Fat points schemes on a smooth quadric

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Abstract

We study zero-dimensional fat points schemes on a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and we characterize those schemes which are arithmetically Cohen–Macaulay (aCM for short) as subschemes of Q giving their Hilbert matrix and bigraded Betti numbers. In particular, we can compute the Hilbert matrix and the bigraded Betti numbers for fat points schemes with homogeneous multiplicities and whose support is a complete intersection (CI for short). Moreover, we find a minimal set of generators for schemes of double points whose support is aCM. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

In the last few years many papers have investigated the Hilbert functions, minimal free resolutions, Betti numbers,... for fat points ideals in $\mathbb{P}^2$, i.e. $I \subset k[x_0, x_1, x_2]$ with $I = \wp_1^{m_1} \cap \cdots \cap \wp_r^{m_r}$ where the $\wp_i$ are prime homogeneous ideals of height 2 (or ideal of a point P_i) and m_i are positive integers. In $\mathbb{P}^2$ our knowledge is very "thin" and a lot of questions are still unanswered; we can see the paper [10] that summarizes works on the topic "fat points". In [11], Giuffrida completely analyzed the case $r = 6$ where the m_i are arbitrary and [8] considers the case where the $\wp_i$ correspond to points on a line of $\mathbb{P}^2$ and the m_i are arbitrary. The Hilbert function of I has been studied in $\mathbb{P}^2$ by many authors, like Gimigliano [10], Harbourne [15–19], Hirschowitz [23], and by Iarrobino in $\mathbb{P}^n$ in [24], but much remains conjectural. More can be said in the case of subschemes of $\mathbb{P}^2$ involving small numbers of points or points in special

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position. For example in [20], the Hilbert function is completely determinated for any scheme of fat points where the points lie on a plane cubic (possibly reducible and not reduced); in [5], Catalisano determines a minimal homogeneous set of generators for I in the case the points P_i lie on a smooth plane conic; using [9], one can determine a minimal homogeneous set of generators for I where the points P_i are $i \leq 9$ general points of $\mathbb{P}^2$ and he also conjectures a result for $i > 9$. In [21], the author finds the degrees of minimal homogeneous sets of generators for ideals I where the points P_i lie on a plane curve of degree at most 3 and we can regard [22] as a continuation of the previous work. In [6], Catalisano and Gimigliano show that there is an algorithm which computes the Hilbert function for ideals I when the points P_i lie on a rational normal cubic. On the other hand, [1–4] determine the Hilbert function for any number of general points in $\mathbb{P}^n$ if the coefficients m_i are at most 2 and [7] for any number of general points in $\mathbb{P}^2$ if the coefficients m_i are small and nearly constant.

In this paper we want to study the behaviour of fat points subschemes of a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, with special regard to their behaviour with respect to the divisors of the quadric itself. This kind of topic seems to be unexplored, so the results in this paper represent a starting point in this field. In [12–14] Giuffrida, Maggioni and Ragusa gave a complete description of the arithmetically Cohen–Macaulay zero-dimensional subschemes of a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ in terms of their Hilbert Matrix.

First, we fix some preliminaries and notation, then we give the definition of fat points schemes on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Section 2 is devoted to the classification of fat points schemes which are aCM subschemes of Q and in Section 3 we complete the study of double points schemes whose support is aCM giving both Hilbert matrix and graded Betti numbers.

This paper will be part of my Ph.D. Thesis.

1. Preliminaries and notation

Let $\mathbb{P}^1 = \mathbb{P}^1_k$ (k an algebraically closed field), and let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric and $\mathcal{O}_Q$ its structure sheaf. Since $\operatorname{Pic} Q \cong \mathbb{Z} \times \mathbb{Z}$, as usual we can assume the classes of the two rulings as basis of $\operatorname{Pic} Q$. If $D \subset Q$ is a divisor of type (a, b), we denote by $\mathcal{O}_Q(a, b)$ the corresponding sheaf, and for any sheaf $\mathscr{F}$ on Q, we set $\mathscr{F}(a, b) = \mathscr{F} \otimes \mathcal{O}_Q(a, b)$. We also assume the notation

$$H^i(a, b) = H^i(Q, \mathcal{O}_Q(a, b)),$$

$$h^i(a, b) = \dim_k H^i(a, b),$$

$$H^i(\mathscr{F}(a, b)) = H^i(Q, \mathscr{F}(a, b)),$$

$$h^i(\mathscr{F}(a, b)) = \dim_k H^i(\mathscr{F}(a, b)).$$

The dimensions $h^i(a,b)$ for $i = 0, 1, 2$ are easily computed [12, Section 1]. Moreover, we will use the following ring:

$$S = H^0_\star(a,b) = \bigoplus_{\substack{a \geq 0 \\ b \geq 0}} H^0(a,b)$$

which is generated by $H^0(1,0)$ and $H^0(0,1)$; let u, u' and v and v' be bases for $H^0(1,0)$ and $H^0(0,1)$, then we have a bigraded ring

$$S \cong k[u,u'] \otimes k[v,v'].$$

For any zero-dimensional subscheme $X \subset Q$ one can consider the Hilbert matrix of X, defined as the function

$$M_X : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N},$$

$$M_X(r,s) = h^0(r,s) - h^0(\mathscr{I}_X(r,s)), \tag{1}$$

where $\mathscr{I}_X$ is the ideal sheaf of X on Q. In the sequel, we will use the first difference matrix $\Delta M_X(r,s) = (C_X(r,s))$ with

$$C_X(r,s) = M_X(r,s) + M_X(r-1,s-1) - M_X(r,s-1) - M_X(r-1,s).$$

In the sequel, it will be useful to consider in $\mathbb{Z} \times \mathbb{Z}$ and in $\mathbb{N} \times \cdots \times \mathbb{N}$ the partial (lexicographic) ordering induced by the usual one in $\mathbb{Z}$ and in $\mathbb{N}$ respectively; we will denote it by "$\leq$".

Definition 1.1. Let $M = (M(r,s))$ be a matrix such that $M(r,s) = 0$ for $r < 0$ or $s < 0$. We say that M is *admissible* when its first difference $\Delta M = (C(r,s))$ satisfies the following conditions:

 (i) $C(r,s) \leq 1$ and $C(r,s) = 0$ for $r \gg 0$ or $s \gg 0$;
 (ii) if $C(r,s) \leq 0$ then $C(h,k) \leq 0$ for any $(r,s) \leq (h,k)$;
(iii) for every (r,s) $0 \leq \sum_{t=0}^s C(r,t) \leq \sum_{t=0}^s C(r-1,t)$ and $0 \leq \sum_{t=0}^r C(t,s) \leq \sum_{t=0}^r C(t,s-1)$.

When M is an admissible matrix the non-zero part of ΔM is contained in a rectangle with opposite vertices $(0,0),(a,b)$ and the elements of the first row (resp. of the first column) are:

$$C(0,s) = 1 \text{ if } s \leq b, \quad \text{and} \quad C(0,s) = 0 \quad \text{if } s > b$$

$$(\text{resp. } C(r,0) = 1 \text{ if } r \leq a, \quad \text{and} \quad C(r,0) = 0 \quad \text{if } r > a).$$

In this case we say M or ΔM is of *size* (a,b).

Let M_X be the Hilbert matrix of a zero-dimensional scheme $X \subset Q$.

Definition 1.2. For every $r \geq 0$ we set:

$$j_X(r) = \min\{t \in \mathbb{N} \mid M_X(r,t) = M_X(r,t+1)\}$$

and for every $s \geq 0$ we set

$$i_X(s) = \min\{t \in \mathbb{N} \mid M_X(t,s) = M_X(t+1,s)\}.$$

Remark 1.3. The sequences $i_X(s)$ and $j_X(r)$ are non-increasing [12, Proposition 2.7], and the meaningful part of the matrix M_X is inside the rectangle with opposite vertices $(0,0),(i_X(0),j_X(0))$; this means that for every $r > i_X(0)$ the rth row is equal to the $i_X(0)$th row, and for every $s > j_X(0)$ the sth column is equal to the $j_X(0)$th column. Of course, for $(r,s) \geq (i_X(0),j_X(0))$ we have $M_X(r,s) = \deg X$, and outside the above rectangle ΔM_X has null entries.

For properties of these matrices see [12, Section 2].

The scheme X is represented in S by a bigraded homogeneous saturated ideal I_X, which has a minimal free resolution as a bigraded S-module of type:

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i})$$

$$\to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \to I_X \to 0, \tag{2}$$

where the morphisms are of bidegree $(0,0)$. From this, taking sheaves, one gets an $\mathcal{O}_Q$-free resolution of the ideal sheaf $\mathscr{I}_X$

$$0 \to \bigoplus_{i=1}^{p} \mathcal{O}_Q(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} \mathcal{O}_Q(-a_{2i}, -a'_{2i})$$

$$\to \bigoplus_{i=1}^{m} \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \to \mathscr{I}_X \to 0. \tag{3}$$

We will refer to (3) as the minimal free resolution of $\mathscr{I}_X$.

Put $S(X) = S/I_X$,

Definition 1.4. If resolution (2) has length 2, i.e. when depth $S(X) = 2$, then $S(X)$ is a Cohen–Macaulay ring and X is called arithmetically Cohen–Macaulay (aCM for short).

Definition 1.5. An admissible matrix M will be called an aCM matrix if ΔM has only non-negative entries.

Definition 1.6. Let M be an aCM matrix of size (a,b). We say (r,s) is a *corner for* ΔM if $(r,s) = (0, b+1)$ or $(r,s) = (a+1, 0)$, or even if $C(r,s) = 0$ and $C(r-1,s) = C(r,s-1) = 1$. We say that (r,s) is a *vertex for* ΔM if $C(r-1,s) = C(r,s-1) = 0$ and $C(r-1,s-1) = 1$; in this case, of course $C(r,s) = 0$.

For more facts about the ring $S(X)$, the Hilbert matrix and properties of zero-dimensional subschemes of Q we refer to [12].

Let P a point on Q and $\wp = (l(u, u') \otimes 1, 1 \otimes l'(v, v'))$, where l and l' are linear forms, its defining ideal. The element $(a, a'; b, b') \in k^2 \times k^2$ homogeneous in a, a' and b, b' with $l(a, a') = 0$ and $l'(b, b') = 0$ gives the coordinates of P as subvariety of Q, with respect to the chosen basis.

Definition 1.7. Let $P \in Q$ and let P correspond to the prime ideal $\wp \subset S$. If t is any positive integer then the subscheme of Q defined by the ideal $\wp^t$ is called a *fat point in S supported on P* and it is denoted by $(P; t)$.

On Q, let R_i be lines of type $(1, 0)$ for $i = 1, \ldots, a$ and let L_j be lines of type $(0, 1)$ for $j = 1, \ldots, b$; denote $P_{ij} = R_i \cap L_j$ for $i = 1, \ldots, a$ and $j = 1, \ldots, b$ and let P_{ij} correspond to the ideal $\wp_{ij}$. Let m_{ij} be positive integers and let α_i be the largest index such that $m_{i\alpha_i} > 0$ for $i = 1, \ldots, a$.

From now on we suppose that

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_a. \tag{4}$$

Definition 1.8. The subscheme X of Q defined by the ideal

$$I_X = \bigcap_{i,j} \wp_{ij}^{m_{ij}}$$

is called *a fat points subscheme* of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ with support $\{P_{ij}\}$ and its ideal I_X is called a *fat points ideal* on Q. The scheme X is denoted by

$$X = \{P_{ij}; m_{ij} \ \forall i = 1, \ldots, a \text{ and } j = 1, \ldots, \alpha_i\} \tag{5}$$

and its support is denoted by X_{red}.

In the case

$$m_{ij} = m \quad \forall i = 1, \ldots, a \text{ and } j = 1, \ldots, \alpha_i$$

we call X a *homogeneous fat points subscheme* of Q.

If $\alpha_1 = \cdots = \alpha_a$ and

$$\forall i = 1, \ldots, a \quad \exists \beta_i \in \mathbb{N}: \alpha_1 = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_a \geq 1,$$

such that

$$m_{ij} = m \quad \text{for } i = 1, \ldots, a \quad \text{and} \quad j = 1, \ldots, \beta_i$$

and

$$m_{ij} = m - 1 \quad \text{for } i = 1, \ldots, a \quad \text{and} \quad j = \beta_i + 1, \ldots, \alpha_1,$$

then X is called a *quasi-homogeneous fat points subscheme* of Q and its support X_{red} is a CI of type $((a, 0), (0, \alpha_1))$.

We have

$$\deg X = \sum_{i,j} \binom{m_{ij} + 1}{2}.$$

Example 1.9. Let us take on Q two non-collinear points (i.e. not contained on a line of Q), say P_1, P_2, and let $\wp_1 = (u \otimes 1, 1 \otimes v)$ and $\wp_2 = (u' \otimes 1, 1 \otimes v')$ their defining ideals. If $X = \{P_1, P_2\}$ we have $I_X = (uu' \otimes 1, u \otimes v', u' \otimes v, 1 \otimes vv')$. Then we have the following ΔM_X:

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 1 & 1 & 0 \\
1 & 1 & -1 & 0
\end{array}
\tag{6}
$$

Hence, in this case the scheme X is not aCM. This follows from the next theorem which gives the characterization for aCM schemes in terms of their Hilbert matrix:

Theorem 1.10. *Let $X \subset Q$ be a zero-dimensional subscheme, and let M_X be its Hilbert matrix. X is an aCM scheme if and only if M_X is an aCM matrix. Furthermore, in this case, the minimal free resolution of $\mathscr{I}_X$ looks like*:

$$
0 \to \bigoplus_{i=1}^{m-1} \mathcal{O}_Q(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} \mathcal{O}_Q(-a_{1i}, -a'_{1i}) \to \mathscr{I}_X \to 0
\tag{7}
$$

where $(-a_{2i}, -a'_{2i})$ runs over all the vertices and $(-a_{1i}, -a'_{1i})$ runs over all the corners of ΔM_X.

Proof. See [12]. $\square$

Remark 1.11. From Theorem 1.10, it follows that a homogeneous fat points scheme with all the multiplicities $m_{ij} = 1$ is equivalent to an aCM scheme of simple points.

Let us consider the following example:

Example 1.12. Let

$$
X = \{P_{11}, P_{22}; m_{11} = m_{22} = 2\}
$$

be the scheme of two non-collinear double points, then X has the following ΔM_X:

$$
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & -1 & 0 \\
2 & 1 & 1 & -2 & 0 & 0 \\
3 & 1 & -1 & 0 & 0 & 0
\end{array}
\tag{8}
$$

Hence, in this case both X and X_{red} (its support) are not aCM.

Now, consider the fat points scheme X'

$$
X' = \{P_{11}, P_{12}, P_{13}, P_{21}; m_{11} = 3, m_{12} = 2, m_{13} = 1, m_{21} = 3\}.
$$

In this case X'_{red} is aCM. After an easy computation, $\Delta M_{X'}$ is the following:

	0	1	2	3	4	5	6
0	1	1	1	1	1	1	0
1	1	1	1	1	0	0	0
2	1	1	1	−1	0	0	0
3	1	1	0	0	0	0	0
4	1	0	0	0	0	0	0
5	1	0	0	0	0	0	0

$$(9)$$

So, even if the support of a scheme of fat points X' is an aCM scheme, X' could be not aCM.

Hence, it seems natural to ask under which conditions such fat points schemes are aCM.

2. The main result

Let $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a smooth quadric and X a fat points scheme on Q of type:

$$X = \{P_{ij};\ m_{ij}\ \forall i = 1,\ldots,a \text{ and } j = 1,\ldots,\alpha_i\} \tag{10}$$

and we use the convention that $m_{ij} = 0$ for $j > \alpha_i$.

Put $\forall h \in \mathbb{N}_0$, $\forall i = 1,\ldots,a$ and $\forall j = 1,\ldots,\alpha_1$

$$t_{ij}(h) = \max(0, m_{ij} - h).$$

Let us consider the following set:

$$\mathscr{S}_X = \{(t_{i1}(h),\ldots,t_{i\alpha_1}(h))\} \tag{11}$$

$\forall i = 1,\ldots,a$ and $\forall h \in \mathbb{N}_0$.

When $\mathscr{S}_X$ is a totally ordered set with respect to the previously defined ordering in $\mathbb{N}_0^{\alpha_1}$ (see Section 1), we define $\forall i = 1,\ldots,a$ and $\forall h \in \mathbb{N}_0$

$$z_{i,h} = \sum_{j=1}^{\alpha_1} t_{ij}(h),$$

$$u_1 = \max_{i,h}\{z_{i,h}\}$$

and

$$u_t = \max_{i,h}\{\{z_{i,h}\} \setminus \{u_1,\ldots,u_{t-1}\}\} \quad \text{for } t = 2,\ldots,\rho$$

where $\rho = \max_{j=1,\ldots,\alpha_1}\{\sum_{i=1}^{a} m_{ij}\}$.

Obviously u_1 is obtained for $h = 0$, hence

$$u_1 = \max_i\{z_{i,0}\} = \max_i\left\{\sum_{j=1}^{\alpha_1} m_{ij}\right\}.$$

Then, with the above notation, we have the following:

Theorem 2.1. *Let X be a fat points scheme on Q as in* (10). *$\mathscr{S}_X$ is totally ordered if and only if X is aCM with the difference Hilbert matrix ΔM_X of X of type*:

$$
\begin{array}{c|cccccc}
 & 0 & 1 & \ldots\ldots & u_1-1 & u_1 \\
\hline
0 & \underbrace{1\ \ 1\ \ \ 1\ldots 1}_{u_1} & & & & 0 \\
1 & \underbrace{1\ \ 1\ldots 1}_{u_2}0\cdots & & & & 0 \\
\vdots & \cdots & & & & \cdots \\
\rho-1 & \underbrace{1\ \ 1\ldots 1}_{u_\rho}0\cdots & & & & 0
\end{array}
\tag{12}
$$

where $\rho = \max_{j=1,\ldots,\alpha_1}\{\sum_{i=1}^a m_{ij}\}$.

Proof. Let us suppose that S_X is totally ordered and let us show that X is aCM.

We work by induction on ρ. If $\rho = 1$, then X is a set of collinear simple points and hence the conclusion is true (Theorem 1.10 and Remark 1.11).

Let us suppose that the theorem is true for fat points schemes $\bar{X} = (P_{ij}; \bar{m}_{ij})$ on Q for which $\bar{\rho} = \max_j\{\sum_i \bar{m}_{ij}\} < \rho$ and let us prove it for X.

Let $1 \leq k \leq a$ be an integer such that we obtain

$$
u_1 = \max_i\{z_{i,0}\} = z_{k,0} = \sum_{j=1}^{\alpha_k} t_{kj}(0) = \sum_{j=1}^{\alpha_k} m_{kj}
$$

and define the following fat points scheme:

$$
X' := \left\{
\begin{array}{ll}
P_{ij};\ m'_{kj} = m_{kj} - 1 & \text{for } j = 1,\ldots,\alpha_k; \\
\quad\quad m'_{ij} = m_{ij} & \text{for } i = 1,\ldots,\hat{k},\ldots,a \\
& \text{and } j = 1,\ldots,\alpha_i
\end{array}
\right\}
$$

where $\hat{\ }$ means omitted.

Let us check that X' satisfies the inductive hypotheses:

(i) $\rho' < \rho$. In fact, $\forall j = 1,\ldots,\alpha_1$

$$
\sum_{i=1}^a m'_{ij} = m'_{kj} + \sum_{\substack{1 \leq i \leq a \\ i \neq k}} m'_{ij}
$$

$$
= m_{kj} - 1 + \sum_{\substack{1 \leq i \leq a \\ i \neq k}} m_{ij} = \sum_{i=1}^a m_{ij} - 1.
$$

Then $\rho' = \rho - 1 < \rho$.

(ii) The set $\mathscr{S}_{X'}$ is totally ordered. In fact, put for all $l \in \mathbb{N}_0$, $i=1,\ldots,a$ and $j=1,\ldots,\alpha_1$

$$
t'_{ij}(l) = \max(0, m'_{ij} - l).
\tag{13}
$$

We have for all $j = 1, \ldots, \alpha_k$

$$t'_{kj}(l) = \max(0, m'_{kj} - l)$$
$$= \max(0, m_{kj} - (l + 1)),$$

then

$$t'_{kj}(l) = t_{kj}(l + 1) \quad \forall j = 1, \ldots, \alpha_k \text{ and } \forall l \in \mathbb{N}_0. \tag{14}$$

Moreover $\forall i \neq k$ and $j = 1, \ldots, \alpha_1$

$$t'_{ij}(l) = \max(0, m'_{ij} - l)$$
$$= \max(0, m_{ij} - l)$$
$$= t_{ij}(l) \quad \forall l \in \mathbb{N}_0. \tag{15}$$

Hence, the set $\mathscr{S}_{X'}$ is obtained eliminating the maximum element from the set $\mathscr{S}_X$, and so $\mathscr{S}_{X'}$ is totally ordered.

Then, by inductive hypotheses, X' is aCM and, from (14) and (15), we have

$$z'_{i,l} = z_{i,h} \quad \forall i = 1, \ldots, \hat{k}, \ldots, a \text{ and } \forall l = h \in \mathbb{N}_0$$

and

$$z'_{k,0} = z_{k,1}.$$

Therefore $\forall h \in \mathbb{N}_0$

$$u'_1 = \max_{i=1,\ldots,a} \{z'_{i,h}\} = \max_{i=1,\ldots,a} \{\{z_{i,h}\} \setminus u_1\} = u_2.$$

So we get

$$u'_1 = u_2,$$
$$u'_2 = u_3,$$
$$\vdots$$
$$u'_{\rho'} = u'_{\rho-1} = u_\rho$$

and hence $\Delta M_{X'}$ is of type:

	$0\,1\,\ldots\ldots u_2 - 1\ \ u_2$
0	$\underbrace{1\quad 1\quad 1\ldots 1}_{u_2}\quad 0$
1	$\underbrace{1\quad 1\ldots 1}_{u_3}0\cdots\quad 0$
$\vdots$	$\ldots\qquad\qquad\qquad\ldots$
$\rho' - 1$	$\underbrace{1\quad 1\ldots 1}_{u_\rho}0\cdots\quad 0$

$$\tag{16}$$

Claim. ΔM_X is obtained from $\Delta M_{X'}$ just adding a first row consisting of u_1 "1" entries, with $u_1 = \sum_{j=1}^{\alpha_k} m_{kj}$.

In fact, since we added only 1 and 0 entries, $\Delta M_X(r,s)$ will be aCM and of type (12).

Proof. Our claim is equivalent to

$$M_X(0,s) = \begin{cases} s+1 & \text{if } 0 \le s \le u_1 - 1, \\ u_1 & \text{if } s \ge u_1 \end{cases} \tag{17}$$

and

$$M_X(r+1,s) = \begin{cases} M_{X'}(r,s) + s + 1 & \text{if } r \ge 0 \text{ and } 0 \le s \le u_1 - 1, \\ M_{X'}(r,s) + u_1 & \text{if } r \ge 0 \text{ and } s \ge u_1 \end{cases} \tag{18}$$

and in terms of the first difference matrices $\Delta M_X(r,s) = (C_X(r,s))$ and $\Delta M_{X'}(r,s) = (C_{X'}(r,s))$, that

$$C_X(0,s) = \begin{cases} 1 & \text{if } 0 \le s \le u_1 - 1, \\ 0 & \text{if } s \ge u_1 \end{cases} \tag{19}$$

and

$$C_X(r+1,s) = C_{X'}(r,s) \quad \text{for } (r,s) \ge (0,0). \tag{20}$$

Let us prove (17) and hence (19).

We have

$$\begin{aligned} \deg(X) &= \sum_{i,j} \binom{m_{ij}+1}{2} \\ &= \sum_{\substack{i,j \\ i \ne k}} \binom{m_{ij}+1}{2} + \sum_{j=1}^{\alpha_k} \binom{m_{kj}+1}{2} \\ &= \sum_{\substack{i,j \\ i \ne k}} \binom{m_{ij}+1}{2} + \sum_{j=1}^{\alpha_k} \binom{m_{kj}}{2} + \sum_{j=1}^{\alpha_k} \binom{m_{kj}}{1} \\ &= \deg(X') + \sum_{j=1}^{\alpha_k} m_{kj} \end{aligned} \tag{21}$$

and we know that

$$\sum_{j=1}^{\alpha_k} m_{kj} = u_1 = \max_{i=1,\dots,a} \{z_{i,0}\} = z_{k,0}$$

hence, for the ideal sheaf $\mathscr{I}_X$ associated to X, we have

$$h^0 \left(\mathscr{I}_X \left(0, \sum_{i=1}^{\alpha_k} m_{kj} - 1 \right) \right) = 0 \tag{22}$$

then, from definition (1),

$$M_X \left(0, \sum_{i=1}^{\alpha_k} m_{kj} - 1 \right) = \sum_{i=1}^{\alpha_k} m_{kj} = u_1.$$

Obviously, for $s < \sum_{j=1}^{\alpha_k} m_{kj} = u_1$

$$h^0(\mathscr{I}_X(0,s)) = 0,$$

then

$$M_X(0,s) = s + 1$$

for $s < u_1$.

Moreover,

$$h^0 \left(\mathscr{I}_X \left(0, \sum_{i=1}^{\alpha_k} m_{kj} \right) \right) = 1$$

hence,

$$M_X \left(0, \sum_{i=1}^{\alpha_k} m_{kj} \right) = \sum_{i=1}^{\alpha_k} m_{kj} = u_1;$$

from Remark 1.3, it is $j_X(0) = u_1 - 1$, so for every $s > (u_1 - 1)$ the sth column is equal to the $(u_1 - 1)$th column, then $M_X(0,s) = u_1$ for $s \geq u_1$.

Now, let us show (18). We may observe that for $s \leq u_1 - 1$ we have

$$h^0(\mathscr{I}_{X'}(r,s)) = h^0(\mathscr{I}_X(r+1,s)),$$

since every curve of type $(r+1,s)$ through X splits into the line R_k and a curve of type (r,s) through X', hence

$$
\begin{aligned}
M_X(r+1,s) &= (r+2)(s+1) - h^0(\mathscr{I}_X(r+1,s)) \\
&= (r+1)(s+1) - h^0(\mathscr{I}_{X'}(r,s)) + s + 1 \\
&= M_{X'}(r,s) + s + 1.
\end{aligned}
\tag{23}
$$

Let us check (20). Since $j_X(0) = u_1 - 1$ and $j_{X'}(0) = u_2 - 1$, from Remark 1.3 we have

$$C_X(r+1,s) = C_{X'}(r,s) = 0 \quad \forall s \geq u_1 \tag{24}$$

and we are done.

Vice versa, let us suppose that X is aCM and let us show that S_X is totally ordered with respect to the previously defined ordering in $\mathbb{N}_0^{\alpha_1}$.

Since X is aCM, ΔM_X is aCM and it is of size (ρ, b).

Let us work by induction on ρ. If $\rho = 1$, then X is a set of collinear simple points and, hence,

$$\mathscr{S}_X = \{(\underbrace{1,1,\ldots,1}_{b+1})\}$$

and it is totally ordered.

Let us suppose $\rho > 1$ and that the theorem is true for aCM fat points schemes X' of size (ρ', b') with $\rho' < \rho$ and $b' \leq b$ and let us prove it for X.

From [12, Theorem 2.12], there exists a line R of type $(1,0)$ such that $|R \cap X| = b + 1$. Let $1 \leq k \leq a$ be an integer such that $\sum_j m_{kj} = b + 1$.

Let us call X' the following fat points scheme:

$$X':= \left\{ \begin{array}{ll} P_{ij}; \ m'_{kj} = m_{kj} - 1 & \text{for } j = 1, \ldots, \alpha_k; \\ \qquad m'_{ij} = m_{ij} & \text{for } i = 1, \ldots, \hat{k}, \ldots, a \\ & \text{and } j = 1, \ldots, \alpha_i \end{array} \right\}$$

where ˆ means omitted.

We have

(i) $\rho' < \rho$;

(ii) $\Delta M_{X'}(r,s) = \Delta M_X(r+1,s)$ (see [12, Lemma 2.15 or Corollary 2.16]), then X' is aCM. Hence, by inductive hypothesis, $\mathscr{S}_{X'}$ is totally ordered;

(iii) moreover,

$$\mathscr{S}_X = \mathscr{S}_{X'} \cup \{(m_{k1}, \ldots, m_{k\alpha_1})\}. \tag{25}$$

Claim. $(m_{k1}, \ldots, m_{k\alpha_1}) \geq (m_{i1}, \ldots, m_{i\alpha_1})$ *for all* $i = 1, \ldots, a$.
In fact, from (25), $\mathscr{S}_X$ *will be totally ordered.*

Let us define

$$m_j = \max_i \{m_{ij}\} \quad \forall 1 \leq j \leq \alpha_1. \tag{26}$$

If $m_{kh} < m_{lh}$ for suitable $h \in \{1, \ldots, \alpha_1\}$ and $l \in \{1, \ldots, a\}$, since $b + 1 = \sum_j m_j$, we would have

$$b + 1 = \sum_j m_j > \sum_j m_{kj} = b + 1,$$

a contradiction.

Hence the theorem is completely proved. $\quad\square$

Remark 2.2. Given a zero-dimensional subscheme $X \subset Q$ we can also consider X as a subscheme of $\mathbb{P}^3$ via the usual embedding $Q \hookrightarrow \mathbb{P}^3$, therefore we can determine both the Hilbert function and the graded Betti numbers for any aCM subscheme $X \subset Q$ as subscheme of $\mathbb{P}^3$ (see [13, Section 2, Proposition 2.3]).

Example 2.3. Let us consider the following fat points scheme:

$$X := \{(P_{11}; 7), \quad (P_{12}; 5), \quad (P_{13}; 3) \quad (P_{21}; 5) \quad (P_{22}; 2) \quad (P_{23}; 1)\}.$$

Here, $\alpha_1 = \alpha_2 = 3$ and $a = 2$. The set $\mathscr{S}_X$ is

$$\{(7, 5, 3), \quad (6, 4, 2), \quad (5, 3, 1), \quad (5, 2, 1), \quad (4, 2, 0), \quad (4, 1, 0),$$

$$(3, 1, 0), \quad (3, 0, 0), \quad (2, 0, 0), \quad (2, 0, 0), \quad (1, 0, 0), \quad (1, 0, 0)\}$$

and it is totally ordered. We have

$$z_{1,0} = \sum_{j=1}^3 t_{1j}(0) = 15, \quad z_{1,1} = \sum_{j=1}^3 t_{1j}(1) = 12,$$

$$z_{1,2} = \sum_{j=1}^{3} t_{1j}(2) = 9, \quad z_{1,3} = \sum_{j=1}^{3} t_{1j}(3) = 6,$$

$$z_{1,4} = \sum_{j=1}^{3} t_{1j}(4) = 4, \quad z_{1,5} = \sum_{j=1}^{3} t_{1j}(5) = 2,$$

$$z_{1,6} = \sum_{j=1}^{3} t_{1j}(6) = 1, \quad z_{2,0} = \sum_{j=1}^{3} t_{2j}(0) = 8,$$

$$z_{2,1} = \sum_{j=1}^{3} t_{2j}(1) = 5, \quad z_{2,2} = \sum_{j=1}^{3} t_{2j}(2) = 3,$$

$$z_{2,3} = \sum_{j=1}^{3} t_{2j}(3) = 2, \quad z_{2,4} = \sum_{j=1}^{3} t_{2j}(4) = 1$$

Hence $k = 1$, $\rho = m_{11} + m_{21} = 12$ and

$$u_1 = z_{1,0} = 15, \quad u_2 = z_{1,1} = 12,$$

$$u_3 = z_{1,2} = 9, \quad u_4 = z_{2,0} = 8,$$

$$u_5 = z_{1,3} = 6, \quad u_6 = z_{2,1} = 5,$$

$$u_7 = z_{1,4} = 4, \quad u_8 = z_{2,2} = 3,$$

$$u_9 = z_{1,5} = 2, \quad u_{10} = z_{2,3} = 2,$$

$$u_{11} = z_{1,6} = 1, \quad u_{12} = z_{2,4} = 1$$

so, ΔM_X is

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0
2	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
3	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
4	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
5	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
6	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
7	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(27)

Remark 2.4. Not all the aCM matrices are the Hilbert matrices of fat points schemes with some $m_{ij} > 1$; in fact, let us consider the following ΔM:

	0	1	2	3	4
0	1	1	1	1	0
1	1	1	1	0	0
2	1	1	0	0	0
3	0	0	0	0	0

(28)

ΔM is an aCM matrix but we can easily check that there is no fat points subscheme of Q with some $m_{ij} > 1$ which has ΔM as first different matrix.

Corollary 2.5. *If*

$$X = \{P_{ij}; \ m_{ij} = m \ \forall i = 1, \ldots, a \ and \ j = 1, \ldots, \alpha_1\}, \tag{29}$$

i.e. X is a homogeneous fat points subscheme of Q whose support is a CI, then X is aCM.

Proof. It easy to check that, for such an X, $\mathcal{S}_X$ is totally ordered. $\quad\square$

Corollary 2.6. *If $\alpha_1 = \cdots = \alpha_a$, let Y be the following:*

$$Y = \left\{ \begin{array}{ll} P_{ij}; & m_{ij} = m \quad for \ i = 1, \ldots, a \ and \ j = 1, \ldots, \beta_i \\ & m_{ij} = m - 1 \quad for \ i = 1, \ldots, a \ and \ j = \beta_i + 1, \ldots, \alpha_1 \end{array} \right\}$$

with $\alpha_1 = \beta_1 \geq \beta_2 \geq \cdots \beta_a$, i.e. Y is a quasi-homogeneous fat points subscheme of Q whose support is the CI of type $((a,0),(0,\alpha_1))$. Then ΔM_Y is aCM; hence Y is aCM.
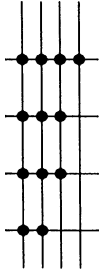
Proof. The set $\mathcal{S}_Y$ is of type

$$\{(\underbrace{m,\ldots,m}_{\alpha_1}),(\underbrace{m,\ldots,m}_{\beta_2},\underbrace{m-1,\ldots,m-1}_{\alpha_1-\beta_2}),\ldots,(\underbrace{m,\ldots,m}_{\beta_a},\underbrace{m-1,\ldots,m-1}_{\alpha_1-\beta_a}),$$

$$(\underbrace{m-1,\ldots,m-1}_{\alpha_1}),(\underbrace{m-1,\ldots,m-1}_{\beta_2},\underbrace{m-2,\ldots,m-2}_{\alpha_1-\beta_2}),\ldots,$$

$$(\underbrace{m-1,\ldots,m-1}_{\beta_a},\underbrace{m-2,\ldots,m-2}_{\alpha_1-\beta_a}),\ldots,$$

$$(\underbrace{1,\ldots,1}_{\alpha_1}),(\underbrace{1,\ldots,1}_{\beta_2},\underbrace{0,\ldots,0}_{\alpha_1-\beta_2}),\ldots,(\underbrace{1,\ldots,1}_{\beta_a},\underbrace{0,\ldots,0}_{\alpha_1-\beta_a})\},$$

therefore it is totally ordered. $\quad\square$

3. Schemes of double points

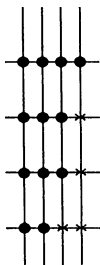**Example 3.1.** Let X be the homogeneous scheme of double points:

where • means a double point. Here

$$\alpha_1 = 4,$$
$$\alpha_2 = \alpha_3 = 3,$$
$$\alpha_4 = 2. \tag{30}$$

$\mathscr{S}_X$ is not totally ordered, then X is not aCM and ΔM_X is the following:

	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	0	0
2	1	1	1	1	1	1	0	0	0
3	1	1	1	1	1	1	−1	0	0
4	1	1	1	1	0	−1	0	0	0
5	1	1	1	0	0	0	0	0	0
6	1	1	1	0	−1	0	0	0	0
7	1	1	0	0	0	0	0	0	0

$$\tag{31}$$

Let Y be the following quasi-homogeneous fat points scheme:

where • means a double point and ∗ means a simple point. Here

$$\alpha_1 = \beta_1 = 4,$$
$$\beta_2 = \beta_3 = 3,$$
$$\beta_4 = 2. \tag{32}$$

$\mathscr{S}_Y$ is totally ordered, then Y is aCM and ΔM_Y is

	0	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1	0
1	1	1	1	1	1	1	1	0	0
2	1	1	1	1	1	1	1	0	0
3	1	1	1	1	1	1	0	0	0
4	1	1	1	1	0	0	0	0	0
5	1	1	1	0	0	0	0	0	0
6	1	1	1	0	0	0	0	0	0
7	1	1	0	0	0	0	0	0	0

$$(33)$$

Let us describe a method how to compute the Hilbert matrix of a homogeneous scheme of double points X with an aCM support.

Let X be the following homogeneous scheme of double points:

$$X := \{ P_{ij}; \ \bar{m}_{ij} = 2 \ \forall i = 1, \ldots, a \text{ and } j = 1, \ldots, \alpha_i \} \tag{34}$$

with $\alpha_1 \geq \cdots \geq \alpha_a$.

Let us define the following quasi-homogeneous scheme Y:

$$Y := \left\{ \begin{array}{ll} P_{ij}; \ m_{ij} = 2 & \text{for } i = 1, \ldots, a \text{ and } j = 1, \ldots, \alpha_i \\ \quad\quad m_{ij} = 1 & \text{for } i = 1, \ldots, a \text{ and } j = \alpha_i + 1, \ldots, \alpha_1 \end{array} \right\}, \tag{35}$$

where Y_{red} is a CI of type $((a, 0), (0, \alpha_1))$.

We remark that, according to the definition of a quasi-homogeneous scheme of fat points, for all $i = 1, \ldots, a$ it is

$$\alpha_i'(Y) = \alpha_1$$

and

$$m_{ij} = \bar{m}_{ij} \quad \text{whenever } j \leq \alpha_i.$$

It means that we complete an aCM support X_{red} of a homogeneous scheme of double points X to a support Y_{red} of a quasi-homogeneous scheme Y and Y_{red} is a CI of type $((a, 0), (0, \alpha_1))$.

In the sequel, we will call X a homogeneous scheme of double points and Y the *completion* of the scheme X to the CI $((a, 0), (0, \alpha_1))$.

Remark 3.2. Let Y be a scheme of type (35). From Theorem 1.10 we know that if (r, s) is a corner of ΔM_Y then $h^0(\mathscr{I}_Y(r, s)) = 1$. Moreover, this generator is a curve of bidegree (r, s) of type

$$R_1 \cdot \ldots \cdot R_r \cdot L_1^2 \cdot L_2^2 \cdot \ldots \cdot L_{s-\alpha_1}^2 \cdot L_{(s-\alpha_1)+1} \cdot \ldots \cdot L_{\alpha_1}$$

if $0 \leq r < a$ and $\alpha_1 < s \leq 2\alpha_1$ (where we put $R_0 = 1$);

$$R_1^2 \cdot \ldots \cdot R_{r-a}^2 \cdot R_{r-a+1} \cdot \ldots \cdot R_a \cdot L_1 \cdot \ldots \cdot L_s$$

if $a \leq r \leq 2a$ and $0 \leq s < \alpha_1$ (where we put $L_0 = 1$).

Let X be a homogeneous scheme of double points and Y its completion.

Definition 3.3. For all $i = 1, \ldots, a$ and $j = \alpha_i + 1, \ldots, \alpha_1$, define

$$X_{a\alpha_1} = Y,$$

$$X_{(i-1)\alpha_1} = X_{i\alpha_i}$$

and, inductively,

$$X_{ij} = X_{i(j+1)} \setminus P_{i(j+1)}$$

that is, proceeding from the right to the left and from the bottom to the top, we take away one simple point in each step from the scheme Y.

Theorem 3.4. *Let X be a homogeneous scheme of double points, Y its completion to a CI and X_{ij} as before. Denoted by*

$$m_{1j} + \cdots + m_{(i-1)j} = u_{ij}$$

and

$$m_{i1} + \cdots + m_{i(j-1)} = v_{ij},$$

then

$$\Delta M_{X_{i(j-1)}}(r,s) = \begin{cases} \Delta M_{X_{ij}}(r,s) & \text{for } (r,s) \neq (u_{ij}, v_{ij}), \\ \Delta M_{X_{ij}}(r,s) - 1 & \text{for } (r,s) = (u_{ij}, v_{ij}) \end{cases} \tag{36}$$

$\forall i = 1, \cdots, a$ *and* $\forall j = \alpha_i + 1, \ldots, \alpha_1$.

Proof. If $\alpha_1 = \cdots = \alpha_a$, then $X = Y$.

Let us suppose that we are in the other cases. The set $\mathscr{S}_Y$ is totally ordered and then, by Theorem 2.1, Y is aCM and ΔM_Y is of type:

$$(37)$$

	$0\,1\,\ldots\ldots\,2\alpha_1-1$	$2\alpha_1$
0	$\underbrace{1\ \ 1\ \ \ 1\cdots 1}_{2\alpha_1}$	0
1	$\underbrace{1\ \ \ 1\ldots 1}_{\alpha_1+\alpha_2}\ \ 0\cdots 0$	
$\vdots$	$\cdots$	$\cdots$
$a-1$	$\underbrace{1\ \ \ 1\ldots 1}_{\alpha_1+\alpha_a}\ \ 0\cdots 0$	
a	$\underbrace{1\ \ \ 1\ldots 1}_{\alpha_1}\ 0\cdots$	0
$\vdots$	$\cdots$	$\cdots$
$2a-1$	$\underbrace{1\ \ \ 1\ldots 1}_{\alpha_a}\ 0\cdots$	0

We may observe that

$$M_{X_{ij}}(r,s) - 1 \leq M_{X_{i(j-1)}}(r,s) \leq M_{X_{ij}}(r,s).$$

Our theorem is equivalent to prove that

$$h^0 \mathscr{I}_{X_{i(j-1)}}(r,s) = h^0 \mathscr{I}_{X_{ij}}(r,s) + 1 \quad \text{for } (r,s) \geq (u_{ij}, v_{ij}), \tag{38}$$

$$h^0 \mathscr{I}_{X_{i(j-1)}}(r,s) = h^0 \mathscr{I}_{X_{ij}}(r,s) \quad \text{otherwise.} \tag{39}$$

We prove only (38) because (39) is trivial. First, we need to find a form F of bidegree (u_{ij}, v_{ij}) such that the curve defined by F passes through $X_{i(j-1)}$ but not through X_{ij}. The form

$$F := R_1^{m_{1j}} \cdot R_2^{m_{2j}} \cdot \ldots \cdot R_{i-1}^{m_{(i-1)j}} \cdot L_1^{m_{i1}} \cdot \ldots \cdot L_{j-1}^{m_{i(j-1)}}$$

$\forall i = 1, \ldots, a$ and $j = \alpha_i + 1, \ldots, \alpha_1$, is a form of bidegree (u_{ij}, v_{ij}) which vanishes at all the points of $X_{i(j-1)}$ but does not vanish in $P_{ij} = R_i \cap L_j$.

Now, we prove that

$$h^0 \mathscr{I}_{X_{i(j-1)}}(r,s) = h^0 \mathscr{I}_{X_{ij}}(r,s) + 1 \quad \text{for } (r,s) > (u_{ij}, v_{ij}).$$

Take a basis $f_1, \ldots, f_t$ for $H^0 \mathscr{I}_{X_{ij}}(r,s)$, and denote by f an element in $H^0 \mathscr{I}_{X_{i(j-1)}}(u_{ij}, v_{ij})$ but not in $H^0 \mathscr{I}_{X_{ij}}(u_{ij}, v_{ij})$, such an element f exists for what we said before. Let C be a curve of type $(r - u_{ij}, s - v_{ij})$ that does not pass through P_{ij}. The elements $f_1, \ldots, f_t, f \cdot C$ are a basis for $H^0 \mathscr{I}_{X_{i(j-1)}}(r,s)$. In fact,

$$f \cdot C \neq \sum_{z=1}^{t} f_z a_z$$

because $f_z(P_{ij}) = 0 \ \forall z = 1, \ldots, t$ and $(f \cdot C)(P_{ij}) \neq 0$. $\square$

Next, we would like to compute a minimal set of generators (for short m.s.o.g.) for a homogeneous scheme $X \subset Q$ of double points.

Thus, for a homogeneous scheme of double points X and its completion Y to a CI, we need few new definitions.

Definition 3.5. We call *base corner of a fat points scheme X* a pair (i,j) for $i = 2, \ldots, a$ and $1 \leq j \leq \alpha_1$ such that $m_{ij} = 0$ and $m_{(i-1)j} = m_{i(j-1)} = 2$.

Definition 3.6. We call *corner of a fat points scheme X* a pair (r,s) such that (r,k) and (h,s) are base corners for some h and k.

Definition 3.7. If we put

$$(u_{ij}, v_{ij}) = (m_{1j} + \cdots + m_{(i-1)j}, m_{i1} + \cdots + m_{i(j-1)}),$$

we call *form of bidegree (u_{ij}, v_{ij}) relative to the pair (i,j)* the following form:

$$F_{ij} := R_1^{m_{1j}} \cdot R_2^{m_{2j}} \cdot \ldots \cdot R_{i-1}^{m_{(i-1)j}} \cdot L_1^{m_{i1}} \cdot \ldots \cdot L_{j-1}^{m_{i(j-1)}}.$$

Using the previous terminology, we also define the corners for the scheme X_{ij}.

Definition 3.8. We call *corners of a fat points scheme X_{ij}*:
(a) the pair $(i, j+1)$;
(b) the corners (r, s) of X such that $m_{rs} = 0$ for X_{ij};
(c) the pairs (h, k) such that $m_{hk} = 0$ for which there exist a (base) corner (t, k) of X with $t < h$, such that $m_{lk} \neq 0$ for all $l \leq h$.

Remark 3.9. We notice that the corners of X_{ij} are those of $X_{i(j+1)}$ plus the pair $(i, j+1)$ and except $(i+1, j+1)$ if it was not of type (b), or $(i, j+2)$ if it was not of type (b) or (c).

Hence, if $(i+1, j+1)$ is a corner of $X_{i(j+1)}$ and is not a corner of X, then it is not a corner for X_{ij}.

Remark 3.10. We may observe that the base corners are of type $(i, \alpha_i + 1)$ for some $i = 2, \ldots, a$. Let us call them

$$(a_1, b_1), \ldots, (a_n, b_n),$$

where (a_1, b_1) is obtained starting from the top of X. Then, all the other corners are of type

$$(a_2, b_1), (a_3, b_2), (a_3, b_1), \ldots, (a_n, b_{n-1}), \ldots, (a_n, b_2), (a_n, b_1).$$

For short, we will use the term corner both for base corners and corners; hence all the corners are $\binom{n+1}{2}$.

Moreover, the forms relative to the corners (a_t, b_z) with $t \geq z$ are of type:

$$F_{(a_t, b_z)} := R_1^2 \cdot \ldots \cdot R_{a_z - 1}^2 \cdot R_{a_z} \cdot \ldots \cdot R_{a_t - 1} \cdot L_1^2 \cdot \ldots \cdot L_{b_z - 1}^2 \cdot L_{b_t} \cdot \ldots \cdot L_{b_z - 1}.$$

Of course, for $t = z$ we have

$$F_{(a_t, b_t)} := R_1^2 \cdot \ldots \cdot R_{a_t - 1}^2 \cdot L_1^2 \cdot \ldots \cdot L_{b_t - 1}^2.$$

Remark 3.11. Let X be a homogeneous scheme of double points, Y its completion to a CI and X_{ij} as before. We may observe that if $m_{i(j+1)} = 1$, then

$$R_i \cdot F_{i(j+1)} = H \cdot F_{(i+1)(j+1)}, \tag{40}$$

$$L_{j+1} \cdot F_{i(j+1)} = K \cdot F_{i(j+2)}, \tag{41}$$

where H and K are suitable bigraded forms.

In fact, from definition we have

$$F_{i(j+1)} := R_1^{m_{1(j+1)}} \cdot R_2^{m_{2(j+1)}} \cdot \ldots \cdot R_{i-1}^{m_{(i-1)(j+1)}} \cdot L_1^{m_{i1}} \cdot \ldots \cdot L_j^{m_{ij}}$$

$$F_{i(j+2)} := R_1^{m_{1(j+2)}} \cdot R_2^{m_{2(j+2)}} \cdot \ldots \cdot R_{i-1}^{m_{(i-1)(j+2)}} \cdot L_1^{m_{i1}} \cdot \ldots \cdot L_{j+1}^{m_{i(j+1)}}$$

and

$$F_{(i+1)(j+1)} := R_1^{m_1(j+1)} \cdot R_2^{m_2(j+1)} \cdot \ldots \cdot R_i^{m_{i(j+1)}} \cdot L_1^{m_{(i+1)1}} \cdot \ldots \cdot L_j^{m_{(i+1)j}}.$$

But, for such fat points schemes, it is $m_{hs} = 2$ or 1 for all (h, s); hence,

$$m_{hs} \geq m_{h(s+1)}$$

and

$$m_{hs} \geq m_{(h+1)s}.$$

So,

$$R_i \cdot F_{i(j+1)} = H \cdot F_{(i+1)(j+1)}$$

where H is equal to 1 or is a form of type $(0, p)$ for some positive integer p, and

$$L_{j+1} \cdot F_{i(j+1)} = K \cdot F_{i(j+2)}$$

where K is equal to 1 or is a form of type $(q, 0)$ for some positive integer q.

Lemma 3.12. *Let X be a homogeneous scheme of double points, Y its completion to a CI and X_{ij} as in Definition 3.3. Then*

1. $F_{i(j+2)}$ *is multiple of* $F_{i(j+1)}$ *if and only if* $(i, j+2)$ *is not a corner of* $X_{i(j+1)}$ *of type (b) or (c).*
2. *If* $(i+1, j+1)$ *is a corner of* $X_{i(j+1)}$ *and is not a corner of X, then* $F_{(i+1)(j+1)}$ *is a multiple of* $F_{i(j+1)}$.

Proof. (1) Let us suppose that $(i, j+2)$ is such that there exist corners $(t, j+2)$ of X with $t \leq i$. Let $(h, j+2)$ and (r, k) with $h < r \leq i$ and $k < j+2$ be base corners of X. We have

$$F_{i(j+2)} := R_1^2 \cdot \ldots \cdot R_{h-1}^2 \cdot R_h \cdot \ldots \cdot R_{i-1} \cdot L_1^2 \cdot \ldots \cdot L_{k-1}^2 \cdot L_k \cdot \ldots \cdot L_{j+1}$$

and

$$F_{i(j+1)} := R_1^2 \cdot \ldots \cdot R_{r-1}^2 \cdot R_r \cdot \ldots \cdot R_{i-1} \cdot L_1^2 \cdot \ldots \cdot L_{k-1}^2 \cdot L_k \cdot \ldots \cdot L_j.$$

Then $F_{i(j+2)}$ is not multiple of $F_{i(j+1)}$.

Let us suppose that there are no corners $(t, j+2)$ of X with $t < i$.

Put $l = \min\{r \mid m_{r(j+2)} = 1\}$ and consider (l, k) with $k < j+2$ the base corner of X. We get

$$F_{i(j+2)} := R_1^2 \cdot \ldots \cdot R_{l-1}^2 \cdot R_l \cdot \ldots \cdot R_{i-1} \cdot L_1^2 \cdot \ldots \cdot L_{k-1}^2 \cdot L_k \cdot \ldots \cdot L_{j+1}$$

and

$$F_{i(j+1)} := R_1^2 \cdot \ldots \cdot R_{l-1}^2 \cdot R_l \cdot \ldots \cdot R_{i-1} \cdot L_1^2 \cdot \ldots \cdot L_{k-1}^2 \cdot L_k \cdot \ldots \cdot L_j.$$

Then $F_{i(j+2)} = F_{i(j+1)} \cdot L_{j+1}$.

(2) Since $(i+1, j+1)$ is a corner of $X_{i(j+1)}$ and is not a corner of X, there exist corners of X of type $(t, j+1)$ with $t \leq i$ but there is no corner in the $(1,0)$-line $i+1$, then $\alpha_i = \alpha_{i+1}$.

Let us take the base corner $(h, j+1)$ with $h < i$, we have

$$F_{(i+1)(j+1)} := R_1^2 \cdot \ldots \cdot R_{h-1}^2 \cdot R_h \cdot \ldots \cdot R_i \cdot L_1^2 \cdot \ldots \cdot L_{\alpha_{i+1}}^2 \cdot L_{\alpha_{i+1}+1} \cdot \ldots \cdot L_j$$

and

$$F_{i(j+1)} := R_1^2 \cdot \ldots \cdot R_{h-1}^2 \cdot R_h \cdot \ldots \cdot R_{i-1} \cdot L_1^2 \cdot \ldots \cdot L_{\alpha_i}^2 \cdot L_{\alpha_i+1} \cdot \ldots \cdot L_j.$$

Then $F_{(i+1)(j+1)} = F_{i(j+1)} \cdot R_i$. $\square$

We need an other general result:

Proposition 3.13. *Let X be a set of points in $\mathbb{P}_k^r$ and P a point of X.*
Set $X' = X \setminus P$ and let u be the integer such that

$$\Delta H(X', u) = \Delta H(X, u) - 1.$$

Let $f \in (I(X'))_u \setminus (I(X))_u$, then

$$I(X') = (I(X), f).$$

Proof. We know that

$$(I(X), f) \subseteq I(X'),$$

it remains to prove that

$$I(X') \subseteq (I(X), f) := J.$$

Claim. $J = I(X')$.

Since $J \subseteq I(X')$, our claim is equivalent to prove that

$$\dim J_n = \dim (I(X'))_n \quad \forall n \in \mathbb{N}$$

From the hypothesis we have that for all $n < u$

$$J_n = (I(X))_n = (I(X'))_n.$$

If $n \geq u$

$$\dim (I(X))_n \leq \dim J_n \leq \dim (I(X'))_n$$

Moreover, if $(I(X))_n = \langle f_1, \ldots, f_r \rangle$ then $f_1, \ldots, f_r, f \cdot h$, for a suitable form $h \in R_{n-u}$, are l.i., so

$$\dim J_n \geq \dim (I(X))_n + 1 = \dim (I(X'))_n$$

hence we are done. $\square$

Remark 3.14. We may observe that Proposition 3.13 is true for every scheme $X' \subseteq X$ such that $\deg X' = \deg X - 1$.

Let X, Y and X_{ij} as above. Then

Theorem 3.15. *A m.s.o.g. for a homogeneous scheme of double points X consists of the generators of the fat points scheme Y, completion of X to a C.I., plus the $\binom{n+1}{2}$ forms F_{hs}, where (h, s) runs among all the corners of X.*

Proof. From Definition 3.3, we have that

$$|X_{ij}| = |Y| - N$$

where $N = \sum_{t=i+1}^{a} (\alpha_1 - \alpha_t) - (\alpha_1 - j)$.

Claim. *A m.s.o.g. for X_{ij} consists of the generators of Y and the forms F_{hs} where (h, s) runs among the corners of X_{ij} for $i = 2, \ldots, a$ and $j = \alpha_i + 1, \ldots, \alpha_1$.*

Proof. We use induction on N. If $N = 0$, that is for $i = a$ and $j = \alpha_1$, then $X_{ij} = X_{a\alpha_1} = Y$ and the conclusion is true (see Remark 3.2).

Let us suppose $N > 0$ and the claim true for fat points schemes X_{hk} for which $N_{hk} < N$ and let us prove it for X_{ij}.

We have

$$X_{ij} = X_{i(j+1)} \setminus P_{i(j+1)} \tag{42}$$

and

$$
\begin{aligned}
|X_{i(j+1)}| &= |Y| - \sum_{t=i+1}^{a} (\alpha_1 - \alpha_t) - (\alpha_1 - j - 1) \\
&= |Y| - \sum_{t=i+1}^{a} (\alpha_1 - \alpha_t) - (\alpha_1 - j) + 1 \\
&= |Y| - (N - 1),
\end{aligned}
$$

hence

$$N_{X_{i(j+1)}} = N - 1$$

and the inductive hypothesis holds for $X_{i(j+1)}$, then a m.s.o.g. for $X_{i(j+1)}$ consists of the generators G_p of Y and the forms F_{hs} where (h, s) runs among the corners of $X_{i(j+1)}$.

Using Remark 3.9 and Lemma 3.12 we can show that a m.s.o.g. for X_{ij} is

$$\{G_p\}, \{F_{hs}\} \quad \text{and} \quad F_{i(j+1)} \tag{43}$$

if $(i + 1, j + 1)$ is a corner of type (b) and $(i, j + 2)$ is a corner of type (b) or (c).

If $(i + 1, j + 1)$ is not of type (b) then a m.s.o.g. can be obtained by deleting $F_{(i+1)(j+1)}$ from (43).

Analogously, if $(i, j + 2)$ is not of type (b) or (c) then a m.s.o.g. can be obtained by deleting $F_{i(j+2)}$ from (43).

In fact, from Proposition 3.13, we know that the generators of $X_{i(j+1)}$ plus $F_{i(j+1)}$ are a system of generators of X_{ij}. It remains to prove the minimality of this set of generators.

Let us suppose that

$$G_q = \sum_{p \neq q} A_p G_p + \sum_{(h,s)} C_{hs} F_{hs} + B F_{i(j+1)} \tag{44}$$

or

$$F_{rt} = \sum_{p} A_p G_p + \sum_{(h,s) \neq (r,t)} C_{hs} F_{hs} + B F_{i(j+1)}, \tag{45}$$

where $(r,t) \neq (i, j+2)$ and $(r,t) \neq (i+1, j+1)$.

Computing (44) and (45) in $P_{i(j+1)}$ we have

$$G_p(P_{i(j+1)}) = 0 \quad \forall p,$$

$$F_{hs}(P_{i(j+1)}) = 0 \quad \forall (h,s),$$

since $P_{i(j+1)} = R_i \cap L_{j+1}$ and F_{hs} is of type

$$F_{hs} := R_1^{m_{1s}} \cdot \ldots \cdot R_i^{m_{is}} \cdot \ldots \cdot R_{h-1}^{m_{(h-1)s}} \cdot L_1^{m_{h1}} \cdot \ldots \cdot L_{s-1}^{m_{h(s-1)}}.$$

Hence,

$$B(P_{i(j+1)}) F_{i(j+1)}(P_{i(j+1)}) = 0 \tag{46}$$

and, since

$$F_{i(j+1)}(P_{i(j+1)}) \neq 0,$$

it must be

$$B(P_{i(j+1)}) = 0,$$

that is

$$B \in (R_i, L_{j+1}).$$

It means that B is of type

$$B = f_1 R_i + f_2 L_{j+1}$$

with f_1 and f_2 bigraded forms.

Using Remark 3.11,

$$R_i F_{i(j+1)} = H F_{(i+1)(j+1)}$$

and

$$L_{j+1} F_{i(j+1)} = K F_{i(j+2)}$$

for suitable H and K bigraded forms.

Hence from (44) and (45) we have that

$$\{G_p\}, \{F_{hs}\}$$

should not be a m.s.o.g. for $X_{i(j+1)}$ and this is not possible.

Let us suppose that

$$F_{(i+1)(j+1)} = \sum_p A_p G_p + \sum_{(h,s)\neq(i+1,j+1)} C_{hs} F_{hs} + B F_{i(j+1)}, \tag{47}$$

where G_p are the generators of Y, A_p and B are opportune bigraded forms and (h,s) runs among the corners of X_{ij}.

Computing (47) in $P_{i(j+1)}$ and using Remark 3.11, we get

$$(1 - f_1 H)F_{(i+1)(j+1)} = \sum_p A_p G_p + \sum_{(h,s)\neq(i+1,j+1)} C'_{hs} F_{hs} \tag{48}$$

for suitable bigraded forms f_1, H, C'_{hs} and A_p.

By the minimality of the generators of $X_{i(j+1)}$, we get

$$1 - f_1 H = 0,$$

hence

$$H = 1,$$

therefore

$$R_i F_{i(j+1)} = F_{(i+1)(j+1)}$$

and this happens when $(i+1, j+1)$ is a corner of $X_{i(j+1)}$ but not a corner of X, i.e. it is not of type (b) (Lemma 3.12).

Analogously, let us suppose that

$$F_{i(j+2)} = \sum_p A_p G_p + \sum_{(h,s)\neq(i,j+2)} C_{hs} F_{hs} + B F_{i(j+1)}. \tag{49}$$

Computing (49) in $P_{i(j+1)}$ and using Remark 3.11, we get

$$(1 - f_2 K)F_{i(j+2)} = \sum_p A_p G_p + \sum_{(h,s)\neq(i,j+2)} \bar{C}_{hs} F_{hs} \tag{50}$$

for suitable bigraded forms f_2, K, A_p and $\bar{C}_{hs}$.

In this case it is

$$K = 1,$$

hence

$$L_{j+1} F_{i(j+1)} = F_{i(j+2)}$$

and this happens iff $(i, j+2)$ is not a corner of type (b) or (c) (Lemma 3.12).

Furthermore, we observe that

$$X_{(i-1)\alpha_1} = X_{i\alpha_i} = X_{i(\alpha_i+1)} \setminus P_{i(\alpha_i+1)}$$

hence the corners of $X_{(i-1)\alpha_1}$ are the same corners of $X_{i\alpha_i}$ and we can work in an analogous way as before.

Hence the claim is proved.

In this way we have finished, since X is the last step of the induction. $\quad\square$

Example 3.16. If X is the homogeneous scheme of double points and Y its completion to the CI $((4,0),(0,4))$ defined in Example 3.1, then the corners of the scheme X are the pairs:

$$(2,4), (4,3), (4,4),$$

since Y is aCM, the corners of ΔM_Y are the pairs

$$(0,8), (1,7), (3,6), (4,4), (5,3), (7,2), (8,0).$$

The generators of the scheme X are

$$G_1 := L_1^2 \cdot L_2^2 \cdot L_3^2 \cdot L_4^2 \quad \text{of bidegree } (0,8);$$
$$G_2 := R_1 \cdot L_1^2 \cdot L_2^2 \cdot L_3^2 \cdot L_4 \quad \text{of bidegree } (1,7)$$
$$G_3 := R_1 \cdot R_2 \cdot R_3 \cdot R_4 \cdot L_1 \cdot L_2 \cdot L_3 \cdot L_4 \quad \text{of bidegree } (4,4)$$
$$G_4 := R_1^2 \cdot R_2 \cdot R_3 \cdot R_4 \cdot L_1 \cdot L_2 \cdot L_3 \quad \text{of bidegree } (5,3)$$
$$G_5 := R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot R_4^2 \quad \text{of bidegree } (8,0)$$
$$G_6 := R_1 \cdot R_2 \cdot R_3 \cdot L_1^2 \cdot L_2^2 \cdot L_3 \cdot L_4 \quad \text{of bidegree } (3,6)$$
$$G_7 := R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot R_4 \cdot L_1 \cdot L_2 \quad \text{of bidegree } (7,2)$$
$$F_{24} := R_1^2 \cdot L_1^2 \cdot L_2^2 \cdot L_3^2 \quad \text{of bidegree } (2,6)$$
$$F_{43} := R_1^2 \cdot R_2^2 \cdot R_3^2 \cdot L_1^2 \cdot L_2^2 \quad \text{of bidegree } (6,4)$$
$$F_{44} := R_1^2 \cdot R_2 \cdot R_3 \cdot L_1^2 \cdot L_2^2 \cdot L_3 \quad \text{of bidegree } (4,5).$$

Acknowledgements

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