{A,i_1\ldots i_k} = \dfrac{\partial^k u_A}{\partial x_{i_1}\ldots\partial x_{i_k}}$, and $\eta^A_{[i_1\ldots i_k]}$ recursively defined by the relation

$$\eta^A_{[i_1\ldots i_k]} = \frac{D\eta^A_{[i_1\ldots i_{k-1}]}}{Dx_{i_k}} - u_{A,i_1\ldots i_{k-1}j}\frac{D\xi^j}{Dx_{i_k}}. \tag{2.40}$$

In (2.38) and (2.40) the *Lie derivative*

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \frac{\partial u_A}{\partial x_i}\frac{\partial}{\partial u_A} + \frac{\partial^2 u_A}{\partial x_i \partial x_j}\frac{\partial}{\partial u_{A,j}} + \ldots \tag{2.41}$$

has been introduced, and the Einstein convention of summation over repeated indices used.

The infinitesimals of the Lie group of transformations leaving a given system of differential equations invariant can be found by means of the straightforward algorithm that will be described below. Remarkably, the search of one–parameter Lie groups of transformations leaving differential equations invariant leads usually to obtain r–parameter Lie groups of transformations.

2.5 Lie's algorithm

Let

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right) = \mathbf{0} \tag{2.42}$$

($\boldsymbol{\Delta} = (\Delta_1, \ldots, \Delta_q)$) be a system of q differential equations of order k, with independent variables $\mathbf{x} \in \mathbb{R}^n$ and dependent variables $\mathbf{u} \in \mathbb{R}^m$. Suppose that the system (2.42) is written in normal form, *i.e.*, it is solved with respect to some partial derivatives of order k_ν, for $\nu = 1, \ldots, q$:

$$\Delta_\nu\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right) \equiv u_{A_\nu, i_1\ldots i_{k_\nu}} - F_\nu(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}) = 0. \quad (2.43)$$

The equations (2.43) can be considered as characterizing a manifold in the *jet space*, the latter having dimension equal to

$$n + m\sum_{h=0}^{k}\binom{n+h-1}{n-1} = n + m\binom{n+k}{k}. \quad (2.44)$$

It is said that the one–parameter Lie group of transformations (2.34) leaves the system (2.43) *invariant* (*is admitted by* (2.43)) if and only if its k–th prolongation leaves invariant the manifold of the jet space defined by (2.43).

Theorem 2.1.4 allows to prove the following important theorem, which leads directly to the algorithm for the computation of the infinitesimals admitted by a given differential system.

Theorem 2.5.1 (Infinitesimal Criterion for differential equations)**.** *Let*

$$\Xi = \xi^i(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial x_i} + \eta^A(\mathbf{x}, \mathbf{u})\frac{\partial}{\partial u^A} \quad (2.45)$$

be the infinitesimal generator corresponding to (2.34), and $\Xi^{(k)}$ the k–th extended infinitesimal generator. The group (2.34) is admitted by the system (2.43) if and only if

$$\begin{aligned} &\Xi^{(k)}\left(\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right)\right) = \mathbf{0} \\ &\text{when} \qquad \boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right) = \mathbf{0}. \end{aligned} \quad (2.46)$$

If the differential system is in polynomial form in the derivatives, the invariance conditions (2.46) are polynomials in the components of $(\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)})$, with coefficients expressed by linear combinations of the unknowns ξ^i, η^A ($i = 1, \ldots, n$; $A = 1, \ldots, m$) and their partial derivatives. After using (2.43) to eliminate the derivatives $u_{A_\nu, i_1\ldots i_{k_\nu}}$, the equations can be splitted with respect to the components of the remaining derivatives of $\mathbf{u}$ that can be arbitrarily varied. By equating to zero the coefficients of these partial derivatives, one obtains an overdetermined system of linear differential equations for the infinitesimals (the so called system of *determining equations*), whose integration leads to the infinitesimals of the group. The infinitesimals involve arbitrary constants (and in some cases arbitrary functions); therefore, we have *de facto* r–parameter Lie groups (infinite–parameter Lie groups when arbitrary functions are involved).

Remark 2.5.1. *In this thesis we will deal mainly with Lie groups of transformations admitted by differential equations with infinitesimals depending on the independent and dependent variables only. These are called* local *Lie point symmetries. Symmetries in which the infinitesimals may depend also on first (respectively, higher) order derivatives of the dependent variables with respect to the independent variables are* contact *(respectively,* generalized*) symmetries, and symmetries with infinitesimals depending also on integrals of dependent variables are called* nonlocal *symmetries.*

2.6 Lie algebras

Lie groups are intimately connected to Lie algebras [28, 40]. Let us recall some basic notions about Lie algebras realized in terms of generators of Lie groups admitted by differential equations.

Let $\mathbb{K}$ be a field. A *Lie algebra* L over $\mathbb{K}$ is a vector space over $\mathbb{K}$ endowed with a bilinear map (*Lie bracket* or *commutator*)

$$\begin{aligned} [\cdot,\cdot]: L\times L &\longrightarrow L,\\ (x,y) &\longmapsto [x,y], \end{aligned} \tag{2.47}$$

satisfying the following properties:

$$\begin{aligned} &[x,x]=0 \quad \forall x\in L,\\ &[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0 \qquad \text{(Jacobi identity)}. \end{aligned} \tag{2.48}$$

Due to the bilinearity of the Lie bracket, it is

$$0=[x+y,x+y]=[x,x]+[x,y]+[y,x]+[y,y]=[x,y]+[y,x], \tag{2.49}$$

whereupon, using $(2.48)_1$, it follows

$$[x,y]=-[y,x] \quad \forall x,y\in L \qquad \text{(antisymmetry)}. \tag{2.50}$$

Moreover, (2.50) implies $(2.48)_1$ if $\mathbb{K}$ has not characteristic 2.

The *dimension* of a Lie algebra L is its dimension as a vector space; if it has finite dimension r, then it is often denoted by L_r.

A Lie algebra L is *Abelian* if $[x,y]=0$ for all $x,y\in L$.

Given a Lie algebra L, a *Lie subalgebra* of L is a vector space $L'\subseteq L$ such that $[x,y]\in L'$ for all $x,y\in L'$.

An *ideal* of a Lie algebra L is a subspace J of L such that

$$[x,y]\in J \qquad \text{for all} \quad x\in L,\ y\in J. \tag{2.51}$$

The Lie algebra L and $\{0\}$ are ideals of L, called the *trivial ideals* of L. Another important example of ideal of L is the *centre* of L, defined by

$$Z(L)=\{x\in L: [x,y]=0 \quad \forall y\in L\}. \tag{2.52}$$

It can be noted that $L=Z(L)$ if and only if L is Abelian.

Remark 2.6.1. *Real ($\mathbb{K}=\mathbb{R}$) and complex ($\mathbb{K}=\mathbb{C}$) Lie algebras are of special relevance in the application of Lie groups of transformations of differential equations. In the following we shall restrict to consider real Lie algebras.*

The *commutator* of two generators Ξ_α and Ξ_β is the first order operator defined by

$$[\Xi_\alpha,\Xi_\beta]=\Xi_\alpha\Xi_\beta-\Xi_\beta\Xi_\alpha=$$

$$= \sum_{i,j=1}^{N} \left[\left(\zeta_\alpha^i(\mathbf{z}) \frac{\partial}{\partial z_i} \right) \left(\zeta_\beta^j(\mathbf{z}) \frac{\partial}{\partial z_j} \right) - \left(\zeta_\beta^i(\mathbf{z}) \frac{\partial}{\partial z_i} \right) \left(\zeta_\alpha^j(\mathbf{z}) \frac{\partial}{\partial z_j} \right) \right] =$$

$$= \sum_{j=1}^{N} \widetilde{\zeta}^j(\mathbf{z}) \frac{\partial}{\partial z_j}, \tag{2.53}$$

where

$$\widetilde{\zeta}^j(\mathbf{z}) = \sum_{i=1}^{N} \left[\zeta_\alpha^i(\mathbf{z}) \frac{\partial \zeta_\beta^j(\mathbf{z})}{\partial z_i} - \zeta_\beta^i(\mathbf{z}) \frac{\partial \zeta_\alpha^j(\mathbf{z})}{\partial z_i} \right]. \tag{2.54}$$

As a consequence of this definition, the operation of commutation is antisymmetric,

$$[\Xi_\alpha, \Xi_\beta] = -[\Xi_\beta, \Xi_\alpha], \tag{2.55}$$

bilinear,

$$[a\Xi_\alpha + b\Xi_\beta, \Xi_\gamma] = a[\Xi_\alpha, \Xi_\gamma] + b[\Xi_\beta, \Xi_\gamma] \tag{2.56}$$

(a, b constants), and satisfies the Jacobi identity

$$[\Xi_\alpha, [\Xi_\beta, \Xi_\gamma]] + [\Xi_\beta, [\Xi_\gamma, \Xi_\alpha]] + [\Xi_\gamma, [\Xi_\alpha, \Xi_\beta]] = 0. \tag{2.57}$$

It follows that a vector space L of generators is a Lie algebra if the commutator $[\Xi_\alpha, \Xi_\beta]$ of any two generators $\Xi_\alpha \in L$ and $\Xi_\beta \in L$ belongs to L.

Lemma 2.6.1. *The commutator of two generators is invariant with respect to any invertible change of variables.*

Proof. Let $\mathbf{y} = \mathbf{g}(\mathbf{z})$ be a change of variables. It results

$$\Xi_\alpha = \zeta_\alpha^i \frac{\partial}{\partial z_i}, \qquad \widetilde{\Xi}_\alpha = \zeta_\alpha^i \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j},$$
$$\Xi_\beta = \zeta_\beta^i \frac{\partial}{\partial z_i}, \qquad \widetilde{\Xi}_\beta = \zeta_\beta^i \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j}.$$

Hence,

$$\left[\widetilde{\Xi}_\alpha, \widetilde{\Xi}_\beta\right] = \left(\widetilde{\zeta}_\alpha^j \frac{\partial \widetilde{\zeta}_\beta^k}{\partial y_j} - \widetilde{\zeta}_\beta^j \frac{\partial \widetilde{\zeta}_\alpha^k}{\partial y_j} \right) \frac{\partial}{\partial y_k} =$$
$$= \left(\zeta_\alpha^i \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} \left(\zeta_\beta^\ell \frac{\partial y_k}{\partial z_\ell} \right) - \zeta_\beta^i \frac{\partial y_j}{\partial z_i} \frac{\partial}{\partial y_j} \left(\zeta_\alpha^\ell \frac{\partial y_k}{\partial z_\ell} \right) \right) \frac{\partial}{\partial y_k} =$$
$$= \left(\zeta_\alpha^i \frac{\partial y_j}{\partial z_i} \frac{\partial \zeta_\beta^\ell}{\partial z_m} \frac{\partial z_m}{\partial y_j} \frac{\partial y_k}{\partial z_\ell} - \zeta_\beta^i \frac{\partial y_j}{\partial z_i} \frac{\partial \zeta_\alpha^\ell}{\partial z_m} \frac{\partial z_m}{\partial y_j} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} =$$
$$= \left(\left(\zeta_\alpha^i \frac{\partial \zeta_\beta^\ell}{\partial z_m} - \zeta_\beta^i \frac{\partial \zeta_\alpha^\ell}{\partial z_m} \right) \frac{\partial z_m}{\partial y_j} \frac{\partial y_j}{\partial z_i} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} =$$
$$= \left(\left(\zeta_\alpha^i \frac{\partial \zeta_\beta^\ell}{\partial z_m} - \zeta_\beta^i \frac{\partial \zeta_\alpha^\ell}{\partial z_m} \right) \delta_{mi} \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} =$$
$$= \left(\left(\zeta_\alpha^i \frac{\partial \zeta_\beta^\ell}{\partial z_i} - \zeta_\beta^i \frac{\partial \zeta_\alpha^\ell}{\partial z_i} \right) \frac{\partial y_k}{\partial z_\ell} \right) \frac{\partial}{\partial y_k} = \widetilde{[\Xi_\alpha, \Xi_\beta]}.$$

□

Theorem 2.6.1. *If a regularly assigned manifold $F(\mathbf{z}) = 0$, $\mathbf{z} \in \mathbb{R}^N$, is invariant with respect to generators Ξ_α and Ξ_β, then it is invariant with respect to their commutator $[\Xi_\alpha, \Xi_\beta]$.*

Proof. Since the manifold $F(\mathbf{z}) = 0$ is invariant with respect to the Lie groups generated by Ξ_α and Ξ_β, it is

$$\begin{aligned} &\Xi_\alpha(F(\mathbf{z})) = \Lambda_\alpha(\mathbf{z})F(\mathbf{z}),\\ &\Xi_\beta(F(\mathbf{z})) = \Lambda_\beta(\mathbf{z})F(\mathbf{z}), \end{aligned} \tag{2.58}$$

where $\Lambda_\alpha(\mathbf{z})$ and $\Lambda_\beta(\mathbf{z})$ are suitable Lagrange multipliers. Then,

$$\begin{aligned} [\Xi_\alpha, \Xi_\beta]\,(F(\mathbf{z})) &= \Xi_\alpha\Xi_\beta(F(\mathbf{z})) - \Xi_\beta\Xi_\alpha(F(\mathbf{z})) =\\ &= \Xi_\alpha(\Lambda_\beta(\mathbf{z})F(\mathbf{z})) - \Xi_\beta(\Lambda_\alpha(\mathbf{z})F(\mathbf{z})) =\\ &= \Xi_\alpha(\Lambda_\beta(\mathbf{z}))F(\mathbf{z}) + \Lambda_\beta(\mathbf{z})\Xi_\alpha(F(\mathbf{z}))\\ &- \Xi_\beta(\Lambda_\alpha(\mathbf{z}))F(\mathbf{z}) - \Lambda_\alpha(\mathbf{z})\Xi_\beta(F(\mathbf{z})) =\\ &= \left(\Xi_\alpha(\Lambda_\beta(\mathbf{z})) - \Xi_\beta(\Lambda_\alpha(\mathbf{z}))\right)F(\mathbf{z}) = \Lambda(\mathbf{z})F(\mathbf{z}), \end{aligned} \tag{2.59}$$

so proving that the manifold $F(\mathbf{z})$ is invariant with respect to $[\Xi_\alpha, \Xi_\beta]$. □

To extend the previous theorem to generators of Lie groups admitted by differential equations we need the following theorem.

Theorem 2.6.2. *The operation of prolongation commutes with the operation of taking a commutator.*

Proof. To prove the theorem it is sufficient, due to the recursive definition of the higher order prolongations, to limit ourselves to first order prolongations. Let us consider the generators Ξ_α and Ξ_β involving the variables $\mathbf{x}$ and $\mathbf{u}$. Since the operations of commutation and prolongation are invariant with respect to a change of variables, let us introduce the canonical variables of the generator Ξ_α (still denoting them with $\mathbf{x}$ and $\mathbf{u}$ to simplify the notation), whereupon the generator Ξ_α may be written as $\dfrac{\partial}{\partial x_1}$, whereas let us write the generator Ξ_β as

$$\Xi_\beta = \xi^i \frac{\partial}{\partial x_i} + \eta^A \frac{\partial}{\partial u_A}. \tag{2.60}$$

It results

$$[\Xi_\alpha, \Xi_\beta] = \frac{\partial \xi^i}{\partial x_1}\frac{\partial}{\partial x_i} + \frac{\partial \eta^A}{\partial x_1}\frac{\partial}{\partial u_A}, \tag{2.61}$$

whereas the first order prolonged operators become

$$\begin{aligned} &\Xi_\alpha^{(1)} = \Xi_\alpha,\\ &\Xi_\beta^{(1)} = \Xi_\beta + \eta^A_{[k]}\frac{\partial}{\partial u_{A,k}}, \end{aligned} \tag{2.62}$$

where $u_{A,k}$ denotes the partial derivative of u_A with respect to the variable x_k and we recall that it is

$$\eta^A_{[k]} = \frac{D\eta^A}{Dx_k} - \frac{D\xi^j}{Dx_k}u_{A,j}. \tag{2.63}$$

Therefore,

$$\left[\Xi_\alpha^{(1)}, \Xi_\beta^{(1)}\right] = [\Xi_\alpha, \Xi_\beta] + \left[\Xi_\alpha, \eta^A_{[k]}\frac{\partial}{\partial u_{A,k}}\right] = \\ = \frac{\partial \xi^i}{\partial x_1}\frac{\partial}{\partial x_i} + \frac{\partial \eta^A}{\partial x_1}\frac{\partial}{\partial u_A} + \frac{\partial \eta^A_{[k]}}{\partial x_1}\frac{\partial}{\partial u_{A,k}}. \tag{2.64}$$

On the other hand, the prolongation of the commutator $[\Xi_\alpha, \Xi_\beta]$ is

$$[\Xi_\alpha, \Xi_\beta]^{(1)} = [\Xi_\alpha, \Xi_\beta] + \widehat{\eta}^A_{[k]}\frac{\partial}{\partial u_{A,k}}, \tag{2.65}$$

where

$$\widehat{\eta}^A_{[k]} = \frac{D}{Dx_k}\left(\frac{\partial \eta^A}{\partial x_1}\right) - \frac{D}{Dx_k}\left(\frac{\partial \xi^j}{\partial x_1}\right)u_{A,j} = \\ = \frac{\partial}{\partial x_1}\left(\frac{D\eta^A}{Dx_k} - \frac{D\xi^j}{Dx_k}u_{A,j}\right) = \frac{\partial \eta^A_{[k]}}{\partial x_1}. \tag{2.66}$$

So it is

$$[\Xi_\alpha, \Xi_\beta]^{(1)} = \left[\Xi_\alpha^{(1)}, \Xi_\beta^{(1)}\right], \tag{2.67}$$

which completes the proof. □

In virtue of the previous results, the following theorem can be stated.

Theorem 2.6.3. *If a differential equation $\mathcal{E}$ admits the generators Ξ_α and Ξ_β, then it admits also the generator $[\Xi_\alpha, \Xi_\beta]$.*

Proof. Immediate. □

Remark 2.6.2. *The set of the generators admitted by a differential equation $\mathcal{E}$ is a vector space, because it is the space of solutions of the determining equations (that are linear and homogeneous differential equations). The previous theorem implies that such a vector space is also a Lie algebra. This algebra is called the* principal Lie algebra*. The knowledge of the subalgebras of the principal Lie algebra of partial differential equations allows one to construct and classify invariant solutions. In the case of ordinary differential equations it permits the reduction of order.*

If L_r is an r–dimensional Lie algebra of generators with a basis $\{\Xi_1, \ldots, \Xi_r\}$, any generator X can be represented as

$$X = f^\alpha \Xi_\alpha, \qquad \alpha = 1, \ldots, r, \tag{2.68}$$

where the f^α's are constant. Since L_r is closed under commutation, the commutator of any two generators Ξ_α and Ξ_β in the basis is a linear combination of the basis generators

$$[\Xi_\alpha, \Xi_\beta] = C^\gamma_{\alpha\beta}\Xi_\gamma, \tag{2.69}$$

where the constants $C^\gamma_{\alpha\beta}$ are called *structure constants*, and equations (2.69) are known as *commutation relations*.

Because of the antisymmetry of the commutator, the structure constants are antisymmetric in the two lower indices,

$$C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}, \tag{2.70}$$

and, because of the Jacobi identity, they satisfy the *Lie identity*

$$C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma} + C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha} + C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta} = 0. \tag{2.71}$$

The structure constants do not change under a coordinate transformation, but they change if the basis is changed.

It is convenient to display the commutators of a Lie algebra through its *commutator table*, whose (α, β)–th entry is $[\Xi_\alpha, \Xi_\beta]$: the commutator table is an antisymmetric matrix. Moreover, the structure constants are easily deduced from the commutator table.

The properties of Lie algebras are useful in the analysis of differential equations. By knowing the structure constants of a Lie algebra we may determine the so–called *derived* algebras. If L is a Lie algebra, then $L^{(1)} = [L, L]$ (spanned by all possible Lie brackets of elements of L) is the first derived algebra of L. By construction, $L^{(1)}$ is an ideal of L. The higher order derived algebras are recursively defined as

$$L^{(n)} = [L^{(n-1)}, L^{(n-1)}]. \tag{2.72}$$

An Abelian Lie algebra L verifies the condition $L^{(1)} = 0$ or, in terms of the commutators of two generators,

$$[\Xi_\alpha, \Xi_\beta] = 0, \qquad \alpha, \beta = 1, \dots r. \tag{2.73}$$

The notion of derived Lie algebra is useful to define *solvable* algebras. The Lie algebra L_r is said to be solvable if there is a series

$$L_r \supset L_{r-1} \supset \dots \supset L_1, \tag{2.74}$$

of subalgebras of respective dimensions $r, r-1, \dots, 1$, such that L_s is an ideal of L_{s+1}, $s = 1, \dots, r-1$.

The Lie algebra L_r is solvable if and only if its derived algebra of a finite order vanishes, *i.e.*, $L_r^{(n)} = 0$, $0 < n < \infty$. Any two–dimensional Lie algebra is solvable.

The Lie algebra L is said to be *simple* if it has no proper ideals.

A Lie algebra is said to be *semi–simple* if it has no solvable ideals different from $\{0\}$. A Lie algebra is semi–simple if and only if it contains no abelian ideals different from $\{0\}$. According to Cartan's criterion [28], the Lie algebra L_r with the structure constants $c^{\lambda}_{\mu\nu}$ is semi–simple if and only if

$$\det \|g_{\mu\nu}\| \neq 0, \tag{2.75}$$

where $\|g_{\mu\nu}\|$ is the matrix with entries

$$g_{\mu\nu} = c^{\lambda}_{\mu\nu}c^{\gamma}_{\nu\lambda}, \qquad \mu, \nu = 1, \dots, r. \tag{2.76}$$

An r–dimensional Lie algebra spanned by the infinitesimal generators $\{\Xi_1, \dots . \Xi_r\}$ is solvable if it is possible to arrange the elements of the basis in such a way

$$[\Xi_i, \Xi_j] = \sum_{k=1}^{j-1} C^k_{ij}\Xi_k, \qquad 1 \le i < j \le r. \tag{2.77}$$

2.7 Examples of Lie symmetries admitted by ODE's and PDE's

In this Section, some examples of the procedure leading to the determination of the Lie point symmetries admitted by ordinary and partial differential equations are given.

2.7.1 Symmetries of Blasius' equation

Let us consider the nonlinear third order ordinary differential equation

$$\Delta \equiv \frac{d^3u}{dx^3} + \frac{1}{2}u\frac{d^2u}{dx^2} = 0, \tag{2.78}$$

known as Blasius's equation, and let us determine its Lie point symmetries. We need the third order prolongation of the infinitesimal operator, say

$$\begin{aligned}\Xi^{(3)} &= \xi(x,u)\frac{\partial}{\partial x} + \eta(x,u)\frac{\partial}{\partial u} + \eta_{[1]}(x,u,u_{,1})\frac{\partial}{\partial u_{,1}} \\ &\quad + \eta_{[2]}(x,u,u_{,1},u_{,2})\frac{\partial}{\partial u_{,2}} + \eta_{[3]}(x,u,u_{,1},u_{,2},u_{,3})\frac{\partial}{\partial u_{,3}},\end{aligned} \tag{2.79}$$

where $u_{,k} = \dfrac{d^k u}{dx^k}$, and

$$\begin{aligned}\eta_{[1]} &= \frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial u}u_{,1} - \left(\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial u}u_{,1}\right)u_{,1}, \\ \eta_{[2]} &= \frac{\partial^2 \eta}{\partial x^2} + \left(2\frac{\partial^2 \eta}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right)u_{,1} + \left(\frac{\partial^2 \eta}{\partial u^2} - 2\frac{\partial^2 \xi}{\partial x \partial u}\right)u_{,1}^2 \\ &\quad - \frac{\partial^2 \xi}{\partial u^2}u_{,1}^3 + \left(\frac{\partial \eta}{\partial u} - 2\frac{\partial \xi}{\partial x}\right)u_{,2} - 3\frac{\partial \xi}{\partial u}u_{,1}u_{,2}, \\ \eta_{[3]} &= \frac{\partial^3 \eta}{\partial x^3} + \left(3\frac{\partial^3 \eta}{\partial x^2 \partial u} - \frac{\partial^3 \xi}{\partial x^3}\right)u_{,1} + 3\left(\frac{\partial^3 \eta}{\partial x \partial u^2} - \frac{\partial^3 \xi}{\partial x^2 \partial u}\right)u_{,1}^2 \\ &\quad + \left(\frac{\partial^3 \eta}{\partial u^3} - 3\frac{\partial^3 \xi}{\partial x \partial u^2}\right)u_{,1}^3 - \frac{\partial^3 \xi}{\partial u^3}u_{,1}^4 + 3\left(\frac{\partial^2 \eta}{\partial x \partial u} - \frac{\partial^2 \xi}{\partial x^2}\right)u_{,2} \\ &\quad - 3\frac{\partial \xi}{\partial u}u_{,2}^2 + \left(\frac{\partial \eta}{\partial u} - 3\frac{\partial \xi}{\partial x}\right)u_{,3} + 3\left(\frac{\partial^2 \eta}{\partial u^2} - 3\frac{\partial^2 \eta}{\partial x \partial u}\right)u_{,1}u_{,2} \\ &\quad - 6\frac{\partial^2 \xi}{\partial u^2}u_{,1}^2u_{,2} - 4\frac{\partial \xi}{\partial u}u_{,1}u_{,3}.\end{aligned} \tag{2.80}$$

By requiring the invariance condition

$$\Xi^{(3)}(\Delta)\Big|_{\Delta=0} = 0 \tag{2.81}$$

of (2.78), the following condition arises

$$\begin{aligned}
&2\frac{\partial^3\eta}{\partial x^3}+\frac{\partial^2\eta}{\partial x^2}u+\left(6\frac{\partial^3\eta}{\partial x^2\partial u}+2\frac{\partial^2\eta}{\partial x\partial u}u-\frac{\partial^2\xi}{\partial x^2}u-2\frac{\partial^3\xi}{\partial x^3}\right)u_{,1}\\
&\quad+\left(6\frac{\partial^3\eta}{\partial x\partial u^2}-6\frac{\partial^3\xi}{\partial x^2\partial u}+\frac{\partial^2\eta}{\partial u^2}u-2\frac{\partial^2\xi}{\partial x\partial u}u\right)u_{,1}^2\\
&\quad+\left(2\frac{\partial^3\eta}{\partial u^3}-6\frac{\partial^3\xi}{\partial x\partial u^2}-\frac{\partial^2\xi}{\partial u^2}u\right)u_{,1}^3-2\frac{\partial^3\xi}{\partial u^3}u_{,1}^4\\
&\quad+\left(6\frac{\partial^2\eta}{\partial x\partial u}-6\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\xi}{\partial x}u+\eta\right)u_{,2}-6\frac{\partial\xi}{\partial u}u_{,2}^2\\
&\quad+\left(6\frac{\partial^2\eta}{\partial u^2}-18\frac{\partial^2\xi}{\partial x\partial u}+\frac{\partial\xi}{\partial u}u\right)u_{,1}u_{,2}-12\frac{\partial^2\xi}{\partial u^2}u_{,1}^2u_{,2},
\end{aligned} \tag{2.82}$$

where the constraint $\Delta = 0$ has been taken into account to eliminate $u_{,3}$. The invariance condition (2.82) is a polynomial in the derivatives $u_{,1}$ and $u_{,2}$ whose coefficients, involving the infinitesimals and their partial derivatives, must vanish. Hence, a set of *determining equations* arises; these determining equations constitute a set of overdetermined linear partial differential equations that integrated provide the expression of the infinitesimals ξ and η:

$$\begin{aligned}
\xi(x,u) &= c_1 + c_2 x,\\
\eta(x,u) &= -c_2 u,
\end{aligned} \tag{2.83}$$

where c_1 and c_2 are arbitrary constants. Due to this arbitrariness, we may say that Blasius' equation admits a 2–parameter Lie group of point transformations spanned by the following vector fields:

$$\Xi_1 = \frac{\partial}{\partial x}, \qquad \Xi_2 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{2.84}$$

The infinitesimal operators (2.84) provide a basis of a vector space of dimensions 2, which is also a solvable Lie algebra, as it can be verified by computing their Lie bracket,

$$[\Xi_1, \Xi_2] = \Xi_1. \tag{2.85}$$

2.7.2 Symmetries of linear heat equation

Let us consider the linear second order partial differential equation

$$\Delta \equiv \frac{\partial u}{\partial x_1} - \frac{\partial^2 u}{\partial x_2^2} = 0 \tag{2.86}$$

(interpreting x_1 as the time, x_2 as a space coordinate, and $u(x_1, x_2)$ as a temperature, (2.86) is the Fourier equation for heat conduction), and determine its Lie point symmetries. We need the second order prolongation of the infinitesimal generator, say

$$
\begin{aligned}
\Xi^{(2)} = {} & \xi^1(x_1, x_2, u)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2, u)\frac{\partial}{\partial x_2} + \eta(x_1, x_2, u)\frac{\partial}{\partial u} \\
& + \eta_{[1]}(x_1, x_2, \mathbf{u}^{(1)})\frac{\partial}{\partial u_{,1}} + \eta_{[2]}(x_1, x_2, u, \mathbf{u}^{(1)})\frac{\partial}{\partial u_{,2}} \\
& + \eta_{[11]}(x_1, x_2, u, \mathbf{u}^{(1)}, \mathbf{u}^{(2)})\frac{\partial}{\partial u_{,11}} + \eta_{[12]}(x_1, x_2, u, \mathbf{u}^{(1)}, \mathbf{u}^{(2)})\frac{\partial}{\partial u_{,12}} \\
& + \eta_{[22]}(x_1, x_2, u, \mathbf{u}^{(1)}, \mathbf{u}^{(2)})\frac{\partial}{\partial u_{,22}},
\end{aligned}
\tag{2.87}
$$

where $u_{,i} = \dfrac{\partial u}{\partial x_i}$, $u_{,ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, and

$$
\begin{aligned}
\eta_{[i]} = {} & \frac{\partial \eta}{\partial x_i} + \frac{\partial \eta}{\partial u}u_{,i} - \left(\frac{\partial \xi^1}{\partial x_i} + \frac{\partial \xi^1}{\partial u}u_{,i}\right)u_{,1} - \left(\frac{\partial \xi^2}{\partial x_i} + \frac{\partial \xi^2}{\partial u}u_{,i}\right)u_{,2}, \\
\eta_{[ij]} = {} & \frac{\partial^2 \eta}{\partial x_i \partial x_j} + \frac{\partial^2 \eta}{\partial x_i \partial u}u_{,j} + \frac{\partial^2 \eta}{\partial x_j \partial u}u_{,i} - \frac{\partial^2 \xi^1}{\partial x_i \partial x_j}u_{,1} - \frac{\partial^2 \xi^2}{\partial x_i \partial x_j}u_{,2} \\
& + \frac{\partial^2 \eta}{\partial u^2}u_{,i}u_{,j} - \left(\frac{\partial^2 \xi^1}{\partial x_i \partial u}u_{,1} + \frac{\partial^2 \xi^2}{\partial x_i \partial u}u_{,2}\right)u_{,j} \\
& - \left(\frac{\partial^2 \xi^1}{\partial x_j \partial u}u_{,1} + \frac{\partial^2 \xi^2}{\partial x_j \partial u}u_{,2}\right)u_{,i} - \left(\frac{\partial \xi^1}{\partial u}u_{,1} + \frac{\partial \xi^2}{\partial u}u_{,2}\right)u_{,ij} \\
& - \left(\frac{\partial^2 \xi^1}{\partial u^2}u_{,1} + \frac{\partial^2 \xi^2}{\partial u^2}u_{,2}\right)u_{,i}u_{,j} + \frac{\partial \eta}{\partial u}u_{ij} - \frac{\partial \xi^1}{\partial x_j}u_{,1i} - \frac{\partial \xi^1}{\partial x_i}u_{,1j} \\
& - \frac{\partial \xi^2}{\partial x_j}u_{,2i} - \frac{\partial \xi^2}{\partial x_i}u_{,2j} - \frac{\partial \xi^1}{\partial u}(u_{,i}u_{,1j} + u_{,j}u_{,1i}) \\
& - \frac{\partial \xi^2}{\partial u}(u_{,i}u_{,2j} + u_{,j}u_{,2i})
\end{aligned}
\tag{2.88}
$$

$(i, j = 1, 2)$.

The invariance condition reads

$$
\begin{aligned}
\Xi^{(2)}(\Delta)\Big|_{\Delta=0} = {} & \\
= {} & \frac{\partial \eta}{\partial x_1} - \frac{\partial^2 \eta}{\partial x_2^2} - \left(2\frac{\partial^2 \eta}{\partial x_2 \partial u} + \frac{\partial \xi^2}{\partial x_1} - \frac{\partial^2 \xi^2}{\partial x_2^2}\right)u_{,2} \\
& + \left(2\frac{\partial^2 \xi^2}{\partial x_2 \partial u} - \frac{\partial^2 \eta}{\partial u^2}\right)u_{,2}^2 + \frac{\partial^2 \xi^2}{\partial u^2}u_{,2}^3 + 2\frac{\partial \xi^1}{\partial x_2}u_{,12} \\
& + 2\frac{\partial \xi^1}{\partial u}u_{,2}u_{,12} + \left(2\frac{\partial \xi^2}{\partial x_2} - \frac{\partial \xi^1}{\partial x_1} + \frac{\partial^2 \xi^1}{\partial x_2^2}\right)u_{,22} \\
& + \left(2\frac{\partial^2 \xi^1}{\partial x_2 \partial u} + 2\frac{\partial \xi^2}{\partial u}\right)u_{,2}u_{,22} + \frac{\partial^2 \xi^1}{\partial u^2}u_{,2}^2 u_{,22} = 0,
\end{aligned}
\tag{2.89}
$$

where the equation (2.86) has been used to express $u_{,1}$ in terms of $u_{,22}$. A polynomial in the derivatives $u_{,2}$, $u_{,12}$ and $u_{,22}$ whose coefficients, involving the infinitesimals and their partial derivatives, must vanish. By integrating the corresponding determining equations, we obtain the solution:

$$\begin{aligned}
\xi^1(x_1, x_2, u) &= a_1 + 2a_3x_1 + a_4x_1^2,\\
\xi^2(x_1, x_2, u) &= a_2 + a_3x_2 + a_4x_1x_2 + a_5x_1,\\
\eta(x_1, x_2, u) &= -\frac{a_4}{4}(x_2^2 + 2x_1)u - \frac{a_5}{2}x_2u + a_6u + f(x_1, x_2),
\end{aligned} \tag{2.90}$$

where $a_1, \dots, a_6$ are arbitrary constants, and $f(x_1, x_2)$ a solution of equation (2.86), *i.e.*,

$$f_{,1} - f_{,22} = 0. \tag{2.91}$$

In this case, the generators live in an infinite–dimensional vector space (because of the occurrence of the function f) that it is also a Lie algebra. It is spanned by the infinitesimal operators:

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2},\\
&\Xi_3 = 2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},\\
&\Xi_4 = x_1^2\frac{\partial}{\partial x_1} + x_1x_2\frac{\partial}{\partial x_2} - \frac{x_2^2 + 2x_1}{4}u\frac{\partial}{\partial u},\\
&\Xi_5 = x_1\frac{\partial}{\partial x_2} - \frac{x_2u}{2}\frac{\partial}{\partial u},\\
&\Xi_6 = u\frac{\partial}{\partial u}, \qquad \Xi_f = f(x_1, x_2)\frac{\partial}{\partial u}.
\end{aligned} \tag{2.92}$$

By computing the Lie brackets of all the possible couples of generators, we get the following list of non–zero commutators:

$$\begin{aligned}
&[\Xi_1, \Xi_3] = 2\Xi_1, && [\Xi_1, \Xi_4] = \Xi_3 - \frac{1}{2}\Xi_6\\
&[\Xi_1, \Xi_5] = \Xi_2, && [\Xi_2, \Xi_3] = \Xi_2,\\
&[\Xi_2, \Xi_4] = \Xi_5, && [\Xi_2, \Xi_6] = -\frac{1}{2}\Xi_6,\\
&[\Xi_3, \Xi_4] = 2\Xi_4, && [\Xi_3, \Xi_5] = \Xi_5,
\end{aligned} \tag{2.93}$$

and

$$\begin{aligned}
&[\Xi_1, \Xi_f] = \Xi_{f_{,1}}, && [\Xi_2, \Xi_f] = \Xi_{f_{,2}},\\
&[\Xi_3, \Xi_f] = \Xi_g, && [\Xi_4, \Xi_f] = \Xi_h,\\
&[\Xi_5, \Xi_f] = -\Xi_m, && [\Xi_6, \Xi_f] = \Xi_{-f},
\end{aligned} \tag{2.94}$$

where

$$\begin{aligned}
g(x_1, x_2) &= 2x_1f_{,1} + x_2f_{,2},\\
h(x_1, x_2) &= x_1^2f_{,1} + x_1x_2f_{,2} + \frac{x_2^2 + 2x_1}{4}f,\\
m(x_1, x_2) &= x_1f_{,2} + \frac{x}{2}f.
\end{aligned} \tag{2.95}$$

The Lie brackets in (2.94) provide generators still belonging to the Lie algebra, since $-f$, $f_{,1}$, $f_{,2}$, and the functions $g(x_1, x_2)$, $h(x_1, x_2)$, $m(x_1, x_2)$, defined by (2.95), satisfy the Fourier equation.

2.8 Use of Lie symmetries of differential equations

The knowledge of Lie groups of transformations admitted by a given system of differential equations can be used

- to lower the order or possibly reduce the equation to quadrature, in the case of ordinary differential equations;
- to determine particular solutions, called *invariant solutions*, or generate new solutions, once a special solution is known, in the case of ordinary or partial differential equations.

2.8.1 Lowering the order of ODE's

Lie showed that if a given ordinary differential equation admits a one–parameter Lie group of point transformations then the order of the equation can be lowered by one. Hence, the solution of the reduced equation and a quadrature provide the solution of the original ordinary differential equation. If a given ordinary differential equation admits an r–parameter Lie group of point transformations, the order of the equation can be lowered by r if the corresponding Lie algebra is solvable: in this case the solution of the original ordinary differential equation is found by solving the reduced equation plus r quadratures [8]; remarkably, one does not need to determine all the intermediate ordinary differential equations of decreasing order. For a first order ordinary differential equation the Lie's method yields the quadrature of the ordinary differential equation, or, equivalently, enables us to find a first integral or an integrating factor.

Now, let us consider the application of Lie groups of point transformations to the study of a second or higher order ordinary differential equation

$$u_{,n} = f(x, u, u_{,1}, \dots, u_{,n-1}), \qquad n \geq 2, \tag{2.96}$$

which, from a geometrical point of view, defines an $(n+1)$–dimensional manifold in the space $(x, u, u_{,1}, \dots, u_{,n})$.

Let us assume that the ordinary differential equation (2.96) admits a one–parameter Lie group of point transformations

$$\begin{aligned} x^\star &= x + a\xi(x, u) + O(a^2), \\ u^\star &= u + a\eta(x, u) + O(a^2) \end{aligned} \tag{2.97}$$

with infinitesimal operator

$$\Xi = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u}. \tag{2.98}$$

In general, the lowering of the order of the n–th order ordinary differential equation (2.96) can be obtained either by introducing the canonical variables or by constructing the differential invariants, the latter being invariants of the prolonged infinitesimal generators.

Reduction of order through canonical variables

Theorem 2.8.1. *Suppose a nontrivial one–parameter Lie group of transformations (2.97), with infinitesimal generator (2.98), is admitted by an n–th order ordinary* *differential equation* (2.96)*,* $n \geq 2$*. Let* $r(x,u)$*,* $s(x,u)$ *be the corresponding canonical coordinates satisfying* $\Xi(r) = 0$*,* $\Xi(s) = 1$*. Then the* n*–th order ordinary differential equation* (2.96) *reduces to an* $(n-1)$*–th order ordinary differential equation,*

$$\frac{d^{n-1}z}{dr^{n-1}} = G\left(r, z, \frac{dz}{dr}, \ldots, \frac{d^{n-2}z}{dr^{n-2}}\right), \tag{2.99}$$

where

$$z = \frac{ds}{dr}. \tag{2.100}$$

Proof. In terms of the canonical coordinates $r(x,u)$ and $s(x,u)$ it is

$$\frac{ds}{dr} = \frac{s_x + s_u u_{,1}}{r_x + r_u u_{,1}}, \tag{2.101}$$

where the subscripts x and u denote the partial derivatives.

Due to

$$\Xi(r) = \xi r_x + \eta r_u = 0,$$

relation (2.101) is nonsingular if $u_{,1} \neq \eta/\xi$.

By differentiating (2.101), we get

$$\begin{aligned}\frac{d^2s}{dr^2} &= \frac{1}{r_x + r_u u_{,1}} \frac{d\left(\frac{s_x + s_u u_{,1}}{r_x + r_u u_{,1}}\right)}{dx} = \\ &= u_{,2} f_1\left(r, s, \frac{ds}{dr}\right) + g_1\left(r, s, \frac{ds}{dr}\right),\end{aligned} \tag{2.102}$$

where

$$\begin{aligned}f_1\left(r, s, \frac{ds}{dr}\right) &= \frac{s_u r_x - s_x r_u}{(r_x + r_u u_{,1})^3},\\ g_1\left(r, s, \frac{ds}{dr}\right) &= \frac{1}{r_x + r_u u_{,1}}\left(u_{,1}^3 (r_u s_{uu} - s_u r_{uu})\right.\\ &\quad + u_{,1}^2 (2 r_u s_{xu} + r_x s_{uu} - 2 s_u r_{xu} - s_x r_{uu})\\ &\quad \left.+ u_{,1}(2 r_x s_{xu} + r_u s_{xu} - s_u r_{xx}) + (r_x s_{xx} - s_x r_{xx})\right).\end{aligned}$$

Solving (2.101) with respect to $u_{,1}$, we have

$$u_{,1} = \frac{s_x - r_x \frac{ds}{dr}}{r_u \frac{ds}{dr} - s_u},$$

that, used in (2.102), gives

$$u_{,2} = \frac{d^2s}{dr^2} F_1\left(r, s, \frac{ds}{dr}\right) + G_1\left(r, s, \frac{ds}{dr}\right),$$

with $F_1 = 1/f_1$ and $G_1 = -g_1/f_1$. Since r and s are canonical coordinates, it is $r_x s_u - r_u s_x \neq 0$ and hence $f_1 \neq 0$. Proceeding inductively, we have

$$\frac{d^k s}{dr^k} = u_{,k} f_{k-1}\left(r, s, \frac{ds}{dr}\right) + g_{k-1}\left(r, s, \frac{ds}{dr}, \ldots, \frac{d^{k-1}s}{dr^{k-1}}\right)$$

for some function $g_{k-1}\left(r, s, \dfrac{ds}{dr}, \dots, \dfrac{d^{k-1}s}{dr^{k-1}}\right)$ with

$$f_{k-1}\left(r, s, \frac{ds}{dr}\right) = \frac{r_x s_u - r_u s_x}{(r_x + r_u u_{,1})^{k+1}}, \qquad k \geq 2.$$

This leads us to write

$$u_{,k} = \frac{d^k s}{dr^k} = F_{k-1}\left(r, s, \frac{ds}{dr}\right) + G_{k-1}\left(r, s, \frac{ds}{dr}, \dots, \frac{d^{k-1}s}{dr^{k-1}}\right),$$

where

$$F_{k-1} = \frac{1}{f_{k-1}}, \qquad G_{k-1} = -\frac{g_{k-1}}{f_{k-1}}, \qquad k \geq 2.$$

Finally, the original ordinary differential equation may be written in the normal form

$$\frac{ds^n}{dr^n} = F\left(r, s, \frac{ds}{dr}, \dots, \frac{d^{n-1}s}{dr^{n-1}}\right)$$

for some function $F\left(r, s, \dfrac{ds}{dr}, \dots, \dfrac{d^{n-1}s}{dr^{n-1}}\right)$. Since (2.99) is invariant with respect to the translation in s, the function F does not depend on s. This allows us to introduce $z = \dfrac{ds}{dr}$ and write (2.96) in the form (2.99) as an $(n-1)$–th order ordinary differential equation. □

Remark 2.8.1. *If*

$$z = \phi(r; C_1, \dots, C_{n-1})$$

($C_1, \dots, C_{n-1}$ arbitrary constants) is the general solution of the equation (2.99), then the quadrature

$$s(x, u) = \int^{r(x,u)} \phi(t; C_1, \dots, C_{n-1}) dt + C_n$$

gives the general solution of (2.96), where also C_n is an arbitrary constant.

Reduction of order through differential invariants

The equation

$$F(x, u, u_{,1}, \dots, u_{,n}) \equiv u_{,n} - f(x, u, u_{,1}, \dots, u_{,n-1}) = 0$$

admits the Lie point symmetry corresponding to the infinitesimal generator (2.98) if and only if

$$\Xi^{(n)}(F)\Big|_{F=0} = 0.$$

Therefore, the function F has to depend on the group's invariants

$$\omega(x, u), \quad \psi_1(x, u, u_{,1}), \dots, \psi_n(x, u, u_{,1}, \dots, u_{,n}),$$

where

$$\Xi(\omega(x, u)) = 0, \quad \Xi^{(k)}(\psi_k(x, u, u_{,1}, \dots, u_{,k})) = 0,$$

with

$$\frac{\partial \psi_k}{\partial u_{,k}} \neq 0, \qquad k = 1, \dots, n.$$

The group's invariants are found by integrating the characteristic equations

$$\frac{dx}{\xi(x,u)} = \frac{du}{\eta(x,u)} = \frac{du_{,1}}{\eta^{(1)}(x,u,u_{,1})} = \ldots = \frac{du_{,k}}{\eta^{(k)}(x,u,u_{,1},\ldots,u_{,k})}.$$

In order to find the group's invariants we need to determine only $\omega(x,u)$ and $\psi_1(x,u,u_{,1})$ by solving

$$\frac{dx}{\xi(x,u)} = \frac{du}{\eta(x,u)} = \frac{du_{,1}}{\eta_x + (\eta_u - \xi_x)u_{,1} - \xi_u u_{,1}^2}.$$

In fact, since $\omega(x,u)$ and $\psi_1(x,u,u_{,1})$ are invariants under the action of the n–th extended group, it follows that $d\psi_1/d\omega$ is an invariant, and so are $d^2\psi_1/d\omega^2, \ldots, d^{n-1}\psi_1/d\omega^{n-1}$.

In terms of the differential invariants the n–th order ordinary differential equation has order $(n-1)$, *i.e.*,

$$\frac{d^{n-1}\psi}{d\omega^{n-1}} = H\left(\omega, \psi, \frac{d\psi}{d\omega}, \ldots, \frac{d^{n-2}\psi}{d\omega^{n-2}}\right)$$

where we set $\psi = \psi_1$, for some function H.

2.8.2 Invariant solutions of PDE's

The function $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$, with components $u_A = \Theta_A(\mathbf{x})$ $(A = 1,\ldots,m)$, is said to be an *invariant solution* of

$$\boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}\right) = \mathbf{0} \tag{2.103}$$

if $u_A = \Theta_A(\mathbf{x})$ is an invariant surface of (2.34), and is a solution of (2.103), *i.e.*, a solution is invariant if and only if:

$$\begin{aligned} &\Xi\,(u_A - \Theta_A(\mathbf{x})) = 0 \quad \text{for} \quad u_A = \Theta_A(\mathbf{x}), \qquad A = 1,\ldots,m, \\ &\text{when} \quad \boldsymbol{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(k)}\right) = \mathbf{0}. \end{aligned} \tag{2.104}$$

The equations $(2.104)_1$, called *invariant surface conditions*, have the form

$$\xi^1(\mathbf{x},\mathbf{u})\frac{\partial u_A}{\partial x_1} + \cdots + \xi^n(\mathbf{x},\mathbf{u})\frac{\partial u_A}{\partial x_n} = \eta^A(\mathbf{x},\mathbf{u}), \quad A = 1,\ldots,m, \tag{2.105}$$

and are solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi^1(\mathbf{x},\mathbf{u})} = \cdots = \frac{dx_n}{\xi^n(\mathbf{x},\mathbf{u})} = \frac{du_1}{\eta^1(\mathbf{x},\mathbf{u})} = \cdots = \frac{du_m}{\eta^m(\mathbf{x},\mathbf{u})}. \tag{2.106}$$

This allows to express the solution $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ (that may be given in implicit form if some of the infinitesimals ξ^i depend on $\mathbf{u}$) as

$$u_A = \psi_A(I_1(\mathbf{x},\mathbf{u}),\ldots,I_{n-1}(\mathbf{x},\mathbf{u})), \qquad A = 1,\ldots,m, \tag{2.107}$$

by substituting (2.107) into $(2.104)_2$, a reduced system of differential equations involving $(n-1)$ independent variables (often called *similarity variables*) is obtained. The name *similarity variables* is due to the fact that the scaling invariance, *i.e.*, the invariance under similarity transformations, was one of the first examples where this procedure has been used systematically.

Note that $I_1, I_2, \dots, I_{n-1}, u_1, \dots, u_m$ are invariants of the given group, and belong to the set of canonical coordinates. By considering also the variable I_n, satisfying

$$\Xi(I_n) = 1, \tag{2.108}$$

we have a complete set of canonical variables for the group characterized by the generator Ξ; nevertheless, the determination of I_n is not required in this context, because it does not appear explicitly in the reduced system.

If $n = 2$, *i.e.*, the system has two independent variables, the reduced system involves only one independent variable, whereupon it is an ordinary differential system.

Since differential equations can admit more than one symmetry, there are different ways to choose a set of similarity variables by starting from different symmetries. It is also possible to achieve a multiple reduction of variables by using multiple–parameter groups of transformations. When this is possible, there are essentially two ways to obtain such a multiple reduction of independent variables: repeating step by step the procedure used in the case of one–parameter Lie groups for each subgroup considered, or performing the reduction all–in–one.

Reducing step by step the number of variables means performing the following:

1. take a generator of a symmetry (say, Ξ, written in terms of the variables involved in the system $\boldsymbol{\Delta} = \mathbf{0}$) and build the associated reduction;

2. write the original system of differential equations $\boldsymbol{\Delta} = \mathbf{0}$ in terms of the invariants, thus obtaining the reduced system $\widehat{\boldsymbol{\Delta}} = \mathbf{0}$;

3. if a further reduction is wanted, set $\boldsymbol{\Delta} = \widehat{\boldsymbol{\Delta}}$, and go to step 1.

This method works only if each considered symmetry is admitted by the system where the reduction is performed; of course, this is true for the first symmetry considered, but for the subsequent steps this is true only if the symmetry (written in terms of the invariants of previous symmetry) is inherited by the reduced system. In the next example, we will show the procedure to obtain a multiple reduction of variables for a second order partial differential equation involving four independent variables.

Example 2.8.1. *The linear wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

(interpreting t as the time and x_i $(i = 1, 2, 3)$ the spatial coordinates) admits, among the others, the scaling invariance

$$\Xi_1 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}.$$

The integration of (2.105) leads to the similarity variables

$$y_1 = \frac{x_1}{t}, \quad y_2 = \frac{x_2}{t}, \quad y_3 = \frac{x_3}{t}.$$

Taking $u = u(y_1, y_2, y_3)$, the wave equation becomes a differential equation involving three independent variables:

$$\frac{\partial^2 u}{\partial y_i \partial y_k}(\delta_{ik} - y_i y_k) - 2\frac{\partial u}{\partial y_i} y_i = 0.$$

Now, let us consider the Lie point symmetry

$$\Xi_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},$$

representing the rotation in the $x_1 x_2$ plane. In terms of the new variables y_i, it reads

$$\Xi_2 = \Xi_2(y_1)\frac{\partial}{\partial y_1} + \Xi_2(y_2)\frac{\partial}{\partial y_2} = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2},$$

which is a symmetry of the reduced wave equation. Its similarity variables are $s = y_3$ and $v = y_1^2 + y_2^2 = (x_1^2 + x_2^2)/t^2$ so that solutions $w = w(s, v)$ may exist and are governed by the reduced equation

$$4v(1 - v)\frac{\partial^2 w}{\partial v^2} - 4vs\frac{\partial^2 w}{\partial v \partial s} - (1 - s^2)\frac{\partial^2 w}{\partial s^2} + (4 - 6v)\frac{\partial w}{\partial v} - 2s\frac{\partial w}{\partial s} = 0.$$

The infinitesimal operator

$$\Xi_3 = t\frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial t}$$

in terms of variables v, s and w reads

$$\Xi_3 = \Xi_3(s)\frac{\partial}{\partial s} + \Xi_3(v)\frac{\partial}{\partial v} + \Xi_3(w)\frac{\partial}{\partial w} = (1 - s^2)\frac{\partial}{\partial s} - 2vs\frac{\partial}{\partial v}.$$

Its similarity variable is $\sigma = v/(1-s^2) = (x_1^2+x_2^2)/(t^2-x_3^2)$ and, in this variable, the similarity solution $w(\sigma)$ satisfies the equation

$$\sigma\frac{\partial^2 w}{\partial \sigma^2} + \frac{\partial w}{\partial \sigma} = 0.$$

This ordinary differential equation admits the generator $\Xi_4 = \partial/\partial w$ (but also the symmetry $\sigma\partial/\partial\sigma$ not inherited from the wave equation), so it can be integrated yielding the particular solution

$$u = w = a_1 + a_2 \log \sigma = a_1 + a_2 \log \frac{x_1^2 + x_2^2}{t^2 - x_3^2}$$

of the wave equation.

2.8.3 New solutions from a known solution

The consideration that, under the action of a Lie group of transformations admitted by a differential equation, a solution, which is not invariant with respect to the group, is mapped into a family of solutions, suggests a way of generating new solutions from a known solution. This is especially interesting when one can obtain nontrivial solutions from trivial ones.

Let us consider a one–parameter Lie group of transformations

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \qquad \mathbf{u}^\star = \mathbf{U}(\mathbf{x}, \mathbf{u}; a), \tag{2.109}$$

admitted by a system of differential equations $\mathcal{E}$, and let

$$\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x}) \tag{2.110}$$

be a solution of the given system $\mathcal{E}$, which is not invariant with respect to the group (2.109).

The transformation (2.109) maps a point $(\mathbf{x}, \boldsymbol{\Theta}(\mathbf{x}))$ of the solution $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ into the point $(\mathbf{x}^\star, \mathbf{u}^\star)$ characterized by:

$$\mathbf{x}^\star = \mathbf{X}(\mathbf{x}, \boldsymbol{\Theta}(\mathbf{x}); a), \qquad \mathbf{u}^\star = \mathbf{U}(\mathbf{x}, \boldsymbol{\Theta}(\mathbf{x}); a). \tag{2.111}$$

For a fixed value of the parameter a, one can eliminate $\mathbf{x}$ from (2.111) by substituting the inverse transformation of $(2.111)_1$,

$$\mathbf{x} = \mathbf{X}(\mathbf{x}^\star, \mathbf{u}^\star; -a), \tag{2.112}$$

into $(2.111)_2$ thus obtaining

$$\mathbf{u}^\star = \mathbf{U}(\mathbf{X}(\mathbf{x}^\star, \mathbf{u}^\star; -a), \boldsymbol{\Theta}(\mathbf{X}(\mathbf{x}^\star, \mathbf{u}^\star; -a)); a). \tag{2.113}$$

Finally, by substituting $(\mathbf{x}^\star, \mathbf{u}^\star; -a)$ with $(\mathbf{x}, \mathbf{u}; a)$ in (2.113), one may state the following theorem.

Theorem 2.8.2. *If $\mathbf{u} = \boldsymbol{\Theta}(\mathbf{x})$ is not an invariant solution of a system $\mathcal{E}$ of differential equations, admitting the group (2.109), then*

$$\mathbf{u} = \mathbf{U}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a), \boldsymbol{\Theta}(\mathbf{X}(\mathbf{x}, \mathbf{u}; a)); -a) \tag{2.114}$$

implicitly defines a one–parameter family of solutions of the given system.

In the next example, a simple application of the previous procedure is given.

Example 2.8.2. *The linear heat equation (2.86) admits the group with the generator*

$$\Xi = x_1^2 \frac{\partial}{\partial x_1} + x_1 x_2 \frac{\partial}{\partial x_2} - \frac{x_2^2 + 2x_1}{4} u \frac{\partial}{\partial u}.$$

By integrating the Lie equations, the related finite transformation is:

$$x_1^\star = \frac{x_1}{1 - a x_1}, \qquad x_2^\star = \frac{x_2}{1 - a x_1},$$
$$u^\star = u\sqrt{1 - a x_1}\, \exp\left(-\frac{a x_2^2}{4(1 - a x_1)}\right).$$

One can obtain the inverse transformation by exchanging (x_1, x_2, u) and $(x_1^\star, x_2^\star, u^\star)$ and replacing a by $-a$:

$$x_1 = \frac{x_1^\star}{1 + a x_1^\star}, \qquad x_2 = \frac{x_2^\star}{1 + a x_1^\star},$$
$$u = u^\star\sqrt{1 + a x_1^\star}\, \exp\left(\frac{a x_2^{\star 2}}{4(1 + a x_1^\star)}\right).$$

By applying this transformation to the trivial solution $u = A$ (A constant), the nontrivial solution

$$u = \frac{A}{\sqrt{1 + ax_1}} \exp\left(-\frac{ax_2^2}{4(1 + ax_1)}\right)$$

is immediately generated.

Besides reducing the order of an ordinary differential equation or finding invariant solutions, the Lie symmetries of partial differential equations, having a suitable algebraic structure, can be used to construct invertible point transformations in order to map a source system of partial differential equations into equivalent forms [6, 8, 22, 66]; in this thesis, we will use such an approach, and prove some theorems that extend some well known results.

3 First order systems of PDE's

In this Chapter, we briefly recall some introductory facts about first order systems of partial differential equations with special emphasis on quasilinear ones. We review some results related to the Lie symmetries admitted by such systems allowing for their transformation into autonomous and/or homogeneous or linear form. All these procedures are constructive and the new independent and dependent variables are obtained by introducing the canonical variables of the admitted Lie symmetries. Some of these results are extended in Chapter 5 of this thesis to introduce invertible point transformations allowing one to map nonlinear systems of first order partial differential equations which are polynomial in the derivatives to first order systems being polynomially homogeneous in the derivatives.

3.1 General considerations on first order PDE's

Many physical problems are often expressed mathematically by systems of partial differential equations (PDE's) in the form of balance laws [15, 20],

$$\sum_{i=1}^{n} \frac{\partial \mathbf{F}^i(\mathbf{u})}{\partial x_i} = \mathbf{B}(\mathbf{u}), \tag{3.1}$$

where $\mathbf{u} \in \mathbb{R}^m$ denotes the set of unknown fields, $\mathbf{x} \in \mathbb{R}^n$ the set of independent variables, $\mathbf{F}^i(\mathbf{u})$ the components of a flux, and $\mathbf{B}(\mathbf{u})$ the production term; when $\mathbf{B}(\mathbf{u}) \equiv \mathbf{0}$, we have a system of conservation laws. In dynamical systems, the first component x_1 of the independent variables is the time, and the components of $\mathbf{F}^1$ are the densities of some physical quantities. The presence of the source terms in systems in divergence form implies additional mathematical difficulties in solving various problems. For instance, from a numerical point of view, the presence of source terms may require fractional step splitting methods where one alternates between solving a homogeneous system of conservation laws and an ordinary differential system obtained from the system of balance laws by dropping the terms involving space derivatives. It is known [53] that for some type of problems fractional step splitting methods perform quite poorly. Systems like (3.1) fall in the more general class of nonhomogeneous quasilinear first order systems of partial differential equations:

$$\sum_{i=1}^{n} A^i(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{u}), \tag{3.2}$$

where A^i $(i = 1, \dots, n)$ are $m \times m$ matrices with entries depending on the field $\mathbf{u}$.

Special problems of physical interest (see [2, 18, 19, 48, 65, 78, 81] for some examples) may require to consider systems where the coefficients may depend also on the independent variables $\mathbf{x}$, accounting for material inhomogeneities, or particular geometric assumptions, or external actions, so that in some applications one may need to consider nonautonomous and/or nonhomogeneous quasilinear systems of the form

$$\sum_{i=1}^{n} A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}). \tag{3.3}$$

In dealing with conservation laws (or, more in general, with autonomous and homogeneous quasilinear first order systems of partial differential equations), one has the invariance with respect to uniform stretching of the independent variables, and this induces the existence of self–similar solutions.

For instance, the system of conservation laws in $(1+1)$–dimensions

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} = \mathbf{0} \tag{3.4}$$

(where we denote the independent variables with t and x) is invariant under uniform stretching of coordinates: $(x,t) \mapsto (ax, at)$ $(a \in \mathbb{R}^+)$; hence, it admits self–similar solutions, defined on the space–time plane and constant along straight–line rays emanating from the origin. Since (3.4) is also invariant under translations of coordinates, $(x,t) \mapsto (x + x_0, t + t_0)$, the focal point of self–similar solutions may be translated from the origin to any fixed point $(\overline{x}, \overline{t})$ in space–time. If $\mathbf{u}$ is a self–similar solution of (3.4), focused at the origin, it admits the representation

$$\mathbf{u}(x,t) = \mathbf{U}(\xi), \quad \xi = \frac{x}{t}, \qquad -\infty < x < \infty, \quad t > 0, \tag{3.5}$$

where $\mathbf{U}(\xi)$ satisfies the system of ordinary differential equations

$$[\mathbf{F}(\mathbf{U}(\xi)) - \xi \mathbf{U}(\xi)]' + \mathbf{U}(\xi) = \mathbf{0}, \tag{3.6}$$

and the prime denotes differentiation with respect to ξ.

Simple instances of first order quasilinear systems are 2×2 homogeneous and autonomous systems, widely used to model one–dimensional nonlinear wave processes through non–dissipative and homogeneous media:

$$A^0(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial t} + A^1(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{3.7}$$

where A^0 and A^1 are 2×2 matrices.

The investigation of systems of the form (3.7) which are hyperbolic in the t–direction may involve the consideration of Riemann invariants [21, 51], as well as the use of the hodograph transformation; remarkably, in the conservative case, the Riemann problem, *i.e.*, the solution with piecewise constant initial data having a single discontinuity (see [20, 86] for details), may be solved.

On the contrary, in dealing with nonhomogeneous and/or dissipative media, we often have to deal with 2×2 quasilinear systems like

$$A^0(t, x, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial t} + A^1(t, x, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{B}(t, x, \mathbf{u}), \tag{3.8}$$

where $\mathbf{B}(t, x, \mathbf{u})$ is a known column vector.

Different physical contexts involve models of the class (3.8): viscoelastic materials [18], nonlinear elastic rods with variable cross–section [48], nonlinear heat conduction problems [81], flows in fluid filled elastic tubes [2, 78], problems with cylindrical or spherical symmetry.

Due to the nonhomogeneous and/or nonautonomous form, one loses the possibility to have in general the hodograph transformation, the Riemann invariants, the invariance with respect to homogeneous scaling of independent variables (no centered waves!), the latter implying that the Riemann problem can not be solved analytically.

Various special techniques are known to map some nonhomogeneous 2×2 systems to homogeneous form, involving Bäcklund transformations [82], hodograph–like transformations [92], transformations via a solution [83].

3.2 Transformations of differential equations

Lie point symmetries of differential equations can be used to construct a mapping from a given (*source*) system of differential equations to another (*target*) suitable system of differential equations that turns out to be equivalent [6, 19, 22, 23, 25, 26, 39, 65, 66, 80]. Such a mapping (if it exists) needs not be a group transformation; moreover, any infinitesimal generator admitted by the source system of differential equations has to be mapped to an infinitesimal generator admitted by the target system of differential equations [8].

If the mapping is one–to–one (invertible) then the mapping must establish a one–to–one correspondence between infinitesimal generators of the source and target system of differential equations. In other words, the Lie algebra of infinitesimal operators of the target system of differential equations has to be isomorphic to the Lie algebra of infinitesimal operators of the source system of differential equations. On the contrary, if the mapping from the source system to the target one is allowed to be non–invertible, then it is not necessary that a one–to–one correspondence between Lie algebras of infinitesimal operators of source and target systems of differential equations exists.

But such a non–invertible mapping must take any infinitesimal operator admitted by the source system into an infinitesimal operator (which could be a null operator) admitted by the target one. More precisely, the mapping must establish a homomorphism between any Lie algebra of infinitesimal operators of the source system and a Lie algebra of infinitesimal operators of the target system.

In this context, the algebraic structure of the admitted Lie symmetries is crucial. Here, we present some well known results related to the transformation of a system of differential equations in autonomous form, the transformation of a nonlinear first order system of partial differential equations to linear form, and the transformation of quasilinear first order systems of partial differential equations to homogeneous and autonomous form. From all the results that are recalled below, the fundamental role of the canonical variables for the admitted Lie symmetries clearly emerges. Similar techniques will be used in Chapter 5 for proving a couple of new theorems concerned with the transformation of general first order nonlinear partial differential equations.

3.2.1 Reduction to autonomous form

Let us consider the case of transformation to autonomous form, *i.e.*, a form in which the independent variables do not appear explicitly. In [22, 26] necessary and sufficient conditions for reducing a system of partial differential equations to autonomous form have been given. The problem requires to map a general system of differential equations of order k

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right) = \mathbf{0}, \tag{3.9}$$

into an equivalent autonomous system, say

$$\widetilde{\mathbf{\Delta}}\left(\mathbf{w}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}\right) = \mathbf{0}, \tag{3.10}$$

where $\mathbf{w} \equiv (w_1, \ldots, w_m)$ denote the new dependent variables. This reduction, when it is possible, is performed by an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{3.11}$$

whose construction is algorithmically suggested by the Lie symmetries admitted by (3.9).

Every autonomous system of differential equations of the form (3.10) is invariant with respect to the n translations of the independent variables, *i.e.*, it admits the Lie point symmetries generated by the following vector fields

$$\Xi_i = \frac{\partial}{\partial z_i}, \qquad i = 1, \ldots, n, \tag{3.12}$$

spanning an n–dimensional Abelian Lie algebra. Since the Lie bracket of two infinitesimal generators of symmetries is not affected by an invertible change of coordinates, it follows that if a general system of the form (3.9) can be mapped by (3.11) to the autonomous form (3.10), it has to admit, as subalgebra of the Lie algebra of its point symmetries, an n–dimensional Lie algebra with a suitable algebraic structure. These conditions are also sufficient as stated by the next theorem.

Theorem 3.2.1. *[22] The system of differential equations of order k*

$$\mathbf{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}\right) = \mathbf{0}, \tag{3.13}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^m$, can be transformed by an invertible point transformation, say

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{3.14}$$

to the autonomous equivalent form

$$\widetilde{\mathbf{\Delta}}\left(\mathbf{w}, \mathbf{w}^{(1)}, \ldots, \mathbf{w}^{(k)}\right) = 0, \tag{3.15}$$

if and only if it is left invariant by n Lie groups of point transformations whose infinitesimal operators Ξ_i $(i = 1, \ldots, n)$ give a distribution of rank n, and satisfy *the conditions:*

$$[\Xi_i, \Xi_j] = 0, \qquad i, j = 1, \dots, n, \tag{3.16}$$

that is, the operators Ξ_i span an n–dimensional Abelian Lie algebra.

This theorem provides useful, for instance, when one is facing nonlinear propagation of discontinuity waves [11] in states which are not constant, when special geometrical assumptions (spherical or cylindrical symmetry) are intrinsic to the studied problem (for instance, in the case of Navier–Stokes–Fourier equations for a gas in rotation about a fixed axis with a constant angular velocity). Applications can be found, for instance, in [25, 70, 71, 85].

3.2.2 Reduction to linear form

Lie symmetries spanning an infinite–dimensional Lie algebra are essential to transform nonlinear first order partial differential equations to linear form. Here, we restrict ourselves to the case in which:

- both the source and target system of partial differential equations are first order systems;
- the mapping is a one–to–one transformation;
- the target system of partial differential equations is *linear*.

Necessary and sufficient conditions for the existence of invertible mappings linking a nonlinear system of first order partial differential equations with a linear system of differential equations have been given by Kumei and Bluman [39] (see also [8]). However, the proof they give does not seem to involve "natural" conditions. A more "natural" proof has been given in [23], and it involves the introduction of the canonical variables related to some infinitesimal operators whose linear combination (with multipliers given by arbitrary functions that are solution of a linear system of partial differential equations) is an admitted group of the source system. The next theorem provides necessary and sufficient conditions for the transformation to the linear form.

Theorem 3.2.2 ([23])**.** *The nonlinear first order system of partial differential equations*

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = 0, \tag{3.17}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, can be transformed to the linear form

$$L(\mathbf{z})[\mathbf{w}] = \mathbf{B}(\mathbf{z}), \tag{3.18}$$

where $L(\mathbf{z})$ is a linear first order differential operator, by means of the invertible point transformation

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{3.19}$$

if and only if it is left invariant by a Lie group of point transformations whose infinitesimal operator has the form

$$\Xi = \sum_{A=1}^{m} F_A(\mathbf{z})\Xi_A, \tag{3.20}$$

where

$$\Xi_A = \sum_{\alpha=1}^{n} \xi_A^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_\alpha} + \sum_{C=1}^{m} \eta_A^C(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_C}, \tag{3.21}$$

with $\xi_A^\alpha(\mathbf{x}, \mathbf{u})$, $\eta_A^C(\mathbf{x}, \mathbf{u})$ *being specific functions of their arguments, and the functions* $F_A(\mathbf{z})$ *satisfying the linear system*

$$L(\mathbf{z})[F_A] = 0 \tag{3.22}$$

along with the conditions

$$\Xi_A(\mathbf{z}) = \mathbf{0}, \qquad [\Xi_A, \Xi_B] = 0, \qquad A, B = 1, \ldots, m. \tag{3.23}$$

Applications of the Theorem 3.2.2 can be found in [29, 39, 67].

3.2.3 Reduction of quasilinear systems to autonomous and homogeneous form

Another important application of Lie symmetries concerns the reduction of a general quasilinear first order system of partial differential equations to a quasilinear one that is also autonomous and homogeneous. By considering the 2×2 quasilinear systems of the form

$$A^0\left(t, x, \mathbf{u}\right) \frac{\partial \mathbf{u}}{\partial t} + A^1\left(t, x, \mathbf{u}\right) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{B}\left(t, x, \mathbf{u}\right), \tag{3.24}$$

in [19], it has been proved a theorem giving necessary and sufficient conditions in order to map a system of the form (3.24), under the action of the one–to–one point variable transformation like

$$\tau = T\left(t, x\right), \qquad \xi = X\left(t, x\right), \qquad \mathbf{v} = \mathbf{V}\left(t, x, \mathbf{u}\right), \tag{3.25}$$

to the autonomous and homogeneous form

$$\widehat{A}^0\left(\mathbf{v}\right) \frac{\partial \mathbf{v}}{\partial \tau} + \widehat{A}^1\left(\mathbf{v}\right) \frac{\partial \mathbf{v}}{\partial \xi} = \mathbf{0}, \qquad \mathbf{v} = \left[v_1, v_2\right]^T. \tag{3.26}$$

The possibility of reducing (3.24) to autonomous and homogeneous form (3.26) is strictly related to the symmetry properties of the model under investigation. In fact, necessary and sufficient conditions allowing for such a reduction are obtained; remarkably, when the approach here considered is applicable, it is possible to construct explicitly the map transforming nonhomogeneous and nonautonomous 2×2 quasilinear systems to homogeneous and autonomous form. The key idea is that any 2×2 homogeneous and autonomous first order quasilinear system is left invariant by an infinite–parameter Lie group of point transformations [8]. Therefore, if a nonhomogeneous and nonautonomous system can be reduced to homogeneous and autonomous form by an invertible point transformation, an infinite–parameter Lie group has to be admitted by the original system; conversely, if a nonhomogeneous and nonautonomous 2×2 quasilinear system admits a suitable infinite–parameter Lie group of point symmetries, then an invertible map exists transforming it to homogeneous and autonomous form [19]. The following theorem holds.

Theorem 3.2.3 ([19]). *The nonhomogeneous and nonautonomous 2×2 quasilinear system*

$$A^0(t,x,\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + A^1(t,x,\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{B}(t,x,\mathbf{u}) \tag{3.27}$$

transforms, under the action of the one–to–one point variable transformation

$$\tau = T(t,x), \qquad \xi = X(t,x), \qquad \mathbf{v} = \mathbf{V}(t,x,\mathbf{u}), \tag{3.28}$$

to the homogeneous and autonomous form, say

$$\widehat{A}^0(\mathbf{v})\frac{\partial \mathbf{v}}{\partial \tau} + \widehat{A}^1(\mathbf{v})\frac{\partial \mathbf{v}}{\partial \xi} = \mathbf{0}, \qquad \mathbf{v} = [v_1, v_2]^T, \tag{3.29}$$

if and only if the system (3.27) is left invariant by the Lie group of point symmetries

$$\Xi = F_1(\mathbf{v})\Xi_1 + F_2(\mathbf{v})\Xi_2, \tag{3.30}$$

where

$$\Xi_i = \tau_i\frac{\partial}{\partial t} + \xi_i\frac{\partial}{\partial x} + \eta_i^{(1)}\frac{\partial}{\partial u_1} + \eta_i^{(2)}\frac{\partial}{\partial u_2}, \qquad i = 1,2$$

are commuting infinitesimal operators, i.e., $[\Xi_1, \Xi_2] = 0$, *the infinitesimal generators* τ_i, ξ_i, $\eta_i^{(1)}$ *and* $\eta_i^{(2)}$ *may depend on* t, x, u_1 *and* u_2, *whereas* $F_1(\mathbf{v})$ *and* $F_2(\mathbf{v})$ *are solutions of the linear system of partial differential equations*

$$\widehat{A}^1(\mathbf{v})\,J\,\nabla_{\mathbf{v}}F_1(\mathbf{v}) - \widehat{A}^0(\mathbf{v})\,J\,\nabla_{\mathbf{v}}F_2(\mathbf{v}) = \mathbf{0}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{3.31}$$

where $\nabla_{\mathbf{v}}$ *is the gradient operator with respect to the components of the indicated subscript, and it is*

$$\Xi_1 T = 1, \quad \Xi_1 X = 0, \quad \Xi_2 T = 0, \quad \Xi_2 X = 1, \quad \Xi_1\mathbf{v} = \Xi_2\mathbf{v} = 0. \tag{3.32}$$

It is worth of underlining that if the reduction of 2×2 quasilinear first order systems of partial differential equations to homogeneous and autonomous form is intimately related to the possibility of their transformation to linear form, on the contrary, for general first order quasilinear systems involving more than two independent variables and/or more than two dependent variables, this link can not be invoked. Nevertheless, also in this case it is possible to recover the necessary and sufficient conditions allowing for the transformation to homogeneous and autonomous form within the framework of Lie groups analysis of differential equations.

In particular, in [65], it has been proved a theorem providing the necessary and sufficient conditions in order to map a general first order quasilinear system of partial differential equations, say

$$\sum_{i=1}^{n} A^i(\mathbf{x},\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x},\mathbf{u}), \tag{3.33}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, A^i are $m \times m$ matrices with entries depending at most on $\mathbf{x}$ and $\mathbf{u}$, and the source term $\mathbf{B} \in \mathbb{R}^m$ depends at most on $\mathbf{x}$ and $\mathbf{u}$ too, into a first order quasilinear homogeneous and autonomous system. This reduction, when it is possible, is performed by an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{3.34}$$

which preserves the quasilinear structure of the system, and whose construction is algorithmically suggested by the Lie point symmetries admitted by (3.33). We need that the new independent variables depend at most on the old independent variables to be sure that the quasilinear structure is preserved. In fact, if the new independent variables $\mathbf{z}$ are allowed to depend also on the old dependent variables $\mathbf{u}$, then in general it may occur that the target system is not in quasilinear form.

Every first order quasilinear homogeneous and autonomous system is invariant with respect to the independent translations and to a uniform scaling of all the n independent variables, *i.e.*, it admits (whatever is the functional form of the entries of matrices A^i) the Lie point symmetries generated by the following vector fields

$$\begin{aligned} \Xi_i &= \frac{\partial}{\partial z_i}, \qquad i = 1, \dots, n, \\ \Xi_{n+1} &= \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}, \end{aligned} \tag{3.35}$$

spanning an $(n+1)$–dimensional solvable Lie algebra where the only non–zero commutators are

$$[\Xi_i, \Xi_{n+1}] = \Xi_i, \qquad i = 1, \dots, n. \tag{3.36}$$

Since the Lie bracket of two infinitesimal generators of symmetries is not affected by an invertible change of coordinates, it follows that if a system of the form (3.33) can be mapped by (3.34) to the autonomous and homogeneous form, it has to admit, as subalgebra of the Lie algebra of its point symmetries, an $(n+1)$–dimensional Lie algebra with a suitable algebraic structure.

All these considerations can be summarized in the following theorem.

Theorem 3.2.4 ([65]). *A nonhomogeneous and/or nonautonomous first order quasilinear system of the form*

$$\sum_{i=1}^{n} A^i(\mathbf{x}, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{B}(\mathbf{x}, \mathbf{u}), \tag{3.37}$$

can be transformed by the invertible map like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{3.38}$$

into an autonomous and homogeneous first order quasilinear system, say

$$\sum_{i=1}^{n} \widehat{A}^i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}, \tag{3.39}$$

if and only if it admits as subalgebra of the algebra of its Lie point symmetries an $(n+1)$–dimensional Lie algebra spanned by the vector fields

$$\begin{aligned} \Xi_i &= \sum_{j=1}^{n} \xi_i^j(\mathbf{x})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_i^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A}, \qquad i=1,\ldots,n,\\ \Xi_{n+1} &= \sum_{j=1}^{n} \xi_{n+1}^j(\mathbf{x})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_{n+1}^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A}, \end{aligned} \tag{3.40}$$

such that

$$[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_{n+1}] = \Xi_i, \qquad i,j = 1,\ldots,n. \tag{3.41}$$

Furthermore, it has to be verified that all minors of order n extracted from the $(n+1)\times n$ matrix with entries ξ_α^i ($\alpha = 1,\ldots,n+1$; $i = 1,\ldots,n$) are non–vanishing, and the new dependent variables $\mathbf{w}$, which by construction are invariants of $\Xi_1,\ldots,\Xi_n$, are invariants of Ξ_{n+1} too.

Of course, the Theorem 3.2.4 proved in [65] generalizes the Theorem 3.2.3 established in [19] for 2×2 quasilinear first order systems. Both theorems may be applied when we consider a given system of quasilinear partial differential equations and the required hypotheses are fulfilled. The new independent and dependent variables can be found by solving the overdetermined system of first order partial differential equations:

$$\Xi_i z_j = \delta_{ij}, \qquad \Xi_i w_A = 0, \qquad i,j = 1,\ldots,n, \qquad A = 1,\ldots,m, \tag{3.42}$$

for the unknowns $\mathbf{z}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x},\mathbf{u})$; because the operators Ξ_i ($i = 1,\ldots,n$) are commuting, this overdetermined system always admits a solution.

Application of Theorem 3.2.4 requires the following steps:

1. determine the Lie algebra L of point symmetries of system (3.37) (various computer algebra packages are available [3, 38, 42, 43, 44, 68]);
2. if $\dim(L) \geq n+1$:
    - determine the $(n+1)$–dimensional Lie subalgebras (an optimal system [62, 73, 76] suffices);
    - check if among the Lie subalgebras there is a Lie algebra having the required structure;
    - find the canonical variables of the symmetries, and reduce the system to homogeneous and autonomous form.

The reduction of a nonhomogeneous quasilinear system to homogeneous form, besides its intrinsic interest, may reveal useful in investigating a well known problem connected with a system of hyperbolic conservation laws, say the Riemann problem [20, 86], where one takes a piecewise constant initial datum with a single discontinuity. As well known, there is an existence and uniqueness theorem for the Riemann problem for a system of conservation laws; on the contrary, analogous results have not been obtained yet for a system of balance laws, even for a generalized (*i.e.*, piecewise non–constant initial data) Riemann problem [4].

In the case in which we have a system of balance laws and the application of Theorem 3.2.4 leads to a system of conservation laws, then one may investigate a Riemann problem (classical or generalized) for the original system of balance laws by studying an associated Riemann problem (which can be classical or generalized) for a system of conservation laws [65]. Once the latter problem has been solved, thanks to the inverse transformation, it is possible to obtain the corresponding solution of the original system.

4 Equivalence transformations

In this Chapter, a class of partial differential equations (a conservation law and four balance laws), with four independent variables, and involving sixteen arbitrary continuously differentiable functions, is considered in the framework of equivalence transformations. These are point transformations of differential equations involving arbitrary elements, defined in an augmented space of independent, dependent and additional variables representing values taken by the arbitrary elements. Projecting the admitted symmetries into the space of independent and dependent variables, we determine some finite transformations mapping the system of balance laws to an equivalent one with the same differential structure but involving different arbitrary elements; in particular, the target system we want to recover is an autonomous system of conservation laws. An example of physical interest (3D Euler equations for an ideal gas in a non–inertial frame and subject to gravity) is also considered.

The results here presented are contained in [35].

4.1 A brief sketch of equivalence transformations

In Chapter 3, it has been shown that the transformation of a general nonautonomous and/or nonhomogeneous first order quasilinear system of partial differential equations (which every system of first order balance laws reduces to) into autonomous and homogeneous quasilinear form is possible if and only if a suitable algebra of point symmetries is admitted. This procedure works well for a given system of partial differential equations.

Nevertheless, if one is interested to identify the systems of balance laws (possibly nonautonomous) that can be transformed by an invertible point transformation to an autonomous system of conservation laws, a convenient approach consists in using equivalence transformations [46, 47, 56, 61, 76, 77, 88, 89].

To fix the notation, we briefly recall the main elements of equivalence transformations of differential equations.

In many applications, we have differential equations involving arbitrary elements (constants or functions), so that one has a *class of differential equations*. Here, we limit ourselves to consider a class $\mathcal{E}(\mathbf{p})$ of first order partial differential equations involving some arbitrary continuously differentiable functions $p_k(\mathbf{x},\mathbf{u})$ $(k=1,\dots,\ell)$,

$$\mathbf{\Delta}\left(\mathbf{x},\mathbf{u},\mathbf{u}^{(1)};\mathbf{p},\mathbf{p}^{(1)}\right)=\mathbf{0}, \tag{4.1}$$

whose elements are given once we fix the functions p_k ($\mathbf{p}^{(1)}$ denotes the set of first order partial derivatives of the $\mathbf{p}$'s with respect to their arguments).

To study the invariance of a class of differential equations, it is convenient to consider equivalence transformations, *i.e.*, transformations that preserve the differential structure of the equations in the class but may change the form of the constitutive functions and/or parameters [46, 47, 56, 69, 72, 76, 77, 88, 89].

Definition 4.1.1 (Equivalence transformations [76]). *A one–parameter Lie group of equivalence transformations of a family $\mathcal{E}(\mathbf{p})$ of PDE's is a one–parameter Lie group of transformations given by*

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a), \qquad \mathbf{U} = \mathbf{U}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a), \qquad \mathbf{P} = \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{p}; a), \tag{4.2}$$

a being the parameter, which is locally a C^∞ diffeomorphism and maps a class $\mathcal{E}(\mathbf{p})$ of differential equations into itself; thus, it may change the differential equations (the form of the arbitrary elements therein involved) but preserves their differential structure.

In the following, we shall assume that the transformations of the independent and dependent variables do not involve the arbitrary elements $\mathbf{p}$.

In an augmented space $\mathcal{A} \equiv \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell$ [56, 76], where the independent variables, the dependent variables and the arbitrary functions are defined, the generator of the equivalence transformation,

$$\Xi = \sum_{i=1}^{n} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{A=1}^{m} \eta^A(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_A} + \sum_{k=1}^{\ell} \mu^k(\mathbf{x}, \mathbf{u}, \mathbf{p}) \frac{\partial}{\partial p_k}, \tag{4.3}$$

involves also the infinitesimals $\mu^k(\mathbf{x}, \mathbf{u}, \mathbf{p})$ accounting for the arbitrary functions p_k. The search for continuous equivalence transformations can be exploited by using the Lie infinitesimal criterion [76].

The first prolongation of Ξ writes as

$$\Xi^{(1)} = \Xi + \sum_{A=1}^{m} \sum_{i=1}^{n} \eta^A_{[i]} \frac{\partial}{\partial u_{A,i}} + \sum_{k=1}^{\ell} \sum_{\alpha=1}^{n+m} \mu^k_{[\alpha]} \frac{\partial}{\partial p_{k,\alpha}}, \tag{4.4}$$

with

$$\eta^A_{[i]} = \frac{D\eta^A}{Dx_i} - \sum_{j=1}^{n} u_{A,j} \frac{D\xi^j}{Dx_i}, \qquad \mu^k_{[\alpha]} = \frac{\widetilde{D}\mu^k}{\widetilde{D}z_\alpha} - \sum_{\beta=1}^{n+m} p_{k,\beta} \frac{\widetilde{D}\zeta^\beta}{\widetilde{D}z_\alpha}, \tag{4.5}$$

($u_{A,j} = \dfrac{\partial u_A}{\partial x_j}$, $p_{k,\alpha} = \dfrac{\partial p_k}{\partial z_\alpha}$, $\mathbf{z} = (\mathbf{x}, \mathbf{u})$, $\boldsymbol{\zeta} = (\boldsymbol{\xi}, \boldsymbol{\eta})$), where the *Lie derivatives* are

$$\frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + \sum_{A=1}^{m} u_{A,i} \frac{\partial}{\partial u_A}, \qquad \frac{\widetilde{D}}{\widetilde{D}z_\alpha} = \frac{\partial}{\partial z_\alpha} + \sum_{k=1}^{\ell} p_{k,\alpha} \frac{\partial}{\partial p_k}. \tag{4.6}$$

In the augmented space $\mathcal{A}$, the arbitrary functions determining the class of differential equations are assumed as dependent variables, and we require the invariance of the class in this augmented space. If we project the symmetries on the space $\mathcal{Z} \equiv \mathbb{R}^n \times \mathbb{R}^m$ of the independent and dependent variables (this is possible because the infinitesimals of independent and dependent variables are assumed to be independent of $\mathbf{p}$), we obtain a transformation changing an element of the class of differential equations to another element in the same class (same differential structure but in general different arbitrary elements). Such projected transformations map solutions of a system in the class to solutions of a transformed system in the same class.

Thus, in the augmented space $\mathcal{A}$, given the equivalence generator (4.3) the integration of Lie's equations

$$\begin{aligned} \frac{d\mathbf{X}}{da} &= \boldsymbol{\xi}(\mathbf{X},\mathbf{U}), & \mathbf{X}(0) &= \mathbf{x}, \\ \frac{d\mathbf{U}}{da} &= \boldsymbol{\eta}(\mathbf{X},\mathbf{U}), & \mathbf{U}(0) &= \mathbf{u}, \\ \frac{d\mathbf{P}}{da} &= \boldsymbol{\mu}(\mathbf{X},\mathbf{U},\mathbf{P}), & \mathbf{P}(0) &= \mathbf{p}, \end{aligned} \tag{4.7}$$

provides the finite transformation which maps the class into itself. On the contrary, the integration of the Lie's equations (4.7) in the projected space $\mathcal{Z}$ gives an equivalence transformation mapping a system in the class into another system in the same class.

4.2 The model and the admitted equivalence transformations

In this Section, we consider a $(3+1)$–dimensional system of first order partial differential equations consisting of a linear conservation law and four general balance laws involving some arbitrary functions. The aim is to identify classes of systems that can be mapped through an invertible point transformation to a system of autonomous conservation laws. A similar approach has been used recently in [69] for a 2×2 first order quasilinear system of partial differential equations, and in [72] for a system of three balance laws in three independent variables.

More in detail, the considered class of differential equations, with four independent and five dependent variables, involves sixteen arbitrary functions of the independent and dependent variables. The equivalence transformations are determined, and the finite transformations corresponding to the admitted generators are constructed. As a consequence, the equivalent conservation laws are characterized.

Consider the class $\mathcal{E}(\mathbf{p})$ with $\mathbf{p} = (p_1, \dots, p_{16})$ of systems

$$\begin{aligned}
&\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + \frac{\partial u_4}{\partial x_4} = 0,\\
&\frac{\partial u_2}{\partial x_1} + \frac{\partial p_1}{\partial x_2} + \frac{\partial p_2}{\partial x_3} + \frac{\partial p_3}{\partial x_4} = p_{13},\\
&\frac{\partial u_3}{\partial x_1} + \frac{\partial p_4}{\partial x_2} + \frac{\partial p_5}{\partial x_3} + \frac{\partial p_6}{\partial x_4} = p_{14},\\
&\frac{\partial u_4}{\partial x_1} + \frac{\partial p_7}{\partial x_2} + \frac{\partial p_8}{\partial x_3} + \frac{\partial p_9}{\partial x_4} = p_{15},\\
&\frac{\partial u_5}{\partial x_1} + \frac{\partial p_{10}}{\partial x_2} + \frac{\partial p_{11}}{\partial x_3} + \frac{\partial p_{12}}{\partial x_4} = p_{16},
\end{aligned} \tag{4.8}$$

$\mathbf{x} \equiv (x_1, x_2, x_3, x_4)$ being the independent variables, $\mathbf{u} \equiv (u_1, u_2, u_3, u_4, u_5)$ the dependent variables, whereas $\mathbf{p} \equiv (p_1, \dots, p_{16})$ stand for arbitrary continuously differentiable functions of $\mathbf{x}$ and $\mathbf{u}$. For instance, three–dimensional Euler equations of ideal gas–dynamics fall into the class (4.8).

By requiring the invariance of the class $\mathcal{E}(\mathbf{p})$ in the augmented space $\mathcal{A} \equiv \mathbb{R}^4 \times \mathbb{R}^5 \times \mathbb{R}^{16}$, through the Lie's infinitesimal criterion [76], we determine 24 infinitesimal operators:

$$\begin{aligned}
\Xi_1^e = {}& f^{(1)}\partial_{x_2} - \partial_{x_2} f^{(1)} u_1 \partial_{u_1} + \left(\partial_{x_1} f^{(1)} u_1 + \partial_{x_3} f^{(1)} u_3 + \partial_{x_4} f^{(1)} u_4\right)\partial_{u_2}\\
&- \partial_{x_2} f^{(1)} \left(u_3 \partial_{u_3} + u_4 \partial_{u_4}\right)\\
&+ \left(2\partial_{x_1} f^{(1)} u_2 + \partial_{x_2} f^{(1)} p_1 + \partial_{x_3} f^{(1)}(p_2 + p_4) + \partial_{x_4} f^{(1)}(p_3 + p_7)\right)\partial_{p_1}\\
&+ \left(\partial_{x_1} f^{(1)} u_3 + \partial_{x_3} f^{(1)} p_5 + \partial_{x_4} f^{(1)} p_8\right)\partial_{p_2}\\
&+ \left(\partial_{x_1} f^{(1)} u_4 + \partial_{x_3} f^{(1)} p_6 + \partial_{x_4} f^{(1)} p_9\right)\partial_{p_3}\\
&+ \left(\partial_{x_1} f^{(1)} u_3 + \partial_{x_3} f^{(1)} p_5 + \partial_{x_4} f^{(1)} p_6\right)\partial_{p_4}\\
&- \partial_{x_2} f^{(1)} \left(p_5 \partial_{p_5} + p_6 \partial_{p_6}\right)\\
&+ \left(\partial_{x_1} f^{(1)} u_4 + \partial_{x_3} f^{(1)} p_8 + \partial_{x_4} f^{(1)} p_9\right)\partial_{p_7}\\
&- \partial_{x_2} f^{(1)} \left(p_8 \partial_{p_8} + p_9 \partial_{p_9}\right)\\
&+ \left(\partial_{x_1} f^{(1)} u_5 + \partial_{x_2} f^{(1)} p_{10} + \partial_{x_3} f^{(1)} p_{11} + \partial_{x_4} f^{(1)} p_{12}\right)\partial_{p_{10}}\\
&+ \left(\partial^2_{x_1x_1} f^{(1)} u_1 + 2\partial^2_{x_1x_2} f^{(1)} u_2 + 2\partial^2_{x_1x_3} f^{(1)} u_3 + 2\partial^2_{x_1x_4} f^{(1)} u_4\right.\\
&+ \partial^2_{x_2x_2} f^{(1)} p_1 + \partial^2_{x_2x_3} f^{(1)}(p_2 + p_4) + \partial^2_{x_2x_4} f^{(1)}(p_3 + p_7) + \partial^2_{x_3x_3} f^{(1)} p_5\\
&\left.+ \partial^2_{x_3x_4} f^{(1)}(p_6 + p_8) + \partial^2_{x_4x_4} f^{(1)} p_9 + \partial_{x_3} f^{(1)} p_{14} + \partial_{x_4} f^{(1)} p_{15}\right)\partial_{p_{13}}\\
&- \partial_{x_2} f^{(1)} \left(p_{14} \partial_{p_{14}} + p_{15} \partial_{p_{15}}\right)\\
&+ \left(\partial^2_{x_1x_2} f^{(1)} u_5 + \partial^2_{x_2x_2} f^{(1)} p_{10} + \partial^2_{x_2x_3} f^{(1)} p_{11} + \partial^2_{x_2x_4} f^{(1)} p_{12}\right)\partial_{p_{16}},
\end{aligned}$$

$$
\begin{aligned}
\Xi_2^e = {} & f^{(2)}\partial_{x_3} - \partial_{x_3} f^{(2)} \left(u_1\partial_{u_1} + u_2\partial_{u_2}\right) \\
& + \left(\partial_{x_1} f^{(2)} u_1 + \partial_{x_2} f^{(2)} u_2 + \partial_{x_4} f^{(2)} u_4\right) \partial_{u_3} \\
& - \partial_{x_3} f^{(2)} \left(u_4\partial_{u_4} + p_1\partial_{p_1}\right) \\
& + \left(\partial_{x_1} f^{(2)} u_2 + \partial_{x_2} f^{(2)} p_1 + \partial_{x_4} f^{(2)} p_3\right) \partial_{p_2} - \partial_{x_3} f^{(2)} p_3 \partial_{p_3} \\
& + \left(\partial_{x_1} f^{(2)} u_2 + \partial_{x_2} f^{(2)} p_1 + \partial_{x_4} f^{(2)} p_7\right) \partial_{p_4} \\
& + \left(2\partial_{x_1} f^{(2)} u_3 + \partial_{x_2} f^{(2)} (p_2 + p_4) + \partial_{x_3} f^{(2)} p_5 + \partial_{x_4} f^{(2)} (p_6 + p_8)\right) \partial_{p_5} \\
& + \left(\partial_{x_1} f^{(2)} u_4 + \partial_{x_2} f^{(2)} p_3 + \partial_{x_4} f^{(2)} p_9\right) \partial_{p_6} - \partial_{x_3} f^{(2)} p_7 \partial_{p_7} \\
& + \left(\partial_{x_1} f^{(2)} u_4 + \partial_{x_2} f^{(2)} p_7 + \partial_{x_4} f^{(2)} p_9\right) \partial_{p_8} - \partial_{x_3} f^{(2)} p_9 \partial_{p_9} \\
& + \left(\partial_{x_1} f^{(2)} u_5 + \partial_{x_2} f^{(2)} p_{10} + \partial_{x_3} f^{(2)} p_{11} + \partial_{x_4} f^{(2)} p_{12}\right) \partial_{p_{11}} \\
& - \partial_{x_3} f^{(2)} p_{13} \partial_{p_{13}} \\
& + \Big(\partial^2_{x_1x_1} f^{(2)} u_1 + 2\partial^2_{x_1x_2} f^{(2)} u_2 + 2\partial^2_{x_1x_3} f^{(2)} u_3 + 2\partial^2_{x_1x_4} f^{(2)} u_4 \\
& + \partial^2_{x_2x_2} f^{(2)} p_1 + \partial^2_{x_2x_3} f^{(2)} (p_2 + p_4) + \partial^2_{x_2x_4} f^{(2)} (p_3 + p_7) + \partial^2_{x_3x_3} f^{(2)} p_5 \\
& + \partial^2_{x_3x_4} f^{(2)} (p_6 + p_8) + \partial^2_{x_4x_4} f^{(2)} p_9 + \partial_{x_2} f^{(2)} p_{13} + \partial_{x_4} f^{(2)} p_{15}\Big) \partial_{p_{14}} \\
& - \partial_{x_3} f^{(2)} p_{15} \partial_{p_{15}} \\
& + \left(\partial^2_{x_1x_3} f^{(2)} u_5 + \partial^2_{x_2x_3} f^{(2)} p_{10} + \partial^2_{x_3x_3} f^{(2)} p_{11} + \partial^2_{x_3x_4} f^{(2)} p_{12}\right) \partial_{p_{16}}, \\
\Xi_3^e = {} & f^{(3)}\partial_{x_4} - \partial_{x_4} f^{(3)} \left(u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}\right) \\
& + \left(\partial_{x_1} f^{(3)} u_1 + \partial_{x_2} f^{(3)} u_2 + \partial_{x_3} f^{(3)} u_3\right) \partial_{u_4} \\
& - \partial_{x_4} f^{(3)} \left(p_1\partial_{p_1} + p_2\partial_{p_2}\right) \\
& + \left(\partial_{x_1} f^{(3)} u_2 + \partial_{x_2} f^{(3)} p_1 + \partial_{x_3} f^{(3)} p_2\right) \partial_{p_3} \\
& - \partial_{x_4} f^{(3)} \left(p_4\partial_{p_4} + p_5\partial_{p_5}\right) \\
& + \left(\partial_{x_1} f^{(3)} u_3 + \partial_{x_2} f^{(3)} p_4 + \partial_{x_3} f^{(3)} p_5\right) \partial_{p_6} \\
& + \left(\partial_{x_1} f^{(3)} u_2 + \partial_{x_2} f^{(3)} p_1 + \partial_{x_3} f^{(3)} p_4\right) \partial_{p_7} \\
& + \left(\partial_{x_1} f^{(3)} u_3 + \partial_{x_2} f^{(3)} p_2 + \partial_{x_3} f^{(3)} p_5\right) \partial_{p_8} \\
& + \left(2\partial_{x_1} f^{(3)} u_4 + \partial_{x_2} f^{(3)} (p_3 + p_7) + \partial_{x_3} f^{(3)} (p_6 + p_8) + \partial_{x_4} f^{(3)} p_9\right) \partial_{p_9} \\
& + \left(\partial_{x_1} f^{(3)} u_5 + \partial_{x_2} f^{(3)} p_{10} + \partial_{x_3} f^{(3)} p_{11} + \partial_{x_4} f^{(3)} p_{12}\right) \partial_{p_{12}} \\
& - \partial_{x_4} f^{(3)} \left(p_{13}\partial_{p_{13}} + p_{14}\partial_{p_{14}}\right) \\
& + \Big(\partial^2_{x_1x_1} f^{(3)} u_1 + 2\partial^2_{x_1x_2} f^{(3)} u_2 + 2\partial^2_{x_1x_3} f^{(3)} u_3 + 2\partial^2_{x_1x_4} f^{(3)} u_4 \\
& + \partial^2_{x_2x_2} f^{(3)} p_1 + \partial^2_{x_2x_3} f^{(3)} (p_2 + p_4) + \partial^2_{x_2x_4} f^{(3)} (p_3 + p_7) + \partial^2_{x_3x_3} f^{(3)} p_5 \\
& + \partial^2_{x_3x_4} f^{(3)} (p_6 + p_8) + \partial^2_{x_4x_4} f^{(3)} p_9 + \partial_{x_2} f^{(3)} p_{13} + \partial_{x_3} f^{(3)} p_{14}\Big) \partial_{p_{15}} \\
& + \left(\partial^2_{x_1x_4} f^{(3)} u_5 + \partial^2_{x_2x_4} f^{(3)} p_{10} + \partial^2_{x_3x_4} f^{(3)} p_{11} + \partial^2_{x_4x_4} f^{(3)} p_{12}\right) \partial_{p_{16}},
\end{aligned}
$$

$$
\begin{aligned}
\Xi^e_4 &= f^{(4)}u_1\partial_{u_5} + f^{(4)}u_2\partial_{p_{10}} + f^{(4)}u_3\partial_{p_{11}} + f^{(4)}u_4\partial_{p_{12}}\\
&\quad + \left(\partial_{x_1}f^{(4)}u_1 + \partial_{x_2}f^{(4)}u_2 + \partial_{x_3}f^{(4)}u_3 + \partial_{x_4}f^{(4)}u_4\right)\partial_{p_{16}},\\
\Xi^e_5 &= f^{(5)}u_2\partial_{u_5} + f^{(5)}p_1\partial_{p_{10}} + f^{(5)}p_2\partial_{p_{11}} + f^{(5)}p_3\partial_{p_{12}}\\
&\quad + \left(\partial_{x_1}f^{(5)}u_2 + \partial_{x_2}f^{(5)}p_1 + \partial_{x_3}f^{(5)}p_2 + \partial_{x_4}f^{(5)}p_3 + f^{(5)}p_{13}\right)\partial_{p_{16}},\\
\Xi^e_6 &= f^{(6)}u_3\partial_{u_5} + f^{(6)}p_4\partial_{p_{10}} + f^{(6)}p_5\partial_{p_{11}} + f^{(6)}p_6\partial_{p_{12}}\\
&\quad + \left(\partial_{x_1}f^{(6)}u_3 + \partial_{x_2}f^{(6)}p_4 + \partial_{x_3}f^{(6)}p_5 + \partial_{x_4}f^{(6)}p_6 + f^{(6)}p_{14}\right)\partial_{p_{16}},\\
\Xi^e_7 &= f^{(7)}u_4\partial_{u_5} + f^{(7)}p_7\partial_{p_{10}} + f^{(7)}p_8\partial_{p_{11}} + f^{(7)}p_9\partial_{p_{12}}\\
&\quad + \left(\partial_{x_1}f^{(7)}u_4 + \partial_{x_2}f^{(7)}p_7 + \partial_{x_3}f^{(7)}p_8 + \partial_{x_4}f^{(7)}p_9 + f^{(7)}p_{15}\right)\partial_{p_{16}},\\
\Xi^e_8 &= f^{(8)}u_5\partial_{u_5} + f^{(8)}p_{10}\partial_{p_{10}} + f^{(8)}p_{11}\partial_{p_{11}} + f^{(8)}p_{12}\partial_{p_{12}}\\
&\quad + \left(\partial_{x_1}f^{(8)}u_5 + \partial_{x_2}f^{(8)}p_{10} + \partial_{x_3}f^{(8)}p_{11} + \partial_{x_4}f^{(8)}p_{12} + f^{(8)}p_{16}\right)\partial_{p_{16}},\\
\Xi^e_9 &= f^{(9)}\partial_{u_5} + \partial_{x_1}f^{(9)}\partial_{p_{16}}, \qquad \Xi^e_{10} = f^{(10)}\partial_{p_1} + \partial_{x_2}f^{(10)}\partial_{p_{13}},\\
\Xi^e_{11} &= f^{(11)}\partial_{p_2} + \partial_{x_3}f^{(11)}\partial_{p_{13}}, \qquad \Xi^e_{12} = f^{(12)}\partial_{p_3} + \partial_{x_4}f^{(12)}\partial_{p_{13}},\\
\Xi^e_{13} &= f^{(13)}\partial_{p_4} + \partial_{x_2}f^{(13)}\partial_{p_{14}}, \qquad \Xi^e_{14} = f^{(14)}\partial_{p_5} + \partial_{x_3}f^{(14)}\partial_{p_{14}},\\
\Xi^e_{15} &= f^{(15)}\partial_{p_6} + \partial_{x_4}f^{(15)}\partial_{p_{14}}, \qquad \Xi^e_{16} = f^{(16)}\partial_{p_7} + \partial_{x_2}f^{(16)}\partial_{p_{15}},\\
\Xi^e_{17} &= f^{(17)}\partial_{p_8} + \partial_{x_3}f^{(17)}\partial_{p_{15}}, \qquad \Xi^e_{18} = f^{(18)}\partial_{p_9} + \partial_{x_4}f^{(18)}\partial_{p_{15}},\\
\Xi^e_{19} &= f^{(19)}\partial_{p_{10}} + \partial_{x_2}f^{(19)}\partial_{p_{16}}, \qquad \Xi^e_{20} = f^{(20)}\partial_{p_{11}} + \partial_{x_3}f^{(20)}\partial_{p_{16}},\\
\Xi^e_{21} &= f^{(21)}\partial_{p_{12}} + \partial_{x_4}f^{(21)}\partial_{p_{16}},\\
\Xi^e_{22} &= f^{(22)}\partial_{u_1} + f^{(23)}\partial_{u_2} + f^{(24)}\partial_{u_3} + f^{(25)}\partial_{u_4} + \partial_{x_1}f^{(23)}\partial_{p_{13}}\\
&\quad + \partial_{x_1}f^{(24)}\partial_{p_{14}} + \partial_{x_1}f^{(25)}\partial_{p_{15}},\\
\Xi^e_{23} &= f(x_1)\partial_{x_1} - f'(x_1)\left(\sum_{k=2}^{4} u_k\partial_{u_k} + 2\sum_{k=1}^{9} p_k\partial_{p_k} + \sum_{k=10}^{12} p_k\partial_{p_k}\right)\\
&\quad - \left(f''(x_1)u_2 + 2f'(x_1)p_{13}\right)\partial_{p_{13}} - \left(f''(x_1)u_3 + 2f'(x_1)p_{14}\right)\partial_{p_{14}}\\
&\quad - \left(f''(x_1)u_4 + 2f'(x_1)p_{15}\right)\partial_{p_{15}} - f'(x_1)p_{16}\partial_{p_{16}},\\
\Xi^e_{24} &= \sum_{k=1}^{4} u_k\partial_{u_k} + \sum_{k=1}^{9} p_k\partial_{p_k} + \sum_{k=13}^{15} p_k\partial_{p_k},
\end{aligned}
$$

where $f^{(i)} \equiv f^{(i)}(\mathbf{x})$ $(i = 1, \dots, 25)$ are arbitrary functions depending on $\mathbf{x}$, with $f^{(22)}$, $f^{(23)}$, $f^{(24)}$ and $f^{(25)}$ subjected to the constraint

$$
\sum_{k=1}^{4} \partial_{x_k} f_{21+k}(\mathbf{x}) = 0, \tag{4.9}
$$

and $f(x_1)$ is an arbitrary function of its argument; the prime denotes the derivative with respect to the argument.

In view of the results we want to achieve, we need to consider the non–vanishing projections of the admitted operators on the space $\mathcal{Z} \equiv \mathbb{R}^4 \times \mathbb{R}^5$:

$$\begin{aligned}
&\Xi_1 = f_1(x_1)\partial_{x_1} - f_1'(x_1)(u_2\partial_{u_2} + u_3\partial_{u_3} + u_4\partial_{u_4}),\\
&\Xi_i = f_i(\mathbf{x})\partial_{x_i} + \sum_{k=1}^{4}\left(u_k\partial_{x_k} f_i(\mathbf{x})\partial_{u_i} - u_k\partial_{x_i} f_i(\mathbf{x})\partial_{u_k}\right), \quad i = 2,\dots,4,\\
&\Xi_{4+i} = u_i f_{4+i}(\mathbf{x})\partial_{u_5}, \qquad i = 1,\dots,5,\\
&\Xi_{10} = f_{10}(\mathbf{x})\partial_{u_5}, \qquad \Xi_{11} = \sum_{k=1}^{4} f_{10+k}(\mathbf{x})\partial_{u_k}, \qquad \Xi_{12} = \sum_{k=1}^{4} u_k\partial_{u_k},
\end{aligned} \tag{4.10}$$

where we suitably relabelled the functions occurring in the equivalence operators; moreover, the functions f_i ($i = 11,\dots,14$) satisfy the condition

$$\sum_{k=1}^{4} \partial_{x_k} f_{10+k}(\mathbf{x}) = 0. \tag{4.11}$$

By considering the corresponding Lie's equations, we are able to compute the finite corresponding transformations, say

$$\mathbf{X} = \mathbf{X}(\mathbf{x},\mathbf{u};a), \qquad \mathbf{U} = \mathbf{U}(\mathbf{x},\mathbf{u};a), \tag{4.12}$$

allowing us to map the original system (4.8) to a different system with the same differential structure; in particular, we are interested to the case where the target system is an autonomous system of conservation laws, *i.e.*,

$$\begin{aligned}
&\frac{\partial U_1}{\partial X_1} + \frac{\partial U_2}{\partial X_2} + \frac{\partial U_3}{\partial X_3} + \frac{\partial U_4}{\partial X_4} = 0,\\
&\frac{\partial U_2}{\partial X_1} + \frac{\partial P_1}{\partial X_2} + \frac{\partial P_2}{\partial X_3} + \frac{\partial P_3}{\partial X_4} = 0,\\
&\frac{\partial U_3}{\partial X_1} + \frac{\partial P_4}{\partial X_2} + \frac{\partial P_5}{\partial X_3} + \frac{\partial P_6}{\partial X_4} = 0,\\
&\frac{\partial U_4}{\partial X_1} + \frac{\partial P_7}{\partial X_2} + \frac{\partial P_8}{\partial X_3} + \frac{\partial P_9}{\partial X_4} = 0,\\
&\frac{\partial U_5}{\partial X_1} + \frac{\partial P_{10}}{\partial X_2} + \frac{\partial P_{11}}{\partial X_3} + \frac{\partial P_{12}}{\partial X_4} = 0,
\end{aligned} \tag{4.13}$$

where $P_i \equiv P_i(U_1,U_2,U_3,U_4,U_5)$ ($i = 1,\dots,12$). Of course, a given system falling in the class (4.8) can be mapped by an equivalence transformation to a system having the form (4.13) provided that the functions $p_i(\mathbf{x},\mathbf{u})$ ($i = 1,\dots,16$) have special functional forms. To simplify the computation, we exchange source and target system; in fact, taking the inverse transformation of (4.12) (which is obtained by exchanging lower and capital letters and replacing a with $-a$), and starting from the autonomous system (4.13) of conservation laws, we are able to obtain the equivalent nonautonomous system of balance laws. In such a way, we are able to identify, for a given equivalence transformation, the elements of the class (4.8) that can be mapped to a system of autonomous conservation laws.

Now, since we start from an autonomous system of conservation laws to arrive to a nonautonomous system of balance laws, let us write the operators (4.10) in terms of the capital letters; then, we build the corresponding finite transformations.

4.3 Finite transformations

In this Section, we integrate the Lie's equations corresponding to the operators $\Xi_1, \ldots, \Xi_{12}$ (4.10), and construct the finite transformations leading us to obtain systems of the form (4.8) that can be mapped to the form (4.13). The most general finite transformation can be recovered by composition of the finite transformations induced by each generator.

4.3.1 Finite transformations generated by Ξ_1

By considering the generator Ξ_1,

$$\Xi_1 = f(X_1)\partial_{X_1} - f'(X_1)\left(U_2\partial_{U_2} + U_3\partial_{U_3} + U_4\partial_{U_4}\right), \tag{4.14}$$

where we renamed the function f_1 as f, we get the finite transformation

$$\begin{aligned}
&x_1 = \tilde{x}_1(X_1;a), \qquad x_2 = X_2, \qquad x_3 = X_3, \qquad x_4 = X_4,\\
&u_1 = U_1, \qquad u_2 = U_2\frac{f(X_1)}{f(\tilde{x}_1)}, \qquad u_3 = U_3\frac{f(X_1)}{f(\tilde{x}_1)},\\
&u_4 = U_4\frac{f(X_1)}{f(\tilde{x}_1)}, \qquad u_5 = U_5,
\end{aligned} \tag{4.15}$$

$\tilde{x}_1(X_1;a)$ being such that $\tilde{x}_1(X_1;0) = X_1$ and $\partial_{X_1}\tilde{x}_1 = \dfrac{f(\tilde{x}_1)}{f(X_1)}$, whereupon we may write

$$\begin{aligned}
&U_1 = u_1, \qquad U_2 = u_2\partial_{X_1}\tilde{x}_1, \qquad U_3 = u_3\partial_{X_1}\tilde{x}_1,\\
&U_4 = u_4\partial_{X_1}\tilde{x}_1, \qquad U_5 = u_5,
\end{aligned} \tag{4.16}$$

and system (4.8) is equivalent to system (4.13) with

$$\begin{aligned}
&p_k = \frac{P_k}{(\partial_{X_1}\tilde{x}_1)^2}, && k = 1, \ldots, 12,\\
&p_{13} = -u_2\frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}, && p_{14} = -u_3\frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2},\\
&p_{15} = -u_4\frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}, && p_{16} = 0,
\end{aligned} \tag{4.17}$$

where $P_k = P_k(u_1, u_2\partial_{X_1}\tilde{x}_1, u_3\partial_{X_1}\tilde{x}_1, u_4\partial_{X_1}\tilde{x}_1, u_5)$ $(k = 1, \ldots, 12)$.

4.3.2 Finite transformations generated by Ξ_2, Ξ_3, Ξ_4

By considering the generators Ξ_i $(i = 2, \ldots, 4)$,

$$\Xi_i = f(\mathbf{X})\partial_{X_i} + \sum_{k=1}^{4}\left(U_k\partial_{X_k}f(\mathbf{X})\partial_{U_i} - U_k\partial_{X_i}f(\mathbf{X})\partial_{U_k}\right), \tag{4.18}$$

where we renamed the function f_i as f, we may write the general finite transformation arising from the integration of Lie's equations in the three cases in a unified form:

$$
\begin{aligned}
x_k &= \begin{cases} X_k, & k=1,\dots,4, \quad k\neq i \\ \tilde{x}_k(\mathbf{X};a), & k=i, \end{cases}\\
u_k &= \begin{cases} U_k\dfrac{f(\mathbf{X})}{f(\tilde{\mathbf{x}})}, & k=1,\dots,4, \quad k\neq i,\\ U_k+f(\mathbf{X})\displaystyle\sum_{j=1,j\neq i}^{4}U_j\int_0^a\frac{\partial_{X_j}f(\tilde{\mathbf{x}})}{f(\tilde{\mathbf{x}})}da, & k=i,\\ U_k, & k=5, \end{cases}
\end{aligned}
\tag{4.19}
$$

where $\tilde{x}_i(\mathbf{X};a)$ is such that $\tilde{x}_i(\mathbf{X};0)=X_i$ and

$$
\partial_{X_k}\tilde{x}_i = \begin{cases} f(\tilde{\mathbf{x}})\displaystyle\int_0^a\frac{\partial_{X_k}f(\tilde{\mathbf{x}})}{f(\tilde{\mathbf{x}})}da, & k\neq i,\\ \dfrac{f(\tilde{\mathbf{x}})}{f(\mathbf{X})}, & k=i. \end{cases}
\tag{4.20}
$$

By introducing the matrix J with the (j,k)–entry equal to $\partial_{X_k}\tilde{x}_j$ $(j,k=1,\dots,4)$, the $(5,5)$–entry equal to $\partial_{X_i}\tilde{x}_i$ and all remaining entries vanishing, we may write

$$
\mathbf{u}=A\mathbf{U}, \qquad A=\frac{J}{\partial_{X_i}\tilde{x}_i};
\tag{4.21}
$$

moreover, by defining the matrices

$$
q=\begin{bmatrix} u_1 & u_2 & u_3 & u_4 & 0\\ u_2 & p_1 & p_2 & p_3 & 0\\ u_3 & p_4 & p_5 & p_6 & 0\\ u_4 & p_7 & p_8 & p_9 & 0\\ u_5 & p_{10} & p_{11} & p_{12} & 0 \end{bmatrix}, \qquad Q=\begin{bmatrix} U_1 & U_2 & U_3 & U_4 & 0\\ U_2 & P_1 & P_2 & P_3 & 0\\ U_3 & P_4 & P_5 & P_6 & 0\\ U_4 & P_7 & P_8 & P_9 & 0\\ U_5 & P_{10} & P_{11} & P_{12} & 0 \end{bmatrix},
\tag{4.22}
$$

system (4.13) is mapped to system (4.8) with

$$
\begin{aligned}
q &= AQJ^T,\\
p_{11+m} &= \sum_{j=1}^{5}A_{mj}\sum_{\ell=1}^{5}\left(\sum_{k=1}^{4}u_k\frac{\partial^2 R_{\ell j}}{\partial U_\ell\partial X_k}-\sum_{k=1}^{5}u_k\frac{\partial^2 R_{\ell j}}{\partial U_k\partial X_\ell}\right)
\end{aligned}
\tag{4.23}
$$

$(m=2,\dots,5)$, where $R_{\ell j}$ is the generic entry of the matrix JQ^T, and it is $P_k=P_k(\mathbf{U})$ $(k=1,\dots 12)$, with $\mathbf{U}$ defined by (4.21); note that the right hand side of $(4.23)_2$ is vanishing for $m=1$.

4.3.3 Finite transformations generated by Ξ_5, Ξ_6, Ξ_7, Ξ_8

By considering the generator Ξ_{4+i} $(i=1,\dots,4)$,

$$
\Xi_{4+i}=U_i f(\mathbf{X})\partial_{U_5},
\tag{4.24}
$$

where we renamed the function f_{4+i} as f, we get from Lie's equations the finite transformation

$$x_k = X_k, \qquad u_k = U_K, \qquad k = 1, \ldots, 4, \qquad u_5 = U_5 - aU_i f(\mathbf{X}). \tag{4.25}$$

System (4.13) is equivalent to system (4.8) if

$$\begin{aligned}
&p_k = P_k, \qquad k = 1, \ldots 9, \\
&p_{9+k} = \begin{cases} P_{9+k} + au_{k+1} f(\mathbf{x}), & i = 1, \\ P_{9+k} + aP_{3i+k-6} f(\mathbf{x}), & i = 2, \ldots, 4, \end{cases} \qquad k = 1, \ldots, 3, \\
&p_{12+k} = au_i \sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{3k+j-4}, \qquad k = 1, \ldots, 3, \\
&p_{16} = \begin{cases} au_i \left(\partial_{X_1} f(\mathbf{x}) + \displaystyle\sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{8+j} \right), \qquad i = 1, \\ au_i \left(\partial_{X_1} f(\mathbf{x}) + \displaystyle\sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \left(\partial_{U_5} P_{8+j} + af(\mathbf{x}) \partial_{U_5} P_{3i+j-7} \right) \right), \\ \qquad i = 2, \ldots, 4, \end{cases}
\end{aligned} \tag{4.26}$$

where $P_k = P_k(u_1, u_2, u_3, u_4, u_5 + au_i f(\mathbf{x}))$ $(k = 1, \ldots, 12)$.

4.3.4 Finite transformations generated by Ξ_9

By considering the generator Ξ_9,

$$\Xi_9 = U_5 f(\mathbf{X}) \partial_{U_5}, \tag{4.27}$$

where we renamed the function f_9 as f, we get from Lie's equations the finite transformation

$$x_k = X_k, \qquad u_k = U_k, \qquad k = 1, \ldots, 4, \qquad u_5 = U_5 \exp(af(\mathbf{X})). \tag{4.28}$$

System (4.13) is equivalent to system (4.8) provided that:

$$\begin{aligned}
&p_k = P_k, \\
&p_{12+i} = a \exp\left(-af(\mathbf{x})\right) \left(\sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{3i+j-4} \right) u_5, \\
&p_{16} = a \left(\partial_{X_1} f(\mathbf{x}) + \sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{8+j} \right) u_5,
\end{aligned} \tag{4.29}$$

where $i = 1, \ldots, 3$, and $P_k = P_k(u_1, u_2, u_3, u_4, u_5 \exp(-af(\mathbf{x})))$ with $k = 1, \ldots, 12$.

4.3.5 Finite transformations generated by Ξ_{10}

By considering the generator Ξ_{10},

$$\Xi_{10} = f(\mathbf{X})\partial_{U_5}, \tag{4.30}$$

where we renamed the function f_{10} as f, we get from Lie's equations the finite transformation

$$x_k = X_k, \qquad u_k = U_k, \qquad k = 1, \dots, 4, \qquad u_5 = U_5 + af(\mathbf{X}), \tag{4.31}$$

and the equivalence between (4.8) and (4.13) is recovered provided that

$$\begin{aligned}
&p_k = P_k, \qquad k = 1, \dots, 12,\\
&p_{12+i} = a \sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{3i+j-4}, \qquad i = 1, \dots, 3,\\
&p_{16} = a \left(\partial_{X_1} f(\mathbf{x}) + \sum_{j=2}^{4} \partial_{X_j} f(\mathbf{x}) \partial_{U_5} P_{8+j} \right),
\end{aligned} \tag{4.32}$$

where $P_k = P_k(u_1, u_2, u_3, u_4, u_5 - af(\mathbf{x}))$.

4.3.6 Finite transformations generated by Ξ_{11}

By considering the generator Ξ_{11},

$$\Xi_{11} = \sum_{k=1}^{4} g_k(\mathbf{X}) \partial_{U_k}, \tag{4.33}$$

where we renamed the functions f_{10+k} as g_k, along with the constraint $\sum_{k=1}^{4} \partial_{X_k} g_k(\mathbf{X}) = 0$, and integrating Lie's equations, the following finite transformation arises:

$$x_k = X_k, \qquad u_k = U_k + ag_k(\mathbf{x}), \qquad k = 1, \dots, 4, \qquad u_5 = U_5. \tag{4.34}$$

System (4.13) is equivalent to system (4.8) provided that:

$$\begin{aligned}
&p_k = P_k, \qquad k = 1, \dots, 12,\\
&p_{12+\ell} = a \left(\partial_{X_1} g_{\ell+1}(\mathbf{x}) + \sum_{i=1}^{4} \sum_{j=2}^{4} \partial_{X_j} g_i(\mathbf{x}) \partial_{U_i} P_{3\ell+j-4} \right), \qquad \ell = 1, \dots, 3,\\
&p_{16} = a \left(\sum_{i=1}^{4} \sum_{j=2}^{4} \partial_{X_j} g_i(\mathbf{x}) \partial_{U_i} P_{8+j} \right),
\end{aligned} \tag{4.35}$$

where $P_k = P_k(u_1 - ag_1(\mathbf{x}), u_2 - ag_2(\mathbf{x}), u_3 - ag_3(\mathbf{x}), u_4 - ag_4(\mathbf{x}), u_5)$.

4.3.7 Finite transformations generated by Ξ_{12}

In this case the finite transformation consists of a uniform scaling of the dependent variables,

$$\mathbf{x} = \mathbf{X}, \qquad \mathbf{u} = \exp(a)\mathbf{U}, \tag{4.36}$$

and for such a transformation there are no balance laws equivalent to conservation laws.

4.4 Physical application

In this Section, we make some assumptions on the form of the functions involved in the generators of equivalence transformations in order to deal with physically relevant systems of differential equations. In particular, we construct the finite transformations corresponding to the infinitesimal generator $\sum_{i=1}^{4} \Xi_i$, where we assume

$$\begin{aligned} f_2 &= n_1(X_1)X_2 + n_2(X_1)X_3, \\ f_3 &= -n_2(X_1)X_2 + n_1(X_1)X_3, \\ f_4 &= n_3(X_1), \end{aligned} \tag{4.37}$$

with $n_i(X_1)$ ($i = 1, \ldots, 3$) arbitrary functions depending on X_1.

Integration of Lie's equations provides:

$$\begin{aligned} x_1 &= \tilde{x}_1(X_1; a), \\ x_2 &= \tilde{x}_2(X_1, X_2, X_3; a) = \\ &= \exp\left(m_1(X_1; a)\right)\left(X_2 \cos\left(m_2(X_1; a)\right) + X_3 \sin\left(m_2(X_1; a)\right)\right), \\ x_3 &= \tilde{x}_3(X_1, X_2, X_3; a) = \\ &= \exp\left(m_1(X_1; a)\right)\left(-X_2 \sin\left(m_2(X_1; a)\right) + X_3 \cos\left(m_2(X_1; a)\right)\right), \\ x_4 &= \tilde{x}_4(X_1, X_4; a) = X_4 + m_3(X_1; a), \\ U_1 &= \exp\left(2m_1(X_1; a)\right) u_1, \\ U_2 &= \exp\left(m_1(X_1; a)\right)\left[\left(u_2 \cos\left(m_2(X_1; a)\right) - u_3 \sin\left(m_2(X_1; a)\right)\right) \partial_{X_1}\tilde{x}_1\right. \\ &\quad \left. -u_1\left(\partial_{X_1}\tilde{x}_2 \cos\left(m_2(X_1; a)\right) - \partial_{X_1}\tilde{x}_3 \sin\left(m_2(X_1; a)\right)\right)\right], \\ U_3 &= \exp\left(m_1(X_1; a)\right)\left[\left(u_2 \sin\left(m_2(X_1; a)\right) + u_3 \cos\left(m_2(X_1; a)\right)\right) \partial_{X_1}\tilde{x}_1\right. \\ &\quad \left. -u_1\left(\partial_{X_1}\tilde{x}_2 \sin\left(m_2(X_1; a)\right) + \partial_{X_1}\tilde{x}_3 \cos\left(m_2(X_1; a)\right)\right)\right], \\ U_4 &= \exp\left(2m_1(X_1; a)\right)\left(u_4 \partial_{X_1}\tilde{x}_1 - u_1 \partial_{X_1}\tilde{x}_4\right), \\ U_5 &= u_5, \end{aligned} \tag{4.38}$$

where

$$
\begin{aligned}
&m_i(X_1;a) = \int_{X_1}^{x_1} \frac{n_i(s)}{f_1(s)} ds, \qquad \tilde{n}_i(X_1;a) = n_i(x_1) - n_i(X_1), \quad i = 1,\dots,3,\\
&\partial_{X_1}\tilde{x}_1 = \frac{f_1(x_1)}{f_1(X_1)}, \qquad \partial_{X_1}\tilde{x}_2 = \frac{\tilde{n}_1(X_1;a)x_2 + \tilde{n}_2(X_1;a)x_3}{f_1(X_1)},\\
&\partial_{X_1}\tilde{x}_3 = \frac{-\tilde{n}_2(X_1;a)x_2 + \tilde{n}_1(X_1;a)x_3}{f_1(X_1)}, \qquad \partial_{X_1}\tilde{x}_4 = \frac{\tilde{n}_3(X_1;a)}{f_1(X_1)}.
\end{aligned}
\tag{4.39}
$$

System (4.13) describes the 3D unsteady flow of an ideal fluid subject to no extraneous force along with the choices

$$
\begin{aligned}
&U_1 = \rho, \qquad U_2 = \rho u, \qquad U_3 = \rho v, \qquad U_4 = \rho w, \qquad U_5 = \rho S,\\
&P_1 = \frac{U_2^2}{U_1} + p(U_1, U_5), \qquad P_2 = P_4 = \frac{U_2U_3}{U_1}, \qquad P_3 = P_7 = \frac{U_2U_4}{U_1},\\
&P_5 = \frac{U_3^2}{U_1} + p(U_1, U_5), \qquad P_6 = P_8 = \frac{U_3U_4}{U_1}, \qquad P_9 = \frac{U_4^2}{U_1} + p(U_1, U_5),\\
&P_{10} = \frac{U_2U_5}{U_1}, \qquad P_{11} = \frac{U_3U_5}{U_1}, \qquad P_{12} = \frac{U_4U_5}{U_1},
\end{aligned}
\tag{4.40}
$$

ρ being the fluid mass density, (u, v, w) the components of its velocity, S the entropy, and $p(\rho, S)$ the pressure. Thorugh the transformation (4.38), we get the system (4.8) with

$$
\begin{aligned}
&p_1 = \frac{u_2^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{X_1}\tilde{x}_1)^2}, \qquad p_2 = p_4 = \frac{u_2u_3}{u_1}, \qquad p_3 = p_7 = \frac{u_2u_4}{u_1},\\
&p_5 = \frac{u_3^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{X_1}\tilde{x}_1)^2}, \qquad p_6 = p_8 = \frac{u_3u_4}{u_1}, \qquad p_{10} = \frac{u_2u_5}{u_1},\\
&p_9 = \frac{u_4^2}{u_1} + \frac{p(\exp(-2m_1)u_1, u_5)}{(\partial_{X_1}\tilde{x}_1)^2}, \qquad p_{11} = \frac{u_3u_5}{u_1}, \qquad p_{12} = \frac{u_4u_5}{u_1},\\
&p_{13} = 2\frac{\partial_{X_1}m_1u_2 + \partial_{X_1}m_2u_3}{\partial_{X_1}\tilde{x}_1} + \frac{x_2(\partial^2_{X_1X_1}m_1 - (\partial_{X_1}m_1)^2 + (\partial_{X_1}m_2)^2)}{(\partial_{X_1}\tilde{x}_1)^2}u_1\\
&\qquad + \frac{x_3(\partial^2_{X_1X_1}m_2 - 2\partial_{X_1}m_1\partial_{X_1}m_2)}{(\partial_{X_1}\tilde{x}_1)^2}u_1 - \frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}u_2,\\
&p_{14} = 2\frac{\partial_{X_1}m_1u_3 - \partial_{X_1}m_2u_2}{\partial_{X_1}\tilde{x}_1} - \frac{x_2(\partial^2_{X_1X_1}m_2 - 2\partial_{X_1}m_1\partial_{X_1}m_2)}{(\partial_{X_1}\tilde{x}_1)^2}u_1\\
&\qquad + \frac{x_3(\partial^2_{X_1X_1}m_1 - (\partial_{X_1}m_1)^2 + (\partial_{X_1}m_2)^2)}{(\partial_{X_1}\tilde{x}_1)^2}u_1 - \frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}u_3,\\
&p_{15} = -\frac{\partial^2_{X_1X_1}\tilde{x}_1}{(\partial_{X_1}\tilde{x}_1)^2}u_4 + \frac{\partial^2_{X_1X_1}\tilde{x}_4}{(\partial_{X_1}\tilde{x}_1)^2}u_1, \qquad p_{16} = 2\frac{\partial_{X_1}m_1}{\partial_{X_1}\tilde{x}_1}u_5.
\end{aligned}
$$

By choosing $\partial_{X_1}\tilde{x}_1 = 1$ (whereupon $x_1 = X_1 + a$), $m_1 = 0$, $m_2 = \omega X_1 + X_{1_0}$, $m_3 = \dfrac{gX_1^2}{2} + a_1X_1 + a_0$, where ω, a_0, a_1, a_2, g and X_{1_0} are constants, we recover the system

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}+\frac{\partial u_4}{\partial x_4}=0,$$

$$\frac{\partial u_2}{\partial x_1}+\frac{\partial}{\partial x_2}\left(\frac{u_2^2}{u_1}+p(u_1,u_5)\right)+\frac{\partial}{\partial x_3}\left(\frac{u_2u_3}{u_1}\right)+\frac{\partial}{\partial x_4}\left(\frac{u_2u_4}{u_1}\right)=$$

$$=2\omega u_3-\omega^2x_2u_1,$$

$$\frac{\partial u_3}{\partial x_1}+\frac{\partial}{\partial x_2}\left(\frac{u_2u_3}{u_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{u_3^2}{u_1}+p(u_1,u_5)\right)+\frac{\partial}{\partial x_4}\left(\frac{u_3u_4}{u_1}\right)=$$

$$=-2\omega u_2+\omega^2x_3u_1,$$

$$\frac{\partial u_4}{\partial x_1}+\frac{\partial}{\partial x_2}\left(\frac{u_2u_4}{u_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{u_3u_4}{u_1}\right)+\frac{\partial}{\partial x_4}\left(\frac{u_4^2}{u_1}+p(u_1,u_5)\right)=gu_1,$$

$$\frac{\partial u_5}{\partial x_1}+\frac{\partial}{\partial x_2}\left(\frac{u_2u_5}{u_1}\right)+\frac{\partial}{\partial x_3}\left(\frac{u_3u_5}{u_1}\right)+\frac{\partial}{\partial x_4}\left(\frac{u_4u_5}{u_1}\right)=0.$$

With obvious identifications, we recognize the equations of an ideal gas in a non–inertial frame rotating with constant angular velocity ω around the vertical x_4–axis and subject to gravity. This implies that the Euler equations for an ideal gas in a non–inertial frame rotating with constant angular velocity around a vertical axis and subject to gravity can be transformed in a form where the gravity and apparent forces disappear.

4.5 Conclusions

In this Chapter, we have characterized a class of partial differential equations in four independent variables expressed under the form of a linear conservation law and four nonautonomous nonlinear balance laws that can be transformed by an invertible point transformation into an autonomous system of conservation laws. This has been accomplished through the use of equivalence transformations. The general results obtained have been specialized in order to deal with a model of physical interest. In particular, it has been shown the equivalence of the 3D unsteady Euler equations of an ideal gas subject to gravity and Coriolis forces with the corresponding system where forces are absent. The approach used, and based on equivalence transformations, allows one to determine the members of a general class of partial differential equations that can be mapped by an invertible point transformation to a target system with suitable properties, *e.g.*, a system of autonomous conservation laws.

5 Transformations of nonlinear first order systems

In this Chapter, we present some new results about the use of Lie symmetries for transforming a given nonlinear first order system of differential equations into an equivalent form. At first, a theorem providing necessary conditions enabling one to map a nonlinear system of first order partial differential equations to an equivalent first order autonomous and homogeneous quasilinear system is given. The reduction to quasilinear form is performed by constructing the canonical variables associated to the Lie point symmetries admitted by the nonlinear system. Some applications to relevant partial differential equations (second order Monge–Ampère equations) are given. An application related to a partial differential equation defining a surface in $\mathbb{R}^3$ whose Gaussian curvature is proportional to the mean curvature showing that the conditions for transforming to quasilinear form are only necessary is given.

In the second part of this Chapter, a theorem providing necessary and sufficient conditions enabling one to map a nonlinear system of first order partial differential equations, polynomial in the derivatives, to an equivalent autonomous first order system polynomially homogeneous in the derivatives is proved. Moreover, several classes of first order Monge–Ampère systems, either with constant coefficients or with coefficients depending on the field variables, are considered. The results presented in this Chapter are contained in [31, 32].

5.1 Reduction to quasilinear autonomous and homogeneous form

In this Section, we shall consider nonlinear systems of first order partial differential equations; nevertheless, it is worth of being recalled that higher order partial differential equations can always be rewritten (though not in a unique way!) as first order systems.

Among the first order systems of partial differential equations, a special role is played by quasilinear systems either for their mathematical properties or for their ubiquity in the applications.

Here, we consider nonlinear first order systems and investigate whether they can be reduced by an invertible point transformation to an equivalent first order system of autonomous and homogeneous quasilinear equations. First order autonomous and homogeneous quasilinear systems possess many relevant features: for instance, they admit self–similar solutions suitable to describe rarefaction waves. Also, many efficient numerical schemes useful for investigating physically relevant problems are available for such a kind of systems. We give necessary conditions for such a reduction.

5.1.1 Necessary conditions

Let us consider a general nonlinear first order system of partial differential equations,

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0}, \tag{5.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ are the independent variables, $\mathbf{u} \in \mathbb{R}^m$ the dependent variables, and $\mathbf{u}^{(1)}$ denotes the set of all first order derivatives of $\mathbf{u}$ with respect to $\mathbf{x}$. We want to exploit the possibility of constructing an invertible mapping like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{5.2}$$

allowing us to map it to a quasilinear homogeneous and autonomous system. When this is possible then *necessarily* the nonlinear system has to possess a suitable $(n+1)$–dimensional solvable Lie algebra as subalgebra of the algebra of its Lie point symmetries.

Let us suppose that system (5.1) can be mapped through an invertible point transformation like (5.2) into an autonomous and homogeneous quasilinear system, say

$$\sum_{i=1}^{n} A^i(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial z_i} = \mathbf{0}, \tag{5.3}$$

where A^i $(i = 1, \dots, n)$ are $m \times m$ matrices with entries depending at most on $\mathbf{w}$. Every system like (5.3) admits the Lie point symmetries generated by the following vector fields:

$$\begin{aligned} \Xi_i &= \frac{\partial}{\partial z_i}, \qquad i = 1, \dots, n, \\ \Xi_{n+1} &= \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}; \end{aligned} \tag{5.4}$$

these vector fields span an $(n + 1)$–dimensional solvable Lie algebra where the only non–zero commutators are

$$[\Xi_i, \Xi_{n+1}] = \Xi_i, \qquad i = 1, \dots, n. \tag{5.5}$$

Moreover, the first n vector fields span an n–dimensional Abelian Lie algebra, and generate a distribution of rank n.

Since neither the rank of a distribution nor the Lie bracket of two infinitesimal generators of symmetries is affected by an invertible change of coordinates, it follows that if a system of the form (5.1) can be mapped by the invertible point transformation (5.2) to the form (5.3) it has to admit, as subalgebra of the Lie algebra of its point symmetries, an $(n+1)$–dimensional Lie algebra with a suitable algebraic structure.

Therefore, the following theorem is proved.

Theorem 5.1.1. *A necessary condition in order the nonlinear first order system*

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0} \tag{5.6}$$

be transformed by the invertible map

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{5.7}$$

into an autonomous and homogeneous quasilinear first order system is that it admits as subalgebra of its Lie point symmetries an $(n+1)$–dimensional Lie algebra spanned by

$$\Xi_i = \sum_{j=1}^{n} \xi_i^j(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_j} + \sum_{A=1}^{m} \eta_i^A(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u_A}, \qquad i = 1,\dots,n+1, \tag{5.8}$$

such that

$$[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_{n+1}] = \Xi_i, \qquad i,j = 1,\dots,n. \tag{5.9}$$

Furthermore, the vector fields $\Xi_1,\dots,\Xi_n$ have to generate a distribution of rank n. The new independent and dependent variables are the canonical variables associated to the symmetries generated by $\Xi_1,\dots,\Xi_n$, say

$$\Xi_i(z_j) = \delta_{ij}, \qquad \Xi_i(\mathbf{w}) = \mathbf{0}, \qquad i,j = 1,\dots,n, \tag{5.10}$$

where δ_{ij} is the Kronecker symbol. Finally, the variables $\mathbf{w}$, which by construction are invariants of $\Xi_1,\dots,\Xi_n$, must result invariant with respect to Ξ_{n+1} too.

Proof. The proof follows from the above considerations. The construction of the new independent and dependent variables, which are the canonical variables of the admitted symmetries, is the same as the one given below in Theorem 5.3.1. □

The conditions required by Theorem 5.1.1 are not sufficient to guarantee the transformation to quasilinear form, as shown by next example.

Example 5.1.1. *Let us consider the first order system made by the equations*

$$\begin{aligned}
&\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0,\\
&\kappa_1\left(\frac{\partial u_1}{\partial x_2}\right)^4 + \left(\kappa_2\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \kappa_3\left(\frac{\partial u_1}{\partial x_2}\right)^2\right)\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\\
&+\left(\kappa_4\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \kappa_5\left(\frac{\partial u_1}{\partial x_2}\right)^2\right)\frac{\partial u_1}{\partial x_1} + \left(\kappa_6\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \kappa_7\left(\frac{\partial u_1}{\partial x_2}\right)^2\right)\frac{\partial u_1}{\partial x_2}\\
&+\left(\kappa_8\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \kappa_9\left(\frac{\partial u_1}{\partial x_2}\right)^2\right)\frac{\partial u_2}{\partial x_2} + \kappa_{10}\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \kappa_{11}\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2}\\
&+\kappa_{12}\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \kappa_{13}\left(\frac{\partial u_1}{\partial x_2}\right)^2 + \kappa_{14}\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \kappa_{15}\left(\frac{\partial u_2}{\partial x_2}\right)^2 = 0,
\end{aligned} \tag{5.11}$$

with $u_1(x_1,x_2)$, $u_2(x_1,x_2)$ scalar functions, and $\kappa_i\,(u_1,u_2)$ ($i = 1,\dots,15$) arbitrary smooth functions of the indicated arguments.

It can be easily ascertained that system (5.11) admits the Lie point symmetries spanned by the operators

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2},\\
&\Xi_3 = (x_1 - au_1 - bu_2)\frac{\partial}{\partial x_1} + (x_2 - bu_1 - cu_2)\frac{\partial}{\partial x_2},
\end{aligned} \tag{5.12}$$

a, b, c being constants, provided that the conditions

$$
\begin{aligned}
&\kappa_1 - c^2\kappa_{10} + bc\kappa_{11} - ac\kappa_{12} - b^2\kappa_{13} + ab\kappa_{14} - a^2\kappa_{15} = 0,\\
&\kappa_2 - c^2\kappa_{10} + bc\kappa_{11} - ac\kappa_{12} - b^2\kappa_{13} + ab\kappa_{14} - a^2\kappa_{15} = 0,\\
&\kappa_3 + 2(c^2\kappa_{10} - bc\kappa_{11} + ac\kappa_{12} + b^2\kappa_{13} - ab\kappa_{14} + a^2\kappa_{15}) = 0,\\
&\kappa_4 + 2c\kappa_{10} - b\kappa_{11} + a\kappa_{12} = 0,\\
&\kappa_5 - 2c\kappa_{10} + b\kappa_{11} - a\kappa_{12} = 0,\\
&\kappa_6 + c\kappa_{11} - 2b\kappa_{13} + a\kappa_{14} = 0,\\
&\kappa_7 - c\kappa_{11} + 2b\kappa_{13} - a\kappa_{14} = 0,\\
&\kappa_8 + c\kappa_{12} - b\kappa_{14} + 2a\kappa_{15} = 0,\\
&\kappa_9 - c\kappa_{12} + b\kappa_{14} - 2a\kappa_{15} = 0
\end{aligned}
\tag{5.13}
$$

are satisfied.

Since

$$
[\Xi_1, \Xi_2] = 0, \qquad [\Xi_1, \Xi_3] = \Xi_1, \qquad [\Xi_2, \Xi_3] = \Xi_2, \tag{5.14}
$$

applying the theorem, we introduce the new independent and dependent variables

$$
\begin{aligned}
&z_1 = x_1 - au_1 - bu_2, \qquad && z_2 = x_2 - bu_1 - cu_2,\\
&w_1 = u_1, && w_2 = u_2,
\end{aligned}
\tag{5.15}
$$

and the nonlinear system (5.11) reduces to

$$
\begin{aligned}
&\frac{\partial w_1}{\partial z_2} - \frac{\partial w_2}{\partial z_1} = 0,\\
&\kappa_{10}\left(\frac{\partial w_1}{\partial z_1}\right)^2 + \kappa_{11}\frac{\partial w_1}{\partial z_1}\frac{\partial w_1}{\partial z_2} + \kappa_{12}\frac{\partial w_1}{\partial z_1}\frac{\partial w_2}{\partial z_2}\\
&\quad + \kappa_{13}\left(\frac{\partial w_1}{\partial z_2}\right)^2 + \kappa_{14}\frac{\partial w_1}{\partial z_2}\frac{\partial w_2}{\partial z_2} + \kappa_{15}\left(\frac{\partial w_2}{\partial z_2}\right)^2 = 0,
\end{aligned}
\tag{5.16}
$$

i.e., *reads as an autonomous system polynomially homogeneous in the derivatives.*

We notice that system (5.16), by specializing the functions $\kappa_{10}, \ldots, \kappa_{15}$ as follows,

$$
\begin{aligned}
&\kappa_{10} = -\kappa(1 + w_2^2)^2,\\
&\kappa_{11} = 4\kappa w_1 w_2(1 + w_2^2),\\
&\kappa_{12} = 2((2 - \kappa)(1 + w_1^2 + w_2^2) - \kappa w_1^2 w_2^2),\\
&\kappa_{13} = -4(1 + w_1^2 + w_2^2 + \kappa w_1^2 w_2^2),\\
&\kappa_{14} = 4\kappa w_1 w_2(1 + w_1^2),\\
&\kappa_{15} = -\kappa(1 + w_1^2)^2,
\end{aligned}
\tag{5.17}
$$

is equivalent to the second order partial differential equation

$$
\begin{aligned}
&\kappa\left(1 + w_{z_2}^2\right)^2 w_{z_1 z_1}^2 - 4\kappa w_{z_1} w_{z_2}\left(1 + w_{z_2}^2\right) w_{z_1 z_1} w_{z_1 z_2}\\
&\quad - 2\left((2 - \kappa)\left(1 + w_{z_1}^2 + w_{z_2}^2\right) - \kappa w_{z_1}^2 w_{z_2}^2\right) w_{z_1 z_1} w_{z_2 z_2}\\
&\quad + 4\left(1 + w_{z_1}^2 + w_{z_2}^2 + \kappa w_{z_1}^2 w_{z_2}^2\right) w_{z_1 z_2}^2\\
&\quad - 4\kappa w_{z_1} w_{z_2}\left(1 + w_{z_1}^2\right) w_{z_1 z_2} w_{z_2 z_2} + \kappa\left(1 + w_{z_1}^2\right)^2 w_{z_2 z_2}^2 = 0,
\end{aligned}
\tag{5.18}
$$

where $w_{z_1} = \dfrac{\partial w}{\partial z_1} = w_1$, $w_{z_2} = \dfrac{\partial w}{\partial z_2} = w_2$, *and* κ *is an arbitrary function of* w_{z_1} *and* w_{z_2}.

Considering a smooth surface in $\mathbb{R}^3$ *with the metric* $ds^2 = dz_1^2 + dz_2^2 + dw^2$, *and its Gaussian and mean curvature,*

$$\begin{aligned} G &= \frac{w_{z_1 z_1} w_{z_2 z_2} - w_{z_1 z_2}^2}{(1 + w_{z_1}^2 + w_{z_2}^2)^2}, \\ H &= \frac{1}{2}\frac{(1 + w_{z_2}^2) w_{z_1 z_1} - 2 w_{z_1} w_{z_2} w_{z_1 z_2} + (1 + w_{z_1}^2) w_{z_2 z_2}}{(1 + w_{z_1}^2 + w_{z_2}^2)^{3/2}}, \end{aligned} \tag{5.19}$$

respectively, equation (5.18) can be written as

$$G = \kappa H^2, \tag{5.20}$$

whereupon it should be $\kappa(w_{z_1}, w_{z_2}) \leq 1$. *In the limit case* $\kappa \equiv 1$, *Eq. (5.20) characterizes a surface with all its points umbilic; it is known that a surface with all its points umbilic is a (open) domain of a plane or a sphere [79]. It is worth of being remarked that Eq. (5.20) with* $\kappa \equiv 1$ *is* strongly Lie remarkable *[60], since it is the unique second order partial differential equation uniquely characterized by the conformal Lie algebra in* $\mathbb{R}^3$ *[59].*

Remark 5.1.1. *Notice that the Example 5.1.1 provides a system polynomial in the derivatives which is transformed into a system polynomially homogeneous of degree 2 in the derivatives.*

Remark 5.1.2. *Theorem 5.1.1 can be used also when the nonlinear source system is autonomous. In such a way, when the hypotheses of the theorem are satisfied, the target system should be autonomous too; in fact, only in this case the invariance with respect to the homogeneous scaling of the independent variables of the target system implies that the system is a homogeneous polynomial in the derivatives (a quasilinear system if the degree of the homogeneous polynomial is 1).*

As we will show later, if the nonlinear system of partial differential equations involves the derivatives in polynomial form, then the conditions of Theorem 5.1.1 are necessary and sufficient for the mapping into a system where each equation is a homogeneous polynomial in the derivatives.

5.2 Applications

In this Section, we provide some examples of systems of first order nonlinear partial differential equations, whose Lie symmetries satisfy the conditions of Theorem 5.1.1, and prove that they can be transformed under suitable conditions to quasilinear autonomous and homogeneous systems. The nonlinear first order systems are obtained from second order $(1+1)$–, $(2+1)$– and $(3+1)$–dimensional Monge–Ampère equations (see [50] for details). It is worth of noticing that some classes of these systems have been proved to be linearizable by invertible point transformations in [67].

Hereafter, to shorten the formulas, we use the notation $u_{,i}$ and $u_{,ij}$ to indicate the first order partial derivative of u with respect to x_i, and the second order partial derivative of u with respect to x_i and x_j, respectively. Moreover, we shall denote with $f_{;i}$ and $f_{;ij}$ the first order partial derivative of the function f with respect to u_i and the second order partial derivative of f with respect to u_i and u_j, respectively.

5.2.1 Monge–Ampère equation in $(1+1)$ dimensions

In 1968, Boillat [12] proved that the most general completely exceptional second order equation in (1 + 1) dimensions is the well known Monge–Ampère equation,

$$\kappa_1\left(u_{,11}u_{,22}-u_{,12}^2\right)+\kappa_2 u_{,11}+\kappa_3 u_{,12}+\kappa_4 u_{,22}+\kappa_5=0, \tag{5.21}$$

with $u(x_1,x_2)$ a scalar function, and $\kappa_i\,(x_1,x_2,u,u_{,1},u_{,2})$ $(i=1,\dots,5)$ arbitrary smooth functions of the indicated arguments.

By introducing

$$u_1=u_{,1}, \qquad u_2=u_{,2}, \tag{5.22}$$

along with the assumptions that the functions κ_i $(i=1,\dots,5)$ depend at most on (u_1,u_2), we obtain the following nonlinear first order system:

$$\begin{aligned}&u_{2,1}-u_{1,2}=0,\\&\kappa_1\left(u_{1,1}u_{2,2}-u_{1,2}^2\right)+\kappa_2 u_{1,1}+\kappa_3 u_{1,2}+\kappa_4 u_{2,2}+\kappa_5=0.\end{aligned} \tag{5.23}$$

Through the substitutions

$$u_1\to u_1+\alpha_1 x_1+\alpha_2 x_2, \qquad u_2\to u_2+\alpha_2 x_1+\alpha_3 x_2, \tag{5.24}$$

α_i $(i=1,\dots,3)$ being constant, we get the system

$$\begin{aligned}&u_{2,1}-u_{1,2}=0,\\&\kappa_1\left(u_{1,1}u_{2,2}-u_{1,2}^2\right)+(\alpha_3\kappa_1+\kappa_2)u_{1,1}\\&\quad+(-2\alpha_2\kappa_1+\kappa_3)u_{1,2}+(\alpha_1\kappa_1+\kappa_4)u_{2,2}\\&\quad+((\alpha_1\alpha_3-\alpha_2^2)\kappa_1+\alpha_1\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_4+\kappa_5)=0.\end{aligned} \tag{5.25}$$

The nonlinear system (5.25) becomes homogeneous if

$$\kappa_5=-((\alpha_1\alpha_3-\alpha_2^2)\kappa_1+\alpha_1\kappa_2+\alpha_2\kappa_3+\alpha_3\kappa_4), \tag{5.26}$$

and in such a case it is straightforward to recognize that it admits the Lie point symmetries spanned by the operators

$$\Xi_1=\frac{\partial}{\partial x_1}, \qquad \Xi_2=\frac{\partial}{\partial x_2}, \qquad \Xi_3=(x_1-f_{;1})\frac{\partial}{\partial x_1}+(x_2-f_{;2})\frac{\partial}{\partial x_2}, \tag{5.27}$$

where $f(u_1,u_2)$ is an arbitrary smooth function of its arguments, provided that

$$\kappa_1=\frac{-\kappa_2 f_{;22}+\kappa_3 f_{;12}-\kappa_4 f_{;11}}{1+\alpha_3 f_{;22}+2\alpha_2 f_{;12}+\alpha_1 f_{;11}}. \tag{5.28}$$

Since

$$[\Xi_1,\Xi_2]=0, \qquad [\Xi_1,\Xi_3]=\Xi_1, \qquad [\Xi_2,\Xi_3]=\Xi_2, \tag{5.29}$$

we introduce the new variables

$$z_1=x_1-f_{;1}, \qquad z_2=x_2-f_{;2}, \qquad w_1=u_1, \qquad w_2=u_2, \tag{5.30}$$

and the generators of the point symmetries write as

$$\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_3 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}. \tag{5.31}$$

In terms of the new variables (5.30), the nonlinear system (5.25) becomes

$$\begin{aligned} & w_{2,1} - w_{1,2} = 0, \\ & (\alpha_3\kappa_1 + \kappa_2)w_{1,1} + (-2\alpha_2\kappa_1 + \kappa_3)w_{1,2} + (\alpha_1\kappa_1 + \kappa_4)w_{2,2} = 0, \end{aligned} \tag{5.32}$$

i.e., reads as an autonomous and homogeneous quasilinear system.

The following example provides a physical system leading to a Monge–Ampère equation.

Example 5.2.1 (One–dimensional Euler equations of isentropic fluids)**.** *Let us consider the Euler equations for an isentropic fluid*

$$\begin{aligned} & \rho_{,1} + (\rho v)_{,2} = 0, \\ & (\rho v)_{,1} + (\rho v^2 + p(\rho, s))_{,2} = 0, \\ & s_{,1} + v s_{,2} = 0, \end{aligned} \tag{5.33}$$

where ρ is the fluid density, v the velocity, s the entropy, $p(\rho, s)$ the pressure which is a function of the density and the entropy, x_1 the time, and x_2 the space variable.

By introducing a potential *function ϕ such that*

$$\rho = \phi_{,2}, \qquad \rho v = -\phi_{,1} \tag{5.34}$$

it results $s = s(\phi)$. Moreover, by introducing ψ and u such that

$$\begin{aligned} & \rho v = \psi_{,2} = -\phi_{,1}, \qquad && \rho v^2 + p = -\psi_{,1}, \\ & \psi = -u_{,1}, && \phi = u_{,2}, \end{aligned} \tag{5.35}$$

we arrive to the nonlinear equation

$$u_{,11} = \frac{u_{,12}^2}{u_{,22}} + p(u_{,22}, s(u_{,1})). \tag{5.36}$$

This equation becomes of Monge–Ampère type for the class of fluids characterized by the constitutive law of Von Kármán [93]

$$p = -\frac{\kappa^2(s)}{\rho} + b(s), \tag{5.37}$$

where $\kappa(s)$ and $b(s)$ are functions of the entropy. What we get is

$$u_{,11}u_{,22} - u_{,12}^2 + \kappa^2(s(u_{,2})) - b(s(u_{,2}))u_{,22} = 0. \tag{5.38}$$

The nonlinear first order system derived from equation (5.38) belongs to the class of equations (5.23) and is mapped to a homogeneous and autonomous quasilinear system provided that

$$\begin{aligned} & \kappa^2(s(u_2)) = \alpha_2^2 - \alpha_3(\alpha_1 + b(s(u_2))), \\ & b(s(u_2)) = \frac{1 + \alpha_3 f_{;22} + 2\alpha_2 f_{;12} + \alpha_1 f_{;11}}{f_{;11}}, \end{aligned} \tag{5.39}$$

and $f(u_1,u_2)$ is such that

$$\frac{\partial}{\partial u_1}\left(\frac{1+\alpha_3 f_{;22}+2\alpha_2 f_{;12}}{f_{;11}}\right)=0. \tag{5.40}$$

5.2.2 Monge–Ampère equation in $(2+1)$ dimensions

The most general second order hyperbolic completely exceptional equation in $(2+1)$ dimensions has been determined in 1973 by Ruggeri [84]; it is a linear combination of the determinant and all minors extracted from the 3×3 Hessian matrix of $u(x_1,x_2,x_3)$ with coefficients κ_i $(i=1,\ldots,14)$ depending on the independent variables, the dependent variable and its first order derivatives. This equation can be written in the following form:

$$\begin{aligned}&\kappa_1 H+\kappa_2\frac{\partial H}{\partial u_{,11}}+\kappa_3\frac{\partial H}{\partial u_{,12}}+\kappa_4\frac{\partial H}{\partial u_{,13}}+\kappa_5\frac{\partial H}{\partial u_{,22}}+\kappa_6\frac{\partial H}{\partial u_{,23}}+\kappa_7\frac{\partial H}{\partial u_{,33}}\\&\quad+\kappa_8 u_{,11}+\kappa_9 u_{,12}+\kappa_{10}u_{,13}+\kappa_{11}u_{,22}+\kappa_{12}u_{,23}+\kappa_{13}u_{,33}+\kappa_{14}=0,\end{aligned} \tag{5.41}$$

where H is the determinant of the 3×3 Hessian matrix of u.

Let us assume κ_i $(i=1,\ldots,14)$ depending at most on first order derivatives of u. By introducing

$$u_1=u_{,1},\qquad u_2=u_{,2},\qquad u_3=u_{,3}, \tag{5.42}$$

the following nonlinear first order system is obtained:

$$\begin{aligned}&u_{2,1}-u_{1,2}=0,\qquad u_{3,1}-u_{1,3}=0,\qquad u_{3,2}-u_{2,3}=0,\\&\kappa_1 H+\kappa_2\frac{\partial H}{\partial u_{1,1}}+\kappa_3\frac{\partial H}{\partial u_{1,2}}+\kappa_4\frac{\partial H}{\partial u_{1,3}}+\kappa_5\frac{\partial H}{\partial u_{2,2}}+\kappa_6\frac{\partial H}{\partial u_{2,3}}+\kappa_7\frac{\partial H}{\partial u_{3,3}}\\&\quad+\kappa_8 u_{1,1}+\kappa_9 u_{1,2}+\kappa_{10}u_{1,3}+\kappa_{11}u_{2,2}+\kappa_{12}u_{2,3}+\kappa_{13}u_{3,3}+\kappa_{14}=0.\end{aligned} \tag{5.43}$$

As done in the previous subsection, the substitutions

$$\begin{aligned}u_1&\to u_1+\alpha_1x_1+\alpha_2x_2+\alpha_3x_3,\\u_2&\to u_2+\alpha_2x_1+\alpha_4x_2+\alpha_5x_3,\\u_3&\to u_3+\alpha_3x_1+\alpha_5x_2+\alpha_6x_3,\end{aligned} \tag{5.44}$$

α_i $(i=1,\ldots,6)$ being constant, provided that

$$\begin{aligned}\kappa_{14}=&(\alpha_1\alpha_5^2+\alpha_2^2\alpha_6-\alpha_1\alpha_4\alpha_6-2\alpha_2\alpha_3\alpha_5+\alpha_3^2\alpha_4)\kappa_1-(\alpha_5^2-\alpha_4\alpha_6)\kappa_2\\&+2(\alpha_2\alpha_6-\alpha_3\alpha_5)\kappa_3+2(\alpha_3\alpha_4-\alpha_2\alpha_5)\kappa_4+(\alpha_3^2-\alpha_1\alpha_6)\kappa_5\\&+2(\alpha_1\alpha_5-\alpha_2\alpha_3)\kappa_6+(\alpha_2^2-\alpha_1\alpha_4)\kappa_7-\alpha_1\kappa_8-\alpha_2\kappa_9-\alpha_3\kappa_{10}\\&-\alpha_4\kappa_{11}-\alpha_5\kappa_{12}-\alpha_6\kappa_{13},\end{aligned} \tag{5.45}$$

allow us to get a homogeneous system with the same differential structure as (5.43), say

$$
\begin{aligned}
&u_{2,1}-u_{1,2}=0, \qquad u_{3,1}-u_{1,3}=0, \qquad u_{3,2}-u_{2,3}=0,\\
&\widehat{\kappa}_1 H+\widehat{\kappa}_2\frac{\partial H}{\partial u_{1,1}}+\widehat{\kappa}_3\frac{\partial H}{\partial u_{1,2}}+\widehat{\kappa}_4\frac{\partial H}{\partial u_{1,3}}+\widehat{\kappa}_5\frac{\partial H}{\partial u_{2,2}}+\widehat{\kappa}_6\frac{\partial H}{\partial u_{2,3}}+\widehat{\kappa}_7\frac{\partial H}{\partial u_{3,3}}\\
&\quad+\widehat{\kappa}_8 u_{1,1}+\widehat{\kappa}_9 u_{1,2}+\widehat{\kappa}_{10}u_{1,3}+\widehat{\kappa}_{11}u_{2,2}+\widehat{\kappa}_{12}u_{2,3}+\widehat{\kappa}_{13}u_{3,3}=0,
\end{aligned}
\tag{5.46}
$$

where the expression of $\widehat{\kappa}_i$ in terms of the coefficients κ_i ($i=1,\dots,13$) and the constants α_j ($j=1,\dots,6$) can be easily found.

The nonlinear system (5.46) admits the Lie point symmetries spanned by the operators

$$
\begin{aligned}
&\Xi_1=\frac{\partial}{\partial x_1}, \qquad \Xi_2=\frac{\partial}{\partial x_2}, \qquad \Xi_3=\frac{\partial}{\partial x_3},\\
&\Xi_4=(x_1-f_{;1})\frac{\partial}{\partial x_1}+(x_2-f_{;2})\frac{\partial}{\partial x_2}+(x_3-f_{;3})\frac{\partial}{\partial x_3},
\end{aligned}
\tag{5.47}
$$

where $f(u_1,u_2,u_3)$ is a smooth arbitrary function of its arguments, provided that the following relations hold true:

$$
\begin{aligned}
&\widehat{\kappa}_1-H_f^{11}\widehat{\kappa}_8-H_f^{12}\widehat{\kappa}_9-H_f^{13}\widehat{\kappa}_{10}-H_f^{22}\widehat{\kappa}_{11}-H_f^{23}\widehat{\kappa}_{12}-H_f^{33}\widehat{\kappa}_{13}=0,\\
&\widehat{\kappa}_2+f_{;33}\widehat{\kappa}_{11}-f_{;23}\widehat{\kappa}_{12}+f_{;22}\widehat{\kappa}_{13}=0,\\
&2\widehat{\kappa}_3-f_{;33}\widehat{\kappa}_9+f_{;23}\widehat{\kappa}_{10}+f_{;13}\widehat{\kappa}_{12}-2f_{;12}\kappa_{13}=0,\\
&2\widehat{\kappa}_4+f_{;23}\widehat{\kappa}_9-f_{;22}\widehat{\kappa}_{10}-2f_{;13}\widehat{\kappa}_{11}+f_{;12}\widehat{\kappa}_{12}=0,\\
&\widehat{\kappa}_5+f_{;33}\widehat{\kappa}_8-f_{;13}\widehat{\kappa}_{10}+f_{;11}\widehat{\kappa}_{13}=0,\\
&2\widehat{\kappa}_6-2f_{;23}\widehat{\kappa}_8+f_{;13}\widehat{\kappa}_9+f_{;12}\widehat{\kappa}_{10}-f_{;11}\widehat{\kappa}_{12}=0,\\
&\widehat{\kappa}_7+f_{;22}\widehat{\kappa}_8-f_{;12}\widehat{\kappa}_9+f_{;11}\widehat{\kappa}_{11}=0,
\end{aligned}
\tag{5.48}
$$

H_f^{ij} denoting the cofactor of the (i,j)–entry of the Hessian matrix H_f of the function $f(u_1,u_2,u_3)$. It is evident that conditions (5.48) place severe restrictions on the coefficients of system (5.46); in fact, they state that the functions $\widehat{\kappa}_i$ ($i=1,\dots,7$) have to be expressed in terms of the coefficients $\widehat{\kappa}_i$ ($i=8,\dots,13$) and the function f. These symmetries generate a 4–dimensional solvable Lie algebra,

$$
[\Xi_i,\Xi_j]=0, \qquad [\Xi_i,\Xi_4]=\Xi_i, \qquad i,j=1,\dots,3, \tag{5.49}
$$

whereupon we may introduce the new variables

$$
\begin{aligned}
&z_1=x_1-f_{;1}, \quad z_2=x_2-f_{;2}, \quad z_3=x_3-f_{;3},\\
&w_1=u_1, \qquad\quad w_2=u_2, \qquad\quad w_3=u_3,
\end{aligned}
\tag{5.50}
$$

and the generators of the point symmetries write as

$$
\begin{aligned}
&\Xi_1=\frac{\partial}{\partial z_1}, \qquad \Xi_2=\frac{\partial}{\partial z_2}, \qquad \Xi_3=\frac{\partial}{\partial z_3},\\
&\Xi_4=z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+z_3\frac{\partial}{\partial z_3}.
\end{aligned}
\tag{5.51}
$$

In terms of the new variables (5.50), the nonlinear system (5.46) reduces to

$$\begin{aligned} &w_{2,1} - w_{1,2} = 0, \qquad w_{3,1} - w_{1,3} = 0, \qquad w_{3,2} - w_{2,3} = 0, \\ &\widehat{\kappa}_8 w_{1,1} + \widehat{\kappa}_9 w_{1,2} + \widehat{\kappa}_{10} w_{1,3} + \widehat{\kappa}_{11} w_{2,2} + \widehat{\kappa}_{12} w_{2,3} + \widehat{\kappa}_{13} w_{3,3} = 0, \end{aligned} \tag{5.52}$$

i.e., to an autonomous and homogeneous quasilinear system.

5.2.3 Monge–Ampère equation in $(3+1)$ dimensions

The most general second order completely exceptional equation in $(3+1)$ dimensions has been characterized by Donato *et al.* [27] and once again it is given as a linear combination of the determinant and all minors extracted from the 4×4 Hessian matrix of $u(x_1, x_2, x_3, x_4)$ with coefficients κ_i ($i = 1, \ldots, 43$) depending on the independent variables, the dependent variable and its first order derivatives:

$$\begin{aligned} &\kappa_1 H + \sum_r \kappa_r \frac{\partial H}{\partial u_{,ij}} + \sum_s \kappa_s \frac{\partial^2 H}{\partial u_{,kl} \partial u_{,mn}} + \sum_r \kappa_{r+31} u_{,ij} + \kappa_{43} = 0, \\ &i, j, k, l, m, n = 1, \ldots, 4, \quad i \leq j, \quad k < l, \quad k \leq m < n, \\ &r = \frac{i(9-i)}{2} + j - 3, \quad s = \sigma_{mn} + \frac{\sigma_{kl}(13 - \sigma_{kl})}{2} + 6, \\ &\sigma_{ab} = 4(a-1) - \frac{a(a+1)}{2} + b, \end{aligned} \tag{5.53}$$

where H is the determinant of the 4×4 Hessian matrix of u; actually, the Monge–Ampère equation in $(3+1)$ dimensions involves only 42 independent coefficients because

$$\frac{\partial^2 H}{\partial u_{,12} \partial u_{,34}} + \frac{\partial^2 H}{\partial u_{,13} \partial u_{,24}} + \frac{\partial^2 H}{\partial u_{,14} \partial u_{,23}} = 0. \tag{5.54}$$

Hereafter, we assume, without loss of generality, $\kappa_{24} = 0$, and the remaining functions κ_i depending at most on first order derivatives.

By introducing

$$u_1 = u_{,1}, \qquad u_2 = u_{,2}, \qquad u_3 = u_{,3}, \qquad u_4 = u_{,4}, \tag{5.55}$$

the following nonlinear first order system is obtained:

$$\begin{aligned} &u_{2,1} - u_{1,2} = 0, \qquad u_{3,1} - u_{1,3} = 0, \qquad u_{4,1} - u_{1,4} = 0, \\ &u_{3,2} - u_{2,3} = 0, \qquad u_{4,2} - u_{2,4} = 0, \qquad u_{4,3} - u_{3,4} = 0, \\ &\kappa_1 H + \sum_r \kappa_r \frac{\partial H}{\partial u_{i,j}} + \sum_s \kappa_s \frac{\partial^2 H}{\partial u_{k,l} \partial u_{m,n}} + \sum_r \kappa_{r+31} u_{i,j} + \kappa_{43} = 0. \end{aligned} \tag{5.56}$$

As done in the previous subsection, the substitutions

$$\begin{aligned} u_1 &\to u_1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4, \\ u_2 &\to u_2 + \alpha_2 x_1 + \alpha_5 x_2 + \alpha_6 x_3 + \alpha_7 x_4, \\ u_3 &\to u_3 + \alpha_3 x_1 + \alpha_6 x_2 + \alpha_8 x_3 + \alpha_9 x_4, \\ u_4 &\to u_4 + \alpha_4 x_1 + \alpha_7 x_2 + \alpha_9 x_3 + \alpha_{10} x_4, \end{aligned} \tag{5.57}$$

provided that κ_{43} can be suitably expressed in terms of the remaining coefficients and of the constants α_i ($i = 1, \ldots, 10$), lead to a homogeneous system like (5.56) where we can assume $\kappa_{43} = 0$.

This nonlinear system admits the Lie point symmetries spanned by the operators

$$\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial x_3}, \qquad \Xi_4 = \frac{\partial}{\partial x_4},\\
&\Xi_5 = (x_1 - f_{;1})\frac{\partial}{\partial x_1} + (x_2 - f_{;2})\frac{\partial}{\partial x_2} + (x_3 - f_{;3})\frac{\partial}{\partial x_3} + (x_4 - f_{;4})\frac{\partial}{\partial x_4},
\end{aligned} \tag{5.58}$$

where $f(u_1, u_2, u_3, u_4)$ is a smooth arbitrary function of the indicated arguments, provided that κ_i ($i = 1, \ldots, 32$) can be expressed suitably in terms of κ_j ($j = 33, \ldots, 42$) and $f(u_1, u_2, u_3, u_4)$:

$$\begin{aligned}
&\kappa_1 + H_f^{11}\kappa_{33} + H_f^{12}\kappa_{34} + H_f^{13}\kappa_{35} + H_f^{14}\kappa_{36} + H_f^{22}\kappa_{37} + H_f^{23}\kappa_{38} + H_f^{24}\kappa_{39}\\
&\quad + H_f^{33}\kappa_{40} + H_f^{34}\kappa_{41} + H_f^{44}\kappa_{42} = 0,\\
&\kappa_2 + (f_{;34}^2 - f_{;33}f_{;44})\kappa_{37} + (f_{;23}f_{;44} - f_{;24}f_{;34})\kappa_{38} + (f_{;24}f_{;33} - f_{;23}f_{;34})\kappa_{39}\\
&\quad + (f_{;24}^2 - f_{;22}f_{;44})\kappa_{40} + (f_{;22}f_{;34} - f_{;23}f_{;24})\kappa_{41} + (f_{;23}^2 - f_{;22}f_{;33})\kappa_{42} = 0,\\
&2\kappa_3 + (f_{;33}f_{;44} - f_{;34}^2)\kappa_{34} + (f_{;24}f_{;34} - f_{;23}f_{;44})\kappa_{35} + (f_{;23}f_{;34} - f_{;24}f_{;33})\kappa_{36}\\
&\quad + (f_{;14}f_{;34} - f_{;13}f_{;44})\kappa_{38} + (f_{;13}f_{;34} - f_{;14}f_{;33})\kappa_{39}\\
&\quad + 2(f_{;12}f_{;44} - f_{;14}f_{;24})\kappa_{40} + (f_{;13}f_{;24} - 2f_{;12}f_{;34} + f_{;14}f_{;23})\kappa_{41}\\
&\quad + 2(f_{;12}f_{;33} - f_{;13}f_{;23})\kappa_{42} = 0,\\
&2\kappa_4 + (f_{;24}f_{;34} - f_{;23}f_{;44})\kappa_{34} + (f_{;22}f_{;44} - f_{;24}^2) + (f_{;23}f_{;24} - f_{;22}f_{;34})\kappa_{36}\\
&\quad + 2(f_{;13}f_{;44} - f_{;14}f_{;34})\kappa_{37} + (f_{;14}f_{;24} - f_{;12}f_{;44})\kappa_{38}\\
&\quad + (f_{;12}f_{;34} + f_{;14}f_{;23} - 2f_{;13}f_{;24})\kappa_{39} + (f_{;12}f_{;24} - f_{;14}f_{;22})\kappa_{41}\\
&\quad + 2(f_{;13}f_{;22} - f_{;12}f_{;23})\kappa_{42} = 0,\\
&2\kappa_5 + (f_{;23}f_{;34} - f_{;24}f_{;33})\kappa_{34} + (f_{;23}f_{;24} - f_{;22}f_{;34})\kappa_{35} + (f_{;22}f_{;33} - f_{;23}^2)\kappa_{36}\\
&\quad + 2(f_{;14}f_{;33} - f_{;13}f_{;34})\kappa_{37} + (f_{;12}f_{;34} + f_{;13}f_{;24} - 2f_{;14}f_{;23})\kappa_{38}\\
&\quad + (f_{;13}f_{;23} - f_{;12}f_{;33})\kappa_{39} + 2(f_{;14}f_{;22} - f_{;12}f_{;24})\kappa_{40}\\
&\quad + (f_{;12}f_{;23} - f_{;13}f_{;22})\kappa_{41} = 0,\\
&\kappa_6 + (f_{;34}^2 - f_{;33}f_{;44})\kappa_{33} + (f_{;13}f_{;44} - f_{;14}f_{;34})\kappa_{35} + (f_{;14}f_{;33} - f_{;13}f_{;34})\kappa_{36}\\
&\quad + (f_{;14}^2 - f_{;11}f_{;44})\kappa_{40} + (f_{;11}f_{;34} - f_{;13}f_{;14})\kappa_{41} + (f_{;13}^2 - f_{;11}f_{;33})\kappa_{42} = 0,\\
&2\kappa_7 + 2(f_{;23}f_{;44} - f_{;24}f_{;34})\kappa_{33} + (f_{;14}f_{;34} - f_{;13}f_{;44})\kappa_{34}\\
&\quad + (f_{;14}f_{;24} - f_{;12}f_{;44})\kappa_{35} + (f_{;12}f_{;34} + f_{;13}f_{;24} - 2f_{;14}f_{;23})\kappa_{36}\\
&\quad + (f_{;11}f_{;44} - f_{;14}^2)\kappa_{38} + (f_{;13}f_{;14} - f_{;11}f_{;34})\kappa_{39} + (f_{;12}f_{;14} - f_{;11}f_{;24})\kappa_{41}\\
&\quad + 2(f_{;11}f_{;23} - f_{;12}f_{;13})\kappa_{42} = 0,\\
&2\kappa_8 + 2(f_{;24}f_{;33} - f_{;23}f_{;34})\kappa_{33} + (f_{;13}f_{;34} - f_{;14}f_{;33})\kappa_{34}\\
&\quad + (f_{;12}f_{;34} + f_{;14}f_{;23} - 2f_{;13}f_{;24})\kappa_{35} + (f_{;13}f_{;23} - f_{;12}f_{;33})\kappa_{36}\\
&\quad + (f_{;13}f_{;14} - f_{;11}f_{;34})\kappa_{38} + (f_{;11}f_{;33} - f_{;13}^2)\kappa_{39} + 2(f_{;11}f_{;24} - f_{;12}f_{;14})\kappa_{40}\\
&\quad + (f_{;12}f_{;13} - f_{;11}f_{;23})\kappa_{41} = 0,\\
&\kappa_9 + (f_{;24}^2 - f_{;22}f_{;44})\kappa_{33} + (f_{;12}f_{;44} - f_{;14}f_{;24})\kappa_{34} + (f_{;14}f_{;22} - f_{;12}f_{;24})\kappa_{36}\\
&\quad + (f_{;14}^2 - f_{;11}f_{;44})\kappa_{37} + (f_{;11}f_{;24} - f_{;12}f_{;14})\kappa_{39} + (f_{;12}^2 - f_{;11}f_{;22})\kappa_{42} = 0,
\end{aligned}$$

$$
\begin{aligned}
&2\kappa_{10} + 2(f_{;22}f_{;34} - f_{;23}f_{;24})\kappa_{33} + (f_{;13}f_{;24} + f_{;14}f_{;23} - 2f_{;12}f_{;34})\kappa_{34}\\
&\quad + (f_{;12}f_{;24} - f_{;14}f_{;22})\kappa_{35} + (f_{;12}f_{;23} - f_{;13}f_{;22})\kappa_{36}\\
&\quad + 2(f_{;11}f_{;34} - f_{;13}f_{;14})\kappa_{37} + (f_{;12}f_{;14} - f_{;11}f_{;24})\kappa_{38}\\
&\quad + (f_{;12}f_{;13} - f_{;11}f_{;23})\kappa_{39} + (f_{;11}f_{;22} - f_{;12}^2)\kappa_{41} = 0,\\
&\kappa_{11} + (f_{;23}^2 - f_{;22}f_{;33})\kappa_{33} + (f_{;12}f_{;33} - f_{;13}f_{;23})\kappa_{34} + (f_{;13}f_{;22} - f_{;12}f_{;23})\kappa_{35}\\
&\quad + (f_{;13}^2 - f_{;11}f_{;33})\kappa_{37} + (f_{;11}f_{;23} - f_{;12}f_{;13})\kappa_{38} + (f_{;12}^2 - f_{;11}f_{;22})\kappa_{40} = 0,\\
&2\kappa_{12} - f_{;44}\kappa_{40} + f_{;34}\kappa_{41} - f_{;33}\kappa_{42} = 0,\\
&2\kappa_{13} + f_{;44}\kappa_{38} - f_{;34}\kappa_{39} - f_{;24}\kappa_{41} + 2f_{;23}\kappa_{42} = 0,\\
&2\kappa_{14} - f_{;34}\kappa_{38} + f_{;33}\kappa_{39} + 2f_{;24}\kappa_{40} - f_{;23}\kappa_{41} = 0,\\
&2\kappa_{15} + f_{;44}\kappa_{35} - f_{;34}\kappa_{36} - f_{;14}\kappa_{41} + 2f_{;13}\kappa_{42} = 0,\\
&2\kappa_{16} - f_{;34}\kappa_{35} + f_{;33}\kappa_{36} + 2f_{;14}\kappa_{40} - f_{;13}\kappa_{41} = 0,\\
&2\kappa_{17} + f_{;34}\kappa_{34} - f_{;23}\kappa_{36} - f_{;14}\kappa_{38} + f_{;12}\kappa_{41} = 0,\\
&2\kappa_{18} - f_{;44}\kappa_{37} + f_{;24}\kappa_{39} - f_{;22}\kappa_{42} = 0,\\
&2\kappa_{19} + 2f_{;34}\kappa_{37} - f_{;24}\kappa_{38} - f_{;23}\kappa_{39} + f_{;22}\kappa_{41} = 0,\\
&2\kappa_{20} + f_{;44}\kappa_{34} - f_{;24}\kappa_{36} - f_{;14}\kappa_{39} + 2f_{;12}\kappa_{42} = 0,\\
&2\kappa_{21} + f_{;24}\kappa_{35} - f_{;23}\kappa_{36} - f_{;14}\kappa_{38} + f_{;13}\kappa_{39} = 0,\\
&2\kappa_{22} - f_{;24}\kappa_{34} + f_{;22}\kappa_{36} + 2f_{;14}\kappa_{37} - f_{;12}\kappa_{39} = 0,\\
&2\kappa_{23} - f_{;33}\kappa_{37} + f_{;23}\kappa_{38} - f_{;22}\kappa_{40} = 0,\\
&2\kappa_{25} + f_{;33}\kappa_{34} - f_{;23}\kappa_{35} - f_{;13}\kappa_{38} + 2f_{;12}\kappa_{40} = 0,\\
&2\kappa_{26} - f_{;23}\kappa_{34} + f_{;22}\kappa_{35} + 2f_{;13}\kappa_{37} - f_{;12}\kappa_{38} = 0,\\
&2\kappa_{27} - f_{;44}\kappa_{33} + f_{;14}\kappa_{36} - f_{;11}\kappa_{42} = 0,\\
&2\kappa_{28} + 2f_{;34}\kappa_{33} - f_{;14}\kappa_{35} - f_{;13}\kappa_{36} + f_{;11}\kappa_{41} = 0,\\
&2\kappa_{29} + 2f_{;24}\kappa_{33} - f_{;14}\kappa_{34} - f_{;12}\kappa_{36} + f_{;11}\kappa_{39} = 0,\\
&2\kappa_{30} - f_{;33}\kappa_{33} + f_{;13}\kappa_{35} - f_{;11}\kappa_{40} = 0,\\
&2\kappa_{31} + 2f_{;23}\kappa_{33} - f_{;13}\kappa_{34} - f_{;12}\kappa_{35} + f_{;11}\kappa_{38} = 0,\\
&2\kappa_{32} - f_{;22}\kappa_{33} + f_{;12}\kappa_{34} - f_{;11}\kappa_{37} = 0.
\end{aligned}
$$

Also in this case, the previous conditions place severe restrictions on the coefficients of system (5.56) since imply that the functions κ_i $(i = 1, \dots, 32)$ have to be expressed in terms of the coefficients κ_i $(i = 33, \dots, 42)$ and the function f.

Due to

$$
[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_5] = \Xi_5, \qquad i, j = 1, \dots, 4, \tag{5.59}
$$

we introduce the new variables

$$
\begin{array}{llll}
z_1 = x_1 - f_{;1}, & z_2 = x_2 - f_{;2}, & z_3 = x_3 - f_{;3}, & z_4 = x_4 - f_{;4},\\
w_1 = u_1, & w_2 = u_2, & w_3 = u_3, & w_4 = u_4,
\end{array} \tag{5.60}
$$

and the generators of the point symmetries write as

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_3 = \frac{\partial}{\partial z_3}, \qquad \Xi_4 = \frac{\partial}{\partial z_4},\\
&\Xi_5 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3} + z_4\frac{\partial}{\partial z_4}.
\end{aligned} \tag{5.61}
$$

In terms of the new variables (5.60), the nonlinear system (5.56) assumes the form of an autonomous and homogeneous quasilinear system,

$$\begin{aligned}
&w_{2,1} - w_{1,2} = 0, \qquad w_{3,1} - w_{1,3} = 0, \qquad w_{4,1} - w_{1,4} = 0,\\
&w_{3,2} - w_{2,3} = 0, \qquad w_{4,2} - w_{2,4} = 0, \qquad w_{4,3} - w_{3,4} = 0,\\
&\kappa_{33}w_{1,1} + \kappa_{34}w_{1,2} + \kappa_{35}w_{1,3} + \kappa_{36}w_{1,4} + \kappa_{37}w_{2,2} + \kappa_{38}w_{2,3}\\
&\quad + \kappa_{39}w_{2,4} + \kappa_{40}w_{3,3} + \kappa_{41}w_{3,4} + \kappa_{42}w_{4,4} = 0,
\end{aligned} \tag{5.62}$$

where $\kappa_i = \kappa_i(w_1, w_2, w_3, w_4)$ $(i = 33, \dots, 42)$.

5.3 Reduction to autonomous polynomially homogeneous in the derivatives form

In this Section, we consider a general nonlinear system of first order partial differential equations involving the derivatives of the unknown variables in polynomial (of degree greater than 1) form, and establish a theorem giving necessary and sufficient conditions in order to map it to an autonomous system which is polynomially homogeneous in the derivatives.

In some relevant situations, *e.g.*, Monge–Ampère systems, the target system results to be quasilinear, but there are cases where the system we obtain is polynomially homogeneous in the derivatives but not quasilinear (see Example 5.1.1). This means that the conditions of the theorem are only necessary for the reduction of a *nonlinear* first order system to autonomous and homogeneous *quasilinear* form [31].

The main difference of the theorem here presented with the similar one proved in [65] (concerned with the transformation of a general first order quasilinear system of partial differential equations into a first order quasilinear homogeneous and autonomous system) consists in the possibility of admitting now an invertible point transformation like

$$\mathbf{z} = \mathbf{Z}(\mathbf{x}, \mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x}, \mathbf{u}), \tag{5.63}$$

i.e., a mapping where the new independent variables $\mathbf{z}$ are allowed to depend also on the old dependent ones. In the next Section, a theorem giving necessary and sufficient conditions for the existence of an invertible mapping linking a nonlinear system of first order partial differential equations which is polynomial in the derivatives to an autonomous system polynomially homogeneous in the derivatives is proved.

5.3.1 Necessary and sufficient conditions

Let us consider a general system of first order partial differential equations

$$\boldsymbol{\Delta}\left(\mathbf{x}, \mathbf{u}, \mathbf{u}^{(1)}\right) = \mathbf{0}, \tag{5.64}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{u}^{(1)} \in \mathbb{R}^{mn}$ are the independent variables, the dependent variables, and the first order partial derivatives, respectively. In particular, in the following we consider systems (5.64) composed by equations which are polynomial in the derivatives, with coefficients depending at most on $\mathbf{x}$ and $\mathbf{u}$, *i.e.*, systems made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = 0, \qquad s = 1,\dots,m, \tag{5.65}$$

where $\boldsymbol{\alpha}$ is the multi–index $(\alpha_1,\dots,\alpha_r)$, $\mathbf{j}$ the multi–index $(j_1,\dots,j_r)$, $\alpha_k = 1,\dots,m$, $j_k = 1,\dots,n$, N_s are integers, and $A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u})$, $B^s(\mathbf{x},\mathbf{u})$ smooth functions of their arguments.

The aim is to determine necessary and sufficient conditions for the construction of an invertible point transformation

$$\mathbf{z} = \mathbf{Z}\left(\mathbf{x},\mathbf{u}\right), \qquad \mathbf{w} = \mathbf{W}\left(\mathbf{x},\mathbf{u}\right), \tag{5.66}$$

mapping system (5.65) into an equivalent autonomous one which is homogeneous polynomial in the derivatives $\mathbf{w}^{(1)}$, *i.e.*, made by equations of the form

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \qquad s = 1,\dots,m, \tag{5.67}$$

for some integers $\overline{N}_s$; of course, it may occur that the target system turns out to be linear in the derivatives, *i.e.*, $\overline{N}_s = 1$ $(s = 1,\dots,m)$, whereupon we have an autonomous and homogeneous quasilinear system.

The following lemma guarantees that an invertible point transformation like (5.66) preserves the polynomial structure in the derivatives.

Lemma 5.3.1. *Given a first order system of partial differential equations like (5.65) which is polynomial in the derivatives, then an invertible point transformation like (5.66) produces a first order system which is still polynomial in the derivatives.*

Proof. Straightforward, by using the chain rule. □

Theorem 5.3.1. *The nonlinear first order system of partial differential equations polynomial in the derivatives*

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=1}^{N_s} A^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{x},\mathbf{u}) \prod_{k=1}^{|\boldsymbol{\alpha}|} \frac{\partial u_{\alpha_k}}{\partial x_{j_k}} + B^s(\mathbf{x},\mathbf{u}) = 0, \qquad s = 1,\dots,m, \tag{5.68}$$

is mapped by an invertible point transformation, say

$$\mathbf{z} = \mathbf{Z}(\mathbf{x},\mathbf{u}), \qquad \mathbf{w} = \mathbf{W}(\mathbf{x},\mathbf{u}), \tag{5.69}$$

to the equivalent nonlinear first order autonomous system having homogeneous polynomial form, say

$$\sum_{|\boldsymbol{\alpha}|,|\mathbf{j}|=\overline{N}_s} \widetilde{A}^s_{\boldsymbol{\alpha}\mathbf{j}}(\mathbf{w}) \prod_{k=1}^{\overline{N}_s} \frac{\partial w_{\alpha_k}}{\partial z_{j_k}} = 0, \qquad s = 1,\dots,m, \tag{5.70}$$

for some integers $\overline{N}_s$, if and only if there exists an $(n+1)$–dimensional subalgebra of the Lie algebra of point symmetries, admitted by system (5.68), spanned by the *vector fields*

$$\Xi_i = \sum_{i=1}^{n} \xi_i^j(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_i^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u_\alpha}, \qquad i = 1, \dots, n+1, \tag{5.71}$$

such that

$$\begin{aligned} &[\Xi_i, \Xi_j] = 0, && i = 1, \dots, n-1, \quad i < j \leq n, \\ &[\Xi_i, \Xi_{n+1}] = \Xi_i, && i = 1, \dots, n, \end{aligned} \tag{5.72}$$

where the Abelian subalgebra spanned by $\Xi_1, \dots, \Xi_n$ *generates a distribution of rank* n*. Moreover, the variables* $\mathbf{w}$*, which by construction are invariants of* $\Xi_1, \dots,$ Ξ_n*, have to be invariant with respect to* Ξ_{n+1} *too.*

Proof. Suppose the conditions of Theorem 5.3.1 are satisfied, and so the system (5.68) admits an $(n+1)$–dimensional algebra as subalgebra of the algebra of its Lie point symmetries generating a distribution of rank $(n+1)$ and verifying the structure conditions (5.72). Let us introduce a set of canonical variables for the vector field Ξ_1, say

$$y_i^1 \quad (i = 1, \dots, n), \qquad v_\alpha^1 \quad (\alpha = 1, \dots, m), \tag{5.73}$$

such that

$$\Xi_1 y_1^1 = 1, \qquad \Xi_1 y_{i_1}^1 = 0, \qquad \Xi_1 v_\alpha^1 = 0 \tag{5.74}$$

$(i_1 = 2, \dots, n)$; as a consequence, Ξ_1 takes the form

$$\Xi_1 = \frac{\partial}{\partial y_1^1}, \tag{5.75}$$

i.e., it corresponds to a translation in the variable y_1^1.

Since $[\Xi_1, \Xi_2] = 0$, it is

$$\Xi_1(\Xi_2 y_i^1) = \Xi_2(\Xi_1 y_i^1) = 0, \qquad \Xi_1(\Xi_2 v_\alpha^1) = \Xi_2(\Xi_1 v_\alpha^1) = 0 \tag{5.76}$$

$(i = 1, \dots, n;\ \alpha = 1, \dots, m)$. Thus, the infinitesimals of Ξ_2, represented in terms of the canonical variables of Ξ_1, will depend upon the invariants of Ξ_1 only, *i.e.*, Ξ_2 writes as

$$\Xi_2 = \sum_{i=1}^{n} \Theta_i^2(y_{j_1}^1, v_\beta^1) \frac{\partial}{\partial y_i^1} + \sum_{\alpha=1}^{m} \Lambda_\alpha^2(y_{j_1}^1, v_\beta^1) \frac{\partial}{\partial v_\alpha^1} \tag{5.77}$$

$(j_1 = 2, \dots, n;\ \beta = 1, \dots, m)$.

If $\Theta_1^2 \neq 0$ we need to replace y_1^1 with

$$y_1^1 + \varphi_1^1(y_{j_1}^1, v_\beta^1), \tag{5.78}$$

where the function φ_1^1 satisfies

$$\Theta_1^2(y_{j_1}^1, v_\beta^1) + \sum_{i_1=2}^{n} \Theta_{i_1}^2(y_{j_1}^1, v_\beta^1) \frac{\partial \varphi_1^1}{\partial y_{i_1}^1} + \sum_{\alpha=1}^{m} \Lambda_\alpha^2(y_{j_1}^1, v_\beta^1) \frac{\partial \varphi_1^1}{\partial v_\alpha^1} = 0. \tag{5.79}$$

That enables us to write Ξ_1 and Ξ_2 as follows:

$$\Xi_1 = \frac{\partial}{\partial y^1_1}, \qquad \Xi_2 = \sum_{i_1=2}^{n} \Theta^2_{i_1}(y^1_{j_1}, v^1_\beta)\frac{\partial}{\partial y^1_{i_1}} + \sum_{\alpha=1}^{m} \Lambda^2_\alpha(y^1_{j_1}, v^1_\beta)\frac{\partial}{\partial v^1_\alpha}, \tag{5.80}$$

where $j_1 = 2, \ldots, n$.

Introducing the canonical variables

$$y^2_1 = y^1_1, \qquad y^2_2, \qquad y^2_{i_2} \quad (i_2 = 3, \ldots, n), \qquad v^2_\alpha \quad (\alpha = 1, \ldots, m), \tag{5.81}$$

such that

$$\Xi_2 y^2_2 = 1, \qquad \Xi_2 y^2_{i_2} = 0, \qquad \Xi_2 v^2_\alpha = 0 \tag{5.82}$$

($i_1 = 2, \ldots, n$), it is obtained

$$\Xi_2 = \frac{\partial}{\partial y^2_2}. \tag{5.83}$$

Continuing inductively for $k = 2, \ldots, n-1$, since Ξ_{k+1} commutes with Ξ_1, …, Ξ_k, in terms of the canonical variables

$$y^k_i \quad (i = 1, \ldots, n), \qquad v^k_\alpha \quad (\alpha = 1, \ldots, m), \tag{5.84}$$

we have

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial y^k_1}, \qquad \Xi_2 = \frac{\partial}{\partial y^k_2}, \qquad \ldots\ldots, \qquad \Xi_k = \frac{\partial}{\partial y^k_k}, \\ &\Xi_{k+1} = \sum_{i=1}^{n} \Theta^{k+1}_i(y^k_{j_k}, v^k_\beta)\frac{\partial}{\partial y^k_i} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_\alpha(y^k_{j_k}, v^k_\beta)\frac{\partial}{\partial v^k_\alpha}, \end{aligned} \tag{5.85}$$

where $j_k = k+1, \ldots, n$. If $\Theta^{k+1}_\ell \neq 0$, for $\ell = 1, \ldots, k$, we need to replace the variable y^k_ℓ with

$$y^k_\ell + \varphi^k_\ell(y^k_{j_k}, v^k_\beta), \tag{5.86}$$

where the function φ^k_ℓ satisfies

$$\Theta^{k+1}_\ell(y^k_{j_k}, v^k_\beta) + \sum_{i_k=k+1}^{n} \Theta^{k+1}_{i_k}(y^k_{j_k}, v^k_\beta)\frac{\partial \varphi^k_\ell}{\partial y^1_{i_k}} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_\alpha(y^k_{j_k}, v^k_\beta)\frac{\partial \varphi^k_\ell}{\partial v^k_\alpha} = 0, \tag{5.87}$$

so that Ξ_{k+1} writes as

$$\Xi_{k+1} = \sum_{i_k=k+1}^{n} \Theta^{k+1}_{i_k}(y^k_{j_k}, v^k_\beta)\frac{\partial}{\partial y^k_{i_k}} + \sum_{\alpha=1}^{m} \Lambda^{k+1}_\alpha(y^k_{j_k}, v^k_\beta)\frac{\partial}{\partial v^k_\alpha}; \tag{5.88}$$

hence, we may construct the canonical variables

$$y^{k+1}_1 = y^k_1, \;\ldots,\; y^{k+1}_k = y^k_k, \quad y^{k+1}_{i_k} \; (i_k = k{+}1, \ldots, n), \quad v^{k+1}_\alpha \; (\alpha = 1, \ldots, m), \tag{5.89}$$

related to the operator Ξ_{k+1}, such that the latter writes as

$$\Xi_{k+1} = \frac{\partial}{\partial y^{k+1}_{k+1}}. \tag{5.90}$$

The complete application of the described algorithm enables us to write each operator Ξ_i in the form

$$\Xi_i = \frac{\partial}{\partial z_i}, \qquad i = 1, \dots, n, \tag{5.91}$$

and the new independent and dependent variables are $z_i = y_i^n$ $(i = 1, \dots, n)$, $w_\alpha = v_\alpha^n$ $(\alpha = 1, \dots, m)$, respectively.

Therefore, what we have obtained is a variable transformation like (5.69) allowing to write the system (5.68) in autonomous form.

Finally, since $[\Xi_i, \Xi_{n+1}] = \Xi_i$ $(i = 1, \dots, n)$, it is

$$\begin{aligned} \Xi_i(\Xi_{n+1} z_j) = \Xi_{n+1}(\Xi_i z_j) + \Xi_i z_j = \delta_{ij}, \\ \Xi_i(\Xi_{n+1} w_\alpha) = \Xi_{n+1}(\Xi_i w_\alpha) + \Xi_i w_\alpha = 0, \end{aligned} \tag{5.92}$$

where δ_{ij} is the Kronecker symbol; these relations, together with the hypothesis that the variables w_α $(\alpha = 1, \dots, m)$ are invariant with respect to Ξ_{n+1}, allow the vector field Ξ_{n+1} to gain the representation

$$\Xi_{n+1} = \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}. \tag{5.93}$$

As a consequence, since the resulting system, written in the variables $\mathbf{z}$ and $\mathbf{w}$, is autonomous and polynomial in the derivatives, and is invariant with respect to a uniform scaling of all independent variables, then it necessarily must be polynomially homogeneous in the derivatives, *i.e.*, it has the form (5.70).

The condition that the n–dimensional Abelian Lie subalgebra of the symmetries generate a distribution of rank n ensures that we may construct the complete set of the new independent variables $\mathbf{z}$.

Conversely, if the nonautonomous and/or nonhomogeneous system (5.68) can be mapped by the invertible point transformation (5.69) to the autonomous system polynomially homogeneous in the derivatives (5.70), then, since the latter admits the n vector fields $\frac{\partial}{\partial z_i}$, spanning an n–dimensional Abelian Lie algebra, and the vector field $\sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j}$, then it follows that also the system (5.68) must admit $(n+1)$ Lie point symmetries with the requested algebraic structure. □

5.4 Applications

In this Section, we provide some examples of nonlinear first order systems polynomial in the derivatives whose Lie symmetries satisfy the conditions of Theorem 5.3.1, and prove that they can be transformed under suitable conditions to autonomous first order systems having homogeneous polynomial form; the systems that will be considered are of Monge–Ampère type, and, remarkably, they are reduced to quasilinear (or linear) form.

In particular, we are concerned with the nonlinear first order systems of Monge–Ampère equations for the unknowns $u_\alpha(x_i)$ $(\alpha = 1, \dots, m;\ i = 1, \dots, n)$. These systems have been characterized by Boillat in 1997 [13] by looking for the nonlinear first order systems possessing, as the quasilinear systems, the property of the linearity of the Cauchy problem. These systems are also completely exceptional [11, 52], and are made by equations which are expressed as linear combinations (with coefficients depending at most on the independent and the dependent variables) of all minors extracted from the gradient matrix of $u_\alpha = u_\alpha(x_i)$.

Hereafter, to shorten the formulas, we denote with $u_{\alpha,i}$ the first order partial derivative of $u_\alpha(x_i)$ with respect to x_i, and with $w_{\alpha,i}$ the first order partial derivative of $w_\alpha(z_i)$ with respect to z_i; moreover, we denote with $f_{i;\alpha}$ the first order partial derivative of the function f_i with respect to u_α (or w_α). In the following we limit ourselves to consider the coefficients of the Monge–Ampère systems at most functions of the field variables.

5.4.1 Case $m = n = 2$

Let us consider the nonlinear first order system of Monge–Ampère made by the equations

$$\kappa_0^i\,(u_{1,1}u_{2,2} - u_{1,2}u_{2,1}) + \kappa_1^i u_{1,1} + \kappa_2^i u_{1,2} + \kappa_3^i u_{2,1} + \kappa_4^i u_{2,2} + \kappa_5^i = 0 \quad (5.94)$$

($i = 1, 2$), with $u_1(x_1, x_2), u_2(x_1, x_2)$ scalar functions, and $\kappa_j^i\,(u_1, u_2)$ ($i = 1, 2;\ j = 0, \ldots, 5$) arbitrary smooth functions of the indicated arguments.

The substitutions

$$u_1 \to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2, \qquad u_2 \to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2, \quad (5.95)$$

where α_{ij} are arbitrary constants, produce a system with $\kappa_5^i = 0$ ($i = 1, 2$) provided that

$$\kappa_0^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_1^i\alpha_{11} + \kappa_2^i\alpha_{12} + \kappa_3^i\alpha_{21} + \kappa_4^i\alpha_{22} + \kappa_5^i = 0 \quad (5.96)$$

($i = 1, 2$). Conditions (5.96) provide two constraints on the functional form of the coefficients so that not all systems can be written in a form where $\kappa_5^i = 0$; however, if the coefficients κ_j^i are constant, due to the arbitrariness of the constants α_{ij}, then (5.96) can always be satisfied whatever the values of the coefficients are.

It is easily recognized that system (5.94), now taken with $\kappa_5^i = 0$, admits the Lie point symmetries spanned by the operators

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = (x_1 - f_1)\,\frac{\partial}{\partial x_1} + (x_2 - f_2)\,\frac{\partial}{\partial x_2}, \quad (5.97)$$

where $f_i(u_1, u_2)$ ($i = 1, 2$) are arbitrary smooth functions of their arguments, provided that

$$\kappa_0^i + \kappa_1^i f_{2;2} - \kappa_2^i f_{1;2} - \kappa_3^i f_{2;1} + \kappa_4^i f_{1;1} = 0, \qquad i = 1, 2. \quad (5.98)$$

The constraints (5.98), once we assign the 10 functions $k_j^i(u_1, u_2)$ ($i = 1, 2$, $j = 0, \ldots, 4$), are the differential equations providing us the functional form of $f_1(u_1, u_2)$ and $f_2(u_1, u_2)$.

In the case where all the coefficients κ^i_j are constant, then the functions f_1 and f_2 are forced to be linear, *i.e.*,

$$f_1 = \beta_{11}u_1 + \beta_{12}u_2, \qquad f_2 = \beta_{21}u_1 + \beta_{22}u_2, \tag{5.99}$$

β_{ij} being constants whose value is determined by the coefficients κ^i_j.

Since

$$[\Xi_1, \Xi_2] = 0, \qquad [\Xi_1, \Xi_3] = \Xi_1, \qquad [\Xi_2, \Xi_3] = \Xi_2, \tag{5.100}$$

we introduce the new variables

$$z_1 = x_1 - f_1, \qquad z_2 = x_2 - f_2, \qquad w_1 = u_1, \qquad w_2 = u_2, \tag{5.101}$$

and the generators of the point symmetries write as

$$\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_3 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}. \tag{5.102}$$

In terms of the new variables (5.101), the nonlinear system (5.94) becomes

$$\kappa^i_1 w_{1,1} + \kappa^i_2 w_{1,2} + \kappa^i_3 w_{2,1} + \kappa^i_4 w_{2,2} = 0, \tag{5.103}$$

i.e., reads as an autonomous and homogeneous quasilinear system. This system is linear if all the coefficients κ^i_j are constant; nevertheless, since it is a 2×2 homogeneous and autonomous quasilinear system, it can be written in linear form by means of the hodograph transformation also when the coefficients κ^i_j depend on u_1 and u_2.

In conclusion, all Monge–Ampère systems with $m = n = 2$ can be reduced to a linear system when the coefficients κ^i_j are constant; on the contrary, when the coefficients depend upon u_1 and u_2 the reduction to the linear form is possible provided that the constraints (5.96) are satisfied.

5.4.2 Case $m = 2$, $n = 3$

By considering the gradient matrix of $u_\alpha(x_i)$ $(\alpha = 1, 2;\ i = 1, \dots, 3)$

$$H = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \end{pmatrix} \tag{5.104}$$

and its extracted minors of order 2,

$$H^1 = \begin{vmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{vmatrix}, \quad H^2 = \begin{vmatrix} u_{1,1} & u_{1,3} \\ u_{2,1} & u_{2,3} \end{vmatrix}, \quad H^3 = \begin{vmatrix} u_{1,2} & u_{1,3} \\ u_{2,2} & u_{2,3} \end{vmatrix}, \tag{5.105}$$

the nonlinear first order system of Monge–Ampère is made by equations like

$$\begin{aligned} &\kappa^i_1 H^1 + \kappa^i_2 H^2 + \kappa^i_3 H^3 \\ &\quad + \kappa^i_4 u_{1,1} + \kappa^i_5 u_{1,2} + \kappa^i_6 u_{1,3} + \kappa^i_7 u_{2,1} + \kappa^i_8 u_{2,2} + \kappa^i_9 u_{2,3} + \kappa^i_{10} = 0, \end{aligned} \tag{5.106}$$

with $\kappa^i_j(u_\alpha)$ $(i = 1, 2;\ j = 1, \dots, 10)$ arbitrary smooth functions of the indicated arguments.

The substitutions

$$
\begin{aligned}
&u_1 \to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3,\\
&u_2 \to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3,
\end{aligned}
\tag{5.107}
$$

where α_{ij} are arbitrary constants, produce a system with $\kappa^i_{10} = 0$ $(i = 1, 2)$ provided that

$$
\begin{aligned}
&\kappa^i_1(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa^i_2(\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21}) + \kappa^i_3(\alpha_{12}\alpha_{23} - \alpha_{13}\alpha_{22})\\
&\quad + \kappa^i_4\alpha_{11} + \kappa^i_5\alpha_{12} + \kappa^i_6\alpha_{13} + \kappa^i_7\alpha_{21} + \kappa^i_8\alpha_{22} + \kappa^i_9\alpha_{23} + \kappa^i_{10} = 0
\end{aligned}
\tag{5.108}
$$

$(i = 1, 2)$. Actually, conditions (5.108) can always be satisfied when the coefficients κ^i_j are constant because of the arbitrariness of the constants α_{ij}.

The nonlinear system (5.106), now taken with $\kappa^i_{10} = 0$, admits the Lie point symmetries spanned by the operators

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial x_3},\\
&\Xi_4 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2} + (x_3 - f_3)\frac{\partial}{\partial x_3},
\end{aligned}
\tag{5.109}
$$

where $f_i(u_1, u_2)$ $(i = 1, \dots, 3)$ are arbitrary smooth functions of their arguments, provided that

$$
\begin{aligned}
&\kappa^i_1 + \kappa^i_4 f_{2;2} - \kappa^i_5 f_{1;2} - \kappa^i_7 f_{2;1} + \kappa^i_8 f_{1;1} = 0,\\
&\kappa^i_2 + \kappa^i_4 f_{3;2} - \kappa^i_6 f_{1;2} - \kappa^i_7 f_{3;1} + \kappa^i_9 f_{1;1} = 0,\\
&\kappa^i_3 + \kappa^i_5 f_{3;2} - \kappa^i_6 f_{2;2} - \kappa^i_8 f_{3;1} + \kappa^i_9 f_{2;1} = 0.
\end{aligned}
\tag{5.110}
$$

The six conditions (5.110) cannot be fulfilled for an arbitrary choice of the coefficients κ^i_j. In the simplest case, where the coefficients κ^i_j are constant, they can be always satisfied and the functions f_i must be linear:

$$
f_1 = \beta_{11}u_1 + \beta_{12}u_2, \quad f_2 = \beta_{21}u_1 + \beta_{22}u_2, \quad f_3 = \beta_{31}u_1 + \beta_{32}u_2, \tag{5.111}
$$

β_{ij} being arbitrary constants.

The Lie point symmetries (5.109) generate a 4–dimensional solvable Lie algebra,

$$
[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_4] = \Xi_i, \qquad i, j = 1, \dots, 3, \tag{5.112}
$$

whereupon we may introduce the new variables

$$
\begin{aligned}
&z_1 = x_1 - f_1, \quad z_2 = x_2 - f_2, \quad z_3 = x_3 - f_3,\\
&w_1 = u_1, \qquad\;\; w_2 = u_2,
\end{aligned}
\tag{5.113}
$$

and the generators of the point symmetries write as

$$
\Xi_1 = \frac{\partial}{\partial z_1}, \quad \Xi_2 = \frac{\partial}{\partial z_2}, \quad \Xi_3 = \frac{\partial}{\partial z_3}, \quad \Xi_4 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + z_3\frac{\partial}{\partial z_3}. \tag{5.114}
$$

In terms of the new variables (5.113), the nonlinear system (5.106) reduces to

$$\kappa_4^i w_{1,1} + \kappa_5^i w_{1,2} + \kappa_6^i w_{1,3} + \kappa_7^i w_{2,1} + \kappa_8^i w_{2,2} + \kappa_9^i w_{2,3} = 0, \tag{5.115}$$

i.e., reads as an autonomous and homogeneous quasilinear system.

5.4.3 Case $m = 3$, $n = 2$

By considering the gradient matrix of $u_\alpha(x_i)$ $(\alpha = 1, \dots, 3;\ i = 1, 2)$

$$H = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \\ u_{3,1} & u_{3,2} \end{pmatrix} \tag{5.116}$$

and its extracted minors of order 2,

$$H^1 = \begin{vmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{vmatrix}, \qquad H^2 = \begin{vmatrix} u_{1,1} & u_{1,2} \\ u_{3,1} & u_{3,2} \end{vmatrix}, \qquad H^3 = \begin{vmatrix} u_{2,1} & u_{2,2} \\ u_{3,1} & u_{3,2} \end{vmatrix}, \tag{5.117}$$

the nonlinear first order system of Monge–Ampère is made by equations like

$$\begin{aligned} &\kappa_1^i H^1 + \kappa_2^i H^2 + \kappa_3^i H^3 \\ &\quad + \kappa_4^i u_{1,1} + \kappa_5^i u_{1,2} + \kappa_6^i u_{2,1} + \kappa_7^i u_{2,2} + \kappa_8^i u_{3,1} + \kappa_9^i u_{3,2} + \kappa_{10}^i = 0, \end{aligned} \tag{5.118}$$

with $\kappa_j^i(u_\alpha)$ $(i = 1, \dots, 3;\ j = 1, \dots, 10)$ arbitrary smooth functions of the indicated arguments.

The substitutions

$$\begin{aligned} u_1 &\to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2, \\ u_2 &\to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2, \\ u_3 &\to u_3 + \alpha_{31}x_1 + \alpha_{32}x_2, \end{aligned} \tag{5.119}$$

where α_{ij} are arbitrary constants, produce a system with $\kappa_{10}^i = 0$ $(i = 1, \dots, 3)$ provided that

$$\begin{aligned} &\kappa_1^i(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) + \kappa_2^i(\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31}) + \kappa_3^i(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}) \\ &\quad + \kappa_4^i\alpha_{11} + \kappa_5^i\alpha_{12} + \kappa_6^i\alpha_{21} + \kappa_7^i\alpha_{22} + \kappa_8^i\alpha_{31} + \kappa_9^i\alpha_{32} + \kappa_{10}^i = 0. \end{aligned} \tag{5.120}$$

Also in this case, conditions (5.120) can always be satisfied when the coefficients κ_j^i are constant because of the arbitrariness of the constants α_{ij}.

The nonlinear system (5.118), now taken with $\kappa_{10}^i = 0$, admits the Lie point symmetries spanned by the operators

$$\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2}, \tag{5.121}$$

where $f_i(u_1, u_2, u_3)$ $(i = 1, 2)$ are arbitrary smooth functions of their arguments, provided that

$$\begin{aligned}
&\kappa_1^i + \kappa_4^i f_{2;2} - \kappa_5^i f_{1;2} - \kappa_6^i f_{2;1} + \kappa_7^i f_{1;1} = 0,\\
&\kappa_2^i + \kappa_4^i f_{2;3} - \kappa_5^i f_{1;3} - \kappa_8^i f_{2;1} + \kappa_9^i f_{1;1} = 0,\\
&\kappa_3^i + \kappa_6^i f_{2;3} - \kappa_7^i f_{1;3} - \kappa_8^i f_{2;2} + \kappa_9^i f_{1;2} = 0.
\end{aligned} \tag{5.122}$$

The six conditions (5.122) cannot be fulfilled for an arbitrary choice of the coefficients κ_j^i. In the simplest case, where the coefficients κ_j^i are constant, they can be always satisfied and the functions f_i must be linear:

$$f_1 = \beta_{11}u_1 + \beta_{12}u_2 + \beta_{13}u_3, \qquad f_2 = \beta_{21}u_1 + \beta_{22}u_2 + \beta_{23}u_3, \tag{5.123}$$

β_{ij} being arbitrary constants.

The Lie point symmetries (5.121) generate a 3–dimensional solvable Lie algebra,

$$[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_3] = \Xi_i, \qquad i, j = 1, 2, \tag{5.124}$$

whereupon we may introduce the new variables

$$\begin{aligned}
&z_1 = x_1 - f_1, \quad z_2 = x_2 - f_2,\\
&w_1 = u_1, \qquad\ \ w_2 = u_2, \qquad w_3 = u_3,
\end{aligned} \tag{5.125}$$

and the generators of the point symmetries write as

$$\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_4 = z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}. \tag{5.126}$$

In terms of the new variables (5.125), the nonlinear system (5.118) reduces to

$$\kappa_4^i w_{1,1} + \kappa_5^i w_{1,2} + \kappa_6^i w_{2,1} + \kappa_7^i w_{2,2} + \kappa_8^i w_{3,1} + \kappa_9^i w_{3,2} = 0, \tag{5.127}$$

i.e., reads as an autonomous and homogeneous quasilinear system.

5.4.4 Case $m = n = 3$

By considering the gradient matrix of $u_\alpha(x_i)$ $(i = 1, \dots, 3;\ \alpha = 1, \dots, 3)$

$$H = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3}\\ u_{2,1} & u_{2,2} & u_{2,3}\\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix} \tag{5.128}$$

and its extracted minors of order 2,

$$\begin{aligned}
&H^1 = \begin{vmatrix} u_{2,2} & u_{2,3}\\ u_{3,2} & u_{3,3} \end{vmatrix}, \quad H^2 = \begin{vmatrix} u_{2,1} & u_{2,3}\\ u_{3,1} & u_{3,3} \end{vmatrix}, \quad H^3 = \begin{vmatrix} u_{2,1} & u_{2,2}\\ u_{3,1} & u_{3,2} \end{vmatrix},\\
&H^4 = \begin{vmatrix} u_{1,2} & u_{1,3}\\ u_{3,2} & u_{3,3} \end{vmatrix}, \quad H^5 = \begin{vmatrix} u_{1,1} & u_{1,3}\\ u_{3,1} & u_{3,3} \end{vmatrix}, \quad H^6 = \begin{vmatrix} u_{1,1} & u_{1,2}\\ u_{3,1} & u_{3,2} \end{vmatrix},\\
&H^7 = \begin{vmatrix} u_{1,2} & u_{1,3}\\ u_{2,2} & u_{2,3} \end{vmatrix}, \quad H^8 = \begin{vmatrix} u_{1,1} & u_{1,3}\\ u_{2,1} & u_{2,3} \end{vmatrix}, \quad H^9 = \begin{vmatrix} u_{1,1} & u_{1,2}\\ u_{2,1} & u_{2,2} \end{vmatrix},
\end{aligned} \tag{5.129}$$

the nonlinear first order system of Monge–Ampère results composed by equations like

$$\begin{aligned}&\kappa_0^i \det(H) + \kappa_1^i H^1 + \kappa_2^i H^2 + \kappa_3^i H^3 + \kappa_4^i H^4 + \kappa_5^i H^5 + \kappa_6^i H^6 \\ &\quad + \kappa_7^i H^7 + \kappa_8^i H^8 + \kappa_9^i H^9 + \kappa_{10}^i u_{1,1} + \kappa_{11}^i u_{1,2} + \kappa_{12}^i u_{1,3} + \kappa_{13}^i u_{2,1} \\ &\quad + \kappa_{14}^i u_{2,2} + \kappa_{15}^i u_{2,3} + \kappa_{16}^i u_{3,1} + \kappa_{17}^i u_{3,2} + \kappa_{18}^i u_{3,3} + \kappa_{19}^i = 0,\end{aligned} \tag{5.130}$$

with $\kappa_j^i(u_\alpha)$ $(i = 1, \dots, 3;\ j = 0, \dots, 19;\ \alpha = 1, \dots, 3)$ arbitrary smooth functions of the indicated arguments.

Also in this case, the substitutions

$$\begin{aligned}u_1 &\to u_1 + \alpha_{11}x_1 + \alpha_{12}x_2 + \alpha_{13}x_3,\\ u_2 &\to u_2 + \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3,\\ u_3 &\to u_3 + \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3,\end{aligned} \tag{5.131}$$

where α_{ij} are arbitrary constants, allow us to obtain a system with $\kappa_{19}^i = 0$ provided that $u_{i,j} = \alpha_{ij}$ is a solution of equations (5.130). This requirement implies some constraints on the coefficients κ_j^i in the general case; on the contrary, no limitation to the values of the coefficients exists if they are assumed to be constant.

The system (5.130), with $\kappa_{19}^i = 0$, admits the Lie point symmetries spanned by the operators

$$\begin{aligned}&\Xi_1 = \frac{\partial}{\partial x_1}, \qquad \Xi_2 = \frac{\partial}{\partial x_2}, \qquad \Xi_3 = \frac{\partial}{\partial x_3},\\ &\Xi_4 = (x_1 - f_1)\frac{\partial}{\partial x_1} + (x_2 - f_2)\frac{\partial}{\partial x_2} + (x_3 - f_3)\frac{\partial}{\partial x_3},\end{aligned} \tag{5.132}$$

where $f_i(u_1, u_2, u_3)$ $(i = 1, \dots, 3)$ are arbitrary smooth functions of their arguments, provided that

$$\begin{aligned}&\kappa_0^i - (f_{2;2}f_{3;3} - f_{2;3}f_{3;2})\kappa_{10}^i - (f_{1;3}f_{3;2} - f_{1;2}f_{3;3})\kappa_{11}^i\\ &\quad - (f_{1;2}f_{2;3} - f_{1;3}f_{2;2})\kappa_{12}^i - (f_{2;3}f_{3;1} - f_{2;1}f_{3;3})\kappa_{13}^i\\ &\quad - (f_{1;1}f_{3;3} - f_{1;3}f_{3;1})\kappa_{14}^i - (f_{1;3}f_{2;1} - f_{1;1}f_{2;3})\kappa_{15}^i\\ &\quad - (f_{2;1}f_{3;2} - f_{2;2}f_{3;1})\kappa_{16}^i - (f_{1;2}f_{3;1} - f_{1;1}f_{3;2})\kappa_{17}^i\\ &\quad - (f_{1;1}f_{2;2} - f_{1;2}f_{2;1})\kappa_{18}^i = 0,\\ &\kappa_1^i + \kappa_{14}^i f_{3;3} - \kappa_{15}^i f_{2;3} - \kappa_{17}^i f_{3;2} + \kappa_{18}^i f_{2;2} = 0,\\ &\kappa_2^i + \kappa_{13}^i f_{3;3} - \kappa_{15}^i f_{1;3} - \kappa_{16}^i f_{3;2} + \kappa_{18}^i f_{1;2} = 0,\\ &\kappa_3^i + \kappa_{13}^i f_{2;3} - \kappa_{14}^i f_{1;3} - \kappa_{16}^i f_{2;2} + \kappa_{17}^i f_{1;2} = 0,\\ &\kappa_4^i + \kappa_{11}^i f_{3;3} - \kappa_{12}^i f_{2;3} - \kappa_{17}^i f_{3;1} + \kappa_{18}^i f_{2;1} = 0,\\ &\kappa_5^i + \kappa_{10}^i f_{3;3} - \kappa_{12}^i f_{1;3} - \kappa_{16}^i f_{3;1} + \kappa_{18}^i f_{1;1} = 0,\\ &\kappa_6^i + \kappa_{10}^i f_{2;3} - \kappa_{11}^i f_{1;3} - \kappa_{16}^i f_{2;1} + \kappa_{17}^i f_{1;1} = 0,\\ &\kappa_7^i + \kappa_{11}^i f_{3;2} - \kappa_{12}^i f_{2;2} - \kappa_{14}^i f_{3;1} + \kappa_{15}^i f_{2;1} = 0,\\ &\kappa_8^i + \kappa_{10}^i f_{3;2} - \kappa_{12}^i f_{1;2} - \kappa_{13}^i f_{3;1} + \kappa_{15}^i f_{1;1} = 0,\\ &\kappa_9^i + \kappa_{10}^i f_{2;2} - \kappa_{11}^i f_{1;2} - \kappa_{13}^i f_{2;1} + \kappa_{14}^i f_{1;1} = 0.\end{aligned} \tag{5.133}$$

The vector fields (5.132) span a 4–dimensional solvable Lie algebra,

$$[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_4] = \Xi_i, \qquad i, j = 1, \dots, 3, \tag{5.134}$$

whereupon we may introduce the new variables

$$\begin{aligned} z_1 &= x_1 - f_1, & z_2 &= x_2 - f_2, & z_3 &= x_3 - f_3, \\ w_1 &= u_1, & w_2 &= u_2, & w_3 &= u_3, \end{aligned} \tag{5.135}$$

and the generators of the point symmetries write as

$$\begin{aligned} &\Xi_1 = \frac{\partial}{\partial z_1}, \qquad \Xi_2 = \frac{\partial}{\partial z_2}, \qquad \Xi_3 = \frac{\partial}{\partial z_3}, \\ &\Xi_4 = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}. \end{aligned} \tag{5.136}$$

In terms of the new variables (5.135), Eqs. (5.130) write as

$$\begin{aligned} &\kappa^i_{10} w_{1,1} + \kappa^i_{11} w_{1,2} + \kappa^i_{12} w_{1,3} + \kappa^i_{13} w_{2,1} + \kappa^i_{14} w_{2,2} + \kappa^i_{15} w_{2,3} \\ &\quad + \kappa^i_{16} w_{3,1} + \kappa^i_{17} w_{3,2} + \kappa^i_{18} w_{3,3} = 0, \end{aligned} \tag{5.137}$$

i.e., they are in autonomous and homogeneous quasilinear (linear, if the coefficients are constant) form.

Conditions (5.133) play severe restrictions to the expression of the coefficients κ^i_j. When these coefficients are assumed to be constant, we are forced to take

$$\begin{aligned} f_1 &= \beta_{11} u_1 + \beta_{12} u_2 + \beta_{13} u_3, \\ f_2 &= \beta_{21} u_1 + \beta_{22} u_2 + \beta_{23} u_3, \\ f_3 &= \beta_{31} u_1 + \beta_{32} u_2 + \beta_{33} u_3, \end{aligned} \tag{5.138}$$

where β_{ij} are arbitrary constants; also in such simple case, the reduction to linear form is not always possible due to (5.133).

5.4.5 Case m and n arbitrary

It is easily recognized that, a general Monge–Ampère system with m dependent variables and n independent variables, provided that some suitable conditions on the coefficients (at most depending on the field variables) are satisfied, is invariant with respect to the Lie groups generated by the vector fields

$$\begin{aligned} &\Xi_i = \frac{\partial}{\partial x_i}, \qquad i = 1, \dots, n, \\ &\Xi_{n+1} = \sum_{i=1}^{n} (x_i - f_i(u_\alpha)) \frac{\partial}{\partial x_i}, \end{aligned} \tag{5.139}$$

where $f_i(u_\alpha)$ are smooth functions of $(u_1, \dots, u_m)$ which have to be linear in their arguments when the coefficients of the Monge–Ampère system are constant.

As one expects, for $m > 3$ or $n > 3$, we have a situation similar to the case $m = n = 3$, *i.e.*, even in the case of constant coefficients, not all Monge–Ampère systems can be reduced to (quasi) linear form.

5.5 Conclusions

In the first part of this Chapter, a theorem giving only necessary conditions for the transformation of a nonlinear first order system of partial differential equations to autonomous and homogeneous quasilinear form has been given. The reduction to quasilinear form is performed by constructing the canonical variables associated to the Lie point symmetries admitted by the nonlinear source system. Some examples concerned with the first order systems related to second order Monge–Ampère equations in $(1+1)$, $(2+1)$ and $(3+1)$ dimensions where the procedure works are also discussed. As physical example, it has been shown that the one–dimensional Euler equations for isentropic fluids can be equivalent to a nonlinear second order Monge–Ampère equation with a special constitutive law and, under suitable conditions, can be mapped into autonomous and homogeneous form. Moreover, an example of a first order system polynomial in the derivatives that can be reduced to a system polynomially homogeneous of degree 2 in the derivatives (equivalent to a second order partial differential equation for a surface in $\mathbb{R}^3$ such that its Gaussian curvature is proportional to the square of its mean curvature) is provided.

In the second part of this Chapter, a theorem giving necessary and sufficient conditions for transforming a nonlinear first order system of partial differential equations involving the derivatives in polynomial form to an equivalent autonomous system polynomially homogeneous in the derivatives has been proved. The theorem is based on the Lie point symmetries admitted by the nonlinear system, and the proof is constructive, in the sense that it leads to the algorithmic construction of the invertible mapping performing the task.

The theorem is applied to a class of nonlinear first order systems belonging to the family of Monge–Ampère systems that have been characterized by Boillat in 1997 [13]. These systems share with the quasilinear systems the property of the linearity of the Cauchy problem. They are also completely exceptional [11, 52], and are made by equations which are expressed as linear combinations (with coefficients depending at most on the independent and the dependent variables) of all minors extracted from the gradient matrix of $u_\alpha = u_\alpha(x_i)$ $(\alpha = 1, \ldots, m;\; i = 1, \ldots, n)$. We considered explicitly either the case of constant coefficients or the case of coefficients depending on the field variables, for $m = 2, 3$ and $n = 2, 3$. If $m = 2$ and $n = 2, 3$, or $n = 2$ and $m = 2, 3$, and the coefficients are assumed to be constant, we proved that the Monge–Ampère systems can always be transformed to linear form.

Nevertheless, for arbitrary m and n, Monge–Ampère systems, provided that the coefficients entering their equations satisfy some constraints, can be mapped to first order quasilinear autonomous and homogeneous systems. This, in some sense, casts new light on the fact, underlined by Boillat [13], that Monge–Ampère systems, because of the linearity of the Cauchy problem, are the closest to quasilinear systems, which are Monge systems.

6 Decoupling of quasilinear first order systems

IN this Chapter, we deal with the decoupling problem of general quasilinear first order systems in two independent variables. After introducing the problem, we start by considering hyperbolic quasilinear first order systems in two and three dependent variables; these systems can be in principle nonautonomous and/or nonhomogeneous. By means of a direct approach, we determine the conditions guaranteeing the decoupling, and write these conditions in terms of the eigenvalues and eigenvectors of the coefficient matrix. Then, we discuss general autonomous and homogeneous quasilinear first order systems (either hyperbolic or not), and prove the necessary and sufficient conditions for the decoupling. The results are also extended to the case of nonhomogeneous and/or nonautonomous systems. The proofs of the theorems presented in this Chapter are mainly based on some quantities, built with the eigenvalues and the eigenvectors of the coefficient matrix, as well as (if any) with the source terms, which are *invariant* under an invertible change of variables. Some examples of physical interest where the procedure can be applied are also given. The original results here presented are contained in [33, 34].

6.1 The problem

Let us consider first order systems of partial differential equations in the form of balance laws [15, 20], that in one space dimension read as

$$\frac{\partial \mathbf{F}^0(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{F}^1(\mathbf{u})}{\partial x} = \mathbf{g}(\mathbf{u}), \tag{6.1}$$

where $\mathbf{u} \in \mathbb{R}^n$ denotes the unknown vector field, $\mathbf{F}^0(\mathbf{u}) \in \mathbb{R}^n$ collects the components of the densities of some physical quantities, $\mathbf{F}^1(\mathbf{u}) \in \mathbb{R}^n$ the components of the corresponding fluxes, and $\mathbf{g}(\mathbf{u}) \in \mathbb{R}^n$ the production terms; when $\mathbf{g}(\mathbf{u}) \equiv \mathbf{0}$, we have a system of conservation laws.

Systems like (6.1) fall in the more general class of nonhomogeneous quasilinear first order systems of partial differential equations,

$$A^0(\mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + A^1(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(\mathbf{u}), \tag{6.2}$$

where $A^0(\mathbf{u})$ and $A^1(\mathbf{u})$ are $n \times n$ matrices (the gradient matrices of $\mathbf{F}^0(\mathbf{u})$ and $\mathbf{F}^1(\mathbf{u})$, respectively, in the case of conservative systems). As shown in Chapter 3, special problems of physical interest may require to consider nonautonomous and/or nonhomogeneous quasilinear systems of the form

$$A^0(t, x, \mathbf{u})\frac{\partial \mathbf{u}}{\partial t} + A^1(t, x, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}). \tag{6.3}$$

The analytical, as well as numerical, treatment of quasilinear systems of conservation laws is in general a difficult task. In the case of hyperbolic systems, the generalized eigenvalues of the matrix pair $\{A^0, A^1\}$, giving the wave speeds, depend on $\mathbf{u}$, whereupon the shape of the various components in the solution will vary in time: rarefaction waves will decay, and compression waves will become steeper, possibly leading to shock formation in a finite time [15, 20]. Since also the eigenvectors, determining the approximate change of field variables across a wave, depend on $\mathbf{u}$, nontrivial interactions between different waves will occur; the strength of the interacting waves may change, and new waves of different families can be created, as a result of the interaction.

For a strictly hyperbolic system of conservation laws we have n families of waves, each corresponding to an eigenvalue of the system. The nonlinearity of wavespeeds leads to the formation of shocks, so that solutions must be understood in the weak sense. The existence and stability of global weak solutions for Cauchy data with small total variation was established by Glimm [30]. For systems of more than two equations nonlocal resonant interaction effects between different families of waves are observed, leading to a variety of new phenomena [57] such as blowup of solutions [41], and delay in the onset of shocks [58]. The resonance determines the occurrence of solutions exhibiting a strong nonlinear instability in the form of catastrophic blowup of solutions [41, 49]. In fact, there are systems where Cauchy data with arbitrarily small oscillation can grow arbitrarily large in a finite time [95, 96, 98].

In dealing with quasilinear systems, it may be interesting to look for the conditions (if any) leading to their possible decoupling into smaller non–interacting subsystems (full decoupling), or their reduction to a set of smaller subsystems that can be solved separately in hierarchy (partial decoupling).

For homogeneous and autonomous first order quasilinear systems of partial differential equations in two independent variables, the decoupling problem can be formulated as follows [9, 10].

Problem 6.1.1. *When can a system like*

$$\frac{\partial u_\ell}{\partial t} = \sum_{j=1}^{n} A_{\ell j}(u_1, \ldots, u_n)\frac{\partial u_j}{\partial x}, \qquad \ell = 1, \ldots, n, \tag{6.4}$$

be locally decoupled in some coordinates $v_1(\mathbf{u}), \ldots, v_n(\mathbf{u})$ into k non–interacting subsystems, say

$$\frac{\partial v_{m_j+i}}{\partial t} = \sum_{\ell=1}^{n_j} \widetilde{A}_{m_j+i, m_j+\ell}(v_{m_j+1}, \ldots, v_{m_j+n_j})\frac{\partial v_{m_j+\ell}}{\partial x}, \tag{6.5}$$

of some orders n_1, …, n_k with $n_1 + \ldots + n_k = n$, where $j = 1, \ldots, k$, $i = 1, \ldots, n_j$, and $m_j = n_1 + \ldots + n_j$?

A first result has been obtained by Nijenhuis [63] in the case of a strictly hyperbolic system; the necessary and sufficient conditions for the complete decoupling of system (6.4) into n non–interacting one–dimensional subsystems require the vanishing of the corresponding Nijenhuis tensor

$$N_{jik} = A_{\alpha i}\frac{\partial A_{jk}}{\partial u_\alpha} - A_{\alpha k}\frac{\partial A_{ji}}{\partial u_\alpha} + A_{j\alpha}\frac{\partial A_{\alpha i}}{\partial u_k} - A_{j\alpha}\frac{\partial A_{\alpha k}}{\partial u_i}. \tag{6.6}$$

The decoupling problem has been considered by Bogoyavlenskij [9, 10], who provided necessary and sufficient conditions by using Nijenhuis [63] and Haantjes [36] tensors. More in detail, to reduce system (6.4) into block–diagonal form with k mutually interacting blocks of dimensions $n_i \times n_j$ [9, 10] it is necessary and sufficient that in the tangent spaces $T_{\mathbf{x}}(\mathbb{R}^n)$ there exist k smooth distributions $L_{1\mathbf{x}}, \dots, L_{k\mathbf{x}}$ of dimensions $n_1, \dots, n_k$ such that $L_{1\mathbf{x}} \oplus \cdots \oplus L_{k\mathbf{x}} = T_{\mathbf{x}}(\mathbb{R}^n)$ and the conditions

$$A(L_{i\mathbf{x}}) \subset L_{i\mathbf{x}}, \qquad N(L_{i\mathbf{x}}, L_{i\mathbf{x}}) \subset L_{i\mathbf{x}}, \qquad N(L_{i\mathbf{x}}, L_{j\mathbf{x}}) \subset L_{i\mathbf{x}} + L_{j\mathbf{x}}, \tag{6.7}$$

hold, provided that the eigenvalues of the operator A in any two different subspaces $L_{i\mathbf{x}}$ and $L_{j\mathbf{x}}$ are different almost everywhere for $\mathbf{x} \in \mathbb{R}^n$ ($i \neq j$; $i, j \in \{1, \dots, k\}$). In the more restrictive case of the decoupling into k non–interacting blocks, the different blocks $\widetilde{A}_{m_j+i,m_j+\ell}$ depend on different variables; hence, for the generic case the eigenvalues corresponding to any two blocks $\widetilde{A}_{m_j+i,m_j+\ell}$ do not coincide with each other almost everywhere for $\mathbf{x} \in \mathbb{R}^n$ (while inside a given block some eigenvalues can coincide). In this case, the necessary and sufficient conditions for the reducibility of the systems (6.4) into k non–interacting subsystems have the form

$$A(L_{i\mathbf{x}}) \subset L_{i\mathbf{x}}, \qquad N(L_{i\mathbf{x}}, L_{i\mathbf{x}}) \subset L_{i\mathbf{x}}, \qquad N(L_{i\mathbf{x}}, L_{j\mathbf{x}}) = 0. \tag{6.8}$$

Within this theoretical framework, a couple of recent papers by Tunitsky [90, 91], who established necessary and sufficient conditions for transforming quasilinear first order systems into block triangular systems by using a geometric formalism for such equations, based on Nijenhuis and Haantjes tensors, are worth of being quoted.

In this Chapter, we shall consider either autonomous and homogeneous first order quasilinear systems like

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.9}$$

or general nonhomogeneous and/or nonautonomous ones, say

$$\frac{\partial \mathbf{u}}{\partial t} + A(t, x, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}) \tag{6.10}$$

(possibly coming from systems in conservative form), and obtain the necessary and sufficient conditions allowing for the partial decoupling in two or more subsystems, as we shall precise below. When such a *partial* decoupling is possible, we may solve the various subsystems separately in hierarchy. Also, we shall prove how to extend the conditions to be satisfied in order to characterize the systems that can be fully decoupled into non–interacting subsystems. The conditions we shall discuss later involve, as one expects, just the properties of the eigenvalues, the eigenvectors (together with the generalized eigenvectors, if needed) of the coefficient matrix; in particular, the conditions for the full decoupling of a hyperbolic system in k non–interacting subsystems require the vanishing both of the change of characteristic speeds of a subsystem across a wave of the other subsystems, and of the interaction coefficients between waves of different subsystems. Even if the computation of eigenvalues and eigenvectors of the coefficient matrix may be hard (especially for large matrices), the conditions we derived have a simple interpretation, as we shall discuss later. Moreover, when the required decoupling conditions are satisfied, we have also the differential constraints whose integration provides the variable transformation leading to the (partially or fully) decoupled system.

6.2 Direct approach for the decoupling of hyperbolic systems

Let us consider a general first order quasilinear hyperbolic (in the t–direction) system of partial differential equations

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} A_{ij}(t, x, \mathbf{u}) \frac{\partial u_j}{\partial x} = g_i(t, x, \mathbf{u}), \tag{6.11}$$

or, in compact form,

$$\frac{\partial \mathbf{u}}{\partial t} + A(t, x, \mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}), \tag{6.12}$$

where $\mathbf{u} \equiv (u_1, \dots, u_n)^T$, A being an $n \times n$ real matrix whose entries A_{ij} are smooth functions depending at most on the independent and dependent variables, and $\mathbf{g} \in \mathbb{R}^n$ being the source term with components smooth functions of t, x and $\mathbf{u}$. Because of the hyperbolicity, the matrix A has real eigenvalues, and a corresponding set of eigenvectors spanning $\mathbb{R}^n$.

By denoting with $\mathbf{x} \equiv (t, x)$ the original independent variables, and introducing new independent variables $\mathbf{X} \equiv (T, X)$ depending on t and x, and new dependent variables $\mathbf{U} \equiv (U_1, \dots, U_n)^T$ as functions of t, x and $\mathbf{u}$, through a locally invertible map like

$$\mathbf{X} = \mathbf{Z}(\mathbf{x}), \qquad \mathbf{u} = \mathbf{h}(\mathbf{x}, \mathbf{U}), \tag{6.13}$$

or, equivalently,

$$\mathbf{x} = \mathbf{z}(\mathbf{X}), \qquad \mathbf{U} = \mathbf{H}(\mathbf{x}, \mathbf{u}), \tag{6.14}$$

the quasilinear form of (6.12) is preserved.

Remark 6.2.1. *The previous result is coherent with the equivalence transformations admitted by system (6.12) [69]. In fact, their projection on the space $(t, x, \mathbf{u})$ is generated by the following vector fields:*

$$\begin{aligned} &\Xi_1 = f_1(t, x) \frac{\partial}{\partial t}, \qquad \Xi_2 = f_2(t, x) \frac{\partial}{\partial x}, \\ &\Xi_{i+2} = f_{i+2}(t, x, \mathbf{u}) \frac{\partial}{\partial u_i}, \qquad i = 1, \dots, n, \end{aligned} \tag{6.15}$$

where f_i $(i = 1, \dots, n+2)$ are arbitrary functions of the indicated arguments. The finite transformations corresponding to (6.15) provide a map like (6.13) preserving the differential structure of (6.12).

By defining the (invertible) matrices

$$\nabla_{\mathbf{U}}\mathbf{h} = \begin{bmatrix} \frac{\partial h_1}{\partial U_1} & \cdots & \frac{\partial h_1}{\partial U_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial U_1} & \cdots & \frac{\partial h_n}{\partial U_n} \end{bmatrix}, \qquad J = \begin{bmatrix} \frac{\partial Z_1}{\partial t} & \frac{\partial Z_1}{\partial x} \\ \frac{\partial Z_2}{\partial t} & \frac{\partial Z_2}{\partial x} \end{bmatrix}, \tag{6.16}$$

the system (6.12) writes as

$$\begin{aligned} &(J_{11}I + J_{12}A)\,(\nabla_{\mathbf{U}}\mathbf{h})\frac{\partial \mathbf{U}}{\partial T} + (J_{21}I + J_{22}A)\,(\nabla_{\mathbf{U}}\mathbf{h})\frac{\partial \mathbf{U}}{\partial X} = \\ &\quad = \left(\mathbf{g} - \frac{\partial \mathbf{h}}{\partial t} - A\frac{\partial \mathbf{h}}{\partial x}\right), \end{aligned} \tag{6.17}$$

whereupon, if $\det\,(J_{11}I + J_{12}A) \neq 0$, we obtain

$$\frac{\partial \mathbf{U}}{\partial T} + \widetilde{A}(T, X, \mathbf{U})\frac{\partial \mathbf{U}}{\partial X} = \mathbf{G}(T, X, \mathbf{U}), \tag{6.18}$$

where the entries of matrix $\widetilde{A}$ and the components of $\mathbf{G}$ now depend on T, X and $\mathbf{U}$.

The matrix $\widetilde{A}$ and the vector $\mathbf{G}$ involved in (6.18) have the form

$$\begin{aligned} &\widetilde{A} = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}\,(J_{11}I + J_{12}A)^{-1}\,(J_{21}I + J_{22}A)\,(\nabla_{\mathbf{U}}\mathbf{h}), \\ &\mathbf{G} = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}\,(J_{11}I + J_{12}A)^{-1}\left(\mathbf{g} - \frac{\partial \mathbf{h}}{\partial t} - A\frac{\partial \mathbf{h}}{\partial x}\right). \end{aligned} \tag{6.19}$$

Let $\mathbf{l}^{(i)} \equiv \left(l_1^{(i)}, \dots, l_n^{(i)}\right)$ and $\mathbf{r}^{(i)} \equiv \left(r_1^{(i)}, \dots, r_n^{(i)}\right)^T$ be the left and right eigenvectors of matrix A corresponding to the eigenvalue λ_i, respectively; then, $\mathbf{L}^{(i)} = \mathbf{l}^{(i)}(\nabla_{\mathbf{U}}\mathbf{h})$ and $\mathbf{R}^{(i)} = (\nabla_{\mathbf{U}}\mathbf{h})^{-1}\mathbf{r}^{(i)}$ are the left and right eigenvectors of $\widetilde{A}$ corresponding to the eigenvalue $\Lambda_i = \dfrac{J_{21} + J_{22}\lambda_i}{J_{11} + J_{12}\lambda_i}$, respectively.

In the next two Sections, we shall consider the cases $n = 2$ and $n = 3$, and derive the conditions on the source system and the transformation allowing us to obtain a system like (6.18) that results partially (or fully) decoupled in some subsystems.

It is worth of being remarked that the introduction of a transformation of the independent variables as well as the possibility that the new field variables depend not only on the old dependent variables but also on the independent variables does not have effects on the decoupling process, but may be useful if we want to get a somehow decoupled system which is autonomous [22, 26, 66] or homogeneous and autonomous [19, 65]: in these last cases the Lie point symmetries admitted by the source system play a central role [65, 66].

6.3 Direct approach for the decoupling of hyperbolic systems in two dependent variables

With the aim of identifying the 2×2 systems that can be decoupled by means of an invertible point transformation like (6.13) (or (6.14)), two main cases can be distinguished:

(I) Partial Decoupling:

$$\widetilde{A}_{12} = \frac{\partial \widetilde{A}_{11}}{\partial U_2} = \frac{\partial G_1}{\partial U_2} = 0; \tag{6.20}$$

(II) Full Decoupling:

$$\widetilde{A}_{12} = \widetilde{A}_{21} = \frac{\partial \widetilde{A}_{11}}{\partial U_2} = \frac{\partial \widetilde{A}_{22}}{\partial U_1} = \frac{\partial G_1}{\partial U_2} = \frac{\partial G_2}{\partial U_1} = 0. \tag{6.21}$$

In addition, we may be interested to get a (partially or fully) decoupled system which is autonomous and/or homogeneous.

When a partially decoupled system is recovered, the first equation can be solved for U_1; inserting this solution in the second equation, we get an equation for U_2. When a fully decoupled system is recovered, the two resulting equations can be solved independently from each other.

(I) Partial decoupling.
Let us require that $\widetilde{A}_{12}$ is vanishing and that $\widetilde{A}_{11}$ and G_1 are independent of U_2, thus obtaining some constraints on the structure of (6.12).
From $\widetilde{A}_{12} = 0$, it follows that

$$0 = R_1^{(2)} = \frac{\partial h_2}{\partial U_2} r_1^{(2)} - \frac{\partial h_1}{\partial U_2} r_2^{(2)}, \tag{6.22}$$

whereupon, considering the inverse transformation of (6.13) as $\mathbf{U} = \mathbf{H}(\mathbf{x}, \mathbf{u})$, it follows

$$(\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}^{(2)} = 0; \tag{6.23}$$

condition (6.23) means that we may write

$$\nabla_{\mathbf{u}} H_1 = \alpha(t, x, \mathbf{u})\, \mathbf{l}^{(1)}, \tag{6.24}$$

with a suitable function α of the indicated arguments.

Since

$$\widetilde{A}_{11} = \Lambda_1 = \frac{J_{21} + J_{22}\lambda_1}{J_{11} + J_{12}\lambda_1}, \tag{6.25}$$

the condition

$$\frac{\partial \widetilde{A}_{11}}{\partial U_2} = 0 \tag{6.26}$$

is equivalent to

$$(\nabla_{\mathbf{u}} \lambda_1) \cdot \frac{\partial \mathbf{h}}{\partial U_2} = 0. \tag{6.27}$$

Furthermore, condition (6.22) means that the vectors $\dfrac{\partial \mathbf{h}}{\partial U_2}$ and $\mathbf{r}^{(2)}$ are parallel, so that the relation (6.27) can be rewritten as

$$(\nabla_{\mathbf{u}}\lambda_1)\cdot \mathbf{r}^{(2)} = 0, \tag{6.28}$$

which is a structure condition on the original system.

Moreover, by requiring G_1 to be independent of U_2, after some algebra, we get

$$\left(\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_1)\cdot \mathbf{g}\right) + \frac{\partial(\nabla_{\mathbf{u}}H_1)}{\partial t} + \lambda_1 \frac{\partial(\nabla_{\mathbf{u}}H_1)}{\partial x}\right)\cdot \mathbf{r}^{(2)} = 0. \tag{6.29}$$

Therefore, the first equation can be decoupled from the second one by introducing the new dependent variables (U_1, U_2), where

$$U_1 = H_1(\mathbf{x}, \mathbf{u}) \tag{6.30}$$

is found by solving the linear first order partial differential equation

$$(\nabla_{\mathbf{u}}H_1)\cdot \mathbf{r}^{(2)} = 0, \tag{6.31}$$

provided that the following conditions are satisfied:

$$\begin{aligned} &(\nabla_{\mathbf{u}}\lambda_1)\cdot \mathbf{r}^{(2)} = 0,\\ &\left(\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_1)\cdot \mathbf{g}\right) + \frac{\partial(\nabla_{\mathbf{u}}H_1)}{\partial t} + \lambda_1 \frac{\partial(\nabla_{\mathbf{u}}H_1)}{\partial x}\right)\cdot \mathbf{r}^{(2)} = 0. \end{aligned} \tag{6.32}$$

Remark 6.3.1. *If H_1 is independent of t and x, then condition (6.32)$_2$ reduces to*

$$\left(\nabla_{\mathbf{u}}\left(\mathbf{l}^{(1)}\cdot \mathbf{g}\right)\right)\cdot \mathbf{r}^{(2)} = 0. \tag{6.33}$$

(II) Full decoupling.

Actually, the full decoupling can be considered as a subcase of partial decoupling. In fact, in addition to the constraints necessary to partially decouple the system (6.12), we need to require also that $\widetilde{A}_{21}$ is vanishing, and $\widetilde{A}_{22}$ and G_2 are independent of U_1. The same steps as in previous subsection provide the following result. We need to introduce the new dependent variables $U_i = H_i(\mathbf{x}, \mathbf{u})$ $(i = 1, 2)$ by solving

$$(\nabla_{\mathbf{u}}H_1)\cdot \mathbf{r}^{(2)} = 0, \qquad (\nabla_{\mathbf{u}}H_2)\cdot \mathbf{r}^{(1)} = 0, \tag{6.34}$$

and a fully decoupled system is recovered provided that the conditions

$$\begin{aligned} &(\nabla_{\mathbf{u}}\lambda_i)\cdot \mathbf{r}^{(j)} = 0,\\ &\left(\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}H_i)\cdot \mathbf{g}\right) + \frac{\partial(\nabla_{\mathbf{u}}H_i)}{\partial t} + \lambda_i \frac{\partial(\nabla_{\mathbf{u}}H_i)}{\partial x}\right)\cdot \mathbf{r}^{(j)} = 0, \end{aligned} \tag{6.35}$$

where $i, j = 1, 2,\ i \neq j$, are fulfilled.

Remark 6.3.2. *If H_1 and H_2 are independent of t and x, then the structure conditions leading to the full decoupling become*

$$\begin{aligned}(\nabla_{\mathbf{u}}\lambda_i)\cdot\mathbf{r}^{(j)} &= 0,\\ \left(\nabla_{\mathbf{u}}\left(\mathbf{l}^{(i)}\cdot\mathbf{g}\right)\right)\cdot\mathbf{r}^{(j)} &= 0,\end{aligned} \tag{6.36}$$

where $i,j=1,2,\ i\neq j$.

Example 6.3.1. *Let us consider a 2×2 system where one of the field variables (u_1) is a velocity, and assume the system to be invariant with respect to the Galilean group,* i.e., *let us take the system*

$$\begin{aligned}&\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+f_1(u_2)\frac{\partial u_2}{\partial x}=g_1(u_2),\\ &\frac{\partial u_2}{\partial t}+f_2(u_2)\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_2}{\partial x}=g_2(u_2),\end{aligned} \tag{6.37}$$

where f_i, g_i ($i=1,2$) are arbitrary functions of their argument such that $f_1f_2>0$.

The characteristic velocities are

$$\lambda_1=u_1+\sqrt{f_1f_2},\qquad \lambda_2=u_1-\sqrt{f_1f_2}, \tag{6.38}$$

with associated left and right eigenvectors

$$\begin{aligned}&\mathbf{l}^{(1)}=\left(\sqrt{f_2},\sqrt{f_1}\right),\quad \mathbf{l}^{(2)}=\left(\sqrt{f_2},-\sqrt{f_1}\right),\\ &\mathbf{r}^{(1)}=\begin{pmatrix}\sqrt{f_1}\\ \sqrt{f_2}\end{pmatrix},\qquad \mathbf{r}^{(2)}=\begin{pmatrix}\sqrt{f_1}\\ -\sqrt{f_2}\end{pmatrix}.\end{aligned} \tag{6.39}$$

The conditions (6.28), (6.29) and (6.31) provide

$$\begin{aligned}&2f_1-(f_1f_2)'=0,\\ &g_1'+\left(\sqrt{\frac{f_1}{f_2}}g_2\right)'=0,\\ &\sqrt{f_1}\frac{\partial H_1}{\partial u_1}-\sqrt{f_2}\frac{\partial H_1}{\partial u_2}=0,\end{aligned} \tag{6.40}$$

(the prime denoting the differentiation with respect to the argument) whose integration gives

$$\begin{aligned}&\int\sqrt{\frac{f_1}{f_2}}\,du_2=\sqrt{f_1f_2}-\kappa_1,\\ &g_1=\kappa_2-\sqrt{\frac{f_1}{f_2}}g_2,\\ &H_1=H_1\left(u_1+\int\sqrt{\frac{f_1}{f_2}}\,du_2\right),\end{aligned} \tag{6.41}$$

where κ_1 and κ_2 are arbitrary constants.

By choosing the new dependent variables as

$$U_1=u_1+\sqrt{f_1f_2}-\kappa_1,\qquad U_2=u_2, \tag{6.42}$$

and replacing the inverse transformation of (6.42) in (6.37), i.e.,

$$u_1 = U_1 - \sqrt{f_1 f_2} + \kappa_1, \qquad u_2 = U_2, \tag{6.43}$$

we get the following partially decoupled system

$$\begin{aligned} &\frac{\partial U_1}{\partial t} + (U_1 + \kappa_1)\frac{\partial U_1}{\partial x} = \kappa_2, \\ &\frac{\partial U_2}{\partial t} + f_2(U_2)\frac{\partial U_1}{\partial x} + (U_1 - 2\sqrt{f_1(U_2) f_2(U_2)} + \kappa_1)\frac{\partial U_2}{\partial x} = g_2(U_2). \end{aligned} \tag{6.44}$$

The system so considered, when

$$f_2(u_2) = u_2, \qquad g_1(u_2) = g_2(u_2) = 0, \tag{6.45}$$

represents the one–dimensional Euler equations of a barotropic fluid with density u_2 and pressure

$$p(u_2) = \int u_2 f_1(u_2)\, du_2. \tag{6.46}$$

The constitutive law

$$p(u_2) = \frac{\kappa}{3} u_2^3, \qquad \kappa \quad \textit{constant}, \tag{6.47}$$

allows to satisfy the decoupling conditions and, in the new variables

$$U_1 = u_1 + \sqrt{\kappa} u_2, \qquad U_2 = u_2, \tag{6.48}$$

we have a partially decoupled system

$$\begin{aligned} &\frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x} = 0, \\ &\frac{\partial U_2}{\partial t} + U_2 \frac{\partial U_1}{\partial x} + (U_1 - 2\sqrt{\kappa}\, U_2)\frac{\partial U_2}{\partial x} = 0. \end{aligned} \tag{6.49}$$

6.4 Direct approach for the decoupling of hyperbolic systems in three dependent variables

To decouple systems like (6.12) with $n = 3$ by means of an invertible point transformation like (6.13), three main cases can be distinguished:

1. Partial Decoupling:

$$\begin{aligned} \widetilde{A}_{13} &= \frac{\partial \widetilde{A}_{11}}{\partial U_3} = \frac{\partial \widetilde{A}_{12}}{\partial U_3} = \frac{\partial G_1}{\partial U_3} = 0, \\ \widetilde{A}_{23} &= \frac{\partial \widetilde{A}_{21}}{\partial U_3} = \frac{\partial \widetilde{A}_{22}}{\partial U_3} = \frac{\partial G_2}{\partial U_3} = 0; \end{aligned} \tag{6.50}$$

2. Partial Decoupling:

$$\widetilde{A}_{31} = \widetilde{A}_{32} = \frac{\partial \widetilde{A}_{33}}{\partial U_1} = \frac{\partial \widetilde{A}_{33}}{\partial U_2} = \frac{\partial G_3}{\partial U_1} = \frac{\partial G_3}{\partial U_2} = 0; \tag{6.51}$$

3. Full Decoupling:

A combination of both previous cases:

$$
\begin{aligned}
&\widetilde{A}_{13} = \frac{\partial \widetilde{A}_{11}}{\partial U_3} = \frac{\partial \widetilde{A}_{12}}{\partial U_3} = \frac{\partial G_1}{\partial U_3} = 0,\\
&\widetilde{A}_{23} = \frac{\partial \widetilde{A}_{21}}{\partial U_3} = \frac{\partial \widetilde{A}_{22}}{\partial U_3} = \frac{\partial G_2}{\partial U_3} = 0,\\
&\widetilde{A}_{31} = \widetilde{A}_{32} = \frac{\partial \widetilde{A}_{33}}{\partial U_1} = \frac{\partial \widetilde{A}_{33}}{\partial U_2} = \frac{\partial G_3}{\partial U_1} = \frac{\partial G_3}{\partial U_2} = 0.
\end{aligned}
\tag{6.52}
$$

In the case 1, the first two equations can be solved for U_1 and U_2; insertion of these solutions in the third equation leads to an equation for U_3. In the case 2, the third equation can be solved for U_3; insertion of this solution in the first two equations provides two equations for U_1 and U_2. Of course, having a 2×2 system, we can check the conditions derived in the previous paragraph enabling us to perform a further decoupling.

6.4.1 Partial decoupling: case 1

Let us require that $\widetilde{A}_{13} = \widetilde{A}_{23} = 0$ and that $\widetilde{A}_{11}$, $\widetilde{A}_{12}$, $\widetilde{A}_{21}$, $\widetilde{A}_{22}$, and G_1, G_2 are independent of U_3; after some algebra, we get some constraints on the structure of (6.12) and the conditions leading to the variable transformation.

From $\widetilde{A}_{13} = \widetilde{A}_{23} = 0$, we obtain

$$
\begin{aligned}
0 = R_1^{(3)} &= r_1^{(3)} C_{11} + r_2^{(3)} C_{21} + r_3^{(3)} C_{31},\\
0 = R_2^{(3)} &= r_1^{(3)} C_{12} + r_2^{(3)} C_{22} + r_3^{(3)} C_{32},
\end{aligned}
\tag{6.53}
$$

where we denoted with C_{ij} the cofactor of the (i, j)–entry of matrix $\nabla_{\mathbf{U}} \mathbf{h}$, whereupon it follows

$$
\begin{aligned}
(\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}^{(3)} = 0,\\
(\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}^{(3)} = 0.
\end{aligned}
\tag{6.54}
$$

Therefore, we have

$$
\begin{aligned}
&\widetilde{A}_{11} = (\Lambda_1 - \Lambda_2) \frac{R_2^{(1)} R_1^{(2)}}{\Delta} + \Lambda_1, \quad &&\widetilde{A}_{12} = -(\Lambda_1 - \Lambda_2) \frac{R_1^{(1)} R_1^{(2)}}{\Delta},\\
&\widetilde{A}_{21} = (\Lambda_1 - \Lambda_2) \frac{R_2^{(1)} R_2^{(2)}}{\Delta}, &&\widetilde{A}_{22} = -(\Lambda_1 - \Lambda_2) \frac{R_2^{(1)} R_1^{(2)}}{\Delta} + \Lambda_2,
\end{aligned}
\tag{6.55}
$$

where $\Delta = R_1^{(1)} R_2^{(2)} - R_2^{(1)} R_1^{(2)}$.

It is easily ascertained that the entries $\widetilde{A}_{ij}$ $(i, j = 1, 2)$ are independent of U_3 if Λ_1 and Λ_2 do not depend on U_3, and

$$
\frac{\partial}{\partial U_3} \left(\frac{R_2^{(1)}}{R_1^{(1)}} \right) = 0, \qquad \frac{\partial}{\partial U_3} \left(\frac{R_1^{(2)}}{R_2^{(2)}} \right) = 0.
\tag{6.56}
$$

Therefore, we obtain

$$
(\nabla_{\mathbf{u}} \lambda_1) \cdot \frac{\partial \mathbf{h}}{\partial U_3} = 0, \qquad (\nabla_{\mathbf{u}} \lambda_2) \cdot \frac{\partial \mathbf{h}}{\partial U_3} = 0,
\tag{6.57}
$$

that, taking into account (6.53), which means that the vectors $\dfrac{\partial \mathbf{h}}{\partial U_3}$ and $\mathbf{r}^{(3)}$ are parallel, can be rewritten as

$$(\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}^{(3)} = 0, \qquad (\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}^{(3)} = 0. \tag{6.58}$$

From (6.56), through the use of the relation $\mathbf{R}^{(i)} = (\nabla_{\mathbf{u}}\mathbf{H})\mathbf{r}^{(i)}$, it follows

$$\nabla_{\mathbf{u}}\left(\frac{(\nabla_{\mathbf{u}}H_2)\cdot\mathbf{r}^{(1)}}{(\nabla_{\mathbf{u}}H_1)\cdot\mathbf{r}^{(1)}}\right)\cdot\mathbf{r}^{(3)} = 0, \qquad \nabla_{\mathbf{u}}\left(\frac{(\nabla_{\mathbf{u}}H_1)\cdot\mathbf{r}^{(2)}}{(\nabla_{\mathbf{u}}H_2)\cdot\mathbf{r}^{(2)}}\right)\cdot\mathbf{r}^{(3)} = 0. \tag{6.59}$$

Because of (6.54) we can write

$$\nabla_{\mathbf{u}}H_k = \alpha_k(t,x,\mathbf{u})\,\mathbf{l}^{(1)} + \beta_k(t,x,\mathbf{u})\,\mathbf{l}^{(2)}, \qquad k=1,2, \tag{6.60}$$

where α_k and β_k are suitable functions such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

Inserting (6.60) in (6.59), we get

$$\begin{aligned}
&(\alpha_1\beta_2-\alpha_2\beta_1)(\mathbf{l}^{(1)}\cdot\mathbf{r}^{(1)})\mathbf{l}^{(2)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(1)})\mathbf{r}^{(3)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(3)})\mathbf{r}^{(1)}\right) = 0,\\
&(\alpha_1\beta_2-\alpha_2\beta_1)(\mathbf{l}^{(2)}\cdot\mathbf{r}^{(2)})\mathbf{l}^{(1)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(2)})\mathbf{r}^{(3)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(3)})\mathbf{r}^{(2)}\right) = 0;
\end{aligned} \tag{6.61}$$

therefore, we obtain the following additional structure conditions:

$$\begin{aligned}
&\mathbf{l}^{(2)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(1)})\mathbf{r}^{(3)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(3)})\mathbf{r}^{(1)}\right) = 0,\\
&\mathbf{l}^{(1)}\cdot\left((\nabla_{\mathbf{u}}\mathbf{r}^{(2)})\mathbf{r}^{(3)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(3)})\mathbf{r}^{(2)}\right) = 0.
\end{aligned} \tag{6.62}$$

Moreover, by requiring $\dfrac{\partial G_1}{\partial U_3} = \dfrac{\partial G_2}{\partial U_3} = 0$, after some algebra we get

$$\left(\nabla_{\mathbf{u}}((\nabla_{\mathbf{u}}H_i)\cdot\mathbf{g}) + \frac{\partial(\nabla_{\mathbf{u}}H_i)}{\partial t} + \lambda_i\frac{\partial(\nabla_{\mathbf{u}}H_i)}{\partial x}\right)\cdot\mathbf{r}^{(3)} = 0, \qquad i=1,2. \tag{6.63}$$

6.4.2 Partial decoupling: case 2

From $\widetilde{A}_{31} = \widetilde{A}_{32} = 0$, it follows that

$$\begin{aligned}
0 = R_3^{(1)} &= r_1^{(1)}C_{13} + r_2^{(1)}C_{23} + r_3^{(1)}C_{33},\\
0 = R_3^{(2)} &= r_1^{(2)}C_{13} + r_2^{(2)}C_{23} + r_3^{(2)}C_{33},
\end{aligned} \tag{6.64}$$

i.e.,

$$\begin{aligned}
(\nabla_{\mathbf{u}}H_3)\cdot\mathbf{r}^{(1)} &= 0,\\
(\nabla_{\mathbf{u}}H_3)\cdot\mathbf{r}^{(2)} &= 0.
\end{aligned} \tag{6.65}$$

In this case, the expression of entries of $\widetilde{A}$ reads

$$
\begin{aligned}
&\widetilde{A}_{11} = (\Lambda_1-\Lambda_2)\frac{R_2^{(1)}R_1^{(2)}}{\Delta}+\Lambda_1, \qquad \widetilde{A}_{12} = -(\Lambda_1-\Lambda_2)\frac{R_1^{(1)}R_1^{(2)}}{\Delta},\\
&\widetilde{A}_{13} = \frac{(\Lambda_1-\Lambda_2)R_1^{(1)}R_1^{(2)}R_2^{(3)}-(\Lambda_1-\Lambda_3)R_1^{(1)}R_2^{(2)}R_1^{(3)}}{R_3^{(3)}\Delta}\\
&\qquad + \frac{(\Lambda_2-\Lambda_3)R_2^{(1)}R_1^{(2)}R_1^{(3)}}{R_3^{(3)}\Delta},\\
&\widetilde{A}_{21} = (\Lambda_1-\Lambda_2)\frac{R_2^{(1)}R_2^{(2)}}{\Delta}, \qquad \widetilde{A}_{22} = -(\Lambda_1-\Lambda_2)\frac{R_2^{(1)}R_1^{(2)}}{\Delta}+\Lambda_2,\\
&\widetilde{A}_{23} = \frac{-(\Lambda_1-\Lambda_2)R_2^{(1)}R_2^{(2)}R_1^{(3)}+(\Lambda_1-\Lambda_3)R_2^{(1)}R_1^{(2)}R_2^{(3)}}{R_3^{(3)}\Delta}\\
&\qquad - \frac{(\Lambda_2-\Lambda_3)R_1^{(1)}R_2^{(2)}R_2^{(3)}}{R_3^{(3)}\Delta}, \qquad \widetilde{A}_{33} = \Lambda_3,
\end{aligned}
\tag{6.66}
$$

where $\Delta = R_1^{(1)}R_2^{(2)} - R_2^{(1)}R_1^{(2)}$.

The independence of $\widetilde{A}_{33}$ from U_1 and U_2 leads us to get

$$\frac{\partial\Lambda_3}{\partial U_1} = \frac{\partial\Lambda_3}{\partial U_2} = 0, \tag{6.67}$$

i.e.,

$$(\nabla_{\mathbf{u}}\lambda_3)\cdot\frac{\partial\mathbf{h}}{\partial U_1} = 0, \qquad (\nabla_{\mathbf{u}}\lambda_3)\cdot\frac{\partial\mathbf{h}}{\partial U_2} = 0. \tag{6.68}$$

Recalling that (6.64) means that the vectors $\frac{\partial\mathbf{h}}{\partial U_1}$ and $\frac{\partial\mathbf{h}}{\partial U_2}$ belong to the plane spanned by $\mathbf{r}^{(1)}$ and $\mathbf{r}^{(2)}$, the conditions (6.68) can be rewritten as

$$(\nabla_{\mathbf{u}}\lambda_3)\cdot\mathbf{r}^{(1)} = 0, \qquad (\nabla_{\mathbf{u}}\lambda_3)\cdot\mathbf{r}^{(2)} = 0, \tag{6.69}$$

that are the structure conditions on the original system.

Finally, by requiring $\frac{\partial G_3}{\partial U_1} = \frac{\partial G_3}{\partial U_2} = 0$ we get

$$\left(\nabla_{\mathbf{u}}((\nabla_{\mathbf{u}}H_3)\cdot\mathbf{g}) + \frac{\partial(\nabla_{\mathbf{u}}H_3)}{\partial t} + \lambda_3\frac{\partial(\nabla_{\mathbf{u}}H_3)}{\partial x}\right)\cdot\mathbf{r}^{(i)} = 0, \tag{6.70}$$

for $i = 1, 2$.

6.4.3 Full decoupling

It is now clear that we may decouple the system into two non–interacting blocks if the conditions of both cases 1 and 2 are verified.

Example 6.4.1. *Let us consider the following Galilean first order system:*

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} + (u_1 + p_{11})\frac{\partial u_1}{\partial x} + p_{12}\frac{\partial u_2}{\partial x} + p_{13}\frac{\partial u_3}{\partial x} = g_1, \\
&\frac{\partial u_2}{\partial t} + \frac{p_{11}p_{22}}{p_{12}}\frac{\partial u_1}{\partial x} + (u_1 + p_{22})\frac{\partial u_2}{\partial x} + p_{23}\frac{\partial u_3}{\partial x} = g_2, \\
&\frac{\partial u_3}{\partial t} + (u_1 + p_{33})\frac{\partial u_3}{\partial x} = g_3,
\end{aligned} \tag{6.71}$$

where p_{ij}, g_i $(i, j = 1, \dots, 3)$ are arbitrary functions of u_2 and u_3.

The characteristic velocities are

$$\lambda_1 = u_1, \qquad \lambda_2 = u_1 + p_{11} + p_{22}, \qquad \lambda_3 = u_1 + p_{33}, \tag{6.72}$$

with associated left and right eigenvectors

$$\begin{aligned}
&\mathbf{l}^{(1)} = (p_{22}p_{33}, -p_{12}p_{33}, p_{12}p_{23} - p_{13}p_{22}), \qquad \mathbf{l}^{(3)} = (0, 0, 1), \\
&\mathbf{l}^{(2)} = (p_{11}(p_{11} + p_{22}, -p_{33}), p_{12}(p_{11} + p_{22}, -p_{33}), p_{11}p_{13} + p_{12}p_{23}), \\
&\mathbf{r}^{(1)} = \begin{pmatrix} p_{12} \\ -p_{11} \\ 0 \end{pmatrix}, \qquad \mathbf{r}^{(2)} = \begin{pmatrix} p_{12} \\ p_{22} \\ 0 \end{pmatrix}, \\
&\mathbf{r}^{(3)} = \begin{pmatrix} p_{12}(p_{13}(p_{22} - p_{33}) - p_{12}p_{23}) \\ p_{11}(p_{12}p_{23} - p_{13}p_{22}) - p_{12}p_{23}p_{33}) \\ p_{12}p_{33}(p_{11} + p_{22} - p_{33}) \end{pmatrix}.
\end{aligned} \tag{6.73}$$

By taking $p_{13} = \dfrac{u_2}{u_3}p_{12}$, conditions (6.54), (6.58), (6.62) and (6.63) give us the following relations

$$\begin{aligned}
&H_1 = H_1(u_1, u_2u_3), && H_2 = H_2(u_1, u_2u_3), \\
&p_{11} = q_{11} - q_{22}, && p_{12} = \frac{q_{12}}{u_2}, \\
&p_{22} = q_{22}, && p_{23} = \frac{u_2}{u_3}(q_{22} - p_{33}), \\
&g_1 = c_1, && g_2 = u_2\left(c_2 - \frac{g_3}{u_3}\right),
\end{aligned} \tag{6.74}$$

where q_{11}, q_{12}, q_{22}, c_1 and c_2 are arbitrary functions of (u_2u_3).

By choosing the new dependent variables as

$$U_1 = u_1 + u_2u_3, \qquad U_2 = u_2u_3, \qquad U_3 = u_3, \tag{6.75}$$

and replacing the inverse transformation of (6.75), i.e.*,*

$$u_1 = U_1 - U_2, \qquad u_2 = \frac{U_2}{U_3}, \qquad u_3 = U_3, \tag{6.76}$$

in (6.71), we get the following partially decoupled system

$$\begin{aligned}
&\frac{\partial U_1}{\partial t}+\left(U_1+U_2\left(\frac{q_{22}}{q_{12}}(q_{11}-q_{22})-1\right)+q_{11}-q_{22}\right)\frac{\partial U_1}{\partial x}\\
&\qquad+\left(U_2\frac{q_{22}}{q_{12}}(q_{22}-q_{11})+\frac{q_{12}}{U_2}+2q_{22}-q_{11}\right)\frac{\partial U_2}{\partial x}=c_1+U_2\,c_2,\\
&\frac{\partial U_2}{\partial t}+\left(U_2\frac{q_{22}}{q_{12}}(q_{11}-q_{22})\right)\frac{\partial U_1}{\partial x}\\
&\qquad+\left(U_1+U_2\left(\frac{q_{22}}{q_{12}}(q_{22}-q_{11})-1\right)+q_{22}\right)\frac{\partial U_2}{\partial x}=U_2\,c_2,\\
&\frac{\partial U_3}{\partial t}+\left(U_1-U_2+p_{33}\left(\frac{U_2}{U_3},U_3\right)\right)\frac{\partial U_3}{\partial x}=g_3\left(\frac{U_2}{U_3},U_3\right),
\end{aligned}\tag{6.77}$$

where the first two equations do not depend on U_3.

6.5 Decoupling of hyperbolic homogeneous and autonomous systems

The results of previous Sections suggest us the form of the necessary and sufficient conditions for the decoupling problem in the general case of quasilinear first order systems in two independent variables and an arbitrary number n of dependent variables.

Let us start by considering the case of a hyperbolic first order homogeneous and autonomous quasilinear system of partial differential equations in two independent variables, and provide the necessary and sufficient conditions for decoupling it.

Let us introduce the notation we will use, and formally define the meaning of partially and fully decoupled systems.

Definition 6.5.1 (Notation)**.** *Given $\mathbf{U}\equiv(U_1,\ldots,U_n)^T\in\mathbb{R}^n$, and a set of $k\geq 2$ integers $n_1,\ldots,n_k$ such that $n_1+\ldots+n_k=n$, let us relabel and group the components of $\mathbf{U}$ as follows:*

$$\left\{\{U^{(1,1)},\ldots,U^{(1,n_1)}\},\ldots,\{U^{(k,1)},\ldots,U^{(k,n_k)}\}\right\}.\tag{6.78}$$

Moreover, let us set

$$\mathcal{U}_i=\bigcup_{r=1}^{i}\left\{U^{(r,1)},\ldots,U^{(r,n_r)}\right\},\qquad \overline{\mathcal{U}}_i=\bigcup_{r=i+1}^{k}\left\{U^{(r,1)},\ldots,U^{(r,n_r)}\right\};\tag{6.79}$$

the cardinality of the set $\mathcal{U}_i$ is m_i, whereas the cardinality of the set $\overline{\mathcal{U}}_i$ is $n-m_i$, where $m_i=n_1+\ldots+n_i$.

Definition 6.5.2 (Partially decoupled systems)**.** *The first order quasilinear system*

$$\frac{\partial\mathbf{U}}{\partial t}+T(\mathbf{U})\frac{\partial\mathbf{U}}{\partial x}=\mathbf{0},\tag{6.80}$$

$\mathbf{U}\equiv(U_1,\ldots,U_n)^T\in\mathbb{R}^n$, T being an $n\times n$ real matrix with entries smooth functions depending on $\mathbf{U}$, is partially decoupled in $2\leq k\leq n$ subsystems of some orders $n_1,\ldots,n_k$ ($n_1+\ldots+n_k=n$) if, relabelling and suitably collecting

the components of $\mathbf{U}$ *in* k *subgroups, say*

$$\left\{\{U^{(1,1)},\dots,U^{(1,n_1)}\},\dots,\{U^{(k,1)},\dots,U^{(k,n_k)}\}\right\}, \tag{6.81}$$

we recognize k *subsystems such that the* i*–th subsystem (*$i=1,\dots,k$*) involves at most the* m_i *field variables of the set* $\mathcal{U}_i$*.*

Definition 6.5.3 (Fully decoupled systems)**.** *The first order quasilinear system*

$$\frac{\partial\mathbf{U}}{\partial t}+T(\mathbf{U})\frac{\partial\mathbf{U}}{\partial x}=\mathbf{0}, \tag{6.82}$$

$\mathbf{U}\equiv(U_1,\dots,U_n)^T\in\mathbb{R}^n$*,* T *being an* $n\times n$ *real matrix with entries smooth functions depending on* $\mathbf{U}$*, is fully decoupled in* $2\leq k\leq n$ *subsystems of some orders* $n_1,\dots,n_k$ *(*$n_1+\dots+n_k=n$*) if, relabelling and suitably collecting the components of* $\mathbf{U}$ *in* k *subgroups, say*

$$\left\{\{U^{(1,1)},\dots,U^{(1,n_1)}\},\dots,\{U^{(k,1)},\dots,U^{(k,n_k)}\}\right\}, \tag{6.83}$$

we recognize k *subsystems such that the* i*–th subsystem (*$i=1,\dots,k$*) involves exactly the* n_i *field variables* $\{U^{(i,1)},\dots,U^{(i,n_i)}\}$*.*

The following lemma will direct us to prove a theorem providing necessary and sufficient conditions for the partial decoupling of a hyperbolic first order quasilinear system in two independent variables.

Lemma 6.5.1. *Let* T *be an* $n\times n$ *lower triangular block real matrix, say*

$$T=\begin{bmatrix} T^1_1 & 0^1_2 & 0^1_3 & \dots & 0^1_{k-1} & 0^1_k \\ T^2_1 & T^2_2 & 0^2_3 & \dots & 0^2_{k-1} & 0^2_k \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T^{k-1}_1 & T^{k-1}_2 & T^{k-1}_3 & \dots & T^{k-1}_{k-1} & 0^{k-1}_k \\ T^k_1 & T^k_2 & T^k_3 & \dots & T^k_{k-1} & T^k_k \end{bmatrix}, \tag{6.84}$$

T^i_j *being* $n_i\times n_j$ *matrices with entries smooth functions depending on* $\mathbf{U}\equiv(U_1,\dots,U_n)^T$*, and* 0^i_j $n_i\times n_j$ *matrices of zeros (*$n_1+\dots+n_k=n$*;* $2\leq k\leq n$*). Let us assume that matrix* T *has real eigenvalues and a complete set of eigenvectors. By relabelling and suitably collecting the components of* $\mathbf{U}$ *in* k *subgroups, say*

$$\left\{\{U^{(1,1)},\dots,U^{(1,n_1)}\},\dots,\{U^{(k,1)},\dots,U^{(k,n_k)}\}\right\}, \tag{6.85}$$

it can be stated that the entries of matrices T^i_j *(*$i=1,\dots,k$*;* $j=1,\dots,i$*) depend at most on the* m_i *variables of the set* $\mathcal{U}_i$ *if and only if:*

1. *the set of the eigenvalues of* T *(counted with their multiplicity) and the corresponding left and right eigenvectors can be divided into* k *subsets each containing* n_i *(*$i=1,\dots,k$*) elements, say*

$$\begin{aligned}&\left\{\{\Lambda^{(1,1)},\dots,\Lambda^{(1,n_1)}\},\dots,\{\Lambda^{(k,1)},\dots,\Lambda^{(k,n_k)}\}\right\},\\ &\left\{\{\mathbf{L}^{(1,1)},\dots,\mathbf{L}^{(1,n_1)}\},\dots,\{\mathbf{L}^{(k,1)},\dots,\mathbf{L}^{(k,n_k)}\}\right\},\\ &\left\{\{\mathbf{R}^{(1,1)},\dots,\mathbf{R}^{(1,n_1)}\},\dots,\{\mathbf{R}^{(k,1)},\dots,\mathbf{R}^{(k,n_k)}\}\right\},\end{aligned} \tag{6.86}$$

where

$$\{\Lambda^{(i,1)}, \dots, \Lambda^{(i,n_i)}\} \tag{6.87}$$

are the eigenvalues (counted with their multiplicity) of matrix T_i^i*;*

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot \mathbf{R}^{(j,\gamma)} = 0,\\
&\mathbf{L}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right) = 0,\\
&i = 1,\dots,k-1, \quad \ell = 1,\dots,i,\\
&\alpha = 1,\dots,n_i, \quad \beta = 1,\dots,n_\ell, \quad \alpha\neq\beta \textit{ if } i=\ell,\\
&j = i+1,\dots,k, \quad \gamma = 1,\dots,n_j,
\end{aligned} \tag{6.88}$$

where

$$\nabla_{\mathbf{U}} \equiv \left(\frac{\partial}{\partial U^{(1,1)}},\dots,\frac{\partial}{\partial U^{(1,n_1)}},\dots,\frac{\partial}{\partial U^{(k,1)}},\dots,\frac{\partial}{\partial U^{(k,n_k)}}\right). \tag{6.89}$$

Proof. Let the lower triangular block matrix T be such that the entries of matrices T_j^i ($i = 1,\dots,k-1$; $j = 1,\dots,i$) depend at most on the elements of the set $\mathcal{U}_i$.

The set of the n eigenvalues of T is the union of the sets of the n_i eigenvalues of T_i^i ($i = 1,\dots,k$); since the entries of matrix T_i^i depend at most on the elements of the set $\mathcal{U}_i$, the same can be said for its n_i eigenvalues.

Let us denote the set of left and right eigenvectors of matrix T as in (6.86), and group the components of a right (left, respectively) eigenvector $\mathbf{R}^{(r,\alpha)}$ ($\mathbf{L}^{(r,\alpha)}$, respectively) as follows:

$$\mathbf{R}^{(r,\alpha)} = \begin{pmatrix}\mathbf{R}_1^{(r,\alpha)}\\ \mathbf{R}_2^{(r,\alpha)}\\ \dots\\ \mathbf{R}_k^{(r,\alpha)}\end{pmatrix}, \qquad \mathbf{L}^{(r,\alpha)} = \left(\mathbf{L}_1^{(r,\alpha)}, \mathbf{L}_2^{(r,\alpha)}, \dots, \mathbf{L}_k^{(r,\alpha)}\right), \tag{6.90}$$

where $\mathbf{R}_i^{(r,\alpha)}$ ($\mathbf{L}_i^{(r,\alpha)}$, respectively) are column (row, respectively) vectors with n_i components.

Taking into account the relations for the left eigenvectors,

$$\begin{aligned}
\mathbf{L}_1^{(r,\alpha)}T_1^1 + \mathbf{L}_2^{(r,\alpha)}T_1^2 + \dots + \mathbf{L}_{k-1}^{(r,\alpha)}T_1^{k-1} + \mathbf{L}_k^{(r,\alpha)}T_1^k &= \Lambda^{(r,\alpha)}\mathbf{L}_1^{(r,\alpha)},\\
\mathbf{L}_2^{(r,\alpha)}T_2^2 + \dots + \mathbf{L}_{k-1}^{(r,\alpha)}T_2^{k-1} + \mathbf{L}_k^{(r,\alpha)}T_2^k &= \Lambda^{(r,\alpha)}\mathbf{L}_2^{(r,\alpha)},\\
\dots\dots&\\
\mathbf{L}_{k-1}^{(r,\alpha)}T_{k-1}^{k-1} + \mathbf{L}_k^{(r,\alpha)}T_{k-1}^k &= \Lambda^{(r,\alpha)}\mathbf{L}_{k-1}^{(r,\alpha)},\\
\mathbf{L}_k^{(r,\alpha)}T_k^k &= \Lambda^{(r,\alpha)}\mathbf{L}_k^{(r,\alpha)},
\end{aligned} \tag{6.91}$$

since $\Lambda^{(r,\alpha)}$ is an eigenvalue of T_r^r, if $r<k$, we can choose $\mathbf{L}_{r+1}^{(r,\alpha)},\dots,\mathbf{L}_k^{(r,\alpha)}$ as zero row vectors; this means that the left eigenvectors $\mathbf{L}^{(r,\alpha)}$ ($\alpha = 1,\dots,n_r$) may have non–vanishing only the first m_r components ($m_r = n_1 + \dots + n_r$); moreover, due to the hypotheses of the functional dependence

of matrices T^i_j, the components of $\mathbf{L}_1^{(r,\alpha)}, \dots, \mathbf{L}_r^{(r,\alpha)}$ depend at most on the elements of the set $\mathcal{U}_r$.

Analogously, by considering the relations for the right eigenvectors,

$$\begin{aligned}
&T_1^1\mathbf{R}_1^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_1^{(r,\alpha)},\\
&T_1^2\mathbf{R}_1^{(r,\alpha)} + T_2^2\mathbf{R}_2^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_2^{(r,\alpha)},\\
&\dots\dots\\
&T_1^{k-1}\mathbf{R}_1^{(r,\alpha)} + T_2^{k-1}\mathbf{R}_2^{(r,\alpha)} + \dots + T_{k-1}^{k-1}\mathbf{R}_{k-1}^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_{k-1}^{(r,\alpha)},\\
&T_1^{k}\mathbf{R}_1^{(r,\alpha)} + T_2^{k}\mathbf{R}_2^{(r,\alpha)} + \dots + T_{k-1}^{k}\mathbf{R}_{k-1}^{(r,\alpha)} + T_k^k\mathbf{R}_k^{(r,\alpha)} = \Lambda^{(r,\alpha)}\mathbf{R}_{k}^{(r,\alpha)},
\end{aligned} \tag{6.92}$$

since $\Lambda^{(r,\alpha)}$ is an eigenvalue of T^r_r, if $r > 1$, we can choose $\mathbf{R}_1^{(r,\alpha)}, \dots, \mathbf{R}_{r-1}^{(r,\alpha)}$ as zero column vectors; this means that the right eigenvectors $\mathbf{R}^{(r,\alpha)}$ ($\alpha = 1, \dots, n_r$) for $r > 1$ may have non–vanishing only the last $n - m_{r-1}$ components; moreover, due to the hypotheses of the functional dependence of matrices T^r_r, the components of $\mathbf{R}_s^{(r,\alpha)}$ ($s = r, \dots, k$) depend at most on the elements of the set $\mathcal{U}_s$. Notice that, because of the hyperbolicity assumption, the vectors $\mathbf{R}_r^{(r,\alpha)}$, as well as the vectors $\mathbf{L}_r^{(r,\alpha)}$ ($r = 1, \dots, k;\ \alpha = 1, \dots, n_r$) are linearly independent.

As a consequence, conditions (6.88) are trivially satisfied. In fact, at most the first m_i components of the vector $\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}$ may be non–vanishing, whereas the first m_{j-1} components of $\mathbf{R}^{(j,\gamma)}$ are zero: since $j > i$,

$$\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)} = 0. \tag{6.93}$$

Moreover, since the first $m_{\ell-1}$ components of $\mathbf{R}^{(\ell,\beta)}$ are vanishing, the components of $\mathbf{R}_s^{(\ell,\beta)}$ ($s = \ell, \dots, k$) depend at most on $\mathcal{U}_s$, the first m_{j-1} of $\mathbf{R}^{(j,\gamma)}$ are vanishing, and $j > \ell$, it follows that the first m_j components of the vector $(\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}$ are vanishing; therefore, it is

$$\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right) = 0. \tag{6.94}$$

Vice versa, if conditions (6.88) hold true, then it can be proved that all entries of matrices T^i_j ($i = 1, \dots, k-1;\ j = 1, \dots, i$) depend at most on the elements of the set $\mathcal{U}_i$.

At first, let us prove that from $(6.88)_1$ it follows that $\Lambda^{(r,\alpha)}$ ($1 \le r < k;\ \alpha = 1, \dots, n_r$) can at most depend on the elements of the set $\mathcal{U}_r$.

Let us denote with Λ one of the eigenvalues of the matrix T^r_r ($1 \le r < k$), and let us set

$$\nabla_{\mathbf{U}} \equiv (\nabla_1, \dots, \nabla_k), \tag{6.95}$$

where

$$\nabla_i \equiv \left(\frac{\partial}{\partial U^{(i,1)}}, \dots, \frac{\partial}{\partial U^{(i,n_i)}}\right), \qquad i = 1, \dots, k. \tag{6.96}$$

Since $\mathbf{R}^{(j,\gamma)}$ for $j > r$ may have non–vanishing only the last $n - m_{j-1}$ components, conditions $(6.88)_1$ read

$$
\begin{aligned}
(\nabla_{r+1}\Lambda)\cdot\mathbf{R}^{(r+1,\gamma)}_{r+1} + (\nabla_{r+2}\Lambda)\cdot\mathbf{R}^{(r+1,\gamma)}_{r+2} + \ldots + (\nabla_k\Lambda)\cdot\mathbf{R}^{(r+1,\gamma)}_{k} &= 0,\\
(\nabla_{r+2}\Lambda)\cdot\mathbf{R}^{(r+2,\gamma)}_{r+2} + \ldots + (\nabla_k\Lambda)\cdot\mathbf{R}^{(r+2,\gamma)}_{k} &= 0,\\
&\ldots\\
(\nabla_k\Lambda)\cdot\mathbf{R}^{(k,\gamma)}_{k} &= 0,
\end{aligned}
\tag{6.97}
$$

whereupon it immediately follows that

$$
\frac{\partial\Lambda}{\partial U^{(r+1,1)}} = \ldots = \frac{\partial\Lambda}{\partial U^{(r+1,n_{r+1})}} = \ldots = \frac{\partial\Lambda}{\partial U^{(k,1)}} = \ldots = \frac{\partial\Lambda}{\partial U^{(k,n_k)}} = 0. \tag{6.98}
$$

Moreover, because of the lower triangular block structure of matrix T, the left eigenvectors $\mathbf{L}^{(i,\alpha)}$ may have non–vanishing only the first m_i components, and the right eigenvectors $\mathbf{R}^{(j,\gamma)}$ may have non–vanishing only the last $n - m_{j-1}$ components, $(6.88)_2$ can be written as:

$$
\sum_{r=\ell}^{i}\left(\mathbf{L}^{(i,\alpha)}_r\cdot\left((\nabla_j\mathbf{R}^{(\ell,\beta)}_r)\mathbf{R}^{(j,\gamma)}_j + \ldots + (\nabla_k\mathbf{R}^{(\ell,\beta)}_r)\mathbf{R}^{(j,\gamma)}_k\right)\right) = 0, \tag{6.99}
$$

for $i = 1,\ldots,k-1,\ \ell \le i,\ j = i+1,\ldots,k$, and $\alpha \neq \beta$ for $\ell = i$.

From the relations

$$
T^r_r\mathbf{R}^{(r,\alpha)}_r = \Lambda^{(r,\alpha)}\mathbf{R}^{(r,\alpha)}_r, \qquad r = 1,\ldots,k-1, \tag{6.100}
$$

for $j > r$, we obtain

$$
\left(\sum_{s=j}^{k}(\nabla_s T^r_r)\mathbf{R}^{(j,\gamma)}_s\right)\mathbf{R}^{(r,\alpha)}_r + (T^r_r - \Lambda^{(r,\alpha)}\mathbb{I}_r)\left(\sum_{s=j}^{k}(\nabla_s\mathbf{R}^{(r,\alpha)}_r)\mathbf{R}^{(j,\gamma)}_s\right) = \mathbf{0}, \tag{6.101}
$$

$\mathbb{I}_r$ being the $r \times r$ identity matrix, whereupon

$$
\begin{aligned}
&\mathbf{L}^{(r,\beta)}_r\cdot\left(\sum_{s=j}^{k}(\nabla_s A^r_r)\mathbf{R}^{(j,\gamma)}_s\right)\mathbf{R}^{(r,\alpha)}_r\\
&\quad + (\Lambda^{(r,\beta)} - \Lambda^{(r,\alpha)})\mathbf{L}^{(r,\beta)}_r\cdot\left(\sum_{s=j}^{k}(\nabla_s\mathbf{R}^{(r,\alpha)}_r)\mathbf{R}^{(j,\gamma)}_s\right) = 0.
\end{aligned}
\tag{6.102}
$$

As a consequence, either when $\alpha = \beta$, or using (6.99) when $\alpha \neq \beta$, it is

$$
\mathbf{L}^{(r,\beta)}_r\cdot\left(\sum_{s=j}^{k}(\nabla_s T^r_r)\mathbf{R}^{(j,\gamma)}_s\right)\mathbf{R}^{(r,\alpha)}_r = 0 \tag{6.103}
$$

for $\alpha,\beta = 1,\ldots,n_r$. Therefore, using (6.103) for $j = k, k-1,\ldots,r+1$, it remains proved that all entries of the matrix T^r_r ($r = 1,\ldots,k-1$), and so all the components of $\mathbf{R}^{(r,\alpha)}_r$, depend at most on the elements of the set $\mathcal{U}_r$.

Let us now take the relations (6.99); by specializing them neatly for $\ell = i-1, i-2,\ldots,1$ and $j = k, k-1,\ldots,i+1$, it is immediately deduced that the components of $\mathbf{R}_s^{(r,\alpha)}$ for $s \geq r$ depend at most on the elements of the set $\mathcal{U}_s$.

Finally, from the relations

$$\sum_{s=r}^{i} T_s^i \mathbf{R}_s^{(r,\alpha)} = \Lambda^{(r,\alpha)} \mathbf{R}_i^{(r,\alpha)}, \qquad r = 1, \dots, i, \quad i < k, \tag{6.104}$$

for $j > i$, we obtain

$$\sum_{s=r}^{i} \left(\left(\sum_{t=j}^{k} (\nabla_t T_s^i) \mathbf{R}_t^{(j,\gamma)} \right) \mathbf{R}_s^{(r,\alpha)} \right) = \mathbf{0}. \tag{6.105}$$

By neatly specializing the relations (6.105) for $r = i-1, i-2, \dots, 1$ and $j = k, k-1, \dots, i+1$, it follows that the entries of the matrices T_s^i ($s = 1, \dots, i$) depend at most on the m_i variables of the set $\mathcal{U}_i$, and this completes the proof.

□

Remark 6.5.1. *Relations (6.88) provide* $\sum_{i=1}^{k} n_i m_i (n - m_i)$ *constraints, and this is exactly the number of conditions required to ensure that the entries of matrices* T_j^i ($i = 1, \dots, k$; $j = 1, \dots, i$) *are independent of the elements of the set* $\overline{\mathcal{U}}_i$. *In fact, the number of entries of the matrices* T_j^i *are* $n_i m_i$, *and the cardinality of the set* $\overline{\mathcal{U}}_i$ *is* $n - m_i$.

Remark 6.5.2. *If matrix* T *has the lower triangular block structure (6.84) then, since the first* m_{j-1} *components of* $\mathbf{R}^{(j,\gamma)}$ *are vanishing, and* $j > i \geq \ell$, *the first* m_i *components of the vector* $(\nabla_{\mathbf{U}} \mathbf{R}^{(j,\gamma)}) \mathbf{R}^{(\ell,\beta)}$ *can not be different from zero; therefore, it is identically*

$$\begin{aligned}
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(j,\gamma)}) \mathbf{R}^{(\ell,\beta)} \right) = 0,\\
&i = 1, \dots, k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \text{ if } i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned} \tag{6.106}$$

Consequently, conditions (6.88) may be written as well as

$$\begin{aligned}
&\left(\nabla_{\mathbf{U}} \Lambda^{(i,\alpha)} \right) \cdot \mathbf{R}^{(j,\gamma)} = 0,\\
&\mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}} \mathbf{R}^{(\ell,\beta)}) \mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}} \mathbf{R}^{(j,\gamma)}) \mathbf{R}^{(\ell,\beta)} \right) = 0,\\
&i = 1, \dots, k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \text{ if } i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned} \tag{6.107}$$

This result reveals useful in the proof of next theorem.

By using Lemma 6.5.1, it is immediately proved the following theorem.

Theorem 6.5.1. *The first order quasilinear system*

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.108}$$

with $\mathbf{u} \in \mathbb{R}^n$*,* A $n \times n$ *real matrix with entries smooth functions of* $\mathbf{u}$*, assumed to be hyperbolic in the* t*–direction, can be transformed by a smooth (locally) invertible transformation*

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \textit{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u}), \tag{6.109}$$

into a system like

$$\frac{\partial \mathbf{U}}{\partial t} + T(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{0}, \tag{6.110}$$

in the unknowns

$$\mathbf{U} \equiv \left(U^{(1,1)}, \dots, U^{(1,n_1)}, \dots, U^{(k,1)}, \dots, U^{(k,n_k)}\right)^T, \tag{6.111}$$

where $T = (\nabla_{\mathbf{u}}\mathbf{H})\, A\, (\nabla_{\mathbf{u}}\mathbf{H})^{-1}$ *is a lower triangular block matrix having the form (6.84) with* T^i_j *(*$i = 1, \dots, k$*;* $j = 1, \dots, i$*)* $n_i \times n_j$ *matrices such that their entries are smooth functions depending at most on the elements of the set* $\mathcal{U}_i$*, whereas* 0^i_j *are* $n_i \times n_j$ *matrices of zeros, respectively, if and only if:*

1. *the set of the eigenvalues of matrix* A *(counted with their multiplicity), and the associated left and right eigenvectors can be divided into* k *subsets each containing* n_i *(*$i = 1, \dots, k$*) elements, say*

$$\begin{aligned}
&\left\{\{\lambda^{(1,1)}, \dots, \lambda^{(1,n_1)}\}, \dots, \{\lambda^{(k,1)}, \dots, \lambda^{(k,n_k)}\}\right\},\\
&\left\{\{\mathbf{l}^{(1,1)}, \dots, \mathbf{l}^{(1,n_1)}\}, \dots, \{\mathbf{l}^{(k,1)}, \dots, \mathbf{l}^{(k,n_k)}\}\right\},\\
&\left\{\{\mathbf{r}^{(1,1)}, \dots, \mathbf{r}^{(1,n_1)}\}, \dots, \{\mathbf{r}^{(k,1)}, \dots, \mathbf{r}^{(k,n_k)}\}\right\};
\end{aligned} \tag{6.112}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0,\\
&\mathbf{l}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right) = 0,\\
&i = 1, \dots, k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \textit{ if } i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned} \tag{6.113}$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$ *(*$i = 1, \dots, k-1$*;* $\alpha = 1, \dots, n_i$*) are found from*

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0, \tag{6.114}$$

where $j = i+1, \dots, k$*,* $\gamma = 1, \dots, n_j$*.*

Proof. Let us consider the hyperbolic system

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.115}$$

and denote the k subsets each containing n_i ($i = 1, \dots, k$) eigenvalues (counted with their multiplicity), together with their associated left and right eigenvectors, as in (6.112). The hyperbolicity condition implies that the eigenvalues $\lambda^{(i,\alpha)}$ $(i = 1, \dots, k;\ \alpha = 1, \dots, n_i)$ are real, whereas the corresponding left (right, respectively) eigenvectors are linearly independent and span $\mathbb{R}^n$.

Let us assume that the conditions (6.113) are satisfied. Then, by introducing a smooth (locally) invertible transformation like (6.109) such that

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}} H^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0, \\
&i = 1, \dots, k-1, \quad \alpha = 1, \dots, n_i, \\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j,
\end{aligned} \tag{6.116}$$

we obtain the system (6.110), where T is a lower triangular block matrix like (6.84).

It remains to prove that the entries of the matrices T^i_j $(i = 1, \dots, k-1;\ j = 1, \dots, i)$ do not depend on the elements of the set $\overline{\mathcal{U}}_i$.

It is

$$\lambda^{(i,\alpha)} = \Lambda^{(i,\alpha)}, \qquad \mathbf{l}^{(i,\alpha)} = \mathbf{L}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H}), \qquad \mathbf{r}^{(i,\alpha)} = (\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(i,\alpha)}, \tag{6.117}$$

and also

$$\nabla_{\mathbf{u}}(\cdot) = \nabla_{\mathbf{U}}(\cdot)(\nabla_{\mathbf{u}}\mathbf{H}). \tag{6.118}$$

As a consequence, we have:

$$\begin{aligned}
0 &= \left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = \\
&= \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) (\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{U}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)} = \\
&= \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)},
\end{aligned} \tag{6.119}$$

whereupon

$$\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0 \quad \Leftrightarrow \quad \left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right) \cdot \mathbf{R}^{(j,\gamma)} = 0. \tag{6.120}$$

Furthermore, it is

$$
\begin{aligned}
0 = \mathbf{l}^{(i,\alpha)} \cdot & \left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right) = \\
= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}}\mathbf{H}) & \left(\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right. \\
& \left. -\nabla_{\mathbf{u}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right) = \\
= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}}\mathbf{H}) & \left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right. \\
& \left. -\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right) = \\
= \mathbf{L}^{(i,\alpha)} (\nabla_{\mathbf{u}}\mathbf{H}) & \left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(\ell,\beta)}\right)\mathbf{R}^{(j,\gamma)}\right. \\
& \left. -\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{R}^{(j,\gamma)}\right)\mathbf{R}^{(\ell,\beta)}\right) = \\
= \mathbf{L}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H}) & \left(\nabla_{\mathbf{U}}\left((\nabla_{\mathbf{u}}\mathbf{H})^{-1}\right)\left(\mathbf{R}^{(\ell,\beta)}\mathbf{R}^{(j,\gamma)} - \mathbf{R}^{(j,\gamma)}\mathbf{R}^{(\ell,\beta)}\right)\right. \\
& \left. +(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right)\right) = \\
= \mathbf{L}^{(i,\alpha)} \cdot & \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)} - (\nabla_{\mathbf{U}}\mathbf{R}^{(j,\gamma)})\mathbf{R}^{(\ell,\beta)}\right) = \\
= \mathbf{L}^{(i,\alpha)} \cdot & \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right),
\end{aligned}
\tag{6.121}
$$

whereupon

$$
\begin{aligned}
& \mathbf{L}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{U}}\mathbf{R}^{(\ell,\beta)})\mathbf{R}^{(j,\gamma)}\right) = 0 \quad \Leftrightarrow \\
& \quad \Leftrightarrow \quad \mathbf{l}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(\ell,\beta)})\mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(\ell,\beta)}\right) = 0.
\end{aligned}
\tag{6.122}
$$

As a result, by using Lemma 6.5.1, the entries of the matrices T^i_j ($i = 1, \ldots, k-1$; $j = 1, \ldots, i$) do not depend on the elements of the set $\overline{\mathcal{U}}_i$.

Vice versa, if the entries of the matrices T^i_j ($i = 1, \ldots, k-1$; $j = 1, \ldots, i$) do not depend on the elements of the set $\overline{\mathcal{U}}_i$, *i.e.*, the system is partially decoupled in k subsystems, then, because of (6.120) and (6.122), conditions (6.113) must hold, and this concludes the proof. □

As a byproduct of Theorem 6.5.1 we may recover immediately the conditions for the full decoupling problem in the case of hyperbolic first order quasilinear systems.

Theorem 6.5.2 (Full decoupling of hyperbolic systems)**.** *For a hyperbolic system of first order homogeneous and autonomous quasilinear partial differential equations like*

$$
\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.123}
$$

$\mathbf{u} \in \mathbb{R}^n$, *$A$ $n \times n$ real matrix with entries smooth functions of $\mathbf{u}$, to be locally reducible into k non–interacting subsystems of some orders $n_1, \ldots, n_k$, with $n_1 + \ldots + n_k = n$, in the unknowns*

$$
\mathbf{U} \equiv \left(U^{(1,1)}, \ldots, U^{(1,n_1)}, \ldots, U^{(k,1)}, \ldots, U^{(k,n_k)}\right)^T, \tag{6.124}
$$

respectively, it is necessary and sufficient that:

1. *the characteristic velocities (counted with their multiplicity), and the corresponding left and right eigenvectors can be divided into k subsets each containing n_i ($i = 1, \dots, k$) elements, say*

$$\begin{aligned}
&\left\{\{\lambda^{(1,1)}, \dots, \lambda^{(1,n_1)}\}, \dots, \{\lambda^{(k,1)}, \dots, \lambda^{(k,n_k)}\}\right\},\\
&\left\{\{\mathbf{l}^{(1,1)}, \dots, \mathbf{l}^{(1,n_1)}\}, \dots, \{\mathbf{l}^{(k,1)}, \dots, \mathbf{l}^{(k,n_k)}\}\right\},\\
&\left\{\{\mathbf{r}^{(1,1)}, \dots, \mathbf{r}^{(1,n_1)}\}, \dots, \{\mathbf{r}^{(k,1)}, \dots, \mathbf{r}^{(k,n_k)}\}\right\};
\end{aligned} \tag{6.125}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0,\\
&\mathbf{l}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(i,\beta)})\mathbf{r}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(j,\gamma)})\mathbf{r}^{(i,\beta)}\right) = 0,\\
&\forall\, i, j = 1, \dots, k, \quad i \neq j,\\
&\alpha, \beta = 1, \dots, n_i, \quad \alpha \neq \beta, \quad \gamma = 1, \dots, n_j.
\end{aligned} \tag{6.126}$$

Moreover, the decoupling variables

$$U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u}), \tag{6.127}$$

are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right) \cdot \mathbf{r}^{(j,\gamma)} = 0, \tag{6.128}$$

where

$$i, j = 1, \dots, k, \quad i \neq j, \quad \alpha = 1, \dots, n_i, \quad \gamma = 1, \dots, n_j.$$

The coefficient matrix for a fully decoupled system results in block diagonal form (diagonal if $k = n$).

Proof. It immediately follows from Theorem 6.5.1. □

Some comments about the decoupling conditions of previous theorem and some well known facts concerning wave solutions of hyperbolic quasilinear systems are needed. For such systems, it is relevant to quantify the dependence of the wave speeds (the eigenvalues of coefficient matrix) upon the field variables (see [96]). More precisely, we may compute the change of the characteristic speed $\lambda^{(i)}$ across a wave with speed $\lambda^{(j)}$, say $\left(\nabla_{\mathbf{u}}\lambda^{(i)}\right) \cdot \mathbf{r}^{(j)}$. For $j = i$ we have the decay coefficient $\left(\nabla_{\mathbf{u}}\lambda^{(i)}\right) \cdot \mathbf{r}^{(i)}$; this is of special importance since it determines the genuine nonlinearity or linear degeneracy of the wave [11, 52]. In the case of completely exceptional systems, *i.e.*, systems where all admitted waves are linearly degenerate, weak waves do not give rise to shock formation. Another important effect, which can have dramatic consequences on the behavior of solutions, is due to the interaction of two waves leading to the formation of waves in other families. In particular, for incident waves belonging to different families, the reflected wave is determined to leading order by the interaction coefficient $\mathbf{l}^{(i)} \cdot \left((\nabla_{\mathbf{u}}\mathbf{r}^{(j)})\mathbf{r}^{(\ell)} - (\nabla_{\mathbf{u}}\mathbf{r}^{(\ell)})\mathbf{r}^{(i)}\right)$ [97].

Therefore, the conditions (6.126), guaranteeing the decoupling of a hyperbolic first order quasilinear system in k non–interacting subsystems, have the following (obvious) meaning:

1. the change in the characteristic speeds of a subsystem across a wave of a different subsystem must be vanishing;

2. waves of different subsystems do not interact.

6.6 Decoupling of general homogeneous and autonomous systems

In Section 6.5, we considered the decoupling problem for hyperbolic homogeneous and autonomous quasilinear systems. Here, we investigate the case where the coefficient matrix does not possess a complete set of eigenvectors and/or has complex–valued eigenvalues. Also in this general case we give necessary and sufficient conditions for the partial or full decoupling.

Definition 6.6.1. *Let A be an $n \times n$ real matrix whose entries are smooth functions depending on $\mathbf{u} \in \mathbb{R}^n$. If the matrix A has not a complete set of eigenvectors and/or has complex–valued eigenvalues, let us associate:*

- *to each real eigenvalue its (left and right) eigenvectors and, if needed, its generalized (left and right) eigenvectors in such a way we have as many linearly independent vectors as the multiplicity of the eigenvalue;*

- *to each couple of conjugate complex eigenvalues the real part and the imaginary part of its (left and right) eigenvectors (or generalized eigenvectors, if needed) in such a way we have as many couples of linearly independent vectors as the multiplicity of the conjugate complex eigenvalues.*

Let us denote such vectors with a superposed hat, and let us call them for simplicity (left and right) autovectors.

Lemma 6.6.1. *Let T be an $n \times n$ lower triangular block real matrix of the form (6.84) whose entries are smooth functions depending on $\mathbf{U}$,*

$$\mathbf{U} \equiv \left(U^{(1,1)}, \ldots, U^{(1,n_1)}, \ldots, U^{(k,1)}, \ldots, U^{(k,n_k)}\right)^T \tag{6.129}$$

($n_1 + \ldots + n_k = n$), where T^i_j are $n_i \times n_j$ matrices, and 0^i_j $n_i \times n_j$ matrices of zeros.

The entries of matrices T^i_j ($i = 1, \ldots, k$; $j = 1, \ldots, i$) depend at most on the m_i variables of the set $\mathcal{U}_i$ if and only if:

1. *the set of the eigenvalues of T (counted with their multiplicity) with corresponding left and right autovectors can be divided into k subsets each containing n_i ($i = 1, \ldots, k$) elements, say*

$$\begin{gathered}
\left\{\{\Lambda^{(1,1)}, \ldots, \Lambda^{(1,n_1)}\}, \ldots, \{\Lambda^{(k,1)}, \ldots, \Lambda^{(k,n_k)}\}\right\}, \\
\left\{\{\widehat{\mathbf{L}}^{(1,1)}, \ldots, \widehat{\mathbf{L}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{L}}^{(k,1)}, \ldots, \widehat{\mathbf{L}}^{(k,n_k)}\}\right\}, \\
\left\{\{\widehat{\mathbf{R}}^{(1,1)}, \ldots, \widehat{\mathbf{R}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{R}}^{(k,1)}, \ldots, \widehat{\mathbf{R}}^{(k,n_k)}\}\right\};
\end{gathered} \tag{6.130}$$

2. *the following structure conditions hold true:*

$$
\begin{aligned}
&\left(\nabla_{\mathbf{U}}\Lambda^{(i,\alpha)}\right)\cdot\widehat{\mathbf{R}}^{(j,\gamma)}=0,\\
&\widehat{\mathbf{L}}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{U}}\widehat{\mathbf{R}}^{(\ell,\beta)})\widehat{\mathbf{R}}^{(j,\gamma)}\right)=0,\\
&i=1,\dots,k-1,\quad \ell=1,\dots,i,\\
&\alpha=1,\dots,n_i,\quad \beta=1,\dots,n_\ell,\quad \alpha\neq\beta \text{ if } i=\ell,\\
&j=i+1,\dots,k,\quad \gamma=1,\dots,n_j,
\end{aligned}
\tag{6.131}
$$

where

$$
\nabla_{\mathbf{U}}\equiv\left(\frac{\partial}{\partial U^{(1,1)}},\dots,\frac{\partial}{\partial U^{(1,n_1)}},\dots,\frac{\partial}{\partial U^{(k,1)}},\dots,\frac{\partial}{\partial U^{(k,n_k)}}\right). \tag{6.132}
$$

Proof. The proof is as that of Lemma 6.5.1 taking into account that:

1. the left autovectors $\widehat{\mathbf{L}}^{(r,\alpha)}$ ($\alpha=1,\dots,n_r$) may have non–vanishing only the first m_r components;
2. the right autovectors $\widehat{\mathbf{R}}^{(r,\alpha)}$ ($\alpha=1,\dots,n_r$) for $r>1$ may have non–vanishing only the last $n-m_{r-1}$ components.

Moreover, using (6.131) and the relations defining the generalized eigenvectors, it is also proved that the entries of matrices T^i_r ($r=1,\dots,i$) are independent of the elements in the set $\overline{\mathcal{U}}_i$. □

Because of Lemma 6.6.1, Theorem 6.5.1 can be generalized to all first order autonomous and homogeneous quasilinear systems.

Theorem 6.6.1. *The first order quasilinear system (not necessarily hyperbolic)*

$$
\frac{\partial\mathbf{u}}{\partial t}+A(\mathbf{u})\frac{\partial\mathbf{u}}{\partial x}=\mathbf{0}, \tag{6.133}
$$

$\mathbf{u}\in\mathbb{R}^n$, *$A$ $n\times n$ real matrix with entries smooth functions of $\mathbf{u}$, can be transformed by a smooth (locally) invertible transformation*

$$
\mathbf{u}=\mathbf{h}(\mathbf{U}),\quad \textit{or, equivalently,}\quad \mathbf{U}=\mathbf{H}(\mathbf{u}), \tag{6.134}
$$

into a system like

$$
\frac{\partial\mathbf{U}}{\partial t}+T(\mathbf{U})\frac{\partial\mathbf{U}}{\partial x}=\mathbf{0}, \tag{6.135}
$$

in the unknowns

$$
\mathbf{U}\equiv\left(U^{(1,1)},\dots,U^{(1,n_1)},\dots,U^{(k,1)},\dots,U^{(k,n_k)}\right)^T, \tag{6.136}
$$

where $T=(\nabla_{\mathbf{u}}\mathbf{H})\,A\,(\nabla_{\mathbf{u}}\mathbf{H})^{-1}$ is a lower triangular block matrix having the form (6.84), with T^i_j ($i=1,\dots,k$; $j=1,\dots,i$) $n_i\times n_j$ matrices such that their entries are smooth functions depending at most on the elements of the set $\mathcal{U}_i$, whereas 0^i_j are $n_i\times n_j$ matrices of zeros, respectively, if and only if:

1. *the set of the eigenvalues (counted with their multiplicity) of matrix A, and the associated left and right autovectors can be divided into k subsets each*

containing n_i *(*$i = 1, \ldots, k$*) elements, say*

$$\begin{aligned}
&\left\{\{\lambda^{(1,1)}, \ldots, \lambda^{(1,n_1)}\}, \ldots, \{\lambda^{(k,1)}, \ldots, \lambda^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{l}}^{(1,1)}, \ldots, \widehat{\mathbf{l}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{l}}^{(k,1)}, \ldots, \widehat{\mathbf{l}}^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{r}}^{(1,1)}, \ldots, \widehat{\mathbf{r}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{r}}^{(k,1)}, \ldots, \widehat{\mathbf{r}}^{(k,n_k)}\}\right\};
\end{aligned} \tag{6.137}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0,\\
&\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(\ell,\beta)})\widehat{\mathbf{r}}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(j,\gamma)})\widehat{\mathbf{r}}^{(\ell,\beta)}\right) = 0,\\
&i = 1, \ldots, k-1, \quad \ell = 1, \ldots, i,\\
&\alpha = 1, \ldots, n_i, \quad \beta = 1, \ldots, n_\ell, \quad \alpha \neq \beta \text{ if } i = \ell,\\
&j = i+1, \ldots, k, \quad \gamma = 1, \ldots, n_j.
\end{aligned} \tag{6.138}$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$ *(*$i = 1, \ldots, k-1$*;* $\alpha = 1, \ldots, n_i$*) are found from*

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0, \tag{6.139}$$

where $j = i+1, \ldots, k$*,* $\gamma = 1, \ldots, n_j$*.*

Proof. The proof, due to Lemma 6.6.1, is as that of Theorem 6.5.1. □

Finally, we are able to state the following theorem providing the conditions for the full decoupling of general first order quasilinear systems.

Theorem 6.6.2 (Full decoupling of general systems)**.** *For the first order homogeneous and autonomous quasilinear system (not necessarily hyperbolic)*

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.140}$$

$\mathbf{u} \in \mathbb{R}^n$*,* A $n \times n$ *real matrix with entries smooth functions of* $\mathbf{u}$*, to be locally reducible into* k *non–interacting subsystems of some orders* $n_1, \ldots, n_k$*, with* $n_1 + \ldots + n_k = n$*, in the unknowns*

$$\mathbf{U} \equiv \left(U^{(1,1)}, \ldots, U^{(1,n_1)}, \ldots, U^{(k,1)}, \ldots, U^{(k,n_k)}\right)^T, \tag{6.141}$$

respectively, it is necessary and sufficient that:

1. *the eigenvalues of matrix* A *(counted with their multiplicity), and the corresponding left and right autovectors can be divided into* k *subsets each containing* n_i *(*$i = 1, \ldots, k$*) elements, say*

$$\begin{aligned}
&\left\{\{\lambda^{(1,1)}, \ldots, \lambda^{(1,n_1)}\}, \ldots, \{\lambda^{(k,1)}, \ldots, \lambda^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{l}}^{(1,1)}, \ldots, \widehat{\mathbf{l}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{l}}^{(k,1)}, \ldots, \widehat{\mathbf{l}}^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{r}}^{(1,1)}, \ldots, \widehat{\mathbf{r}}^{(1,n_1)}\}, \ldots, \{\widehat{\mathbf{r}}^{(k,1)}, \ldots, \widehat{\mathbf{r}}^{(k,n_k)}\}\right\};
\end{aligned} \tag{6.142}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)} = 0,\\
&\widehat{\mathbf{l}}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(i,\beta)})\widehat{\mathbf{r}}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(j,\gamma)})\widehat{\mathbf{r}}^{(i,\beta)}\right) = 0,\\
&\forall\, i,j = 1,\dots,k, \quad i\neq j,\\
&\alpha,\beta = 1,\dots,n_i, \quad \alpha\neq\beta, \quad \gamma = 1,\dots,n_j.
\end{aligned} \tag{6.143}$$

Moreover, the decoupling variables

$$U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u}), \tag{6.144}$$

are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)} = 0, \tag{6.145}$$

where

$$i,j = 1,\dots,k, \quad i\neq j, \quad \alpha = 1,\dots,n_i, \quad \gamma = 1,\dots,n_j.$$

The coefficient matrix for a fully decoupled system results in block diagonal form (diagonal if $k = n$).

Proof. It immediately follows from Theorem 6.6.1. □

6.7 Decoupling of nonhomogeneous and/or nonautonomous systems

In some physical applications it may occur to consider systems involving source terms, and/or systems where the coefficients may depend also on the independent variables, accounting for material inhomogeneities, or special geometric assumptions, or external actions [2, 18, 19, 48, 65, 78, 81]. Therefore, one has to deal with nonhomogeneous and/or nonautonomous first order quasilinear systems of the form (6.10).

In some cases, systems like (6.10) may be transformed by a (locally) invertible transformation to autonomous and homogeneous form or only to autonomous form preserving the quasilinear structure. This is possible if and only if the system (6.10) admits suitable algebras of Lie point symmetries. In [65] it has been proved a theorem stating necessary and sufficient conditions in order to map systems like (6.10) to autonomous and homogeneous form. By relaxing the hypotheses, the same theorem can be used to map systems (6.10) into autonomous and nonhomogeneous first order quasilinear systems [22, 66].

Therefore, three different situations may occur:

1. System (6.10) can be mapped by an invertible point transformation to an equivalent autonomous and homogeneous first order quasilinear system in the independent variables $\widehat{t}(t,x)$, $\widehat{x}(t,x)$ and the dependent variables $\mathbf{U} = \mathbf{U}(t,x,\mathbf{u})$. It is required that it admits as subalgebra of its algebra of Lie point symmetries a three–dimensional Lie algebra spanned by the vector fields

$$\Xi_i = \tau_i(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x} + \sum_{A=1}^{n}\eta_i^A(t,x,\mathbf{u})\frac{\partial}{\partial u_A} \tag{6.146}$$

($i = 1, \ldots, 3$), such that

$$[\Xi_i, \Xi_j] = 0, \qquad [\Xi_i, \Xi_3] = \Xi_i, \qquad i, j = 1, 2; \tag{6.147}$$

moreover, it has to be verified that all minors of order two extracted from the 3×2 matrix with rows (τ_i, ξ_i) ($i = 1, \ldots, 3$) are non–vanishing, and the variables $\mathbf{U}$, which by construction are invariants of Ξ_1 and Ξ_2, result invariant with respect to Ξ_3 too.

2. System (6.10) can be transformed to an equivalent autonomous and nonhomogeneous first order quasilinear system in the independent variables $\widehat{t}(t, x)$, $\widehat{x}(t, x)$ and the dependent variables $\mathbf{U} = \mathbf{U}(t, x, \mathbf{u})$. It is required that it admits as subalgebra of its algebra of Lie point symmetries a two–dimensional Lie algebra spanned by the vector fields

$$\Xi_i = \tau_i(t, x)\frac{\partial}{\partial t} + \xi_i(t, x)\frac{\partial}{\partial x} + \sum_{A=1}^{n} \eta_i^A(t, x, \mathbf{u})\frac{\partial}{\partial u_A} \tag{6.148}$$

($i = 1, 2$), such that the 2×2 matrix with rows (τ_i, ξ_i) ($i = 1, 2$) is non–singular and

$$[\Xi_i, \Xi_j] = 0, \qquad i, j = 1, 2. \tag{6.149}$$

3. System (6.10) can not be transformed to autonomous form.

In the first case, the decoupling problem can be faced by using the results of previous Sections, whereas cases 2 and 3 can be managed together.

It is worth of being observed that the decoupling of the system (6.10) is not affected by a variable change of the independent variables provided that the new independent variables depend only on the old independent variables. Therefore, we can manage in a unified way nonhomogeneous quasilinear systems either when they are autonomous or not. So, we introduce only new dependent variables $\mathbf{U}$, as suitable functions of the old dependent variables, and state the following two theorems for the partial and the full decoupling.

Theorem 6.7.1 (Partial decoupling for quasilinear systems)**.** *The first order quasilinear system*

$$\frac{\partial \mathbf{u}}{\partial t} + A(t, x, \mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{g}(t, x, \mathbf{u}), \tag{6.150}$$

$\mathbf{u}, \mathbf{g} \in \mathbb{R}^n$, *$A$ $n \times n$ real matrix (the components of $\mathbf{g}$ as well as the entries of matrix A are smooth functions depending on t, x and $\mathbf{u}$), can be transformed by a smooth (locally) invertible transformation*

$$\mathbf{u} = \mathbf{h}(\mathbf{U}), \quad \textit{or, equivalently,} \quad \mathbf{U} = \mathbf{H}(\mathbf{u}), \tag{6.151}$$

into a system like

$$\frac{\partial \mathbf{U}}{\partial t} + T(t, x, \mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = \mathbf{G}(t, x, \mathbf{U}), \tag{6.152}$$

in the unknowns

$$\mathbf{U} \equiv \left(U^{(1,1)}, \ldots, U^{(1,n_1)}, \ldots, U^{(k,1)}, \ldots, U^{(k,n_k)}\right)^T, \tag{6.153}$$

where

$$\mathbf{G} \equiv \left(G^{(1,1)}, \dots, G^{(1,n_1)}, \dots, G^{(k,1)}, \dots, G^{(k,n_k)}\right)^T, \tag{6.154}$$

$T = (\nabla_{\mathbf{u}}\mathbf{H})\, A\, (\nabla_{\mathbf{u}}\mathbf{H})^{-1}$ *being a lower triangular block matrix having the form (6.84),* $\mathbf{G} = (\nabla_{\mathbf{u}}\mathbf{H})\mathbf{g}$*, such that* T^i_j *and* $G^{(i,\alpha)}$ $(i = 1, \dots, k;\ j = 1, \dots, i;\ \alpha = 1, \dots, n_i)$ *depend at most on* t*,* x *and the elements of the set* $\mathcal{U}_i$*, whereas* 0^i_j *are* $n_i \times n_j$ *matrices of zeros, respectively, if and only if:*

1. *the set of the eigenvalues of matrix* A *(counted with their multiplicity), and the associated left and right autovectors can be divided into* k *subsets each containing* n_i $(i = 1, \dots, k)$ *elements, say*

$$\begin{aligned}
&\left\{\{\lambda^{(1,1)}, \dots, \lambda^{(1,n_1)}\}, \dots, \{\lambda^{(k,1)}, \dots, \lambda^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{l}}^{(1,1)}, \dots, \widehat{\mathbf{l}}^{(1,n_1)}\}, \dots, \{\widehat{\mathbf{l}}^{(k,1)}, \dots, \widehat{\mathbf{l}}^{(k,n_k)}\}\right\},\\
&\left\{\{\widehat{\mathbf{r}}^{(1,1)}, \dots, \widehat{\mathbf{r}}^{(1,n_1)}\}, \dots, \{\widehat{\mathbf{r}}^{(k,1)}, \dots, \widehat{\mathbf{r}}^{(k,n_k)}\}\right\};
\end{aligned} \tag{6.155}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}
&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0,\\
&\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \left((\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(\ell,\beta)})\widehat{\mathbf{r}}^{(j,\gamma)} - (\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(j,\gamma)})\widehat{\mathbf{r}}^{(\ell,\beta)}\right) = 0,\\
&\left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0,\\
&i = 1, \dots, k-1, \quad \ell = 1, \dots, i,\\
&\alpha = 1, \dots, n_i, \quad \beta = 1, \dots, n_\ell, \quad \alpha \neq \beta \textit{ if } i = \ell,\\
&j = i+1, \dots, k, \quad \gamma = 1, \dots, n_j.
\end{aligned} \tag{6.156}$$

Moreover, the decoupling variables $U^{(i,\alpha)} = H^{(i,\alpha)}(\mathbf{u})$ $(i = 1, \dots, k-1;\ \alpha = 1, \dots, n_i)$ *are found from*

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0, \tag{6.157}$$

where $j = i+1, \dots, k$*,* $\gamma = 1, \dots, n_j$*.*

Proof. The proof, due to Lemma 6.6.1, follows the same steps as those of Theorem 6.5.1, the only difference being in the additional requirement expressed by $(6.156)_3$.

It is

$$\begin{aligned}
0 &= \left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} =\\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) (\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\widehat{\mathbf{R}}^{(j,\gamma)} =\\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)}(\nabla_{\mathbf{u}}\mathbf{H})(\nabla_{\mathbf{u}}\mathbf{H})^{-1}\mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)} =\\
&= \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)} \cdot \mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)}.
\end{aligned} \tag{6.158}$$

Therefore,

$$\left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)} \cdot \mathbf{g})\right) \cdot \widehat{\mathbf{r}}^{(j,\gamma)} = 0 \quad \Leftrightarrow \quad \left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)} \cdot \mathbf{G})\right) \cdot \widehat{\mathbf{R}}^{(j,\gamma)} = 0. \tag{6.159}$$

Since $\widehat{\mathbf{L}}^{(i,\alpha)}$ may have non–vanishing only the first m_i components, whereas $\widehat{\mathbf{R}}^{(j,\gamma)}$ may have non–vanishing only the last $n-m_{j-1}$ components, conditions

$$\left(\nabla_{\mathbf{U}}(\widehat{\mathbf{L}}^{(i,\alpha)}\cdot\mathbf{G})\right)\cdot\widehat{\mathbf{R}}^{(j,\gamma)}=0 \tag{6.160}$$

are necessary and sufficient in order the components $\{\mathbf{G}^{(r,1)},\dots,\mathbf{G}^{(r,n_r)}\}$ to be dependent at most on t, x and the elements of the set $\mathcal{U}_r$, and this concludes the proof. □

Finally, we are able to state the following theorem providing a solution to the full decoupling problem for general nonhomogeneous and/or nonautonomous first order quasilinear systems.

Theorem 6.7.2 (Full decoupling for nonhomogeneous quasilinear systems)**.** *For the first order nonhomogeneous and/or nonautonomous quasilinear system, say*

$$\frac{\partial\mathbf{u}}{\partial t}+A(t,x,\mathbf{u})\frac{\partial\mathbf{u}}{\partial x}=\mathbf{g}(t,x,\mathbf{u}), \tag{6.161}$$

$\mathbf{u},\mathbf{g}\in\mathbb{R}^n$, *$A$ $n\times n$ real matrix (the components of* $\mathbf{g}$ *as well as the entries of matrix A are smooth functions depending on t, x and* $\mathbf{u}$*), to be locally reducible into k non–interacting subsystems of some orders $n_1,\dots,n_k$, with $n_1+\ldots+n_k=n$, in the unknowns*

$$\mathbf{U}\equiv\left(U^{(1,1)},\dots,U^{(1,n_1)},\dots,U^{(k,1)},\dots,U^{(k,n_k)}\right)^T, \tag{6.162}$$

respectively, it is necessary and sufficient that:

1. *the eigenvalues of matrix A (counted with their multiplicity), and the corresponding left and right autovectors can be divided into k subsets each containing n_i ($i=1,\dots,k$) elements, say*

$$\begin{aligned}&\left\{\{\lambda^{(1,1)},\dots,\lambda^{(1,n_1)}\},\dots,\{\lambda^{(k,1)},\dots,\lambda^{(k,n_k)}\}\right\},\\&\left\{\{\widehat{\mathbf{l}}^{(1,1)},\dots,\widehat{\mathbf{l}}^{(1,n_1)}\},\dots,\{\widehat{\mathbf{l}}^{(k,1)},\dots,\widehat{\mathbf{l}}^{(k,n_k)}\}\right\},\\&\left\{\{\widehat{\mathbf{r}}^{(1,1)},\dots,\widehat{\mathbf{r}}^{(1,n_1)}\},\dots,\{\widehat{\mathbf{r}}^{(k,1)},\dots,\widehat{\mathbf{r}}^{(k,n_k)}\}\right\};\end{aligned} \tag{6.163}$$

2. *the following structure conditions hold true:*

$$\begin{aligned}&\left(\nabla_{\mathbf{u}}\lambda^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)}=0,\\&\widehat{\mathbf{l}}^{(i,\alpha)}\cdot\left((\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(i,\beta)})\widehat{\mathbf{r}}^{(j,\gamma)}-(\nabla_{\mathbf{u}}\widehat{\mathbf{r}}^{(j,\gamma)})\widehat{\mathbf{r}}^{(i,\beta)}\right)=0,\\&\left(\nabla_{\mathbf{u}}(\widehat{\mathbf{l}}^{(i,\alpha)}\cdot\mathbf{g})\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)}=0,\\&\forall\, i,j=1,\dots,k,\quad i\neq j,\\&\alpha,\beta=1,\dots,n_i,\quad \alpha\neq\beta,\quad \gamma=1,\dots,n_j.\end{aligned} \tag{6.164}$$

Moreover, the decoupling variables

$$U^{(i,\alpha)}=H^{(i,\alpha)}(\mathbf{u}), \tag{6.165}$$

are found from

$$\left(\nabla_{\mathbf{u}}H^{(i,\alpha)}\right)\cdot\widehat{\mathbf{r}}^{(j,\gamma)}=0, \tag{6.166}$$

where

$$i, j = 1, \dots, k, \quad i \neq j, \quad \alpha = 1, \dots, n_i, \quad \gamma = 1, \dots, n_j.$$

Proof. It immediately follows from Theorem 6.7.1. □

6.8 Applications

In this Section, we consider some applications of the results above derived. As far as the notation is concerned, the components of the field $\mathbf{u}$ are denoted with the symbols typically used in the applications.

The first two examples are related to the Euler equations of an ideal gas with the special value $\Gamma = 3$ [17, p. 88] for the adiabatic index, whereas the third example concerns the equations of a model of travelling threadline with a particular constitutive law for the tension.

Example 6.8.1. *One–dimensional Euler equations of barotropic fluids.*

Let us consider the one–dimensional Euler equations of a barotropic fluid

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.167}$$

with

$$\mathbf{u} = \begin{bmatrix} \rho \\ v \end{bmatrix}, \qquad A = \begin{bmatrix} v & \rho \\ \frac{p'(\rho)}{\rho} & v \end{bmatrix}, \tag{6.168}$$

where $\rho(t,x)$ is the mass density, $v(t,x)$ the velocity, and $p(\rho)$ the pressure. This system is strictly hyperbolic provided that $p'(\rho) > 0$ (the prime denoting the differentiation with respect to the argument), with characteristic velocities

$$\lambda_{1,2} = v \pm \sqrt{p'(\rho)}, \tag{6.169}$$

to which correspond the left and right eigenvectors

$$\mathbf{l}_{1,2} = \left(\sqrt{p'(\rho)}, \pm\rho\right), \qquad \mathbf{r}_{1,2} = \begin{pmatrix} \rho \\ \pm\sqrt{p'(\rho)} \end{pmatrix}. \tag{6.170}$$

The conditions for the possible decoupling provide the constraint

$$\rho p''(\rho) - 2p'(\rho) = 0, \tag{6.171}$$

which is satisfied by the special constitutive law

$$p(\rho) = p_0\rho^3, \qquad p_0 \quad \text{constant}. \tag{6.172}$$

Thus, if the adiabatic index is equal to 3 (in this case the characteristics are straight lines, [17, p. 88]), we may introduce the variable transformation

$$\mathbf{U} = \mathbf{H}(\mathbf{u}) \tag{6.173}$$

such that

$$(\nabla_{\mathbf{u}} H_1) \cdot \mathbf{r}_2 = 0, \qquad (\nabla_{\mathbf{u}} H_2) \cdot \mathbf{r}_1 = 0. \tag{6.174}$$

As a consequence, by choosing

$$U_1 = H_1(\rho, v) = v + \sqrt{3p_0}\rho, \qquad U_2 = H_2(\rho, v) = v - \sqrt{3p_0}\rho, \tag{6.175}$$

we obtain the following fully decoupled system

$$\begin{aligned}\frac{\partial U_1}{\partial t}+U_1\frac{\partial U_1}{\partial x}&=0,\\ \frac{\partial U_2}{\partial t}+U_2\frac{\partial U_2}{\partial x}&=0.\end{aligned}\tag{6.176}$$

Example 6.8.2. *One–dimensional isentropic gas dynamics equations.*

Let us consider the one–dimensional Euler equations for the isentropic flow of an ideal fluid subject to no external forces,

$$\frac{\partial \mathbf{u}}{\partial t}+A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x}=\mathbf{0},\tag{6.177}$$

with

$$\mathbf{u}=\begin{bmatrix}\rho\\ v\\ s\end{bmatrix},\qquad A=\begin{bmatrix}v & \rho & 0\\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & v & \frac{1}{\rho}\frac{\partial p}{\partial s}\\ 0 & 0 & v\end{bmatrix},\tag{6.178}$$

where $\rho(t,x)$ is the mass density, $v(t,x)$ the velocity, $s(t,x)$ the entropy, and $p(\rho,s)$ the pressure.

The eigenvalues of matrix A are

$$\lambda_{1,2}=v\pm\sqrt{\frac{\partial p}{\partial \rho}},\qquad \lambda_3=v,\tag{6.179}$$

with associated left and right eigenvectors

$$\begin{aligned}\mathbf{l}_{1,2}&=\left(\sqrt{\frac{\partial p}{\partial \rho}},\pm\rho,\frac{\rho}{s}\sqrt{\frac{\partial p}{\partial \rho}}\right), &\qquad \mathbf{l}_3&=(0,0,1),\\ \mathbf{r}_{1,2}&=\begin{pmatrix}\rho\\ \pm\sqrt{\frac{\partial p}{\partial \rho}}\\ 0\end{pmatrix}, &\qquad \mathbf{r}_3&=\begin{pmatrix}\rho\\ 0\\ -s\end{pmatrix}.\end{aligned}\tag{6.180}$$

The constraints

$$\begin{aligned}(\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_2=0,&\qquad (\nabla_{\mathbf{u}}\lambda_1)\cdot\mathbf{r}_3=0,\\ (\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_1=0,&\qquad (\nabla_{\mathbf{u}}\lambda_2)\cdot\mathbf{r}_3=0,\end{aligned}\tag{6.181}$$

are satisfied with the constitutive law

$$p(\rho,s)=p_0\rho^3 s^2+f(s),\tag{6.182}$$

where p_0 is constant and $f(s)$ a function of its argument; therefore, we may introduce the variable transformation

$$\mathbf{U}=\mathbf{H}(\mathbf{u})\tag{6.183}$$

such that

$$\begin{aligned}(\nabla_{\mathbf{u}}H_1)\cdot\mathbf{r}_2=0,&\qquad (\nabla_{\mathbf{u}}H_1)\cdot\mathbf{r}_3=0,\\ (\nabla_{\mathbf{u}}H_2)\cdot\mathbf{r}_1=0,&\qquad (\nabla_{\mathbf{u}}H_2)\cdot\mathbf{r}_3=0.\end{aligned}\tag{6.184}$$

As a consequence, by choosing

$$\begin{aligned} U_1 &= H_1(\rho, v, s) = v + \sqrt{3p_0}\rho s, \\ U_2 &= H_2(\rho, v, s) = v - \sqrt{3p_0}\rho s, \\ U_3 &= H_3(\rho, v, s) = s, \end{aligned} \tag{6.185}$$

we obtain the following partially decoupled system

$$\begin{aligned} &\frac{\partial U_1}{\partial t} + U_1 \frac{\partial U_1}{\partial x} = 0, \\ &\frac{\partial U_2}{\partial t} + U_2 \frac{\partial U_2}{\partial x} = 0, \\ &\frac{\partial U_3}{\partial t} + \frac{1}{2}(U_1 + U_2) \frac{\partial U_3}{\partial x} = 0, \end{aligned} \tag{6.186}$$

where the first two equations can be solved independently from each other and the third one.

Example 6.8.3. *Model of travelling threadline.*

Let us consider the nonlinear model describing the motion of a moving threadline [1, 24] taking into account both geometric and material nonlinearities.

Based upon the following hypotheses:

- *the motion is two–dimensional;*
- *the string is elastic and always in tension;*
- *the string is perfectly flexible;*
- *the effects of gravity and air drag are neglected;*

in [1] the following equations have been derived:

$$\begin{aligned} m(1 + u_x^2)^{1/2} \frac{dV^x}{dt} &= \frac{\partial}{\partial x}(T \sin\theta), \\ m(1 + u_x^2)^{1/2} \frac{dV^y}{dt} &= \frac{\partial}{\partial x}(T \cos\theta), \end{aligned} \tag{6.187}$$

supplemented by the continuity equation

$$\frac{d}{dt}\left(m(1 + u_x^2)^{1/2}\right) + m(1 + u_x^2)^{1/2} \frac{\partial V^x}{\partial x} = 0, \tag{6.188}$$

and a constitutive law in the form

$$T = T(m, m_t). \tag{6.189}$$

In the previous equations, m is the mass per unit length, V^x the axial component of the velocity, V^y the transverse component of the velocity, T the tension, and u the transverse displacement; moreover, the subscripts t and x denote partial derivatives with respect to the indicated variables.

Upon introduction of the quantity

$$\rho = m(1 + u_x^2)^{1/2}, \tag{6.190}$$

and taking into account (see [1] for details) that

$$\sin\theta = \frac{u_x}{(1+u_x^2)^{1/2}}, \qquad \cos\theta = \frac{1}{(1+u_x^2)^{1/2}}, \\ V^y = u_t + V^x u_x = v + V^x \varepsilon, \tag{6.191}$$

assuming the constitutive equation for the tension in the form $T = T(m)$ [24], the governing equations can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}, \tag{6.192}$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ V^x \\ v \\ \varepsilon \end{pmatrix}, \qquad A = \begin{bmatrix} V^x & \rho & 0 & 0 \\ \frac{-T'}{\rho(1+\varepsilon^2)} & V^x & 0 & \frac{\varepsilon}{1+\varepsilon^2)^2}\left(T' + \frac{T}{m}\right) \\ 0 & 0 & 2V^x & (V^x)^2 - \frac{T}{m(1+\varepsilon^2)} \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{6.193}$$

The eigenvalues of matrix A are

$$\lambda_{1,2} = V^x \pm \left(\frac{-T'}{1+\varepsilon^2)}\right)^{1/2}, \qquad \lambda_{3,4} = V^x \pm \left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2}, \tag{6.194}$$

with associated left and right eigenvectors

$$\begin{aligned} \mathbf{l}_{1,2} &= \left(\pm\frac{\sqrt{-(1+\varepsilon^2)T'}}{\rho\varepsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}\right)}, \frac{1+\varepsilon^2}{\varepsilon\left(V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}\right)}, \frac{1}{V^x \mp \sqrt{\frac{-T'}{1+\varepsilon^2}}}, 1\right), \\ \mathbf{l}_{3,4} &= \left(0, 0, \rho, \rho V^x \pm \sqrt{\frac{\rho T}{(1+\varepsilon^2)^{1/2}}}\right), \\ \mathbf{r}_{1,2} &= \begin{pmatrix} \rho \\ \pm\left(\frac{-T'}{1+\varepsilon^2}\right)^{1/2} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_{3,4} = \begin{pmatrix} \frac{\rho\varepsilon}{1+\varepsilon^2} \\ \pm\left(\frac{T}{m(1+\varepsilon^2)}\right)^{1/2} \frac{\varepsilon}{1+\varepsilon^2} \\ -\left(V^x \pm \frac{T}{m(1+\varepsilon^2)}\right)^{1/2} \\ 1 \end{pmatrix}. \end{aligned} \tag{6.195}$$

The structure conditions for the partial decoupling

$$\begin{aligned} &(\nabla_{\mathbf{u}}\lambda_i)\cdot\mathbf{r}_j = 0, \\ &\mathbf{l}_i\cdot((\nabla_{\mathbf{u}}\mathbf{r}_\ell)\mathbf{r}_j - (\nabla_{\mathbf{u}}\mathbf{r}_j)\mathbf{r}_\ell) = 0, \qquad i,\ell = 1,2, \quad i \neq \ell, \quad j = 3,4, \end{aligned} \tag{6.196}$$

are satisfied with the following constitutive law

$$T(m) = \frac{k}{m}, \qquad k \quad \textit{constant}. \tag{6.197}$$

Then, we may introduce the variable transformation

$$\mathbf{U} = \mathbf{H}(\mathbf{u}) \tag{6.198}$$

such that

$$(\nabla_{\mathbf{u}} H_i) \cdot \mathbf{r}_j = 0, \qquad i, = 1, 2, \quad j = 3, 4. \tag{6.199}$$

By integrating relations (6.199), i.e,

$$\begin{aligned} &\left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \varepsilon} = 0, \quad \left(V^x + \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \varepsilon} = 0, \\ &\left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_1}{\partial v} - \frac{\partial H_1}{\partial \varepsilon} = 0, \quad \left(V^x - \frac{\sqrt{k}}{\rho}\right)\frac{\partial H_2}{\partial v} - \frac{\partial H_2}{\partial \varepsilon} = 0, \end{aligned} \tag{6.200}$$

it follows that

$$H_1 = H_1(\rho, V^x), \qquad H_2 = H_2(\rho, V^x). \tag{6.201}$$

As a consequence, by choosing the identity transformation, we obtain this partially decoupled system

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + V^x \frac{\partial \rho}{\partial x} + \rho \frac{\partial V^x}{\partial x} = 0, \\ &\frac{\partial V^x}{\partial t} + \frac{k}{\rho^3}\frac{\partial \rho}{\partial x} + V^x \frac{\partial V^x}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + 2V^x \frac{\partial v}{\partial x} + \left((V^x)^2 - \frac{k}{\rho^2}\right)\frac{\partial \varepsilon}{\partial x} = 0, \\ &\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0. \end{aligned} \tag{6.202}$$

It is worth of being observed that with the constitutive relation (6.197) the system (6.192) has two distinct eigenvalues each with multiplicity 2, and is completely exceptional [11, 52].

6.9 Conclusions

In the beginning of this Chapter, we faced the decoupling problem of hyperbolic quasilinear first order systems in two independent variables and two or three dependent variables. The considered systems can be in principle nonautonomous and/or nonhomogeneous and we provided the conditions leading to the partial or full decoupling of the systems. As physical applications, the partial decoupling of Galilean first order systems in two and three dependent variables has been considered. In the family of 2×2 Galilean first order systems the one–dimensional Euler equations of barotropic fluids, which, under suitable conditions, can be mapped to the partially decoupled form, have been characterized. Then, a generalization of the results found by a direct approach in the case of the decoupling problem for quasilinear first order systems involving two or three dependent variables is presented. After introducing the definitions of partially and fully decoupled systems, a theorem establishing the necessary and sufficient conditions for the decoupling of hyperbolic first order homogeneous and autonomous quasilinear systems has been given; remarkably, the proof involves only the properties of eigenvalues and eigenvectors of the coefficient matrix, and is constructive in the sense that it gives the differential constraints whose integration leads to the decoupling transformation.

The results were extended, at first, to general first order homogeneous and autonomous quasilinear systems and, then, to general nonhomogeneous and/or nonautonomous ones. Some examples of systems of physical interest that can be, under suitable conditions, partially or fully decoupled are presented. In particular, the theorem is applied to the one–dimensional isentropic gas dynamics equations that have been decoupled into two non–interacting blocks, and to a nonlinear model describing the motion of a moving threadline which, with a special costitutive law rendering it linearly degenerate, has been partially decoupled.


I HAVE WHAT MAY BE A RATHER DIFFICULT QUESTION FOR YOU...
WHAT'S THE DIFFERENCE BETWEEN BEING DEPRESSED AND JUST FEELING BAD?
WHO CARES?

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