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On obstacle problems for non coercive linear operators

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Abstract
In this paper we prove the existence and uniqueness of the solution to the one and the two obstacles problems associated with a linear elliptic operator, which is non coercive due to the presence of a convection term. We show that the operator is *weakly T-monotone* and, as a consequence, we establish the Lewy–Stampacchia dual estimates and we study the comparison and the continuous dependence of the solutions as the obstacles vary. As an application, we prove also the existence of solutions for a class of non coercive implicit obstacle problems.

Keywords Obstacle problems · Variational inequalities · Linear elliptic operators · Non coercive problems.

Mathematics Subject Classification 35J86 · 35R35 · 49J40

1 Introduction and main results

We consider various obstacle problems associated with the partial differential operator, from $H_0^1(\Omega)$ into its dual $(H_0^1(\Omega))' = H^{-1}(\Omega)$, defined by

$$Av = Lv + D \cdot (vE) = -D \cdot (M\,Dv - vE)$$

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where Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$, $M = M(x)$ is an elliptic matrix with bounded and measurable coefficients satisfying

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N \tag{1}$$

with positive constants α and β and $E = E(x)$ is a convection vector field such that

$$E \in [L^N(\Omega)]^N. \tag{2}$$

Here, given a function v and a vector field G, we denote $Dv = \text{grad}\, v$ and $D \cdot G = \text{div}\, G$.

Let $\mathbb{K}$ be a non empty convex set of $H_0^1(\Omega)$ and

$$F \in H^{-1}(\Omega). \tag{3}$$

We consider the following variational inequality, briefly denoted by $(A, \mathbb{K}, F)$

$$u \in \mathbb{K} : \quad \langle Lu - F, u - v \rangle \leq \int_\Omega u\, E \cdot D(u - v), \quad \forall\, v \in \mathbb{K} \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H_0^1(\Omega)$ and its dual, so that

$$\langle Au, v \rangle = \int_\Omega MDu \cdot Dv - \int_\Omega u\, E \cdot Dv, \quad \forall\, u, v \in H_0^1(\Omega).$$

Here we shall consider the cases where $\mathbb{K}$ is the convex set related to the lower or the upper or the two obstacles problem, by setting

$$\mathbb{K} = \mathbb{K}_\psi = \left\{ v \in H_0^1(\Omega) : v(x) \geq \psi(x) \text{ in } \Omega \right\}$$

with

$$\psi \in H_0^1(\Omega) \tag{5}$$

when the lower obstacle problem is considered, or

$$\mathbb{K} = \mathbb{K}^\varphi = \left\{ v \in H_0^1(\Omega) : v(x) \leq \varphi(x) \text{ in } \Omega \right\}$$

with

$$\varphi \in H_0^1(\Omega) \tag{6}$$

when the constraint is the upper obstacle, while for the two obstacles problem we set

$$\mathbb{K} = \mathbb{K}_\psi^\varphi = \left\{ v \in H_0^1(\Omega) : \psi(x) \leq v(x) \leq \varphi(x) \text{ in } \Omega \right\}$$

with

$$\psi, \varphi \in H_0^1(\Omega) \quad \text{and} \quad \psi \leq \varphi \quad \text{a.e.} \quad \text{in } \Omega. \tag{7}$$

The Dirichlet problem associated with general elliptic operators of second order with discontinuous coefficients has been studied in the well known paper [18] by G. Stampacchia who proved the existence, uniqueness and regularity of the weak solution in the coercive case assuming more summability on E than (2) or a smallness condition of $\|E\|_{L^N}$ with respect to the ellipticity constant in the bounded domain Ω. We recall that for dimensions greater than two the power N is optimal, by the Sobolev inequality $\|v\|_{L^{2^*}} \le C_* \|Dv\|_{L^2}$, for $v \in H_0^1(\Omega)$ where $2^* = 2N/(N-2)$. In the framework of variational inequalities associated with the non coercive operator A with other lower order terms, existence and comparison results were proven in [4] and in Section 4.7 of [15], under the assumption $D \cdot E \ge 0$ in addition to (2).

Without additional assumptions on E, only with the general integrability condition (2) the operator A is not monotone and the coerciveness fails; nevertheless, the existence and uniqueness results of G. Stampacchia have been retrieved in [1] for the Dirichlet problem using a nonlinear approach.

Here assuming only (2) and developing the nonlinear approximation of [1], which produces the key a priori estimate (21) below, we prove first the existence of a solution of the problem (4) with one or two obstacles.

Moreover, proving an important property of the operator A, the so-called *weak T-monotonicity* (see below for the definition), we derive some comparison principles and the dual estimates of Lewy–Stampacchia, extending [10] (see also [14, 15, 17] and their references).

Finally, we extend the continuous dependence of the solutions with respect to the Mosco-convergence of the convex sets to these non coercive obstacle problems. This allows us to consider, as an application motivated by a semiconductors problem that can be modelled as an implicit obstacle problem, the existence of solutions to certain quasi-variational inequalities of obstacle type, when the convex $\mathbb{K} = \mathbb{K}[u]$ depends on the solution u through appropriate nonlinear mappings $\Psi : u \mapsto \psi = \Psi(u)$ and $\Phi : u \mapsto \varphi = \Phi(u)$.

We observe that, by Sobolev embeddings, all ours results are still valid in dimension $N = 2$ with $E \in [L^p(\Omega)]^2$, with any $p > 1$, and in dimension $N = 1$ with $E \in L^1(\Omega)$.

1.1 Existence and comparison theorems

The first goal of the paper is to prove the following existence result

Theorem 1 *Assume that hypotheses* (1), (2) *and* (3) *hold. If assumption* (5), *or* (6) *or* (7) *is satisfied then there exists a unique solution u to the lower obstacle*, *or to the upper obstacle or to the two obstacles problem* (4), *respectively*.

Next, we will highlight a property of the pseudomonotone operator A, which in general is not monotone due to the presence of a general convection term. To this aim we need to specify some notations.

Given $h > 0$, $T_h(s)$ denotes the standard truncation function defined by

$$T_h(s) = \begin{cases} s & \text{if } |s| \le h \\ h\frac{s}{|s|} & \text{if } |s| > h, \end{cases}$$

and, as usual, we set

$$s^+ = \sup\{s, 0\} = s \vee 0 \quad \text{and} \quad s^- = -\inf\{s, 0\} = -(s \wedge 0), \quad \forall s \in \mathbb{R}.$$

Definition 1 An operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is called *weakly T-monotone* if it satisfies:

$$v \in H_0^1(\Omega) : \quad \langle Av, T_h(v^+)\rangle \le 0 \quad \forall h > 0 \quad \text{implies} \quad v \le 0 \quad \text{a.e. in } \Omega. \tag{8}$$

We will prove the following

Theorem 2 *Assume that hypothesis (1) holds and let*

$$E \in [L^2(\Omega)]^N.$$

Then the operator A is weakly T-monotone.

We point out that the new notion of weakly T-monotonicity given above plays, in our framework, the role of the T-monotonicity property introduced by Brézis and Stampacchia in [6] and allows us to obtain some new comparison results and duality inequalities already known for the coercive obstacle problems.

First of all we derive the following comparison principles, which, in turn, imply the uniqueness of the solution of the problem $(A, \mathbb{K}, F)$ in the cases of one (lower or upper) and two obstacles.

Corollary 1 *Assume that hypotheses (1), (2) hold and let*

$$F_1, F_2 \in H^{-1}(\Omega)$$

such that

$$\langle F_2, v\rangle \ge \langle F_1, v\rangle, \quad \forall v \in H_0^1(\Omega), \quad v \ge 0$$

and

$$\psi_1, \psi_2 \in H_0^1(\Omega), \quad \psi_2 \ge \psi_1 \quad \text{a.e. in } \Omega, \ \text{(lower obstacle)}$$

or

$$\varphi_1, \varphi_2 \in H_0^1(\Omega), \quad \varphi_2 \ge \varphi_1 \quad \text{a.e. in } \Omega, \ \text{(upper obstacle)}$$

or, for the case of the two obstacles, for $i = 1, 2$, *let also*

$$\varphi_i \le \psi_i, \quad a.e. \quad in \quad \Omega.$$

Let u_i denote the solutions of one of the following obstacle problems $(A, \mathbb{K}_{\psi_i}, F_i)$, $(A, \mathbb{K}^{\varphi_i}, F_i)$ *or* $(A, \mathbb{K}_{\psi_i}^{\varphi_i}, F_i)$, *corresponding respectively, to the lower obstacle, to the upper obstacle or to the two obstacles case. Then*

$$u_2 \geq u_1, \quad \text{a.e. in } \Omega.$$

1.2 Lewy–Stampacchia inequalities

As already pointed out, the notion of weakly T-monotonicity of the operator A introduced above also allows the extension to our framework of some dual estimates already known for solutions of variational inequalities related to coercive and T-monotone operators (see [6, 14, 15, 17]). For this purpose we recall some definitions.

Let X be an Hilbert space, which is a vector lattice with respect to a partial order relation $\leq$ (that is, $u \vee v = \sup(u, v)$ and $u \wedge v = \inf(u, v) \in X$ for all vectors $u, v \in X$). Thus

$$X = P - P$$

where P is the closed positive cone

$$P = \{v \in X : \quad v \geq 0\}.$$

If V is a sublattice (that is, V is a subspace of X such that $u \wedge v, u \vee v \in V$ for all vectors $u, v \in V$) we denote by V^* the subspace of the dual space V' generated by the positive cone

$$P' = \left\{v \in V' : \quad \langle v', v\rangle \geq 0 \quad \forall v \in P\right\}.$$

that is, $V^* = P' - P'$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and V'. The space V^* is called the *order dual* of V.

It is well known that the space $H^1(\Omega)$ is a vector lattice under the ordering

$$u \leq v \quad \text{iff} \quad u(x) \leq v(x) \quad \text{a.e.}\, x \in \Omega.$$

Moreover, $H_0^1(\Omega)$ is a sublattice and $F \in H^{-1}(\Omega)$ is a positive element for the dual ordering if and only if F is a positive distribution, hence a positive measure belonging to $H^{-1}(\Omega)$. Thus, the order dual $[H_0^1(\Omega)]^*$ is the space of all distributions in $H^{-1}(\Omega)$ which can be written as the difference of two positive measures belonging to $H^{-1}(\Omega)$.

The following result follows from the comparison corollary stated above and extends a property already known for coercive and strictly T-monotone operator (see Theorem 3.2 of [14]) to the non coercive and weakly T-monotone operator A.

Corollary 2 *Assume that hypotheses* (1) *and* (2) *hold and let* $u, v \in H_0^1(\Omega)$ *such that*

$$Au, Av \in [H_0^1(\Omega)]^*.$$

Then, $A(u \wedge v), A(u \vee v) \in [H^1_0(\Omega)]^*$,

$$A(u \wedge v) \geq Au \wedge Av \quad \text{and} \quad A(u \vee v) \leq Au \vee Av. \tag{9}$$

In particular, *since* $A0 = 0$, *we also have*

$$A(u^-) \geq (Au)^- \quad \text{and} \quad A(u^+) \leq (Au)^+.$$

We say that $w \in H^1(\Omega)$ is a *supersolution* to the lower obstacle problem $(A, \mathbb{K}_\psi, F)$ if $w \geq \psi$ in Ω, $Aw \geq F$ in $H^{-1}(\Omega)$ and $w \geq 0$ on $\partial\Omega$, in the sense that $w^- \in H^1_0(\Omega)$. We can also extend to non coercive operators the Stampacchia's result which establishes that the solution to $(A, \mathbb{K}_\psi, F)$ is the lowest of its supersolutions (see [15] and its references). Defining similarly *subsolutions*, using instead $\leq$, to the upper obstacle problem $(A, \mathbb{K}^\varphi, F)$ we leave to the reader the formulation of the symmetrical case of the following interesting properties for the one obstacle problem.

Corollary 3 *Assume that hypotheses* (1), (2), (3) *and* (5) *hold.*

(i) *If* u *is the solution to the lower obstacle problem* $(A, \mathbb{K}_\psi, F)$ *and* w *is any supersolution then* $u \leq w$.
(ii) *If* $v, w \in H^1_0(\Omega)$ *are two supersolutions to* $(A, \mathbb{K}_\psi, F)$ *then also* $v \wedge w$ *is a supersolution to* $(A, \mathbb{K}_\psi, F)$.

Next we state the following Lewy–Stampacchia inequalities for the one and the two obstacle problems associated to weakly T-monotone operators.

Theorem 3 *Assume that the hypotheses* (1), (2), (3) *and* (5) *hold and let*

$$F, A\psi \in [H^1_0(\Omega)]^*. \tag{10}$$

Let u *be the solution of the problem* $(A, \mathbb{K}_\psi, F)$. *Then the following inequalities hold*

$$F \leq Au \leq F \vee A\psi. \tag{11}$$

***Remark* 1** For the upper obstacle problem we have a symmetric result, since we observe that u is the solution of $(A, \mathbb{K}^\varphi, F)$ iff $-u$ solves the lower obstacle problem $(A, \mathbb{K}_{-\varphi}, -F)$ and we may apply the inequality (11) to the function $-u$. Therefore, under the hypotheses (1), (2), (3), (6) and $F, A\varphi \in [H^1_0(\Omega)]^*$ the solution u of problem $(A, \mathbb{K}^\varphi, F)$ satisfies the symmetrical Lewy–Stampacchia inequalities

$$F \wedge A\varphi \leq Au \leq F, \text{ in } [H^1_0(\Omega)]^*.$$

Theorem 4 *Assume that the hypotheses* (1), (2), (3) *and* (7) *hold and let*

$$F, A\psi, A\varphi \in [H^1_0(\Omega)]^*. \tag{12}$$

Let u *be the solution of the problem* $(A, \mathbb{K}^\varphi_\psi, F)$. *Then the following inequalities hold*

$$F \wedge A\varphi \leq Au \leq F \vee A\psi. \tag{13}$$

***Remark* 2** The Lewy–Stampacchia inequalities for the two obstacles problem in abstract form for T-monotone coercive operators have been proved in [17].

1.3 Mosco convergence

Next, we consider the continuous dependence of the solution u with respect to the Mosco convergence of the convex sets, which definition we recall for convenience.

Definition 2 Let X be a Banach space and K_n, K_0 closed, convex subsets of X. The sequence $\{K_n\}$ Mosco-converges to K_0 (we briefly write $K_n \xrightarrow{M} K_0$) if

(i) K_0 is the set of all $v_0 \in X$ such that $\|v_0 - v_n\|_X \to 0$, with $v_n \in K_n$;
(ii) for every subsequence $\{v_{n_i}\}$, with $v_{n_i} \in K_{n_i}$, weakly convergent to v_0, we have $v_0 \in K_0$.

The first result we will state concerns the stability of the solutions of lower (or upper) obstacle problems, as the obstacles vary.

Theorem 5 *Assume that hypotheses* (1), (2), (3) *hold and let*

$$\psi_n, \psi_0 \in H_0^1(\Omega), \tag{14}$$

with

$$\left\{\psi_n\right\} \text{ bounded in } H_0^1(\Omega).$$

Let u_n and u_0 be the solutions of the obstacle problems $(A, \mathbb{K}_{\psi_n}, F)$ and $(A, \mathbb{K}_{\psi_0}, F)$, respectively. If

$$\mathbb{K}_{\psi_n} \xrightarrow{M} \mathbb{K}_{\psi_0},$$

then

$$u_n \to u_0 \quad \text{strongly in } H_0^1(\Omega). \tag{15}$$

***Remark* 3** We recall that in order to have

$$\mathbb{K}_{\psi_n} \xrightarrow{M} \mathbb{K}_{\psi_0}$$

if suffices that one of the following assertions holds (see [2, 3, 12, 13]):

1. the sequence $\{\psi_n\}$ is weakly convergent in $H_0^1(\Omega)$ to ψ_0, and $\psi_n \leq \psi_0$, $\forall\ n$;
2. the sequence $\{\psi_n\}$ is weakly convergent in $L^p(\Omega)$ to ψ_0 and decreasing;
3. the sequence $\{\psi_n\}$ strongly converges in $H_0^1(\Omega)$ to ψ_0;

4. the sequence $\{\psi_n\}$ strongly converges in $L^\infty(\Omega)$ to ψ_0 and there exists $\Psi \in W_0^{1,p}(\Omega)$, such that $\Psi \geq \psi_n$, $\forall\ n$;
5. the sequence $\{\psi_n\}$ weakly converges in $W_0^{1,q}(\Omega)$ to ψ_0, with $q > 2$.

***Remark* 4** The convergence of the solutions of the upper obstacles problems, as the obstacles vary, reads exactly in the same manner, just by replacing ψ_n by φ_n and $\mathbb{K}_{\psi_n}$ by $\mathbb{K}^{\varphi_n}$. Also the sufficient conditions for the Mosco convergence of the convex sets hold with the obvious adaptations to upper obstacles.

To conclude, we state the similar result on the convergence of the solutions of the two obstacles problem.

Theorem 6 *Assume that hypotheses* (1), (2) *and* (3) *hold with given obstacles in* $H_0^1(\Omega)$, *satisfying* $\varphi_0 \leq \psi_0, \varphi_n \leq \psi_n \quad \forall n \in \mathbb{N}$, *such that*

$$\{\psi_n\}, \{\varphi_n\} \text{ are bounded in } H_0^1(\Omega).$$

Let u_n *and* u_0 *be the solutions of the obstacle problems* $(A, \mathbb{K}_{\psi_n}^{\varphi_n}, F)$ *and* $(A, \mathbb{K}_{\psi_0}^{\varphi_0}, F)$, *respectively. If*

$$\mathbb{K}_{\psi_n}^{\varphi_n} \xrightarrow{M} \mathbb{K}_{\psi_0}^{\varphi_0}$$

then

$$u_n \to u_0 \quad \text{strongly in } H_0^1(\Omega). \tag{16}$$

***Remark* 5** For convex sets with two obstacles, in order to have $\mathbb{K}_{\psi_n}^{\varphi_n} \xrightarrow{M} \mathbb{K}_{\psi_0}^{\varphi_0}$ with $\psi_n, \varphi_n \in H_0^1(\Omega)$, it is sufficient to assume

$$\varphi_n \leq \psi_n, \quad \varphi_n \to \varphi_0 \quad \text{and} \quad \psi_n \to \psi_0 \quad \text{strongly in } H_0^1(\Omega).$$

Indeed, for any sequence $v_n \in \mathbb{K}_{\psi_n}^{\varphi_n}$ such that $v_n \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$, for a subsequence, we have $v_n \to v_0$ in $L^2(\Omega)$ and a.e. in Ω, so it is clear that $v_0 \in \mathbb{K}_{\psi_0}^{\varphi_0}$. On the other hand, for every $v_0 \in \mathbb{K}_{\psi_0}^{\varphi_0}$ we have $v_n = (\psi_n \vee v_0) \wedge \varphi_n \in \mathbb{K}_{\psi_n}^{\varphi_n}$ and the strong convergence of the obstacles implies $v_n \to v_0$ in $H_0^1(\Omega)$.

1.4 Implicit obstacle problems

Suppose that the obstacles depend, through some functional relation, from the solution u, by assuming given the mappings

$$\Psi : u \mapsto \psi = \Psi(u) \quad \text{and} \quad \Phi : u \mapsto \varphi = \Phi(u), \tag{17}$$

and therefore, in each one of the three cases (5), (6) and (7), we also have a functional dependence $w \mapsto \mathbb{K} = \mathbb{K}[w]$.

Then each implicit obstacle problem may be formulated as a quasivariational inequality, denoted by $(A, \mathbb{K}[u], F)$

$$u \in \mathbb{K}[u], \qquad \langle Au - F, u - v\rangle \le 0 \qquad \forall v \in \mathbb{K}[u]. \tag{18}$$

Theorem 7 *Assume that the assumptions* (1), (2) *and* (3) *hold and the obstacle mappings* $\Psi : L^2(\Omega) \to H^1_0(\Omega)$ *and* $\Phi : L^2(\Omega) \to H^1_0(\Omega)$ *are continuous and have bounded ranges. Then, there exists at least a solution to the implicit lower* (*resp. upper*) *obstacle problem* (18) *with* $\mathbb{K}[u] = \mathbb{K}_{\Psi(u)}$ (*resp.* $\mathbb{K}[u] = \mathbb{K}^{\Phi(u)}$). *Moreover if, in addition,* $\Psi(w) \le \Phi(w)$ *for all* $w \in L^2(\Omega)$, *then there exists at least a solution to the implicit non coercive two obstacles problem* (18) *with* $\mathbb{K}[u] = \mathbb{K}^{\Phi(u)}_{\Psi(u)}$.

***Remark* 6** Other examples of implicit obstacle problems are discussed for instance in [14]. In general the uniqueness of the solution cannot be expected. However, in special situations, the uniqueness of certain coercive implicit obstacle problems can be obtained under smallness conditions on the data as, for instance, in the case of a semiconductor model [16]. In Sect. 5 we also give an application of Theorem 7.

2 Proof of Theorems 1 and 2

For each $n \in \mathbb{N}$ let u_n be a solution of the following approximated obstacle problem

$$u_n \in \mathbb{K},\ \langle Lu_n - F, u_n - v\rangle \le \int_\Omega \frac{u_n}{1 + \frac{1}{n}|u_n|} E \cdot D(u_n - v) \quad \forall v \in \mathbb{K}. \tag{19}$$

The existence of u_n follows by the well known results by [8, 11], since, for every $n \in \mathbb{N}$, the nonlinear composition is bounded with respect to u_n.

***Proof of Theorem* 1** We begin with the proof in the case $\mathbb{K} = \mathbb{K}_\psi$.

Step 1: We claim that there exists a positive constant $C_0 = C_{\alpha,\beta,E,\psi,F,N,\Omega}$, independent of n, such that $\forall k > 0$

$$\operatorname{meas}(\{x : k < u_n(x) - \psi(x)\}) \le \frac{C_0}{[\log(1+k)]^2} \tag{20}$$

Let

$$F = f_0 - D \cdot f, \qquad \text{with } f_0 \in L^{\frac{2N}{N+2}}, \quad f \in \left[L^2(\Omega)\right]^N.$$

Put $v = u_n - \frac{u_n - \psi}{1 + u_n - \psi}$ in (19). Note that $v \ge \psi$. Then

$$\begin{aligned}
&\alpha \int_\Omega \frac{|D(u_n-\psi)|^2}{(1+u_n-\psi)^2} \\
&\quad\le \int_\Omega |f_0| + \int f\cdot \frac{D(u_n-\psi)}{(1+u_n-\psi)^2} + \int_\Omega \frac{u_n-\psi}{1+\frac{1}{n}|u_n|} E\cdot \frac{D(u_n-\psi)}{(1+u_n-\psi)^2} \\
&\qquad + \int_\Omega \frac{\psi}{1+\frac{1}{n}|u_n|} E\cdot \frac{D(u_n-\psi)}{(1+u_n-\psi)^2} + \int_\Omega \frac{MD\psi\cdot D(u_n-\psi)}{(1+u_n-\psi)^2} \\
&\quad\le \int_\Omega |f_0| + \int |f| \frac{D(u_n-\psi)}{(1+u_n-\psi)^2} + \int_\Omega \frac{u_n-\psi}{(1+u_n-\psi)} |E| \frac{|D(u_n-\psi)|}{(1+u_n-\psi)} \\
&\qquad + \int_\Omega \frac{|\psi|}{(1+u_n-\psi)} |E| \frac{|D(u_n-\psi)|}{(1+u_n-\psi)} + \int_\Omega \frac{|MD\psi\cdot D(u_n-\psi)|}{(1+u_n-\psi)}.
\end{aligned}$$

Then the use of the Young inequality in the right-hand side yields to

$$\begin{aligned}
&\frac{\alpha}{2} \int_\Omega \frac{|D(u_n-\psi)|^2}{(1+u_n-\psi)^2} \\
&\quad\le C_{\alpha,\beta}\Big[\int_\Omega |E|^2 + \int_\Omega |E|^N + \int_\Omega |\psi|^{2^*} + \int_\Omega |D\psi|^2 + \int_\Omega |f|^2\Big] + \int_\Omega |f_0|
\end{aligned}$$

and by the Poincaré inequality ($\sqrt{P}\|v\|_{L^2} \le C_*\|Dv\|_{L^2}$, for $v \in H_0^1(\Omega)$ with constant $P > 0$) in the left-hand side, we obtain

$$P\int_\Omega [\log(1+u_n-\psi)]^2 \le C_{\alpha,\beta,E,\psi,F,N}.$$

Recall that $0 \le u_n - \psi \in H_0^1(\Omega)$ implies also $\log(1+u_n-\psi) \in H_0^1(\Omega)$. Since

$$\begin{aligned}
&[\log(1+k)]^2 \mathrm{meas}(\{x : k < u_n(x) - \psi(x)\}) \\
&\quad\le \int_{\{k\le u_n-\psi\}} [\log(1+u_n-\psi)]^2
\end{aligned}$$

we deduce that

$$\mathrm{meas}(\{x : k+\psi(x) < u_n(x)\}) \le \frac{C_{\alpha,\beta,E,\psi,F,N}}{P[\log(1+k)]^2}$$

and (20) follows.

Step 2: We prove that there exists a positive constant C_1, independent of n such that

$$\int_\Omega |Du_n|^2 \le C_1. \tag{21}$$

Let $k > 0$ and $G_k(s)$ be the function defined by

$$G_k(s) = s - T_k(s), \quad \forall\, s \in \mathbb{R}.$$

In (19) we take $v = u_n - T_k(u_n - \psi)$ (note that $v \geq \psi$)

$$\langle L(u_n - \psi), T_k(u_n - \psi)\rangle \leq \int_\Omega \frac{u_n - \psi}{1 + \frac{1}{n}|u_n|} E \cdot DT_k(u_n - \psi)$$
$$+ \int_\Omega \frac{\psi}{1 + \frac{1}{n}|u_n|} E \cdot DT_k(u_n - \psi) + \langle F - L\psi, T_k(u_n - \psi)\rangle \leq k\|f_0\|_{L^1}$$
$$+ \left[\beta\|D\psi\|_{L^2} + \|f\|_{L^2} + k\|E\|_{L^2} + \|\psi\|_{L^{2*}}\|E\|_{L^N}\right]\left[\int_\Omega |DT_k(u_n - \psi)|^2\right]^{\frac{1}{2}}.$$

Using the ellipticity condition in the left hand side and the Young inequality in the right one, we obtain

$$\int_\Omega |DT_k(u_n - \psi)|^2 \leq C_{k,\alpha,\beta,F,E,\psi,N}. \tag{22}$$

Note that the function $v = u_n - G_k(u_n - \psi)$ is an admissible test function in (19) and that

$$u_n = T_k(u_n - \psi) + G_k(u_n - \psi) + \psi, \quad \forall\, n \in \mathbb{N}.$$

This choice implies

$$\langle L(u_n - \psi), G_k(u_n - \psi)\rangle$$
$$\leq \int_\Omega \frac{G_k(u_n - \psi)}{1 + \frac{1}{n}|u_n|} E \cdot DG_k(u_n - \psi) + \langle F - L\psi, G_k(u_n - \psi)\rangle$$
$$+ \int_\Omega \frac{\psi}{1 + \frac{1}{n}|u_n|} E \cdot DG_k(u_n - \psi) + \int_\Omega \frac{T_k(u_n - \psi)}{1 + \frac{1}{n}|u_n|} E \cdot DG_k(u_n - \psi)$$
$$\leq \int_{\{k \leq u_n - \psi\}} |G_k(u_n - \psi)|\,|E|\,|DG_k(u_n - \psi)| + \langle F - L\psi, G_k(u_n - \psi)\rangle$$
$$+ \int_\Omega |\psi||E|\,|DG_k(u_n - \psi)| + k\int_\Omega |E|\,|DG_k(u_n - \psi)|.$$

Then the use of the Sobolev inequality with constant C_* yields

$$\left(\alpha - C_*\left[\int_{\{k \leq u_n - \psi\}} |E|^N\right]^{\frac{1}{N}}\right)\int_\Omega |DG_k(u_n - \psi])^2$$
$$\leq \langle F - L\psi, G_k(u_n - \psi)\rangle$$
$$+ \|\psi\|_{L^{2*}}\|E\|_{L^N}\|DG_k(u_n - \psi)\|_{L^2} + k\int_\Omega |E|\,|DG_k(u_n - \psi)|.$$

Now, by virtue of (20) we can choose $k > 0$ such that

$$C_*\left[\int_{\{k\le u_n-\psi\}}|E|^N\right]^{\frac{1}{N}}\le\frac{\alpha}{2}$$

and we prove an estimate on $|DG_k(u_n-\psi)$ in $L^2(\Omega)$, since the left hand side grows quadratically, while the right hand side grows linearly, with respect to $\|DG_k(u_n-\psi)\|_{L^2}$. At last, taking into account the estimate (22) we obtain (21).

Step 3: As a consequence of (21) there exists a subsequence, still denoted by $\{u_n\}$, and a function u such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_0^1(\Omega),\\ u_n \to u & \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega.\end{cases} \tag{23}$$

Note that $u\in\mathbb{K}_\psi$.

We prove that u is a solution of the problem (4). Given $w\in\mathbb{K}_\psi$ we take in (19) $v=u_n-T_k(u_n-w)$ and we get

$$\begin{aligned}&\langle L(u_n-w),T_k(u_n-w)\rangle+\langle Lw,T_k(u_n-w)\rangle\\&\quad\le\int_\Omega\frac{T_k(u_n-w)}{1+\frac{1}{n}|u_n|}E\cdot DT_k(u_n-w)+\int_\Omega\frac{w}{1+\frac{1}{n}|u_n|}E\cdot DT_k(u_n-w)+\langle F,T_k(u_n-w)\rangle.\end{aligned}$$

Here we have used the property $u_n=T_k(u_n-w)+w$ if $|u_n-w|<k$ and $DT_k(u_n-w)=0$ if $|u_n-w|\ge k$. Thanks to the weak lower semicontinuity of the quadratic form $H_0^1(\Omega)\ni\varphi\mapsto\langle L\varphi,\varphi\rangle$ we pass to the limit as $n\to+\infty$ in the left hand side; moreover, since the sequence $\left\{\frac{T_k(u_n-w)}{1+\frac{1}{n}|u_n|}\right\}$ is bounded we use the Lebesgue theorem in the right hand side and we have

$$\begin{aligned}&\langle Lu,T_k(u-w)\rangle\\&\quad\le\int_\Omega T_k(u-w)\,E\cdot DT_k(u-w)+\int_\Omega w\,E\cdot DT_k(u-w)+\langle F,T_k(u-w)\rangle\end{aligned}$$

that is

$$\langle Lu-F,T_k(u-w)\rangle\le\int_\Omega u\,E\cdot DT_k(u-w).$$

Taking the limit as $k\to\infty$ and observing that $T_k(u-w)\to u-w$ in $H_0^1(\Omega)$, we obtain

$$\langle Lu-F,u-w\rangle\le\int_\Omega u\,E\cdot D(u-w),$$

that is, u is a solution of the obstacle problem $(A,\mathbb{K}_\psi,F)$. Finally, the uniqueness will be a consequence of the next Corollary 1.

In order to prove the existence of solution for the upper obstacle problem, we just observe that u is solution of $(A,\mathbb{K}^\varphi,F)$ iff $-u$ solves the lower obstacle problem $(A,\mathbb{K}_{-\varphi},-F)$.

Finally, the existence of solution of the two obstacles problem $(A, \mathbb{K}_{\psi}^{\varphi}, F)$ can be achieved essentially by repeating the proof made for the lower obstacle problem. Indeed, first we remark that u is the solution of the two obstacles problem $(A, \mathbb{K}_{\psi}^{\varphi}, F)$ iff $u^* = u - \varphi$ is the solution of $(A, \mathbb{K}_{\psi^*}^{0}, F^*)$, with $\psi^* = \psi - \varphi \in H_0^1(\Omega)$ and $F^* = F - L\varphi + D \cdot (\varphi E) \in H^{-1}(\Omega)$. Secondly, we repeat the three steps of the proof noting that in the corresponding approximating problem for $u_n^* \in \mathbb{K}_{\psi^*}^{0}$ the test functions $v = u_n^* - \frac{u_n^* - \psi^*}{1 + u_n^* - \psi^*}$, $v = u_n^* - T_k(u_n^* - \psi^*)$ and $v = u_n^* - T_k(u_n^* - w)$, with any $w \in \mathbb{K}_{\psi^*}^{0}$ are negative and therefore are admissible, since they belong also to $\mathbb{K}_{\psi^*}^{0}$.

□

Proof of Theorem 2 Let $v \in H_0^1(\Omega)$ such that

$$\langle Av, T_h(v^+) \rangle \leq 0, \quad \forall h > 0.$$

Then

$$\langle Lv, T_h(v^+) \rangle \leq \int_{\Omega} v\, E \cdot DT_h(v^+)$$

and

$$\alpha \int_{\Omega} |DT_h(v^+)|^2 \leq h \left[\int_{\{0<v<h\}} |E|^2 \right]^{\frac{1}{2}} \left[\int_{\Omega} |DT_h(v^+)|^2 \right]^{\frac{1}{2}}$$

Observe that the inequality

$$\alpha \left[\int_{\Omega} |DT_h(v^+)|^2 \right]^{\frac{1}{2}} \leq h \left[\int_{\{0<v<h\}} |E|^2 \right]^{\frac{1}{2}}$$

and the Poincaré inequality (with constant $P > 0$) yield

$$P\alpha \left[\int_{\Omega} |T_h(v^+)|^2 \right]^{\frac{1}{2}} \leq h \left[\int_{\{0<v<h\}} |E|^2 \right]^{\frac{1}{2}}.$$

Now, we fix $\delta > 0$ and let $0 < h < \delta$. We note that

$$[\text{meas}\{\delta < v\}]^{\frac{1}{2}} \leq \frac{1}{h} \left[\int_{\{h<v\}} |T_h(v^+)|^2 \right]^{\frac{1}{2}} \leq \frac{1}{P\alpha} \left[\int_{\{0<v<h\}} |E|^2 \right]^{\frac{1}{2}}. \tag{24}$$

Since the last integral goes to zero as $h \to 0$ we obtain

$$\text{meas}\{\delta < v\} = 0, \quad \forall\, \delta > 0,$$

which implies $v \leq 0$ a.e. in Ω.

□

Remark 7 The main point of the above proof is the inequality (24). It is worth noting that, it is only needed the L^2-summability of E, instead of the L^N-summability required in the proof of the existence result.

***Proof of Corollary* 1** Let u_1, u_2 be solutions of the obstacle problems $(A, \mathbb{K}_{\psi_1}, F_1)$ and $(A, \mathbb{K}_{\psi_2}, F_2)$, respectively.

Given $h > 0$, we may choose $v = u_1 - T_h((u_1 - u_2)^+)$ as test function in the formulation of the problem $(A, \mathbb{K}_{\psi_1}, F_1)$ and $v = u_2 + T_h((u_1 - u_2)^+)$ in the formulation of the problem $(A, \mathbb{K}_{\psi_2}, F_2)$. Thus, we have

$$\langle Lu_1, T_h((u_1 - u_2))^+\rangle \le \int_\Omega u_1\, E \cdot DT_h((u_1 - u_2)^+) + \langle F_1, T_h((u_1 - u_2)^+)\rangle$$
$$\langle Lu_2, -T_h((u_1 - u_2)^+)\rangle \le -\int_\Omega u_2\, E \cdot DT_h((u_1 - u_2)^+) - \langle F_2, T_h((u_1 - u_2)^+)\rangle$$

Adding the two above inequalities and using the assumption $F_1 \le F_2$ in $H^{-1}(\Omega)$ we deduce

$$\langle A(u_1 - u_2), T_h((u_1 - u_2)^+)\rangle \le 0$$

and the thesis of Corollary 1 for the lower obstacle problem easily follows applying the weakly T-monotonicity property of the operator A.

For the two obstacles problem the proof is similar. □

3 Proof of the Lewy–Stampacchia inequalities

In this section we will prove the Lewy–Stampacchia inequalities (11) and (13).

***Proof of Corollary* 2** The proof is similar to the case of coercive T-monotone operators of [14].

Let $z \in H_0^1(\Omega)$ be the unique solution of $(A, \mathbb{K}_{u\wedge v}, Au \wedge Av)$, that is the lower obstacle problem

$$z \ge u \wedge v, \quad \langle Az - (Au \wedge Av), z - w\rangle \le 0 \quad \forall w \in H_0^1(\Omega), \quad w \ge u \wedge v \tag{25}$$

The existence and uniqueness of z follows by Theorem 1 and Corollary 1. Let $\zeta \in H_0^1(\Omega)$ such that $\zeta \ge 0$. The function $w = z + \zeta$ belongs to $\mathbb{K}_{u\wedge v}$ and choosing w as test function in (25) we obtain

$$\langle Az - (Au \wedge Av), \zeta\rangle \ge 0$$

which, in turn, implies

$$Az \ge Au \wedge Av. \tag{26}$$

Since u and v are the solutions of the obstacle problems $(A, \mathbb{K}_{u\wedge v}, Au)$ and $(A, \mathbb{K}_{u\wedge v}, Av)$, respectively, and $Au \wedge Av \leq Au$ and $Au \wedge Av \leq Av$, applying twice Corollary (1) we deduce

$$z \leq u \quad \text{and} \quad z \leq v$$

and then

$$z \leq u \wedge v.$$

Thus $z = u \wedge v$ and the conclusion follows by (26).

The proof of the second inequality in (9) can be performed in a similar way, using the existence, uniqueness and comparison results for the upper obstacle problem.

□

***Proof of Theorem* 3** The inequality $F \leq Au$ follows easily by taking as test function in (4) $v = u + w$, with $w \in H_0^1(\Omega)$ and $w \geq 0$.

Now, let us prove the right-hand side of the inequality (11). Let $z \in H_0^1(\Omega)$ be the unique solution of the upper obstacle problem $(A, \mathbb{K}^u, F \vee A\psi)$, that is

$$z \leq u, \quad \langle Az, z - w\rangle \leq \langle F \vee A\psi, z - w\rangle \quad \forall\, w \in H_0^1(\Omega), \quad w \leq u. \tag{27}$$

Let $\zeta \in H_0^1(\Omega)$, $\zeta \geq 0$. Testing (27) with the admissible test function $w = z - \zeta$ we obtain

$$\langle Az - (F \vee A\psi), \zeta\rangle \leq 0$$

which implies

$$Az \leq F \vee A\psi. \tag{28}$$

Now, we claim that

$$z \geq \psi. \tag{29}$$

Given $h > 0$ the function

$$w = z + T_h((\psi - z)^+)$$

belongs to the convex $\mathbb{K}^u$ and putting w in (27) we obtain

$$-\langle Az, T_h((\psi - z)^+)\rangle \leq -\langle F \vee A\psi, T_h((\psi - z)^+)\rangle \leq -\langle A\psi, T_h((\psi - z)^+)\rangle$$

that is

$$\langle A\psi - Az, T_h((\psi - z)^+)\rangle \leq 0$$

and using the weakly T-monotonicity property of the operator A we get the inequality (29). Consequently, for any $h > 0$ the function $v = u - T_h(u - z)$ satisfies $v \geq \psi$ (note that $u \geq z$). Testing the problem (4) with this v we obtain

$$\langle Au, T_h((u - z)^+)\rangle \leq \langle F, T_h((u - z)^+)\rangle.$$

Now, taking $w = z + T_h((u-z)^+)$ as test function in (27) we have

$$-\langle Az, T_h((u-z)^+)\rangle \le -\langle F \vee A\psi, T_h((u-z)^+)\rangle$$

and adding the last two inequalities we get

$$\langle Au - Az, T_h((u-z)^+)\rangle \le 0.$$

Again, by virtue of the weakly T-monotonicity of the operator A we have $u \le z$. Thus, $z = u$ and (28) yields the conclusion. □

***Proof of Theorem* 4** The proof is an extension to non coercive weakly T-monotone operators of the one in [17].

Let $u \in \mathbb{K}_\psi^\varphi$ be the solution of the problem $(A, \mathbb{K}_\psi^\varphi, F)$. Thus $u \in \mathbb{K}_\psi^\varphi$ and it satisfies the following inequality

$$\langle Au, u - v\rangle \le \langle F, u - v\rangle, \quad \forall v \in \mathbb{K}_\psi^\varphi. \tag{30}$$

The upper bound in the inequality (13) can be achieved as in the proof of Theorem 3. In order to prove the lower bound in (13), let $z \in H_0^1(\Omega)$ be the unique solution of the one obstacle problem $(A, \mathbb{K}_u, F \wedge A\varphi)$, that is

$$z \ge u, \quad \langle Az, z - w\rangle \le \langle F \wedge A\varphi, z - w\rangle \quad \forall\, w \in H_0^1(\Omega), \quad w \ge u. \tag{31}$$

For any $\zeta \in H_0^1(\Omega)$, $\zeta \ge 0$ we test problem (31) with the function $w = z + \zeta$ (note that $w \ge u$) and we obtain

$$Az \ge F \wedge A\varphi.$$

Our goal is to prove that $z = u$ and then it is enough to show that $z \le u$. First of all, we claim that $z \le \varphi$. As a matter of the fact, for any $h > 0$ the choice $w = z - T_h((z-\varphi)^+)$ (note that $w \ge u$) in (31) gives

$$\langle Az, T_h(z-\varphi)^+\rangle \le \langle F \wedge A\varphi, T_h((z-\varphi)^+)\rangle \le \langle A\varphi, T_h((z-\varphi)^+)\rangle$$

which in turn implies

$$\langle Az - A\varphi, T_h((z-\varphi)^+)\rangle \le 0.$$

By virtue of the weakly T-monotonicity of the operator A we conclude that $z \le \varphi$ and, since $z \ge u \ge \psi$, we get $z \in \mathbb{K}_\psi^\varphi$.

Let $h > 0$. Putting in (31) $w = z - T_h((z-u)^+)$ (note that $w \in \mathbb{K}_u$) we obtain

$$\langle Az, T_h((z-u)^+)\rangle \le \langle F \wedge A\varphi, T_h((z-u)^+)\rangle$$

while choosing $v = u + T_h((z-u)^+)$ in the inequality (30) (note that $v \in \mathbb{K}_\psi^\varphi$, since $u, z \in \mathbb{K}_\psi^\varphi$) we deduce

$$-\langle Au, T_h((z-u)^+)\rangle \le -\langle F, T_h((z-u)^+)\rangle.$$

Adding the last two inequalities we have

$$\langle A(z-u), T_h((z-u)^+)\rangle \le \langle (F \wedge A\varphi) - F, T_h((z-u)^+)\rangle \le 0$$

which implies $z \le u$. □

***Remark* 8** It is well known, since [9] for the one obstacle problem and [19] for the two obstacles problem, that some regularity results on the solution of the obstacle problems $(A, \mathbb{K}, F)$ can be obtained as a consequence of the Lewy–Stampacchia inequalities as they are a direct consequence of the regularity results already known for the solutions of the Dirichlet problem.

For instance, let us focus on the lower obstacle problem under the hypotheses (1), (2) with $F = f$ and the obstacle satisfying

$$f, (A\psi - f)^+ \in L^\rho(\Omega), \quad \rho \ge \frac{2N}{N+2}, \tag{32}$$

and let u be the solution of the obstacle problem $(A, \mathbb{K}_\psi, f)$. Then the inequality (11) becomes

$$f \le Au \le f \vee A\psi = f + (A\psi - f)^+ \quad \text{a.e. in } \Omega \tag{33}$$

and u solves the Dirichlet problem for some $h = h(x,u)$

$$u \in H_0^1(\Omega) : \; Au = Lu + D \cdot (uE) = h \in L^\rho(\Omega).$$

Thus, by the linear theory, we may conclude (see [1]):

(i) if $\frac{2N}{N+2} < \rho < \frac{N}{2}$ then $u \in L^{\rho^{**}}(\Omega)$, where $\rho^{**} = \frac{N\rho}{N-2\rho}$;
(ii) if $\rho > \frac{N}{2}$ and $E \in (L^\mu(\Omega))^N$, $\mu > N$, then $u \in L^\infty(\Omega)$.

***Remark* 9** Under the assumption (32) the Lewy–Stampacchia inequality (33) can be obtained through an alternative proof based on an approximation based in the bounded penalization with a family of monotone increasing Lipschitz functions ϑ_n introduced in [9], for instance with $\vartheta_n(s) = 0$ for $s \le 0$ and $\vartheta_n(s) = 1$ for $s \ge 1/n$ (see [15], for instance) or with a homographic function $\vartheta_n(s) = \frac{s}{|s|+1/n}$ (see [5]). This method, both of theoretical and numerical interest, represents an alternative to the classical unbounded penalty method.

We set

$$g(x) = (A\psi - f)^+ = (f - A\psi)^-. \tag{34}$$

For each $n \in \mathbb{N}$, let u_n be a weak solution of the following approximating Dirichlet problem: $u_n \in H_0^1(\Omega)$,

$$Lu_n + D \cdot \left(\frac{u_n - \psi}{1 + \frac{1}{n}|u_n - \psi|} E + \psi E \right) = f + g\,(1 - \vartheta_n(u_n - \psi)). \tag{35}$$

The existence of u_n follows by the well known results by [8, 11], since, for every n, the nonlinear composition is bounded with respect to u_n.

Given $h > 0$ let us take $T_h((u_n - \psi)^-)$ as test function in the weak formulation of (35) and we have

$$\langle L(u_n - \psi), T_h((u_n - \psi)^-)\rangle + \int_\Omega g\,\vartheta_n(u_n - \psi)T_h((u_n - \psi)^-)$$
$$= \langle f + g - L\psi - D\cdot(\psi E), T_h((u_n - \psi)^-)\rangle + \int_\Omega \frac{u_n - \psi}{1 + \frac{1}{n}|u_n - \psi|} E \cdot DT_h((u_n - \psi)^-).$$

Since

$$\int_\Omega g\,\vartheta_n(u_n - \psi)T_h((u_n - \psi)^-) \le 0,$$

and

$$\langle f + g - L\psi - D\cdot(\psi E), T_h((u_n - \psi)^-)\rangle \ge 0$$

we obtain

$$\langle L(u_n - \psi), T_h((u_n - \psi)^-)\rangle \ge \int_\Omega \frac{u_n - \psi}{1 + \frac{1}{n}|u_n - \psi|} E \cdot DT_h((u_n - \psi)^-)$$

that is

$$\langle LT_h((u_n - \psi)^-), T_h((u_n - \psi)^-)\rangle \le \int_\Omega \frac{T_h((u_n - \psi)^-)}{1 + \frac{1}{n}|u_n - \psi|} E \cdot DT_h((u_n - \psi)^-).$$

Now, working as in the proof of Theorem 2 we get $(u_n - \psi)^- = 0 \quad$ a.e. in Ω. Consequently, u_n being a weak solution of the Dirichlet problem, the positivity of g yields, in the sense of distributions, the following inequalities

$$f \le Lu_n + D\cdot\left(\frac{u_n - \psi}{1 + \frac{1}{n}(u_n - \psi)} E + \psi E\right) \le f + g\,. \tag{36}$$

The proof of the boundedness of the sequence $\{u_n\}$ in $H_0^1(\Omega)$ can be carried on as in the proof of Theorem 1 (see *step 2*). Thus, there exists a subsequence, still denoted by u_n and a function u such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in} H_0^1(\Omega) \\ u_n \to u & \text{strongly in} L^2(\Omega) \text{and a.e. in} \Omega. \end{cases} \tag{37}$$

Note that $u \in \mathbb{K}_\psi$; moreover, letting $n \to +\infty$ in the distributional sense in (35) and in the inequalities (36), we deduce that u is the solution of the problem (4) and satisfies the Lewy–Stampacchia estimate

$$f \le Lu + D\cdot(uE) \le f + g \quad \text{a.e. in } \Omega.$$

4 Proof of the Mosco convergence

This section is devoted only to the proof of Theorem 5, since the proof of Theorem 6 can be performed exactly in the same way.

Let u_n and u_0 be the solutions of the obstacle problems $(A, \mathbb{K}_{\psi_n}, F)$ and $(A, \mathbb{K}_{\psi_0}, F)$, respectively, that is

$$u_n \in \mathbb{K}_{\psi_n},\ \langle Lu_n, u_n - v\rangle \leq \int_\Omega u_n\, E \cdot D(u_n - v) + \langle F,\, u_n - v\rangle,\quad \forall v \in \mathbb{K}_{\psi_n} \tag{38}$$

$$u_0 \in \mathbb{K}_{\psi_0},\ \langle Lu_0, u_0 - v\rangle \leq \int_\Omega u_0\, E \cdot D(u_0 - v) + \langle F,\, u_0 - v\rangle,\ \forall v \in \mathbb{K}_{\psi_0}. \tag{39}$$

Working as in the proof of Theorem 1 (step 1 and step 2) we deduce that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega)$; thus, there exists a subsequence, which we denote also by $\{u_n\}$, which weakly converges to some u_* in $H_0^1(\Omega)$. Since $u_n \in \mathbb{K}_{\psi_n}$, the condition (ii) of the definition of Mosco convergence implies that $u_* \in \mathbb{K}_{\psi_0}$, that is $u_* \geq \psi_0$, a.e. in Ω.

Now, we claim that

$$\{u_n^2\} \text{ converges weakly to } u_*^2 \text{ in } W_0^{1,q}(\Omega) \text{ with } q = \frac{N}{N-1}. \tag{40}$$

We note that, the sequence $\{u_n^2\}$ is bounded in $W_0^{1,q}(\Omega)$. Thus, up another subsequence, we can say that $\{u_n^2\}$ weakly converges in $W_0^{1,q}(\Omega)$ to some $w \in W_0^{1,q}(\Omega)$. Let $\Phi \in (L^\sigma(\Omega))^N$, $\sigma > q'$. We have

$$\frac{1}{2}\int_\Omega D(u_n^2) \cdot \Phi = \int_\Omega u_n D u_n \cdot \Phi.$$

Since u_n strongly converges to u in $L^\mu(\Omega)$, $\mu < 2^*$, and Du_n weakly converges to Du in $(L^2(\Omega))^N$, we may to pass to the limit and we deduce that

$$\frac{1}{2}\int_\Omega Dw \cdot \Phi = \int_\Omega u_* D u_* \cdot \Phi$$

and then $w = u_*^2$. Moreover

$$\int_\Omega u_n\, E \cdot Du_n \to \int_\Omega u_*\, E \cdot Du_*. \tag{41}$$

Now we need to prove that $u_* = u_0$. By (i) of the definition of Mosco-convergence, for every $w_0 \in \mathbb{K}_{\psi_0}$, there exists a sequence $\{w_n\}$, with $w_n \in \mathbb{K}_{\psi_n}$ and $\|w_n - w_0\|_{H_0^1(\Omega)} \to 0$. We take $v = w_n$ as test function in (38) and we have

$$\langle Lu_n, u_n - w_n\rangle \leq \int_\Omega u_n\, E \cdot D(u_n - w_n) + \langle F,\, u_n - w_n\rangle.$$

Then

$$\begin{aligned}&\langle L(u_n - w_n), u_n - w_n\rangle + \langle Lw_n, u_n - w_n\rangle \\ &\quad \le \int_\Omega u_n E\cdot Du_n - \int_\Omega u_n E\cdot Dw_n + \langle F, u_n - w_n\rangle.\end{aligned}$$

We pass to the limit thanks to the weak lower semicontinuity of the quadratic form $\langle Lv, v\rangle$, (41) and $\|w_n - w_0\|_{H_0^1(\Omega)} \to 0$. Thus we obtain

$$\langle Lu_*, u_* - w_0\rangle \le \int_\Omega u_* E\cdot D(u_* - w_0) + \langle F, u_* - w_0\rangle.$$

Then the uniqueness of the solution of the unilateral problem on the convex K_{ψ_0} implies that $u_* = u_0$ and the full sequence $\{u_n\}$ weakly converges to u_0.

Finally, we prove that the sequence $\{u_n\}$ strongly converges to u_0. Since $u_0 \in \mathbb{K}_{\psi_0}$, due to the condition (i) of the definition of Mosco convergence, there exists a sequence $\{z_n\}$, with $z_n \in \mathbb{K}_{\psi_n}$, $\|z_n - u_0\|_{H_0^1(\Omega)} \to 0$, . We can take $v = z_n$ in (38) and we have

$$\begin{aligned}&\langle L(u_n - z_n), u_n - z_n\rangle \\ &\quad \le -\langle Lz_n, u_n - z_n\rangle + \int_\Omega u_n E\cdot Du_n - \int_\Omega u_n E\cdot Dz_n + \langle F, u_n - z_n\rangle,\end{aligned}$$

where the right hand side converges to zero also thanks (41). Thus we have

$$\alpha \limsup \|u_n - z_n\|^2_{H_0^1(\Omega)} \le 0,$$

which says that the sequence $\{u_n\}$ converges strongly in $H_0^1(\Omega)$ to the same limit of the sequence $\{z_n\}$; that is $\{u_n\}$ converges strongly in $H_0^1(\Omega)$ to u_0. □

5 Application to quasi-variational inequalities of obstacle type

In this section we prove Theorem 7 and we provide an example of application inspired in a semiconductor model (see [7, 16] for references of the physical free boundary problem).

***Proof of Theorem* 7** We limit ourselves to the two obstacles problem, since the other two cases are similar.

For any $w \in L^2(\Omega)$, denote by $u_w = S(w)$ the unique solution to $(A, \mathbb{K}_\psi^\varphi, F)$. By the Theorem 1, it is clear that the solution map $S : L^2(\Omega) \mapsto \mathbb{K}_{\Psi(w)}^{\Phi(w)} \subset H_0^1(\Omega)$ is well defined and its range is bounded, since the ranges of Ψ and of Φ are also bounded in $H_0^1(\Omega)$ by hypothesis.

Since Ψ and of Φ are also continuous and compatible, i.e. $\Psi(w) \le \Phi(w)$ for all $w \in L^2(\Omega)$, if we take a sequence $w_n \to w$ in $L^2(\Omega)$, by Theorem 6 we have $\mathbb{K}_{\Psi(w_n)}^{\Phi(w_n)} \xrightarrow{M} \mathbb{K}_{\Psi(w)}^{\Phi(w)}$ and consequently $S(w_n) \to S(w)$ in $H_0^1(\Omega)$.

Therefore the solution map S is also continuous from $L^2(\Omega)$ into $H_0^1(\Omega)$. Finally, since the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, the image of S, $S(L^2(\Omega))$, is not only bounded but in fact also compact in $L^2(\Omega)$. Then by the Schauder fixed point theorem, there is a $u = S(u) \in \mathbb{K}^{\Phi(u)}_{\Psi(u)}$ solving (18) for the implicit two obstacles problem. □

A simplified variant of a model with a free boundary for the determination of the depletion zone in certain semiconductor diodes, which is the non coincidence set of a two obstacles problem, corresponds to an asymptotic coupled system (44)–(45)–(46) below for the electrostatic potential $u = u(x)$, subjected to a drift by the vector field $E = E(x)$, $x \in \Omega$, and lying in between two Fermi quasi-potentials $\psi = \psi(x)$ and $\varphi = \varphi(x)$, which are both functions depending implicitly of the potential u through two Dirichlet problems depending on a function $v \in L^2(\Omega)$ of the following type:

$$w \in H^1_0(\Omega), \qquad -D \cdot (B(v)\, Dw) = G \quad \text{in } \Omega. \tag{42}$$

Here the coefficients $B(v)(x) = B(x, v(x))$ are given by a Carathéodory matrix $B(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, i.e., it is measurable in x for each $s \in \mathbb{R}$ and continuous in s for a.e. $x \in \Omega$, satisfying for positive constants α^* and β^*:

$$\alpha^* |\xi|^2 \le B(x,s)\xi \cdot \xi, \quad |B(x,s)| \le \beta^*, \quad \text{a.e.} \quad x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N. \tag{43}$$

It is clear that for $G \in H^{-1}(\Omega)$ and any $v \in L^2(\Omega)$ the problem (42) has a unique solution $w = T(v, G) \in H^1_0(\Omega)$. Fixing G, we have the continuity of the map $T_G : L^2(\Omega) \ni v \mapsto w = T(v, G) \in H^1_0(\Omega)$ for the strong topologies. Indeed $v_n \to v$ in $L^2(\Omega)$, then by (43) and Lebesgue theorem, $B_n = B(v_n) \to B(v) = B$ in $L^p(\Omega)^{N^2}$, for all $1 \le p < \infty$, and for a subsequence also a.e. in Ω. Since by (43) we have $\alpha^* \|Dw_n\|_{L^2} \le \|G\|_{H^{-1}}$, we may assume that $w_n \rightharpoonup w_*$ in $H^1_0(\Omega)$-weak and the strong convergence of $B_n \to B$ clearly implies $w_* = w$.

Also by Lebesgue theorem, $\|(B_n - B)Dw\|_{L^2} \to 0$, and comparing the equations (42) in variational form for $w_n = T(v_n, G)$ and for $w = T(v, G)$, by addition, we obtain

$$\int_\Omega B_n\, D(w - w_n) \cdot D(w - w_n) = \int_\Omega (B - B_n)\, Dw \cdot D(w_n - w),$$

which by (43) yields

$$\alpha^* \|D(w - w_n)\|^2_{L^2} \le \int_\Omega (B - B_n)\, Dw \cdot D(w_n - w) \;\to 0$$

and therefore $w_n \to w$ in $H^1_0(\Omega)$-strong. So the continuity of T_G holds. By the comparison principle, for any fixed $v \in L^2(\Omega)$, and given $G, H \in H^{-1}(\Omega)$, such that

$$G \le H \in H^{-1}(\Omega), \quad \text{i.e.} \quad \langle G - F,\, z \rangle \le 0 \qquad \forall z \in H^1_0(\Omega),$$

it is clear that $\psi' = T(v, G) \le \varphi' = T(v, G)$ a.e. in Ω.

Now we can formulate the following implicit obstacle problem:

$$u \in \mathbb{K}_{\psi(u)}^{\varphi(u)} : \quad \langle Lu - F, u - v\rangle \le \int_\Omega uE \cdot D(u - v) \qquad \forall v \in \mathbb{K}_{\psi(u)}^{\varphi(u)}, \tag{44}$$

where $\psi = \psi(u)$ and $\varphi = \varphi(u)$ are given by

$$\psi \in H_0^1(\Omega) : \int_\Omega B(u)\, D\psi \cdot Dv = \langle G, v\rangle, \qquad \forall v \in H_0^1(\Omega), \tag{45}$$

$$\varphi \in H_0^1(\Omega) : \int_\Omega B(u)\, D\varphi \cdot Dv = \langle H, v\rangle \qquad \forall v \in H_0^1(\Omega). \tag{46}$$

With the considerations above it is clear that, exactly as in the Theorem 7, we have proved the following result.

Corollary 4 *Assuming* (1), (2) *and* (43), *for any* $F, G, H \in H^{-1}(\Omega)$ *with* $G \le H$, *there exists at least a solution* $(u, \psi, \varphi) \in [H_0^1(\Omega)]^3$ *solving the coupled system* (44)–(45)–(46).

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