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NEW NON-ADDITIVE INTEGRALS IN MULTIPLE CRITERIA DECISION ANALYSIS

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Preface

Decision has inspired reflection of many thinkers since the ancient times. Often decision is strongly related to the comparison of different points of view, some in favor and some against a certain decision. This means that decision is intrinsically related to a plurality of points of view, which technically are defined criteria. Contrary to this very natural observation, for many years the only way to state a decision problem was considered to be the definition of a single criterion, which amalgamates the multidimensional aspects of the decision situation into a single scale of measure. For example, even today this approach can be found in any textbooks of Operations Research. This is a very reductive, and in some sense also unnatural, way to look at a decision problem. Thus, for at least thirty years, a new way to look at decision problems has more and more gained the attention of researchers and practitioners. This approach explicitly takes into account the pros and the cons of a plurality of points of view, in other words the domain of Multiple Criteria Decision Analysis (MCDA). Therefore, MCDA intuition is closely related to the way humans have always been making decisions. Consequently, despite the diversity of MCDA approaches, methods and techniques, the basic ingredients of MCDA are very simple: a finite or infinite set of actions (alternatives, solutions, courses of action, ...), at least two criteria, and, obviously, at least one decision-maker (DM). Given these basic elements, MCDA is an activity which helps making decisions mainly in terms of choosing, ranking or sorting the actions.

MCDA is not just a collection of theories, methodologies, and techniques, but a specific perspective to deal with decision problems. Losing this perspective, even the most rigorous theoretical developments and applications of the most refined methodologies are at risk of being meaningless, because they miss an adequate consideration of the aims and of the role of MCDA. A fundamental problem of MCDA is the representation of preferences. Classically, for example in economics, it is supposed that preference can be represented by a utility function assigning a numerical value to each action such that the more preferable an action, the larger its numerical value. Moreover, it is very often assumed that the comprehensive evaluation of an action can be seen as the sum of its numerical values for the considered criteria. Let us call this the classical model. It is very simple but not too realistic. Indeed, there is a lot of research studying under which conditions the classical model holds. These conditions are very often quite strict and it is not reasonable to assume that they are satisfied in all real world situations. In the last years many non-classical approaches have been proposed in MCDA. This thesis focuses on MCDA methods based on fuzzy integrals. These methods are known in MCDA for the last two decades. In very simple words this methodology permits a flexible modeling of the importance of criteria. Indeed, fuzzy integrals are based on a capacity which assigns an importance to each subset of criteria and not only to each single criterion. Thus, the importance of a given set of criteria is not necessarily equal to the sum of the importance of the criteria from the considered subset. Consequently, if the importance of the whole subset of criteria is smaller than the sum of the importances of its individual criteria, then we observe an average redundancy between criteria, which in some way represents overlapping points of view. On the other hand, if the importance of the whole subset of criteria is larger than the sum of the importances of its members, then we observe an average synergy between criteria, the evaluations of which reinforce one another. On the basis of the importance of criteria measured by means of a capacity, the criteria are aggregated by means of specific fuzzy integrals, the most important of which are the Choquet integral (for cardinal evaluations) and the Sugeno integral (for ordinal evaluations).

The proposal and the axiomatization of new fuzzy integrals has a central role in modern MCDA. In this thesis we propose some generalizations of well known fuzzy integrals (Choquet, Shilkret and Sugeno). This thesis is thought to make each chapter independent of the others, so they can be read in any order or selected to suit different interests. No general conclusion are given since any chapter contains proper conclusions.

Chapter 1 is a brief survey of the methodology based on fuzzy integrals in MCDA. In chapter 2 we propose and characterize bipolar fuzzy integrals, which are generalization of the most famous fuzzy integrals to the case of bipolar scale, i.e. those symmetric scale where it is possible for each value to find the opposite. Cardinal bipolar scales are intervals $[-a, a]$, $]-\infty, +\infty[$, while an example of an ordinal bipolar scale is: *very bad, bad, neutral, good, very good*. In chapter 3 we deal with the generalization of the concept of universal integral (recently proposed to generalize several fuzzy integrals) to the case of bipolar scales. We also provide the characterization of the bipolar universal integral with respect to a level dependent bi-capacity. Finally, in chapter 4 we consider the problem to adapt classical definitions of fuzzy integrals to the case of imprecise interval evaluations. More precisely, standard fuzzy integrals used in MCDA request that the starting evaluations of a choice on various criteria must be expressed in terms of exact-evaluations. In this last chapter we present the *robust* Choquet, Shilkret and Sugeno integrals, computed with respect to an *interval capacity*. These are quite natural generalizations of the Choquet, Shilkret and Sugeno integrals, useful to aggregate *interval-evaluations* of choice alternatives into a single overall evaluation. We show that, when the interval-evaluations collapse into exact-evaluations, our definitions of robust integrals collapse into the previous definitions. We also provide an axiomatic characterization of the robust Choquet integral. The approach of robust integral promises interesting developments for future researches, this further improvement is based to the generalization of the concept of interval to $h-$interval. We shall close the thesis by briefly discussing this last approach.

Contents

Chapter 1

Fuzzy measures and integrals in MCDA

Grabisch and Labreuche have exhaustively discussed the use of fuzzy measures and integrals in MCDA in literature [16, 17], to which we refer for this chapter.

The aim of MCDA is to model the preferences of a Decision Maker (DM) over a set of possible alternatives $X = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots\}$ described by several points of view, called criteria $N = \{1, 2, \ldots, n\}$. Thus, an alternative $\boldsymbol{x}$ is characterized by an evaluation $x_i \in X_i,\ i = 1, \ldots, n$ (not necessarily numerical) w.r.t. each point of view and can be identified with a score vector $\boldsymbol{x} = (x_1, \ldots, x_n)$. We denote by $\succeq$ the preference relation of the DM over alternatives, then $\boldsymbol{x} \succeq \boldsymbol{y}$ means that the DM prefers the alternative $\boldsymbol{x}$ to $\boldsymbol{y}$. In order to come up with the knowledge of $\succeq$ on $X \times X$, some informations must be elicited from the DM. This elicitation process should request a relatively small amount of questions asked to the DM. The DM provides informations by means of examples of comparisons between alternatives, as well as more qualitative judgments. A *numerical representation* [33] of the preference $\succeq$ is obtained whenever there exists a function $u : X \to \mathbb{R}$ such that

$$\forall \boldsymbol{x}, \boldsymbol{y} \in X, \qquad \boldsymbol{x} \succeq \boldsymbol{y} \quad iff \quad u(\boldsymbol{x}) \geq u(\boldsymbol{y}). \tag{1.1}$$

We focus on a special model of (1.1) called *decomposable* [29] given by:

$$u(\boldsymbol{x}) = F(u_1(x_1), \ldots, u_n(x_n)), \tag{1.2}$$

where the u_i are the *utility functions* and $F : \mathbb{R}^n \to \mathbb{R}$ is an *aggregation function.* Krantz et al. [33] (see also [23]) gave the axioms that characterize the representation of $\succeq$ by (1.2). The weighted sums $F(u_1, \ldots, u_n) = \sum_1^n \alpha_i u_i$ are the most classical functions used to aggregate the criteria. These family of aggregation operators are characterized by an independence axiom [29, 49]. This property implies some limitations in the way the weighted sum can model typical decision behaviors. To make this more precise, we shall provide an example. The construction of the utility functions and the determination of the parameters of the aggregation function are often carried out in two separate steps.

The determination of the utility function is also concerned with commensurateness between criteria, i.e. the possibility to compare any element of one point of view with any element of any other point of view. This is inter-criteria comparability:

For $x_i \in X_i$ and $x_j \in X_j$, we have $u_i(x_i) \geq u_j(x_j)$ iff x_i is considered at least as good as x_j by the DM.

This assumption is very strong. By the way of an example, assuming as criteria to buy a car consumption and maximal speed, the DM should be able to say if she prefers a consumption of 5 liters/100km to a maximum speed of 200 km/h. This does not generally make sense to the DM, so that he or she is not generally able to make this comparison directly.

1.1 Notion of Interaction – A Motivating Example

In [16] the authors provide the following example to explain the importance of interaction of criteria and to show some flaws of the weighted sum. The director of a university decides on students who are applying for graduate studies in management where some prerequisites from school are required. Students are indeed evaluated according to mathematics, statistics and language skills. All the marks with respect to the scores are given on the same scale from 0 to 20. These three criteria serve as a basis for a preselection of the candidates. The applicants have generally speaking a strong scientific background so that mathematics and statistics have a big importance to the director. However, he does not wish to favor too much students that have a scientific profile with some flaws in languages. Besides, mathematics and statistics are in some sense *redundant*, since, usually, students good at mathematics are also good at statistics. As a consequence, for students good in mathematics, the director prefers a student good at languages to one good at statistics. Consider the following four students

	Mathematics	Statistics	Language
student A	16	13	7
student B	16	11	9
student C	6	13	7
student D	6	11	9

Student A is highly penalized by his performance in languages. Henceforth, the director would prefer the student B which has the same mark in mathematics but is a little bit better in languages even if he is a little bit worse in statistics. Consider now students C and D. Both of them have a weakness in mathematics. In this case, since the applicants are supposed to have strong scientific skills, the student C which is good in statistics is now preferred to the student D, good in languages. The director preferences, $B \geq A$ and $C \geq D$, lead to the following requirement

$$F(16, 13, 7) < F(16, 11, 9) \quad \text{and} \quad F(6, 13, 7) > F(6, 11, 9).$$

No weighted sum can model such preferences, since the preference of B over A implies that languages is more important than statistics whereas the preference of C over D tells exactly the contrary.

1.2 Capacities and Choquet Integral

The above example suggests that to explain the director preferences we should assign weights not only to the single criteria, but also to the *coalitions* (i.e. groups, subsets) of criteria. This can be achieved by introducing particular functions on $\mathcal{P}(N)$, called fuzzy measures [47] or capacities [43]. A fuzzy measure or capacity is a set function $\mu : 2^N \to \mathbb{R}$ such that $\mu(\emptyset) = 0$, $\mu(N) = 1$ and satisfying the monotonicity condition: if $A \subseteq B$, then $\mu(A) \leq \mu(B)$, for all $A, B \in \mathcal{P}(N)$. The capacity is said to be *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$, whenever $A \cap B = \emptyset$. The Choquet integral [7] of $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ w.r.t. a capacity μ has the following expression :

$$Ch(\boldsymbol{x}, \mu) = \int_{-\infty}^{0} [\mu\{i \mid x_i \geq t\} - 1] dt + \int_{0}^{\infty} \mu\{i \mid x_i \geq t\}. \tag{1.3}$$

Note that when the capacity is additive, the Choquet integral reduces to a weighted sum. The preference of the DM are modeled via the Choquet integral if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in X, \qquad \boldsymbol{x} \geq \boldsymbol{y} \quad iff \quad Ch(\boldsymbol{x}, \mu) \geq Ch(\boldsymbol{y}, \mu). \tag{1.4}$$

Obviously in the (1.4) and according with the (1.2) we have supposed that the component x_i of the vector $\boldsymbol{x}$ are expressed n terms of utility. It is easy to see that the use of the (1.4) applied to the above example of student evaluation allows for a simple explanation. Indeed, the preferences of the director correspond to 2μ(Mat,Sta)> μ(Mat) + 1 and 2μ(Stat) > μ(Stat,Lang). There is no contradiction between previous two inequalities, hence the Choquet integral can model the preferences of the DM. For other properties and characterizations of the Choquet integral, we refer the reader to survey papers [35].

1.3 Conclusions

In the last thirty years several non-additive fuzzy integrals have been developed in MCDA. We recall the Choquet and Shilkret integral (for the cardinal case) and the Sugeno integral (for the ordinal case) among others. In this chapter we have described the Choquet integral in order to show its potentiality in the context of MCDA. In the next chapters we shall present other fuzzy integrals together with their relevant, old and new, generalizations.

Chapter 2

Bipolar Fuzzy Integrals

2.1 Introduction

The basic reference for this chapter is [25]. In decision analysis and especially in multiple criteria decision analysis, several non additive integrals have been introduced in the last sixty years [8, 10, 16]. Among them, we remember the Choquet integral [7], the Shilkret integral [45] and the Sugeno integral [47]. Recently the bipolar Choquet integral [14, 15, 22] has been proposed for the case in which the underlying scale is bipolar. A further generalization is that of level dependent integrals, which has lead to the definition of the level dependent Choquet integral [21], the level dependent Shilkret integral [4], the level dependent Sugeno integral [37] and the bipolar level dependent Choquet integral [21]. Very recently, on the basis of a minimal set of axioms, one concept of universal integral giving a common framework to many of the above integrals have been proposed [32]. In this chapter we aim to provide a general framework for the case of bipolar fuzzy integrals, i.e. those integrals whose underlying scale is bipolar. For this purpose we propose the definition of bipolar Shilkret integral and bipolar Sugeno integral. Then, in order to provide a mathematical characterization of the three mentioned bipolar integrals, we give necessary and sufficient conditions for an aggregation function to be the bipolar Choquet integral or the bipolar Shilkret integral or the bipolar Sugeno integral. As we said, the bipolar fuzzy integrals admit a further generalization if the fuzzy measure (capacity) with respect to which the integrals are calculated can change from a level to another [21, 20]. For the sake of clarity, we shall remind the characterization of the bipolar Shilkret and Sugeno integral with respect to a level dependent capacity in a forthcoming paper (we wish to remember as such results have just been presented in [20]). The chapter is organized as follows. In section 2.2 we give the preliminaries and list some properties of an aggregation function useful to the characterization of the bipolar fuzzy integrals we shall propose in this chapter. In section 2.3 we review the definitions and characterizations of the classical Choquet integral, Shilkret integral, Sugeno integral and some of their symmetric extensions on a bipolar scale. In section 2.4 we give our main results: first we propose the bipolar version of the Shilkret integral and of the Sugeno integral; next we characterize the bipolar Choquet, Shilkret and Sugeno integrals. All the proofs are presented in section 2.5. Section 2.6 contains conclusions.

2.2 Preliminaries

Let us consider a set of criteria $N = \{1, \dots, n\}$ and let us suppose that the range of evaluation of given criteria is a real numbers interval $\mathcal{I}$. We denote $\alpha = \inf \mathcal{I}$ and $\beta = \sup \mathcal{I}$. An *alternative* can be identified with a score vector $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathcal{I}^n$, being x_i the evaluation of such an alternative $\boldsymbol{x}$ with respect to the i^{th} criterion. An alternative $\boldsymbol{x}$ dominates another $\boldsymbol{y}$ if on each criterion the evaluation of $\boldsymbol{x}$ is not smaller than the evaluation of $\boldsymbol{y}$, i.e. for all $i \in N$, $x_i \geq y_i$ and in this case we simply write $\boldsymbol{x} \geq \boldsymbol{y}$. The indicator function of any $A \subseteq N$ is the function which attains 1 on A and 0 on $N \setminus A$ and can be identified with the vector $\mathbf{1}_A$ whose i^{th} component is equal to 1 if $i \in A$ and 0 otherwise.

In general, an aggregation function is a function $G : \mathcal{I}^n \to \mathcal{I}$ such that

1. $G(\alpha, \dots, \alpha) = \alpha$ if $\alpha \in \mathcal{I}$ and $\lim_{x \to \alpha^+} G(x, \dots, x) = \alpha$ if $\alpha \notin \mathcal{I}$;

2. $G(\beta, \dots, \beta) = \beta$ if $\beta \in \mathcal{I}$ and $\lim_{x \to \beta^-} G(x, \dots, x) = \beta$ if $\beta \notin \mathcal{I}$;

3. for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ such that $\boldsymbol{x} \geq \boldsymbol{y}$, $G(\boldsymbol{x}) \geq G(\boldsymbol{y})$.

In this chapter we often denote the maximum and the minimum of a set X respectively with $\bigvee X$ and $\bigwedge X$. For any two alternatives $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, the following definitions hold

- $\boldsymbol{x} \wedge \boldsymbol{y}$ is the vector whose i^{th} component is $(x \wedge y)_i = \bigwedge \{x_i, y_i\}$ for all $i = 1, \dots, n$ (in case $\boldsymbol{y} = (h, \dots, h)$ is a constant, then we can write $\boldsymbol{x} \wedge h$);

- $\boldsymbol{x} \vee \boldsymbol{y}$ is the vector whose i^{th} component is $(x \vee y)_i = \bigvee \{x_i, y_i\}$ for all

$i = 1, \ldots, n$ (in case $\boldsymbol{y} = (h, \ldots, h)$ is a constant, then we can write $\boldsymbol{x} \vee h$);

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are comonotone (or comonotonic) if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j \in N$;

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are bipolar comonotone if $(|x_i| - |x_j|)(|y_i| - |y_j|) \geq 0$ and $x_i y_i \geq 0$, for all $i, j \in N$.

The following properties of an aggregation function $G : \mathcal{I}^n \to \mathcal{I}$ are useful to characterize several integrals:

- idempotency: for all $\mathbf{a} \in \mathcal{I}^n$ such that $\mathbf{a} = (a, \ldots, a)$, $G(\mathbf{a}) = a$;

- homogeneity: for all $\boldsymbol{x} \in \mathcal{I}^n$ and $c > 0$ such that $c \cdot \boldsymbol{x} \in \mathcal{I}^n$, $G(c \cdot \boldsymbol{x}) = c \cdot G(\boldsymbol{x})$;

- stability w.r.t. the minimum: for all $\boldsymbol{x} \in \mathcal{I}^n$ and $\gamma \in \mathcal{I}$, $G(\boldsymbol{x} \wedge \gamma) = \bigwedge\{G(\boldsymbol{x}), \gamma\}$;

- additivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ such that $\boldsymbol{x}+\boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x}+\boldsymbol{y}) = G(\boldsymbol{x})+G(\boldsymbol{y})$;

- maxitivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, with $\alpha \geq 0$, $G(\boldsymbol{x} \vee \boldsymbol{y}) = \bigvee\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$;

- minitivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, with $\beta \leq 0$, $G(\boldsymbol{x} \wedge \boldsymbol{y}) = \bigwedge\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$;

- comonotonic additivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y})$;

- comonotonic maxitivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} \vee \boldsymbol{y}) = \bigvee\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$;

- comonotonic minitivity: for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$, $G(\boldsymbol{x} \wedge \boldsymbol{y}) = \bigwedge\{G(\boldsymbol{x}), G(\boldsymbol{y})\}$;

2.3 Fuzzy integrals

In this section we briefly review the three most famous fuzzy integrals, i.e. the Choquet, Shilkret and Sugeno integrals and some of their symmetric extensions. For each of them we shall discuss the restrictions to be imposed on the scale $\mathcal{I}$.

2.3.1 The Choquet integral

Definition 1. *A capacity (or fuzzy measure) is function $\mu : 2^N \to [0, 1]$ satisfying the following properties:*

1. $\mu(\varnothing) = 0$, $\mu(N) = 1$,
2. *for all* $A \subseteq B \subseteq N$, $\mu(A) \leq \mu(B)$.

Definition 2. *The Choquet integral* [7] *of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n \subseteq [0, +\infty[^n$ *with respect to the capacity μ is given by*

$$Ch(\boldsymbol{x}, \mu) = \int_0^\infty \mu(\{i \in N : x_i \geq t\})\, dt. \tag{2.1}$$

Schmeidler [43] extended the above definition to negative values too, moreover he characterized the Choquet integral in terms of comonotonic additivity and idempotency.

Definition 3. [43] *The Choquet integral of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n$ *with respect to the capacity* μ *is given by*

$$Ch(\boldsymbol{x}, \mu) = \int_{-\infty}^{0} (\mu(\{i \in N : x_i \geq t\}) - 1)\, dt + \int_{0}^{\infty} \mu(\{i \in N : x_i \geq t\})\, dt. \quad (2.2)$$

Alternatively (2.2) can be written as [21]

$$Ch(\boldsymbol{x}, \mu) = \int_{\min_i x_i}^{\max_i x_i} \mu(\{i \in N : x_i \geq t\})\, dt + \min_i x_i. \quad (2.3)$$

Another formulation of (2.2) can be obtained, by using the summation, as

$$Ch(\boldsymbol{x}, \mu) = \sum_{i=2}^{n} \left(x_{\sigma(i)} - x_{\sigma(i-1)}\right) \cdot \mu\left(\{j \in N : x_j \geq x_{\sigma(i)}\}\right) \ + \ x_{\sigma(1)}, \quad (2.4)$$

being $\sigma : N \to N$ any permutation of indexes such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$.

Theorem 1. [43] *An aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent and comonotone additive if and only if there exists a capacity* μ *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Ch(\boldsymbol{x}, \mu).$$

The Šipoš integral [46] (or symmetric Choquet integral) of $\boldsymbol{x} \in \mathcal{I}^n$ with respect to the capacity μ is defined by

$$\check{\mathrm{C}}\mathrm{h}(\boldsymbol{x}, \mu) = Ch(\boldsymbol{x} \vee 0, \mu) - Ch(-(\boldsymbol{x} \wedge 0), \mu). \quad (2.5)$$

More in general, a functional $L : \mathcal{I}^n \to \mathcal{I}$ is a rank and sign-dependent functional [39] if there exist two fuzzy measures μ^+ and μ^- such that for all $\boldsymbol{x} \in \mathcal{I}^n$

$$L(\boldsymbol{x}) = Ch(\boldsymbol{x} \vee 0, \mu^+) - Ch(-(\boldsymbol{x} \wedge 0), \mu^-).$$

This functional is used in the cumulative prospect theory [48]. Clearly when $\mu^+ = \mu^-$, the rank and sign-dependent functional L is exactly the symmetric Choquet integral. For further details on the rank and sign-dependent functional and its use in cumulative prospect theory, we refer the reader to [48, 39]. We wish also to remember that Choquet integral is generalized and characterized in [2, 3].

2.3.2 The Shilkret integral

Definition 4. *The Shilkret integral* [45] *of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n \subseteq [0, +\infty[^n$ *with respect to the capacity* μ *is given by*

$$Sh(\boldsymbol{x}, \mu) = \bigvee_{i \in N} \{x_i \cdot \mu(\{j \in N : x_j \geq x_i\})\}. \tag{2.6}$$

A generalization of the Shilkret integral is introduced and characterized in [2, 3]. From the cited papers we can get a characterization of the Shilkret integral in terms of idempotency, comonotonic maxitivity and homogeneity. For the sake of completeness we report the proof of such a characterization (Theorem 2) in section 2.5.

Theorem 2. *Suppose that* $\alpha = \inf \mathcal{I} \geq 0$*, then an aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent, comonotone maxitive and homogeneous if and only if* *there exists a capacity* μ *on* N *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Sh(\boldsymbol{x}, \mu).$$

Although in [45] the Shilkret integral was formulated for nonnegative functions, however (2.6) works also for a generic $\boldsymbol{x} \in \mathcal{I}^n \subseteq \mathbb{R}^n$. But, in our opinion, if we allow for negative values too, the essence of the Shilkret integral is lost. Let us stress this point with some examples. Suppose that an alternative is strongly negatively evaluated on each criterion except on the last, where it has a low nonnegative evaluation, e.g. $\boldsymbol{x} = (-100, -100, -100, 1)$. By applying (2.6), $Sh(\boldsymbol{x}, \mu) = \mu(\{4\})$, for every capacity μ. Thus, the negative evaluations and the weights that the capacity assigns to the relative criteria with respect to which these negative evaluations are given, are ininfluent on the evaluation of $\boldsymbol{x}$. In general, if for a given alternative $\boldsymbol{x}$ we have simultaneously negative and positive evaluations on the various criteria, the negative ones are ininfluent and the Shilkret integral of $\boldsymbol{x}$ coincides with the Shilkret integral of $\boldsymbol{x} \vee 0$. In the case of $\boldsymbol{x} \in \,]-\infty, 0[^n$ it is straightforward noting that $Sh(\boldsymbol{x}, \mu) = (\max_{i \in N} x_i) \cdot \mu(\{j \in N \mid x_j \geq \max_{i \in N} x_i\})$. Again, we note how for all capacities only the maximum evaluation of $\boldsymbol{x}$ matters. For vectors with non-positive evaluation on each criterion, the logic of the Shilkret integral can be recovered if in the (2.6) we substitute the maximum with the minimum and $\geq$ with $\leq$.

Definition 5. *The negative Shilkret integral of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in$

$\mathcal{I}^n \subseteq \,]-\infty, 0]^n$ *with respect to the capacity* μ *is given by*

$$Sh^-(\boldsymbol{x}, \mu) = \bigwedge_{i \in N} \{x_i \cdot \mu(\{j \in N : x_j \le x_i\})\} =$$

$$-\bigvee_{i \in N} \{-x_i \cdot \mu(\{j \in N : -x_j \ge -x_i\})\} = -Sh(-\boldsymbol{x}, \mu). \tag{2.7}$$

Obviously, from theorem 2, the characterization of the negative Shilkret integral is in terms of idempotency, comonotonic minitivity and homogeneity.

Corollary 1. *Suppose that* $\beta = \sup \mathcal{I} \le 0$*, then an aggregation function* $G : \mathcal{I}^n \to \mathcal{I}$ *is idempotent, comonotone minitive and homogeneous if and only if there exists a capacity* μ *on* N *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$*,*

$$G(\boldsymbol{x}) = Sh^-(\boldsymbol{x}, \mu).$$

So far, we have a Shilkret integral for alternatives with all non-negative evaluations and one for alternatives with all non-positive evaluations. To obtain a suitable definition of the Shilkret integral for the mixed case we propose two different approaches. In the first approach we define a *symmetric Shilkret integral* by applying a logic à la Šipoš [46], i.e. for all $\boldsymbol{x} \in \mathcal{I}$

$$\check{\mathrm{S}}\mathrm{h}\,(\boldsymbol{x}, \mu) = Sh(\boldsymbol{x} \vee 0, \mu) + Sh^-(\boldsymbol{x} \wedge 0, \mu). \tag{2.8}$$

Note that the (2.8) is called symmetric since $\check{\mathrm{S}}\mathrm{h}\,(\boldsymbol{x}, \mu) = -\check{\mathrm{S}}\mathrm{h}\,(-\boldsymbol{x}, \mu)$. A second, more general, approach will be to define a *bipolar Shilkret integral* (see next section). This would be used directly for the bipolar scale, while restricted on $\mathbb{R}^+$ and on $\mathbb{R}^-$ it would coincide respectively with the Shilkret integral and the negative Shilkret integral.

2.3.3 The Sugeno integral

Definition 6. *A measure on N with a scale $\mathcal{I}$ is any function $\nu : 2^N \to \mathcal{I}$ such that:*

1. $\nu(\varnothing) = \alpha = \inf \mathcal{I},\ \nu(N) = \beta = \sup \mathcal{I}$,
2. *for all* $A \subseteq B \subseteq N,\ \nu(A) \leq \nu(B)$.

Definition 7. *The Sugeno integral* [47] *of a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n$ with respect to the measure ν on N with scale $\mathcal{I}$ is given by*

$$Su(\boldsymbol{x}, \nu) = \bigvee_{i \in N} \bigwedge \{x_i, \nu(\{j \in N \mid x_j \geq x_i\})\}. \tag{2.9}$$

Alternatively the Sugeno integral can be written as

$$Su(\boldsymbol{x}, \nu) = \bigvee_{A \subseteq N} \bigwedge \left\{\nu(A), \bigwedge_{i \in A} x_i\right\}. \tag{2.10}$$

Next theorem gives necessary and sufficient conditions for an aggregation function to be the Sugeno integral.

Theorem 3. [36] *An aggregation function $G : \mathcal{I}^n \to \mathcal{I}$ is idempotent, comonotone maxitive and stable with respect to the minimum if and only if there* *exists a measure* ν *on* N *with a scale* $\mathcal{I}$ *such that, for all* $\boldsymbol{x} \in \mathcal{I}^n$,

$$G(\boldsymbol{x}) = Su(\boldsymbol{x}, \nu).$$

Let us observe that the definition of the Sugeno integral only imposes that the x_i and the $\nu(A)$ are measured on the same (possible only ordinal) scale $\mathcal{I}$. For further generalization and characterization of the Sugeno integral see [2, 3].

Let us consider the symmetric scale $[-1, 1]$. The *symmetric maximum* of two elements $a, b \in [-1, 1]$ - introduced and discussed in [11, 12] - is defined by the following binary operation:

$$a \circledvee b = \begin{cases} -(|a| \vee |b|) & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases}$$

Alternatively the symmetric maximum can be written as

$$a \circledvee b = \text{sign}(a + b)(|a| \vee |b|).$$

The *symmetric minimum* of two elements [11, 12] is defined as:

$$a \circledwedge b = \begin{cases} -(|a| \wedge |b|) & \text{if } sign(b) \neq sign(a) \\ |a| \wedge |b| & \text{else.} \end{cases}$$

Alternatively the symmetric minimum of $a, b \in \mathbb{R}$ can be written as

$$a \otimes b = sign(a \cdot b)(|a| \wedge |b|).$$

Suppose that $\mu : 2^N \to [0,1]$ is a capacity and $\boldsymbol{x} \in [-1,1]^n$ is a vector evaluated on each criterion on the symmetric scale $[-1,1]$. The *symmetric Sugeno integral* [11] of $\boldsymbol{x}$ is defined as

$$\check{\mathrm{S}}\mathrm{u}\,(\boldsymbol{x},\mu) = (Su(\boldsymbol{x} \vee 0,\mu)) \otimes (-Su((-\boldsymbol{x}) \vee 0,\mu))\,. \tag{2.11}$$

In (2.11), as before in (2.8), symmetric means that $\check{\mathrm{S}}\mathrm{u}\,(\boldsymbol{x},\mu) = -\check{\mathrm{S}}\mathrm{u}\,(-\boldsymbol{x},\mu)$. Clearly if $x_i \geq 0$ for all $i \in N$, $\check{\mathrm{S}}\mathrm{u}\,(\boldsymbol{x},\mu) = Su(\boldsymbol{x},\mu)$, while if $x_i \leq 0$ for all $i \in N$,

$$\check{\mathrm{S}}\mathrm{u}\,(\boldsymbol{x},\mu) = \bigwedge_{i \in N} \bigvee \{x_i, -\nu\left(\{j \in N \mid x_j \leq x_i\}\right)\}\,. \tag{2.12}$$

(2.12) can be considered as a definition of a negative Sugeno integral, for the case in which $\boldsymbol{x}$ is negatively evaluated on each criterion.
In [41] the notion of symmetric Sugeno integral has been extended.

Definition 8. *A functional $L : [-1,1]^n \to [-1,1]$ is a fuzzy rank and sign-dependent functional if there exist two fuzzy measures μ^+ and μ^- such that for all $\boldsymbol{x} \in [-1,1]^n$*

$$L(\boldsymbol{x}) = (Su(\boldsymbol{x} \vee 0,\mu^+)) \otimes (-Su((-\boldsymbol{x}) \vee 0,\mu^-))\,. \tag{2.13}$$

Clearly when $\mu^+ = \mu^-$, the fuzzy rank and sign-dependent functional L is exactly the symmetric Sugeno integral. For further details on the fuzzy rank and sign-dependent functional and on the symmetric Sugeno integral we refer the reader to [11, 41].

In the next section we shall propose a more general approach, defining a *bipolar Sugeno integral*, which restricted on $\mathbb{R}^+$ and on $\mathbb{R}^-$ coincides respectively with the (7) and the (2.12).

2.4 Bipolar fuzzy integrals on the scale [-1,1]

The present work is devoted to the study of bipolar fuzzy integrals, i.e. those integrals useful when the scale underlying the alternatives evaluation is bipolar. For the sake of simplicity, trough this section we shall adopt the bipolar scale $[-1,1]$ to present our results. However, without loss of the generality, they can be extended to every other symmetric interval of $\mathbb{R}$, i.e. any of $[-\alpha,\alpha],]-\alpha,\alpha[, \;]-\infty,+\infty[$, where $\alpha \in \mathbb{R}^+$.

Let us consider the set $\mathcal{Q} = \{(A,B) \in 2^N \times 2^N \; : \; A \cap B = \emptyset\}$ of all disjoint pairs of subsets of N. With respect to the binary relation $(A,B) \precsim (C,D)$ iff $A \subseteq C$ and $B \supseteq D$, $\mathcal{Q}$ is a lattice, i.e. a partial ordered set in which any two elements have a unique supremum, $(A,B) \vee (C,D) = (A \cup C, B \cap D)$, and a unique infimum, $(A,B) \wedge (C,D) = (A \cap C, B \cup D)$. For all $(A,B),(C,D) \in \mathcal{Q}$ if $A \subseteq C$ and $B \subseteq D$, we simply write $(A,B) \subseteq (C,D)$. For all $(A,B) \in \mathcal{Q}$ the indicator function $1_{(A,B)} : N \to \{-1,0,1\}$ is the function which attains 1 on A, -1 on B and 0 on $(A \cup B)^c$. Such a function can be identified with the vector $\mathbf{1}_{(A,B)}$ whose i^{th} component is equal to 1 if $i \in A$, is equal to -1 if $i \in B$ and is equal to 0 otherwise.

In [38] it has been shown that the symmetric maximum $\circledvee : [-1,1] \times [-1,1] \to [-1,1]$ coincides with two recent symmetric extensions of the Choquet integral, the *balancing Choquet integral* and the *fusion Choquet integral*, when they are computed with respect to the strongest capacity (i.e. the capacity $\nu : 2^N \to [0,1]$ which attains zero on the empty set and one elsewhere). However, the symmetric maximum of a set X cannot be defined, being $\oslash$ non associative; e.g, suppose that $X = \{3, -3, 2\}$, then $(3 \oslash -3) \oslash 2 = 2$ or $3 \oslash (-3 \oslash 2) = 0$, depending on the order. Several possible extensions of the symmetric maximum for dimension $n, n > 2$ have been proposed (see [12, 18] and also the related discussion in [38]). One of these extensions is based on the splitting rule applied to the maximum and to the minimum as described in the following. Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the *bipolar maximum* of X, shortly $\bigvee^b X$, is defined as:

$$\bigvee^b X = \bigvee_i^m{}^b x_i = \left(\bigvee_i^m x_i\right) \oslash \left(\bigwedge_i^m x_i\right). \tag{2.14}$$

The following definitions are closely related to the above discussion.

Definition 9. *Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the positive bipolar maximum of X, shortly $\bigvee^{b^+} X$, is the element with the greatest absolute value, with the convention that, in the case of two different opposite elements with this property, we choose the non-negative.*

Definition 10. *Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the negative bipolar maximum of X, shortly $\bigvee^{b^-} X$, is the element with the greatest absolute value, with the convention that, in the case of two different opposite elements with this property, we choose the non-positive.*

Following these definitions, if $X = \{9, -9, 7, -3\}$ thus, $\bigvee^b X = 0$, $\bigvee^{b^+} X = 9$ and $\bigvee^{b^-} X = -9$. Clearly the three operators just defined are linked by means of the relation: $\bigvee^b X = \bigvee^b \left\{\bigvee^{b^+} X, \bigvee^{b^-} X\right\}$.

Given the vectors $\boldsymbol{x}^1, \dots, \boldsymbol{x}^k \in [-1.1]^n$ with $K = \{1, \dots, k\}$, $\bigvee\limits_{j\in K}^b \boldsymbol{x}_j$ is the vector whose i^{th} component is $\bigvee^b \{x_i^1, \dots, x_i^k\}$ for all $i = 1, \dots, n$.

The following properties of an aggregation function $G : [-1,1]^n \to [-1,1]$ are useful to characterize several bipolar integrals.

- bipolar comonotonic additivity: for all bipolar comonotone $\boldsymbol{x}$, $\boldsymbol{y} \in [-1,1]^n$,

$$G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y});$$

- bipolar stability of the sign: for all $r, s \in]0,1]$ and for all $(A,B) \in \mathcal{Q}$,

$$G(r\mathbf{1}_{A,B})G(s\mathbf{1}_{A,B}) > 0 \qquad \text{or} \qquad G(r\mathbf{1}_{A,B}) = G(s\mathbf{1}_{A,B}) = 0,$$

  i.e., in simple words, $G(r\mathbf{1}_{(A,B)})$ and $G(s\mathbf{1}_{(A,B)})$ have the same sign;

- bipolar stability with respect to the minimum: for all $r, s \in]0,1]$ such that $r > s$, and for all $(A,B) \in \mathcal{Q}$, $|G(r\mathbf{1}_{(A,B)})| \geq |G(s\mathbf{1}_{(A,B)})|$ and, moreover,

$$\text{if} \qquad |G(r\mathbf{1}_{(A,B)})| > |G(s\mathbf{1}_{(A,B)})| \qquad \text{then} \qquad |G(s\mathbf{1}_{(A,B)})| = s.$$

2.4.1 A specific property: bipolar comonotone maxitivity

With a slight abuse of notation we extend the relation of set inclusion to $\mathcal{Q}$, by defining $(A,B) \subseteq (C,D)$ if and only if $A \subseteq C$ and $B \subseteq D$, for all

$(A, B), (C, D) \in \mathcal{Q}$. Let us suppose to have k different levels $l_1, \ldots, l_k \in \mathbb{R}$ with $0 < l_1 < l_2 < \ldots < l_k \leq 1$ and a sequence $\{(A_i, B_i)\}_{i=1,\ldots,k}$ such that $(A_i, B_i) \in \mathcal{Q}$ for all $i = 1, \ldots, k$ and $(A_{i+1}, B_{i+1}) \subseteq (A_i, B_i)$ for all $i = 1, \ldots, k-1$. The vectors $l_i \cdot \mathbf{1}_{(A_i, B_i)}$, $i = 1, \ldots, k$ are bipolar comonotonic and, moreover, by ordering them with respect to the level l_i, then in the vector $l_i \cdot \mathbf{1}_{(A_i, B_i)}$, for each component the elements under the level l_i are the opposite of that under the level $-l_i$. See for example the four vectors

$$\begin{aligned}
\boldsymbol{x} &= (7, -7, \ 0, \ \ 0) \\
\boldsymbol{y} &= (5, -5, \ 5, \ \ 0) \\
\mathbf{w} &= (3, -3, \ 3, -3) \\
\boldsymbol{z} &= (2, -2, \ 2, -2).
\end{aligned}$$

An aggregation function G is said to be bipolar comonotone maxitive if it is maxitive on such a type of bipolar comonotonic *bi-constants*, i.e. if fixed $K = \{1, \ldots, k\}$ it holds:

$$G\left(\bigvee_{i \in K}^{b} l_i \cdot \mathbf{1}_{(A_i, B_i)}\right) = \bigvee_{i \in K}^{b} G\left(l_i \cdot \mathbf{1}_{(A_i, B_i)}\right). \tag{2.15}$$

G is said to be right bipolar comonotone maxitive if

$$G\left(\bigvee_{i \in K}^{b^+} l_i \cdot \mathbf{1}_{(A_i, B_i)}\right) = \bigvee_{i \in K}^{b^+} G\left(l_i \cdot \mathbf{1}_{(A_i, B_i)}\right). \tag{2.16}$$

G is said to be left bipolar comonotone maxitive if

$$G\left(\bigvee_{i \in K}^{b^-} l_i \cdot \mathbf{1}_{(A_i, B_i)}\right) = \bigvee_{i \in K}^{b^-} G\left(l_i \cdot \mathbf{1}_{(A_i, B_i)}\right). \tag{2.17}$$

Clearly, due to bipolar comonotonicity, in equations (2.15)-(2.17):

$$\bigvee_{i\in K}^{b} l_i \cdot \mathbf{1}_{(A_i,B_i)} = \bigvee_{i\in K}^{b^+} l_i \cdot \mathbf{1}_{(A_i,B_i)} = \bigvee_{i\in K}^{b^-} l_i \cdot \mathbf{1}_{(A_i,B_i)}.$$

2.4.2 The bipolar Choquet integral

Definition 11. *A function $\mu_b : \mathcal{Q} \to [-1,1]$ is a bi-capacity [14, 15, 22] on N if*

- $\mu_b(\emptyset,\emptyset) = 0$, $\mu_b(N,\emptyset) = 1$ *and* $\mu_b(\emptyset, N) = -1$;
- $\mu_b(A,B) \le \mu_b(C,D)\ \forall\ (A,B),(C,D) \in \mathcal{Q}$ *such that* $(A,B) \precsim (C,D)$.

Definition 12. *The bipolar Choquet integral of $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b is given by* [14, 15, 22, 21]*:*

$$Ch_b(\boldsymbol{x},\mu_b) = \int_0^\infty \mu_b(\{i\in N : x_i > t\},\{i \in N : x_i < -t\})dt. \tag{2.18}$$

The bipolar Choquet integral of $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b can be rewritten as

$$Ch_b(\boldsymbol{x},\mu_b) = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \mu_b(\{j \mid x_j \ge |x_{\sigma(i)}|\},\{j \mid x_j \le -|x_{\sigma(i)}|\}), \tag{2.19}$$

being $\sigma : N \to N$ any permutation of index such that $0 = |x_{\sigma(0)}| \le |x_{\sigma(1)}| \le \ldots \le |x_{\sigma(n)}|$. Note that to ensure that the pair $(\{j \in N : x_j \ge |t|\},\{j\in N : x_j \le -|t|\})$ is an element of $\mathcal{Q}$ for all $t \in \mathbb{R}$, we adopt the convention - which will be maintained trough all the chapter - that in the case of $t = 0$ the inequality $x_j \leq 0$ is to be understood as $x_j < 0$. The formulation (2.19) will be useful in proving some results, like that exposed in the next representation theorem.

Theorem 4. [22] *An aggregation function $G : [-1,1]^n \to [-1,1]$ is idempotent and bipolar comonotonic additive if and only if there exists a bi-capacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$,*

$$G(\boldsymbol{x}) = Ch_b(\boldsymbol{x}, \mu_b).$$

Remark 1. *Although the bipolar Choquet integral is trivially homogeneous, this condition does not appear in the theorem, since an aggregation function which is idempotent and bipolar comonotone additive is also homogeneous. Observe also that we could relax idempotency with the conditions $G(\boldsymbol{1}_{(N,\varnothing)}) = 1$ and $G(\boldsymbol{1}_{(\varnothing,N)}) = -1$.*

2.4.3 The bipolar Shilkret integral

Definition 13. *The bipolar Shilkret integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b is given by:*

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \quad (2.20)$$

Definition 14. *The right bipolar Shilkret integral of*

$$\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$$

with respect to the bi-capacity μ_b *is given by:*

$$Sh_b^+(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^+} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \quad (2.21)$$

Definition 15. *The left bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ *with respect to the bi-capacity* μ_b *is given by:*

$$Sh_b^-(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^-} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \quad (2.22)$$

Clearly the three definitions are linked via the

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee^b \{Sh_b^+(\boldsymbol{x}, \mu_b), Sh_b^-(\boldsymbol{x}, \mu_b)\}.$$

The condition $Sh_b(\boldsymbol{x}, \mu_b) = 0$ is equivalent to the $Sh_b^+(\boldsymbol{x}, \mu_b) = -Sh_b^-(\boldsymbol{x}, \mu_b)$ and, in this case, either the three integrals are all zero or they give three different results, one zero, one positive and one negative. We can think about them in terms of a neutral, an optimistic and a pessimistic aggregate evaluation of $\boldsymbol{x}$. The condition $Sh_b(\boldsymbol{x}, \mu_b) \neq 0$ implies that $Sh_b^+(\boldsymbol{x}, \mu_b) = Sh_b^-(\boldsymbol{x}, \mu_b) = Sh_b(\boldsymbol{x}, \mu_b)$.

The following theorems characterize the bipolar Shilkret integral.

Theorem 5. *An aggregation function* $G : [-1, 1]^n \to [-1, 1]$ *is idempotent, bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1, 1]^n$,

$$G(\boldsymbol{x}) = Sh_b(\boldsymbol{x}, \mu_b).$$

Remark 2. *Let us note that theorem 5 implies, as corollary, theorem 2 since bipolar comonotone maxitivity restricted on* $\mathbb{R}^+$ *implies comonotone maxitivity.*

Theorem 6. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, positive bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Sh_b^+(\boldsymbol{x}, \mu_b).$$

Theorem 7. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, negative bipolar comonotone maxitive and homogeneous if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Sh_b^-(\boldsymbol{x}, \mu_b).$$

Remark 3. *Idempotency could be relaxed with the conditions* $G(\boldsymbol{1}_{(N,\varnothing)}) = 1$ *and* $G(\boldsymbol{1}_{(\varnothing,N)}) = -1$, *in fact from these and from homogeneity idempotency can be elicited.*

2.4.4 The bipolar Sugeno integral

Definition 16. *The bipolar Sugeno integral of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *on* N *is given by:*

$$Su_b(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b} \Big\{ |x_i| \oslash \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}) \Big\}. \quad (2.23)$$

Definition 17. *The right bipolar Sugeno integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *on* N *is given by:*

$$Su_b^+(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^+} \Big\{ |x_i| \oslash \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}) \Big\}. \quad (2.24)$$

Definition 18. *The left bipolar Sugeno integral of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to the bi-capacity* μ_b *on* N *is given by:*

$$Su_b^-(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b^-} \Big\{ |x_i| \oslash \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}) \Big\}. \quad (2.25)$$

Clearly the three definitions are linked via the

$$Su_b(\boldsymbol{x}, \mu_b) = \bigvee^{b} \left\{ Su_b^+(\boldsymbol{x}, \mu_b), Su_b^-(\boldsymbol{x}, \mu_b) \right\}.$$

The condition $Su_b(\boldsymbol{x}, \mu_b) = 0$ is equivalent to the $Su_b^+(\boldsymbol{x}, \mu_b) = -Su_b^-(\boldsymbol{x}, \mu_b)$ and, in this case, either the three integrals are all zero or they give three different results, one zero (neutral), one positive (optimistic) and one negative (pessimistic). The condition $Su_b(\boldsymbol{x}, \mu_b) \neq 0$ implies that $Su_b^+(\boldsymbol{x}, \mu_b) = Su_b^-(\boldsymbol{x}, \mu_b) = Su_b(\boldsymbol{x}, \mu_b)$.

The following theorems characterize the bipolar Sugeno integral.

Theorem 8. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Su_b(\boldsymbol{x}, \mu_b).$$

Theorem 9. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, positive bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Su_b^+(\boldsymbol{x}, \mu_b).$$

Theorem 10. *An aggregation function* $G : [-1,1]^n \to [-1,1]$ *is idempotent, negative bipolar comonotone maxitive, bipolar stable with respect to the sign and bipolar stable with respect to the minimum if and only if there exists a bi-capacity* μ_b *on* N *such that, for all* $\boldsymbol{x} \in [-1,1]^n$,

$$G(\boldsymbol{x}) = Su_b^-(\boldsymbol{x}, \mu_b).$$

2.5 Proofs of theorems

Proof of Theorem 2.

First we prove the necessary part. Let us suppose there exists a capacity μ on N such that, for all $\boldsymbol{x} \in \mathcal{I}^n$, $G(\boldsymbol{x}) = Sh(\boldsymbol{x}, \mu)$. In this case it is trivial to prove that the Shilkret integral is idempotent, comonotone maxitive and homogeneous by definition and we leave the proof to the reader. Now we prove the sufficient part of the theorem. Let us define

$$\mu(A) = G(\mathbf{1}_A), \quad \text{for all} \quad A \in 2^N. \tag{2.26}$$

Because G is an idempotent aggregation function, we get $\mu(\varnothing) = 0$, $\mu(N) = 1$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. Thus μ is a capacity on N. Every $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathcal{I}^n$ can be written as

$$\boldsymbol{x} = \bigvee_{i \in N} x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \geq x_{\sigma(i)}\}}$$

being $\sigma : N \to N$ any permutation of index such that $x_{\sigma(1)} \leq \ldots \leq x_{\sigma(n)}$. Because vectors $x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \geq x_{\sigma(i)}\}}$ are comonotonic, we get the thesis by applying comonotonic maxitivity, homogeneity of G and the definition of μ according to (2.26):

$$G(\boldsymbol{x}) = G\left(\bigvee_{i \in N} x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \geq x_{\sigma(i)}\}}\right) = \bigvee_{i \in N} G\left(x_{\sigma(i)} \cdot \mathbf{1}_{\{j \in N \mid x_j \geq x_{\sigma(i)}\}}\right) =$$

$$= \bigvee_{i \in N} x_{\sigma(i)} \cdot G\left(\mathbf{1}_{\{j \in N \mid x_j \geq x_{\sigma(i)}\}}\right) = \bigvee_{i \in N} x_{\sigma(i)} \cdot \mu\left(\{j \in N \mid x_j \geq x_{\sigma(i)}\}\right) = Sh(\boldsymbol{x}, \mu)$$

□

Proof of Theorem 4.

First we prove the necessary part. Let us suppose that there exists a bicapacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$, $G(\boldsymbol{x}) = Ch_b(\boldsymbol{x},\mu_b)$. Idempotency of the bipolar Choquet integral follows from definition, because if $\lambda \geq 0$, then $Ch_b\left(\lambda \cdot \mathbf{1}_{(N,\varnothing)},\mu_b\right) = \int_0^{\lambda} \mu_b\left(N,\varnothing\right) dt = \lambda$, while if $\lambda < 0$, then $Ch_b\left(\lambda \cdot \mathbf{1}_{(N,\varnothing)},\mu_b\right) = \int_0^{-\lambda} \mu_b\left(\varnothing,N\right) dt = \lambda$. If $\boldsymbol{x}$, $\boldsymbol{y} \in [-1,1]^n$ are bipolar comonotone, then there exists a permutation of indexes $\sigma : N \to N$ such that $0 = |x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq \ldots \leq |x_{\sigma(n)}|$ and $0 = |y_{\sigma(0)}| \leq |y_{\sigma(1)}| \leq \ldots \leq |y_{\sigma(n)}|$, and then

$$Ch_b\left(\boldsymbol{x},\mu_b\right) = \sum_{i=1}^{n}\left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot \mu_b\left(\{j \mid x_j \geq |x_{\sigma(i)}|\}, \{j \mid x_j \leq -|x_{\sigma(i)}|\}\right),$$

and

$$Ch_b\left(\boldsymbol{y},\mu_b\right) = \sum_{i=1}^{n}\left(|y_{\sigma(i)}| - |y_{\sigma(i-1)}|\right) \cdot \mu_b\left(\{j \mid y_j \geq |y_{\sigma(i)}|\}, \{j \mid y_j \leq -|y_{\sigma(i)}|\}\right).$$

Since $\boldsymbol{x}$ and $\boldsymbol{y}$ are absolutely comonotonic an cosigned, for every $i = 1,\ldots,n$

$$\mu_b\left(\{j \in N : x_j \geq |x_{\sigma(i)}|\}, \{j \in N : x_j \leq -|x_{\sigma(i)}|\}\right) =$$

$$\mu_b\left(\{j \in N : y_j \geq |y_{\sigma(i)}|\}, \{j \in N : y_j \leq -|y_{\sigma(i)}|\}\right). \tag{2.27}$$

Moreover, again because $\boldsymbol{x}$ and $\boldsymbol{y}$ are absolutely comonotonic and cosigned, for every $i = 1, \dots, n$, $|x_{\sigma(i)} + y_{\sigma(i)}| = |x_{\sigma(i)}| + |y_{\sigma(i)}|$ and consequently

$$0 = |x_{\sigma(0)} + y_{\sigma(0)}| \le |x_{\sigma(1)} + y_{\sigma(i)}| \le \dots \le |x_{\sigma(n)} + y_{\sigma(n)}| \quad \text{for every} \quad i = 1, \dots, n. \tag{2.28}$$

By (2.27) and (2.28) we get $Ch_b(\boldsymbol{x}, \mu_b) + Ch_b(\boldsymbol{y}, \mu_b) = Ch_b(\boldsymbol{x} + \boldsymbol{y}, \mu_b)$.

Now we prove the sufficient part of the theorem. Let us define

$$\mu_b(A, B) = G\left(\mathbf{1}_{(A,B)}\right), \quad \text{for all} \quad (A, B) \in \mathcal{Q}. \tag{2.29}$$

μ_b represents a bi-capacity, since by idempotency of G we get that $\mu_b(N, \varnothing) = G\left(\mathbf{1}_{(N,\varnothing)}\right) = 1$, $\mu_b(\varnothing, N) = G\left(\mathbf{1}_{(\varnothing,N)}\right) = -1$, $\mu_b(\varnothing, \varnothing) = G\left(\mathbf{1}_{(\varnothing,\varnothing)}\right) = 0$. Moreover, if $(A, B) \precsim (A', B')$, being for all $i \in N$, the i^{th} component of the vector $\mathbf{1}_{(A,B)}$ not greater than the i^{th} component of the vector $\mathbf{1}_{(A',B')}$ and being G an aggregation function (then monotone), thus $\mu_b(A, B) \le \mu_b(A', B')$. Observe now that any vector $\boldsymbol{x} = (x_1, \dots, x_n) \in [-1, 1]^n$ can be rewritten as

$$\boldsymbol{x} = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \cdot \mathbf{1}_{(\{j \in N : x_j \ge |x_{\sigma(i)}|\}, \{j \in N : x_j \le -|x_{\sigma(i)}|\})}, \tag{2.30}$$

being $\sigma : N \to N$ any permutation of indexes such that $0 = |x_{\sigma(0)}| \le |x_{\sigma(1)}| \le \dots \le |x_{\sigma(n)}|$. Let us note that for all $(A, B),\ (A', B') \in \mathcal{Q}$ such that $(A, B) \subseteq (A', B')$ and for all $a, b \in [0, 1]$, vectors $a \cdot \mathbf{1}_{(A,B)}$ and $b \cdot \mathbf{1}_{(A',B')}$ are bipolar comonotone. Consequently, (2.30) shows that any vector $\boldsymbol{x} \in [-1, 1]^n$ can be decomposed as a sum of bipolar comonotonic vectors. Remembering that an aggregation function which is idempotent and bipolar comonotone additive is also homogeneous, thus to get the thesis it is sufficient to apply, respectively, bipolar comonotone additivity, homogeneity of G and definition of bi-capacity μ_b according to (2.29):

$$G(\boldsymbol{x}) = G\left(\sum_{i=1}^{n}\left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right)\cdot \mathbf{1}_{\left(\{j\in N: x_j\geq |x_{\sigma(i)}|\},\{j\in N: x_j\leq -|x_{\sigma(i)}|\}\right)}\right) =$$

$$= \sum_{i=1}^{n}\left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right)\cdot G\left(\mathbf{1}_{\left(\{j\in N: x_j\geq |x_{\sigma(i)}|\},\{j\in N: x_j\leq -|x_{\sigma(i)}|\}\right)}\right) = Ch_b(\boldsymbol{x},\mu_b).$$

□

Proof of Theorem 5.

First we prove the necessary part. Let us suppose there exists a bi-capacity μ_b such that, for all $\boldsymbol{x}\in[-1,1]^n$, $G(\boldsymbol{x}) = Sh_b(\boldsymbol{x},\mu_b)$. The bipolar Shilkret integral is, trivially, idempotent and homogeneous and we only need to demonstrate the bipolar comonotonic maxitivity. Let us consider a set of indexes $K=\{1,\ldots,k\}$, k increasing levels $l_1,\ldots,l_k\in\mathbb{R}$ with $0<l_1<l_2<\ldots<l_k\leq 1$ and a sequence $\{(A_i,B_i)\}_{i\in K}$ such that $(A_i,B_i)\in\mathcal{Q}$ and $(A_{i+1},B_{i+1})\subseteq(A_i,B_i)$ for all $i\in K$. The j^{th} component of the vector $\bigvee_{i\in K}^{b}\{l_i\cdot\mathbf{1}_{(A_i,B_i)}\}$ is equal to l_i if $j\in A_i\smallsetminus A_{i+1}$, is equal to $-l_i$ if $j\in B_i\smallsetminus B_{i+1}$ and is equal to zero if $j\in N\smallsetminus(A_1\cup B_1)$ for all $i\in K$ and taking $A_{k+1}=B_{k+1}=\varnothing$. Clearly, such a vector has a component greater or equal to l_i for indexes in A_i and has component smaller or equal to $-l_i$ for indexes in B_i. Thus, by definition

$$Sh_b\left(\bigvee_{i\in K}^{b}\{l_i\cdot\mathbf{1}_{(A_i,B_i)}\},\mu_b\right) = \bigvee_{i\in K}^{b}\{l_i\cdot\mu_b\left((A_i,B_i)\right)\} = \bigvee_{i\in K}^{b}\{Sh_b\left(l_i\cdot\mathbf{1}_{(A_i,B_i)},\mu_b\right)\}. \tag{2.31}$$

Now we prove the sufficient part of the theorem. Let us define

$$\mu_b(A,B) = G\left(\mathbf{1}_{(A,B)}\right), \quad \text{for all} \quad (A,B) \in \mathcal{Q}. \tag{2.32}$$

μ_b represents a bi-capacity (see proof of theorem 4). Notice that each $\boldsymbol{x} \in [-1,1]^n$ can be rewritten as

$$\boldsymbol{x} = \bigvee_{i \in N}^{b} |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})} \tag{2.33}$$

and observe that vectors $|x_i| \cdot \mathbf{1}_{(\{j \in N \mid x_j \geq |x_i|\}, \{j \in N \mid x_j \leq -|x_i|\})}$, $i = 1 \ldots, n$ are bipolar comonotone. Consequently, for any $\boldsymbol{x} \in [-1,1]^n$ by bipolar comonotone maxitivity, homogeneity and definition of bi-capacity μ_b according to the (2.32) we get

$$G(\boldsymbol{x}) = G\left(\bigvee_{i \in N}^{b} |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)$$

$$= \bigvee_{i \in N}^{b} G\left(|x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right) =$$

$$= \bigvee_{i \in N}^{b} |x_i| \cdot G\left(\mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)$$

$$= \bigvee_{i \in N}^{b} |x_i| \cdot \mu_b\left(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\}\right) = Sh_b(\boldsymbol{x}, \mu_b)$$

□

Proof of Theorems 6 and 7. They are analogous to the proof of previous *Theorem* 5.

□

Proof of Theorem 8. First we prove the necessary part. Let us suppose there exists a bi-capacity μ_b such that, for all $\boldsymbol{x} \in [-1,1]^n$, $G(\boldsymbol{x}) = Su_b(\boldsymbol{x}, \mu_b)$. The Sugeno integral is idempotent by definition. Bipolar stability with respect to the sign and with respect to the minimum are trivially verified once we consider that for all $r > 0$ and for all $(A,B) \in \mathcal{Q}$

$$Su_b\left(r \cdot \mathbf{1}_{(A,B)}, \mu_b\right) = sign\left(\mu_b(A,B)\right) \bigwedge \{r, |\mu_b(A,B)|\}.$$

Let us consider a set of indexes $K = \{1,\dots,k\}$, k increasing levels $l_1,\dots,l_k \in \mathbb{R}$ with $0 < l_1 < l_2 < \dots < l_k \leq 1$ and a sequence $\{(A_i,B_i)\}_{i\in K}$ such that $(A_i,B_i) \in \mathcal{Q}$ and $(A_{i+1},B_{i+1}) \subseteq (A_i,B_i)$ for all $i \in K$. Thus, by definition

$$\begin{aligned} Su_b\left(\bigvee_{i\in K}^{b}\{l_i \cdot \mathbf{1}_{(A_i,B_i)}\}, \mu_b\right) &= \bigvee_{i\in K}^{b}\{sign\left[\mu_b\left((A_i,B_i)\right)\right] \bigwedge \{l_i, |\mu_b\left((A_i,B_i)\right)|\}\} = \\ &= \bigvee_{i\in K}^{b}\{Su_b\left(l_i \cdot \mathbf{1}_{(A_i,B_i)}, \mu_b\right)\}. \end{aligned} \tag{2.34}$$

Now we prove the sufficient part of the theorem. Let us define $\mu_b(A,B) = G\left(\mathbf{1}_{(A,B)}\right)$ for all $(A,B) \in \mathcal{Q}$. μ_b represents a bi-capacity (see proof of theorem 4). Let us note that using bipolar stability with respect to the minimum and idempotency of G we have that for all $r > 0$ and for all $(A,B) \in \mathcal{Q}$,

$$\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| = \bigwedge \left\{r,\ \left|G\left(\mathbf{1}_{(A,B)}\right)\right|\right\}. \tag{2.35}$$

The (2.35) is obvious if $r = 0$ or $r = 1$. If $0 < r < 1$ and $\left|G\left(\mathbf{1}_{(A,B)}\right)\right| > \left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right|$, then using stability w.r.t. the minimum, $\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| = r$ and the (2.35) is true again. If $\left|G\left(\mathbf{1}_{(A,B)}\right)\right| = \left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right|$ observe that by monotonicity and idempotency of G, $\left|G\left(r \cdot \mathbf{1}_{(A,B)}\right)\right| \leq \left|G\left(r \cdot \mathbf{1}_{(N,\varnothing)}\right)\right| = r$, which means that also in this last case the (2.35) is true. Finally, notice that each $\boldsymbol{x} \in [-1,1]^n$ can be rewritten as

$$\boldsymbol{x} = \bigvee_{i \in N}{}^{b} |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})} \tag{2.36}$$

and observe that vectors $|x_i| \cdot \mathbf{1}_{(\{j \in N \mid x_j \geq |x_i|\}, \{j \in N \mid x_j \leq -|x_i|\})}$, $i = 1 \ldots, n$ are bipolar comonotone.

Consequently, for any $\boldsymbol{x} \in [-1, 1]^n$ by bipolar comonotone maxitivity

$$G(\boldsymbol{x}) = G\left(\bigvee_{i \in N}{}^{b} |x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right) = \bigvee_{i \in N}{}^{b} G\left(|x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right) =$$

(by bipolar stability with respect to the sign)

$$= \bigvee_{i \in N}{}^{b} \left\{ \text{sign}\left[G\left(\mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)\right] \left|G\left(|x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)\right| \right\} =$$

$$= \bigvee_{i \in N}{}^{b} \left\{ \text{sign}\left[\mu_b\left(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\}\right)\right] \left|G\left(|x_i| \cdot \mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)\right| \right\} =$$

(by bipolar stability with respect to the minimum)

$$= \bigvee_{i \in N}{}^{b} \left\{ \text{sign}\left[\mu_b\left(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\}\right)\right] \bigwedge \left\{ |x_i|, \left|G\left(\mathbf{1}_{(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\})}\right)\right| \right\} \right\} =$$

$$= \bigvee_{i \in N}{}^{b} \left\{ \text{sign}\left[\mu_b\left(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\}\right)\right] \bigwedge \left\{ |x_i|, \left|\mu_b\left(\{j \mid x_j \geq |x_i|\}, \{j \mid x_j \leq -|x_i|\}\right)\right| \right\} \right\}$$

that is the Sugeno integral $Su_b(\boldsymbol{x}, \mu_b)$.

□

Proof of Theorems 9 and 10. They are analogous to the proof of previous *Theorem* 8.

□

2.6 Concluding remarks

In recent years there has been an increasing interest in development of new integrals useful in decision analysis process or in modeling engineering problems. An interesting line of research is that of bipolar fuzzy integrals, that considers the case in which the underling scale is bipolar. For an exhaustive presentation of bipolarity and its possible applications, a recent survey is [9]. In this chapter we have axiomatically characterized the bipolar Choquet integral and defined and axiomatically characterized the bipolar Shilkret integral and the bipolar Sugeno integral. Thus, the scenario of bipolar fuzzy integrals appears clearer and richer. A further direction of research in this field is that of level dependent bipolar fuzzy integrals. In this case, the fuzzy measure with respect to which the bipolar integrals are calculated can change from a level to another [21, 20]. Observe also that in [24] it has been introduced the concept of bipolar universal integral, which generalizes the Choquet, Shilkret and Sugeno bipolar integrals presented in this chapter.

Chapter 3

The bipolar universal integral

3.1 Introduction

The basic reference for this chapter is [24]. Recently a concept of universal integral has been proposed [32]. The universal integral generalizes the Choquet integral [7], the Sugeno integral [47] and the Shilkret integral [45]. Moreover, in [30], [31] a formulation of the universal integral with respect to a level dependent capacity has been proposed, in order to generalize the level-dependent Choquet integral [21], the level-dependent Shilkret integral [4] and the level-dependent Sugeno integral [37]. The Choquet, Shilkret and Sugeno integrals admit a bipolar formulation, useful in those situations where the underlying scale is bipolar ([14], [15], [22], [20]). In this chapter we introduce and characterize the bipolar universal integral, which generalizes the Choquet, Shilkret and Sugeno bipolar integrals.
The chapter is organized as follows. In section 3.2 we introduce the basic concepts. In section 3.3 we define and characterize the bipolar universal integral. In section 3.4 we give an illustrative example of a bipolar universal integral which is neither the Choquet nor Sugeno or Shilkret type. Finally, in section 3.6, we

present conclusions.

3.2 Basic concepts

Given a set of criteria $N = \{1, \ldots, n\}$, an *alternative* $\boldsymbol{x}$ can be identified with a score vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-\infty, +\infty]^n$, being x_i the evaluation of $\boldsymbol{x}$ with respect to the i^{th} criterion. For the sake of simplicity, without loss of generality, in the following we consider the bipolar scale $[-1, 1]$ to expose our results, so that $\boldsymbol{x} \in [-1, 1]^n$. Let us consider the set of all disjoint pairs of subsets of N, i.e. $\mathcal{Q} = \left\{(A, B) \in 2^N \times 2^N \; : \; A \cap B = \varnothing\right\}$. With respect to the binary relation $\lesssim$ on $\mathcal{Q}$ defined as $(A, B) \lesssim (C, D)$ iff $A \subseteq C$ and $B \supseteq D$, $\mathcal{Q}$ is a lattice, i.e. a partial ordered set in which any two elements have a unique supremum $(A, B) \vee (C, D) = (A \cup C, B \cap D)$ and a unique infimum $(A, B) \wedge (C, D) = (A \cap C, B \cup D)$. For all $(A, B) \in \mathcal{Q}$ the indicator function $1_{(A,B)} : N \to \{-1, 0, 1\}$ is the function which attains 1 on A, -1 on B and 0 on $(A \cup B)^c$.

Definition 19. *A function $\mu_b : \mathcal{Q} \to [-1, 1]$ is a normalized bi-capacity ([14], [15], [22]) on N if*

- $\mu_b(\varnothing, \varnothing) = 0$, $\mu_b(N, \varnothing) = 1$ *and* $\mu_b(\varnothing, N) = -1$;
- $\mu_b(A, B) \leq \mu_b(C, D) \;\; \forall \;\; (A, B), (C, D) \in \mathcal{Q} \; : \; (A, B) \lesssim (C, D)$.

Definition 20. *The bipolar Choquet integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ *with respect to a bi-capacity* μ_b *is given by ([14], [15], [22], [21]):*

$$Ch_b(\boldsymbol{x}, \mu_b) = \int_0^{\infty} \mu_b(\{i \in N : x_i > t\}, \{i \in N : x_i < -t\})dt. \tag{3.1}$$

The bipolar Choquet integral of $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1,1]^n$ with respect to the bi-capacity μ_b can be rewritten as

$$Ch_b(\boldsymbol{x}, \mu_b) = \sum_{i=1}^{n} \left(|x_{\sigma(i)}| - |x_{\sigma(i-1)}|\right) \mu_b(\{j \in N : x_j \geq |x_{\sigma(i)}|\}, \{j \in N : x_j \leq -|x_{\sigma(i)}|\}), \tag{3.2}$$

being $\sigma : N \to N$ any permutation of indexes such that $0 = |x_{\sigma(0)}| \leq |x_{\sigma(1)}| \leq \ldots \leq |x_{\sigma(n)}|$. Let us note that to ensure that $(\{j \in N : x_j \geq |t|\}, \{j \in N : x_j \leq -|t|\}) \in \mathcal{Q}$ for all $t \in \mathbb{R}$, we adopt the convention - which will be maintained trough all the chapter - that in the case of $t = 0$ the inequality $x_j \leq 0$ is to be understood as $x_j < 0$.

In this chapter we use the symbol $\bigvee$ to indicate the maximum and $\bigwedge$ to indicate the minimum. The *symmetric maximum* of two elements - introduced and discussed in [11], [12] - is defined by the following binary operation:

$$a \circledvee b = \begin{cases} -\left(|a| \vee |b|\right) & \text{if } b \neq -a \text{ and either } |a| \vee |b| = -a \text{ or } = -b \\ 0 & \text{if } b = -a \\ |a| \vee |b| & \text{else.} \end{cases}$$

Alternatively the symmetric maximum of $a, b \in \mathbb{R}$ can be written as

$$a \circledvee b = sign(a+b)(|a| \vee |b|).$$

The *symmetric minimum* of two elements [11, 12] is defined as:

$$a \oslash b = \begin{cases} -(|a| \wedge |b|) & \text{if } sign(b) \neq sign(a) \\ |a| \wedge |b| & \text{else.} \end{cases}$$

Alternatively the symmetric minimum of $a, b \in \mathbb{R}$ can be written as

$$a \oslash b = sign(a \cdot b)(|a| \wedge |b|).$$

In [38] it has been showed as on the domain $[-1, 1]$ the symmetric maximum coincides with two recent symmetric extensions of the Choquet integral, the *balancing Choquet integral* and the *fusion Choquet integral*, when they are computed with respect to the strongest capacity (i.e. the capacity which attains zero on the empty set and one elsewhere). However, the symmetric maximum of a set X cannot be defined, being $\varovee$ non associative. Suppose that $X = \{3, -3, 2\}$, then $(3 \varovee -3) \varovee 2 = 2$ or $3 \varovee (-3 \varovee 2) = 0$, depending on the order. Several possible extensions of the symmetric maximum for dimension $n, n > 2$ have been proposed (see [12], [18] and also the relative discussion in [38]). One of these extensions is based on the splitting rule applied to the maximum and to the minimum as described in the following. Given $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{R}$, the *bipolar maximum* of X, shortly $\bigvee^b X$, is defined as

$$\bigvee^b X = (\bigvee X) \varovee (\bigwedge X). \tag{3.3}$$

In the same way and for an infinite set X, it is possible to define the concept of $\sup^{bip} X$ as the symmetric maximum applied to the supremum and the infimum of X, with the convention that $\bigvee^b \{\pm\infty, l\} = \pm\infty$ and $\bigvee^b \{+\infty, -\infty\} = 0$.

Definition 21. *The bipolar Shilkret integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [-1, 1]^n$ *with* *respect to a bi-capacity* μ_b *is given by [20]:*

$$Sh_b(\boldsymbol{x}, \mu_b) = \bigvee_{i \in N}^{b} \{|x_i| \cdot \mu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})\}. \qquad (3.4)$$

Definition 22. *A bipolar measure on* N *with a scale* $(-\alpha, \alpha)$, $\alpha > 0$, *is any function* $\nu_b : Q \to (-\alpha, \alpha)$ *satisfying the following properties:*

1. $\nu_b(\varnothing, \varnothing) = 0$;

2. $\nu_b(N, \varnothing) = \alpha$, $\nu_b(\varnothing, N) = -\alpha$;

3. $\nu_b(A, B) \leq \nu_b(C, D) \ \forall \ (A, B), (C, D) \in \mathcal{Q} \ : \ (A, B) \precsim (C, D)$.

Definition 23. *The bipolar Sugeno integral of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in (-\alpha, \alpha)^n$ *with respect to the bipolar measure* ν_b *on* N *with scale* $(-\alpha, \alpha)$ *is given by [20]:*

$$\begin{aligned} Su_b(\boldsymbol{x}, \nu_b) \quad = \quad & \bigvee_{i \in N}^{b} \Big\{ \operatorname{sign}\left(\nu_b\left(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}\right)\right) \cdot \\ & \cdot \bigwedge \{|\nu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\})|, |x_i|\} \Big\}. \qquad (3.5) \end{aligned}$$

The bipolar Sugeno integral can be written using the symmetric minimum as

$$Su_b(\boldsymbol{x}, \nu_b) = \bigvee_{i \in N}^{b} \Big\{ |x_i| \oslash \nu_b(\{j \in N : x_j \geq |x_i|\}, \{j \in N : x_j \leq -|x_i|\}) \Big\}. \qquad (3.6)$$

3.3 The universal integral and the bipolar universal integral

In order to define the universal integral it is necessary to introduce the concept of pseudomultiplication. This is a function $\otimes : [0, 1] \times [0, 1] \to [0, 1]$, which is nondecreasing in each component (i.e. for all $a_1,\ a_2,\ b_1,\ b_2 \in [0,1]$ with $a_1 \leq a_2$ and $b_1 \leq b_2$, $a_1 \otimes b_1 \leq a_2 \otimes b_2$), has 0 as annihilator (i.e. for all $a \in [0,1]$, $a \otimes 0 = 0 \otimes a = 0$) and has a neutral element $e \in]0,1]$ (i.e. for all $a \in [0,1]$, $a \otimes e = e \otimes a = a$). If $e = 1$ then $\otimes$ is a *semicopula*, i.e. a binary operation $\otimes : [0,1]^2 \to [0,1]$ that is nondecreasing in both components and has 1 as neutral element (thus 0 is a annihilator).

A semicopula $\otimes : [0,1]^2 \to [0,1]$ which is associative and commutative is called a *triangular norm.*

The concept of semicopula can be generalized to the symmetric interval $[-1,1]$

Definition 24. *A bipolar semicopula is a function*

$$\otimes_b : [-1,1]^2 \to [-1,1]$$

that is "absolute-nondecreasing", has 1 as neutral element and -1 *as opposite-neutral element, and preserves the sign rule, i.e*

(A1) if $|a_1| \leq |a_2|$ *and* $|b_1| \leq |b_2|$ *then* $|a_1 \otimes_b b_1| \leq |a_2 \otimes_b b_2|$*;*

(A2) $a \otimes_b \pm 1 = \pm 1 \otimes_b a = \pm a$*; and*

(A3) $sign\,(a \otimes_b b) = sign(a) \otimes_b sign(b)$*.*

Let us note that a bipolar semicopula also satisfies the following additional properties

(A4) $a \otimes_b 0 = 0 \otimes_b a = 0$; and

(A5) $sign(a) \otimes_b sign(b) = sign(a \cdot b)$.

Indeed, $0 \leq |a \otimes_b 0| \leq |\pm 1 \otimes_b 0| = |\pm 0| = 0$ and $0 \leq |0 \otimes_b a| \leq |0 \otimes_b \pm 1| = |\pm 0| = 0$.
$(A5)$ is true by $(A4)$ if $a = sign(a) = 0$ or $b = sign(b) = 0$, while is true by $(A2)$

and $(A3)$ if $a = sign(a), b = sign(b) \in \{-1, 1\}$.

Let us consider the binary operation $*$ on $[-1,1]$ given by

$$a * b = \begin{cases} -ab & if \ (a,b) \ \in]-1,1[^2 \\ ab & else. \end{cases}$$

This satisfies axioms $(A1)$ and$(A2)$, but not $(A3)$ (think to $a = -1/3 = b$), then the additional axiom $(A3)$ is necessary in order to consider bipolar semicopulas as symmetric extensions of standard semicopulas in the sense of product. Note that this approach preserves commutativity and associativity.

Notable examples of bipolar semicopulas are the standard product, $a \cdot b$ and the symmetric minimum [11],[12]

$$a \oslash b = sign(a \cdot b)(|a| \wedge |b|).$$

Proposition 1. $\otimes_b : [-1,1]^2 \to [-1,1]$ *is a bipolar semicopula if and only if there exists a semicopula* $\otimes : [0,1]^2 \to [0,1]$ *such that for all* $a, b \in [-1,1]$

$$a \otimes_b b = sign(a \cdot b) \left(|a| \otimes |b| \right). \tag{3.7}$$

We call $\otimes_b$ the bipolar semicopula induced by the semicopula $\otimes$ whenever the (3.7) holds. For example, the semicopula product induces the bipolar semicopula product, the semicopula minimum induces the bipolar semicopula symmetric minimum. Finally let us note that the concept of bipolar semicopula is closely related to that of *symmetric pseudo-multiplication* in [13]

A capacity [7] or fuzzy measure [47] on N is a non decreasing set function $m : 2^N \to [0,1]$ such that $m(\varnothing) = 0$ and $m(N) = 1$.

Definition 25. *[32] Let F be the set of functions $f : N \to [0,1]$ and M the set of capacities on N. A function $I : M \times F \to [0,1]$ is a universal integral on the scale $[0,1]$ (or fuzzy integral) if the following axioms hold:*

(I1) $I(m,f)$ *is nondecreasing with respect to* m *and with respect to* f*;*

(I2) there exists a semicopula $\otimes$ *such that for any* $m \in M$*,* $c \in [0,1]$ *and* $A \subseteq N$*,*

$I(m, c \cdot 1_A) = c \otimes m(A)$*;*

(I3) for all pairs $(m_1, f_1), (m_2, f_2) \in M \times F$*, such that for all* $t \in [0,1]$*,*

$m_1(\{i \in N : f_1(i) \geq t\}) = m_2(\{i \in N : f_2(i) \geq t\})$*,* $I(m_1, f_1) = I(m_2, f_2)$*.*

We can generalize the concept of universal integral from the scale $[0,1]$ to the symmetric scale $[-1,1]$ by extending definition 25.

Definition 26. *Let F_b be the set of functions $f : N \to [-1,1]$ and M_b the set of bi-capacities on Q. A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ (or bipolar fuzzy integral) if the following axioms hold:*

(I1) $I_b(m_b, f)$ *is nondecreasing with respect to* m_b *and with respect to* f*;*

(I2) there exists a semicopula $\otimes$ *such that for any* $m_b \in M_b$*,* $c \in [0,1]$ *and* $(A,B) \in Q$*,* $I(m_b, c \cdot 1_{(A,B)}) = sign(m_b(A,B))\ (c \otimes |m_b(A,B)|)$*;*

(I3) for all pairs $(m_{b_1}, f_1), (m_{b_2}, f_2) \in M_b \times F_b$*, such that for all* $t \in [0,1]$*,*

$m_{b_1}(\{i \in N : f_1(i) \geq t\}, \{i \in N : f_1(i) \leq -t\}) =$

$= m_{b_2}(\{i \in N : f_2(i) \geq t\}, \{i \in N : f_2(i) \leq -t\})$*,* $I(m_{b_1}, f_1) = I(m_{b_2}, f_2)$*.*

Clearly, in definition 25, F can be identified with $[0,1]^n$ and in definition 26, F_b can be identified with $[-1,1]^n$, such that a function $f : N \to [-1,1]$ can be regarded as a vector $\boldsymbol{x} \in [-1,1]^n$. Note that the bipolar Choquet, Shilkret and Sugeno integrals are bipolar universal integrals in the sense of Definition 26. Observe that the underlying semicopula $\otimes$ is the standard product in the case of the bipolar Choquet and Shilkret integrals, while $\otimes$ is the minimum (with neutral element $\beta = 1$) for the bipolar Sugeno integral.

The concept of bipolar universal integral can also be defined using the concept of bipolar semicopula. To this extent, in definition 26 axioms $(I2)$ must be replaced with the following axioms

(I2') There exists a bipolar semicopula $\otimes_b$ such that for any $m_b \in M_b$, $c \in [0,1]$ and $(A,B) \in Q$, $I(m_b, c \cdot 1_{(A,B)}) = c \otimes_b m_b(A,B)$.

Again we observe that the underlying bipolar semicopula $\otimes_b$ is the standard product in the case of the bipolar Choquet and Shilkret integrals, while $\otimes_b$ is the symmetric minimum for the bipolar Sugeno integral.

3.3.1 Representation Theorem

Now we turn our attention to the characterization of the bipolar universal integral. Due to axiom $(I3)$ for each universal integral I_b and for each pair $(m_b, \boldsymbol{x}) \in M_b \times F_b$, the value $I_b(m_b, \boldsymbol{x})$ depends only on the function

$$h^{(m_b,\boldsymbol{x})} : [0,1] \to [-1,1],$$

defined for all $t \in [0,1]$ by

$$h^{(m,\boldsymbol{x})}(t) = m_b\left(\{i \in N : x_i \geq t\}, \{i \in N : x_i \leq -t\}\right). \tag{3.8}$$

Note that for each $(m_b, \boldsymbol{x}) \in M_b \times F_b$ such a function is not in general monotone but it is Borel measurable, since it is a step function, i.e. a finite linear combination of indicator functions of intervals. To see this, suppose that $\sigma : N \to N$ is a permutation of criteria such that $|x_{\sigma(1)}| \leq \ldots \leq |x_{\sigma(n)}|$ and let us consider the following intervals decomposition of $[0,1]$: $A_1 = [0, |x_{\sigma(1)}|]$, $A_{j+1} =]|x_{\sigma(j)}|, |x_{\sigma(j+1)}|]$ for all $j = 1, \ldots, n-1$ and $A_{n+1} =]|x_{\sigma(n)}|, 1]$. Thus, we can rewrite the function h as

$$h^{(m,\boldsymbol{x})}(t) = \sum_{j=1}^{n} m_b\left(\left\{i \in N : x_i \geq |x_{\sigma(j)}|\right\}, \left\{i \in N : x_i \leq -|x_{\sigma(j)}|\right\}\right) \cdot 1_{A_j}(t). \tag{3.9}$$

Let $\mathcal{H}_n$ be the subset of all step functions in $\mathcal{F}_{[-1,1]}^{([0,1],\mathcal{B}([0,1]))}$ with no more than n-values.

Proposition 2. *A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ related to some semicopula $\otimes$ if and only if there is a function $J : \mathcal{H}_n \to \mathbb{R}$ satisfying the following conditions:*

(J1) J is nondecreasing;

(J2) $J(d \cdot 1_{[x,x+c]}) = sign(d)(c \otimes |d|)$ for all $[x, x+c] \subseteq [0,1]$ and for all $d \in [-1,1]$;

(J3) $I(m_b, f) = J\left(h^{(m_b,f)}\right)$ for all $(m_b, f) \in M_b \times F_b$.

3.4 An illustrative example

The following is an example of a bipolar universal integral (which is neither the Choquet nor Sugeno or Shilkret type), and illustrates the interrelationship between the functions I, J and the semicopula $\otimes$. Let $I_b : M_b \times F_b \to \mathbb{R}$ be given by

$$I(m_b, f) = \sup{}^{bip}\left\{\frac{t \cdot m_b\left(\{f \geq t\}, \{f \leq -t\}\right)}{1 - (1-t)\left(1 - |m_b\left(\{f \geq t\}, \{f \leq -t\}\right)|\right)} \mid t \in]0,1]\right\}. \tag{3.10}$$

Note that (3.10) defines a bipolar universal integral, indeed if $m_b \geq m_b'$ and $f \geq f'$ then $h^{(m_b,f)} \geq h^{(m_b',f')}$ and being the function $t \cdot h/[1-(1-t)(1-|h|)]$ non decreasing in $h \in \mathbb{R}$, we conclude that $I(m_b,f) \geq I(m_b',f')$ using the monotonicity of the bipolar supremum. Moreover

$$\begin{aligned} I(m_b, c \cdot 1_{(A,B)}) &= sign(m_b(A,B)) \frac{t \cdot |m_b(\{f \geq t\}, \{f \leq -t\})|}{1-(1-t)(1-|m_b(\{f \geq t\}, \{f \leq -t\})|)} = \\ &= sign(m_b(A,B))(c \otimes |m_b(A,B)|). \end{aligned} \tag{3.11}$$

This means that the semicopula underlying the bipolar universal integral (3.11) is the Hamacher product

$$a \otimes b = \begin{cases} 0 & \text{if} \quad a = b = 0 \\ \frac{a \cdot b}{1-(1-a)(1-b)} & \text{if} \quad |a| + |b| \neq 0. \end{cases}$$

Now let us compute this integral in the simple situation of $N = \{1,2\}$. In this case the functions we have to integrate can be identified with two dimensional vectors $\boldsymbol{x} = (x_1, x_2) \in [-1,1]^2$ and we should define a bi-capacity on $\mathcal{Q}$. For example

$$m_b(\{1\}, \varnothing) = 0.6, \quad m_b(\{2\}, \varnothing) = 0.2, \quad m_b(\{1\}, \{2\}) = 0.1,$$

$$m_b(\{2\}, \{1\}) = -0.3, \quad m_b(\varnothing, \{1\}) = -0.1 \quad \text{and} \quad m_b(\varnothing, \{2\}) = -0.5.$$

First let us consider the four cases $|x_1| = |x_2|$. If $x \geq 0$:

$$I(m_b, (x,x)) = x, \quad I(m_b, (x,-x)) = \frac{0.1x}{0.1 + 0.9x},$$

$$I(m_b, (-x,x)) = \frac{-0.3x}{0.3 + 0.7x} \quad \text{and} \quad I(m_b, (-x,-x)) = -x.$$

For all the other possible cases, we have the following formula

$$
I(m_b, (x,y)) = \begin{cases}
\bigvee^b \left\{ y \ , \ \frac{0.6x}{0.6+0.4x} \right\} & x > y \geq 0 \\
\bigvee^b \left\{ \frac{0.1|y|}{0.1+0.9|y|} \ , \ \frac{0.6x}{0.6+0.4x} \right\} & x \geq 0 > y > -x \\
\bigvee^b \left\{ \frac{0.1x}{0.1+0.9x} \ , \ \frac{-0.5|y|}{0.5+0.5|y|} \right\} & x \geq 0 \geq -x > y \\
\bigvee^b \left\{ x \ , \ \frac{-0.5|y|}{0.5+0.5|y|} \right\} & 0 > x > y \\
\bigvee^b \left\{ x \ , \ \frac{0.2y}{0.2+0.8y} \right\} & y > x \geq 0 \\
\bigvee^b \left\{ \frac{-0.3|x|}{0.3+0.7|x|} \ , \ \frac{0.2y}{0.2+0.8y} \right\} & y \geq 0 > x > -y \\
\bigvee^b \left\{ \frac{-0.3y}{0.3+0.7y} \ , \ \frac{-0.1|x|}{0.1+0.9|x|} \right\} & y \geq 0 \geq -y > x \\
\bigvee^b \left\{ y \ , \ \frac{-0.1|x|}{0.1+0.9|x|} \right\} & 0 > y > x.
\end{cases} \tag{3.12}
$$

3.5 The bipolar universal integral with respect to a level dependent bi-capacity

All the bipolar fuzzy integrals (3.1), (3.4) and (3.5) as well as the universal integral, admit a further generalization with respect to a *level dependent capacity* ([21], [20], [31]). Next, after remembering previous definitions, we will give the concept of *bipolar universal integral with respect to a level dependent capacity.*

Definition 27. *[21] A bipolar level dependent bi-capacity is a function* $\mu_{bLD} : \mathcal{Q} \times [0,1] \to [-1,1]$ *satisfying the following properties:*

1. *for all* $t \in [0,1]$, $\mu_{bLD}(\emptyset,\emptyset,t) = 0, \mu_{bLD}(N,\emptyset,t) = 1, \mu_{bLD}(\emptyset,N,t) = -1$;

2. *for all* $(A,B,t),(C,D,t) \in \mathcal{Q} \times [0,1]$ *such that* $(A,B) \precsim (C,D)$, $\mu_{bLD}(A,B,t) \leq \mu_{bLD}(C,D,t)$;

3. *for all* $(A,B) \in \mathcal{Q}$, $\mu_{bLD}(A,B,t)$ *considered as a function with respect to* t *is Borel measurable.*

Definition 28. *[21] The bipolar Choquet integral of a vector* $\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$ *with respect to the level dependent bi-capacity* μ_{bLD} *is given by*

$$Ch_{bLD}(\boldsymbol{x}) = \int_0^{max_i|x_i|} \mu_{bLD}(\{i \in N : x_i \geq t\},\{i \in N : x_i \leq -t\},t)dt. \tag{3.13}$$

A level dependent bi-capacity μ_{bLD} is said Shilkret compatible if for for all $t,r \in [-1,1]$ such that $t \leq r$, and $(A,B),(C,D) \in \mathcal{Q}$ with $(A,B) \precsim (C,D)$,

$$t\mu_{bLD}((A,B),t) \leq r\mu_{bLD}((C,D),r).$$

Definition 29. *[20] The bipolar level dependent Shilkret integral of*

$$\boldsymbol{x} = (x_1,\ldots,x_n) \in [-1,1]^n$$

with respect to a Shilkret compatible bi-capacity level dependent, μ_{bLD}*, is given by*

$$Sh_{bLD}(\boldsymbol{x},\mu_{bLD}) = \bigvee_{i \in N}^{b} \left\{ \sup_{t \in]0,|x_i|]} \{t \cdot \mu_{bLD}(\{j \in N : x_j \geq t\},\{j \in N : x_j \leq -t\},t)\} \right\}. \tag{3.14}$$

Definition 30. *[20] A bipolar level dependent measure on N with a scale $[-\alpha, \alpha]$ with $\alpha > 0$, is any function $\nu_{bLD} : \mathcal{Q} \times [-\alpha, \alpha] \to [-\alpha, \alpha]$ satisfying the following properties:*

1. $\nu_{bLD}(\varnothing, \varnothing, t) = 0$ *for all* $t \in [-\alpha, \alpha]$;
2. $\nu_{bLD}(N, \varnothing, t) = \alpha$, $\nu_{bLD}(\varnothing, N, t) = -\alpha$ *for all* $t \in (\alpha, \beta)$;
3. *for all* $(A, B), (C, D) \in \mathcal{Q}$ *such that* $(A, B) \precsim (C, D)$, *and for all* $t \in [-\alpha, \alpha]$, $\nu_{bLD}(A, B, t) \leq \nu_{bLD}(C, D, t)$.

Definition 31. *[20] The bipolar level dependent Sugeno integral of*

$$\boldsymbol{x} = (x_1, \ldots, x_n) \in [-\alpha, \alpha]^n$$

with respect to the bipolar measure ν_{bLD} is given by

$$\begin{aligned} &\bigvee_{i \in N}^{b} \Big\{ \sup_{t \in]0, |x_i|]}^{bip} \{ \text{sign} \left[\nu_{bLD}(\{j \in N : x_j \geq t\}, \{j \in N : x_j \leq -t\}, t) \right] \\ &\cdot \min \left\{ |\nu_{bLD}(\{j \in N : x_j \geq t\}, \{j \in N : x_j \leq -t\}, t)| , t \right\} \} \Big\} = \\ &Su_{bLD}(\boldsymbol{x}, \nu_{bLD}). \end{aligned} \tag{3.15}$$

A level dependent bi-capacity can be, also, indicated as $M_b^t = \left(m_{b,t}\right)_{t \in]0,1]}$ where $m_{b,t}$ is a bi-capacity. Given a level dependent bi-capacity $M_b^t = \left(m_{b,t}\right)_{t \in]0,1]}$ for each alternative $\boldsymbol{x} \in [-1, 1]^n$ we can define the function $h_{M_b^t, f} : [0, 1] \to [-1, 1]$, which accumulates all the information contained in M_b^t and f, by:

$$h_{M_b^t, f}(t) = m_{b,t} \left(\{j \in N : x_j \geq t\}, \{j \in N : x_j \leq -t\} \right) \tag{3.16}$$

In general, the function $h_{M_b^t,f}$ is neither monotone nor Borel measurable. Following the ideas of inner and outer measures in Caratheodory's approach [27], we introduce the two functions $\left(h_{M_b^t,f}\right)^* : [0,1] \to [-1,1]$ and $\left(h_{M_b^t,f}\right)_* : [0,1] \to [-1,1]$ defined by

$$\begin{aligned} \left(h_{M_b^t,f}\right)^* &= \inf\left\{h \in \mathcal{H} \mid h \geq h_{M_b^t,f}\right\}, \\ \left(h_{M_b^t,f}\right)_* &= \sup\left\{h \in \mathcal{H} \mid h \geq h_{M_b^t,f}\right\}. \end{aligned} \tag{3.17}$$

Clearly, both functions (3.17) are non increasing and, therefore, belong to $\mathcal{H}$. If the level dependent bi-capacity M_b^t is constant, then the three functions considered in (3.16), (3.17) coincide.

Let $\mathcal{M}_b$ the set of all level dependent bi-capacities on Q, for a fixed $M_b^t \in \mathcal{M}_b$ a function $f : N \to [-1,1]$ is M_b^t-measurable if the function $h_{M_b^t,f}$ is Borel measurable. Let $F_{[-1,1]}^{M_b^t}$ be the set of all M_b^t measurable functions. Let us consider

$$\mathcal{L}_{[-1,1]} = \bigcup_{M_b^t \in \mathcal{M}_b} M_b^t \times F_{[-1,1]}^{M_b^t}$$

Definition 32. *A function $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ is a level-dependent bipolar universal integral on the scale $[-1,1]$ if the following axioms hold:*

(I1) $I_b(m,f)$ is nondecreasing in each component;

(I2) there is a bipolar universal integral $I_b : M_b \times F_b \to \mathbb{R}$ such that for each bipolar capacity $m_b \in M_b$, for each $\boldsymbol{x} \in [-1,1]^n$ and for each level dependent bipolar capacity $M_b^t \in \mathcal{M}_b$, satisfying $m_{b,t} = m_b$ for all $t \in]0,1]$, we have

$$L_b\left(M_b^t, \boldsymbol{x}\right) = I_b\left(m_b, \boldsymbol{x}\right);$$

(I3) for all pairs (M_{b_1}, f_1), $(M_{b_2}, f_2) \in \mathcal{L}_{[-1,1]}$ *with* $h_{M_{b_1},f_1} = h_{M_{b_2},f_2}$ *we have*

$$L_b\left(M_{b_1}, f_1\right) = L_b\left(M_{b_2}, f_2\right).$$

Obviously the bipolar Choquet, Shilkret and Sugeno integrals with respect to a level dependent capacity are level-dependent bipolar universal integrals in the sense of Definition 32.

Finally, we present the representation theorem which gives necessary and sufficient conditions to be a function $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ a level-dependent bipolar universal integral.

Proposition 3. *A function* $L_b : \mathcal{L}_{[-1,1]} \to [-1,1]$ *is a level-dependent bipolar universal integral if and only if there exist a semicopula* $\otimes : [0,1]^2 \to [0,1]$ *and a function* $J : \mathcal{H} \to \mathbb{R}$ *satisfying the following conditions:*

(J1) J *is nondecreasing;*

(J2) $J(d \cdot 1_{]0,c]}) = sign(d)(c \otimes |d|)$ *for all* $[x, x+c] \subseteq [0,1]$ *and for all* $d \in [-1,1]$*;*

(J3) $L_b\left(M_b, f\right) = J\left(h_{M_b,f}\right)$ *for all* $(M_b^t, f) \in \mathcal{L}_{[-1,1]}$.

3.6 Conclusions

The concept of universal integral generalizes, over all, the Choquet, Shilkret and Sugeno integrals. Those integrals admit a bipolar formulation, helpful for the case in which the underlying scale is bipolar. In this chapter we have defined and characterized the bipolar universal integral, thus providing a common frame including the bipolar Choquet, Shilkret and Sugeno integrals. Moreover, we have also defined and characterized the bipolar universal integral with respect to a level dependent bi-capacity, which includes, as notable examples, the bipolar level dependent Choquet, Shilkret and Sugeno integrals.

Chapter 4

Robust Integrals

4.1 Introduction

In many decision problems a set of alternatives is evaluated with respect to a set of points of view, called criteria. For example, in evaluating a car one can consider criteria such as maximum speed, price, acceleration, fuel consumption. In evaluating a set of students one can consider as criteria the notes in examinations with respect to different subjects such as Mathematics, Physics, Literature and so on. In general, evaluations of an alternative with respect to different criteria can be conflicting with respect to preferences. For example, very often when a car has a good maximum speed, it has also a high price and a high fuel consumption, or if a student is very good in Mathematics, may be not so good in Literature. then, in order to express a decision such as a choice from a given set of cars or a ranking of a set of students, it is necessary to aggregate the evaluations on considered criteria, taking into account the possible interactions. This is the domain of multiple-criteria decision analysis and in this context several methodologies have been proposed (for a collection of extensive state-of-art surveys see [8]). Suppose to have n criteria $N = \{1, \dots, n\}$ and that on each of them the evaluation of a given alternative $\boldsymbol{x}$ is expressed by a single number (on the same scale). then, such an alternative can be identified with a score vector $\boldsymbol{x} = (x_1, \dots, x_n)$, where $x_i \in \mathbb{R}$ represents the evaluation of $\boldsymbol{x}$ with respect to the ith criterion. If the criteria are independent, a natural way to aggregate the score $x_i,\ i = 1, \dots, n$ is using the weighted arithmetic means $E_w(\boldsymbol{x}) = \sum_1^n w_i x_i$ with $\sum_1^n w_i = 1$ and $w_i \geq 0$. When the criteria are interacting the weighted arithmetic means must be substituted with non additive operators. In the last years, several non additive integrals have been developed in order to obtain an aggregated evaluation of $\boldsymbol{x}$, say $E(\boldsymbol{x})$ (for a comprehensive survey see [16]). These include the Choquet integral [7], the Shilkret integral [45] and the Sugeno integral [47], among others. All these integrals are computed with respect to a capacity [7] or fuzzy measure [47] allowing the importance of a set of criteria to be not necessarily the sum of the importance of each criterion in the set. It can be smaller or greater, due respectively to redundancy or synergy among criteria. These integrals can be used if the starting evaluations are exactly expressed (on a numerical or ordinal scale). However, in the real life it is very simple to image situations where we have only partial informations about the possible evaluations on each criterion. Specifically, on this chapter we face the case of *interval-evaluations*. For example, suppose a situation where, considering only two criteria, an alternative $\boldsymbol{x}$ is evaluated between 5 and 10 on the first criterion and between 7 and 20 on the second. Again $\boldsymbol{x}$ can be represented as a score vector $\boldsymbol{x} = ([5, 10], [7, 20])$. Using a generic aggregation operator E, it seems natural to aggregate separately the $\boldsymbol{x}$ "pessimistic" evaluations $\boldsymbol{x}_* = (5, 7)$ and the "optimistic" ones, $\boldsymbol{x}^* = (10, 20)$, in order to obtain an interval $[E(\boldsymbol{x}_*), E(\boldsymbol{x}^*)]$ containing the global evaluation of $\boldsymbol{x}$. If we wish to obtain such a global evaluation, we should furthermore aggregate $E(\boldsymbol{x}_*)$ and $E(\boldsymbol{x}^*)$ into a single number.

then, the aggregation of interval-evaluations into an exact evaluation should necessarily request two steps. In this chapter we aim to synthesize these two processes into one single aggregating process. To this purpose we provide a quite natural generalization of the classical Choquet, Shilkret and Sugeno integrals, which we call the *robust* Choquet, Shilkret and Sugeno integrals computed wit respect to an *interval-capacity.* Roughly speaking, our integrals are special case of integrals of set valued functions [1]. Another question we face is that of order on the set of intervals. It is well known that the philosophy of the Choquet integral applied to a given alternative is based on the ranking of the alternative evaluations on the various criteria. Being these evaluations single numbers, their ranking agrees with the natural order of $\mathbb{R}$. In the case of interval-evaluations, we have not a "natural order" to be preserved, like in $\mathbb{R}$. On the other hand we want that an evaluation on the range $[5, 10]$ is considered better than an evaluation on the range $[1, 4]$ and, then, some assumption about a primitive ordering on intervals must be done. One choice could be to assume the lexicographic order: $[a, b] \prec [a', b']$ iff $a < a'$ or $a = a'$ and $b < b'$. The lexicographic order has the advantage to be a complete order on the set of intervals, but it leads to the conclusion that $[2.99, 100] \prec [3, 4]$, which we do not consider a suitable conclusion in the case of interval-evaluations. Instead, throughout this chapter we shall assume as desirable order on intervals to be preserved that defined by considering an evaluation on the range $[a, b]$ better or equal than an evaluation on the range $[a', b']$ iff $a \geq a'$ and $b \geq b'$.
Finally, we wish to recall as, in contrast to the fact that in real life decisions we often face imprecise evaluations, in multiple-criteria decision analysis little has been developed in order to provide appropriate tools to aggregate such evaluations. In the best of our knowledge this question has been only partially treated in literature [28, 5].

The chapter is organized as follows. Section 2 contains the basic concepts. In Section 3 we give the definition of Robust Choquet Integral (RCI) computed with respect to an interval-capacity. In section 4 we give an illustrative application of the RCI, while in Section 5 we provide a full axiomatic characterization of this integral. In Section 6 we explore the possibility of rewriting the RCI by means of its Möbius inverse. In Section 7 we give the definitions of robust Sugeno and Shilkret integrals and in Section 8 we apply our generalization to other fuzzy integrals, among them to the concave integral of Lehrer [34]. In Section 9 we extend our discussion to the case of m-point intervals [40]. Section 10 concludes.

4.2 Basic concepts

Let us consider a set of *alternative* $A = \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \ldots\}$ to be evaluated with respect to a set of criteria $N = \{1, \cdots, n\}$. Suppose that for every $\boldsymbol{x} \in A$, we have, on each criterion, a numerical imprecise evaluation. Specifically, suppose that for each $i \in N$ we know a range $[\underline{x}_i, \overline{x}_i]$ containing the exact evaluation of $\boldsymbol{x}$ with respect to i. then, being $\mathcal{I} = \{[a,b] \mid a, b \in \mathbb{R},\ a \leq b\}$ the set of bounded and closed intervals of $\mathbb{R}$, any alternative $\boldsymbol{x}$ can be identified with a score vector

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots,\ [\underline{x}_i, \overline{x}_i], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n \tag{4.1}$$

whose ith component, $[\underline{x}_i, \overline{x}_i]$, is the interval containing the evaluation of $\boldsymbol{x}$ with respect to the ith criterion. Vectors of $\mathbb{R}^n$ are considered elements of $\mathcal{I}^n$ by identifying each $x \in \mathbb{R}$ with the degenerate interval (or singleton) $[x,x] = \{x\}$. then, with a slight abuse of notation, we write $[x,x] = x$. We associate to every $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ the vector $\underline{\boldsymbol{x}} = (\underline{x}_1, \ldots, \underline{x}_n)$ of all the worst (or pessimistic) evaluations of $\boldsymbol{x}$ on each criterion and the vector $\overline{\boldsymbol{x}} = (\overline{x}_1, \ldots, \overline{x}_n)$ of all the best (or optimistic) evaluations of $\boldsymbol{x}$ on each criterion. Throughout this chapter, the elements of $\mathcal{I}^n$ will be, indifferently, called alternatives or vectors.

Let us consider the set $\mathcal{Q} = \{(A, B) \mid A \subseteq B \subseteq N\}$ of all pairs of subsets of N in which the first component is included in the second. With a slight abuse of notation we extend to $\mathcal{Q}$ the relation of set inclusion and the operations of union and intersection by defining for all $(A, B), (C, D) \in \mathcal{Q}$

$$(A, B) \subseteq (C, D) \text{ if and only if } A \subseteq C \text{ and } B \subseteq D;$$

$$(A, B) \cup (C, D) = (A \cup C, B \cup D)\,;$$

$$(A, B) \cap (C, D) = (A \cap C, B \cap D)\,.$$

Regarding the algebraic structure of $\mathcal{Q}$, we can observe that with respect to the relation $\subseteq$, $\mathcal{Q}$ is a lattice, i.e. a partial ordered set in which every two elements have a unique supremum and a unique infimum. Those are given, for all $(A, B), (C, D) \in \mathcal{Q}$, respectively, by

$$\sup\{(A, B)\,, (C, D)\} = (A, B) \cup (C, D)\,;$$

$$\inf\{(A, B)\,, (C, D)\} = (A, B) \cap (C, D)\,.$$

Moreover the lattice $(\mathcal{Q}, \subseteq)$ is also distributive. Indeed, due to the distributive property of set union over intersection (and vice versa) we have that

$$(A, B) \cup [(C, D) \cap (E, F)] = [(A, B) \cup (C, D)] \cap [(A, B) \cup (E, F)]\,;$$

$$(A, B) \cap [(C, D) \cup (E, F)] = [(A, B) \cap (C, D)] \cup [(A, B) \cap (E, F)]\,.$$

Regarding the significance of $\mathcal{Q}$ in this work, let us consider

$$\boldsymbol{x} = \left([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\right) \in \mathcal{I}^n$$

and a fixed evaluation level $t \in \mathbb{R}$. We define

$$\left(A(\boldsymbol{x}, t), B(\boldsymbol{x}, t)\right) = \left(\{i \in N \mid \underline{x}_i \geq t\}, \{i \in N \mid \overline{x}_i \geq t\}\right). \tag{4.2}$$

In (4.2) the set $A(\boldsymbol{x}, t)$ aggregates the criteria whose pessimistic evaluation of $\boldsymbol{x}$ is at least t, while $B(\boldsymbol{x}, t)$ aggregates the criteria whose optimistic evaluation of $\boldsymbol{x}$ is at least t. Clearly, $A(\boldsymbol{x}, t) \subseteq B(\boldsymbol{x}, t) \subseteq N$ and then $((A(\boldsymbol{x}, t), B(\boldsymbol{x}, t)) \in \mathcal{Q}$ for all $t \in \mathbb{R}$ and for all $\boldsymbol{x} \in \mathcal{I}^n$. We aim to define a tool allowing for the assignment of a "weight" to such elements of $\mathcal{Q}$.

4.3 The robust Choquet integral

Definition 33. *A function $\mu_r : \mathcal{Q} \to [0,1]$ is an interval-capacity on $\mathcal{Q}$ if*

- $\mu_r((\varnothing, \varnothing)) = 0$ *and* $\mu_r((N, N)) = 1$*; and*
- $\mu_r((A, B)) \leq \mu_r((C, D))$ *for all* $(A, B), (C, D) \in \mathcal{Q}$ *such that* $(A, B) \subseteq (C, D)$*.*

By the sake of simplicity, in the sequel we shall indicate $\mu_r((A, B))$ with $\mu_r(A, B)$.

Definition 34. *The Robust Choquet Integral (RCI) of*

$$\boldsymbol{x} = \left([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]\right) \in \mathcal{I}^n$$

with respect to an interval-capacity $\mu_r : \mathcal{Q} \to [0,1]$ *is given by*

$$Ch_r(\boldsymbol{x}, \mu_r) =: \int_{\min\{\underline{x}_1,\ldots,\underline{x}_n\}}^{\max\{\overline{x}_1,\ldots,\overline{x}_n\}} \mu_r(\{i \in N \mid \underline{x}_i \geq t\}, \{i \in N \mid \overline{x}_i \geq t\})dt + \min\{\underline{x}_1, \ldots, \underline{x}_n\}. \tag{4.3}$$

Note that, being in (4.3) the integrand bounded and not increasing, the integral is the standard Riemann integral.

An alternative formulation of the RCI implies some additional notations. We identify every vector $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ with $\boldsymbol{x}^* = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ defined by setting for all $i = 1, \ldots, 2n$

$$x_i = \begin{cases} \underline{x}_i & i \leq n \\ \overline{x}_{i-n} & i > n. \end{cases} \tag{4.4}$$

This corresponds to identify $\boldsymbol{x} \in \mathcal{I}^n$ with

$$\boldsymbol{x}^* = (x_1, \ldots, x_{2n}) = (\underline{x}_1, \ldots, \underline{x}_n, \overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^{2n}.$$

Let $(\cdot) : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ be a permutation of indices such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(2n)}$ and for all $i = 1, \ldots, 2n$ consider $\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right) = \left(\left\{j \in N \mid \underline{x}_j \geq x_{(i)}\right\}, \left\{j \in N \mid \overline{x}_j \geq x_{(i)}\right\}\right)$. Then, two alternative formulations of (4.3) are:

$$Ch_r(\boldsymbol{x}, \mu_r) = \sum_{i=2}^{2n} \left(x_{(i)} - x_{(i-1)}\right) \mu_r\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right) + x_{(1)} \tag{4.5}$$

and

$$Ch_r(\boldsymbol{x}, \mu_r) = \sum_{i=1}^{2n} x_{(i)} \left[\mu_r\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right) - \mu_r\left(A\left(\boldsymbol{x}, x_{(i+1)}\right), B\left(\boldsymbol{x}, x_{(i+1)}\right)\right)\right]. \tag{4.6}$$

4.3.1 Interpretation

The indicator function of a set $A \subseteq N$ is the function $1_A : N \to \{0,1\}$ which takes the value of 1 on A and 0 elsewhere. Such a function can be identified with the vector $\mathbf{1}_A \in \mathbb{R}^n$ whose ith component equals 1 if $i \in A$ and equals 0 if $i \notin A$. For all $(A,B) \in \mathcal{Q}$ the generalized indicator function $1_{(A,B)} : N \to \{0,1,[0,1]\}$ is defined by setting for all $i \in N$

$$1_{(A,B)}(i) = \begin{cases} [1,1] = 1 & i \in A \\ [0,1] & i \in B \setminus A \\ [0,0] = 0 & i \in N \setminus B. \end{cases} \tag{4.7}$$

The (4.7) can be thought as the function indicating "A for sure and, possibly, $B \setminus A$." Clearly, if $A = B$, $1_{(A,A)} = 1_A$. The function $1_{(A,B)}$ can be identified with the vector $\mathbf{1}_{(A,B)} \in \mathcal{I}^n$ whose ith component equals $[1,1] = 1$ if $i \in A$, equals $[0,1]$ if $i \in B \setminus A$ and equals 0 if $i \notin B$.

It follows by the definition of RCI that for any interval-capacity μ_r,

$$Ch_r(\mathbf{1}_{(A,B)}, \mu_r) = \mu_r\,(A,B)\,. \tag{4.8}$$

This relation offers an appropriate definition of the weights $\mu_r\,(A,B)$. Indeed, provided that the partial score $[\underline{x}_i, \overline{x}_i]$ are contained in $[0,1]$, the (4.8) suggests that the weight of importance of any couple $(A,B) \in \mathcal{Q}$ is defined as the global evaluation of the alternative that

- completely satisfies the criteria from A,
- has an unknown degree of satisfaction (on the scale $[0,1]$) about the criteria from $B \setminus A$, and

- totally fails to satisfy the criteria from $N \setminus B$.

4.3.2 Relation with the Choquet Integral

A capacity [7] or fuzzy measure [47] on N is a non decreasing set function $\nu : 2^N \to [0,1]$ such that $\nu(\emptyset) = 0$ and $\nu(N) = 1$.

Definition 35. *The Choquet integral [7] of a vector*

$$\boldsymbol{x} = (x_1, \ldots, x_n) \in [0, +\infty\,[^{\,n}$$

with respect to the capacity ν is given by

$$Ch(\boldsymbol{x}, \nu) = \int_0^{\infty} \nu\left(\{i \in N : x_i \geq t\}\right) dt. \tag{4.9}$$

Schmeidler [43] extended the above definition to negative values too.

Definition 36. *The Choquet integral of a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with respect to the capacity ν is given by*

$$Ch(\boldsymbol{x}, \nu) = \int_{\min_i x_i}^{\max_i x_i} \nu\left(\{i \in N : x_i \geq t\}\right) dt + \min_i x_i. \tag{4.10}$$

Alternatively (4.10) can be written as

$$Ch(\boldsymbol{x}, \nu) = \sum_{i=2}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot \nu\left(\{j \in N : x_j \geq x_{(i)}\}\right) \; + \; x_{(1)}, \tag{4.11}$$

being $(\cdot) : N \to N$ any permutation of indexes such that $x_{(1)} \leq \ldots \leq x_{(n)}$.

Now, suppose to have $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ such that $\underline{x}_i = \overline{x}_i$ for all $i \in N$, then $\boldsymbol{x} \in \mathbb{R}^n$. Let be given an interval-capacity $\mu_r : \mathcal{Q} \to [0,1]$. It is straightforward to note that $\nu(A) = \mu_r(A, A) : 2^N \to [0,1]$ defines a capacity. In this case the RCI of $\boldsymbol{x}$ with respect to μ_r collapses on the Choquet integral of $\boldsymbol{x}$ with respect to ν, i.e. $Ch_r(\boldsymbol{x}, \mu_r) = Ch(\boldsymbol{x}, \nu)$.

Moreover, the RCI is a monotonic functional (see section 4.5) and then for all $\boldsymbol{x} \in \mathcal{I}^n$,

$$Ch_r(\underline{\boldsymbol{x}}, \mu_r) = Ch(\underline{\boldsymbol{x}}, \nu) \leq Ch_r(\boldsymbol{x}, \mu_r) \leq Ch(\overline{\boldsymbol{x}}, \nu) = Ch_r(\overline{\boldsymbol{x}}, \mu_r). \tag{4.12}$$

If $\mu_r(\varnothing, N) \notin \{0, 1\}$, other two capacities can be elicited from μ_r by setting for all $A \subseteq N$

$$\underline{\nu}(A) = \frac{\mu_r(A, N) - \mu_r(\varnothing, N)}{1 - \mu_r(\varnothing, N)} \quad \text{and} \quad \overline{\nu}(A) = \frac{\mu_r(\varnothing, A)}{\mu_r(\varnothing, N)}.$$

These two capacities naturally arise in the proof of proposition 4.

Now let us examine the relation between the Choquet integral and the RCI in the other verse. Starting from two capacities $\underline{\nu} : 2^N \to [0, 1]$ and $\overline{\nu} : 2^N \to [0, 1]$, we can define an interval-capacity for every $\alpha \in (0, 1)$ by means of

$$\mu_r(A, B) = \alpha\underline{\nu}(A) + (1 - \alpha)\overline{\nu}(B), \text{ for all } (A, B) \in \mathcal{Q}. \tag{4.13}$$

Definition 37. *An interval-capacity $\mu_r(A, B) : \mathcal{Q} \to [0, 1]$ is said separable if there exist an $\alpha \in (0, 1)$ and two capacities, $\underline{\nu} : 2^N \to [0, 1]$ and $\overline{\nu} : 2^N \to [0, 1]$, such that* (4.13) *holds.*

Proposition 4. *An interval-capacity $\mu_r(A, B) : \mathcal{Q} \to [0, 1]$ is separable if and only if for every $A, A', B, B' \in 2^N$ with $A \cup A' \subseteq B \cap B'$, the following condition holds*

$$\mu_r(A, B) - \mu_r(A', B) = \mu_r(A, B') - \mu_r(A', B'). \tag{4.14}$$

Proof. Let us note that the (4.14) can be rewritten as

$$\mu_r(A', B') - \mu_r(A', B) = \mu_r(A, B') - \mu_r(A, B). \tag{4.15}$$

then the condition (4.14) means that the difference between two interval-capacities is independent from common coalitions of criteria in the first or in the second argument. The necessary part of the theorem is trivial, let us prove the sufficient part. Suppose that μ_r is an interval-capacity satisfying the (4.14). then if $A' = \varnothing$ and $B' = N$ and if $\mu_r(\varnothing, N) \notin \{0, 1\}$ we get

$$\mu_r(A, B) = \mu_r(A, N) - \mu_r(\varnothing, N) + \mu_r(\varnothing, B) =$$

$$\frac{\mu_r(A, N) - \mu_r(\varnothing, N)}{1 - \mu_r(\varnothing, N)} (1 - \mu_r(\varnothing, N)) + \frac{\mu_r(\varnothing, B)}{\mu_r(\varnothing, N)} \mu_r(\varnothing, N).$$

In this case μ_r is separable taking for all $A, B \in 2^N$,

$$\alpha = 1 - \mu_r(\varnothing, B), \quad \underline{\nu}(A) = \frac{\mu_r(A, N) - \mu_r(\varnothing, N)}{1 - \mu_r(\varnothing, N)} \quad \text{and} \quad \overline{\nu}(B) = \frac{\mu_r(\varnothing, B)}{\mu_r(\varnothing, N)}.$$

If $\mu_r(\varnothing, N) = 0$ we take $\alpha = 1$ and $\underline{\nu}(A) = \mu_r(A, N)$. Finally, if $\mu_r(\varnothing, N) = 1$ we take $\alpha = 0$ and $\overline{\nu}(B) = \mu_r(\varnothing, B)$. □

It is easy to verify that if μ_r is a separable interval-capacity defined according to (4.13), the RCI of every $\boldsymbol{x} \in \mathcal{I}^n$ is the mixture of the two Choquet integrals of $\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}} \in \mathbb{R}^n$ computed with respect to $\underline{\nu}$ and $\overline{\nu}$:

$$Ch_r(\boldsymbol{x}, \mu_r) = \alpha Ch(\underline{\boldsymbol{x}}, \underline{\nu}) + (1 - \alpha) Ch(\overline{\boldsymbol{x}}, \overline{\nu}). \tag{4.16}$$

In the case of a single capacity $\underline{\nu} = \overline{\nu} = \nu$, one could think to obtain a lower, an intermediate and an upper aggregate evaluation of an alternative $\boldsymbol{x} \in \mathcal{I}^n$ by means of

$$Ch(\underline{\boldsymbol{x}},\nu) \leq \alpha Ch(\underline{\boldsymbol{x}},\nu) + (1-\alpha)Ch(\overline{\boldsymbol{x}},\nu) \leq Ch(\overline{\boldsymbol{x}},\nu). \tag{4.17}$$

The mixture $\alpha Ch(\underline{\boldsymbol{x}},\nu)+(1-\alpha)Ch(\overline{\boldsymbol{x}},\nu)$ is the RCI of $\boldsymbol{x}$ with respect to a separable interval-capacity μ_r. Clearly, our approach is more general since it does not impose the separability of μ_r.

4.4 An illustrative example

Taking inspiration from an example very well known in the specialized literature [10] let us consider a case of evaluation of students. A typical situation, which can arise in the middle of a school year, is that when some teachers, being not sure about the evaluation of a student, express it in terms of an interval. Perhaps it is not a great lack of information to know that a student is evaluated in Mathematics between 5 and 6. But the problems can arise when we must compare several students having imprecise evaluations and, to this scope, we need an aggregated evaluation of each student.

We suppose that the students are evaluated on each subject on a 10 point scale. Let us suppose that we globally evaluate students with respect to evaluations in Mathematics, Physics and Literature. Let us consider three students having the evaluations presented in Table 4.1. As can be seen, some evaluations are imprecise. Suppose also that the dean of the school ranks the students as follows:

$$S_2 \succ S_1 \succ S_3.$$

The rationale of this ranking is that:

- $S_1 \succ S_3$ since the better evaluations of S_3 in scientific subjects, i.e. Mathe-

	Mathematics	Physics	Literature
S_1	8	8	7
S_2	$[7, 8]$	8	$[6, 8]$
S_3	9	9	$[5, 6]$

Table 4.1: Students' evaluations

matics and Physics are redundant, and the dean retains relevant the better evaluation of S_1 in Literature, where S_3 risks an insufficiency. In other words, when the scientific evaluation is fairly high, Literature becomes very important;

- $S_2 \succ S_1$ since the conjoint evaluation in Mathematics and Physics is very similar, also considering the redundancy of the two subjects. However S_2 has the same average in Literature and, then, a greater potential;

- $S_2 \succ S_3$ by transitivity of preferences.

Let us note that, if we consider separately the three averages given by the minimum, central and maximum evaluations of each student for each subject, see Table 4.2, we cannot explain the (rational) preferences of the dean. On the contrary, the evidence of such average evaluations shows how we should consider S_3 the best student. Next we show how the RCI permits to represent the preferences of the dean. Let $N = \{M, Ph, L\}$ be the set of criteria and let us identify the three students (alternative) S_1, S_2 and S_3, respectively with the three vectors:

$$\boldsymbol{x}_1 = ([8,8],[8,8],[7,7]),$$
$$\boldsymbol{x}_2 = ([7,8],[8,8],[6,8]),$$
$$\boldsymbol{x}_3 = ([9,9],[9,9],[5,6]).$$

The RCI represents the preferences of the dean if there exists an interval-capacity μ_r such that

$$Ch_r(\boldsymbol{x}_2, \mu_r) > Ch_r(\boldsymbol{x}_1, \mu_r) > Ch_r(\boldsymbol{x}_3, \mu_r),$$

that is

$$6 + \mu_r(\{M, Ph\}, N) + \mu_r(\{Ph\}, N) > 7 + \mu_r(\{M, Ph\}, \{M, Ph\}) >$$

$$> 5 + \mu_r(\{M, Ph\}, N) + 3\mu_r(\{M, Ph\}, \{M, Ph\}).$$

Which can be explained, for example, by setting

$$\begin{cases} \mu_r(\{M, Ph\}, N) = 0.9 \\ \mu_r(\{Ph\}, N) = 0.7 \\ \mu_r(\{M, Ph\}, \{M, Ph\}) = 0.5. \end{cases}$$

4.5 Axiomatic characterization of the RCI

Let us recall some well known definitions. Consider two vectors (alternatives) of $\mathbb{R}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$. We say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ if for all $i \in N$ $x_i \geq y_i$ and in this case we simply write $\boldsymbol{x} \geq \boldsymbol{y}$. We say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are comonotone if $(x_i - x_j)(y_i - y_j) \geq 0$ for all $i, j \in N$. A monotone function

	minimum	medium	maximum
S_1	7.67	7.67	7.67
S_2	7	7.5	8
S_3	7.67	7.83	8

Table 4.2: Average evaluations

$G : \mathbb{R}^n \to \mathbb{R}$ is a function such that $G(\boldsymbol{x}) \geq G(\boldsymbol{y})$ whenever $\boldsymbol{x} \geq \boldsymbol{y}$. In the context of multiple-criteria decision analysis, a monotone function $G : \mathbb{R}^n \to \mathbb{R}$ which fulfills the boundary conditions $\inf_{\boldsymbol{x}\in\mathbb{R}^n} G(\boldsymbol{x}) = -\infty$ and $\sup_{\boldsymbol{x}\in\mathbb{R}^n} G(\boldsymbol{x}) = +\infty$ is called an aggregation function [19]. Aggregation functions are useful tools to aggregate n evaluations of an alternative into a single overall evaluation. A function $G : \mathbb{R}^n \to \mathbb{R}$ is:

- idempotent, if for all constant vector $\mathbf{a} = (a, \ldots, a) \in \mathbb{R}^n$, $G(\mathbf{a}) = a$;
- homogeneous, if for all $\boldsymbol{x} \in \mathbb{R}^n$ and $c > 0$, $G(c \cdot \boldsymbol{x}) = c \cdot G(\boldsymbol{x})$;
- comonotone additive, if for all comonotone $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$G(\boldsymbol{x} + \boldsymbol{y}) = G(\boldsymbol{x}) + G(\boldsymbol{y}).$$

In [43] it has been showed that the Choquet integral is an idempotent, homogeneous and comonotone additive aggregation function. Moreover, these properties are also characterizing the Choquet integral, as showed by the following theorem.

Theorem 11. *[43] A monotone function $G : \mathbb{R}^n \to \mathbb{R}$ satisfying $G(\mathbf{1}_N) = 1$ is comonotone additive if and only if there exists a capacity ν such that, for all $\boldsymbol{x} \in \mathbb{R}^n$,*

$$G(\boldsymbol{x}) = Ch(\boldsymbol{x}, \nu).$$

Note that homogeneity is not among the hypotheses of the theorem since it can be elicited from monotonicity and comonotone additivity. Moreover from homogeneity and the condition $G(\mathbf{1}_N) = 1$ we also elicit idempotency of G.
Now we turn our attention to the RCI. As we shall soon see, the RCI with respect to an interval-capacity μ_r, can be considered a generalized aggregation function. In order to provide an axiomatic characterization of the RCI we need to extend the notions of monotonicity, idempotency, homogeneity and comonotone additivity for a generic function $G : \mathcal{I}^n \to \mathbb{R}$. To this purpose we introduce on $\mathcal{I}$ and on $\mathcal{I}^n$, a mixture operation and a preference relation.

Definition 38. *For every $a \in \mathbb{R}^+$ and $[x_1, x_2] \in \mathcal{I}$ we define: $a \cdot [x_1, x_2] = [ax_1, ax_2]$. Moreover, for every $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ we define $a \cdot \boldsymbol{x}$ as the element of I^n whose ith component is $a \cdot [\underline{x}_i, \overline{x}_i]$, for all $i = 1, \ldots, n$.*

Definition 39. *For every $[x_1, x_2], [y_1, y_2] \in \mathcal{I}$ we define:*

$$[x_1, x_2] + [y_1, y_2] = [x_1 + y_1, x_2 + y_2].$$

Moreover, for every pair of vectors of $\mathcal{I}^n$, $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ and $\boldsymbol{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$, we define $\boldsymbol{x} + \boldsymbol{y}$ as the element of I^n whose ith component is $[\underline{x}_i, \overline{x}_i] + [\underline{y}_i, \overline{y}_i]$, for all $i = 1, \ldots, n$.

Let us note that the two previous definitions can be summarized as follows. For every $a, b \in \mathbb{R}^+$ and $[x_1, x_2], [y_1, y_2] \in \mathcal{I}$ we have the "mixture operation"

$$a \cdot [x_1, x_2] + b \cdot [y_1, y_2] = [ax_1 + by_1, ax_2 + by_2].$$

Moreover, for every pair of vectors of $\mathcal{I}^n$, $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ and $\mathbf{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$ and for all $a, b \in \mathbb{R}^+$, we have that $a\boldsymbol{x} + b\boldsymbol{y}$ is the element of I^n whose ith component is $a \cdot [\underline{x}_i, \overline{x}_i] + b \cdot [\underline{y}_i, \overline{y}_i]$, for all $i = 1, \ldots, n$.

Definition 40. *For all $[\alpha, \beta], [\alpha_1, \beta_1] \in \mathcal{I}$, we define $[\alpha, \beta] \leq_{\mathcal{I}} [\alpha_1, \beta_1]$ whenever $\alpha \leq \alpha_1$ and $\beta \leq \beta_1$. The symmetric and asymmetric part of $\leq$ on $\mathcal{I}$ are denoted by $=_{\mathcal{I}}$ and $<_{\mathcal{I}}$. Moreover, for every pair of vectors of $\mathcal{I}^n$, $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ and $\boldsymbol{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$ we write $\boldsymbol{x} \leq_{\mathcal{I}} \boldsymbol{y}$ whenever $[\underline{x}_i, \overline{x}_i] \leq_{\mathcal{I}} [\underline{y}_i, \overline{y}_i]$ for all $i \in N$.*

For the sake of simplicity in the remaining part of the chapter the relations $\leq_{\mathcal{I}}$, $=_{\mathcal{I}}$ and $<_{\mathcal{I}}$ shall be simply denoted by $\leq$, $=$ and $<$.

Remark 4. *Alternatively, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ we can say that $\boldsymbol{x} \leq \boldsymbol{y}$ iff $\underline{\boldsymbol{x}} \leq \underline{\boldsymbol{y}}$ and $\overline{\boldsymbol{x}} \leq \overline{\boldsymbol{y}}$.*

Let us note that $(\mathcal{I}, \leq)$ is a partial ordered set, i.e. $\leq$ is reflexive, antisymmetric and transitive. However, this relation is not complete, e.g. we are not able to establish the preference between $[2, 5]$ and $[3, 4]$. Then, generally, the evaluations of an alternative on the various criteria, cannot be ranked.

The notion of comonotonicity can be easily extended to elements of $\mathcal{I}^n$ identifying every vector $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ with the vector $\boldsymbol{x}^* = (x_1, \ldots, x_{2n}) = (\underline{x}_1, \ldots, \underline{x}_n, \overline{x}_1, \ldots, \overline{x}_n) \in \mathbb{R}^{2n}$, according to (4.4).

Definition 41. *The two vectors of* $\mathcal{I}^n$*,*

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \quad \text{and} \quad \boldsymbol{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$$

are comonotone if are comonotone in $\mathbb{R}^{2n}$*, the two vectors*

$$\boldsymbol{x}^* = (\underline{x}_1, \ldots, \underline{x}_n, \ldots, \overline{x}_1, \ldots, \overline{x}_n) \quad \text{and} \quad \boldsymbol{y}^* = (\underline{y}_i, \ldots, \underline{y}_n, \ldots, \overline{y}_1, \ldots, \overline{y}_n).$$

Clearly a constant vector $\boldsymbol{k} = (k, k, \ldots, k) \in \mathbb{R}^n$ with $k \in \mathbb{R}$, is comonotone with every $\boldsymbol{x} \in \mathcal{I}^n$. Suppose that $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ and $\mathbf{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$ are two comonotone vectors of $\mathcal{I}^n$ and consider the correspondent vectors of $\mathbb{R}^{2n}$,

$$\boldsymbol{x}^* = (x_1, \ldots, x_{2n}) = (\underline{x}_1, \ldots, \underline{x}_n, \overline{x}_1, \ldots, \overline{x}_n)$$

and

$$\boldsymbol{y}^* = (y_1, \ldots, y_{2n}) = (\underline{y}_1, \ldots, \underline{y}_n, \ldots, \overline{y}_1, \ldots, \overline{y}_n)-$$

According to Schmeidler [43] if $\boldsymbol{x}$ and $\boldsymbol{y}$ are comonotone, then there exists a permutation of indices $(\cdot) : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(2n)}$ and $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(2n)}$

Remark 5. *If* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are comonotone, then both* $\underline{\boldsymbol{x}}$ *and* $\underline{\boldsymbol{y}}$ *are comonotone as well as* $\overline{\boldsymbol{x}}$ *and* $\overline{\boldsymbol{y}}$*. The reverse is generally false. For example* $\boldsymbol{x} = ([1,3], [2,4])$ *and* $\boldsymbol{y} = ([1,3], [4,5])$ *are non comonotone, although* $\underline{\boldsymbol{x}}$ *is comonotone with* $\underline{\boldsymbol{y}}$ *and* $\overline{\boldsymbol{x}}$ *is comonotone with* $\overline{\boldsymbol{y}}$*.*

Let us note that for all $(A, B), (A', B') \in \mathcal{Q}$, the relation $(A, B) \subseteq (A', B')$, ensures that $\mathbf{1}_{(A,B)}$ and $\mathbf{1}_{(A',B')}$ are comonotone. Note that also the sum $\mathbf{1}_{(A,B)}$ +

$\mathbf{1}_{(A',B')}$ is comonotone with $\mathbf{1}_{(A,B)}$ and $\mathbf{1}_{(A',B')}$ (see Tab 4.3).

	$\mathbf{1}_{(A,B)}$	$\mathbf{1}_{(A',B')}$	$\mathbf{1}_{(A,B)} + \mathbf{1}_{(A',B')}$
A	[1,1]	[1,1]	[2,2]
$(A' \cap B) \setminus A$	$[0,1]$	[1,1]	$[1,2]$
$B \setminus A'$	$[0,1]$	$[0,1]$	$[0,2]$
$A' \setminus B$	$[0,0]$	$[1,1]$	[1,1]
$B' \setminus (A' \cup B)$	$[0,0]$	$[0,1]$	$[0,1]$

Table 4.3: comonotone indicator functions.

Now we are able to study the properties of the RCI and to give the characterization Theorem.

Proposition 5. *Let μ_r be an interval-capacity, then $Ch_r(\cdot,\mu_r)$ satisfies the following properties.*

(P1) ***Idempotency****. For all $\boldsymbol{k} = (k,k,\ldots,k)$ with $k \in \mathbb{R}$, $Ch_r(\boldsymbol{k},\mu_r) = k$.*

(P2) ***Positive homogeneity****. For all $a > 0$ and $\boldsymbol{x} \in \mathcal{I}^n$, $Ch_r(a \cdot \boldsymbol{x},\mu_r) = a \cdot Ch_r(\boldsymbol{x},\mu_r)$.*

(P3) ***Monotonicity****. For all $\boldsymbol{x},\boldsymbol{y} \in \mathcal{I}^n$ with $\boldsymbol{x} \leq \boldsymbol{y}$, $Ch_r(\boldsymbol{x},\mu_r) \leq Ch_r(\boldsymbol{y},\mu_r)$.*

(P4) ***Comonotone additivity****. For all comonotone $\boldsymbol{x},\boldsymbol{y} \in \mathcal{I}^n$,*

$$Ch_r(\boldsymbol{x}+\boldsymbol{y},\mu_r) = Ch_r(\boldsymbol{x},\mu_r) + Ch_r(\boldsymbol{y},\mu_r).$$

Proof. (P1) follows trivially by definition of RCI. Let us prove (P2). Fixed $a > 0$

and $\boldsymbol{x} \in \mathcal{I}^n$, by definition

$$Ch_r\left(a \cdot \boldsymbol{x}, \mu_r\right) = \int_{\min\{a\underline{x}_1,\ldots,a\underline{x}_n\}}^{\max\{a\overline{x}_1,\ldots,a\overline{x}_n\}} \mu_r(\{i \in N \mid a\underline{x}_i \geq t\}, \{i \in N \mid a\overline{x}_i \geq t\})dt +$$

$$+ \min\left\{a\underline{x}_1, a\underline{x}_2, \ldots, a\underline{x}_n\right\} =$$

$$= a \cdot \int_{a \cdot \min\{\underline{x}_1,\ldots,\underline{x}_n\}}^{a \cdot \max\{\overline{x}_1,\ldots,\overline{x}_n\}} \mu_r(\{i \in N \mid \underline{x}_i \geq \frac{t}{a}\}, \{i \in N \mid \overline{x}_i \geq \frac{t}{a}\})d(t/a) +$$

$$a \cdot \min\left\{\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n\right\} = a \cdot Ch_r\left(\boldsymbol{x}, \mu_r\right).$$

In the last passage we change the variable in the integral from $y = t/a$ to $z = y \cdot a$. To prove (P3) let us note that for all $t \in \mathbb{R}$ and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}^n$ with $\boldsymbol{x} \leq \mathbf{y}$, we get that $\{i \in N : \ \underline{x}_i \geq t\} \subseteq \{i \in N : \ \underline{y}_i \geq t\}$ and $\{i \in N : \ \overline{x}_i \geq t\} \subseteq \{i \in N : \ \overline{y}_i \geq t\}$. We conclude that the RCI is a monotonic function by definition and invoking the monotonicity of μ_r and of the Riemann integral.

To prove (P4), suppose that

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \quad \text{and} \quad \mathbf{y} = ([\underline{y}_1, \overline{y}_1], \ldots, [\underline{y}_n, \overline{y}_n])$$

are two comonotone vectors of $\mathcal{I}^n$ and consider the correspondent vectors of $\mathbb{R}^{2n}$, $\boldsymbol{x}^* = (x_1, \ldots, x_{2n})$ and $\boldsymbol{y}^* = (y_1, \ldots, y_{2n})$, defined according to (4.4). then, there exists a permutation of indexes $(\cdot) : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ such that $x_{(1)} \leq \ldots \leq x_{(2n)}$ and $y_{(1)} \leq \ldots \leq y_{(2n)}$ or equivalently (being $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ comonotone), $x_{(1)} + y_{(1)} \leq \ldots \leq x_{(2n)} + y_{(2n)}$. By setting for all $i = 1, \ldots, 2n$

$$\begin{aligned} A_{(i)} &= \left\{j \in N \mid \underline{x}_j \geq x_{(i)}\right\} \bigcap \left\{j \in N \mid \underline{y}_j \geq y_{(i)}\right\}, \\ B_{(i)} &= \left\{j \in N \mid \overline{x}_j \geq x_{(i)}\right\} \bigcap \left\{j \in N \mid \overline{y}_j \geq y_{(i)}\right\}, \end{aligned} \tag{4.18}$$

we have that

$$Ch_r(\boldsymbol{x}, \mu_r) = \sum_{i=2}^{2n} \left(x_{(i)} - x_{(i-1)}\right) \mu_r \left(A_{(i)}, B_{(i)}\right) + x_{(1)},$$
$$Ch_r(\boldsymbol{y}, \mu_r) = \sum_{i=2}^{2n} \left(y_{(i)} - y_{(i-1)}\right) \mu_r \left(A_{(i)}, B_{(i)}\right) + y_{(1)}, \quad (4.19)$$

and also

$$Ch_r(\boldsymbol{x} + \boldsymbol{y}, \mu_r) = \sum_{i=2}^{2n} \left(x_{(i)} + y_{(i)} - x_{(i-1)} - y_{(i-1)}\right) \mu_r \left(A_{(i)}, B_{(i)}\right) + x_{(1)} + y_{(1)}. \quad (4.20)$$

From (4.19) and (4.20), comonotone additivity is obtained. □

Remark 6. *Since the RCI is additive on comonotone vectors and being a constant vector comonotone with all vectors, it follows that the RCI is translational invariant. This means that for all* $\boldsymbol{x} \in \mathcal{I}^n$ *and for all* $\boldsymbol{k} = (k, \ldots, k) \in \mathbb{R}^n$, $Ch_r(\boldsymbol{x} + \boldsymbol{k}, \mu_r) = k + Ch_r(\boldsymbol{x}, \mu_r)$.

The next theorem gives a characterization of the RCI.

Theorem 12. *A function* $G : \mathcal{I}^n \to \mathbb{R}$ *satisfies the properties*

- $G(\boldsymbol{1}_{(N,N)}) = 1$,
- *(P3) Monotonicity, and*
- *(P4) Comonotone additivity*

if and only if there exists an interval capacity $\mu_r : \mathcal{Q} \to [0, 1]$ *such that*

$$G(\boldsymbol{x}) = Ch_r(\boldsymbol{x}, \mu_r), \quad \text{for all} \quad \boldsymbol{x} \in \mathcal{I}^n.$$

Proof. The necessary part is a direct consequence of Proposition 5, let us sufficient part. First let us note that the properties (P1) and (P2), are not among the hypotheses of Theorem 12 since they are implied by comonotone additivity (P4), monotonicity (P3) and the condition $G(\mathbf{1}_{(N,N)}) = 1$. The proof of this claim is obtained by adapting that in [43]. Regarding the homogeneity, if $n \in \mathbb{N}$ is a positive integer, by comonotone additivity we get

$$G(n \cdot \boldsymbol{x}) = G(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{\text{n times}}) = n \cdot G(\boldsymbol{x}), \qquad \text{for every } \boldsymbol{x} \in \mathcal{I}^n.$$

If $a = n/m \in \mathbb{Q}^+$ is a positive razional number, with $n, m \in \mathbb{N}$ we get

$$n \cdot G(\boldsymbol{x}) = G(n \cdot \boldsymbol{x}) = G(\frac{nm}{m} \cdot \boldsymbol{x}) = m \cdot G(\frac{n}{m} \cdot \boldsymbol{x}), \qquad \text{for every } \boldsymbol{x} \in \mathcal{I}^n.$$

Finally, for $a \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ it is sufficient to consider two sequences of rational numbers convergent to a, $\{a_i^-\}$ and $\{a_i^+\}$ such that $a_1^- < a_2^- \ldots < a < \ldots a_2^+ < a_1^+$ and using monotonicity of G we get that $G(a \cdot \boldsymbol{x}) = a \cdot G(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathcal{I}^n$.

Regarding idempotency, if $a \in \mathbb{R}^+$ we get $G(a \cdot \mathbf{1}_{(N,N)}) = a \cdot G(\mathbf{1}_{(N,N)}) = a$. By comonotone additivity $0 = G(0 \cdot \mathbf{1}_{(N,N)})$ and $0 = G((a-a)\mathbf{1}_{(N,N)}) = G(a \cdot \mathbf{1}_{(N,N)}) + G(-a \cdot \mathbf{1}_{(N,N)}) = a + G(-a \cdot \mathbf{1}_{(N,N)})$, then $G(-a \cdot \mathbf{1}_{(N,N)}) = -a$.

The hypotheses of theorem ensure that the

$$\mu_r(A,B) = G\left(\mathbf{1}_{(A,B)}\right) \;\; \forall (A,B) \in \mathcal{Q} \tag{4.21}$$

defines an interval-capacity. Indeed: $\mu_r(N,N) = G(\mathbf{1}_{(N,N)}) = 1$; $\mu_r(\varnothing,\varnothing) = G(\mathbf{1}_{(\varnothing,\varnothing)}) = 0$, since by comonotone additivity $G(\mathbf{1}_{(\varnothing,\varnothing)}) = G(\mathbf{1}_{(\varnothing,\varnothing)} + \mathbf{1}_{(\varnothing,\varnothing)}) = G(\mathbf{1}_{(\varnothing,\varnothing)}) + G(\mathbf{1}_{(\varnothing,\varnothing)})$ and then $G(\mathbf{1}_{(\varnothing,\varnothing)}) = 0$; for all $(A,B),(C,D) \in \mathcal{Q}$ such that $(A,B) \subseteq (C,D)$, $\mu_r(A,B) \leq \mu_r(C,D)$ follows by monotonicity of G. Let $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n])$ be a vector and $(\cdot) : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ be a permutation such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(2n)}$. Since $\left(A\left(\boldsymbol{x}, x_{(i+1)}\right), B\left(\boldsymbol{x}, x_{(i+1)}\right)\right) \subseteq$

$\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)$ for all $i = 1, \ldots, 2n-1$, then the vectors $\mathbf{1}_{\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)}$ are comonotone for all $i = 1, \ldots, 2n$. The vector $\boldsymbol{x}$ can be rewritten as sum of comonotone vectors (take $x_{(0)} = 0$):

$$\boldsymbol{x} = \sum_{i=1}^{2n} \left[x_{(i)} - x_{(i-1)}\right] \cdot \mathbf{1}_{\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)}. \tag{4.22}$$

Finally, the proof follows from (4.22) by using, respectively, comonotone additivity, homogeneity of G and definition of the interval-capacity μ_r according to (4.21):

$$G(\boldsymbol{x}) = G\left(\sum_{i=1}^{2n} \left[x_{(i)} - x_{(i-1)}\right] \cdot \mathbf{1}_{\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)}\right) =$$

$$\sum_{i=1}^{2n} G\left(\left[x_{(i)} - x_{(i-1)}\right] \cdot \mathbf{1}_{\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)}\right) =$$

$$= \sum_{i=1}^{2n} \left[x_{(i)} - x_{(i-1)}\right] \cdot G\left(\mathbf{1}_{\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right)}\right)$$

$$= \sum_{i=1}^{2n} \left[x_{(i)} - x_{(i-1)}\right] \cdot \mu_r\left(A\left(\boldsymbol{x}, x_{(i)}\right), B\left(\boldsymbol{x}, x_{(i)}\right)\right) =$$

$$= Ch_r(\boldsymbol{x}, \mu_r).$$

□

4.6 The RCI and Möbius inverse

The following proposition gives the closed formula of the Möbius inverse [42] of a function on $\mathcal{Q}$.

Proposition 6. *Suppose $f, g : \mathcal{Q} \to \mathbb{R}$ are two real valued functions on $\mathcal{Q}$. Then*

$$f(A,B) = \sum_{\substack{(C,D)\in\mathcal{Q} \\ (C,D)\subseteq(A,B)}} g(C,D) \qquad \text{for all} \quad (A,B) \in \mathcal{Q} \tag{4.23}$$

if and only if

$$g(A,B) = \sum_{\varnothing\subseteq X\subseteq A} (-1)^{|X|} \sum_{\substack{(C,D)\in\mathcal{Q} \\ (C,D)\subseteq(A\smallsetminus X, B\smallsetminus X)}} (-1)^{|B\smallsetminus A|-|D\smallsetminus C|} f(C,D) \qquad \text{for all} \quad (A,B) \in \mathcal{Q}. \tag{4.24}$$

Proof. See Appendix □

Remark 7. *By setting for all $X \subseteq A \subseteq N$ and for all $(A,B) \in \mathcal{Q}$*

$$g^*(A \smallsetminus X, B \smallsetminus X) = \sum_{(C,D)\subseteq(A\smallsetminus X, B\smallsetminus X)} (-1)^{|B\smallsetminus A|-|D\smallsetminus C|} f(C,D),$$

then equation (4.24) *can be rewritten as*

$$g(A,B) = \sum_{\varnothing\subseteq X\subseteq A} (-1)^{|X|} g^*(A \smallsetminus X, B \smallsetminus X).$$

Remark 8. *Let us apply proposition 6 to $\mathcal{Q}_0 = \{(\varnothing, B)) \mid B \subseteq N\} \subseteq \mathcal{Q}$ which we identify with 2^N. then we obtain the well known result, applied to functions $f, g : 2^N \to \mathbb{R}$,*

$$f(B) = \sum_{D\subseteq B} g(D) \qquad \text{for all} \quad B \in 2^N \tag{4.25}$$

if and only if

$$g(B) = \sum_{D\subseteq B} (-1)^{|B\smallsetminus D|} f(D) \qquad \text{for all} \quad B \in 2^N. \tag{4.26}$$

The first of the two following propositions characterizes an interval-capacity by means of its Möbius inverse. The second one allows the RCI with respect to an interval-capacity to be rewritten using the Möbius inverse of such an interval-capacity.

Proposition 7. *$\mu_r : \mathcal{Q} \to \mathbb{R}$ is an interval-capacity if and only if its Möbius inverse $m : \mathcal{Q} \to \mathbb{R}$ satisfies:*

1. $m(\varnothing, \varnothing) = 0$;
2. $\sum_{(A,B) \in \mathcal{Q}} m(A,B) = 1$;
3. $\sum_{\{a\} \subseteq C \subseteq A} \sum_{C \subseteq D \subseteq B} m(C,D) \geq 0, \ \forall a \in A \subseteq B \in 2^N$;
4. $\sum_{\{b\} \subseteq D \subseteq B} \sum_{C \subseteq A \cap D} m(C,D) \geq 0, \ \forall b \in B \supseteq A \in 2^N$.

Proof. See Appendix ☐

Proposition 8. *Let $\mu_r : \mathcal{Q} \to [0,1]$ be an interval-capacity and let $m : \mathcal{Q} \to [0,1]$ be its Möbius inverse, then for all $\boldsymbol{x} \in \mathcal{I}^n$*

$$Ch_r(\boldsymbol{x}, \mu_r) = \sum_{(A,B) \in \mathcal{Q}} m(A,B) \cdot \left(\min \left\{ \bigwedge_{i \in A} \underline{x}_i, \bigwedge_{j \in B} \overline{x}_j \right\} \right). \tag{4.27}$$

Proof. For all $\boldsymbol{x} \in \mathcal{I}^n$,

$$\begin{aligned} Ch_r(\boldsymbol{x}, \mu_r) &= \textstyle\sum_{i=1}^{2n} x_{(i)} \left[\mu_r \left(A_{(i)}, B_{(i)} \right) - \mu_r \left(A_{(i+1)}, B_{(i+1)} \right) \right] = \\ &= \textstyle\sum_{i=1}^{2n} x_{(i)} \sum_{(A,B) \subseteq \left(A_{(i)}, B_{(i)} \right) \setminus \left(A_{(i+1)}, B_{(i+1)} \right)} m(A,B) = \\ &= \textstyle\sum_{(A,B) \in \mathcal{Q}} m(A,B) \cdot \left(\min \left\{ \bigwedge_{i \in A} \underline{x}_i, \bigwedge_{i \in B} \overline{x}_i \right\} \right). \end{aligned} \tag{4.28}$$

☐

Remark 9. *Note that the term* $\bigwedge_{i\in B}\overline{x}_i$ *can also be written* $\bigwedge_{i\in B\setminus A}\overline{x}_i$ *and can have an influence. See, e.g., the following example:* $N=\{1,2\}$, $(A,B)=(\{1\},\{1,2\})$, $x=([3,4],[1,2])$. *In this case, by applying the* (4.27) *the term* $m(\{1\},\{1,2\})$ *must be multiplied by* $2=\min\{3,4,2\}=\min\{3,2\}$.

Using previous proposition the RCI assumes a linear expression with respect to the interval-measure.

Corollary 2. *There exist functions* $f_{(A,B)}:\mathbb{R}^n\to\mathbb{R}$, $(A,B)\in\mathcal{Q}$ *such that*

$$Ch_r(\boldsymbol{x},\mu_r)=\sum_{(A,B)\in\mathcal{Q}}\mu_r(A,B)f_{(A,B)}(\boldsymbol{x}). \tag{4.29}$$

Proof. Indeed, using the (4.24)

$$m(A,B)=\sum_{\emptyset\subseteq X\subseteq A}(-1)^{|X|}\sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A\setminus X,B\setminus X)}}(-1)^{|B\setminus A|-|D\setminus C|}\mu_r(C,D)\qquad\text{for all}\quad(A,B)\in\mathcal{Q}. \tag{4.30}$$

in the (4.27), the (4.29) is verified with

$$f_{(A,B)}(\boldsymbol{x})=\sum_{\emptyset\subseteq X\subseteq N\setminus A}(-1)^{|X|}\sum_{(A\cup X,B\cup X)}(-1)^{|B\setminus A|}\bigwedge\left\{\bigwedge_{i\in A\cup X}\underline{x}_i,\ \bigwedge_{i\in B\cup X}\overline{x}_i\right\} \tag{4.31}$$

□

4.7 The robust Sugeno and Shilkret integrals

Let us consider a set of criteria $N=\{1,2,\ldots,n\}$ and a set of alternatives $A=\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\ldots\}$ to be evaluated, on each criterion, on the scale $[0,1]$. then each

$\boldsymbol{x} \in A$ can be identified with a score vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$, whose ith component, x_i, represents the evaluation of $\boldsymbol{x}$ with respect to the ith criterion.

Definition 42. *The Sugeno Integral [47] of* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$ *with respect to the capacity* $\nu : 2^{[0,1]} \to [0,1]$ *is*

$$S(\boldsymbol{x}, \nu) = \bigvee_{i \in N} \bigwedge \left\{x_{(i)}, \nu\left(A_{(i)}\right)\right\}, \tag{4.32}$$

being $(\cdot) : N \to N$ *an indexes permutation such that* $x_{(1)} \leq \ldots \leq x_{(n)}$ *and* $A_{(i)} = \{(i), \ldots, (n)\}$, $i = 1, \ldots, n$.

It follows from the definition that $S(\boldsymbol{x}, \nu) \in \{x_1, \ldots, x_n\} \bigcup \{\nu(A) \mid A \subseteq N\}$. Moreover the Sugeno integral can also be computed if the elements of the set

$$\{x_1, \ldots, x_n\} \bigcup \{\nu(A) \mid A \subseteq N\}$$

are just ranked on an ordinal scale.
The (4.32) involves n terms but requests a permutation. An equivalent formulation (see [36]) involves 2^n terms but does not request a permutation.

$$S(\boldsymbol{x}, \nu) = \bigvee_{A \subseteq N} \bigwedge \left\{\nu(A), \bigwedge_{i \in A} x_i\right\}. \tag{4.33}$$

Now, suppose that for every $\boldsymbol{x} \in A$, we have, on each criterion, a numerical imprecise evaluation on the scale $[0,1]$. Specifically, suppose that for each $i \in N$ we know a range $[\underline{x}_i, \overline{x}_i] \subseteq [0,1]$ containing the exact evaluation of $\boldsymbol{x}$ with respect to i. then, being $\mathcal{I}_{[0,1]} = \{[a,b] \mid a, b \in [0,1],\ a \leq b\}$ the set of bounded and closed subintervals of $[0,1]$, any alternative $\boldsymbol{x}$ can be identified with a score vector

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots,\ [\underline{x}_i, \overline{x}_i], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n_{[0,1]}, \tag{4.34}$$

whose ith component, $x_i = [\underline{x}_i, \overline{x}_i]$, is the interval containing the evaluation of $\boldsymbol{x}$ with respect to the ith criterion. Vectors of $[0,1]^n$ are considered elements of $\mathcal{I}^n_{[0,1]}$ by identifying each $x \in [0,1]$ with the degenerate interval $[x,x] = \{x\}$. We associate to every $\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$ the vector $\underline{\boldsymbol{x}} = (\underline{x}_1, \ldots, \underline{x}_n)$ of all the worst (or pessimistic) evaluations and the vector $\overline{\boldsymbol{x}} = (\overline{x}_1, \ldots, \overline{x}_n)$ of all the best (or optimistic) evaluations on each criterion.

Definition 43. *The Robust Sugeno Integral (RSI) of* $\boldsymbol{x}$ *with respect to the interval-capacity* μ_r *is*

$$S_r(\boldsymbol{x}, \mu_r) = \bigvee_{(A,B)\in\mathcal{Q}} \bigwedge \left\{ \bigwedge_{i\in A} \underline{x}_i, \bigwedge_{i\in B-A} \overline{x}_i \ , \ \mu_r(A,B) \right\}. \tag{4.35}$$

It follows from the definition that

$$S_r(\boldsymbol{x}, \mu_r) \in \{\underline{x}_1, \ldots, \underline{x}_n\} \bigcup \{\overline{x}_1, \ldots, \overline{x}_n\} \bigcup \{\mu_r(A,B) \mid (A,B) \in \mathcal{Q}\}$$

Moreover the RSI can also be computed if the elements of this set are just ranked on an ordinal scale.
The (4.35) involves $|\mathcal{Q}| = 3^n$ terms. An alternative formulation of the RSI implies some additional notations. We identify every vector

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n_{[0,1]}$$

with the vector

$$\boldsymbol{x}^* = (x_1, \ldots, x_{2n}) \in [0,1]^{2n}$$

defined according to (4.4). Let $(\cdot) : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ be a permutation of indices such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(2n)}$ and for all $i = 1, \ldots, 2n$ let us define $A_{(i)} = \left\{ j \in N \mid \underline{x}_j \geq x_{(i)} \right\}$ and $B_{(i)} = \left\{ j \in N \mid \overline{x}_j \geq x_{(i)} \right\}$. then, the RSI of $\boldsymbol{x}$ with respect to the interval-capacity μ_r is:

$$S_r(\boldsymbol{x}, \mu_r) = \bigvee_{i \in \{1,\dots,2n\}} \bigwedge \left\{x_{(i)}, \mu_r\left(A_{(i)}, B_{(i)}\right)\right\}. \tag{4.36}$$

This requests $2n$ terms and a permutation of indices.

We now give two illustrative examples. The first just shows the equivalence of formulation (4.36) and (4.35), with the scale $[0,1]$ substituted by the scale $[0,10]$. The second is applied to a student evaluation problem with the scale $[0,1]$ substituted by the scale $[0,30]$.

Example 1

Let us suppose that $N = \{1,2\}$ and consider $\boldsymbol{x} = ([5,9],[2,4])$. Let be given the following interval-capacity on $\mathcal{Q}$:

$$\mu_r(\varnothing,\varnothing) = 0,\ \mu_r(\varnothing,\{1\}) = 3,\ \mu_r(\varnothing,\{2\}) = 2,\ \mu_r(\varnothing,N) = 5,\ \mu_r(\{1\},\{1\}) = 4,$$

$$\mu_r(\{1\},N) = 6,\ \mu_r(\{2\},\{2\}) = 4,\ \mu_r(\{2\},N) = 7,\ \mu_r(N,N) = 10.$$

It follows that

$$\mu_r(A(\boldsymbol{x},2), B(\boldsymbol{x},2)) = \mu_r(N,N) = 10,\ \mu_r(A(\boldsymbol{x},4), B(\boldsymbol{x},4)) = \mu_r(\{1\},N) = 6,$$

$$\mu_r(A(\boldsymbol{x},5), B(\boldsymbol{x},5)) = \mu_r(\{1\},\{1\}) = 4,\ \mu_r(A(\boldsymbol{x},9), B(\boldsymbol{x},9)) = \mu_r(\varnothing,\{1\}) = 3.$$

By using the (4.36) we get

$$S_r(\boldsymbol{x}, \mu_r) = \max\{\min\{2,10\}, \min\{4,6\}, \min\{5,4\}, \min\{9,3\}\} =$$

$$\max\{2,4,4,3\} = 4.$$

Alternatively, we can use the (4.35)

$$S_r(\boldsymbol{x}, \mu_r) = \max\{0, \min\{3,9\}, \min\{2,4\}, \min\{5,4\}, \min\{4,5,9\},$$

$$\min\{6,5,4\}, \min\{4,2,4\}, \min\{7,2,4\}, \min\{10,2,4\}\} = 4.$$

Example 2

Suppose we need to evaluate a university student in four economic subjects, $N = \{m_1, m_2, m_3, m_4\}$ of which $\{m_1, m_2\}$ belong to the subcategory of microeconomic. We suppose that the student is evaluated on each subject by a 30 point scale, allowing interval-evaluations. Let us consider the vector $E(Student) = E(S)$ containing the single evaluation in each subject $E(m_i)$:

$$E(S) = (E(m_1), E(m_2)\, E(m_3), E(m_4)) = ([26,30], [28,30], [24,27], [23,27]) \tag{4.37}$$

In order to compute the RSI of $E(S)$ we have to specify some values of an interval-capacity defined on $\mathcal{Q}$. For example the following:

$$\mu_r(N,N) = 30,\ \mu_r(\{m_1,m_2,m_3\}, N) = 29,\ \mu_r(\{m_1,m_2\}, N) = 28,$$
$$\mu_r(\{m_2\}, N) = 24,\ \mu_r(\{m_2\}, \{m_1,m_2\}) = 23\mu_r(\varnothing, \{m_1,m_2\}) = 20. \tag{4.38}$$

These weights reflect the fact that we retain the microeconomic subcategory $\{m_1, m_2\}$ particularly important. Indeed when $\{m_1, m_2\}$ is not included on A the weight assigned to (A, B) is small. The question is: how much should be globally evaluated the student in accordance with the partials evaluations (4.37) and (4.38)? Using the RSI, equation (4.35), such a student should be evaluated

$$S_r(S, \mu_r) = \bigvee \{\bigwedge\{23,30\},\ \bigwedge\{24,29\}, \bigwedge\{26,28\},$$

$$\bigwedge\{27,24\}\ \bigwedge\{28,23\}\ \bigwedge\{30,20\}\} = 26.$$

In this case we cannot assign a greater evaluation, due to the pessimistic evaluation of the student in the relevant subject m_1.

For nonnegative valued alternative, another famous integral useful to aggregate criteria evaluations is the Shilkret integral [45].

Definition 44. *The Shilkret integral [45] of a vector* $\boldsymbol{x} = (x_1, \ldots, x_n) \in [0,1]^n$ *with respect to the capacity* ν *is given by*

$$Sh(\boldsymbol{x}, \nu) = \bigvee_{i \in N} \{x_i \cdot \nu(\{j \in N : x_j \geq x_i\}\} . \tag{4.39}$$

For interval-evaluations on the criteria, the Shilkret integral can be computed with respect to an interval-capacity. Let us define

$$\mathcal{I}[0,1]^n = \{[a,b] \mid a, b \in \mathbb{R},\ 0 \leq a \leq b \leq 1\}$$

then we have the following

Definition 45. *The robust Shilkret integral of* $\boldsymbol{x} \in \mathcal{I}^n_{[0,1]}$ *with respect to the interval-capacity* μ_r *is*

$$Sh_r(x, \mu_r) = \bigvee_{(A,B) \in \mathcal{Q}} \left\{ \bigwedge \left\{ \bigwedge_{i \in A} \underline{x}_i, \bigwedge_{i \in B} \overline{x}_i \right\} \cdot \mu_r(A,B) \right\}.$$

4.8 Other robust integrals

What we have done regarding the Choquet, Shilkret and the Sugeno integrals can be extended to other integrals. Recently, in the context of multiple-criteria decision analysis, the literature on fuzzy integrals has increased very fast. An interesting line of research is that of bipolar fuzzy integrals: the bipolar Choquet integral has been proposed in [14, 15, 22] and the bipolar Shilkret and Sugeno integrals have been proposed in [25]. Here we propose the generalization of the bipolar Choquet integral to the case of interval-evaluations. Let us consider the set

$$\mathcal{Q}_b = \{(A^+, B^+, A^-, B^-) \mid A^+ \subseteq B^+ \subseteq N,\ N \supseteq A^- \supseteq B^- \text{ and } B^+ \cap A^- = \emptyset\}.$$

Definition 46. *A function $\mu_r^b : \mathcal{Q}_b \to [-1,1]$ is a bipolar interval-capacity on $\mathcal{Q}_b$ if*

- $\mu_r^b(\emptyset,\emptyset,\emptyset,\emptyset) = 0$, $\mu_r^b(N,N,\emptyset,\emptyset) = 1$ *and* $\mu_r^b(\emptyset,\emptyset,N,N) = -1$;
- $\mu_r^b(A_1^+, B_1^+, A_1^-, B_1^-) \leq \mu_r^b(A_2^+, B{:}_2^+, A_2^-, B_2^-)$ *for all*

$$(A_1^+, B_1^+, A_1^-, B_1^-), (A_2^+, B{:}_2^+, A_2^-, B_2^-) \in \mathcal{Q}_b$$

such that $A_1^+ \subseteq A_2^+$, $B_1^+ \subseteq B_2^+$, $A_1^- \supseteq A_2^-$ *and* $B_1^- \supseteq B_2^-$.

Definition 47. *The bipolar Robust Choquet Integral (bRCI) of*

$$\boldsymbol{x} = ([\underline{x}_1, \overline{x}_1], \ldots, [\underline{x}_n, \overline{x}_n]) \in \mathcal{I}^n$$

with respect to a bipolar interval-capacity $\mu_r^b : 2^N \to [0,1]$ is given by:

$$Ch_r^b(\boldsymbol{x}, \mu_r b) =: \int_{-\infty}^{\infty} \mu_r^G(\{i \mid \underline{x}_i > t\}, \{i \mid \overline{x}_i > t\}, \{i \mid \underline{x}_i < -t\}, \{i \mid \overline{x}_i < -t\}) dt. \quad (4.40)$$

A further generalization in the field of fuzzy integrals is that of level dependent integrals. This line of research has lead to the definition of the level dependent Choquet integral and the bipolar level dependent Choquet integral [21], the level dependent Shilkret integral [4], the level dependent Sugeno integral [37]. In [21] the *generalized Choquet integral* is defined with respect to a level dependent capacity. Also the RCI can be generalized in this sense.

Definition 48. *Let* $(\alpha, \beta) \subseteq \mathbb{R}$ *be any possible interval of the real line. A level dependent interval-capacity is a function* $\mu_r^G : \mathcal{Q} \times (\alpha, \beta) \to [0, 1]$ *such that*

1. *for all* $t \in (\alpha, \beta)$ *and* $(A, B) \subseteq (C, D) \in \mathcal{Q}$*,*

$$\mu_r^G((A, B), t) \leq \mu_r^G((C, D), t)$$

2. *for all* $t \in (\alpha, \beta)$*,* $\mu_r^G((\varnothing, \varnothing), t) = 0$ *and* $\mu_r^G((N, N), t) = 1$

3. *for all* $(A, B) \in \mathcal{Q}$*,* $\mu_r^G((A, B), t)$ *considered as a function with respect to* t *is Lebesgue measurable.*

Definition 49. *The Robust Choquet Integral (RCIg) of* $\boldsymbol{x} \in \left(\mathcal{I}_{(\alpha,\beta)}\right)^n$ *with respect to the level dependent interval-capacity* $\mu_r^G : \mathcal{Q} \times (\alpha, \beta) \to [0, 1]$ *is given by:*

$$Ch_r^G(\boldsymbol{x}, \mu_r) =: \int_{\min\{\underline{x}_1,\ldots,\underline{x}_n\}}^{\infty} \left(\mu_r^G(\{i \in N \mid \underline{x}_i \geq t\}, \{i \in N \mid \overline{x}_i \geq t\}), t\right) dt$$

$$+ \min\{\underline{x}_1, \ldots, \underline{x}_n\}. \tag{4.41}$$

Definition 50. *The Robust Concave Integral of a non-negative interval-valued alternative* $\boldsymbol{x} \in \mathcal{I}_+^n$ *with respect to the interval-capacity* μ_r *is*

$$\int^{cav} \boldsymbol{x} d\mu_r = \bigvee \left\{ \sum_{(A,B)\in\mathcal{Q}} \alpha_{(A,B)} \mu_r(A, B) \mid \sum_{(A,B)\in\mathcal{Q}} \alpha_{(A,B)} \boldsymbol{1}_{(A,B)} = \boldsymbol{x},\ \alpha_{(A,B)} \geq 0 \right\}. \tag{4.42}$$

Obviously, if on every criterion $\boldsymbol{x}$ receives an exact evaluation, then the (4.42) reduces to the Concave Integral of $\boldsymbol{x} \in \mathbb{R}^n_+$ with respect to the capacity $\nu(A) = \mu_r(A, A)$.

4.9 Generalizing the concept of interval to m–points interval

In [40] the concept of interval has been generalized (allowing the presence of more than two points).

We can image that on every of the n criteria an alternative $\boldsymbol{x}$ is evaluated m times, so that this alternative can be identified with a vector of score vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ being for all $i = 1, \ldots, n$

$$x_i = (f_1(x_i), \ldots f_m(x_i)) \qquad \text{with} \qquad f_j(x_i) \le f_{j+1}(x_i) \text{ for all } j = 1, \ldots, m-1.$$

For example, the case $m = 3$ corresponds to have on each criterion a pessimistic, a realistic and an optimistic evaluation.

The idea to extend the RCI to the case of m–interval based evaluation is simple. Let us define

$$\mathcal{Q}_m = \{(A_1, \ldots, A_m) \mid A_1 \subseteq A_2 \ldots \subseteq A_m \subseteq N\}.$$

Definition 51. *An m–interval-capacity is a function $\mu_m : \mathcal{Q}_m \to [0,1]$ such that*

- $\mu_m(\varnothing, \ldots, \varnothing) = 0,$
- $\mu_m(N, \ldots, N) = 1,$
- $\mu_m(A_1, \ldots, A_m) \le \mu_m(B_1, \ldots, B_m)$, *whenever* $A_i \subseteq B_i \subseteq N, \ \forall i = 1, \ldots, m.$

Definition 52. *The Robust Choquet Integral of* $\boldsymbol{x}$ *(m−points interval-valued) w.r.t. the m−interval-capacity* μ_m *is*

$$Ch_r(\boldsymbol{x}, \mu_m) = \int_{\min_i f_1(x_i)}^{\max_i f_m(x_i)} \mu_m \left(\{j|f_1(x_j) \geq t\}, \ldots, \{j|f_m(x_j) \geq t\}dt\right) + \min_i f_1(x_i). \tag{4.43}$$

4.10 Future researches

We consider h−intervals $[a_1, \ldots, a_h], a_1, \ldots, a_h \in \mathbb{R}$ such that $a_1 \leq \ldots \leq a_h$ that express evaluations with respect to a considered point of view by means of the h values $a_1, \ldots, a_h$. For example, if $h = 2$, then evaluations w.r.t. each criterion are 2-intervals assigning to each alternative two evaluations corresponding to a pessimistic and an optimistic evaluation. If $h = 3$, then evaluations are 3-intervals $[a_1, a_2, a_3]$ assigning to each alternative three evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 corresponds to an average evaluation and a_3 corresponds to an optimistic evaluation. If $h = 4$, then evaluations are 4-intervals $[a_1, a_2, a_3, a_4]$ assigning to each alternative four evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 and a_3 to two evaluations defining an interval $[a_2, a_3]$ of average evaluation and a_4 corresponds to an optimistic evaluation. Observe that 2-interval evaluations can be seen as usual intervals of evaluations, 3-interval evaluations can be seen as triangular fuzzy numbers and 4-intervals evaluations can be seen as trapezoidal fuzzy numbers. Similar situations we have with $h \geq 5$. Let us denote by $\mathcal{I}_h$ the set of all h-intervals, i.e.

$$\mathcal{I}_h = \{[a_1, \ldots, a_h] \mid a_1, \ldots, a_h \in \mathbb{R}\ , a_1 \leq \ldots \leq a_h\}.$$

A general framework for the comparison of $h-$intervals has been presented in [40]. Here we introduce $h-k-$aggregation functions that assigns to vectors

$$\boldsymbol{x} = ([x_{11}, \ldots, x_{1h}], \ldots, [x_{n1} \ldots, x_{nh}]) \in \mathcal{I}_h^n$$

of $h-$interval evaluations with respect to a set $N = \{1, \ldots, n\}$ of considered criteria an overall evaluation in terms of a $k-$interval. Formally an $h-k-$aggregation function is a function $g : \mathcal{I}_h^n \to \mathcal{I}_k$ satisfying the following properties:

- monotonicity: for all $x, y \in \mathcal{I}_h^n$, if $x_{i,j} \geq y_{i,j}$ for all $i \in N$ and for all $j = 1, \ldots, h$, then $g_r(x) \geq g_r(y)$ for all $r = 1, \ldots, k$;
- left boundary condition: if $x_{i,h} \to -\infty$ for all $i = 1, \ldots, n$, then $g_r(x) \to -\infty$ for all $r = 1, \ldots, k$;
- right boundary condition if $x_{i,1} \to +\infty$ for all $i = 1, \ldots, n$, then $g_r(x) \to +\infty$ for all $r = 1, \ldots, k$;

4.11 The $h-k-$weighted average

Let us consider a vector $\mathbf{a} = [a_{i,j,r}], \mathbf{a} \in [0,1]^{n \times h \times k}$ such that

(i) $\sum_{j=h-t}^{h} a_{i,j,r_1} \geq \sum_{j=h-t}^{h} a_{i,j,r_2}$, for all $i = 1, \ldots, n$, $t = 1, \ldots h-1$ and $r_1, r_2 = 1, \ldots, k$, such that $r_1 \geq r_2$;

(ii) $\sum_{i=1}^{n} \sum_{j=1}^{h} a_{i,j,r} = 1$, for all $r = 1, \ldots, k$.

The $h-k-$weighted average w.r.t. weights $\mathbf{a} = [a_{i,j,r}]$ is the $h-k-$aggregation function $WA_{\mathbf{a}} : \mathcal{I}_h^n \to \mathcal{I}_k$ defined as follows: for all $\boldsymbol{x} \in \mathcal{I}_h^n$ and $r = 1, \ldots, k$,

$$WA_{\mathbf{a},r}(x) = \sum_{i=1}^{n} \sum_{j=1}^{h} a_{i,j,r} x_{i,j}. \tag{4.44}$$

The $h-k$–weighted average can be formulated also as follows. Let us consider a vector $\mathbf{a}' = [a'_{i,j,r}], \mathbf{a}' \in [0,1]^{n\times h\times k}$ such that

(i)' $a'_{i,1,r} \geq a'_{i,2,r} \geq \ldots \geq a'_{i,h,r} \geq 0$, for all $i = 1, \ldots, n$ and $r = 1, \ldots, k$;

(ii)' $a'_{i,j,1} \geq a'_{i,j,2} \geq \ldots \geq a'_{i,j,k} \geq 0$, for all $i = 1, \ldots, n$ and $j = 1, \ldots, h$;

(iii)' $\sum_{i=1}^{n} a'_{i,1,r} = 1$, for all $i = 1, \ldots, n$ and $r = 1, \ldots, k$.

The $h-k$–weighted average with respect to weights $\mathbf{a}' = [a'_{i,j,r}]$ is the $h-k$–aggregation function $WA_{\mathbf{a}} : \mathcal{I}_h^n \to \mathcal{I}_k$ defined as follows: for all $\boldsymbol{x} \in \mathcal{I}_h^n$ and $r = 1, \ldots, k$,

$$WA_{\mathbf{a},r}(\boldsymbol{x}) = \sum_{i=1}^{n} a'_{i,1,r} x_{i,1} + \sum_{i=1}^{n}\sum_{j=2}^{h} a'_{i,j,r}(x_{i,j} - x_{i,j-1}). \tag{4.45}$$

There is the following relation between weights $a'_{ij,r}$ and $a_{i,j,r}$: for all $i = 1, \ldots, n$; $j = 1, \ldots, h-1$; and $r = 1, \ldots, k$ it holds the

$$\begin{cases} a_{i,j,r} = a'_{i,j,r} - a'_{i,j+1,r} \\ a_{i,h,r} = a'_{i,h,r}. \end{cases} \tag{4.46}$$

Two very natural conditions for $h-k$–aggregation functions are the following

- additivity: for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{I}_h^n$, $g(\boldsymbol{x} + \boldsymbol{y}) = g(\boldsymbol{x}) + g(\boldsymbol{y})$, where $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z}$ with $z_{i,j} = x_{i,j} + y_{i,j}$ for all $i \in N$ and for all $j = 1, \ldots, h$;

- idempotency: for all $a \in \mathbb{R}$, $g(\boldsymbol{a}) = a$, where $\boldsymbol{a} \in \mathcal{I}_h^n$ is $\boldsymbol{a} = [a, \ldots, a]$.

Theorem 13. *An $h-k$–aggregation function is additive and idempotent if and only if it is the $h-k$–weighed average.*

Proof. The $h-k$–weighed average is additive and idempotent by definition. Let $f : \mathcal{I}_h^n \to \mathcal{I}_k$ be an additive and idempotent $h-k$–aggregation function and let us consider a generic element $\boldsymbol{x} = ([x_{11}, \dots, x_{1h}], \dots, [x_{n1} \dots, x_{nh}]) \in \mathcal{I}_h^n$. Let us consider the following $n \cdot h$ vectors of $\mathcal{I}_h^n$

$$\boldsymbol{x}_{1,1} = ((1, 1, \dots, 1), \dots, (0, \dots, 0)),$$
$$\boldsymbol{x}_{1,2} = ((0, 1, \dots, 1), \dots, (0, \dots, 0)),$$
$$\dots$$
$$\boldsymbol{x}_{1,h} = ((0, \dots, 0, 1), \dots, (0, \dots, 0)),$$
$$\dots$$
$$\boldsymbol{x}_{n,1} = ((0, \dots, 0), \dots, (1, 1, \dots, 1)),$$
$$\boldsymbol{x}_{n,2} = ((0, \dots, 0), \dots, (0, 1, \dots, 1)),$$
$$\dots$$
$$\boldsymbol{x}_{n,h} = ((0, \dots, 0), \dots, (0, \dots, 0, 1)).$$

Consider the following decomposition of $\boldsymbol{x}$

$$\begin{aligned} \boldsymbol{x} = \quad & x_{1,1}\boldsymbol{x}_{1,1} + (x_{1,2} - x_{1,1})\,\boldsymbol{x}_{1,2} + \dots + \left(x_{1,h} - x_{1,h-1}\right)\boldsymbol{x}_{1,h} \quad + \\ & \dots \\ + \quad & x_{n,1}\boldsymbol{x}_{n,1} + (x_{n,2} - x_{n,1})\,\boldsymbol{x}_{n,2} + \dots + \left(x_{n,h} - x_{n,h-1}\right)\boldsymbol{x}_{n,h} \quad . \end{aligned}$$

By additivity and idempotency of f we get

$$\begin{aligned} f(\boldsymbol{x}) = \quad & (x_{1,1} f(\boldsymbol{x}_{1,1}) + (x_{1,2} - x_{1,1})\, f(\boldsymbol{x}_{1,2}) + \dots + \left(x_{1,h} - x_{1,h-1}\right) f(\boldsymbol{x}_{1,h}) \quad + \\ & \dots \\ + \quad & x_{n,1} f(\boldsymbol{x}_{n,1}) + (x_{n,2} - x_{n,1})\, f(\boldsymbol{x}_{n,2}) + \dots + \left(x_{n,h} - x_{n,h-1}\right) f(\boldsymbol{x}_{n,h}) \quad . \end{aligned}$$

Finally, let us consider the following $n \cdot h$ vectors of $\mathcal{I}_k$

$$\mathbf{a}'_{1,1} = f(\boldsymbol{x}_{1,1}) = (a'_{1,1,1}, \dots, a'_{1,1,r}),$$
$$\mathbf{a}'_{1,2} = f(\boldsymbol{x}_{1,2}) = (a'_{1,2,1}, \dots, a'_{1,2,r}),$$
$$\dots$$
$$\mathbf{a}'_{1,h} = f(\boldsymbol{x}_{1,h}) = (a'_{1,h,1}, \dots, a'_{1,h,r}),$$
$$\dots$$
$$\mathbf{a}'_{n,1} = f(\boldsymbol{x}_{n,1}) = (a'_{n,1,1}, \dots, a'_{n,1,r}),$$
$$\mathbf{a}'_{n,2} = f(\boldsymbol{x}_{n,2}) = (a'_{n,2,1}, \dots, a'_{n,2,r}),$$
$$\dots$$
$$\mathbf{a}'_{n,n} = f(\boldsymbol{x}_{n,n}) = (a'_{n,n,1}, \dots, a'_{n,n,r}).$$

These $a'_{i,j,r}$ (with $i = 1, \dots, n;\ j = 1, \dots, h;\ r = 1, \dots, r$) satisfy the $(i)', (ii)', (iii)'$ and equation (4.45) is also satisfied for all $r = 1, \dots, k$

$$f(\boldsymbol{x})_r = \sum_{i=1}^{n} a'_{i,1,r} x_{i,1} + \sum_{i=1}^{n} \sum_{j=2}^{h} a'_{i,j,r}(x_{i,j} - x_{i,j-1}).$$

□

4.12 Non-additive h–k–aggregation functions

Let us consider the set $\mathcal{Q} = \{(A_1, \dots, A_h) \mid A_1 \subseteq A_2 \subseteq \dots \subseteq A_h \subseteq N\}$. With a slight abuse of notation we extend to $\mathcal{Q}$ the relation of set inclusion and the operations of union and intersection by defining for all $(A_1, \dots, A_h)$ and $(B_1, \dots, B_h) \in$

$\mathcal{Q}$,

$$(A_1, \ldots, A_h) \subseteq (B_1, \ldots, B_h) \text{ if and only if } A_i \subseteq B_i \text{ for all } i = 1, \ldots, h;$$

$$(A_1, \ldots, A_h) \cup (B_1, \ldots, B_h) = (A_1 \cup B_1, \ldots, A_h \cup B_h);$$

$$(A_1, \ldots, A_h) \cap (B_1, \ldots, B_h) = (A_1 \cap B_1, \ldots, A_h \cap B_h).$$

Regarding the algebraic structure of $\mathcal{Q}$, we can observe that with respect to the relation $\subseteq$, $\mathcal{Q}$ is a lattice.

Definition 53. *A function $\mu_h : \mathcal{Q} \to [0,1]$ is an h-interval-capacity on $\mathcal{Q}$ if*

- $\mu_r(\emptyset, \ldots \emptyset) = 0$*, and* $\mu_h(N, \ldots, N) = 1$*; and*
- $\mu_h(A_1, \ldots, A_h) \le \mu_h(B_1, \ldots, B_h)$ *for all* $(A_1, \ldots, A_h), (B_1, \ldots, B_h) \in \mathcal{Q}$ *such that* $A_i \subseteq B_i$ *for all* $i = \ldots, h$.

Definition 54. *A $h - k-$capacity is a vector $(\mu_{h_1}, \ldots, \mu_{h_k})$ such that*

- *for every* $i = 1, \ldots, k$, $\mu_{h_i} : \mathcal{Q} \to [0,1]$ *is an h-interval capacity; and*
- *for all* $(A_1, \ldots, A_h) \in \mathcal{Q}$, $\mu_{h_i}(A_1, \ldots, A_h) \le \mu_{h_{i+1}}(A_1, \ldots, A_h)$, *for all* $i = 1, \ldots, k-1$.

Moreover, we say that the $h - k-$capacity $(\mu_{h_1}, \ldots, \mu_{h_k})$ is additive if every capacity μ_{h_i} $i = 1, \ldots, k$ is additive.

Definition 55. *The $h - k-$Choquet Integral of*

$$\boldsymbol{x} = ([x_{1,1}, \ldots, x_{1,h}], \ldots, [x_{n,1} \ldots, x_{n,h}])$$

with respect to the $h-k-$capacity $(\mu_{h_1},\dots,\mu_{h_k})$ is given by

$$Ch_{h-k}\left(\boldsymbol{x},(\mu_{h_1},\dots,\mu_{h_k})\right) =: \left[Ch_h\left(\boldsymbol{x},\mu_{h_1}\right),\dots,Ch_h\left(\boldsymbol{x},\mu_{h_k}\right)\right], \tag{4.47}$$

being for all $r=1,\dots,k$

$$Ch_h\left(\boldsymbol{x},\mu_{h_r}\right) = \int_{\min\limits_{i\in N} x_{1,i}}^{\max\limits_{i\in N} x_{i,h}} \mu_{h_r}(\{i\in N|x_{1,i}\geq t\},\dots,\{i\in N|x_{n,i}\geq t\})dt + \min_{i\in N} x_{1,i}. \tag{4.48}$$

Let us note that the $2-1-$Choquet integral is the Robust Choquet integral presented in [26]. Moreover a $h-k-$Choquet integral computed with respect to an additive $h-k-$capacity becomes an $h-k-$weighting average.

Definition 56. *The two vectors of $\mathcal{I}_h^n$*

$$\boldsymbol{x} = \left(\left[x_{1,1},\dots,x_{1,h}\right],\dots,\left[x_{n,1}\dots,x_{n,h}\right]\right), \boldsymbol{y} = \left(\left[y_{1,1},\dots,y_{1,h}\right],\dots,\left[y_{n,1}\dots,y_{n,h}\right]\right)$$

are comonotone if the two vectors of $\mathbb{R}^{nh}$ $\boldsymbol{x}^* = \left(x_{1,1},\dots,x_{1,h},\dots,x_{n,1},\dots,x_{n,h}\right)$ and $\boldsymbol{y}^* = \left(y_{1,1},\dots,y_{1,h},\dots,y_{n,1}\dots,y_{n,h}\right)$ are comonotone.

We extend the property of comonotone additivity for a standard aggregation function to an $h-k-$aggregation function: an $h-k-$aggregation function is comonotone additive if it is additive for comonotone vectors.

Theorem 14. *An $h-k-$aggregation function is comonotone additive and idempotent if and only if it is the $h-k-$Choquet integral.*

In [26] the robust Shilkret and Sugeno integrals have been presented. These are $2-1-$aggregation functions which can be generalized to the case of $h-k-$aggregation functions.

Definition 57. *Given*

$$\boldsymbol{x} = \left(\left[x_{1,1}, \ldots, x_{1,h}\right], \ldots, \left[x_{n,1} \ldots, x_{n,h}\right]\right) \in \mathcal{I}_h^n$$

the $h-k-$Shilkret integral of $\boldsymbol{x}$ with respect to the the $h-k-$capacity $(\mu_{h_1}, \ldots, \mu_{h_k})$ is given by

$$Sh_{h-k}\left(\boldsymbol{x}, (\mu_{h_1}, \ldots, \mu_{h_k})\right) =: \left[Sh_h\left(\boldsymbol{x}, \mu_{h_1}\right), \ldots, Sh_h\left(\boldsymbol{x}, \mu_{h_k}\right)\right], \tag{4.49}$$

being for all $r = 1, \ldots, k$

$$Sh_h\left(\boldsymbol{x}, \mu_{h_r}\right) = \bigvee_{(A_1,\ldots,A_h)\in\mathcal{Q}} \left\{ \bigwedge \left\{ \bigwedge_{i\in A_1} x_{1,i}, \ldots, \bigwedge_{i\in A_h} x_{h,i} \right\} \cdot \mu_{h,r}(A_1, \ldots, A_h) \right\}. \tag{4.50}$$

Definition 58. *Given*

$$\boldsymbol{x} = \left(\left[x_{1,1}, \ldots, x_{1,h}\right], \ldots, \left[x_{n,1} \ldots, x_{n,h}\right]\right) \in \mathcal{I}_h^n$$

the $h-k-$Sugeno integral of $\boldsymbol{x}$ with respect to the the $h-k-$capacity $(\mu_{h_1}, \ldots, \mu_{h_k})$ is given by

$$Su_{h-k}\left(\boldsymbol{x}, (\mu_{h_1}, \ldots, \mu_{h_k})\right) =: \left[Su_h\left(\boldsymbol{x}, \mu_{h_1}\right), \ldots, Su_h\left(\boldsymbol{x}, \mu_{h_k}\right)\right], \tag{4.51}$$

being for all $r = 1, \ldots, k$

$$Su_h\left(\boldsymbol{x}, \mu_{h_r}\right) = \bigvee_{(A_1,\ldots,A_h)\in\mathcal{Q}} \bigwedge \left\{ \mu_{h,r}(A_1, \ldots, A_h), \bigwedge_{i\in A_1} x_{1,i}, \ldots, \bigwedge_{i\in A_h} x_{h,i} \right\}. \tag{4.52}$$

Finally, in [26] several non-additive $2-1-$aggregation functions have been presented, i.e. the robust Choquet integral with respect to a bipolar interval-capacity, the robust Choquet integral with respect to an interval capacity level dependent, the robust concave integral and the robust universal integral. All these integrals admit a natural generalization to the case of $h-k-$aggregation functions presented here.

4.13 Conclusions

In this chapter we have faced the question regarding the aggregation of interval-evaluations of an alternative on various criteria into a single overall evaluation. To this scope we have introduced the concept of interval-capacity which allows for a quite natural generalizations of the classical Choquet Shilkret and Sugeno integrals to the case of interval-evaluations. We called these generalizations robust integrals. Our analysis shows that, when the interval-evaluations collapse into exact evaluations, our definitions of robust integrals collapse into the original definitions. Situations where we meet imprecise evaluations are very common in the real life (we have provided realistic examples), so the aim of this chapter is to cover the existing gap in the literature for the aggregations of such data.

4.14 Appendix

In order to prove proposition 6, we need some preliminary lemmas.

The following two lemmas have been proved in [44] (see also [6])

Lemma 1. *If A is a finite set then*

$$\sum_{B\subseteq A}(-1)^{|B|}=\begin{cases}1 & \text{if } A=\varnothing\\ 0 & \text{otherwise.}\end{cases} \tag{4.53}$$

Lemma 2. *If A is a finite set and $B \subseteq A$ then*

$$\sum_{B \subseteq C \subseteq A} (-1)^{|C|} = \begin{cases} (-1)^{|A|} & \text{if } A = B \\ 0 & \text{otherwise.} \end{cases} \tag{4.54}$$

With these results we are able to prove the following additional lemmas

Lemma 3. *For all $(A, B) \in \mathcal{Q}$*

$$\sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} (-1)^{|D|} = \begin{cases} (-1)^{|B|} & \text{if } A = B \\ 0 & \text{otherwise.} \end{cases} \tag{4.55}$$

Proof.

$$\sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} (-1)^{|D|} = \sum_{X \subseteq A} \sum_{Y \subseteq B \setminus X} (-1)^{|Y|} = (-1)^{|A|} \sum_{Y \subseteq B \setminus X} (-1)^{|Y|} + \sum_{X \subseteq A} \sum_{Y \subset B \setminus X} (-1)^{|Y|} =$$

$$= (-1)^{|A|} \sum_{Y \subseteq B \setminus X} (-1)^{|Y|} = \text{lemma 1} = \begin{cases} (-1)^{|B|} & \text{if } A = B \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 10. *If $A = \varnothing$ then lemma 3 coincides with lemma 1.*

Corollary 3. *For all $(A, B) \in \mathcal{Q}$*

$$\sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} (-1)^{|B \setminus D|} = \begin{cases} 1 & \text{if } A = B \\ 0 & \text{otherwise.} \end{cases} \tag{4.56}$$

Proof. For all $(A,B) \in \mathcal{Q}$

$$\sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} (-1)^{|B\setminus D|} = (-1)^{|B|} \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} (-1)^{|D|} = \text{lemma 3} = \begin{cases} 1 & \textit{if } A = B \\ 0 & \textit{otherwise.} \end{cases}$$

□

Lemma 4. *Suppose that $(C,D),(A,B) \in \mathcal{Q}$ with $(C,D) \subseteq (A,B)$, then*

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|Y|} = \begin{cases} (-1)^{|B|}|2^{(A\cap D)\setminus C}| & \text{if} \quad A\cup D = B \\ 0 & \text{if} \quad A\cup D \subset B \end{cases} \tag{4.57}$$

Remark 11. *If $A = \varnothing$ and considering that $|2^{\varnothing}| = 1$, lemma 4 reduces to l*

Proof. For all $(C,D),(A,B) \in \mathcal{Q}$ with $(C,D) \subseteq (A,B)$,

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|Y|} = \sum_{C\subseteq X\subseteq A} \sum_{(X\cup D)\subseteq Y\subseteq B} (-1)^{|Y|} = \text{lemma 2} = \sum_{\substack{C\subseteq X\subseteq A\\X\cup D=B}} (-1)^{|B|} =$$

$$= \begin{cases} (-1)^{|B|}|2^{(A\cap D)\setminus C}| & \text{if} \quad A\cup D = B \\ 0 & \text{if} \quad A\cup D \subset B. \end{cases}$$

□

Lemma 5. *For all $(A,B) \in \mathcal{Q}$*

$$\sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} (-1)^{|D|+|C|} = \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} (-1)^{|D\setminus C|} = \begin{cases} 0 & \text{if} \quad A \neq B \text{ i.e. } B\setminus A \neq \varnothing \\ 1 & \text{if} \quad A = B \text{ i.e. } B\setminus A = \varnothing. \end{cases} \tag{4.58}$$

Remark 12. *Note that if* $A = \varnothing$ *lemma 5 states that for all* $(\varnothing, B) \in \mathcal{Q}$

$$\sum_{(\varnothing,D)\subseteq(\varnothing,B)} (-1)^{|D|} = \begin{cases} 0 & \text{if} \quad B \setminus \varnothing \neq \varnothing \\ 1 & \text{if} \quad A = B = \varnothing. \end{cases} \tag{4.59}$$

that is lemma 1.

Proof. For all $(A, B) \in \mathcal{Q}$,

$$\sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} (-1)^{|D|+|C|} = \sum_{C\subseteq A} (-1)^{|C|} \sum_{C\subseteq D\subseteq B} (-1)^{|D|} = \text{lemma 2} = \begin{cases} 0 & \text{if} \quad C \subseteq A \subset B \\ 1 & \text{if} \quad A = B. \end{cases} \tag{4.60}$$

Note that if $A = B$

$$\sum_{C\subseteq B} (-1)^{|C|} \sum_{C\subseteq D\subseteq B} (-1)^{|D|} = (-1)^{|B|} (-1)^{|B|} = 1.$$

□

Lemma 6. *Suppose that* $(C, D), (A, B) \in \mathcal{Q}$ *with* $(C, D) \subseteq (A, B)$*, then*

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|X|+|Y|} = \sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|Y\setminus X|} =$$

$$= \begin{cases} (-1)^{|B\setminus A|} = (-1)^{|D\setminus C|} & \text{if} \quad B \setminus A = D \setminus C \\ 0 & \text{otherwise.} \end{cases} \tag{4.61}$$

Proof. Let us suppose that $(C, D), (A, B) \in \mathcal{Q}$ with $(C, D) \subseteq (A, B)$, then

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|X|+|Y|} = \sum_{C\subseteq X\subseteq A} (-1)^{|X|} \sum_{D\cup X\subseteq Y\subseteq B} (-1)^{|Y|} = \text{lemma 2} =$$

$$= \begin{cases} 0 & \text{if} \quad A\cup D \subset B\,(\text{ and then } D\cup X\subset B \text{ for all } X\subseteq A) \\ (-1)^{|B|} \sum\limits_{\substack{C\subseteq X\subseteq A\\ D\cup X=B}} (-1)^{|X|} & \text{if} \quad A\cup D = B. \end{cases}$$

Now we further examine the case $A\cup D = B$.

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|X|+|Y|} = (-1)^{|B|} \sum_{\substack{C\subseteq X\subseteq A\\ D\cup X=B}} (-1)^{|X|} =$$

$$= (-1)^{|B|} \sum_{X'\subseteq(A\cap D)\smallsetminus C} (-1)^{|C\cup(A\smallsetminus D)|} (-1)^{|X'|} =$$

$$= (-1)^{|B|+|C|+|A\smallsetminus D|} \sum_{X'\subseteq(A\cap D)\smallsetminus C} (-1)^{|X'|} = (\text{lemma 1}) =$$

$$= \begin{cases} (-1)^{|B|+|C|+|A\smallsetminus D|} & \text{if} \quad C = A\cap D \\ 0 & \text{if} \quad C \neq A\cap D. \end{cases}$$

then we have proved that

$$\sum_{\substack{(X,Y)\in\mathcal{Q}\\(C,D)\subseteq(X,Y)\subseteq(A,B)}} (-1)^{|X|+|Y|} =$$

$$= \begin{cases} (-1)^{|B|+|C|+|A\smallsetminus D|} = (-1)^{|B\smallsetminus A|} = (-1)^{|D\smallsetminus C|} & \text{if} \quad D\cup A = B \text{ and } D\cap A = C \\ 0 & \text{otherwise.} \end{cases}$$

To complete the proof we show that $B\smallsetminus A = D\smallsetminus C$ iff ($A\cap B = C$ and $A\cup D = B$). Indeed if ($A\cap B = C$ and $A\cup D = B$) then $B\smallsetminus A = (D\cup A)\smallsetminus A = D\smallsetminus A = D\smallsetminus(D\cap A) = D\smallsetminus C$. Now suppose that $B\smallsetminus A = D\smallsetminus C$. If $D\cup A \neq B$, it exists $x* \in B\smallsetminus(A\cup D)$

then $x* \in B \setminus A$ and $x* \notin D \setminus C$ and we get the contradiction that $B \setminus A \neq D \setminus C$. If $A \cap D \neq C$ it exists $y* \in (A \cap D) \setminus C$ and in this case $y* \in D \setminus C$ and $y* \notin B \setminus A$ contradicting the hypothesis that $B \setminus A = D \setminus C$. □

Proof. of proposition 6.

(4.23) → (4.24). For all $(A, B) \in \mathcal{Q}$,

$$\sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X)}} \left((-1)^{|B \setminus A| - |D \setminus C|} f(C,D) \right) \right] =$$

$$= (-1)^{|B \setminus A|} \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X)}} \left((-1)^{|D \setminus C|} f(C,D) \right) \right] = \text{ (4.23)}$$

$$= (-1)^{|B \setminus A|} \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X)}} \left((-1)^{|D \setminus C|} \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (T,Z) \subseteq (C,D)}} g(T,Z) \right) \right] =$$

$$= (-1)^{|B \setminus A|} \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X)}} \left(g(C,D) \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (C,D) \subseteq (T,Z) \subseteq (A \setminus X, B \setminus X)}} (-1)^{|Z \setminus T|} \right) \right] =$$

$$= \text{lemma 6} = (-1)^{|B \setminus A|} \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X) \\ D \setminus C = (B \setminus X) \setminus (A \setminus X) = B \setminus A}} \left(g(C,D) \, (-1)^{|B \setminus A|} \right) \right] =$$

$$= \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X) \\ D \setminus C = (B \setminus X) \setminus (A \setminus X) = B \setminus A}} g(C,D) \right]$$

$$= \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X) \\ D \cap (A \setminus X) = C \\ D \cup (A \setminus X) = B \setminus X}} g(C,D) \right]$$

$$= \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A \setminus X, B \setminus X) \\ D \setminus C = B \setminus A}} g(C,D) \right] = \sum_{\varnothing \subseteq X \subseteq A} \left[(-1)^{|X|} \sum_{C \subseteq A \setminus X} g(C, C \cup (B \setminus A)) \right] =$$

$$= \text{ (second inversion) } = \sum_{\varnothing \subseteq X \subseteq A} \left[g(X, X \cup (B \setminus A)) \sum_{Y \subseteq A \setminus X} (-1)^{|Y|} \right] =$$

$$(1) \ = g(A, A \cup (B \setminus A)) = g(A,B).$$

(4.24) → (4.23). For all $(A,B) \in \mathcal{Q}$,

$$\sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} g(C,D) = \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} \left[\sum_{\varnothing \subseteq X \subseteq C} (-1)^{|X|} g^*(C \setminus X, D \setminus X) \right] =$$

$$= \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} \left[\sum_{\varnothing \subseteq X \subseteq C} \left((-1)^{|X|} \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (T,Z) \subseteq (C \setminus X, D \setminus X)}} (-1)^{|(D \setminus X) \setminus (C \setminus X)| - |Z \setminus T|} f(T,Z) \right) \right] =$$

$$= \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} \left[\sum_{\varnothing \subseteq X \subseteq C} \left((-1)^{|X|} \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (T,Z) \subseteq (C \setminus X, D \setminus X)}} (-1)^{|D \setminus C| - |Z \setminus T|} f(T,Z) \right) \right] =$$

$$= \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} \left[(-1)^{|D \setminus C|} \sum_{\varnothing \subseteq X \subseteq C} \left((-1)^{|X|} \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (T,Z) \subseteq (C \setminus X, D \setminus X)}} (-1)^{|Z \setminus T|} f(T,Z) \right) \right] =$$

$$= \sum_{\substack{(C,D) \in \mathcal{Q} \\ (C,D) \subseteq (A,B)}} \left[(-1)^{|D \setminus C|} \sum_{\substack{(T,Z) \in \mathcal{Q} \\ (T,Z) \subseteq (C,D)}} \left((-1)^{|Z \setminus T|} f(T,Z) \sum_{\varnothing \subseteq X \subseteq C \setminus Z} (-1)^{|X|} \right) \right] =$$

$$(\text{lemma 1}) = \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} \left[(-1)^{|D\setminus C|} \sum_{\substack{(T,Z)\in\mathcal{Q}\\(T,Z)\subseteq(C,D)\\C\setminus Z=\varnothing}} (-1)^{|Z\setminus T|} f(T,Z)\right] =$$

$$= \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} \left[(-1)^{|D\setminus C|} \sum_{\varnothing\subseteq X\subseteq C} \left(\sum_{C\subseteq Y\subseteq D} (-1)^{|Y\setminus X|} f(X,Y)\right)\right] =$$

$$= \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} \left[(-1)^{|D\setminus C|} f(C,D) \sum_{C\subseteq X\subseteq D\cap A} \left(\sum_{D\subseteq Y\subseteq B} (-1)^{|Y\setminus X|}\right)\right] =$$

(being $X \subseteq D \cap A \subseteq D \subseteq Y$)

$$= \sum_{\substack{(C,D)\in\mathcal{Q}\\(C,D)\subseteq(A,B)}} \left[(-1)^{|D\setminus C|} f(C,D) \sum_{C\subseteq X\subseteq D\cap A} \left((-1)^{|X|} \sum_{D\subseteq Y\subseteq B} (-1)^{|Y|}\right)\right] =$$

$$= (\text{lemma 1}) = \sum_{\substack{(C,B)\in\mathcal{Q}\\(C,B)\subseteq(A,B)}} \left[(-1)^{|B\setminus C|} f(C,B) \sum_{C\subseteq X\subseteq A} (-1)^{|X|} (-1)^{|B|}\right] =$$

$$= \left[(-1)^{|B|}\right]^2 \sum_{C\subseteq A} \left[(-1)^{|C|} f(C,B) \sum_{C\subseteq X\subseteq A} (-1)^{|X|}\right] = (\text{lemma 1}) =$$

$$= (-1)^{|A|} f(A,B)\,(-1)^{|A|} = f(A,B).$$

□

Proof. of proposition 7.

1) and 2) follow directly by the conditions

$$\mu_r(\varnothing,\varnothing) = 0, \quad \mu_r(N,N) = 1, \quad \text{and} \quad \mu_r(A,B) = \sum_{(C,D)\subseteq(A,B)} m(C,D).$$

To prove 3) and 4) it is sufficient to note that for any function $f : \mathcal{Q} \to \mathbb{R}$ and for all $(A,B),(C,D) \in \mathcal{Q}$, the monotonicity condition

$$f(C,D) \leq f(A,B) \quad \text{whenever} \quad (C,D) \subseteq (A,B) \tag{4.62}$$

is equivalent to the following two statements

$$f(A \setminus \{a\}, B) \leq f(A,B) \quad \text{for all} \quad a \in A \tag{4.63}$$

and

$$f(A \setminus \{b\}, B \setminus \{b\}) \leq f(A,B) \quad \text{for all} \quad b \in B. \tag{4.64}$$

(4.62) trivially imply (4.63) and (4.64). Suppose that $(C,D) \subseteq (A,B)$ and note that $C \subseteq A \cap D$. By using respectively (4.63) and (4.64), we get:

$$f(C,D) \leq f(A \cap D, D) = f(A \setminus (B \setminus D), B \setminus (B \setminus D)) \leq f(A,B).$$

□

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