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Continuity and Affine Invariance of Measures of Convexity[1]

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Abstract

Schneider asks [8] whether there exists an affine invariant and continuous measure of convexity on the space $\mathcal{C}$of compact subsets of a finite-dimensional Euclidean space. This question is considered still open. We provide a negative answer to the previous question: any affine invariant measure of convexity on $\mathcal{C}$ cannot be continuous in terms of the Hausdorff metric. We also show that some weaker form of continuity can still be retained for affine invariant measures of convexity by showing that the Schneider's measure proposed in [7] is lower semi continuous.

Keywords: Measures of convexity, convex sets in finite-dimensional spaces

1 Introduction

In reminiscence of Grünbaum's definition of "measure of symmetry" ([4]), Schneider ([7,9]) proposes an affine invariant measure of convexity justified by the fact that "convexity if an affine notion" (([9, p 131]). However, unlike measures of symmetry, which usually are affine invariant and continuous as well,

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the existing measures of convexity are either continuous but not affine invariant (see e.g. [1], [10]) or affine invariant but not continuous (see [7]) (For a survey see [5]). Consequently, Schneider [8] asks whether there exists an affine invariant measure of convexity on the space $\mathcal{C}$ of compact and non-empty subsets in $\Re^d$ which is also continuous with respect to the Hausdorff metric d_H. This problem has recently been referred to as a still open question in [5, p. 32].
By means of an elementary proof we provide a negative answer to the previous question: any affine invariant measure of convexity on $\mathcal{C}$ cannot be continuous in terms of the Hausdorff metric. Intuitively, the reason is that while convexity is an affine invariant concept, non-convexity can be lost under (infinite iteration of) affine transformations. Given the importance of continuity of measures of convexity in applications (see e.g. [6]), one may wonders whether it is possible to maintain some weaker form of continuity for affine invariant measures of convexity. We show that Schneider's measure is lower semi continuous.

2The main result

Recall that for $A,B\in\mathcal{C}$ the Hausdorff distance is defined as follows: $d_H(A,B)=\max\left\{\sup_{x\in A}\inf_{y\in B}|x-y|,\sup_{x\in B}\inf_{y\in A}|x-y|\right\}$ and that a real valued function m defined on the space $\mathcal{C}$ is a measure of convexity if $m(A) = \beta$ for some real number β if and only if A is a convex subsets in $\mathcal{C}$. The measure m is affine invariant if $m(A)=m(TA)$ for any nonsingular affine transformation T *of* $\Re^d$, and it is continuous if for every sequence of sets $\{A_n\}$ in $\Re^d$ such that $A_n \xrightarrow[d_H]{} A$, if $m(A_n)\to m^*$ then $m^*=m(A)$.

THEOREM 1.*Let* $m:\mathcal{C}\to\Re$ *be a measure of convexity on* $\mathcal{C}$*. If m is affine invariant, then it is not continuous.*

Proof. Let m be an affine invariant measure of convexity on $\mathcal{C}$. Suppose for the moment that there exists a sequence $\{A_n\}$ in $\mathcal{C}$ such that: (i) $A_0\in\mathcal{C}$ is a non-convex set, (ii) $A_n=TA_{n-1}$ for every n = 1, 2,….., where T is a nonsingular affine mapping, and (iii) $A_n \xrightarrow[d_H]{} A$,with A convex. By the non-convexity of A_0, we have that $m(A_0)$ = $\alpha\neq\beta$ and, by affine invariance, that $m(A_n)=m(A_{n-1})=\alpha$ for every n = 1, 2, ….. However, by convexity of A, $m(A) = \beta$, then function m is not continuous as $A_n \xrightarrow[d_H]{} A$ and $m(A_n)\to\alpha\neq\beta=m(A)$.

To complete the proof, we have to prove that there exists a sequence $\{A_n\}$ in $\mathcal{C}$ satisfying conditions (i), (ii) and (iii) specified above. To this end consider the sequence $\{A_n\}$ defined as follows: $A_0 = \{0, x\}$ where 0 is the origin of $\Re^d$ and x any vector different from 0, and $T = \frac{1}{2} I$ where I is the identity matrix. It is obvious that the sequence $\{A_n\}$ with $A_n = T^n A_0$ satisfies the desired conditions (i)-(iii).

REMARK. The proof shows that continuity of measures of convexity can fail even if the stronger condition of homothetic invariance is required.

3. Lower semi-continuous affine invariant measures of convexity

In this section we show that Schneider's measure proposed in [7] is lower semi-continuous.
Given a metric space X, function $f : X \rightarrow \Re$ is *lower semi continuous* at point x_0 if for each real number a such that $f(x_0) > a$ there exists a positive number δ such that $d(x,x_0) < \delta$ implies $f(x) > a$, where d is the Euclidean distance function on X (see, e.g.[2,p. 150]). The measure of convexity introduced by Schneider in [7] is the following. For a subset $A \subset \Re^d$, define $m(A) = \inf\{\lambda \geq 0 \,|\, A + \lambda \operatorname{conv} A$ is convex$\}$. Schneider proves that function m maps set $\mathcal{C}$ into the interval $[0,d]$. Obviously, $m(A) = 0$ if and only if A is convex.

THEOREM 2. *Function m is lower semi continuous on* $\mathcal{C}$.

Proof. Consider $\mathcal{C}$ endowed with the Hausdorff metric d_H. Then, suppose that the assertion is not true, hence, there exist a set $A_0 \in \mathcal{C}$ and a number $a \in [0,d]$ satisfying the condition $m(A_0) > a$ such that for every $\delta > 0$ there exists a set A such that $d_H(A,A_0) < \delta$ implies $m(A) \leq a$. So, consider a sequence $\{\delta_n\}$ of numbers in $[0,+\infty]$ such that $\delta_n \rightarrow 0$, therefore there exists a sequence of sets $\{A_n\}$ in $\mathcal{C}$ such that for every n, $d_H(A_n,A_0) < \delta_n$ and $m(A_n) \leq a$. Clearly, $A_n \rightarrow A_0$ and $0 \leq m(A_n) \leq a < m(A_0)$ for every n. The latter inequality implies that there exists a subsequence of sets $\{A_{n'}\}$ such that the associate sequence $\{m(A_{n'})\}$ is convergent with limit m_0, $0 \leq m_0 \leq a < m(A_0)$. By construction, set $A_{n'} + m(A_{n'}) \operatorname{conv} A_{n'}$ is convex and compact for every n', moreover $A_{n'} \rightarrow A_0$, $m(A_{n'}) \rightarrow m_0$ and $\operatorname{conv} A_n \rightarrow \operatorname{conv} A_0$. So, $A_0 + m_0 \operatorname{conv} A_0$ should be convex (see e.g. [11, p. 94]). However, this cannot be true because $m_0 < m(A_0)$.

COROLLARY 1. *Let* $\mathcal{X}$ *be a compact subset of* $\mathcal{C}$, *then there exists a set* $A^* \in \mathcal{X}$ *such*

that $m(A^*) \le m(A)$ *for every* $A \in \mathcal{X}$.
Set A^* in Corollary can be interpreted as a set which is "closest" to be convex with respect to all sets in $\mathcal{X}$. The following corollary is an immediate consequence of Ekeland'sVariational Principle (see [3]) and of the fact that the metric space $(\mathcal{C}, d_H)$ is complete (see, e.g. [9, p. 49]).

COROLLARY 2. *Let* ε, $\mu > 0$ *and let* $A \in \mathcal{C}$ *be such that* $m(A) \le \inf_{\mathcal{C}} m + \varepsilon$. *Then, there exists* $B \in \mathcal{C}$ *with the following properties*:
(a) $m(B) \le m(A)$;
(b) $d_H(A,B) \le \frac{1}{\mu}$;
(c) $m(B) < m(A) + \varepsilon\mu d_H(A,B)$ for all $A \in \mathcal{C} \setminus \{B\}$

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