Explicit expressions of the eigenfrequencies of damaged frames

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Abstract. The presence of damage can strongly affect the residual carrying capacity and the dynamic properties of frame structures. The uncertainty in the position and intensity of damage implies the difficulty in adopting for practical purposes deterministic analyses, which rigorously require complicated calculations. In this paper, multi-cracked frames are studied considering the crack positions as deterministic whereas the intensities are uncertain. Explicit, although approximated, formulas of the main modal parameters as a function of the damage intensities are proposed. The latter expressions, which extend a previous study on beam-like structures, are built on the basis of detailed analyses, here computed combining the Dynamic Stiffness Matrix (DSM) approach with an efficient solution employing the distribution theory to treat the presence of cracks, and applying the Wittrick and Williams algorithm. An extremely low number of configurations of the frame is adopted to build the approximated solution is adopted. The proposed explicit formulas, which are duly verified for several meaningful cases, are then applied for the dynamic analysis of multi-cracked frames.

Introduction

The adoption of explicit expressions for the solution of structural problem can be very useful in many fields of engineering problems, e.g. when different design solutions have to be assessed according to the change of one or more parameters (parametric design), or when sensitivity analyses are needed. In addition, explicit expressions can be easily exploited for solving inverse problems. When closed form solutions have been obtained, the exact parametric expressions can be directly applied without running many numerical simulations. However, there is a wide variety of structural problems for which only numerical solutions are available, then the employment of re-analysis can become cumbersome and practically impossible to pursue. In such cases the availability of explicit solutions can be a powerful tool.

Within this framework, and with regard to free vibrations of damaged beams, in this paper a methodology to provide explicit although approximated expressions of the eigenfrequencies of damaged frames is proposed.

Damaged frame structures are here treated as continuous Euler-Bernoulli models and therefore handled with the Dynamic Stiffness Matrix (DSM) approach [1]. Damage is often assumed as an uncertain parameter [2], since its presence in structures is often hidden and, even when its presence is known, its intensity is hard to quantify. The presence of damage is considered with concentrated patterns, which is approximately realistic, especially for an early damage detection. The modelling of concentrated damage is usually approached with the so called equivalent rotational hinge model [3] where cracks correspond to rotational springs connecting two chunks of 1D elements. The stiffness of the spring is calibrated according to the crack depth following various strategies [4]. To optimize the required effort, in this study cracks are treated by means of a distributional approach rather than enforcing continuity conditions. This approach was successfully adopted for a wide variety of problems [5,6] and is able to provide closed form solutions as a function of the boundary conditions only, irrespectively of the number of along-axis discontinuities. For the specific case at hand, both the dynamics of multi-cracked beam [7] and frame [8], were already solved.

Indeed, when a structure is treated as a continuous system a frequency equation is inferred and then numerically solved to find its roots. One of the strategies that proved to be particularly effective in the frequency equation solution is the Wittrick and Williams (WW) algorithm [9], usually combined with the DSM.

This work aims at an explicit evaluation of the eigenfrequencies of damaged frames by considering deterministic crack locations, whereas a model of uncertainty for their severities is accounted for. The eigenfrequencies are approximated with a simple but effective explicit expression, calibrated on a limited obtained of reference solutions obtained combining the DSM with the WW algorithm for specific damage configurations. The proposed procedure represents a nontrivial extension of a previous study conducted on damaged beams [10]. This explicit solution can be applied to systems with multiple cracks and can be both employed to assess the response variability in direct problems and for damage detection purposes. In spite of the reduction of the computational effort for the parametric analysis with respect to a classic approach, it is shown how the proposed procedure is able to retrieve with good accuracy the results obtained with the DSM approach. A numerical application on a cracked portal frame exploring the potentiality of the method is presented, and the accuracy of the proposed methodology is evaluated.

The DSM approach for the multi-cracked Euler-Bernoulli beam

Reference solutions of the eigenfrequencies of multi-cracked frames are here obtained by means of the approach proposed in [7]. Each beam of a frame, characterised by length *L*, mass per unit length *m*, Young's modulus E_o, moment of inertia I_o, *n* cracks (where the *i*-th crack is located at the dimensionless abscissa ξ_i and is characterized by intensity λ_i, to be related to the crack depth according to one of the models available in the literature [5]). The Euler-Bernoulli multi-cracked beam model can be elaborated to infer the corresponding planar 4x4 DSM as a function of the frequency parameter $\alpha^4 = \omega^2 (mL^4 / E_o I_o)$, irrespectively of the number of cracks along each beam as follows [8]:

$$\mathbf{K} = \frac{E_o I_o}{L^2} \begin{bmatrix} \delta & \vartheta & -\bar{\delta} & \bar{\vartheta} \\ \vartheta & \rho & -\vartheta^* & \varepsilon \\ -\bar{\delta} & -\vartheta^* & \delta^* & \hat{\vartheta} \\ \bar{\vartheta} & \varepsilon & \hat{\vartheta} & \hat{\rho} \end{bmatrix} \tag{1}$$

where

$$\delta = 2\alpha^3 \frac{\alpha\left(cS + sC\right) + \alpha\left(Sg_{0,2} + sg_{0,4}\right) + g_{0,2}g_{I,4} - g_{0,4}g_{I,2} + cg_{I,4} - Cg_{I,2}}{D};$$

$$\vartheta = \alpha^2 \frac{\left\{ \begin{array}{l} \alpha\left[2sS + (c+C)\left(g_{0,2} - g_{0,4}\right) + (s+S)\left(g_{0,1} + g_{0,3}\right)\right] + \\ + \left(g_{I,1} + g_{I,3}\right)\left(c - C + g_{0,2} - g_{0,4}\right) + \left(g_{I,4} - g_{I,2}\right)\left(s + S + g_{0,1} + g_{0,3}\right) \end{array} \right\}}{D};$$

$$\bar{\delta} = 2\alpha^3 \frac{\alpha(s+S) + g_{I,4} - g_{I,2}}{D}; \qquad \bar{\vartheta} = 2\alpha^3 \frac{C - c + g_{0,4} - g_{0,2}}{D};$$

$$\rho = 2\alpha \frac{\left(\alpha C + g_{I,3}\right)\left(g_{0,1} + s\right) - \left(\alpha c + g_{I,1}\right)\left(g_{0,3} + S\right)}{D};$$

$$\vartheta^* = 2\alpha^2 \frac{\alpha(C-c) + g_{I,3} - g_{I,1}}{D}; \qquad \varepsilon = 2\alpha^2 \frac{s - S + g_{0,1} - g_{0,3}}{D}; \tag{2}$$

$$\delta^* = -\frac{\vartheta^*}{2\alpha^2}\left[\alpha^3(s-S) + g_{III,2} - g_{III,4}\right] - \frac{\bar{\delta}}{2\alpha^3}\left[\alpha^3(C+c) - g_{III,1} + g_{III,3}\right];$$

$$\hat{\vartheta} = \frac{\bar{\vartheta}}{2\alpha^3}\left[\alpha^3(C+c) - g_{III,1} + g_{III,3}\right] + \frac{\varepsilon}{2\alpha^2}\left[\alpha^3(S-s) - g_{III,2} + g_{III,4}\right];$$

$$\hat{\rho} = \frac{\bar{\vartheta}}{2\alpha^3}\left[-\alpha^2(s+S) + g_{II,1} - g_{II,3}\right] - \frac{\varepsilon}{2\alpha^2}\left[\alpha^2(C+c) - g_{II,2} + g_{II,4}\right]$$

with

$$\begin{aligned} D = {} & 2\alpha(1 - cC) + \alpha\left[(C-c)\left(g_{0,4} - g_{0,2}\right) - (s+S)\left(g_{0,3} - g_{0,1}\right)\right] + \\ & + \left(g_{0,1} - g_{0,3} + s - S\right)\left(g_{I,4} - g_{I,2}\right) + \left(g_{0,2} - g_{0,4} + c - C\right)\left(g_{I,1} - g_{I,3}\right); \end{aligned} \tag{3}$$

$$g_{P,k} = \frac{1}{2\alpha}\sum_{i=1}^{n} \lambda_i \varepsilon_{ki}(\alpha) S_i^P(\alpha); \quad c = \cos\alpha; \quad s = \sin\alpha; \quad C = \cosh\alpha; \quad S = \sin\alpha$$

where $P = 0, I, II, III$, $k = 1, \ldots, 4$. The values $S_i^P(\alpha)$, obtained by evaluating the P-th derivative of the function $S_i(\alpha, \xi)$ at the dimensionless abscissa $\xi = 1$, are finally given, together with the terms $\varepsilon_{ki}(\alpha)$ and b_{ki}, by:

$$S_i(\xi) = \sin\alpha(\xi - \xi_i) + \sinh\alpha(\xi - \xi_i);$$

$$\varepsilon_{ki}(\alpha) = \frac{\alpha}{2}\sum_{j=1}^{i-1} \lambda_j \varepsilon_{kj}(\alpha)\left[-\sin\alpha\left(\xi_i - \xi_j\right) + \sinh\alpha\left(\xi_i - \xi_j\right)\right] + \alpha^2 b_{ki}; \tag{4}$$

$$b_{1i} = -\sin\alpha\xi_i; \;\; b_{2i} = -\cos\alpha\xi_i; \; b_{3i} = \sinh\alpha\xi_i; \;\; b_{4i} = \cosh\alpha\xi_i$$

Once the global dynamic stiffness matrix of the entire structure is derived, the evaluation of the eigenfrequencies can be conveniently obtained via WW algorithm by evaluating the number J of natural frequencies that are lower than a specified dimensional frequency value ω^*. J is the sum of two terms $J=J_k+J_o$. J_k is the number of negative eigenvalues of the global dynamic stiffness matrix evaluated at ω^*. J_o is the sum of the number J_r of vibration frequencies lower than ω^* of the generic r-th beam with both ends clamped. Considering the number N_b of beams that compose the system, J_o can be obtained that $J_o = \sum_{r=1}^{N_b} J_r$. It is worth noting that J_r can be either evaluated by means of closed form expressions for undamaged beams or, alternatively, with substructuring approach.

Explicit approximate approach for the evaluation of the modal parameters

The procedure summarized in the previous section, although optimized with regard to the numerical algorithm and the size of the problem, significantly increases its computational burden when a considerable number of damage configurations are analyzed. In this section an approximated explicit formulation is proposed to assess the variability of the eigenfrequencies of a damaged frame with deterministic crack locations and uncertain intensities. The procedure is an extension of a study previously applied to cracked beams [10]. The generic crack severity λ_i is treated as a variable parameter ranging in the interval $\lambda_i \in \left[\underline{\lambda}_i, \overline{\lambda}_i\right]$ and expressed as $\lambda_i = \lambda_{o,i} + \beta_i$, where $\beta_i \in \left[-\Delta\lambda_i, \Delta\lambda_i\right]$ being $\Delta\lambda_i$ the deviation amplitude and $\lambda_{o,i}$ a reference damage intensity. The damage configuration of the frame is associated to the crack severities vector $\boldsymbol{\lambda} = \left[\lambda_1, \lambda_2, ..., \lambda_i, ...\lambda_n\right]$, considering the cracks located in one or more beams of the frame. In order to provide an approximated evaluation of the eigenfrequencies for a generic damage configuration of the frame, the correspondence between exact and approximated computations of the considered eigenfrequency parameter $\alpha_p^4(\boldsymbol{\lambda})$ is enforced for $2n$ significant configurations associated to the following damage intensity distributions

$$\boldsymbol{\lambda}_o = \left[\lambda_{o,1}, \lambda_{o,2}, ..., \lambda_{o,i}, ...\lambda_{o,n}\right]; \;\; \overline{\boldsymbol{\lambda}}_s = \left[\lambda_{o,1}, \lambda_{o,2}, ..., \overline{\lambda}_s, ...\lambda_{o,n}\right]; \;\; \underline{\boldsymbol{\lambda}}_s = \left[\lambda_{o,1}, \lambda_{o,2}, ..., \underline{\lambda}_s, ...\lambda_{o,n}\right] \tag{5}$$

with s=1,…,n. The interpolating formula proposed in [11] and already employed in [12] for the inversion of matrices, is adapted for the case at hand providing an estimation of frequency parameters as functions of the damage configuration in the form:

$$\alpha_p^4(\boldsymbol{\lambda}) \cong \alpha_{o,p}^4 + \sum_{i=1}^{n} \beta_i a_{p,i} \Big/ \left(1 + \beta_i b_{p,i}\right), \quad p = 1, ..., n, \qquad \beta_i \in \left[-\Delta\lambda_i, \Delta\lambda_i\right] \tag{6}$$

being $\alpha_{o,p}^4$ the p-th eigenfrequency of the reference damage distribution $\boldsymbol{\lambda}_o = \left[\lambda_{o,1}, \lambda_{o,2}, ..., \lambda_{o,n}\right]$, whereas $a_{p,i}$, $b_{p,i}$ represent a set of coefficients obtained enforcing the correspondence between exact and approximated frequency parameters for the $2n$ damage configurations in Eq. (5):

$$\begin{aligned} a_{p,s} &= \frac{2}{\Delta\lambda_s} \frac{\left[\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right) - \alpha_{o,p}^4\right]\left[\alpha_{o,p}^4 - \alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)\right]}{\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right) - \alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)} \\ b_{p,s} &= \frac{1}{\Delta\lambda_s} \frac{2\alpha_{o,p}^4 - \alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right) - \alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right)}{\alpha_p^4\left(\overline{\boldsymbol{\lambda}}_s\right) - \alpha_p^4\left(\underline{\boldsymbol{\lambda}}_s\right)} \end{aligned} \qquad p = 1, ..., n_m; \quad s = 1, ..., n \tag{7}$$

Numerical applications

The accuracy of the approximated solution is tested on the cracked steel portal frame reported in Fig. 1 [13], composed of two columns of length L_c=800 mm and a beam of length L_b=1000 mm. Constant 40x8 mm rectangular cross section for all the elements, Young's modulus E=2*10^5 N/mm^2, mass density per unit volume m=7.849*10^{-9} N s^2/mm^4, have been assigned. Here, the crack is located at a dimensionless abscissa $\xi_1 = 0.05$ of the left column (40 mm from the clamp), considering a variable crack severity with reference intensity $\lambda_{o,1}$=0.1 and deviation amplitude $\Delta\lambda_1$=0.1.

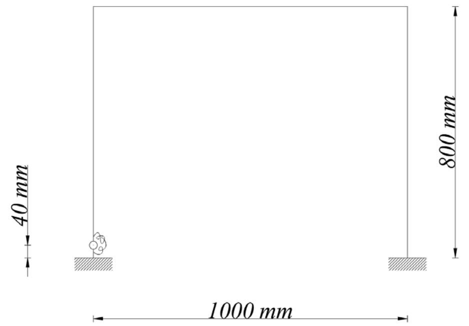

Figure 1 – Investigated damaged portal frame [14].

In Fig. 2 the comparisons between reference and approximated frequency parameters are shown for the first four frequencies; all the terms are normalized by the corresponding reference value. The continuous lines correspond to the exact properties, whereas the dashed lines are relative to the approximated solution. In Fig. 3 the corresponding relative errors are reported.

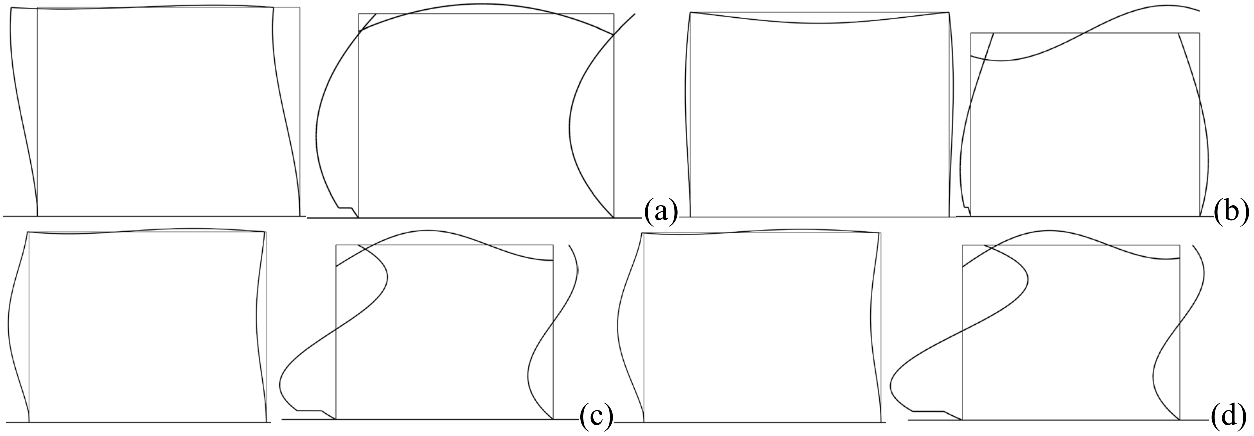

Figure 2 – (a) First, (b) second, (c).third and (d) fourth mode shape in terms of displacement (left) and rotation (right) of the investigated portal frame in the reference damage configuration

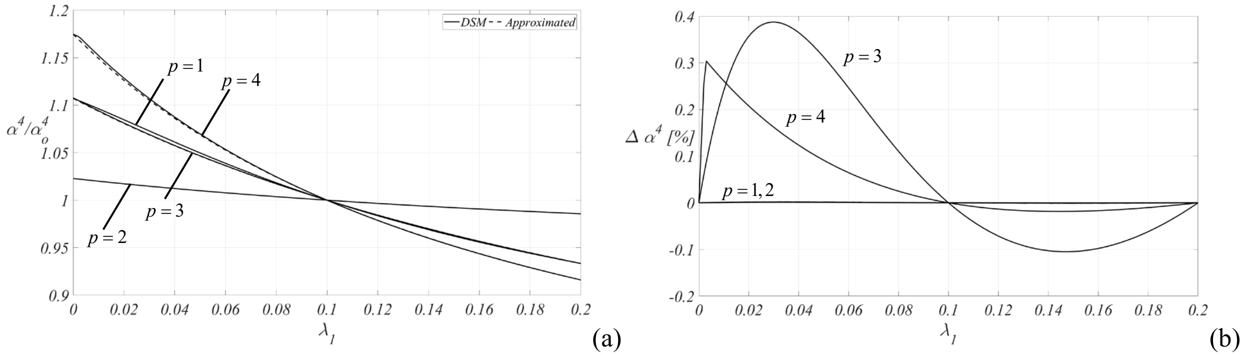

Figure 3 - Comparison between reference and approximated frequency parameter (a) and relative error (b)

For the first two frequencies the discrepancy between exact and approximated values is almost unappreciable (maximum error lower than 0.002%), while for third and fourth frequencies it is higher but always lower than 0.4%.

Conclusions

An explicit although approximated expression for the eigenfrequency of damaged frames was proposed in this study. The damaged was modelled via cracks with known position and uncertain severity, thus the frequencies are expressed as functions of the damage intensities. The proposed expressions, which are inspired to the Sherman and Morrison formula, are calibrated on reference results obtained combining the construction of the Dynamic Stiffness Matrix of multi-cracked Euler-Bernoulli beams with a formulation which takes advantage of the distributional theory with the Wittrick and Williams algorithm; a limited number of reference damage configurations is employed to calibrate the approximated solution. Validation of the proposed formulas, which can be exploited for damage identification problems as well as for sensitivity analyses, was provided for the first four frequencies with regard to a single cracked frame. The accuracy of the proposed approximated solution is satisfactory and assures a considerable computational advantage.

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References

[1] J.R. Banerjee, F.W. Williams, Exact Bernoulli–Euler dynamic stiffness matrix for a range of tapered beams, Int. J. for Numer. Meth. Eng. 21(12) (1985) 2289-2302. https://doi.org/10.1002/nme.1620211212

[2] G. Muscolino, R. Santoro, Dynamics of multiple cracked prismatic beams with uncertain-but-bounded depths under deterministic and stochastic loads J. Sound Vib. 443 (2019) 717-731. https://doi.org/10.1016/j.jsv.2018.11.029

[3] A.D. Dimarogonas, Vibration of cracked structures: a state of the art review Eng. Frac. Mech. 55(5) (1996) 831-857. https://doi.org/10.1016/0013-7944(94)00175-8

[4] C. Bilello, Theoretical and experimental investigation on damaged beams under moving systems, University of Palermo: PhD Thesis, 2001.

[5] S. Caddemi, I. Caliò, F. Cannizzaro, Tensile and compressive buckling of columns with shear deformation singularities Meccanica 50 (2015) 707-720. https://doi.org/10.1007/s11012-014-9964-3

[6] S. Caddemi, I. Caliò, F. Cannizzaro, Influence of an elastic end support on the dynamic stability of Beck's column with multiple weak sections Int. J. Nonin. Mech. 69 (2015) 14-28. https://doi.org/10.1016/j.ijnonlinmec.2014.10.016

[7] S. Caddemi, I. Caliò, Exact closed-form solution for the vibration modes of the Euler-Bernoulli beam with multiple open cracks J. Sound Vib. 327 (2009) 473-489. https://doi.org/10.1016/j.jsv.2009.07.008

[8] S. Caddemi, I. Caliò, The exact explicit dynamic stiffness matrix of multi-cracked Euler-Bernoulli beam and applications to damaged frame structures J. Sound Vib. 332(12) (2013) 3049-3063. https://doi.org/10.1016/j.jsv.2013.01.003

[9] F.W. Williams, W.H. Wittrick, An automatic computational procedure for calculating natural f requencies of skeletal structures, Int. J. Mech. Sci. 12(9) (1970) 781-791. https://doi.org/10.1016/0020-7403(70)90053-6

[10] F. Cannizzaro, N. Impollonia, S. Caddemi, I. Caliò, Explicit dynamic response of damaged beams with application to uncertain and identification problems J. Sound Vib. 487 (2020) 115608. https://doi.org/10.1016/j.jsv.2020.115608

[11] J. Sherman, W. Morrison, Adjustment of an inverse matrix corresponding to changes in the elements of a given column or a given row of the original matrix (abstract) Ann. Math. Stat. 20 (1949) 621.

[12] N. Impollonia, A method to derive approximate explicit solutions for structural mechanics problems Int. J. Solids Struct. 43 (2006) 7082-7098. https://doi.org/10.1016/j.ijsolstr.2006.03.003

[13] A. Greco, A. Pau, Damage identification in Euler frames Comput. Struct. 92-93 (2012) 328-336. https://doi.org/10.1016/j.compstruc.2011.10.007