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Some results on cardinality bounds and covering-type properties of a topological space

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CONTENTS

PRELIMINARY DEFINITIONS AND NOTATIONS

X denotes a non empty topological space. We say that a subset A of X is regular open (regular closed) if $A = int(\overline{A})$ ($A = \overline{int(A)}$). The family $\mathcal{B} = \{U : U \text{ is regular open in } X\}$ is a base for X. X equipped with the topology generated by the base $\mathcal{B}$ is called the *semiregularization* of X and it is denoted by X_s. If $X = X_s$, then X is called *semiregular*. A *clopen* subset of a space X is a subset of X that is both open and closed in X. Let $\mathcal{B}(X)$ be the family of all clopen subsets of X. A space X is *zero-dimensional* if $\mathcal{B}(X)$ is an open base for X. A space X is said to be *extremally disconnected* if the closure of each open subset of X is also open.

X is called: T_1 if for every distinct points $x, y \in X$ there exist two open subsets U and V of X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$; *Hausdorff* if for every distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$; *Urysohn* if for every distinct points $x, y \in X$ there exist open sets U and V with $x \in U$ and $y \in V$ suchthat $\overline{U} \cap \overline{V} = \emptyset$; *Regular* if X is T_1 and for every closed subset F of X and every $x \in X \setminus F$, there exist two disjoint open subsets U and V of X such that $x \in U$ and $F \subseteq V$; *Tychonoff* if X is T_1 and for every closed subset F of X and every $x \in X \setminus F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

The cardinality of a set A is denoted by $|A|$. λ, κ denote infinite cardinals. $\aleph_0$ and ω denote the smallest infinite cardinal and the smallest infinite ordinal. The cardinal successor of κ will be denoted by k^+. We will denote by $[A]^{\leq\lambda}$ the family of all subsets of A of cardinality $\leq \lambda$.

In the following we recall the cardinal functions that we will use in this thesis. The *cellularity* of a space X, denoted by $c(X)$, is the smallest cardinal number $m \geq \aleph_0$ such that every family of pairwise disjoint non empty open subsets of X has cardinality $\leq m$. If $c(X) = \aleph_0$, X is called c.c.c.. The *Urysohn-cellularity* of a space X, denoted by $Uc(X)$, is the smallest cardinal number $m \geq \aleph_0$ such that every family of non empty open subsets of X having pairwise disjoint closure has cardinality $\leq m$ [61]. A family $\mathcal{B}(x)$ of neighbourhoods of x is called a *base for a space X at the point x* if for any neighborhood V of x there exists a $U \in \mathcal{B}(x)$ such that $x \in U \subset V$. The *character of a point x* in the space X, denoted by $\chi(x, X)$, is the smallest cardinality of a base at the point x. The *character of the space X* is $\sup\{\chi(x, X) : x \in X\}$. The *pseudocharacter of a point x* in a T_1-space X, denoted by $\psi(x, X)$, is the smallest cardinality of a family of open sets $\mathcal{U}$ of X such that $\bigcap\mathcal{U} = \{x\}$. The *pseudocharacter of a T_1-space X* is $\psi(X) = \sup\{\psi(x, X) : x \in X\}$. The *weight of X*, denoted by $w(X)$, is the smallest cardinality of a base for X. A family $\mathcal{B}(x)$ of open subsets of X is called a *local π-base at x* if for each neoghborhood R of p, there exists $V \in \mathcal{U}$ such that $V \subseteq R$. The *π-character of a point* x in the space X, denoted by $\pi\chi(x, X)$, is the smallest cardinality of a π-base at the point x. The *π-character of the space* X is $\sup\{\pi\chi(x, X) : x \in X\}$. If X is a Hausdorff space, the *closed pseudocharacter of a point* x in X is $\psi_c(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open neighborhoods of x and $\{x\}$ is the intersection of the closure of elements of $\mathcal{U}\}$; the *closed pseudocharacter of* X is $\psi_c(X) = sup\{\psi_c(x, X) : \ x \in X\}$ (see [62] where it is called $S\psi(X)$). The *tightness* of X at $x \in X$ is $t(x, X) = \min\{\kappa :$ for every $A \subseteq X$ with $x \in \overline{A}$ there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in \overline{B}$; the *tightness of* X is $t(X) = sup\{t(x, X) : \ x \in X\}$. The *$\theta$-closure* of a set A in a space X is the set $cl_\theta(A) = \{x \in X :$ for every neighborhood $U \ni x, \overline{U} \cap A \neq \emptyset\}$; A is said to be θ-closed if $A = cl_\theta(A)$ [65]. The *θ-tightness* of X at $x \in X$ is $t_\theta(x, X) = \min\{\kappa :$ for every $A \subseteq X$ with $x \in cl_\theta(A)$ there exists $B \subseteq A$ such that $|B| \leq \kappa$ and $x \in cl_\theta(B)$; the *θ-tightness of* X is $t_\theta(X) = sup\{t_\theta(x, X) : x \in X\}$ [25]. We have that tightness and θ-tightness are independent, but if X is a regular space then $t(X) = t_\theta(X)$. The density, denoted by $d(X)$ is the smallest cardinality of a dense subset of X. The θ-density of X is $d_\theta(X) = min\{\kappa : \ A \subseteq X$, A is a θ dense subset of $X$and $|A| \leq \kappa\}$. Recall that a subset A of X is *θ-dense* in X if $cl_\theta(A) = X$. Recall that a family $\mathcal{U}$ of open subsets of X is a *cover* for X if $X = \bigcup\mathcal{U}$. If $\mathcal{U}$ is an open cover of X, a subfamily $\mathcal{V} \subseteq \mathcal{U}$ is called a *subcover* of X if $X = \bigcup \mathcal{V}$. A family of sets $\mathcal{A}$ refines a family of sets $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $A \subset B$. For the families of sets $\mathcal{A}$ and $\mathcal{B}$, we will write $\mathcal{A} \approx \mathcal{B}$ if both $\mathcal{A}$ refines $\mathcal{B}$ and $\mathcal{B}$ refines $\mathcal{A}$. The *Lindelöf degree* of a space X, denoted by $L(X)$ is the smallest cardinal κ such that for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X = \bigcup\mathcal{U}$. The *almost Lindelöf degree*, denoted by $wL(X)$, of a space X is the least cardinal κ such that for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X = \bigcup_{V \in \mathcal{V}} \overline{V}$ [67]. The *weak Lindelöf degree*, denoted by $wL(X)$, of a space X is the least cardinal κ such that for every open cover $\mathcal{U}$ of X there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X = cl(\bigcup \mathcal{V})$. We have $wL(X) \leq aL(X) \leq L(X)$.

Recall that a space X is: *compact* (resp. *Lindelöf*) if for every open cover $\mathcal{U}$ of X there exists a finite (countable) subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{V}$; *countably compact* if for every countable open cover $\mathcal{U}$ of X there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{V}$; *σ-compact* if it is the union of countably many compact subsets; *paracompact* if every open cover has a locally finite open refinement; *metacompact* if every open cover has a point-finite open refinement; *metaLindelöf* if every open cover has a point-countable open refinement; *linearly Lindelöf* if for every linearly ordered open cover $\mathcal{U}$ of X there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X = \bigcup \mathcal{V}$; *Menger* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick finite subfamilies $\mathcal{F}(n) \subset \mathcal{U}(n)$, $n \in \omega$, so that $\bigcup\{\mathcal{F}(n) : n \in \omega\}$ covers X; *Hurewicz* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick finite subfamilies $\mathcal{F}(n) \subset \mathcal{U}(n)$, $n \in \omega$, so that every $x \in X$ is contained in $\bigcup \mathcal{F}(n)$ for all but finitely many n; *Rothberger* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick $U(n) \in \mathcal{U}(n)$, $n \in \omega$, so that $\{U(n) : n \in \omega\}$ covers X.

Let $\mathcal{U}$ be a cover of a space X and M be a subset of X; the star of M with respect to $\mathcal{U}$ is the set $St(M,\mathcal{U}) = \bigcup\{U : U \in \mathcal{U} \text{ and } U \cap M \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathcal{U}$ is called the star of the point x with respect to $\mathcal{U}$ and it is denoted by $St(x,\mathcal{U})$. A cover $\mathcal{U}$ is a *star-refinement* of a cover $\mathcal{V}$ if for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $St(U,\mathcal{U}) \subset V$. A space X is called *star-compact* if for every open cover $\mathcal{U}$ of X there is a finite subset F of X such that $St(F,\mathcal{U}) = X$.

For definitions of cardinal invariants and other notions not defined in this thesis, we refer the reader to [34], [47] and [58].

INTRODUCTION

In this study we consider cardinal functions in topology with a primary focus on cardinality bounds, and covering properties.

In the first chapter of this thesis we present results on bounds for the cardinality of a topological space using cardinal functions. In particular, we give variations and improvements of the most known cardinality bounds for topological spaces. Popularly speaking, the cardinality of a given set is the "number" of points in it. Restricting the cardinality of a given topological space by continuum (i.e. 2^{ω}), which is the "number of points" in every Euclidean space, allows us to think about that space as real line, real plane or higher dimensional Euclidean spaces endowed with some specific richer topological structure. The area of cardinality restrictions of topological spaces began with a problem posed by the father of General Topology – P.S. Alexandroff, who asked: Is the cardinality of a Hausdorff, first countable, compact topological spaces less than or equal to continuum? It seemed simple and natural to ask but it remained unsolved for 50 years when a positive solution was given by the famous Russian topologist A.V. Arhangel'skiĭ [7] who proved that if X is a Hausdorff space, then $|X| \leq 2^{L(X)\chi(X)}$ (note that since in a Lindelöf first countable space we have $L(X)\chi(X) = \aleph_0$, using the Arhangel'skiĭ theorem we have a positive answer the previous question). This was a milestone question since it drew the attention of topologists to investigate what kind of topological properties lead to restriction of the cardinality of a given topological space. It gave rise to a completely new area combining ideas, methods and questions from both topology and set-theory – the theory of cardinal invariants of topological spaces which is an active and important part of topology even nowadays and leads to unexpected and beautiful results and recent interactions with model theory, forcing method and the very new approach of proofs and research (for example using the game theory). Even before Arhangel'skiĭ's solution, various results were obtained by other famous mathematicians. In [39], De Groot proved that if X is a Hausdorff space, then $|X| \leq 2^{hL(X)}$. In [40] and [41], Hajnal and Juhász proved, resp., that if X is a Hausdorff space, then $|X| \leq 2^{c(X)\chi(X)}$ and that if X is a T_1 space, then $|X| \leq 2^{s(X)\psi(X)}$. With the above theorems we have a link between the cardinality of a space and other covering and local properties of spaces such hereditary Lindelof number $hL(X)$, cellularity $c(X)$, spread $s(X)$, character $\chi(X)$ and pseudocharacter $\psi(X)$ of a space. The general theory was developed by Juhász (see [47]). After Arhangel'skiĭ's solution, a series of results in that direction were proved by Hodel, Porter, Ponomarev, Stavrova, Dissanayeke and Willard, Šapirovskiĭ, Gryzlov, Pol, Bell, Ginsburg and Woods. To prove cardinality restriction of a topological space, firstly Gryzlov in his PhD thesis (1973) and then Pol, Šapirovskiĭ and Arhangel'skiĭ developed the "closure method" using topological properties that are extendable for general cardinality restrictions.

In Section 1.1 we pose our attention on cardinality bounds for Urysohn spaces. Schröder [61] proved that if X is a Urysohn space, then $|X| \leq 2^{Uc(X)\chi(X)}$, and Šapirovskiĭ proved that if X is a regular space, then $|X| \leq \pi_\chi(X)^{c(X)\psi(X)}$. We define, in the class of Urysohn spaces, the cardinal function called θ-*pseudocharacter of* X in the following way $\psi_\theta(X) = \sup\{\psi_\theta(x, X) : \ x \in X\}$, where $\psi_\theta(x, X)$ is $\min\{\kappa :$ there is a family $\mathcal{B}$ of open neighborhoods of x such that $|\mathcal{B}| \leq \kappa$ and $\{x\} = \bigcap_{U \in \mathcal{B}} cl_\theta(\overline{U})\}$. Recall that if A is a subest of a space X, the θ-closure of X, denoted by $cl_\theta(A)$, is the set of all $x \in X$ such that $\overline{U} \cap A \neq \emptyset$ for every open subset U of X containing x. Using the θ-pseudocharacter of the space X we establish in Theorem 1.1.1 the following:

Theorem. *If X is a Urysohn space, then* $|X| \leq \pi\chi(X)^{Uc(X)\psi_\theta(X)}$.

Note that the previous theorem is a common generalization of both the Schröder's inequality and Šapirovskiĭ's inequality mentioned above.

The Arhangel'skiĭ's inequality, mentioned in the beginning of the introduction, has been widely generalize for Hausdorff spaces using "better" cardinal functions (i.e. cardinal functions strictly smaller then the Lindelöf degree or than the character of the space). Furthermore, many authors proved variations of the Arhangel'skiĭ's inequality for other classes of spaces stronger than Hausdorff using cardinal functions related respectively to the character and to the Lindelöf degree of the space (for example, Bella and Cammaroto [14] proved that if X is a Urysohn space, then $|X| \leq 2^{aL(X)\chi(X)}$). In Section 1.1.3, using a filter argument which represents another technique of proving cardinality bounds for topological spaces, we improve the Bella-Cammaroto inequality mentioned above. Also our result represents a variation of the Arhangel'skiĭ's inequality. In particular we prove the following (Theorem 1.1.4).

Theorem. *If X is a Urysohn space, then* $|X| \leq 2^{\theta\text{-}aL'(X)t_{\bar{c}}(X)\psi_\theta(X)}$.

The cardinal invariant θ-$aL'(X)$ is related to θ-$aL(X)$ and is defined in Definition 1.1.11. As θ-$aL'(X) \leq aL(X)$, and $t_{\bar{c}}(X)\psi_\theta(X) \leq \chi(X)$, the theorem above represents an improvement of the Bella-Cammaroto inequality.

In Section 1.3 we prove cardinality bounds for Hausdorff spaces. To do that, we introduce a new cardinal function called *quasicellularity*. Let X be a space and $A \subseteq X$, we define the *quasicellularity of* $A \subset X$, $qc(A, X)$, as the least infinite cardinal κ such that if there exists a family $\mathcal{U}$ of open sets of X with $A = \bigcup_{U \in \mathcal{U}} \overline{U}$ then there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $A = \overline{\bigcup \mathcal{V}}$. The *quasicellularity* of X is $qc(X) = qc(X, X)$. We have $qc(X) \leq c(X)$ and it is shown that $c(X) = qc(X)dot(X)$, decomposing $c(X)$ into two components, where $dot(X)$ is defined in [38]. Relationships between $qc(X)$ and other cardinal invariants are investigated. We also have that $qc(X) = wL(X)$ for any extremally disconnected space.

In our proofs we make use of the Pol-Šapirovskiĭ technique.

The second chapter of this thesis is dedicated to results obtained in the field of covering properties for topological spaces. Lots of important theorems in general topology make use of covering properties. In particular we consider covering properties defined by stars, neighborhood assignments or as monotone versions of selection principles. Star covering properties have been widely studied in literature (see for example [17], [33], [44], [45], [55]). The use of stars is very important in general topology. In fact, some topological and covering properties are characterized using stars. Recall that if A is a subset of a space X and $\mathcal{B}$ is a family of subsets of X, the *star of A* with respect to $\mathcal{B}$, denoted by $St(A, \mathcal{B})$, is the set $\bigcup\{B \in \mathcal{B} : B \cap A \neq \emptyset\}$. The notion of star appears, for example in the characterization of normality: "A topological space is normal if and only if every finite open cover has a finite open star-refinement" . We also have that countable compactness is equivalent to star-compactness in the class of Hausdorff spaces and paracompactness is equivalent to the following property: every open cover has an open star refinement. It is natural to consider stars of subspaces having particular properties. In this case we will say that a space X has the *star-$\mathcal{P}$* property (briefly St-$\mathcal{P}$) if for every open cover $\mathcal{U}$ of the space X, there exists a subset Y of X having the property $\mathcal{P}$ such that $St(Y, \mathcal{U}) = X$ [45]. We can also consider covering properties not only in terms of stars but also in terms of neighborhood assignments. Recall that a *neighborhood assignment* in a space X is a family $\{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for every $x \in X$. For example the Lindelöf property can be characterized using neighborhood assignments in the following way: a space X is Lindelöf if and only if for every neighborhood assignment $\{O_x : x \in X\}$ there is a countable subset Y of X such that $\{O_x : x \in Y\}$ is a cover of X. If Y is closed and discrete instead of countable we obtain the notion of *D-spaces* defined by van Douwen in [31]. The idea of van Douwen have been generalized in [56] where the authors defined a space to be *neighborhood assignment $\mathcal{P}$* (briefly NA-$\mathcal{P}$) if for any neighborhood assignment $\{O_x : x \in X\}$ there exists a subspace Y of X having the property $\mathcal{P}$ such that $\{O_x : x \in Y\}$ is a cover of X. In particular the authors pose their attention on the following properties: compactness, pseudocompactness, countable compactness and Lindelöfness (see also [1], [3], and [5]). In Section 2.1 we study and compare "star" and "neighborhood assignment" versions of compactness, countable compactness, Lindelöfness, and of the Menger property. Using *expansion operators*, we give a description of the previous properties in terms of cardinal functions and generalize known results. A map $\Phi : P(X) \times \mathsf{NA}(X) \to P(X)$, where $\mathsf{NA}(X)$ is the family of all neighborhood assignments of X, such that: (i) $Y \subseteq \Phi(Y, \mathrm{NA})$ for every $(Y, \mathrm{NA}) \in P(X) \times \mathsf{NA}(X)$; (ii) $Z \subseteq Y \subseteq X$ implies $\Phi(Z, \mathrm{NA}) \subseteq \Phi(Y, \mathrm{NA})$ for every $\mathrm{NA} \in \mathsf{NA}(X)$, will be called an *expansion* *operator on* X. In particular, typical expansion operators are the *neighborhood assignment operator*, and the *star operator*, defined in Definition 2.1.4.

In Section 2.2 we consider monotone versions of some selection principles. When we add monotonicity to a covering property, we obtain a stronger property. The idea of a covering property being monotonic has its roots in the definition of "monotone normality" that has nothing to do with open covers. A space X is called *monotonically normal* if for each pair (H, K) of disjoint closed subsets of X, one can assign an open set $r(H, K)$ such that $H \subset r(H, K) \subset \overline{r(H, K)} \subset X \setminus K$, and if $H_1 \subset H_2$ and $K_1 \supset K_2$ then $r(H_1, K_1) \subset r(H_2, K_2)$ (see [22]). Shortly after, the style of this definition was adapted and applied to other kinds of properties, including covering properties. Gartside and Moody in [36] described a process for obtaining a monotone version of any well-known covering property: "by requiring that there is an operator, r, assigning to every open cover a refinement in such a way that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$". Using this process, any covering property can be "upgrated" into a monotonic property. Our starting point is the class of monotone Lindelöf spaces (see for example [50], [51]). The monotone version of the Lindelöf property introduced by Matveev in [53] is the following: a space X is *monotonically Lindelöf* if there exists an operator r assigning to every open cover $\mathcal{U}$ a countable open refinement such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$. In Section 2.1 we consider all four different ways of defining monotone versions of Menger, Rothberger and Hurewicz properties, and we show that one of this monotone versions introduced is absurd. In the next definition the letters W and S are abbreviations for "weakly" and "strongly".

A space X is

- SS-mM ([18], where it is called *monotonically Menger*, briefly mM) if there exists an operator, called SS-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and for every $n \in \omega$, $\mathcal{U}(n)$ refines $\mathcal{V}(n)$, then for every $n \in \omega$, $r(\mathcal{U})(n)$ refines $r(\mathcal{V})(n)$.
- SW-mM if there exists an operator, called SW-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,

3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and for every $n \in \omega$, $\mathcal{U}(n)$ refines $\mathcal{V}(n)$, then $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(n) : n \in \omega\}$.

- WW-mM if there exists an operator, called WW-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$, then $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(n) : n \in \omega\}$.
- WS-mM if there exists an operator, called SS-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$, then for every $n \in \omega$, $r(\mathcal{U})(n)$ refines $r(\mathcal{V})(n)$.

A specific study of the SS-mM has been done in [18] where it is called *monotone Menger property*. In [18] it was proved that (within ZFC) every separable SS-mM space is first countable. This result contrasts with the previously known fact that under CH there exists countable mL spaces which are not first countable.

The monotone versions of the Hurewicz and Rothberger properties are defined in similar ways. For the Rothberger property, we deal with at most one element refinements rather then with one element refinements, while for Hurewicz property, we replace condition 2. with

$2'$. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$ such that for every $x \in X$, $x \in \bigcup r(\mathcal{U})(n)$, for all but finitely many n.

We prove that WS-mM spaces, in the class of Hausdorff spaces concide with the class of dicrete and countable spaces. Also we prove that the discrete sum of countably many convergent sequences is SS-mH (hence SS-mM) but it is not WW-mM (hence not WW-mH), and that the one point Lindelöfication of the discrete space ω_1, $L(\omega_1)$, is WW-mR and WW-mH (hence WW-mM) but it is not SS-mM (hence neither SS-mR nor not SS-mH). Then SS- and WW-properties are indipendent in Menger and Hurewicz cases. The following question is open: Is there a SS-mR not WW-mR space? Furthermore, $L(\omega_1)$ permitts to distinguish SW- and SS- properties. In Section 2.2.4 we consider the local version of the monotonic Rothberger-type properties. Recall that a space X is mL *at the point* p [50] if one can assign to every non empty family $\mathcal{U}$ of neighborhoods of p a non empty countable family $r(\mathcal{U})$ of neighborhoods of p so that $r(\mathcal{U})$ refines $\mathcal{U}$, and $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. This technical notion was used in [50], [51], [52] to disprove monotone Lindelöfness of certain spaces. One gets the definition of mC at p when replacing "countable" with "finite". We consider the local versions of SS-mR and SW-mR properties. There is one principal difference between the new properties and monotone Lindelofness: obviously, a space X with single non-isolated point p is mL (mC) iff X is Lindelöf (respectively, compact) and mL (respectively, mC) at p; we show that there exists a space X with single non-isolated point p which is Rothberger, SS-mR at p but not SS-mR (Example 2.2.5). In Section 2.2.5, using the notion of thin base of neighborhoods of a point (see Definition 2.2.3), we contruct a monotonically paracompact SW-mH space which is not SS-mM, and distinguish SW-mH property from the local version of SS-mR property.

1 CARDINALITY BOUNDS FOR TOPOLOGICAL SPACES

The theory of bounds on the cardinality of a topological space is an elegant topic with a rich history with many applications in a variety of mathematical fields. The story of cardinality bounds for a topological space began in the early 20^{th} century. The following is an early result: A second countable Hausdorff space has cardinality at most $\mathfrak{c}$. In 1922 (not published until 1929) Alexandroff and Urysohn improved the previous result by proving that a hereditary Lindelöf Hausdorff space has cardinality $\mathfrak{c}$. Two later important results in this theory in the class of Hausdorff spaces are Arhangel'skiĭ's Theorem and Hajnal-Juhász's Theorem. In the proofs of the previous results the "closing-off argument" technique was developed (this technique is still used in this branch of topology). It is important to mention that Arhangel'skiĭ's theorem answered a long standing problem posed by Alexandroff and Urysohn, who asked "Is the cardinality of a Hausdorff first countable Lindelöf space at most $\mathfrak{c}$?". Arhangel'skiĭ proved that if X is a Hausdorff space, then $|X| \leq 2^{L(X)\chi(X)}$. Since for a Lindelöf space $L(X) = \aleph_0$ and for a first countable space $\chi(X) = \aleph_0$, using the Arhangel'skiĭ's Theorem we have an affirmative answer to the problem posed by Alexandroff and Urysohn. Many topologists gave improvements (using "better" cardinal functions) and variations (using different classes of spaces and different cardinal functions) of Arhangel'skiĭ's theorem (for example Bella-Cammaroto [14], Basile-Bonanzinga-Carlson [9]) and of the Hajnal-Juhász's Theorem (for example Shu-Hao [62], Basile-Bonanzinga-Carlson [9]). In this chapter, we continue this process by proving several improvements and variations of well-known cardinality bounds. Some of the results that we discuss are included in [9], [15] and [12].

1.1 CARDINALITY BOUNDS FOR URYSOHN SPACES

In this section we give variations and improvements of known cardinality inequalities in the class of Urysohn spaces.

1.1.1 A generalization of Šapirovskiĭ's inequality for a regular space X: $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$

Šapirovskiĭ [60] proved that $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$, for a regular space X. Later Shu-Hao [62] proved that the previous inequality holds in the class of Hausdorff spaces by replacing the pseudocharacter with the closed pseudocharacter. We introduce the notion of *θ-pseudocharacter* of a Urysohn space X, in order to prove a variation, in the class of Urysohn spaces, of the Shu-Hao inequality.

The following result is trivial and gives a characterization for Urysohn spaces.

Proposition 1.1.1. *X is a Urysohn space if and only if for every $x \in X$, $\{x\}$ is the intersection of the θ-closure of the closure of a family of open neighborood of x.*

Proof. Let X be a Urysohn space and $x \in X$. For every $y \in X \setminus \{x\}$, there exist U_y and V_y open disjoint subsets of X such that $x \in U_y$, $y \in V_y$ and $\overline{U_y} \cap \overline{V_y} = \emptyset$. So, y $\notin cl_\theta(\overline{U_y})$ and $\{x\} = \bigcap_{y \in X \setminus \{x\}} cl_\theta(\overline{U_y})$. For the converse, let x, y be distinct points of X. By hypothesis there exists an open neighbourhood V of x such that $y \notin cl_\theta(\overline{V})$. Then there exists an open subset U of X such that $y \in U$ and $\overline{U} \cap \overline{V} = \emptyset$. So X is Urysohn. □

Using Proposition 1.1.1 we can define the following cardinal function in the class of Urysohn spaces.

Definition 1.1.1. Let X be a Urysohn space. The *θ-pseudocharacter of a point* $x \in X$, denoted by $\psi_\theta(x, X)$, is:

$$\psi_\theta(x, X) = \min\{\kappa : \text{ there is a family } \mathcal{B} \text{ of open neighborhoods of } x$$

$$\text{such that } |\mathcal{B}| \leq \kappa \text{ and } \{x\} = \bigcap_{U \in \mathcal{B}} cl_\theta(\overline{U})\}.$$

The *θ-pseudocharacter of* X is $\psi_\theta(X) = \sup\{\psi_\theta(x, X) : x \in X\}$.

Independently, the notion of θ-pseudocharacter was given in [37] where it is called θ^2-pseudocharacter.

We have that:

$$\psi(X) \leq \psi_c(X) \leq \psi_\theta(X) \leq U\psi(X) \leq \chi(X).$$

Since for a regular space X, $cl_\theta(A) = \overline{A}$ for every $A \subseteq X$, we have that for a regular space X, $\psi_c(X) = \psi_\theta(X)$. In general this need not be true for non regular spaces. Indeed if we consider $\mathbb{R}$ with the countable complement topology we have that $\overline{\mathbb{Q}} \neq cl_\theta(\mathbb{Q})$.

It was proved in [14] that for Urysohn spaces, $|cl_\theta(A)| \leq |A|^{\chi(X)}$ for every $A \subseteq X$ and further this inequality was used for the estimation of cardinality of Lindelöf spaces. Since $t_\theta(X)\psi_\theta(X) \leq \chi(X)$, the following proposition improves the result in [14]. (Note that if $X = \omega \cup \{p\}$ (p an ultrafilter on ω) is the space with the single ultrafilter topology, we have that $\aleph_0 = t_\theta(X)\psi_\theta(X) < \chi(X)$.)

Proposition 1.1.2. *Let X be a Urysohn space such that $t_\theta(X)\psi_\theta(X) \leq \kappa$. Then for every $A \subseteq X$ we have that $|cl_\theta(A)| \leq |A|^\kappa$.*

Proof. Let $x \in cl_\theta(A)$, since $\psi_\theta(X) \leq \kappa$ there exist a family $\{U_\alpha(x)\}_{\alpha<\kappa}$ of neighborhood of x such that $\{x\} = \bigcap_{\alpha<\kappa} cl_\theta(\overline{U_\alpha(x)})$. We want to prove that $x \in cl_\theta(\overline{U_\alpha(x)} \cap A)$, $\forall \alpha < \kappa$. Let U be a neighborhood of x and $\alpha < \kappa$. Then $\emptyset \neq \overline{U \cap U_\alpha(x)} \cap A \subseteq \overline{U} \cap \overline{U_\alpha(x)} \cap A$. This shows that $x \in cl_\theta(\overline{U_\alpha(x)} \cap A)$. Since $t_\theta(X) \leq \kappa$, there exists $A_\alpha \subset \overline{U_\alpha(x)} \cap A$ such that $|A_\alpha| \leq \kappa$ and $x \in cl_\theta(A_\alpha)$. Then $\{x\} = \bigcap_{\alpha<\kappa} cl_\theta(A_\alpha)$ and $\{A_\alpha\}_{\alpha<\kappa} \in [[A]^{\leq\kappa}]^{\leq\kappa}$, so $|cl_\theta(A)| \leq |[[A]^{\leq\kappa}]^{\leq\kappa}| = |A|^\kappa$. □

Corollary 1.1.1. *[14] If X is a Urysohn space then for every $A \subseteq X$ we have that $|cl_\theta(A)| \leq |A|^{\chi(X)}$.*

The following result is the analogue of 2.20 in [47]: if X is a Hausdorff space then $|X| \leq d(X)^{\chi(X)}$.

Corollary 1.1.2. *If X is a Urysohn space then $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$.*

Proof. Let A be a θ-dense subset of X, i.e. $cl_\theta(A) = X$, with $|A| = d_\theta(X)$. From the above theorem we have that $|cl_\theta(A)| \leq |A|^{t_\theta(X)\psi_\theta(X)}$, so $|X| \leq d_\theta(X)^{t_\theta(X)\psi_\theta(X)}$. □

Proposition 1.1.2 and Corollary 1.1.2 have been proved independently in [37].

The following result will be used in the proof of Theorem 1.1.1

Lemma 1.1.1. *Let X be a topological space, $\mathcal{B}$ a π-base for X and $\mathcal{W}$ a family of open sets. Then there exists $\mathcal{M}$ a maximal Urysohn cellular subfamily of $\{U \in \mathcal{B} : U \subseteq W \text{ for some } W \in \mathcal{W}\}$ such that $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \supseteq \bigcup \mathcal{W}$.*

Proof. Using Zorn's Lemma we can say that there exists a maximal Urysohn-cellular subfamily $\mathcal{M}$ of $\{U \in \mathcal{B} : U \subseteq W$ for some $W \in \mathcal{W}\}$. We want to prove that $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \supseteq \bigcup \mathcal{W}$. Assume, by the way of contradiction, that $cl_\theta\left(\bigcup \overline{\mathcal{M}}\right) \not\supseteq \bigcup \mathcal{W}$. Let $x \in \bigcup \mathcal{W}$ such that $x \notin cl_\theta(\bigcup \overline{\mathcal{M}})$. Then there exists an open set U such that $x \in U$ such that $\overline{U} \cap \overline{M} = \emptyset$, $\forall M \in \mathcal{M}$. So $x \notin M$, $\forall M \in \mathcal{M}$. Let $W \in \mathcal{W}$ such that $x \in W$. $\mathcal{M} \cup \{U \cap W\}$ is a Urysohn cellular family. Since $\mathcal{B}$ is a π-base for X and $U \cap W$ is an open set containing x, there exists $B \in \mathcal{B}$ such that $B \subseteq U \cap W$, so $\mathcal{M}' = \mathcal{M} \cup \{B\}$ is a Urysohn cellular subfamily of $\{U \in \mathcal{B} : U \subseteq W$ for some $W \in \mathcal{W}\}$ containing $\mathcal{M}$; a contradiction. □

Shu-Hao in [62] proved that if X is Hausdorff, then $|X| \leq \pi\chi(X)^{c(X)\psi_c(X)}$. This represents an improvement of the Šapirovskiĭ's inequality (if X is a regular space, then $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$ [60]). Here we prove a Urysohn version of the Shu-Hao inequality mentioned above that also gives an improvement of the Šapirovskiĭ's inequality.

Theorem 1.1.1. *Let X be a Urysohn space. Then $|X| \leq \pi\chi(X)^{Uc(X)\psi_\theta(X)}$.*

Proof. Let $\pi\chi(X) = \lambda$ and $Uc(X)\psi_\theta(X) = \kappa$; for each $p \in X$, let $\mathcal{U}_p$ be a local π-base at p such that $|\mathcal{U}_p| \leq \lambda$.

Construct an increasing chain $\{A_\alpha : \alpha < \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : 0 < \alpha < \kappa^+\}$ of open collections in X such that:

1. $|A_\alpha| \leq \lambda^\kappa$, $0 \leq \alpha < \kappa^+$;
2. $\mathcal{U}_\alpha = \{V \in \mathcal{U}_p : p \in \bigcup_{\beta<\alpha} A_\beta\}$, $0 < \alpha < \kappa^+$;
3. for each $\gamma < \kappa$, if $\mathcal{V}_\gamma \in [\mathcal{U}_\alpha]^{\leq\kappa}$ and $W = \bigcup_{\gamma<\kappa} cl_\theta(\overline{\bigcup \mathcal{V}_\gamma}) \neq X$, then $A_\alpha \setminus W \neq \emptyset$.

The construction is by transfinite induction. Let $0 < \alpha < \kappa^+$ and assume that $\{A_\beta : \beta < \alpha\}$ has already been constructed. Then $\mathcal{U}_\alpha$ is defined by (2), i.e., we put $\mathcal{U}_\alpha = \{V : \exists p \in \bigcup_{\beta<\alpha} A_\beta,\ V \in \mathcal{U}_p\}$. It follows that $|\mathcal{U}_\alpha| \leq \lambda^\kappa$. If $\{\mathcal{V}_\gamma\}_{\gamma<\kappa} \in [[\mathcal{U}_\alpha]^{\leq\kappa}]^{\leq\kappa}$ and $W = \bigcup_{\gamma<\kappa} cl_\theta(\overline{\bigcup \mathcal{V}_\gamma}) \neq X$, then we can choose one point of $X \setminus W$. Let S_α be the set of points chosen in this way. Note that $|[[\mathcal{U}_\alpha]^{\leq\kappa}]^{\leq\kappa}| \leq \lambda^\kappa$. Define A_α to be the set $S_\alpha \cup (\bigcup_{\beta<\alpha} A_\beta)$. Then A_α satisfies (1), and (3) is also satisfied if $\beta \leq \alpha$. This completes the construction.

Now let $S = \bigcup_{\alpha<\kappa^+} A_\alpha$; then $|S| \leq \kappa^+\lambda^\kappa = \lambda^\kappa$. The proof is complete if $S = X$. Suppose not and let $p \in X \setminus S$; since $\psi_\theta(X) \leq \kappa$, there exist open neighbourhoods $\{U_\alpha\}_{\alpha<\kappa}$ of p such that $\{p\} = \bigcap_{\alpha<\kappa} cl_\theta(\overline{U_\alpha})$. For each $\alpha < \kappa$, let $V_\alpha = X \setminus cl_\theta(\overline{U_\alpha})$. Then $S = \bigcup_{\alpha<\kappa} V_\alpha \cap S$. Fix $\alpha < \kappa$. For each $q \in V_\alpha \cap S$, there exists V_q an open subsets of X containing q such that $\overline{V_q} \cap \overline{U_\alpha} = \emptyset$ (from the definition of V_α). We have that $\{V \in \mathcal{U}_q : V \subseteq V_q\}$ is a local π-base at q. Since $q \in \overline{\bigcup\{V \in \mathcal{U}_q : V \subseteq V_q\}}$, we have that $S \cap V_\alpha \subseteq \bigcup_{q\in S\cap V_\alpha} \overline{\bigcup\{V \in \mathcal{U}_q : V \subseteq V_q\}} \subseteq \overline{\bigcup\{V : V \in \mathcal{U}_q,\ V \subseteq V_q,\ q \in S \cap V_\alpha\}}$. We put $\mathcal{W}_\alpha = \{V : V \in \mathcal{U}_q,\ V \subseteq V_q,\ q \in S \cap V_\alpha\}$. Since $Uc(X) \leq \kappa$, by Lemma 1.1.1 we have that $\forall \alpha < \kappa$ there exists a maximal Urysohn cellular family $\mathcal{W}_\alpha' \in [\mathcal{W}_\alpha]^{\leq\kappa}$ such that $cl_\theta(\overline{\bigcup \mathcal{W}_\alpha'}) \supseteq \overline{\bigcup \mathcal{W}_\alpha}$. Since $cl_\theta(\overline{\bigcup \mathcal{W}_\alpha'})$ is closed, it follows that $S \cap V_\alpha \subseteq \overline{\bigcup \mathcal{W}_\alpha} \subseteq cl_\theta(\overline{\bigcup \mathcal{W}_\alpha'}) \subseteq cl_\theta(\bigcup_{q\in S\cap V_q} \overline{V_q})$. Then, since $(\bigcup_{q\in S\cap V_\alpha} \overline{V_q}) \cap \overline{U_\alpha} = \emptyset$ and $p \notin cl_\theta(\bigcup_{q\in S\cap V_\alpha} \overline{V_q})$, we have that $p \notin cl_\theta(\overline{\bigcup \mathcal{W}_\alpha'})$. Put $W = \bigcup_{\alpha<\kappa} cl_\theta(\overline{\bigcup \mathcal{W}_\alpha'})$. Since $|\{V : V \in \mathcal{W}_\alpha'$ for some $\alpha < \kappa\}| \leq \kappa \cdot \kappa = \kappa < \kappa^+$, there is an $\alpha_0 < \kappa^+$ such that $\mathcal{W}_\alpha' \in [\mathcal{U}_{\alpha_0}]^{\leq\kappa}$ for each $\alpha < \kappa$. Hence, by (3), one has $A_{\alpha_0} \setminus W \neq \emptyset$. But $W \supseteq \bigcup_{\alpha<\kappa}(V_\alpha \cap S) = S$ and $A_{\alpha_0} \setminus W \subseteq S \setminus W = \emptyset$; a contradiction. □

Corollary 1.1.3. *[60] Let X be a regular space. Then $|X| \leq \pi\chi(X)^{c(X)\psi(X)}$.*

Corollary 1.1.4. *[61] Let X be a Urysohn space. Then $|X| \leq 2^{Uc(X)\chi(X)}$.*

1.1.2 Variation of the de Groot's inequality for a Hausdorff space X: $|X| \leq 2^{hL(X)}$

In this section, new cardinal functions are considered: $UW(X)$, $\psi w_\theta(X)$, θ-$aL(X)$, $h\theta$-$aL(X)$, θ-$aL_c(X)$ and θ-$aL_\theta(X)$ such that $HW(X) \leq UW(X)$, $\psi w(X) \leq \psi w_\theta(X)$ and θ-$aL(X) \leq aL(X)$. Furthermore we introduce a variation of Theorem 2.23 in [21] (if X is a T_1 space, then $|X| \leq HW(X)\psi w(X)^{haL(X)}$).

In Proposition 1.1.1 it was shown that Urysohn axiom is equivalent to $\{x\} = \bigcap\{cl_\theta(\overline{U}) : U \text{ open}, x \in U\}$, for every point x of the space. The following example shows that in spaces which are not Urysohn the previous intersection can be large.

Example 1.1.1. Any infinite space X with the cofinite topology is a T_1, compact, non Hausdorff space for which there is a point x such that $\bigcap\{cl_\theta(\overline{U}) : x \in U\}$ has large cardinality.

The example above gives a motivation to introduce the following definition:

Definition 1.1.2. Let X be a T_1 topological space and for all $x \in X$, let

$$Uw(x) = \bigcap\{cl_\theta(\overline{U}) : x \in U, U \text{ open}\}.$$

The *Urysohn width* is:

$$UW(X) = \sup\{|Uw(x)| : x \in X\}.$$

It is clear that if X is a Urysohn space then $UW(X) = 1$.

Recall that $HW(X) = \sup\{|Hw(x)| : x \in X\}$ is the *Hausdorff width*, where $Hw(x) = \bigcap\{\overline{U} : x \in U, U \text{ open}\}$ [21]. Since the θ-closure of a set contains its closure we have that $HW(X) \leq UW(X)$.

Definition 1.1.3. [21] Let X be a space and $x \in X$.

$$\psi w(x) = \min\{|\mathcal{U}_x| : \bigcap\{\overline{U} : U \in \mathcal{U}_x\} = Hw(x), \mathcal{U}_x \text{ is a family of open neighborhood of } x\};$$

and

$$\psi w(X) = \sup\{\psi w(x) : x \in X\}.$$

Similarly, we introduce the following definition.

Definition 1.1.4. Let X be a space and $x \in X$.

$$\psi w_\theta(x) = \min\{|\mathcal{U}_x| : \bigcap\{cl_\theta(\overline{U}) : U \in \mathcal{U}_x\} = Uw(x), \mathcal{U}_x \text{ is a}$$

family of open neighborhood of $x\}$;

and

$$\psi w_\theta(X) = \sup\{\psi w_\theta(x) : x \in X\}.$$

Of course, if X is a T_1 space then $\psi w(X) \leq \psi w_\theta(X) \leq \chi(X)$; further if X is a Urysohn space then we have that $\psi w_\theta(X) = \psi_\theta(X)$.

The definition below is a generalization of the almost Lindelöf degree of a space defined in [67].

Definition 1.1.5. Let Y be a subset of a space X. The *θ-almost Lindelöf degree* of a subset Y of a space X is

θ-$aL(Y,X) = \min\{\kappa :$ for every cover $\mathcal{V}$ of Y consisting of open subsets of X, there exists $\mathcal{V}' \subseteq \mathcal{V}$ such that $|\mathcal{V}'| \leq \kappa$ and $\bigcup\{cl_\theta(\overline{V}) : V \in \mathcal{V}'\} \subseteq Y\}$.

The cardinal number θ-$aL(X,X)$ is called *θ-almost Lindelöf degree of the space X* and is denoted by θ-$aL(X)$.

The *hereditary θ-almost Lindelöf degree of X*, denoted by $h\theta$-$aL(X)$, is the cardinal $h\theta$-$aL(X) = \min\{\kappa :$ for every family γ of open subsets of X, there exists a family $\gamma' \in [\gamma]^{\leq\kappa}$ such that $\bigcup\gamma \subseteq \bigcup\{\overline{U} : U \in \gamma'\}\}$.

Obviously we have θ-$aL(X) \leq aL(X)$ for every space X. Using a slight modification of Example 2.3 in [13] we prove that the previous inequality can be strict.

Example 1.1.2. A space X such that θ-$aL(X) < aL(X)$.

Proof. Let κ be any uncountable cardinal, let $\mathbb{Q}$ be the set of all the rationals and let $\mathbb{P}$ be the set of the irrationals. Put $X = (\mathbb{Q} \times \kappa) \cup \mathbb{P}$. We topologized X as follows. If $q \in \mathbb{Q}$ and $\alpha < \kappa$ then a neighborhood base at (q,α) is $\mathcal{U}(q,\alpha) = \{U_n(q,\alpha) : n \in \omega\}$ where

$$U_n(q,\alpha) = \{(r,\alpha) : r \in \mathbb{Q} \text{ and } |r\text{-}q| < \frac{1}{n}\}.$$

If $p \in \mathbb{P}$ a neighborhood base at p takes the form:

$$\{\{b \in \mathbb{P} : |b\text{-}p| < \frac{1}{n}\} \cup \{(q,\alpha) : \alpha < \kappa \text{ and } |q\text{-}p| < \frac{1}{n}\} : n \in \omega\}.$$

For every $q \in \mathbb{Q}$, $\alpha < \kappa$ and $n \in \omega$ we have that:

$$\overline{U_n(q,\alpha)} = U_n(q,\alpha) \bigcup \{p \in \mathbb{P} : |q\text{-}p| < \frac{1}{n}\};$$

and:

$$cl_\theta(\overline{U_n(q,\alpha)}) = \overline{U_n(q,\alpha)} \bigcup \{(r,\beta) : |r-q| < \frac{1}{n}, \beta < \kappa\}.$$

Let $\alpha < \kappa$, we have that $X = \bigcup_{q \in \mathbb{Q}} cl_\theta(\overline{\mathcal{U}(q,\alpha)})$ and so θ-$aL(X) = \aleph_0$ but we have that $aL(X) = 2^{\aleph_0}$. □

It is easy to show that the almost Lindelöf degree is hereditary with respect to θ-closed subsets. We have (see Proposition 1.1.3 below) that the θ-almost Lindelöf degree is hereditary with respect to a new class of spaces that we call *γ-closed*.

Definition 1.1.6. Let X be a topological space and $A \subseteq X$. The *γ-closure* of the set A is

$cl_\gamma(A) = \{x : \textit{for every open neighborhood of } X,\ cl_\theta(\overline{U}) \cap A \neq \emptyset\}$.

A is said to be γ-closed if $A = cl_\gamma(A)$.

The following example shows that the γ-closure and the θ-closure of a subset of a topological space can be different.

Example 1.1.3. A Urysohn space X having a subset Y such that $cl_\gamma(Y) \neq cl_\theta(Y)$.

Proof. Let $\mathbb{R} = A \cup B \cup C \cup D$ where A, B, C, D are pairwise disjoint and each is dense in $\mathbb{R}$. Let A' be a topological copy of A; points in A' are denoted as a' where $a \in A$.

Let $a, b \in \mathbb{R}$. A base for X is generated by these families of open sets:
(1)$\{(a,b) \cap A : a, b \in \mathbb{R}, a < b\}$
(2)$\{(a,b) \cap C : a, b \in \mathbb{R}, a < b\}$,
(3)$\{(a,b) \cap A' : a, b \in \mathbb{R}, a < b\}$,
(4)$\{(a,b) \cap (A \cup B \cup C) : a, b \in \mathbb{R}, a < b\}$, and
(5)$\{(a,b) \cap (C \cup D \cup A') : a, b \in \mathbb{R}, a < b\}$.

Note that for every $a,\ b \in \mathbb{R}$, $\overline{(a,b) \cap A} = [a,b] \cap (A \cup B)$, $\overline{(a,b) \cap A'} = [a,b] \cap (A' \cup D)$, $\overline{(a,b) \cap C} = [a,b] \cap (B \cup C \cup D)$, $cl_\theta(\overline{(a,b) \cap A}) = [a,b] \cap (A \cup B \cup C)$ and $cl_\theta(\overline{(a,b) \cap A'}) = [a,b] \cap (A' \cup D \cup C)$. For these reasons we can say that if $a,\ b \in \mathbb{R}$ and if we put $Y = (a,b) \cap C$, we have that $cl_\theta(Y) = [a,b] \cap (B \cup C \cup D)$ and $cl_\gamma(Y) = [a,b] \cap (A \cup B \cup C \cup D \cup A')$. □

We have the following:

Proposition 1.1.3. *The θ-almost Lindelöf degree is hereditary with respect to γ-closed subsets.*

Proof. Let X be a topological space such that $\theta\text{-}aL(X) \leq \kappa$ and let $C \subseteq X$ be γ-closed set. $\forall x \in X \setminus C$ we have that there exists an open neighborhood U_x of x such that $cl_\theta(\overline{U}) \subseteq X \setminus C$. Let $\mathcal{U}$ be a cover of C consisting of open subsets of X. Then $\mathcal{V} = \mathcal{U} \bigcup \{U_x : \ x \in X \setminus C\}$ is an open cover of X and since $\theta\text{-}aL(X) \leq \kappa$, there exists $\mathcal{V}' \in [\mathcal{V}]^{\leq\kappa}$ such that $X = \bigcup\{cl_\theta(\overline{V}) : V \in \mathcal{V}'\}$. Then there exists $\mathcal{V}'' \in [\mathcal{U}]^{\leq\kappa}$ such that $C \subseteq \bigcup\{cl_\theta(\overline{V}) : V \in \mathcal{V}''\}$; this proves that $\theta\text{-}aL(C) \leq \kappa$. □

Now, using $UW(X)$, $\psi w_\theta(X)$ and $h\theta\text{-}aL(X)$, a variation of the Theorem 2.23 in [21] is proved. The proof of Theorem 1.1.2 follows step-by-step the proof of Theorem 2.23 in [21].

Theorem 1.1.2. *If X is a T_1 space then $|X| \le UW(X)\psi w_\theta(X)^{h\theta\text{-}aL(X)}$.*

Proof. Let $UW(X) \le \kappa$, $h\theta\text{-}aL(X) \le \tau$ and $\psi w_\theta(X) \le \lambda$. For all $x \in X$, let $\mathcal{U}_x$ be a family of open neighborhoods of x such that $|\mathcal{U}_x| \le \lambda$ and $Uw(x) = \bigcap\{cl_\theta(\overline{U}) : U \in \mathcal{U}_x\}$. By transfinite induction we construct two families $\{H_\alpha : \alpha \in \tau^+\}$ and $\{\mathcal{B}_\alpha : \alpha \in \tau^+\}$ such that:

1. $\{H_\alpha : \alpha \in \tau^+\}$ is an increasing sequence of subsets of X;
2. $|H_\alpha| \le \kappa\lambda^\tau$ for all $\alpha \in \tau^+$;
3. if $\{H_\beta : \beta \in \alpha\}$ are defined for some $\alpha \in \tau^+$, then $\mathcal{B}_\alpha = \bigcup\{\mathcal{U}_x : x \in \bigcup\{Uw(y) : y \in \bigcup\{H_\beta : \beta \in \alpha\}\}\}$;
4. if $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_\alpha]^{\le\tau}$ is such that $X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$ then $H_\alpha \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$.

Let $\alpha \in \tau^+$ and $\{H_\beta : \beta \in \alpha\}$ be already defined. For all $\mathcal{W}$ as in (4), choose a point $x(\mathcal{W}) \in X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\})$ and let C_α be the set of these points.

Let $H_\alpha = \bigcup\{H_\beta : \beta \in \alpha\} \cup C_\alpha$. Then $|H_\alpha| \le \kappa \cdot \lambda^\tau$.

Let $H = \bigcup\{H_\alpha : \alpha \in \tau^+\}$ and $H^* = \bigcup\{Uw(x) : x \in H\} \supseteq H$. Then $|H^*| \le \kappa \cdot \lambda^\tau$.

We want to prove that $X = H^*$. Suppose that there exists a point $q \in X \setminus H^*$. Then $q \notin Uw(x)$, $\forall x \in H$. Hence for all $x \in H$ there is $U(x) \in \mathcal{U}_x$ such that $q \notin cl_\theta(\overline{U(x)})$. From $h\theta\text{-}aL(X) \le \tau$ choose $H' \in [H]^{\le\tau}$ such that $H \subseteq \bigcup\{cl_\theta(\overline{U(x)}) : x \in H'\}$. Let $\mathcal{W} = \{U(x) : x \in H'\}$. We have that $H' \subseteq H_\alpha$ for some $\alpha \in \tau^+$ and $\mathcal{W} \in [\mathcal{B}_{\alpha+1}]^{\le\tau}$ and $X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\}) \neq \emptyset$. Hence we have already chosen $x(\mathcal{W}) \in X \setminus (\bigcup\{cl_\theta(\overline{U}) : U \in \mathcal{W}\} \subseteq X \setminus H$ and $x(\mathcal{W}) \in H$ a contradiction. Hence $X = H^*$ and $|X| \le \kappa\lambda^\tau$. □

Corollary 1.1.5. *If X is a Urysohn space then $|X| \le \psi_\theta(X)^{h\theta\text{-}aL(X)}$.*

1.1.3 An improvement of the Bella-Cammaroto inequality for a Urysohn space X: $|X| \le 2^{aL(X)\chi(X)}$

In this section we modify a filter construction related to that given in [27]. In that paper an operator $\hat{c}$ was constructed. Here we construct a related operator $\tilde{c}$.

Let X be a topological space, $x \in X$ and $\mathcal{F}_x$ the collection of all finite intersections C of regular closed sets such that $x \in C$. It is easy to prove that $\mathcal{F}_x$ is a filter base which can be extended to a filter $\mathcal{C}_x$ that is maximal in the collection of all finite intersections of regular closed sets, partially ordered by inclusion. It is important to note that the absolute of a space is generated by these fixed maximal filters (see 6.6 in [58]).

The maximal filter $\mathcal{C}_x$ has the following properties:

Proposition 1.1.4. *Let X be a topological space and $x \in X$. Every regular closed subset of X which meets every element of $\mathcal{C}_x$ is an element of $\mathcal{C}_x$.*

Proof. Let U be an open subset of X and $x \in X$. $\overline{U}$ is regular closed. Suppose $\overline{U}$ meets every element of $\mathcal{C}_x$. Then $\{\overline{U}\} \cup \mathcal{C}_x$ is a filter base that contains $\mathcal{C}_x$ which can be extended to a maximal filter $\mathcal{M}$. As $\mathcal{C}_x$ is maximal, we have $\mathcal{C} = \mathcal{M}$ and $\overline{U} \in \mathcal{C}_x$. □

Proposition 1.1.5. *Let X be a Urysohn space, then for every $x, y \in X$ with $x \neq y$ we have that $\mathcal{C}_x \neq \mathcal{C}_y$.*

Proof. Let $x, y \in X$ with $x \neq y$, since X is a Urysohn space, there exist U, V two open subsets of X such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. We have that $\overline{U} \in \mathcal{C}_x$ and $\overline{V} \in \mathcal{C}_y$. If $\mathcal{C}_x = \mathcal{C}_y$, then $\overline{V} \in \mathcal{C}_x$ and $\overline{U} \cap \overline{V} = \emptyset \in \mathcal{C}_x$, a contradiction. □

We now define new operators using the maximal filter $\mathcal{C}_x$.

Definition 1.1.7. For a space X and an open subset U of X, define:

$$\widetilde{U} = \{x \in X : \overline{U} \in \mathcal{C}_x\}.$$

In the following propositions we give several properties of $\widetilde{U}$.

Proposition 1.1.6. *Let X be a space and U an open subset of X. Then $\overline{U} \subseteq \widetilde{U} \subseteq cl_\theta(\overline{U})$.*

Proof. If $x \in \overline{U}$, then $\overline{U} \in \mathcal{C}_x$ and $x \in \widetilde{U}$. Let V be an open subset of X such that $x \in \overline{V}$. By Proposition 1.1.4, $\overline{V} \cap \overline{U} \neq \emptyset$, then $x \in cl_\theta(\overline{U})$. □

Proposition 1.1.7. *If X is a topological space and V, W are open subsets of X, then $\widetilde{V} \cup \widetilde{W} = \widetilde{V \cup W}$. In particular this operator distributes over finite unions.*

Proof. $\widetilde{V} \cup \widetilde{W} = \{x \in X : \overline{V} \in \mathcal{C}_x\} \cup \{x \in X : \overline{W} \in \mathcal{C}_x\} = \{x \in X : \overline{V} \cap \overline{W} = \overline{V \cup W} \in \mathcal{C}_x\} = \widetilde{V \cup W}$. □

The analogue of the following proposition in the case of Hausdorff spaces is contained in the proof of Proposition 4.1 in [27].

Proposition 1.1.8. *X is a Urysohn space if and only if for every $x, y \in X$ with $x \neq y$, there exist U, V open subsets of X such that $\widetilde{U} \cap \widetilde{V} = \emptyset$.*

Proof. Suppose that for every $x,\ y \in X$ with $x \neq y$, there exists $U,\ V$ open subsets of X such that $\widetilde{U} \cap \widetilde{V} = \emptyset$. We have that $\overline{U} \subseteq \widetilde{U}$ and $\overline{V} \subseteq \widetilde{V}$, so $\overline{U} \cap \overline{V} = \emptyset$. This means that X is a Urysohn space.

Conversely, suppose X is Urysohn, then for every $x,\ y \in X$ with $x \neq y$, there exists $U,\ V$ open subsets of X such that $\overline{U} \cap \overline{V} = \emptyset$. We want to show that $\widetilde{U} \cap \widetilde{V} = \emptyset$. In order to have a contradiction, suppose there exists $z \in \widetilde{U} \cap \widetilde{V}$. From the definition of $\widetilde{U}$ and $\widetilde{V}$, we have $\overline{U},\ \overline{V} \in \mathcal{C}_z$. $\mathcal{C}_z$ is a filter, therefore $\overline{U} \cap \overline{V} = \emptyset \in \mathcal{C}_z$, a contradiction. □

For a space X and a subset A of X we define a new operator called *$\tilde{c}$-closure* in this way:

Definition 1.1.8. Let X be a space and A a subset of X,

$$\tilde{c}(A) = \{x \in X : \ \widetilde{U} \cap A \neq \emptyset \textit{ for every open subset } U \textit{ of } X \textit{ containing } x\}.$$

We say that $A \subseteq X$ is *$\tilde{c}$-closed* if $A = \tilde{c}(A)$.

We have the following propositions.

Proposition 1.1.9. *If X is a space and A is a subset of X, then we have $cl_\theta(A) \subseteq \tilde{c}(A) \subseteq cl_\gamma(A)$.*

Proof. If $x \in cl_\theta(A)$, then for every open subset V of X such that $x \in V$, $\overline{V} \cap A \neq \emptyset$. Therefore $\widetilde{V} \cap A \neq \emptyset$ and $x \in \tilde{c}(A)$.

If $x \in \tilde{c}(A)$, then for every open subset U of X such that $x \in U$ we have that $\widetilde{U} \cap A \neq \emptyset$ and by Proposition 1.1.6 we have $\widetilde{U} \subseteq cl_\theta(\overline{U})$. Thus for every open subset U of X such that $x \in U$ we have that $cl_\theta(\overline{U}) \cap A \neq \emptyset$, so $x \in cl_\gamma(A)$. □

Proposition 1.1.10. *If X is a space and U is an open subset of X, then $cl_\theta(\overline{U}) \subseteq \tilde{c}(\widetilde{U})$.*

Proof. If U is an open subset of X, by definitions we have $cl_\theta(\overline{U}) \subseteq \tilde{c}(\overline{U}) \subseteq \tilde{c}(\widetilde{U})$. □

We investigate the relation between Urysohn spaces and the operator $\tilde{c}(\cdot)$ and we prove the following:

Proposition 1.1.11. *If X is a Urysohn space, then for all $x,\ y \in X$ with $x \neq y$ there exists U open subset of X such that $x \in U$ and $y \notin \tilde{c}(\widetilde{U})$.*

Proof. Let $x,\ y \in X$ with $x \neq y$. X is a Urysohn space, so that by Proposition 1.1.8 there exist $U,\ V$ open subsets of X such that $x \in U$, $y \in V$ and $\widetilde{V} \cap \widetilde{U} = \emptyset$. We have that $\tilde{c}(\widetilde{U}) = \{z \in X : \ \widetilde{W} \cap \widetilde{U} \neq \emptyset$ for every open subset U of $X\}$ and $y \in V$ but $\widetilde{V} \cap \widetilde{U} = \emptyset$ so $y \notin \tilde{c}(\widetilde{U})$. □

For a space X we define a new cardinal invariant $t_{\tilde{c}}(X)$ related to the tightness $t(X)$.

Definition 1.1.9. For a space X, the *$\tilde{c}$-tightness of a point $x \in X$*, denoted by $t_{\tilde{c}}(x, X)$, is:

$$t_{\tilde{c}}(x, X) = min\{k : \text{ for every } A \subseteq X, \text{ if } x \in \tilde{c}(A) \text{ , there exists } B \in [A]^{\leq k}$$
$$\text{such that } x \in \tilde{c}(B)\}.$$

The *$\tilde{c}$-tightness of the space X*, denoted by $t_{\tilde{c}}(X)$, is:

$$t_{\tilde{c}}(X) = sup_{x\in X} t_{\tilde{c}}(x, X)$$

Like other variations of tightness, the following proposition we prove that $t_{\tilde{c}}(X)$ is bounded above by the character.

Proposition 1.1.12. *If X is a space, then $t_{\tilde{c}}(X) \leq \chi(X)$.*

Proof. Let $x \in X$, $A \subseteq X$ such that $x \in \tilde{c}(A)$ and $\mathcal{V}_x$ a neighborhood system of x in X with $|\mathcal{V}_x| \leq \chi(x, X)$. Because $x \in \tilde{c}(A)$, for every $V \in \mathcal{V}_x$ we have that $\widetilde{V} \cap A \neq \emptyset$. Let $y_{\widetilde{V}} \in \widetilde{V} \cap A$ for every $V \in \mathcal{V}_x$. We put $B = \{y_{\widetilde{V}} : V \in \mathcal{V}_x\}$, so $B \subseteq A$, $x \in \tilde{c}(B)$ and $|B| \leq \chi(x, X)$. This proves that $t_{\tilde{c}}(x, X) \leq \chi(x, X)$. □

Using the $\tilde{c}$-tightness and the θ-pseudocharacter we find a limitation for the cardinality of the $\tilde{c}$-closure of a subset A of a space X.

Proposition 1.1.13. *Let X be a Urysohn space such that $t_{\tilde{c}}(X)\psi_\theta(X) \leq \kappa$. Then for every $A \subseteq X$ we have that $|\tilde{c}(A)| \leq |A|^\kappa$.*

Proof. Let $x \in \tilde{c}(A)$. Since $\psi_\theta(X) \leq k$, from Proposition 1.1.11 there exists a family $\{U_\alpha(x)\}_{\alpha<\kappa}$ of neighborhood of x such that $\{x\} = \bigcap_{\alpha<k} cl_\theta(\overline{U_\alpha(x)}) = \bigcap_{\alpha<\kappa} \tilde{c}(\widetilde{U_\alpha(x)})$. We want to prove that $x \in \tilde{c}(\widetilde{U_\alpha(x)} \cap A)$, $\forall \alpha < k$. Let U be an open neighborhood of x and $\alpha < \kappa$. Since $x \in \tilde{c}(A)$, we have that $\emptyset \neq U \cap \widetilde{U_\alpha(x)} \cap A \subseteq \widetilde{U} \cap \widetilde{U_\alpha(x)} \cap A$. This shows that $x \in \tilde{c}(\widetilde{U_\alpha(x)} \cap A)$. Since $t_{\tilde{c}}(X) \leq k$, there exists $A_\alpha \subset \widetilde{U_\alpha(x)} \cap A$ such that $|A_\alpha| \leq \kappa$ and $x \in \tilde{c}(A_\alpha) \subseteq \tilde{c}(\widetilde{U_\alpha(x)})$. Then $\{x\} = \bigcap_{\alpha<\kappa} \tilde{c}(A_\alpha)$ and $\{A_\alpha\}_{\alpha<\kappa} \in [[A]^{\leq\kappa}]^{\leq\kappa}$, so $|\tilde{c}(A)| \leq |[[A]^{\leq\kappa}]^{\leq\kappa}| = |A|^\kappa$. □

The following represents another version of the Lindelöf degree and of the θ-almost Lindelöf degree using these new operators.

Definition 1.1.10. Let X be a topological space and Y a subset of X. We define $\widetilde{L}(Y, X)$ in this way:

$$\widetilde{L}(Y, X) = min\{k : \text{ for every cover } \mathcal{U} \text{ of } Y \text{ by sets open in } X$$
$$\text{there exists } \mathcal{V} \in [\mathcal{U}]^{\leq k} \text{such that } X = \bigcup \widetilde{\mathcal{V}}\}.$$

We put $\tilde{L}(X, X) = \tilde{L}(X)$.

We show now that if A is $\tilde{c}$-closed, then $\widetilde{L}(A,X) \leq \tilde{L}(X)$.

Proposition 1.1.14. *If A is $\tilde{c}$-closed subset of X, then $\widetilde{L}(A,X) \leq \tilde{L}(X)$.*

Proof. Suppose that $\widetilde{L}(X) \leq k$ and let C be a $\tilde{c}$-closed subset of X. For every $x \in X \setminus C$ there exists an open subset U_x of X such that $\widetilde{U_x} \subseteq X \setminus C$. Let $\mathcal{U}$ be an open cover of C, then $\mathcal{V} = \mathcal{U} \cup \{U_x : x \in X \setminus C\}$ is an open cover of X. As $\widetilde{L}(X) \leq k$, then there exists $\mathcal{V}' \in [\mathcal{V}]^{\leq k}$ such that $X = \bigcup \widetilde{\mathcal{V}'}$. Thus there exists $\mathcal{V}'' \in [\mathcal{U}]^{\leq k}$ such that $C \subseteq \bigcup \widetilde{\mathcal{V}''}$. This proves that $\widetilde{L}(C,X) \leq k$. □

Definition 1.1.11. For a space X, the θ-almost Lindelöf degree of X with respect to $\hat{c}$-closed subsets is denoted as $\theta\text{-}aL'(X)$ (instead of $\theta\text{-}aL_{\hat{c}}(X)$) and is defined by $\sup\{\theta\text{-}aL(C,X) : C \text{ is } \hat{c}\text{-closed }\}$.

Note that $\theta\text{-}aL(X) \leq \theta\text{-}aL'(X) \leq \widetilde{L}(X)$.

The main result of this section, a new cardinality bound for Urysohn spaces, can be proved using Theorem 3.1 in [43].

Theorem 1.1.3. *[Hodel] Let X be a set, k an infinite cardinal, $f : \mathcal{P}(X) \to \mathcal{P}(X)$ an operator on X and for each $x \in X$ let $\{V(\alpha,x) : \alpha < k\}$ be a collection of subsets of X. Assume the following:*

- *(T) (tightness condition) if $x \in f(H)$, then there exists $A \subseteq H$ with $|A| \leq k$ such that $x \in f(A)$;*
- *(C) (cardinality condition) if $A \subseteq X$ with $|A| \leq k$, then $|f(A)| \leq 2^k$;*
- *(C-S) (cover-separation condition) if $H \neq \emptyset$, $f(H) \subseteq H$ and $q \notin H$, then there exists $A \subseteq H$ with $|A| \leq k$ and a function $f : A \to k$ such that $H \subseteq \bigcup_{x\in A} V(f(x),x)$ and $q \notin \bigcup_{x\in A} V(f(x),x)$.*

Then $|X| \leq 2^k$.

To prove next theorem we use Theorem 1.1.3 and the operator $\tilde{c}(\cdot)$.

Theorem 1.1.4. *If X is a Urysohn space, then $|X| \leq 2^{\theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)}$.*

Proof. Let $k = \theta\text{-}aL'(X)t_{\tilde{c}}(X)\psi_\theta(X)$. As $\psi_\theta(X) \leq k$, for every $x \in X$ there exists a family $\mathcal{W}_x = \{W(\alpha,x) : \alpha < k\}$ of open subsets of X containing x such that $\{x\} = \bigcap_{W\in\mathcal{W}_x} cl_\theta(\overline{W})$.

For every $x \in X$ and $\alpha < k$, we put $V(\alpha,x) = cl_\theta(\overline{W(\alpha,x)})$ and prove the three conditions of Theorem 1.1.3.

For $H \subseteq X$, define $f(H) = \tilde{c}(H)$.

- Condition (T) is true because $t_{\tilde{c}}(X) \leq k$;
- Condition (C) is true by Proposition 1.1.13;

- We prove condition (C-S). Let $\emptyset \neq H \subseteq X$ satisfying $\tilde{c}(H) \subseteq H$. We have that $H \subseteq \tilde{c}(H)$ so $H = \tilde{c}(H)$ and H is $\tilde{c}$-closed. Suppose $q \notin H$. For every $a \in H$ there exists $\alpha_a < k$ such that $q \notin cl_\theta(\overline{W(\alpha_a, a)}) = V(\alpha_a, a)$. Let $f : H \to X$ such that $f(a) = \alpha_a$. The set $\{W(f(a), a) : a \in H\}$ is an open cover of H and since H is $\tilde{c}$-closed and $\theta\text{-}aL'(X) \leq k$, there exists $A \in [H]^{\leq k}$ such that $H \subseteq \bigcup_{a \in A} V(f(a), a)$ and $q \notin \bigcup_{a \in A} V(f(a), a)$. This proves condition (C-S).

Applying Theorem 1.1.3 we have that $|X| \leq 2^k = 2^{\theta\text{-}aL'(X) t_{\tilde{c}}(X) \psi_\theta(X)}$. □

We can observe that every $\tilde{c}$-closed set is also θ-closed. We also know that the almost Lindelöf degree is hereditary with respect to θ-closed sets, so for every space X we have:

$$\theta\text{-}aL'(X) \leq \theta\text{-}aL_\theta(X) \leq aL_\theta(X) = aL(X).$$

Furthermore we know that $\psi_\theta(X) \leq \chi(X)$ and by Proposition 1.1.12 we have that $t_{\tilde{c}}(X) \leq \chi(X)$.

For these motivations we obtain the following:

Corollary 1.1.6. *[14] If X is a Urysohn space, then $|X| \leq 2^{aL(X)\chi(X)}$.*

1.2 CARDINALITY BOUNDS FOR HAUSDORFF SPACES

In the following, the definition of the quasicellularity $qc(X)$ of a space X is given. We have that $wL(X) \leq qc(X) \leq c(X)$ for any space X. It is shown that $c(X) = qc(X)dot(X)$, decomposing $c(X)$ into two components, where $dot(X)$ is defined in [38]. Relationships between $qc(X)$ and other cardinal invariants are investigated. While $qc(X) = wL(X)$ for any extremally disconnected space, a generalization of this fact is given in Theorem 1.2.4 having consequences for extensions of X. Cardinality bounds involving $qc(X)$ are given.

Definition 1.2.1. Let X be a space and $A \subseteq X$, we define the *quasicellularity of $A \subset X$*, $qc(A, X)$, as the least infinite cardinal κ such that if there exists a family of open sets $\mathcal{U}$ with $A \subseteq \bigcup_{U \in \mathcal{U}} \overline{U}$ then there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $A \subseteq \overline{\bigcup \mathcal{V}}$. The *quasicellularity* of X is $qc(X) = qc(X, X)$. A family of open sets $\mathcal{U}$ of X is called a *q-cover* for X if $\bigcup_{U \in \mathcal{U}} \overline{U} = X$.

It is straightforward to see that $wL(X) \leq qc(X) \leq c(X)$ for any space X. Examples 1.2.1 and 1.2.2 below demonstrate these inequalities can be strict. Also, if X is extremally disconnected, then $qc(X) = wL(X)$ as regular closed sets are clopen in such spaces.

The following cardinal invariant was defined by Gotchev, Tkachenko, and Tkachuk in [38].

Definition 1.2.2. *[38]* Let X be a space. The *dense o-tightness of* X, $dot(X)$, is the least infinite cardinal κ such that for any family $\mathcal{U}$ of open sets with dense union and every $x \in X$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $x \in \overline{\bigcup \mathcal{V}}$.

The following straightforward proposition was shown in [38].

Proposition 1.2.1 (Lemma 4.2 in [38])**.** *For any space X, $dot(X) \leq t(X)$, $dot(X) \leq \pi\chi(X)$, and $dot(X) \leq c(X)$.*

The next theorem shows the cardinal functions $qc(X)$ and $dot(X)$ have an interesting connection with the cellularity $c(X)$ of a space X. In fact, $c(X)$ can be decomposed into the product of $qc(X)$ and $dot(X)$, neither of which involve cellular families.

Theorem 1.2.1. *For any space X, $c(X) = dot(X)qc(X)$.*

Proof. To show $c(X) \leq dot(X)qc(X)$, let $\kappa = dot(X)qc(X)$ and suppose $c(X) \geq k^+$. Then there exists a maximal cellular family $\mathcal{C}$ such that $|\mathcal{C}| \geq \kappa^+$ and $\bigcup \mathcal{C}$ is dense in X. Since $dot(X) \leq \kappa$, there exists $\mathcal{C}_x \in [\mathcal{C}]^{\leq\kappa}$ such that $x \in \overline{\bigcup \mathcal{C}_x}$. We have $X = \bigcup_{x\in X} \overline{\bigcup \mathcal{C}_x}$. Since $qc(X) \leq k$, there exists $A \in [X]^{\leq k}$ such that $X = \overline{\bigcup_{x\in A} \bigcup \mathcal{C}_x}$. There exists $C \in \mathcal{C} \setminus \bigcup_{x\in A} \mathcal{C}_x$ and $x \in A$ such that $C \cap \bigcup \mathcal{C}_x \neq \emptyset$. Thus there is $U \in \mathcal{C}_x$ such that $C \cap U \neq \emptyset$, which is a contradiction as the elements of $\mathcal{C}$ are pairwise disjoint. Therefore $c(X) \leq k$. The other inequality follows from $dot(X) \leq c(X)$ and $qc(X) \leq c(X)$. □

One should regard $dot(X)$ as a "small" cardinal invariant, which indicates how "close" $qc(X)$ and $c(X)$ are in a sense. Any space X with countable $dot(X)$ satisfies $c(X) = qc(X)$, including spaces with countable tightness or countable π-character.

Corollary 1.2.1. *For every space X, $c(X) \leq qc(X)t(X)$ and $c(X) \leq qc(X)\pi\chi(X)$.*

As $qc(X) \leq c(X)$ for every space X, we have the following.

Corollary 1.2.2. *For every space X, $c(X)\pi\chi(X) = qc(X)\pi\chi(X)$ and $c(X)t(X) = qc(X)t(X)$.*

Recalling that $wL(X) = qc(X)$ if X is extremally disconnected, we have [16]:

Corollary 1.2.3. *[16] If X is extremally disconnected, then $c(X) \leq wL(X)t(X)$ and $c(X) \leq wL(X)\pi\chi(X)$.*

The following cardinal function seems to be new in the literature.

Definition 1.2.3. Let X be a space. Define $b(X)$ as the least infinite cardinal κ such that X has a π-base $\mathcal{B}$ with $|B| \leq \kappa$ for all $B \in \mathcal{B}$.

Observe that if a the space X has a dense set of isolated points, then $b(X)$ is countable. The following result gives a connection between the cardinal functions used above.

Theorem 1.2.2. *Let X be a space. Then $d(X) \leq qc(X)dot(X)b(X)$.*

Proof. Let $\kappa = qc(X)dot(X)b(X)$. Let $\mathcal{B}$ be a π-base for X such that $|B| \leq \kappa$ for all $B \in \mathcal{B}$, and note $\bigcup \mathcal{B}$ is dense in X. As $dot(X) \leq \kappa$, for all $x \in X$ there exists $\mathcal{B}_x \in [\mathcal{B}]^{\leq\kappa}$ such that $x \in \overline{\bigcup \mathcal{B}_x}$. For all $x \in X$, set $U_x = \bigcup \mathcal{B}_x$ and note that $x \in \overline{U_x}$, $|U_x| \leq \kappa \cdot \kappa = \kappa$, and $\{U_x : x \in X\}$ is a q-cover of X. As $qc(X) \leq \kappa$, there exists $A \in [X]^{\leq\kappa}$ such that $\bigcup_{x\in A} U_x$ is dense in X. As $|\bigcup_{x\in A} U_x| \leq \kappa \cdot \kappa = \kappa$, we have $d(X) \leq \kappa$ as required. □

Corollary 1.2.4. *For every space X, $\pi w(X) \leq qc(X)\pi\chi(X)b(X)$.*

Proof. We have that $\pi w(X) = d(X)\pi\chi(X)$ and using Theorem 1.2.2 we have $\pi w(X) = d(X)\pi\chi(X) \leq qc(X)\pi\chi(X)b(X)$. □

Corollary 1.2.5. *Let X be a space with a dense set of isolated points, then*

(a) $d(X) \leq qc(X)t(X)$;

(b) $\pi w(X) \leq qc(X)\pi\chi(X)$.

Example 1.2.1. *There exists a space X such that $qc(X) < c(X)$.*

Let Y be the one-point compactification of a discrete space of cardinality $\aleph_1$, and let $X = EY$, the Iliadis absolute of Y. Then X is compact and extremally disconnected and $qc(X) = wL(X) = \aleph_0$. But X is not c.c.c. since $c(X) = c(Y) = \aleph_1$. It follows that $qc(X) < c(X)$.

Example 1.2.2. *There exists a space X such that $wL(X) < qc(X)$.*

Consider the space X of Example 1.2.6. We have $wL(X) = \aleph_0$ and $qc(X) > \aleph_0$.

Example 1.2.3. *The cardinal functions $qc(X)$ and $L(X)$ are incomparable.*

There exists c.c.c. not Lindelöf spaces, so $qc(X) < L(X)$. For the space of Example 1.2.2 we have $L(X) < qc(X)$.

It is natural to investigate the relationship between $qc(X)$ and $dot(X)$. We have that:

Example 1.2.4. *There exists a space X such that $dot(X) < qc(X)$.*

Let X be the space of Example 1.2.6. This space has countable π-character since it is first countable, then $dot(X)$ is countable since it is bounded above with $\pi\chi(X)$. Furthermore $qc(X) = c(X) > \aleph_0$.

Example 1.2.5. *There exists a space X such that $dot(X) > qc(X)$.*

Let X be the space of Example 1.2.1. X is compact e.d. with large cellularity, then $qc(X) = wL(X) = \aleph_0$ and by Theorem 1.2.1 $dot(X) > \aleph_0$.

The cardinal function $qc(X)$ is not necessarily hereditary for a space X. Let D be a discrete space of cardinality $\aleph_1$. Observe that $X = \beta D$ is extremally disconnected, thus $qc(X) = wL(X) = \aleph_0$. But D is a subspace which is not weakly Lindelöf and thus $qc(D) \geq wL(D) = \aleph_1 > \aleph_0$. However, the following proposition demonstrates that $qc(Y, X) \leq qc(X)$ if Y is a regular closed subset of X. The proof is similar to the well-known proof that $wL(X)$ is hereditary on regular closed subsets of X.

Proposition 1.2.2. *Let X be a space, W an open subset of X and $qc(X) \leq \kappa$, then $qc(\overline{W}, X) \leq \kappa$.*

Proof. Let $\mathcal{U}$ be a family of open subsets of X such that $\overline{W} \subseteq \bigcup_{U \in \mathcal{U}} \overline{U}$. We have $\bigcup_{U \in \mathcal{U}} \overline{U} \cup (\overline{X \setminus \overline{W}}) = X$. Since $qc(X) \leq \kappa$, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq \kappa}$ such that $X = \overline{\bigcup \mathcal{V}} \cup \overline{X \setminus \overline{W}}$. As $W \cap \overline{X \setminus \overline{W}} = \emptyset$, it follows that $W \subseteq \overline{\bigcup \mathcal{V}}$ and $\overline{W} \subseteq \overline{\bigcup \mathcal{V}}$. Thus $q(\overline{W}, X) \leq \kappa$. □

Example 1.2.3 shows that $qc(X)$ and $L(X)$ are incomparable. With the following we have a relationship between $qc(X)$ and $L(X)$.

Definition 1.2.4. *[13]* For a space X, $\overline{\psi}(X)$ is defined as the smallest cardinal κ such that every closed subset of X is the intersection of no more than κ-many open sets. $r\psi(X)$ is defined as the smallest cardinal κ such that every closed subset of X is the intersection of the closure of κ of its neighborhoods.

Proposition 1.2.3. *Let X be a topological space, then $qc(X) \leq \overline{\psi}(X)L(X)$.*

Proof. Let $\kappa = \overline{\psi}(X)L(X)$ and $\mathcal{U}$ a family of open subsets of X such that $\{\overline{U} : U \in \mathcal{U}\}$ is a cover of X. Let $C = X \setminus \bigcup \mathcal{U}$, a closed set. There exists a family $\{V_\alpha : \alpha \in \kappa\}$ of κ-many open subsets of X such that $C = \bigcap_{\alpha < \kappa} V_\alpha$. For every $\alpha \in \kappa$, $\{V_\alpha\} \cup \mathcal{U}$ is an open cover of X. Since $L(X) \leq \kappa$, there exists $\mathcal{W}_\alpha \in [\mathcal{U}]^{\leq \kappa}$ such that $\{V_\alpha\} \cup \bigcup \mathcal{W}_\alpha = X$. Let $\mathcal{W} = \bigcup_{\alpha < \kappa} \mathcal{W}_\alpha$ and note $|\mathcal{W}| \leq \kappa \cdot \kappa = \kappa$. We show that $\overline{\bigcup \mathcal{W}} = X$. Suppose there exists a non-empty open subset S of X such that $S \cap (\bigcup \mathcal{W}) = \emptyset$. Then $S \cap W = \emptyset$ for every $W \in \mathcal{W}$. This means $S \subseteq V_\alpha$ for every $\alpha < \kappa$ and $S \subseteq \bigcap_{\alpha < \kappa} V_\alpha = C = X \setminus \bigcup \mathcal{U}$. But $\bigcup_{U \in \mathcal{U}} \overline{U} = X$ and S must intersect an element of $\mathcal{U}$, a contradiction. Thus $\overline{\bigcup \mathcal{W}} = X$ and $qc(X) \leq \kappa$. □

The following represents a relation between $qc(X)$ and $wL(X)$ for a space X. It follows from the fact that $c(X) \leq r\psi(X)wL(X)$, which is Theorem 3.1 in [13].

Proposition 1.2.4. *For any space X, $qc(X) \leq r\psi(X)wL(X)$.*

For the remainder of this section bounds involving $qc(X)$ for the cardinality of a topological space are proved.

It was shown in [16] that the cardinality of a Hausdorff space with a dense set of isolated points is $|X| \leq 2^{wL(X)t(X)\psi_c(X)}$. We give a variation of this result below.

Theorem 1.2.3. *Let X be a Hausdorff space with a dense set of isolated points, then $|X| \leq qc(X)^{t(X)\psi_c(X)}$.*

Proof. If X is a Hausdorff space, then $|X| \leq d(X)^{t(X)\psi_c(X)}$. Applying Corollary 1.2.5(a), we have $|X| \leq d(X)^{t(X)\psi_c(X)} \leq qc(X)t(X)^{t(X)\psi_c(X)} \leq qc(X)^{t(X)\psi_c(X)}$. □

In order to find other classes of spaces for which the Bell, Ginsburg, Woods bound [13] holds, in light of Theorem **??** above, it is natural to ask if there exist classes of spaces such that $qc(X) \leq wL(X)\chi(X)$. (Recall however that $qc(X) = wL(X)$ for e.d. spaces). Example 1.2.6 gives a compact T_5 space such that $qc(X) > wL(X)\chi(X)$, and Example 1.2.7 gives a compact zero-dimensional space such that $qc(X) > wL(X)\chi(X)$.

Example 1.2.6. *A compact T_5 space such that $qc(X) > wL(X)\chi(X)$.*

We can consider the unit square X with the lexicographic order topology [64]. This space is first countable, compact, T_5 with uncountable cellularity. As X is first countable, it follows that $\pi\chi(X)$ is countable. Thus, by Corollary 1.2.2, $c(X) = c(X)\pi\chi(X) = qc(X)\pi\chi(X) = qc(X)$. So $qc(X)$ is uncountable and $qc(X) > \aleph_0 = wL(X)\chi(X)$.

Example 1.2.7. *(see 4.1.D pag. 225 in [8]) A compact zero-dimensional space such that $qc(X) > wL(X)\chi(X)$.*

Let X be the Alexandroff duplicate of the Cantor set. Then X is a zero-dimensional, first-countable non-metrizable compact Hausdorff space of cellularity $\mathfrak{c}$. Then $wL(X)\chi(X) = \aleph_0$ but, by Corollary 1.2.2, $qc(X) = \mathfrak{c}$.

As stated above, we have $qc(X) = wL(X)$ for an extremally disconnected space X. The following theorem gives a larger class of spaces for which $qc(X) = wL(X)$ and it gives also a connection between $qc(X)$ and the extension of a space.

Definition 1.2.5. Let X be a space and let $x \in X$. x is an *extremally disconnected point*, briefly *e.d. point*, if $x \notin \overline{U} \cap \overline{V}$ whenever U and V are disjoint open subsets of X.

Recall that, for a cardinal κ, a G^c_κ*-set* is the κ-intersection of its closed neighborhoods.

Theorem 1.2.4. *Let X be a Hausdorff space, let $\kappa = wL(X)$, and let C be a θ-closed, nowhere dense, G^c_κ-set. Suppose further that every $x \in X \setminus C$ is an e.d. point of X. Then $qc(X) = wL(X)$.*

Proof. Let be $\mathcal{U}$ a q-cover of X. Then $X = \bigcup_{U \in \mathcal{U}} \overline{U}$ and, for every $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in \overline{U}_x$. Thus $X = \bigcup_{x \in X} \overline{U}_x$.

We show first that if W is any non-empty open subset of X such that $\overline{W} \subseteq X \setminus C$, then $\overline{W}$ is open. Let $V = X \setminus \overline{W}$. We note that V and W are two disjoint open subsets of X. As every point in $X \setminus C$ is an e.d. point, and $\overline{W} \subseteq X \setminus C$, we have $\emptyset = \overline{V} \cap \overline{W} \cap X \setminus C = \overline{V} \cap \overline{W}$. As $\overline{V} = \overline{X \setminus \overline{W}} = X \setminus int(\overline{W})$, then $\overline{W} \subseteq X \setminus \overline{V} \subseteq int(\overline{W}) \subseteq \overline{W}$. Therefore $\overline{W} = int(\overline{W})$ and $\overline{W}$ is open.

As C is θ-closed, for every $x \in X \setminus C$ there exists an open subset V_x containing x such that $\overline{V}_x \cap C = \emptyset$. Observe that $x \in \overline{U_x \cap V_x}$ for every $x \in X \setminus C$. Also, by what was shown above, $\overline{U_x \cap V_x}$ is open for every $x \in X \setminus C$.

By hypothesis, C is a G^c_κ-set with $\kappa = wL(X)$. Then there exists a family $\mathcal{W}$ of open subsets of X such that $|\mathcal{W}| \leq \kappa$ and $C = \bigcap_{W \in \mathcal{W}} \overline{W} = \bigcap_{W \in \mathcal{W}} W$. We put $\mathcal{V} = \{U_x \cap V_x : x \in X \setminus C\}$. For a fixed $W \in \mathcal{W}$, the family $\mathcal{V}' = \{\overline{U_x \cap V_x} : x \in X \setminus C\} \cup \{W\}$ is an open cover of X. Using $wL(X) \leq \kappa$, we can find $\mathcal{V}_W \in [\mathcal{V}]^{\leq\kappa}$ such that $X = \overline{\bigcup_{V \in \mathcal{V}_W} \overline{V} \cup W} = \overline{\bigcup_{V \in \mathcal{V}_W} \overline{V}} \cup \overline{W}$ (*). The family $\mathcal{B} = \{\mathcal{V}_W : W \in \mathcal{W}\}$ is such that $|\mathcal{B}| \leq \kappa$.

We show now that $X = \overline{\bigcup \mathcal{B}}$. Suppose that there exists a non-empty open subset T of X such that $T \cap \bigcup \mathcal{B} = \emptyset$. Then $T \cap \overline{\bigcup_{B \in \mathcal{B}} \overline{B}} = \emptyset$ and $T \cap \overline{\bigcup_{V \in \mathcal{V}_W} \overline{V}} = \emptyset$ for every $W \in \mathcal{W}$. Using (*), we have $T \subseteq \bigcap_{W \in \mathcal{W}} \overline{W} = C$. This is a contradiction as C is nowhere dense. This shows $X = \overline{\bigcup \mathcal{B}}$. Finally, we note that for every $B \in \mathcal{B}$, there exists $x_B \in X \setminus C$ such that $B = U_{x_B} \cap V_{x_B}$. Then $\{U_{x_B} : B \in \mathcal{B}\}$ is a κ-subfamily of the q-cover $\mathcal{U}$ with dense union. This shows $qc(X) \leq \kappa$. As always $wL(X) \leq qc(X)$, the proof is complete. □

Now we consider an application of Theorem 1.2.4. Recall the following definition.

Definition 1.2.6. A space X is σ-compact if it is the union of countably many compact subspaces.

The following characterizations of σ-compact spaces are well-known.

Theorem 1.2.5. *Let X be a Hausdorff space. The following are equivalent:*

(a) X is σ-compact;

(b) $X = \bigcup_{n \in \omega} U_n$ where, for every $n \in \omega$, U_n is open, $\overline{U_n}$ is compact and $\overline{U_n} \subseteq U_{n+1}$;

(c) X is Lindelöf and locally compact.

Proposition 1.2.5. *If X is a σ-compact Hausdorff space, then $Y \setminus X$ is a G^c_δ-set in any compactification Y of X.*

Proof. Let X be a σ-compact Hausdorff space space and Y be a compactification of X. As X is locally compact, X is open in Y. By Theorem 1.2.5, $X = \bigcup_{n\in\omega} U_n$ where U_n is open in X (and also open in Y), $\overline{U_n}^X$ is compact (and also closed in Y) and $\overline{U_n}^X \subseteq U_{n+1}$ for every $n \in \omega$. As $X = \bigcup_{n\in\omega} U_n = \bigcup_{n\in\omega} \overline{U_n}^X$, then $Y \setminus X = \bigcap_{n\in\omega} Y \setminus U_n = \bigcap_{n\in\omega} Y \setminus \overline{U_n}^X$. This demonstrates that $Y \setminus X$ is a G^c_δ-set in Y. □

Lemma 1.2.1. *If p is an e.d. point of a locally compact Hausdorff space X, then p is an e.d. point in any compactification of X.*

Proof. Let $p \in X$ be an e.d. point of X and let Y be a compactification of X. Suppose p is not an e.d. point of Y, then there exist two open disjoint subsets U, V of Y such that $p \in \overline{U}^Y \cap \overline{V}^Y$. We show $p \in \overline{U \cap X}^Y \cap \overline{V \cap X}^Y$. Let W be an open set in Y containing p. As X is locally compact then X is open in Y. It follows that $W \cap X$ is an open set in Y containing p. Thus $W \cap X \cap U \neq \emptyset$ and $W \cap X \cap V \neq \emptyset$, showing $p \in \overline{U \cap X}^Y \cap \overline{V \cap X}^Y$. Furthermore, $p \in \overline{U \cap X}^Y \cap \overline{V \cap X}^Y \cap X = \overline{U \cap X}^X \cap \overline{V \cap X}^X$. Yet $U \cap X$ and $V \cap X$ are disjoint open sets in X, contradicting that p is an e.d. point in X. □

If Y is a compactification of a σ-compact Hausdorff extremally disconnected space X, then $Y \setminus X$ is closed and, by Proposition 1.2.5, $Y \setminus X$ is a G^c_δ-set. If X is additionally extremally disconnected, then by Lemma 1.2.1 every point of X is e.d. in Y. Thus we have the following corollary of Theorem 1.2.4.

Corollary 1.2.6. *If Y is any compactification of a Hausdorff σ-compact e.d. space, then $qc(Y)$ is countable.*

2 COVERING PROPERTIES FOR TOPOLOGICAL SPACES

A *cover* of a topological space X is a family $\mathcal{U}$ of subsets of X such that $X = \bigcup \mathcal{U}$. A subfamily $\mathcal{V}$ of a cover $\mathcal{U}$ of a space X is a *subcover* if $X = \bigcup \mathcal{V}$. Using particular families of subsets of a space satisfiyng certain conditions we obtain several covering properties like compactness and Lindelöfness. One important and recent branch of topology related to covers of a space is the theory of selection principles. These properties are schemas for generating from one sort of open cover of a topological space a second sort of open cover. Several important topological properties have been described by these schemas. Famous are the Menger, Hurewicz and Rothberger properties. Recall that a space X is *Menger* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick finite subfamilies $\mathcal{F}(n) \subset \mathcal{U}(n)$, $n \in \omega$, so that $\bigcup\{\mathcal{F}(n) : n \in \omega\}$ covers X; X is *Hurewicz* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick finite subfamilies $\mathcal{F}(n) \subset \mathcal{U}(n)$, $n \in \omega$, so that every $x \in X$ is contained in $\bigcup \mathcal{F}(n)$ for all but finitely many n; X is *Rothberger* if for every sequence $(\mathcal{U}(n) : n \in \omega)$ of open covers of X, one can pick $U(n) \in \mathcal{U}(n)$, $n \in \omega$, so that $\{U(n) : n \in \omega\}$ covers X. Lots of important theorems in topology make use of covering properties. In particular we consider covering properties defined by stars, neighborhood assignments or as monotone versions of selection principles. Star covering properties have been widely studied in literature (see for example [17], [33], [44], [45], [55]). The use of stars is very important in general topology. In fact, some topological and covering properties are characterized using stars. Recall that if A is a subset of a space X and $\mathcal{B}$ is a family of subsets of X, the *star of A* with respect to $\mathcal{B}$, denoted by $St(A, \mathcal{B})$, is the set $\bigcup\{B \in \mathcal{B} : B \cap A \neq \emptyset\}$. The notion of star appears, for example, in the characterization of normality using stars: "A topological space is normal if and only if every finite open cover has a finite open star-refinement". Also countable compactness is equivalent to star-compactness in the class of Hausdorff spaces and paracompactness is equivalent to the following property: every open cover has an open star refinement. It is natural to consider stars of a space with respect to subspaces having particular properties. In this case we will say that a space X has the *star-$\mathcal{P}$* property (briefly St-$\mathcal{P}$) if for every open cover $\mathcal{U}$ of the space X, there exists a subset Y of X having the property $\mathcal{P}$ such that $St(Y, \mathcal{U}) = X$ [45]. We can also define covering properties not only in terms of stars but also in terms of neighborhood assignments. Recall that a *neighborhood assignment* in a space X is a family $\{O_x : x \in X\}$ of open subsets of X such that $x \in O_x$ for every $x \in X$. For example the Lindelöf property can be characterized using neighborhood assignments in the following way: a space X is Lindelöf if and only if for every neighborhood assignment $\{O_x : x \in X\}$

there is a countable subset Y of X such that $\{O_x : x \in Y\}$ is a cover of X. If Y is closed and discrete instead of countable we obtain the notion of *D-spaces* defined by van Douwen in [31]. The idea of van Douwen have been generalized in [56] where the authors defined a space to be *neighborhood assignment* $\mathcal{P}$ if for any neighborhood assignment $\{O_x : x \in X\}$ there exists a subspace Y of X having the property $\mathcal{P}$ such that $\{O_x : x \in Y\}$ is a cover of X. In particular the authors pose their attention on the following properties: compactness, pseudocompactness, countable compactness and Lindelöfness (see also [1], [3], and [5]). In the first section of this chapter we consider and compare ”star” and ”neighborhood assignment” versions of compactness, countable compactness, Lindelöfness, and of Menger property. A map $\Phi : P(X) \times \mathsf{NA}(X) \to P(X)$, where $\mathsf{NA}(X)$ is the family of all neighborhood assignments of X, such that: (i) $Y \subseteq \Phi(Y, \mathrm{NA})$ for every $(Y, \mathrm{NA}) \in P(X) \times \mathsf{NA}(X)$; (ii) $Z \subseteq Y \subseteq X$ implies $\Phi(Z, \mathrm{NA}) \subseteq \Phi(Y, \mathrm{NA})$ for every $\mathrm{NA} \in \mathsf{NA}(X)$, will be called an *expansion operator on* X. In particular, typical expansion operators are the *neighborhood assignment operator*, and the *star operator*, defined in Definition 2.1.4.

In the last section of this chapter we consider monotone versions of some selection principles. Monotonic versions of classical topological properties have been widely studied (see for example [50], [51], and [52]). When we add monotonicity to a covering property, it becomes stronger. The idea of a covering property being monotonic has its roots in the definition of ”monotone normality” that has nothing to do with open covers (see [22]). Shortly after, the style of this definition was adapted and applied to other kinds of properties, including covering properties. Gartside and Moody in [36] described a process for obtaining a monotone version of any well-known covering property: “by requiring that there is an operator, r, assigning to every open cover a refinement in such a way that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$ ”. Using this process, any covering property can be ”upgrated” into a monotonic property. Our starting point is the class of monotone Lindelöf spaces (see for example [50], [51]). The monotone version of the Lindelöf property described by Matveev in [53] is the following: a space X is *monotonically Lindelöf* if there exists an operator r assigning to every open cover $\mathcal{U}$ a countable open refinement such that $r(\mathcal{V})$ refines $r(\mathcal{U})$ whenever $\mathcal{V}$ refines $\mathcal{U}$. The monotone version of the Menger property is defined similarly to the monotonically Lindelöf property and has been studied in [18]. Logically there are four different ways of defining monotone versions of selection principles. We focus on all four monotone versions of the Menger, Rothberger and Hurewicz properties, and we show that one of this monotone versions introduced is absurd. We conclude the chapter with Section 2.2.5 in which we give the costruction of a space distinguishing several of the introduced monotone properties. Some of the results that we discuss are included in [10], [19] and [20].

2.1 STAR COVERING PROPERTIES AND NEIGHBORHOOD ASSIGNMENTS

In [56] the authors give the following definition which represents a development of an idea of E. van Douwen used to define D-spaces [31]. Given a topological property (or a class) $\mathcal{P}$, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to neighbourhood assignments) consists of spaces X such that for any neighborhood assignment $\{O_x : x \in X\}$ there is $Y \subset X$ with the property $\mathcal{P}$ such that $\bigcup\{O_x : x \in Y\} = X$; if X is a member of the class $\mathcal{P}^*$, then X is called dually $\mathcal{P}$. We express the idea in [56] in the following way.

Definition 2.1.1. A space X is called *neighborhood assignment* $\mathcal{P}$ (briefly NA-$\mathcal{P}$) if for any neighbourhood assignment $\{O_x : x \in X\}$ there is a subspace $Y \subset X$ such that Y has the property $\mathcal{P}$ and $\bigcup_{x \in Y} O_x = X$.

Definition 2.1.2. [45] A space X has the *star-*$\mathcal{P}$ property (briefly St-$\mathcal{P}$) if for every open cover $\mathcal{U}$ of the space X, there exists a subset Y of X with the property $\mathcal{P}$ such that $St(Y, \mathcal{U}) = X$.

It is obvious that if a space X has a dense subspace with the property $\mathcal{P}$, then it is star-$\mathcal{P}$.

We study the previous classes of spaces for some covering property $\mathcal{P}$. In Section 2.1.1, we give a diagram (Diagram 1) summing up the main relations between the considered properties and present several examples distinguishing almost all of them. In Section 2.1.2, we give a description of the previous properties in terms of cardinal functions and generalize known results.

2.1.1 Neighbourhood assignments and expansion operators

For a set X we use $P(X)$ to denote the family of all subsets of X.

Definition 2.1.3. Let X be a topological space.

(i) A *neighborhood assignment of* X is a map $\mathcal{N} : X \to P(X)$ such that $\mathcal{N}(x)$ is an open neighborhood of x in X for every x.

(ii) For a topological space X, we denote by $\mathsf{NA}(X)$ the family of all neighborhood assignments of X.

Definition 2.1.4. Let X be a topological space. A map $\Phi : P(X) \times \mathsf{NA}(X) \to P(X)$ satisfying properties (i) and (ii) below will be called an expansion operator on neighbourhood assignments of X, or shortly, an expansion operator on X.

(i) $Y \subseteq \Phi(Y, \mathcal{N})$ for every $(Y, \mathcal{N}) \in P(X) \times \mathsf{NA}(X)$;

(ii) $Z \subseteq Y \subseteq X$ implies $\Phi(Z, \mathcal{N}) \subseteq \Phi(Y, \mathcal{N})$ for every $\mathcal{N} \in \mathsf{NA}(X)$.

Typical expansion operators we shall consider are the *neighbourhood assignement operator* NA defined by

$$NA(Y,\mathcal{N}) = \bigcup\{\mathcal{N}(y) : y \in Y\} \text{ for every } (Y,\mathcal{N}) \in P(X) \times \mathsf{NA}(X). \quad (1)$$

and the *star operator* St defined by

$$St(Y,\mathcal{N}) = \bigcup\{\mathcal{N}(x) : x \in X \text{ and } \mathcal{N}(x) \cap Y \neq \emptyset\} \text{ for every } (Y,\mathcal{N}) \in P(X) \times \mathsf{NA}(X). \quad (2)$$

Definition 2.1.5. Given two expansion operators $\Phi : P(X) \times \mathsf{NA}(X) \to P(X)$ and $\Psi : P(X) \times \mathsf{NA}(X) \to P(X)$, we write $\Phi \leq \Psi$ provided that

$$\Phi(Y,\mathcal{N}) \subseteq \Psi(Y,\mathcal{N}) \textit{ for every } (Y,\mathcal{N}) \in P(X) \times NA(X). \quad (3)$$

Lemma 2.1.1. *$NA \leq St$.*

Proof. Let X be a space and $(Y,\mathcal{N}) \in P(X) \times \mathsf{NA}(X)$. Fix an arbitrary $x \in NA(Y,\mathcal{N})$. It follows from (1) that $x \in \mathcal{N}(y)$ for some $y \in Y$. Since $y \in \mathcal{N}(y)$ by Definition 2.1.3 (i), we have $\mathcal{N}(y) \cap Y \neq \emptyset$. Therefore, $\mathcal{N}(y) \subseteq St(Y,\mathcal{N})$ by (2). Since $x \in \mathcal{N}(y)$, we conclude that $x \in St(Y,\mathcal{N})$. Since this inclusion holds for every $x \in NA(Y,\mathcal{N})$, this shows that $NA(Y,\mathcal{N}) \subseteq St(Y,\mathcal{N})$. Since this inclusion holds for every $(Y,\mathcal{N}) \in P(X) \times \mathsf{NA}(X)$, we have $NA \leq St$ by Definition 2.1.5. □

Definition 2.1.6. If X is a space, Φ is an expansion operator on X and $\mathcal{N} \in \mathsf{NA}(X)$, then a subset Y of X is called a *Φ-core of $\mathcal{N}$* provided that $X = \Phi(Y,\mathcal{N})$.

Definition 2.1.7. Given a class $\mathcal{P}$ of topological spaces and an expansion operator Φ on a space X, we shall say that X is a *Φ-$\mathcal{P}$ space* provided that every $\mathcal{N} \in \mathsf{NA}(X)$ has a Φ-core which belongs to the class $\mathcal{P}$.

Notice that NA-$\mathcal{P}$ was first defined with a different terminology in [56] and St-$\mathcal{P}$ was first defined in [45].

The proof of the following proposition is straightforward.

Proposition 2.1.1. *Let Φ be an extension operator on a space X. If $\mathcal{P}$, $\mathcal{Q}$ are classes of spaces such that $\mathcal{P} \subseteq \mathcal{Q}$, and X is a Φ-$\mathcal{P}$ space, then X is an Φ-$\mathcal{Q}$ space.*

Proposition 2.1.2. *Let $\mathcal{P}$ be a class of topological spaces. Suppose that Φ and Ψ are extension opearators satisfying $\Psi \leq \Phi$. Then every Ψ-$\mathcal{P}$ space is also a Φ-$\mathcal{P}$ space.*

Proof. Let X be a Ψ-$\mathcal{P}$ space. Fix $\mathcal{N} \in \mathrm{NA}(X)$. By Definition 2.1.7, X has a Ψ-core Y which belongs to the class $\mathcal{P}$. By Definition 2.1.6, this means that $X = \Psi(Y, \mathcal{N})$. Since $\Psi \leq \Phi$ by our assumption, we have $\Psi(Y, \mathcal{N}) \subseteq \Phi(Y, \mathcal{N})$ by Defnition 2.1.5. Since $\Phi(Y, \mathcal{N}) \in P(X)$, we get $\Phi(Y, \mathcal{N}) \subseteq X$. Combining the above inclusions, we obtain $X = \Phi(Y, \mathcal{N})$. By Definition 2.1.6, this means that Y is Φ-core for $\mathcal{N}$. We have proved that every $\mathcal{N} \in \mathrm{NA}(X)$ has a Φ-core Y which belongs to $\mathcal{P}$. By Definition 2.1.7, X is a Φ-$\mathcal{P}$ space. □

Corollary 2.1.1. *For every class $\mathcal{P}$ of topological spaces, each NA-$\mathcal{P}$ space is a St-$\mathcal{P}$ space.*

Proof. Indeed, $NA \leq St$ by Lemma 2.1.1. Now the conclusion follows from Proposition 2.1.2. □

We will use this notation for the covering properties in Diagram 1. Compact spaces are denoted by C; σ-compact spaces are denoted by σC; Menger spaces are denoted by M; Lindelöf spaces are denoted by L; paracompact spaces are denoted by PC; metacompact spaces are denoted by MC; metalindelöf spaces are denoted by ML.

The implications of the following diagram are obvious.

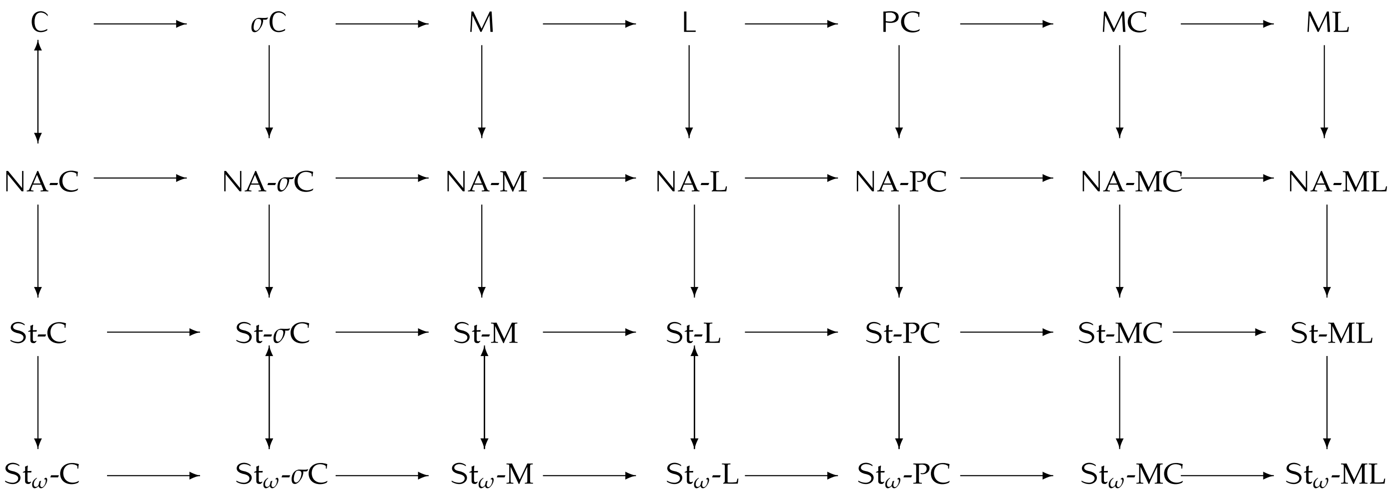


Diagram 1

Recall that (see also [48] and in [49] where a different terminology is used) a space X has the *Star$_\omega$-$\mathcal{P}$* property (briefly St$_\omega$-$\mathcal{P}$) if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of the space X, there exist subsets Y_n of X, for every $n \in \omega$, having property $\mathcal{P}$ such that $\{\bigcup St(Y_n, \mathcal{U}_n) : n \in \omega\}$ is a cover of X. It is obvious that if the property $\mathcal{P}$ is closed under countable unions, then a space is St-$\mathcal{P}$ if and only if the space is St$_\omega$-$\mathcal{P}$.

In [54] Matveev noted that in every T_1 space X for every open cover $\mathcal{U}$ of the space X, there exists a closed and discrete subset Y of X such that $St(Y, \mathcal{U}) = X$. Then, every T_1 space is St-PC (hence St_ω-PC), St-MC (hence St_ω-MC) and St-ML (hence St_ω-ML).

Denoting by D the class of all discrete spaces and with CD the topological property to be closed and discrete, we have the following diagram.

$$\text{NA-C} \overset{CC}{\Longleftrightarrow} \text{NA-CD} \Longrightarrow \text{NA-D} \Longrightarrow \text{NA-PC}.$$

Note that NA-D was introduces in [1] with a different terminology and NA-D spaces are exactely D-spaces defined in [32].

Now we give examples showing that some of the arrows of Diagram 1 can not be reversed. Before doing it we prove some useful results.

Proposition 2.1.3. *Let X be a NA-$\mathcal{P}$ space and C a closed and discrete subset of X, then there exists a subset Y of X having the property $\mathcal{P}$ such that $Y \supset C$.*

Proof. We can consider the following neighborhood assignment. For every $x \in C$ we choose an open subset O_x such that $O_x \cap C = \{x\}$ and we consider $O_x = X \setminus C$ for every $x \in X \setminus C$. We put $\mathcal{O} = \{O_x : x \in X\}$. Since the space is NA-$\mathcal{P}$, there exists $Y \subset X$ having the property $\mathcal{P}$ such that $\bigcup_{x \in Y} O_x = X$. We want to show that $Y \supset C$. Let $x \in C$, then $\exists! O_x \in \mathcal{O}$ such that $x \in O_x$ therefore $x \in Y$. □

In particular by the proposition above we can prove the existence of non NA-$\mathcal{P}$ spaces.

Corollary 2.1.2. *Let X be a NA-$\mathcal{P}$ space. If for every subset Y of X having the property $\mathcal{P}$ we have $e(Y) = \aleph_0$, then $e(X) = \aleph_0$.*

Corollary 2.1.3. *If X is NA-L, then $e(X) = \aleph_0$.*

Following step by step the proof of Theorem 2.4 in [3] we can give an improvement of Theorem 2.4 in [3]:

Theorem 2.1.1. *Let $\mathcal{P}$ be a property that is hereditary with respect to closed subsets and preserved under the union of two disjoint subsets. If a space X is the union of two subspaces Y and Z, where Y is NA-$\mathcal{P}$ and Z is a closed subset of X such that for every open subset U of X containing Z, $X \setminus U$ is NA-$\mathcal{P}$. Then X is NA-$\mathcal{P}$.*

In particular we have:

Example 2.1.1. *Every Isbell-Mrowka space is NA-D.*

Examples

Lemma 2.1.2. *The following hold for any topological space.*

(i) A NA-ML, hereditarily separable space is NA-L.

(ii) A locally countable, NA-L space is countable.

(iii) A locally countable, hereditarily separable, NA-ML space is countable.

Proof. (i) Let X be an NA-ML, hereditarily separable space, and let $\{O_x : x \in X\}$ be a neighbourhood assignment for X. Since X is NA-ML, we can find a metaLindelöf subspace Y of X such that

$$X = \bigcup\{O_y : y \in Y\}. \tag{4}$$

Since X is hereditarily separable, Y has countable extent. Since Y is metaLindelöf, it must be Lindelöf; see for example, [3, Corollary 2.8].

(ii) Let X be a locally countable, NA-L space. Since X is locally countable, we can fix a neighbourhood assignment $\{O_x : x \in X\}$ such that each O_x is countable. Since X is NA-L, there exists a Lindelöf subspace Y of X satisfying (4). Since Y is Lindelöf, the cover $\{O_y : y \in Y\}$ of Y has a countable subcover; that is,

$$Y \subseteq \bigcup\{O_y : y \in Z\} \tag{5}$$

for a countable subset Z of Y. Since each O_y is countable, it follows from (5) that Y is countable. By the same reason, it follows from (4) that X is countable.

(iii) Let X be a locally countable, hereditarily separable, NA-ML space. By item (i), X is NA-L. By item (ii), X is countable. □

Corollary 2.1.4. *ω_1 with the order topology is not NA-L.*

Proof. ω_1 is locally countable but not countable, so the conclusion follows from item (ii) of Lemma 2.1.2. □

Example 2.1.2. *A St-C space which is not NA-L.*

Proof. Indeed, ω_1 with the order topology is countably compact,hence since Hausdorff it is St-C. On the other hand, ω_1 is not NA-L by Corollary 2.1.4. □

Under Jensen's Axiom $\diamond$ (see [46]), we have even an example with stronger properties.

Example 2.1.3. *(Under $\diamond$). A St-C space which is not NA-ML.*

Proof. The Ostaszewski space given in [57] is given a Hausdorff, countably compact (hence St-C), hereditary separable space having cardinality ω_1. Therefore, X is not NA-ML by item (iii) of Lemma 2.1.2. □

A ZFC example of a non NA-ML space is constructed in Proposition 2.1(2) in [23].

Example 2.1.4. *A NA-M not NA-σC space.*

Proof. Let $L(\omega_1)$ be the one point Lindelöfication $\omega_1 \cup \{p\}$ of the discrete space ω_1. Consider the space $X = L(\omega_1) \times [0, \omega]$. The space X, being the product of a Menger space and of a compact space, is Menger. This means that X is NA-M. We want to show that X is not NA-σC. To do this, we can consider the neighborhood assignment $\{O_x : x \in X\}$ where for $x = (l, \omega) \in D \times \{\omega\}$, $O_x = \{l\} \times [0, \omega]$, for $x = (p, \alpha) \in \{p\} \times [0, \omega)$, $O_x = L(\omega_1) \times \{\alpha\}$ and for $x = (p, \omega)$, $O_x = ([0, \omega_1] \setminus C) \times ([0, \omega] \setminus F)$ where C is a fixed countable subset of ω_1 and F is a fixed finite subset of ω. For every σ-C subset Y of X we have $U_{y \in Y} O_y \neq X$. This means X is not NA-σC. □

Example 2.1.5. *A St-PC not St-L space.*

Proof. (This is the space of Example 2.2 in [63], when $\mathfrak{c}$ is taken instead of ω_1) Let $A(\omega_1)$ be the Alexandroff (one-point) compactification of the discrete space ω_1. We may assume that ω_1 is the only non-isolated point of $A(\omega_1)$. Define $X = [0, \omega_1] \times A(\omega_1) \setminus \{(\omega_1, \omega_1)\}$.

(i) *X contains a dense paracompact subspace, so X is* St-PC. Indeed, the subspace $Y = [0, \omega_1] \times \omega_1$ of X is dense in X and homeomorphic to a disjoint sum of ω_1-many copies of the compact space $[0, \omega_1]$, so Y is paracompact.

(ii) *X is not* St-*L.*[1] Before proving this, we shall prove the following.

Claim 2.1.1. *For every Lindelöf subspace Y of X, there exists $\gamma \in \omega_1$ such that*

$$(0, \omega_1) \notin \pi((X \setminus ([0, \gamma) \times A(\omega_1))) \cap Y), \tag{6}$$

where $\pi : [0, \omega_1] \times A(\omega_1) \to A(\omega_1)$ is the projection on the second coordinate.

Proof. Since Y is Lindelöf and $Z = [0, \omega_1) \times \{\omega_1\}$ is a closed subpsce of X, $Y \cap Z$ is Lindelöf as well. Since Z is homeomorphic to the ordinal space $[0, \omega_1)$, the Lindelöf subspace $Y \cap Z$ of Z must be countable. Therefore, there exists $\gamma \in \omega_1$ such that $Y \cap Z \subseteq [0, \gamma) \times \{\omega_1\}$. Finally, note that this γ satisfies (6). □

For every $\alpha \in \omega_1$, the set $U_\alpha = (\alpha, \omega_1] \times \{\alpha\}$ is open in X. Furthermore, $V = [0, \omega_1) \times A(\omega_1)$ is an open subset of X. Therefore, $\mathscr{U} = \{U_\alpha : \alpha \in \omega_1\} \cup \{V\}$ is an open cover of X. Suppose that Y is a Lindelöf subspace of X. We are going to show that $\mathrm{St}(Y, \mathscr{U}) \neq X$. Let $\gamma \in \omega_1$ be the ordinal as in the conclusion of Claim 2.1.1. $Y' = (X \setminus ([0, \gamma) \times A(\omega_1))) \cap Y$ is a closed subset of Y, so it is Lindelöf. Since π is a continuous mapping, $\pi(Y')$ is Lindelöf subspace of $A(\omega_1)$. By the conclusion of Claim 2.1.1, $\omega_1 \notin \pi(Y')$. Therefore,

1 In [63] was only proved that X is not St_ω-C.

$\pi(Y')$ is a Lindelöf subspace of the discrete space $A(\omega_1) \setminus \omega_1$, so $\pi(Y')$ is countable. Therefore, we can find $\beta \in \omega_1$ such that $\pi(Y') \subseteq [0, \beta)$. Now let $\alpha = \max\{\beta, \gamma\}$. Note that $U_\alpha \cap Y = \emptyset$ by our construction.

We claim that $(\omega_1, \alpha) \in X \setminus \mathrm{St}(Y, \mathscr{U})$. To see this, it is sufficient to realize that U_α is the only element of $\mathscr{U}$ containing $(\omega_1, \alpha) \in X$. Since $U_\alpha \cap Y = \emptyset$, this means that $(\omega_1, \alpha) \notin \mathrm{St}(Y, \mathscr{U})$. □

Example 2.1.6. *A NA-D not St-L space.*

Proof. Let S be the set of isolated points in ω_1. Consider the set $X = (\omega_1 \times \omega) \cup (S \times \{\omega\})$ with the subspace topology inherited from the product $\omega_1 \times (\omega + 1)$ of two cardinals ω_1 and $\omega + 1$. This example is included as item (5) on page 623 of [2] and attributed to anonymous referee.

(i) *X is not St-L.* This is proved in [2].

(ii) *X is NA-D, and so NA-PC.* Indeed, the subset $Y = S \times \{\omega\}$ of X is discrete, so NA-D, and its complement $X \setminus Y = \omega_1 \times \omega$ is a disjoint sum of countably many copies of ω_1. Since the latter space is NA-D (see Example 2.3 in [56]), so is $X \setminus Y$. Now the conclusion of item (ii) follows from Theorem 2.1.1. □

Example 2.1.7. *In some model of ZFC, there exists a Urysohn space X which has a dense subspace homeomorphic to the space of irrational numbers (so X is a St-L not St-M space).*

Proof. Let P be the space of irrational numbers in its usual topology. Define $X = P \times (\omega + 1)$. For $(p, n) \in P \times \omega$, we declare

$$\{U \times \{n\} : p \in U \text{ and } U \text{ is open in } P\}$$

to be the neighbourhood base of a point (p, n). Therefore, $Z = P \times \omega$ is a disjoint sum of countably many copes of P, so it is homeomorphic to P itself.

For every $p \in P$, a basic open neighbourhood of (p, ω) in X is of the form

$$O(p, U, n, M) = \{(p, \omega)\} \cup ((U \times (n, \omega)) \setminus M), \tag{7}$$

where U is a clopen subset of P containing p, $n \in \omega$ and M is a Menger subset of $Z = P \times \omega$.

Claim 2.1.2. *Z is dense in X, so X is St-L.*

Proof. Let $p \in P$ be arbitrary and let $O(p, U, n, M)$ be a basic open neighbourhood of (p, ω) as in (7). Note that $U \times (n, \omega)$ is a non-empty clopen subset of Z. Since Z is homeomorphic to P, this implies that $U \times (n, \omega)$ is homeomorphic to P. In particular, $U \times (n, \omega)$ is not Menger. This implies that $(U \times (n, \omega)) \setminus M \neq \emptyset$. (Indeed, otherwise $(U \times (n, \omega))$ be a closed subset of the Menger space M, so would be Menger.) Since $(U \times (n, \omega)) \setminus M \subseteq Z$, it follows from (7) that $O(p, U, n, M) \cap Z \neq \emptyset$. □

Claim 2.1.3. *X is not St-M.*

Proof. It was mention in [59] that in [35] was proved that in some model of ZFC, the family of all Menger subsets of the real line has cardinality of the continuum. Since P is a subset of the reals, the number of Menger subsets of P is at most continuum. Since $|P| = \mathfrak{c}$, we can fix an enumeration $\{M_p : p \in P\}$ of all Menger subsets of Z such that the set $P_M = \{p \in P : M_p = M\}$ has size $\mathfrak{c}$ for every Menger subspace of Z.

Consider the following assignment $\mathcal{N} \in \mathrm{NA}(X)$. For every $p \in P$ define $\mathcal{N}(p,\omega) = O(p,P,0,M_p)$ and $\mathcal{N}(p,n) = Z$ for every $n \in \omega$.

Suppose that Y is a Menger subset of X. Note that $D = P \times \{\omega\}$ is a closed subset of X, so $Y \cap D$ is a Menger subspace of D, so it is Lindelöf. Since D is a discrete subset of X, this means that $|Y \cap D| \leq \omega$.

Since Z is an F_σ-subset of X, $Y \cap Z$ is an F_σ-subset of Y. Since Y is Menger, so is $Y \cap Z$. Clearly, $Y \cap Z$ is a Menger subset of Z. By the property of our enumeration, the set $P_{Y \cap Z}$ has cardinality $\mathfrak{c}$. Since $|Y \cap D| \leq \omega$, there exists $p \in P_{Y\cap Z}$ such that $(p,\omega) \in D \setminus Y$. Note that $M_p = Y \cap Z$, as $p \in P_{Y \cap Z}$. It now follows from (7) that $O(p,P,0,M_p) \cap Y = \emptyset$. Thus, $\mathcal{N}(p,\omega) \cap Y = \emptyset$ by the definition of $\mathcal{N}$. Finally, note that $\mathcal{N}(p,\omega)$ is the only member of the family $\{\mathcal{N}(x) : x \in X\}$ containing (p,ω). From this fact and (2), we conclude that $(p,\omega) \notin St(Y,\mathcal{N})$. Therefore, Y is not a core for $\mathcal{N}$.

We have shown that no Menger subset of X can be a core for $\mathcal{N}$, this means that X is not St-M. □

A St-M not St-σC space is given by Example 2.1.8 in Section 2.1.2.

2.1.2 Cardinal invariants associated with an expansion operator on neighbourhood assignments

Definition 2.1.8. Having a cardinal function φ on a class of topological spaces, a space X, an expansion operator $\Phi : P(X) \times \mathrm{NA}(X) \to P(X)$ and $\mathcal{N} \in \mathrm{NA}(X)$, we define

$$\Phi\text{-}\varphi_{\mathcal{N}}(X) = \min\{\varphi(Y) : Y \textit{ is a } \Phi\textit{-core for } \mathcal{N}\}, \tag{8}$$

and we let

$$\Phi\text{-}\varphi(X) = \sup\{\Phi\text{-}\varphi_{\mathcal{N}}(X) : \mathcal{N} \in NA(X)\}. \tag{9}$$

Proposition 2.1.4. *If φ is a cardinal function on a class of topological spaces, X is a space and $\Phi : P(X) \times NA(X) \to P(X)$ is an expansion operator, then $\Phi\text{-}\varphi(X) \leq \varphi(X)$.*

Proof. Let $\mathcal{N} \in \mathrm{NA}(X)$ be arbitrary. Note that $X \subseteq \Phi(X,\mathcal{N})$ by Definition 2.1.4 (i) and $\Phi(X,\mathcal{N}) \subseteq X$ by the definition of Φ, so $X = \Phi(X,\mathcal{N})$. Therefore, X is Φ-core of $\mathcal{N}$ by Definition 2.1.6. This shows that the minimum in (8) is well defined and $\Phi\text{-}\varphi_{\mathcal{N}}(X) \leq \varphi(X)$. Since this holds for every $\mathcal{N} \in \mathsf{NA}(X)$, from (9) we obtain that $\Phi\text{-}\varphi(X) \leq \varphi(X)$. □

Lemma 2.1.3. *If $\Phi \leq \Psi$, then $\Psi\text{-}\varphi(X) \leq \Phi\text{-}\varphi(X)$ for every cardinal function φ.*

Proof. Let $\mathcal{N} \in \mathsf{NA}(X)$ be arbitrary. Since $\Phi \leq \Psi$, the inequality (3) holds. Therefore, if $X = \Phi(Y, \mathcal{N})$ for some $Y \subseteq X$, then $X = \Psi(Y, \mathcal{N})$ holds as well. This means that every Φ-core for $\mathcal{N}$ is also a Ψ-core. Applying (8), we conclude that $\Psi\text{-}\varphi_{\mathcal{N}}(X) \leq \Phi\text{-}\varphi_{\mathcal{N}}(X)$. Since this inequality holds for every $\mathcal{N} \in \mathsf{NA}(X)$, from (9) we conclude that $\Psi\text{-}\varphi(X) \leq \Phi\text{-}\varphi(X)$. □

Since $NA \leq St$ by Lemma 2.1.1, we obtain the following

Lemma 2.1.4. *$St\text{-}\varphi(X) \leq NA\text{-}\varphi(X) \leq \varphi(X)$ holds for every space X and each cardinal function φ.*

Now we can give an example of St-L not NA-L space.

Definition 2.1.9. [14] Let X be a space. The *compact covering number $\varkappa(X)$ of X* is the least cardinal number τ such that X can be covered by τ-many compact subsets. A space X is σ-compact if and only if $\varkappa(X) \leq \omega$.

Example 2.1.8. *For every uncountable cardinal κ, there exists a space X having the following properties:*

(i) $e(X) = \kappa$;

(ii) X is NA-D, so also NA-PC;

(iii) $NA\text{-}L(X) = \kappa$; in particular, X is not NA-L;

(iv) X is St-M, so also St-L;

(v) $NA\text{-}\varkappa(X) = \kappa$; in particular, X is not St-σC.

Proof. Indeed, fix an uncountable cardinal κ. Let D be a discrete space satisfying $|D| = \kappa$. Let L be the one-point Lindelöfication of D, let p be the unique non-isolated point of L, so that $L \setminus \{p\} = D$. Define $X = L \times [0, \omega] \setminus \{(p, \omega)\}$.

(i) Note that $C = D \times \{\omega\}$ is a closed discrete subspace of X. Since $|C| = |D| = \kappa = |X|$, this shows that the extent of X is equal to κ.

(ii) The closed subspace C of X defined in (i) is discrete, so it is trivially NA-D. Moreover, its completement $X \setminus C = L \times [0, \omega)$ is homeomorphic to a disjount sum of countably many copes of L. Since L is obviously NA-D, so is $X \setminus C$. By Theorem 2.1.1, X is NA-D.

(iii) Let us check that $NA\text{-}L(X) = \kappa$. For $x = (l, \omega) \in D \times \{\omega\}$, define $O_x = \{l\} \times [0, \omega]$. For $x = (p, \alpha) \in \{p\} \times [0, \omega)$, define $O_x = L \times \{\alpha\}$.

Finally, for $x = (l, \alpha) \in D \times [0, \omega)$, define $O_x = \{x\}$. Then $\{O_x : x \in X\}$ is a neighbourhood assignment for X.

Let Y be a subspace of X such that $X = \bigcup_{y \in Y} O_y$. Let $x \in C$ be arbitrary. There exists $y_x \in Y$ such that $x \in O_{y_x}$. On the other hand, by the choice of our assignment, O_x is the only elelement of the assignment $\{O_x : x \in X\}$ which contains x. Therefore, $x = y_x \in Y$. This shows that $C \subseteq Y$. Since C is a closed subset of X, it is also closed in Y, which implies $L(C) \leq L(Y)$. Since C is discrete, we have $L(C) = |C| = \kappa$. Thus, $L(Y) \geq L(C) = \kappa$. This means that $NA - L(X) \geq \kappa$. On the other hand, $NA - L(X) \leq L(X) \leq |X| = \kappa$.

(iv) X has a dense Menger subspace $L \times [0, \omega)$, so X is St-M.

(v) First, we show that *for every compact subset K of X, there exists an at most countable set $S_K \subseteq L$ such that $K \subseteq S_K \times [0, \omega]$*. Let K be a compact subset of X. For every $n \in [0, \omega)$, the set $L_n = L \times \{n\}$ is closed in X, so $K_n = K \cap L_n$ is a closed subset of K, so it must be compact. Since L_n is homeomorphic to L and the latter set has only finite compact subsets, each K_n is finite. Similarly, the set $L_\omega = D \times \{\omega\}$ is closed in X, so $K_\omega = K \cap L_\omega$ is compact as well. Since L_ω is discrete, K_ω must be finite. Therefore, $K = \bigcup\{K_n : n \in [0, \omega]\}$ is an at most countable set. Therefore, we can find an at most countable subset S_K of L such that $K \subseteq S_K \times [0, \omega]$.

Let $\mathscr{U} = \{O_x : x \in X\}$ be the cover of X defined in item (iii). Let $Y = \bigcup\{K_\alpha : \alpha < \tau\}$, where τ is a cardinal and each K_α is a compact subset of X. Suppose that $\tau < \kappa$. Note that by the property of our cover $\mathscr{U}$,

$$C \cap St(K_\alpha, \mathscr{U}) \subseteq S_{K_\alpha},$$

so

$$C \cap St(K, \mathscr{U}) \subseteq \bigcup\{S_{K_\alpha} : \alpha < \tau\},$$

which implies $|C \cap St(K, \mathscr{U})| \leq \max\{\omega, \tau\} < \kappa$. Since $|C| = \kappa$, we have $C \setminus St(K, \mathscr{U}) \neq \emptyset$, and so $X \neq St(K, \mathscr{U})$. This establishes the inequality $St - \varkappa(X) \geq \kappa$. The converse inequality follows from $St - \varkappa(X) \leq \varkappa(X) \leq |X| = \kappa$. □

Remark 2.1.1. *The space X from Example 2.1.8 is not normal.*

Remark 2.1.2. *When $\kappa = \omega_1$, the space X from Example 2.1.8 was considered by Ikenaga [45] who showed that it is not St-σC. When $\kappa = \mathfrak{c}$, this space was considered in [63, Example 2.3].*

In the following we focus on the cardinal functions St-L and NA-L. For a space X:

$$St\text{-}L(X) = min\{\kappa : \text{for every open cover } \mathcal{U} \text{ of } X, \text{ there is } F \subset X \text{ such that}$$

$$L(F) \leq \kappa \text{ and } St(F, \mathcal{U}) = X\} (\text{see also } [26])$$

and

$$NA\text{-}L(X) = \min\{\tau : \text{for every neighbourhood assignment } \{O_x : x \in X\} \text{ of } X, \exists \text{ a subspace } Y \text{of } X \text{ such that } X = \bigcup_{y\in Y} O_y \text{ and } L(Y) \leq \tau\}.$$

Note that a space X is St-L if and only if $St\text{-}L(X) \leq \omega$ and it is NA-L if and only if $NA\text{-}L(X) \leq \omega$. By Lemma 2.1.4 we have that $St\text{-}L(X) \leq NA\text{-}L(X) \leq L(X)$.

In order to prove that if X is a paracompact space and A is a dense subset of X, then $L(X) \leq St\text{-}L(A)$ (Theorem 2.1.3 below), we need the following. Recall that a space X is paracompact if and only if every open cover of X has a star refinement [34]. We introduce the following definition.

Definition 2.1.10. Let $n \in \mathbb{N}$, $n \geq 2$, X be a space, $\mathcal{U}$ and $\mathcal{V}$ two families of subsets of X. We say that $\mathcal{V}$ is a *n-star-refinement* of $\mathcal{U}$ and we write $\mathcal{V} \prec_n^* \mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $St^n(V, \mathcal{U}) \subseteq U$. If $n = 1$ we have the notion of star-refinement.

Lemma 2.1.5. *Let X be a topological space and $\mathcal{U}$ a family of sets of X. If $\mathcal{W} \prec_{n-1}^* \mathcal{V} \prec_1^* \mathcal{U}$, then $\mathcal{W} \prec_n^* \mathcal{U}$.*

Proof. We prove the statement for $n = 2$. Take $W \in \mathcal{W}$. Since $\mathcal{W} \prec_1^* \mathcal{V}$, there exists $V \in \mathcal{V}$ such that $St(W, \mathcal{W}) \subseteq V$. We have to show that for every $W \in \mathcal{W}$, there exists $U \in \mathcal{U}$ such that $St^2(W, \mathcal{W}) \subseteq U$. Let $y \in St^2(W, \mathcal{W})$. Then there exists $W' \in \mathcal{W}$ such that $y \in W'$ and $W' \cap W \neq \emptyset$. Since $\mathcal{W} \prec_1^* \mathcal{V}$, there exists $V' \in \mathcal{V}$ such that $St(W', \mathcal{W}) \subseteq V'$. We can notice that $W' \subseteq St(W', \mathcal{W}) \subseteq V'$ and $W \subseteq St(W, \mathcal{W}) \subseteq V$. So, $\emptyset \neq W \cap W' \subseteq V \cap V'$, then $V \cap V' \neq \emptyset$. This means $V' \subseteq St(V, \mathcal{V})$. We also have $y \in W' \subseteq V' \subseteq St(V, \mathcal{V})$. Therefore, $St^2(W, \mathcal{W}) \subseteq St(V, \mathcal{V})$. Since $\mathcal{V} \prec_1^* \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $St(V, \mathcal{V}) \subseteq U$. Thus, $St^2(W, \mathcal{W}) \subseteq U$ This means $\mathcal{W} \prec_2^* \mathcal{U}$.

□

Theorem 2.1.2. *A space X is paracompact if and only if every open cover of X has a n-star refinement.*

Now we can prove the following.

Theorem 2.1.3. *Let X be a paracompact space and A be a dense subset of X. Then $L(X) \leq St\text{-}L(A)$.*

Proof. Let A be a dense subset of X and let $St\text{-}L(A) \leq \kappa$. Let $\mathcal{U}$ be an open cover of X, $\mathcal{W}$ a closed locally finite refinement of $\mathcal{U}$ and $\mathcal{V}$ a 2-star refinement of $\mathcal{W}$. We consider $\mathcal{V}_A = \{V \in \mathcal{V} : V \cap A \neq \emptyset\}$ that is a cover of A. Since $St\text{-}L(A) \leq \kappa$, there exists $Y \subset A$ such that $L(Y) \leq \kappa$ and $A = St(Y, \mathcal{V}_A) = \bigcup\{V \in \mathcal{V}_A : V \cap Y \neq \emptyset\} = \bigcup_{y\in Y}\{V \in \mathcal{V}_A : y \in V\} = \bigcup_{y\in Y} \bigcup \mathcal{V}_y =$

$\bigcup_{y \in Y} St(y, \mathcal{V}_A)$, where $\mathcal{V}_y = \{V \in \mathcal{V}_A : y \in V\}$. Since $Y \subset A$ and $L(Y) \leq \kappa$, there exixts $Z \in [Y]^{\leq \kappa}$ such that $Y \subseteq \bigcup_{y \in Z} \bigcup \mathcal{V}_y = \bigcup_{y \in Z} St(y, \mathcal{V}_A)$(*).

Claim 2.1.4. *$A \subseteq \bigcup_{y \in Z} St^2(y, \mathcal{V}_A)$.*

Proof. Take $a \in A$. Since $A \subseteq \bigcup_{y \in Y} \mathcal{V}_y$, there exists $y_0 \in Y$ such that $a \in \bigcup \mathcal{V}_{y_0}$, then there exists $V_0 \in \mathcal{V}_A$ such that $a, y_0 \in V_0$. Since $y_0 \in Y$, by (*), there exists $y_1 \in Z$ such that $y_0 \in \bigcup \mathcal{V}_{y_1}$, then there exists $V_1 \in \mathcal{V}_A$ such that $y_0, y_1 \in V_1$. This means $a \in St^2(y_1, \mathcal{V}_A)$. □

Since $\mathcal{V}_A \prec \mathcal{V}$, then $\{St^2(z, \mathcal{V}_A) : z \in Z\} \prec \{St^2(x, \mathcal{V}) : x \in X\} \prec \mathcal{W} \prec \mathcal{U}$. For every $z \in Z$, take W_z such that $St^2(z, \mathcal{V}_A) \subseteq W_Z$. By *Claim*, $A \subseteq \{St^2(z, \mathcal{V}_A) : z \in Z\} \subseteq \bigcup\{W_z : z \in Z\}$. Now $A = \bigcup\{A \cap W_z : z \in Z\}$ and $\{A \cap W_z : z \in Z\}$ is locally finite, furthermore $A = \bigcup\{A \cap W_z : z \in Z\}$. $X = \overline{A} = \overline{\bigcup\{A \cap W_z : z \in Z\}} = \bigcup\{\overline{A \cap W_z} : z \in Z\} \subseteq \bigcup\{\overline{W_z} : z \in Z\} \subseteq \bigcup\{U_z : z \in Z\}$ that is $L(X) \leq \kappa$. □

Corollary 2.1.5. *Let X be a paracompat space with a dense St-L subspace, then it is Lindelöf.*

Corollary 2.1.6. *[34] Let X be a paracompact space with a dense Lindelöf subspace, then it is Lindelöf.*

Remark 2.1.3. *Theorem 2.1.3 does not hold replacing St-L with St-M spaces and L with M. In fact, the space of irrational numbers is Lindelöf and separable so it contains a countable dense subspace that is also Menger but the space is not Menger.*

In order to obtain a generalization of Proposition 3.8 in [5] and Theorem 2.10 in [4],we need to define the following cardinal invariants.

Definition 2.1.11. The *metacompact number* of a space X is

$$MC(X) = min\{\kappa : \textit{every open cover } \mathcal{U} \textit{ of } X \textit{ such that } |\mathcal{U}| \leq \kappa \textit{ has a point-finite open refinement}\}.$$

Definition 2.1.12. The *linearly Lindelöf number* of a space X is

$$LL(X) = min\{\kappa : \textit{for every linearly ordered open cover } \mathcal{U} \textit{ of } X, \exists \mathcal{V} \in [\mathcal{U}]^{\leq \kappa} : X = \bigcup \mathcal{V}\}.$$

We have the following relation:

$$NA\text{-}LL(X) \leq NA\text{-}L(X) \leq L(X).$$

It was proved in [56] that the properties LL and NA-LL are equivalent. More in general, we can prove, following essentially the same proof of Proposition 2.7 in [56], that $NA\text{-}LL(X) = LL(X)$, for every space X.

To prove the following results, we follow step by step, respectively, the proofs of Theorem 2.10 in [4] and of Proposition 3.8 in [5] using cardinal functions.

Theorem 2.1.4. *Let X be a space, then $L(X) \leq MC(X)LL(X)$.*

Proof. Let $\tau = MC(X)LL(X)$ and $\kappa = min\{\mu : \mu$ is a cardinal such that there exists an open cover of cardinality μ that does not have subcovers of cardinality $\tau\}$. For every closed subset F of X and for every family $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| < \kappa$ such that $\bigcup \mathcal{U} \supseteq F$, there is $\mathcal{U}' \in [\mathcal{U}]^{\leq \tau}$ such that $F \subset \bigcup \mathcal{U}'$. Let $\mathcal{V} = \{V_\alpha : \alpha \in \kappa\}$ an open cover of X of size κ. For every $\beta < \kappa$, let $W_\beta = \bigcup_{\alpha<\beta} V_\alpha$. The family of W_β for every $\beta \in \kappa$ is a linearly ordered open cover of X. Since $LL(X) \leq \tau$, there is $\{\beta_\alpha : \alpha \in \tau\}$ such that $\bigcup\{W_{\beta_\alpha} : \alpha \in \tau\} = X$. Considering that $MC(X) \leq \tau$, there exists closed subsets F_α of W_{β_α} such that $X = \bigcup_{\gamma\in\tau} F_\gamma$. The family $\mathcal{V}_\gamma = \{V_\alpha : \alpha < \beta_\gamma\}$ is an open cover of F_γ having cardinality strictly less than κ, so there is $\mathcal{V}'_\gamma \in [\mathcal{V}_\gamma]^{\leq\tau}$ such that $F_\gamma \subset \bigcup {\mathcal{V}_\gamma}'$. The family $\mathcal{V}' = \{\bigcup \mathcal{V}'_\gamma : \gamma \in \tau\}$ has cardinality at most τ and it is a subcover of $\mathcal{V}$, that is a contradiction. □

Corollary 2.1.7 ([3],Theorem 2.10)**.** *A linearly Lindelöf, countably metacompact space is Lindelöf.*

Proposition 2.1.5. *Let X be a space, then $L(X) \leq NA\text{-}L(X)MC(X)$.*

Proof. Let $NA\text{-}L(X)MC(X) = \kappa$, then $NA\text{-}LL(X) = LL(X) \leq \kappa$. Using Theorem 2.1.4 we have $L(X) \leq MC(X)LL(X) \leq \kappa$. □

Corollary 2.1.8 ([4], Proposition 3.8)**.** *A NA-L, countably metacompact space is Lindelöf.*

2.2 MONOTONE VERSIONS OF SOME SELECTION PRINCIPLES: SS, WS, SW, WW FORMS

A space X is *monotonically Lindelöf* (mL for short) if there is an operator r that assigns to every open cover $\mathcal{U}$ a countable open cover $r(\mathcal{U})$ so that $r(\mathcal{U})$ refines $\mathcal{U}$, and $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. The following spaces are monotonically Lindelöf: all second countable spaces, $L(\omega_1)$ that is the one point Lindelöfication of the discrete space of cardinality ω_1; the following spaces are not monotonically Lindelöf: some countable spaces, $L(\omega_2)$ that is the one point Lindelöfication of the discrete space of cardinality ω_2 (see [50], [51]). In this section we consider monotone versions of some stronger forms of the Lindelöf property defined in terms of sequences of covers: Menger, Rothberger and Hurewicz properties. Logically, there are four ways to give the definition of the monotone version of each of the previous properties. In the following we introduce all four versions even if one of this monotone versions introduced is absurd. A specific study of one of these forms of the Menger property (SS-mM, see below) has been done in [18] where it is called *monotone Menger property*. In [18] it was proved that (within ZFC) every separable SS-mM space is first countable. This result contrasts with the previously known fact that under CH there exists countable mL spaces which are not first countable. Some of the results that we discuss are included in [19] and in [20].

In particular in Definition 2.2.1 the letters W and S are abbreviations for "weakly" and "strongly".

Definition 2.2.1. A space X is

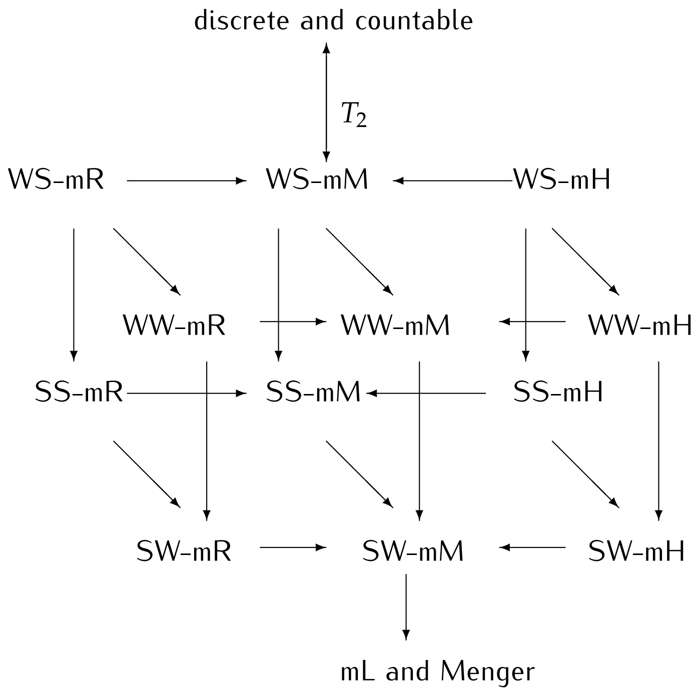
- *SS-mM* ([18], where it is called *monotonically Menger*, briefly mM) if there exists an operator, called SS-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and for every $n \in \omega$, $\mathcal{U}(n)$ refines $\mathcal{V}(n)$, then for every $n \in \omega$, $r(\mathcal{U})(n)$ refines $r(\mathcal{V})(n)$.
- *SW-mM* if there exists an operator, called SW-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and for every $n \in \omega$, $\mathcal{U}(n)$ refines $\mathcal{V}(n)$, then $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(n) : n \in \omega\}$.
- *WW-mM* if there exists an operator, called WW-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,
  2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
  3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$, then $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(n) : n \in \omega\}$.
- *WS-mM* if there exists an operator, called SS-mM operator, that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of open covers of X a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ so that
  1. $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ is an open cover of X,

2. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$,
3. if $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ are two sequences of open covers of X and $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$, then for every $n \in \omega$, $r(\mathcal{U})(n)$ refines $r(\mathcal{V})(n)$.

The monotone versions of the Hurewicz and Rothberger properties are defined in similar ways. For the Rothberger property, we deal with at most one element refinements rather then with one element refinements, while for Hurewicz property, we replace condition 2. with

$2'$. for every $n \in \omega$, $r(\mathcal{U})(n)$ is a finite refinement of $\mathcal{U}(n)$ such that for every $x \in X$, $x \in \bigcup r(\mathcal{U})(n)$, for all but finitely many n.

Then we obtain the definitions of SS-mR, SW-mR, WW-mR, WS-mR, SS-mH, SW-mH, *WW-mH* and *WS-mH* spaces. The implications between the previous forms of the monotone Rothberger, Menger and Hurewicz properties and the monotone Lindelöf property are shown in the following diagram (that Hausdorff WS-mM implies discrete and countable will be shown in Proposition 2.2.5; the rest is obvious).



We study the previous properties and we consider the local version of the monotonic Rothberger-type properties.

2.2.1 Positive results on SS-forms

Proposition 2.2.1. *Every countable, first countable space is SS-mR.*

Proof. Let $X = \{x_n : n \in \omega\}$ and for every $n \in \omega$, let $\{B_m(x_n) : m \in \omega\}$ be a base of neighborhoods of x_n such that $B_{m+1}(x_n) \subset B_m(x_n)$ for each $m \in \omega$. Let $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ be a sequence of open covers of X. For $n \in \omega$, put $m_{\mathcal{U}}(n) = \min\{m \in \omega : \{B_m(x_n)\}$ refines $\mathcal{U}(n)\}$. Put $r(\mathcal{U})(n) = \{B_{m_{\mathcal{U}}(n)}(x_n)\}$. Then r is a SS-mR operator. □

Recall the following result.

Lemma 2.2.1. *[18] Let K be a compact subspace of a metric space M. Then there is a sequence $\{\mathcal{C}_n : n \in \omega\}$ of finite covers of K by open sets in M such that the following two conditions hold:*

(1) $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$ for every $n \in \omega$.

and

(2) If $\mathcal{C}$ is a cover of K by open sets in M, then there is $n \in \omega$ such that $\mathcal{C}_n$ refines $\mathcal{C}$.

Using the previous lemma, in [18], it is proved that every σ-compact metrizable space is SS-mM. Following the same proof and noting that the SS-mM operator construced is a SS-mH operator, we prove that the following result.

Proposition 2.2.2. *Every σ-compact metrizable space is SS-mH.*

Proof. Let $M = \bigcup\{M_i : i \in \omega\}$ be a metric space where each M_i is compact. For $i \in \omega$, put $K_i = \bigcup\{M_j : 0 \le j \le i\}$. Then the sets K_i, $i \in \omega$, are compact. By Lemma 2.2.1, for every $i \in \omega$ there is a sequence $\{\mathcal{C}_{i,n} : n \in \omega\}$ of finite covers of K_i by open sets in M such that $\mathcal{C}_{i,n+1}$ refines $\mathcal{C}_{i,n}$ for every $n \in \omega$, and for every cover $\mathcal{C}$ of K_i by open sets in M there is $n \in \omega$ such that $\mathcal{C}_{i,n}$ refines $\mathcal{C}$. Let $\mathcal{U} = (\mathcal{U}(i) : i \in \omega)$ be a sequence of open covers of M. For $i \in \omega$, put $n_{\mathcal{U}}(i) = \min\{n : \mathcal{C}_{i,n}$ refines $\mathcal{U}(i)\}$. Put $r(\mathcal{U})(i) = \mathcal{C}_{i,n_{\mathcal{U}}(i)}$. Then r is a SS-mH operator. □

Corollary 2.2.1. *[18] Every σ-compact metrizable space is SS-mM.*

Proposition 2.2.3. *Let X be SS-mM and $Y \subset X$ be a closed subset of X. Then Y is SS-mM.*

Proof. Let r be the SS-mM operator on X and let $\mathcal{W} = (\mathcal{W}(n) : n \in \omega)$ be a sequence of open covers of Y. Then, for every $n \in \omega$, $\tilde{\mathcal{W}}(n) = \{W \cup (X \setminus Y) : W \in \mathcal{W}(n)\}$ is an open cover of X. Put $\tilde{\mathcal{W}} = (\tilde{\mathcal{W}}(n) : n \in \omega)$. Then $r(\tilde{\mathcal{W}}) = (r(\tilde{\mathcal{W}})(n) : n \in \omega)$ satisfies conditions 1-3 of the SS-mM definition. Put $s(\mathcal{W}) = (r(\tilde{\mathcal{W}})(n)|_Y : n \in \omega)$. Then s is a SS-mM operator for Y. □

Of course, Proposition 2.2.3 can be restated for the corresponding monotone versions of the Rothberger and Hurewicz properties.

2.2.2 Confirming the absurdness of the WS-forms

Recall that X is a *P-space* if every G_δ-set in X is open. We consider this special class of spaces early because this provides both some examples and some steps in proofs below. Recall that a space is *c.c.c.* if every pairwise disjoint family of nonempty open sets is countable.

Proposition 2.2.4. *If X is a Hausdorff, SS-mM, P-space, then X is discrete and countable.*

Proof. Suppose X is uncountable. Then by the previous lemma it is not ccc. Let $\{O_\alpha : \alpha < \omega_1\}$ be a family of non empty pairwise disjoint open sets. Pick $q_\alpha \in O_\alpha$ and put $H_\alpha = \{q_\gamma : \gamma < \alpha\}$ (thus, in particular, $H_0 = \emptyset$). For $\beta < \omega_1$ put $\mathcal{O}_\beta = \{O_\alpha : \alpha < \omega_1\} \cup \{X \setminus H_\beta\}$. Then $\mathcal{O}_\beta$ is an open cover of X. Consider a "constant" sequence of open covers $\mathcal{U}_\beta = (\mathcal{U}_\beta(n) : n \in \omega)$ where $\mathcal{U}_\beta(n) = \mathcal{O}_\beta$ for all $n \in \omega$. Suppose there is a SS-mM operator r. Put

$$S = \{(n,\alpha) \in \omega \times \omega_1 : \text{ there is } \beta(n,\alpha) \text{ such that } \alpha < \beta(n,\alpha) < \omega_1 \text{ and for every } \beta \text{ with } \beta(n,\alpha) \leq \beta < \omega_1 \text{ there is } O \in r(\mathcal{U}_\beta)(n) \text{ such that } q_\alpha \in O\}.$$

Claim 1: For every $\alpha \in \omega_1$ there is $n \in \omega$ such that $(n,\alpha) \in S$.

Indeed, otherwise for every n there would be a cofinal subset $A_n \subset \omega_1$ such that for every $\gamma \in A_n$ there is no $O \in r(\mathcal{U}_\gamma)(n)$ with $q_\alpha \in O$. Put $\gamma_n = \min(A_n)$. It follows from the monotonicity of r that for every $\gamma \geq \gamma_n$ there is no $O \in r(\mathcal{U}_\gamma)(n)$ with $q_\alpha \in O$. Put $\gamma^* = \sup\{\gamma_n : n \in \omega\}$. Then q_α is not covered by $\cup\{r(\mathcal{U}_{\gamma^*})(n) : n \in \omega\}$, a contradiction.

Claim 2: For each $n \in \omega$, the set $S_n = \{\alpha : (n,\alpha) \in S\}$ is finite.

Indeed, suppose S_n is infinite. Pick a countably infinite $T_n \subset S_n$. Put $\beta^* = \sup\{\beta(n,\alpha) : \alpha \in T_n\}$. Then for each $\alpha \in T_n$, $r(\mathcal{U}_{\beta^*})(n)$ must contain an element $V_\alpha \ni q_\alpha$. Since $r(\mathcal{U}_{\beta^*})(n)$ refines $\mathcal{O}_{\beta^*}$, and O_α is the only element of $\mathcal{O}_{\beta^*}$ containing q_α (because $\beta^* \geq \beta(n,\alpha) > \alpha$), we have $V_\alpha \subset O_\alpha$. But the sets O_α are pairwise disjoint, so $r(\mathcal{U}_{\beta^*})(n)$ must be infinite, a contradiction.

Finally, by Claim 1, $|S| = \omega_1$ while by Claim 2, S is at most countable, a contradiction. So X is countable. Being T_1 and P, it is discrete. □

Then we have the following example.

Example 2.2.1. *The one point Lindelöfication $L(\tau)$ of the discrete space of uncountable cardinality τ is not SS-mM.*

Lemma 2.2.2. *If X is a WS-mM T_1-space, then X is a P-space.*

Proof. Let $p \in X$ and for $k \in \omega$, let U_k be a neighborhood of p. It suffices to show that $\mathrm{Int}(\bigcap\{U_k : k \in \omega\}) \ni p$. Let r be a WS-mM operator for X. For a sequence of open covers $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$, and for $n \in \omega$, put $s(\mathcal{U})(n) = \{W \in r(\mathcal{U})(n) : W \ni p\}$. For $k \in \omega$ consider the open cover $\mathcal{O}_k = \{U_k, X \setminus \{p\}\}$. Denote by Θ the set of all permutations of ω. For $\pi \in \Theta$, define the sequence of open covers $\mathcal{U}_\pi = (\mathcal{U}_\pi(n) : n \in \omega)$ by $\mathcal{U}_\pi(n) = \mathcal{O}_{\pi(n)}$. For any $\pi, \rho \in \Theta$, $\bigcup\{\mathcal{U}_\pi(n) : n \in \omega\} = \bigcup\{\mathcal{U}_\rho(n) : n \in \omega\}$, so for every $n \in \omega$, $r(\mathcal{U}_\pi)(n) \approx r(\mathcal{U}_\rho)(n)$ and therefore $s(\mathcal{U}_\pi)(n) \approx s(\mathcal{U}_\rho)(n)$. Fix π and vary ρ. It follows that for every $n, k \in \omega$, $s(\mathcal{U}_\pi)(n)$ must refine $\{U_k\}$. So for every $n \in \omega$, $s(\mathcal{U}_\pi)(n)$ must refine $\{\bigcap\{U_k : k \in \omega\}\}$. For some n^*, $s(\mathcal{U}_\pi)(n^*) \neq \emptyset$, so we have $p \in \bigcup s(\mathcal{U}_\pi)(n^*) \subset \mathrm{Int}(\bigcap\{U_k : k \in \omega\})$. □

By the previous lemma and Proposition 2.2.4 we have

Proposition 2.2.5. *If X is a WS-mM Hausdorff space, then X is discrete and countable.*

2.2.3 The WW-forms are less exceptional

First, we show that a WW-mR space does not have to be discrete nor a P-space.

Example 2.2.2. *Convergent sequence is WW-mR.*

Proof. Let $S = \omega + 1$ be convergent sequence, and $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ a sequence of open covers of S. For $n \in \omega$, put $l(n) = \min\{l : \{[l, \omega]\}$ refines $\mathcal{U}(n)\}$. Put $l^* = \min\{l(n) : n \in \omega\}$ and pick $n^* \in \omega$ so that $l(n^*) = l^*$. Further, pick $n_0, ..., n_{l^*-1}$ so that these numbers are distinct from each other and from n^*. Put $r(\mathcal{U})(n^*) = \{[l^*, \omega]\}$, $r(\mathcal{U})(n_i) = \{\{i\}\}$ for $0 \leq i < l^* - 1$, and $r(\mathcal{U})(n) = \{\emptyset\}$ for all other $n \in \omega$. Then r is a WW-mR operator. □

Proposition 2.2.6. *Every metrizable compact space is WW-mH.*

Proof. Let K be a compact metrizable space, and let $\{\mathcal{C}_i : i \in \omega\}$ be like in Lemma 2.2.1 (where $M = K$).

Let $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ be a sequence of open covers of K. For $n \in \omega$, put $i_{\mathcal{U}}(n) = \min\{i : \mathcal{C}_i$ refines $\mathcal{U}(n)\}$ and $r(\mathcal{U})(n) = \{C \in \mathcal{C}_i : 0 \leq i \leq i_{\mathcal{U}}(n)$ and $\{C\}$ refines $\mathcal{U}(n)\}$. Then $r(\mathcal{U})(n)$ is a finite cover of X, so the Hurewicz property is satisfied. It remains to check monotonicity.

Let $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ and $\mathcal{V} = (\mathcal{V}(n) : n \in \omega)$ be two sequences of open covers of K such that $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$. Suppose $C \in r(\mathcal{U})(n)$ for some $n \in \omega$. Then there is $U \in \mathcal{U}(n) \subset \bigcup\{\mathcal{U}(n) : n \in \omega\}$ such that $C \subset U$. Since $\bigcup\{\mathcal{U}(n) : n \in \omega\}$ refines $\bigcup\{\mathcal{V}(n) : n \in \omega\}$ there is $V \in \bigcup\{\mathcal{V}(n) : n \in \omega\}$ such that $U \subset V$. Then $V \in r(\mathcal{V})(m)$ for some $m \in \omega$, and since $C \subset U \subset V$, we conclude that (*) $\{C\}$ refines $\mathcal{V}(m)$. We have $C \in \mathcal{C}_i$ for some $i \in \omega$. If $i \leq i_{\mathcal{V}}(m)$, then by (*) and the definition of the operator r, $C \in r(\mathcal{V})(m)$. If $i > i_{\mathcal{V}}(m)$, then there is $C' \in \mathcal{C}_{i_{\mathcal{V}}(m)}$ such that $C' \supset C$. Since $\mathcal{C}_{i_{\mathcal{V}}(m)} \subset r(\mathcal{V})(m)$ we conclude that in both cases ($i \leq i_{\mathcal{V}}(m)$ and $i \leq i_{\mathcal{V}}(m)$) there is $\tilde{C} \in r(\mathcal{V})(m) \subset \bigcup\{r(\mathcal{V})(k) : k \in \omega\}$. So $\bigcup\{r(\mathcal{U})(k) : k \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(k) : k \in \omega\}$. □

Example 2.2.3. *$L(\omega_1)$ is WW-mR and WW-mH.*

Proof. For WW-mR the proof is similar to the case of convergent sequence. To prove WW-mH, let $L(\omega_1) = \omega_1 \cup \{p\}$ where p is the single non isolated point the basic neighorhood of which take s the form $[\alpha, \omega_1) \cup \{p\}$ where $\alpha < \omega_1$. Let $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ be a sequence of open covers of $L(\omega_1)$. For $n \in \omega$, put $l(n) = \min\{l : \{[l, \omega_1) \cup \{p\}\}$ refines $\mathcal{U}(n)\}$. Put $l^* = \sup\{l(n) : n \in \omega\}$ and enumerate $[0, l^*] = \{\alpha_k : k \in \omega\}$. For $n \in \omega$, put $r(\mathcal{U})(n) = \{[l(n), \omega_1) \cup \{p\}\} \cup \{\{\alpha_k\} : k \leq n\}$. Then r is a WW-mH operator for $L(\omega_1)$. □

Remark 2.2.1. By Examples 2.2.1 and 2.2.3 we may conclude that $L(\omega_1)$ is WW-mR and WW-mH but not SS-mM.

We have the following example.

Example 2.2.4. The discrete sum of countably many convergent sequences is not WW-mM.

Proof. We consider the space $X = \omega \times (\omega + 1)$. Suppose r is a WW-mM operator for X. With a function $f : \omega \to \omega$ we associate a cover $\mathcal{O}_f$ of X defined as follows: $\mathcal{O}_f = \{\{n\} \times [f(n), \omega]\} : n \in \omega\} \cup \{\langle n, m \rangle : n \in \omega$ and $m < f(n)\}$. Let $\overline{0} \in \omega^\omega$ denote the function constant zero and let $\mathcal{U} = (\mathcal{U}(k) : k \in \omega)$ be the sequence of covers of X such that $\mathcal{U}(k) = \mathcal{O}_{\overline{0}}$ for all k. For $n \in \omega$, put $f(n) = \min\{m : \{n\} \times [m, \omega]$ refines $\bigcup\{r(\mathcal{U})(k) : k \in \omega\}\}$ and $g(n) = f(n) + 1$. Then $f, g \in {}^\omega\omega$. Consider the sequence of covers $\mathcal{V} = (\mathcal{V}(k) : k \in \omega)$ where $\mathcal{V}(0) = \mathcal{O}_{\overline{0}}$ and $\mathcal{V}(k) = \mathcal{O}_g$ for $k > 0$. Then $\bigcup\{\mathcal{U}(k) : k \in \omega\} \approx \bigcup\{\mathcal{V}(k) : k \in \omega\}$ (because each of these unions is equal to $\mathcal{O}_{\overline{0}}$). By definition of WW-mM we must have $\bigcup\{r(\mathcal{U})(k) : k \in \omega\} \approx \bigcup\{r(\mathcal{V})(k) : k \in \omega\}$. However $\bigcup\{r(\mathcal{U})(k) : k \in \omega\}$ does not refine $\bigcup\{r(\mathcal{V})(k) : k \in \omega\}$. Indeed, since $r(\mathcal{V})(0)$ is finite, so is the set $N = \pi_1(\bigcup r(\mathcal{V})(0))$ where π_1 is the projection to the first factor. Pick $n^* \in \omega \setminus N$. For some $k \in \omega$, some element of $r(\mathcal{U})(k)$ contains the set $\{n^*\} \times [f(n^*), \omega]$, but no element of $\bigcup\{r(\mathcal{V})(k) : k \in \omega\}$ can contain this set. □

Remark 2.2.2. By Proposition 2.2.2 and Example 2.2.4 we have that the discrete sum of countably many convergent sequences is SS-mH but not WW-mM.

2.2.4 The local versions of SS-mR and SW-mR properties

A space X is mL *at the point* p [50] if one can assign to every non empty family $\mathcal{U}$ of neighborhoods of p a non empty countable family $r(\mathcal{U})$ of neighborhoods of p so that $r(\mathcal{U})$ refines $\mathcal{U}$, and $r(\mathcal{U})$ refines $r(\mathcal{V})$ whenever $\mathcal{U}$ refines $\mathcal{V}$. This technical notion was used in [50], [51], [52] to disprove monotone Lindelöfness of certain spaces. One gets the definition of mC at p when replacing "countable" with "finite". We also consider the local versions of SS-mR and SW-mR properties. There is one principal difference: obviously, a space X with single non-isolated point p is mL (mC) iff X is Lindelöf (respectively, compact) and mL (respectively, mC) at p. We show that there exists a space X with single non-isolated point p which is Rothberger, SS-mR at p but not SS-mR (Example 2.2.5). The local version of SS-mM was considered in [18] to show that (within ZFC) every separable SS-mM space is first countable. This contrasts with the previously known fact hat under CH there are countable mL spaces which are not first countable.

Now we consider the local versions of SS-mR and SW-mR properties. Note that a space X with a single non-isolated point p is mL mC) iff X is Lindelöf (respectively, compact) and mL (respectively, mC) at p. We will see that there exists a space X with a single non-isolated point p which is Rothberger, SS-mR at p but not SS-mR.

Definition 2.2.2. Say that X is SS-mR at p (SW-mR at p) if there is an operator r that assigns to every sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of non empty families of neighborhoods of p a sequence $r(\mathcal{U}) = (r(\mathcal{U})(n) : n \in \omega)$ where each $r(\mathcal{U})(n)$ is an (≤ 1)-element family of neighborhoods of p so that:

1. not all families $r(\mathcal{U})(n)$ are empty;
2. $r(\mathcal{U})(n)$ refines $\mathcal{U}(n)$;
3. if $\mathcal{U}(n)$ refines $\mathcal{V}(n)$ for all n, then $r(\mathcal{U})(n)$ refines $r(\mathcal{V})(n)$ for all n (respectively, $\bigcup\{r(\mathcal{U})(n) : n \in \omega\}$ refines $\bigcup\{r(\mathcal{V})(n) : n \in \omega\}$).

The following proposition gives a characterization of SS-mR at p property.

Proposition 2.2.7. *Let $p \in (X, \mathcal{T})$. The following conditions are equivalent:*

(1) There is a well ordered by $\supset$ base of neighborhoods of p.

(2) There is a linearly ordered by $\supset$ base of neighborhoods of p.

(3) There is a monotone neighborhood assignment (abbreviated mNA) at p, that is, an operator r that assigns to every non empty family $\mathcal{U}$ of neighborhoods of p a neighborhood $r(\mathcal{U})$ of p so that $\{r(\mathcal{U})\}$ refines $\mathcal{U}$ and $r(\mathcal{U}) \subset r(\mathcal{V})$ if $\mathcal{U}$ refines $\mathcal{V}$.

(4) $(X, \mathcal{T})$ is SS-mR at p.

Proof. (1) $\Rightarrow$ (2) is trivial; (2) $\Rightarrow$ (1) is easily proved by induction.

(1) $\Rightarrow$ (3) Let $\{B_\alpha : \alpha < \kappa\}$ be a base of neighborhoods of p such that $B_\alpha \supset B_\beta$ whenever $\alpha < \beta < \kappa$. For a non empty family $\mathcal{U}$ of neighborhoods of p, put $\alpha(\mathcal{U}) = \min\{\alpha : B_\alpha$ refines $\mathcal{U}\}$ and $r(\mathcal{U}) = B_{\alpha(\mathcal{U})}$.

(3) $\Rightarrow$ (1) Let r be a mNA operator at p. By induction on α we will define neighborhoods B_α of p such that $B_{\alpha_1} \supset B_{\alpha_2}$ whenever $\alpha_1 < \alpha_2$ and the families $\mathcal{U}_\alpha$ of neighborhoods of p such that $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$ whenever $\alpha < \beta$. Put $\mathcal{U}_0 = \{U \in \mathcal{T} : p \in U\}$ and $B_0 = r(\mathcal{U}_0)$. Suppose $\alpha > 0$ and B_γ and $\mathcal{U}_\gamma$ have been defined for all $\gamma < \alpha$. Put $\mathcal{B}_\alpha = \{B_\gamma : \gamma < \alpha\}$. If $\mathcal{B}_\alpha$ is a base at p then we are done. If it is not, put $\mathcal{U}_\alpha = \{U \in \mathcal{T} : p \in U$ and there is no $V \in \mathcal{B}_\alpha$ with $V \subset U\}$ and $B_\alpha = r(\mathcal{U}_\alpha)$ and continue.

(3) $\Rightarrow$ (4) Let s be a mNA operator at p. For a sequence $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ of non empty families of neighborhoods of p, put $r(\mathcal{U})(n) = \{s(\mathcal{U})(n)\}$. Then r is a SS-mR operator at p.

(4) $\Rightarrow$ (3) Suppose r is a SS-mR operator at p but there is no mNA at p.

We will construct by induction on $m \in \omega$ sequences $\mathcal{U}_m = (\mathcal{U}_m(n) : n \in \omega)$ of nonempty families of neighborhoods of p, and nonempty families $\mathcal{O}_m$ of neighborhoods of p. Let $\mathcal{T}_p$ be the family of all neighborhoods of p.

Let $m = 0$. For a non empty family $\mathcal{O}$ of neighborhoods of p, put $\mathcal{U}_{0,\mathcal{O}} = (\mathcal{U}_{0,\mathcal{O}}(n) : n \in \omega)$ where $\mathcal{U}_{0,\mathcal{O}}(0) = \mathcal{O}$ and $\mathcal{U}_{0,\mathcal{O}}(n) = \mathcal{T}_p$ for all $n > 0$. There is $\mathcal{O}^*$ such that $r(\mathcal{U}_{0,\mathcal{O}^*})(0) = \emptyset$. Indeed, otherwise $\mathcal{O} \mapsto r(\mathcal{U}_{0,\mathcal{O}})(0)$ would be a mNA at p. Put $\mathcal{O}_0 = \mathcal{O}^*$ and define $\mathcal{U}_0$ by $\mathcal{U}_0(0) = \mathcal{O}_0$ and $\mathcal{U}_0(n) = \mathcal{T}_p$ for all $n > 0$.

Now suppose $m > 0$ and $\mathcal{U}_k$ and $\mathcal{O}_k$ have been defined for $0 \leq k < m$. For a non empty family $\mathcal{O}$ of neighborhoods of p, put $\mathcal{U}_{m,\mathcal{O}} = (\mathcal{U}_{m,\mathcal{O}}(n) : n \in \omega)$ where $\mathcal{U}_{m,\mathcal{O}}(n) = \mathcal{O}_n$ for $0 \leq n < m$, $\mathcal{U}_{m,\mathcal{O}}(m) = \mathcal{O}$ and $\mathcal{U}_{0,\mathcal{O}}(n) = \mathcal{T}_p$ for all $n > m$. As in the case $m = 0$, there is $\mathcal{O}^*$ such that $r(\mathcal{U}_{m,\mathcal{O}^*})(m) = \emptyset$. Put $\mathcal{O}_m = \mathcal{O}^*$ and define $\mathcal{U}_m$ by $\mathcal{U}_m(n) = \mathcal{O}_n$ for $0 \leq n \leq m$ and $\mathcal{U}_m(n) = \mathcal{T}_p$ for all $n > m$.

Now that $\mathcal{U}_m$ and $\mathcal{O}_m$ have been defined for all $m \in \omega$. Define $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ by $\mathcal{U}(n) = \mathcal{O}_n$. Then for every $m, n \in \omega$, $\mathcal{U}(n)$ refines $\mathcal{U}_m(n)$. It follows from monotonicity that for all $n \in \omega$, $r(\mathcal{U})(n) = \emptyset$, a contradiction with condition (1) in the definition of SS-mR. □

Example 2.2.5. *The one point Lindelöfication $L(\omega_1) = \omega_1 \cup \{p\}$ of a discrete space of cardinality ω_1 is SS-mR at p.*

Note that $L(\omega_1)$ is Rothberger, SS-mR at p but not SS-mM by Example 2.2.1 (hence not SS-mR) .

Now we introduce the following definition.

Definition 2.2.3. Let p be a point of a space X. A *thin base* of neighborhoods of p is a base of neighborhoods of p of the form $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa\}$ where κ is some ordinal so that the following conditions hold:

(a) $\mathcal{B}_\alpha$ is non empty and finite for every α.

(b) If $\alpha < \beta < \kappa$, then $\mathcal{B}_\beta$ refines $\mathcal{B}_\alpha$.

(c) If $\alpha < \beta < \kappa$, $B_\alpha \in \mathcal{B}_\alpha$, and $B_\beta \in \mathcal{B}_\beta$, then $B_\alpha \not\subset B_\beta$.

Note that if $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa\}$ is a thin a base of neighborhoods of $p \in X$ and $\gamma < \kappa$, then $\bigcup\{\mathcal{B}_\alpha : \alpha < \gamma\}$ is not a base at p.

In [18] it is considered the more general notion of thin poset in which the requirement to be a base is omitted.

Proposition 2.2.8. *Let p be a point of a space X and κ be an ordinal. If $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa\}$ is a thin base of neighborhoods of p satisying the following condition*

(d) For every neighborhood U of p, there is $\alpha < \kappa$ such that $\mathcal{B}_\alpha$ refines $\{U\}$,

then $\mathcal{B}$ is a well-ordered by inclusion base at p.

Proof. It is enough to take $\{\bigcup \mathcal{B}_\alpha : \alpha < \kappa\}$. □

The asymmetric V example (see Section 2.2.5) proves that the existence of a thin base of neighborhoods of p need not imply the existence of a well ordered base.

Theorem 2.2.1. *Let p be a point of a space $(X, \mathcal{T})$. If X is mC at p, then X has a thin base of neighborhoods of p.*

Proof. Let r be a mC operator at p. By induction on α we will define the families $\mathcal{B}_\alpha$ satisfying (a)—(c) in the definition of thin base and the families $\mathcal{U}_\alpha$ of neighborhoods of p such that $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$ whenever $\alpha < \beta$. Put $\mathcal{U}_0 = \{U \in \mathcal{T} : p \in U\}$ and $\mathcal{B}_0 = r(\mathcal{U}_0)$. Suppose $\alpha > 0$, and $\mathcal{U}_\gamma$ and $\mathcal{B}_\gamma$ have been defined for all $\gamma < \alpha$. Put $\mathcal{U}_\alpha = \{U \in \mathcal{T} : p \in U$ and $\bigcup\{\mathcal{B}_\gamma : \gamma < \alpha\}$ does not refine $\{U\}\}$. If $\mathcal{U}_\alpha = \emptyset$, then we stop and put $\kappa = \alpha$. Otherwise we put $\mathcal{B}_\alpha = r(\mathcal{U}_\alpha)$ and continue. □

2.2.5 The "asymmetric V" example

The following example allow us to distinguish some of the properties mentioned above. In particular this space is mC at p, monotonically paracompact, SW-mH and not SS-mM (hence not SS-mH).

Let $V = \omega_1 \cup \{p\} \cup \{a_n : n \in \omega\}$ where all points are distinct, the points of ω_1 and points a_n are isolated, and a basic neighborhood of the point p takes the form

$$B_{\gamma,n} = \{p\} \cup (\gamma, \omega_1) \cup \{a_m : n \leq m < \omega\}.$$

In other words, V is the quotient space of the one-point Lindelöfication of the discrete space of cardinality ω_1 and a convergent sequence. The following is easy to see:

Proposition 2.2.9. *V does not have a well ordered by inclusion base of neighborhoods of p.*

Proposition 2.2.10. *V is mC at p.*

Proof. Let $\mathcal{U}$ be a non empty family of neighborhoods of p. For $n \in \omega$, put $\Gamma_{\mathcal{U}}(n) = \{\gamma \in \omega_1 : B_{\gamma,n} \text{ refines } \mathcal{U}\}$ and

$$\gamma_{\mathcal{U}}(n) = \begin{cases} \min \Gamma_{\mathcal{U}}(n) & \text{if} \quad \Gamma_{\mathcal{U}}(n) \neq \emptyset \\ \omega_1 \text{ if} \quad \Gamma_{\mathcal{U}}(n) = \emptyset \end{cases}$$

Put $s_p(\mathcal{U}) = \{B_{\gamma_{\mathcal{U}}(n),n} : n \in \omega \text{ and } \gamma_{\mathcal{U}}(n) < \omega_1\}$. Then $s_p(\mathcal{U})$ is a non empty countable family of neighborhoods of p and $s_p(\mathcal{U})$ refines $\mathcal{U}$. Obviously, the operator s_p is monotonic with respect to $\mathcal{U}$. Say that an integer $n > 0$ is $\mathcal{U}$-special if $\gamma_{\mathcal{U}}(n) < \gamma_{\mathcal{U}}(n-1)$. Say that the number 0 is $\mathcal{U}$-special if $\gamma_{\mathcal{U}}(0) < \omega_1$. Put $N_{\mathcal{U}} = \{n \in \omega : n \text{ is } \mathcal{U}\text{-special}\}$. The function $n \mapsto \gamma_{\mathcal{U}}(n)$ is non increasing, so the set $N_{\mathcal{U}}$ is finite (and non empty). Then $r_p(\mathcal{U}) = \{B_{\gamma_{\mathcal{U}}(n),n} : n \in N_{\mathcal{U}}\}$ is a finite refinement of $\mathcal{U}$. It is easy to see that $r_p(\mathcal{U}) \approx s_p(\mathcal{U})$. Since s_p is monotonic, so is r_p; so r_p is a mC operator at p. □

Corollary 2.2.2. *A space which is mC at some point does not have to have a well-ordered base at this point.*

Recall that a space X is *monotonically paracompact in the sense of locally finite refinement* if there exists a function r which assigns to every open cover $\mathcal{U}$ a locally finite refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U})$ refines $r(\mathcal{V})$.

Proposition 2.2.11. *V is monotonically paracompact in the sense of locally finite refinement, SW-mH not SS-mM (hence not SS-mH).*

Proof. Let r_p be the operator of monotone compactness at p from the Proposition 2.2.10.

For an open cover $\mathcal{U}$ of V, put $r(\mathcal{U}) = r_p(\mathcal{U}) \cup \{\{y\} : y \in V \setminus \bigcup r_p(\mathcal{U})\}$. Then r witness the monotone paracompactness of V.

Let $\mathcal{U} = (\mathcal{U}(n) : n \in \omega)$ be a sequence of open covers of V. Put $S(\mathcal{U})(n) = r_p(\mathcal{U}(n))$, enumerate $V \setminus \bigcap\{\bigcup r_p(\mathcal{U}(n)) : n \in \omega\} = \{x_n : n \in \omega\}$ and put $R(\mathcal{U})(n) = S(\mathcal{U})(n) \cup \{\{x_m\} : 0 \leq m \leq n\}$. Then R is a SW-mH operator for V. □

Since $L(\omega_1)$ is a closed not SS-mM subspace of V and SS-mM property is hereditarily with respect to closed subsets (see Proposition 2.2.3), we have that V is not SS-mM and hence it is not SS-mH.

Recall that a space X is *monotonically paracompact in the sense of star refinement* if there exists a function r which assigns to every open cover $\mathcal{U}$ an open star-refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ refines $\mathcal{V}$ then $r(\mathcal{U})$ refines $r(\mathcal{V})$ [36].

Proposition 2.2.12. *V is not monotonically paracompact at p in the sense of star refinement.*

Proof. It follows from propositions 2.2.9 and 2.2.7 since obviously monotonically paracompact at p in the sense of star refinement implies mNA. □

Note that the asymmetric V example distinguishes also SW-mH property and local version of SS-mR property.

Example 2.2.6. *V is SW-mH but is is not SS-mR at p.*

Proof. Indeed by Proposition 2.2.11, V is SW-mH and by propositions 2.2.10 and 2.2.7, V is not SS-mR at p. □

The asymmetric V example permitts also to distinguish the two local versions of monotone paracompactness defined respectively in the sense of locally finite refinement and in the sense of star-refinement. Note that in [29] the space V is described in a different way and it is used to distinguish the corresponding two global versions of monotone paracompactness.

BIBLIOGRAPHY

[1] O. Alas, L.R. Junqueira, and R.G. Wilson, *Dually discrete spaces*, Top. Appl., No. **155** (2008), 1420-1425.

[2] O. Alas, L.R. Junqueira, and R.G. Wilson, *Countability and star covering properties*, Top. Appl., No. **158** (2011), 620-626.

[3] O. Alas, V.V. Tkachuk, and R.G. Wilson, *Addition theorems, D-spaces and dually discrete spaces*, Comment. Math. Univ. Carolin., No. **50(1)** (2009), 113-124.

[4] O. Alas, V.V. Tkachuk, and R.G. Wilson, *Closures of discrete sets often reflect global properties*, Top. Proceedings, Vol. **25** (Spring 2000), 27-44.

[5] O. Alas, V.V. Tkachuk, and R.G. Wilson, *Covering properties and neighborhood assignments*, Top. Proceedings, Vol. **30(1)** (2006), 25-38.

[6] G. Baloglou, W.W. Comfort, *Compact covering numbers*, Fund. Math., No. **131** (1988), 69–82.

[7] A. Arhangel'skiĭ, *The cardinality of first countable bicompacta*, Soviet Math. Dokl. **10** (1969), 951-955.

[8] A. Arhangel'skiĭ, M. Tkachenko, *Topological Groups and Related Structures, An Introduction to Topological Algebra*, Atlantis Press (2008).

[9] F.A. Basile, M. Bonanzinga, N. Carlson, *Variations on known and recent cardinality bounds*, Top. Appl. **240** (2018), 228-237.

[10] F.A. Basile, M. Bonanzinga, D. Shakhmatov, *Star covering properties and neighborhood assignments*, submitted.

[11] F.A. Basile, N. Carlson, *On the cardinality of Urysohn spaces and weakly H-closed spaces*, published online in Mathematica Bohemica as doi: 10.21136/MB.2018.0037-18.

[12] F.A. Basile, N. Carlson, *On the quasicellularity of a space*, submitted.

[13] M.Bell, J.Ginsburg, G.Woods, *Cardinal inequalities for topological space involving the weak Lindelöf number*, Pacific J. Math., Vol. **79(1)** (1978), 37-45.

[14] A. Bella, F. Cammaroto, *On the cardinality of Urysohn spaces*, Canad. Math. Bull. Vol. **31(2)** (1988), 153-158.

[15] A. Bella, N. Carlson, *On cardinality bounds involving the weak Lindelöf degree*, Quest. Math., **41(1)** (2018), 99-113.

[16] A. Bella, N. Carlson, I. Gotchev, *More on cardinality bounds involving the weak Lindelöf degree*, preprint.

[17] M. Bonanzinga, *Star Lindelöf and absolutely star Lindelöf spaces*, Q&A in General Topology, Vol. **16** (1998), 79-104.

[18] M. Bonanzinga, M.Matveev, Combinatorics of Thin Posets: Application to Monotone Covering Properties, Order **28** (2011) 173-179.

[19] M. Bonanzinga, M.V. Matveev, F.A. Basile, *Monotone versions of some selection principles*, Top. App. **258** (2019) 177-186.

[20] M. Bonanzinga, M.V. Matveev, F.A. Basile, *Monotone versions of some selection principles II*, submitted.

[21] M. Bonanzinga, D. Stavrova, P. Staynova, *Separation and Cardinality- Some New Results and Old Questions*, Topol. Appl. **221** (2017), 556-569.

[22] C.J. Borger, *On Stratifiable Spaces*, Pacific Journ. of Math. **17** (1966), 1-16.

[23] R.Z. Buzyakova, V.V. Tkachuk, and R.G. Wilson, *A quest for nice kernels of neighbourhood assignments*, Comment. Math. Univ. Carolin., No. **48(4)** (2007), 689-697.

[24] F. Cammaroto, A. Catalioto, J. Porter, *On the cardinality of Urysohn spaces*, Topology Appl., Vol. **160** (2013), 1862-1869.

[25] F. Cammaroto, Lj. Kočinac, *On θ-tightness*, Facta Universitatis (Niš), Ser. Math. Inform. **8** (1993), 77-85.

[26] J. Cao, J. Kim, T. Nogura, and Y. Song, *Cardinal invariants related to star covering properties*, Top. Proceedings, No. **26** (2001-2002), 83-96.

[27] N. Carlson, J. Porter, *On the cardinality of Hausdorff spaces and H-closed spaces*, Topology Appl. **241** (2018), 377-395.

[28] N. Carlson, J. Porter, G. Ridderbos, *On cardinality bounds for homogeneous spaces and the G_k-modification of a space*, Top. and its Appl., **159** (2012), 2932-2941.

[29] T. Chase, *Monotonic covering properties*, Dissertation, Auburn University, 2014 (estimated).

[30] A. Dow, J. Porter, *Cardinalities of H-closed spaces*, Top. Proc., **7(1)** (1982), 27-50.

[31] E.K. van Douwen, *Pixley–Roy topology on spaces of subsets*, Set-Theoretic Topology, Academic Press Inc., New York (1977), 111–134.

[32] E.K. van Douwen, W. Pfeffer, *Some properties of the Sorgenfrey line and related spaces*, Pacific Jou. Math., Vol. **81(2)** (1979), 371-377.

[33] E.K. van Douwen, G.M. Reed, A.W. Roscoe, I.J. Tree, *Star covering properties*, Top. Appl., No. **39** (1991), 71-103.

[34] Ryszard Engelking, *General Topology* (1989), Heldermann Verlag.

[35] P. Gartside, A. Medini, and L. Zdomskyy, *The Tukey Order, Hyperspaces, and Selection Principles*, work in progress.

[36] P.M.Gartside,P.J.Moody, *A note on proto-metrisable spaces*, Topol. and Appl. **52(1)** (1993), 1-9.

[37] I.S. Gotchev, *Cardinal inequalities for Urysohn spaces involving variations of the almost Lindelöf degree*, Serdica Math. J. **44** (2018), 195-212.

[38] I. Gotchev, M.G. Tkachenko, V.V. Tkachuk, *Regular G_δ-diagonals and some upper bounds for cardinality of topological spaces*, Acta Math. Hungar. **149 (2)** (2016), 324-337.

[39] J. de Groot, *Discrete subspaces of Hausdorff spaces*, Bull. Acad. Pol. Sci. **13** (1965), 537-544.

[40] A. Hajnal, and I. Juhász, *Discrete subspaces of topological spaces*, Indag. Math. **29** (1967), 343-356.

[41] A. Hajnal, and I. Juhász, *Having small weight is determined by the small subspaces*, Proc. Amer. Math. Soc. **79** (1980), 657-658.

[42] K.P. Hart, J. Nagata and, J.E. Vaughan, Encyclopedia of General Topology, Elsevier Science B.V., (2004).

[43] R.E. Hodel, *Arhangel'skiĭ's solution to Alexandroff's problem: A survey*, Topology Appl. **153** (2006), 2199-2217.

[44] S. Ikenaga, *A class which contains Lindelöf spaces, separable spaces and countably compact spaces*, Memories of Numazu College of Technology, No. **18** (1983), 105-108.

[45] S. Ikenaga, *Topological concept between Lindelöf and Pseudo-Lindelöf*, Research Reports of Nara National College of Technology, No. **26** (1990), 103-108.

[46] R. Jensen, *The fine structure of the constructible hierarchy*, Annals of Mathematical Logic **4** (1972), 229-308.

[47] I. Juhász, *Cardinal functions in topology*, Math. Centr. (1979).

[48] Lj. Kočinac, *Star-Menger and related spaces*, Publ. Math. Debrecen, No. **55** (1999), 421-431.

[49] Lj. Kočinac, *Star-Menger and related spaces, II*, Filomat (Niš), No. **13** (1999), 129-140.

[50] R. Levy and M. Matveev, *The monotone Lindelöf property for countable spaces*, Comment. Math. Univ. Carol. **49(1)** (2008), 155-161.

[51] R. Levy and M. Matveev, *Some more examples of monotonically Lindelöf and not monotonically Lindelöf spaces*, Topol. and Appl. **154** (2007) 2333-2343.

[52] R. Levy and M. Matveev, *Some questions on monotone Lindelöfness*, Questions and Answers in Gen. Topol. **26** (2008) 13-27.

[53] M. Matveev, *A monotonically Lindelöf space which is not monotonically normal*, preprint (1994).

[54] M. Matveev, *Some questiones on Property (a)*, Q&A in General Topology, No. **15** (1997), 103-111.

[55] M. Matveev, *A survey on star covering properties*, Topology Atlas, April 15, 1998.

[56] J. van Mill, V.V. Tkachuk, and R.G. Wilson, *Classes defined by stars and neighbourhood assignments*, Topology and its Applications, No. **154** (2007), 2127–2134.

[57] A. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc., No. **14(2)** (1976), 505–516.

[58] J. Porter, R. Woods, *Extensions and Absolutes of Hausdorff Spaces* (1980), Springer-Verlag.

[59] D. Repovš, and L. Zdomskyy, *Products of Hurewicz spaces in the Laver Model*, The Bull. of Symb. Logic, Vol. **23(3)** (2017), 324-333.

[60] B. Šapirovskiĭ, *Canonical sets and character, density and weight in compact spaces*, Soviet. Math. Dokl. **15** (1974), 1282-1287.

[61] J. Schröder, *Urysohn cellularity and Urysohn spread*, Math. Japonica, Vol. **38** (1993), 1129-1133.

[62] S. Shu-Hao, *Two new topological cardinal inequalities*, Am. Math. Soc. **104** (1988), 313-316.

[63] Y. Song, and Z. Yin, *On star-C-Menger spaces*, Questiones Mathematicae, No. **37** (2014), 337-347.

[64] Lynn A. Steen, J. Arthur Seebach, *Counterexamples in Topology*, Holt, Rinehart and Winston, Inc (1970).

[65] N.V. Veličko, *H-closed topological spaces*, Mat. Sb. (N.S.) **70(112)** (1966), 98-112 (in Russian).

[66] G. Viglino, $\overline{T}_n$*-spaces*, Kyungpook Math. J. **11** (1971), 33-35.

[67] S. Willard, U.N.B. Dissanayake, *The almost Lindelöf degree*, Canad. Math. Bull. Vol. **27(4)** (1984), 452-455.