Journal of Differential Equations **182**, 121–140 (2002)
doi:10.1006/jdeq.2001.4094

Hölder Continuity for Solutions of Linear Degenerate Elliptic Equations Under Minimal Assumptions

Pietro Zamboni

Dipartimento di Matematica, Universitá di Catania, Viale Andrea Doria 6, 95125 Catania, Italy
E-mail: zamboni@dipmat.unict.it

Received November 17, 2000

In this paper, we study the local regularity of solutions to linear degenerate elliptic equations of the form

$$-\sum_{i,j=1}^{n}(a_{ij}u_{x_i}+d_ju)_{x_j}+\sum_{i=1}^{n}b_iu_{x_i}+cu=f-\sum_{i=1}^{n}(f_i)_{x_i}. \qquad (*)$$

We prove Harnack's inequality for positive solutions and the Hölder continuity of the solutions of $(*)$, assuming on the coefficients of the lower order terms very general hypotheses involving appropriate "degenerate" Morrey spaces.

Key Words: Hölder continuity; Harnack inequality; degenerate elliptic equation; Muckenhoupt class of weights; Kato–Stummel class; Morrey spaces.

1. INTRODUCTION

In [9], Gutiérrez studied the local regularity properties of solutions of degenerate elliptic equations of the form

$$-\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j}+cu=0, \qquad (1.1)$$

where the coefficients $a_{ij}(x)$ are measurable functions such that

$$a_{ij}(x)=a_{ji}(x), \qquad i,j=1,2,\dots,n \qquad (1.2)$$

and satisfy

$$\exists\nu>0\colon\ \nu^{-1}w(x)|\xi|^2\leqslant\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leqslant\nu w(x)|\xi|^2 \quad \text{a.e. } x\in\Omega,\ \ \forall\xi\in\mathbb{R}^n, \qquad (1.3)$$

0022-0396/02 $35.00

where the function w controlling the degeneracy is a weight in the class A_2. Assuming that the function c belongs to the *degenerate* Kato–Stummel class (see Definition 2.3), Gutiérrez proved Harnack's inequality for positive solutions of (1.1) and as a consequence, the local continuity of solutions. In this way, he extended to the degenerate case the sharp result proved by Chiarenza et al. [1].

Subsequently, Vitanza and Zamboni [16] proved the local Hölder continuity of solutions to Eq. (1.1) assuming that the function c is in the *degenerate* Morrey space $M_\sigma(w)$, $\sigma > 0$ (see Definition 2.4), and extending in this way to the degenerate case the results contained in the papers [5, 14].

Recently, De Cicco and Vivaldi [4] studied the regularity of solutions to more general degenerate elliptic equations of the form

$$-\sum_{i,j=1}^{n} (a_{ij}u_{x_i} + d_j u)_{x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = f - \sum_{i=1}^{n} (f_i)_{x_i}, \tag{1.4}$$

proving Harnack's inequality for nonnegative weak solutions and establishing the Hölder continuity of weak solutions. They assumed on the weight w in (1.3) properties typical of the A_2 weights and the coefficients of the lower order terms in some suitable weighted L^p spaces; precisely they supposed

$$\left(\frac{b_i}{w}\right) \in L^{\kappa}(w), \qquad \left(\frac{d_i}{w}\right)^2, \frac{c}{w}, \left(\frac{f_i}{w}\right)^2 \in L^p(w), \qquad p > \frac{\kappa}{2},$$

where κ represents the intrinsic dimension induced on $\mathbb{R}^n$ by weight w (see [3]) and depends on the constant appearing in the duplication formula (see Lemma 2.1(a)). The constant κ coincides with the euclidean dimension n if $w = 1$. We stress that their condition on the coefficient c is stronger than those in [9, 16], see Remark 2.6.

The purpose of this note is to improve the results contained in [4] extending the result obtained in [16] to more general equations like (1.4). Given $w \in A_2$ and Ω an open bounded subset of $\mathbb{R}^n$, that without loss of generality, we assume the ball centered at the origin and radius $R > 0$, we study the local behavior of the solutions to the degenerate elliptic equation (1.4), where the coefficients $a_{ij}(x)$ satisfy the assumptions (1.2) and (1.3). Taking the functions b_i, c, d_i, f and f_i in such a way that

$$\left(\frac{b_i}{w}\right)^2, \frac{c}{w}, \left(\frac{d_i}{w}\right)^2, \frac{f}{w}, \left(\frac{f_i}{w}\right)^2 \in M_\sigma(\Omega, w), \qquad \sigma > 0, \tag{1.5}$$

we prove Harnack's inequality for nonnegative solutions of (1.4) (see Theorem 4.1) and, as a consequence, the Hölder continuity of the solutions of the same equation (see Theorem 5.2). Our hypotheses on the coefficients are more general than those in [4]. In fact, it is shown in Remark 2.6 that $L^p(w) \subset M_\sigma(w)$, for some $\sigma > 0$, if $p > \frac{\kappa}{2}$.

About the technique used, it follows as closely as possible the classical work [13] (see also [17]), and Moser's iteration technique (see [10]). We take, as usual, powers of solutions as test functions; the novelty is that we use Theorem 2.7 to estimate products of the coefficients of the equation times test functions instead of Hölder and Sobolev inequalities.

The organization of the paper is as follows. Section 2 contains some preliminary results. In Section 3, we prove that solutions are locally bounded. Harnack's inequality is proved in Section 4. Finally, in Section 5 we establish the Hölder continuity.

2. FUNCTION SPACES AND PRELIMINARY RESULTS

Let $p > 1$. A function $w : \mathbb{R}^n \to]0, +\infty[$ is an A_p weight, $w \in A_p$, if and only if

$$\sup_B \left(\frac{1}{|B|}\int_B w(x)\,dx\right)\left(\frac{1}{|B|}\int_B [w(x)]^{-\frac{1}{(p-1)}}\,dx\right)^{p-1} = C_0 < +\infty \qquad (2.1)$$

for all balls B in $\mathbb{R}^n$; C_0 is called the A_p constant of w.

We now recall some results about A_p weights (see [2, 8] for the proof).

LEMMA 2.1. *Let $w(x) \in A_p$, $p \in]1, +\infty[$. Then*

(a) *Doubling: There exist positive constants C_d and κ such that*

$$w(B(x_0, tr)) \leqslant C_d t^\kappa w(B(x_0, r))$$

for every $x_0 \in \mathbb{R}^n$, $r > 0$, and $t \geqslant 1$, where $w(B_r) = \int_{B_r} w(x)\,dx$;[1] *and consequently*

(b) *Reverse doubling: There exist positive constants C_a and μ such that*

$$w(B(x_0, tr)) \geqslant C_a t^\mu w(B(x_0, r))$$

for every $x_0 \in \mathbb{R}^n$, $r > 0$, and $t \geqslant 1$.

[1] We write $B(x, r)$ to indicate the Euclidean ball centered at x with radius r. Whenever x is not relevant we will write B_r.

LEMMA 2.2 [See Fabes et al. [7]]. *Let $w \in A_2$, $B \subset \mathbb{R}^n$ an arbitrary ball, then*

(a) *for all $\phi \in C_0^\infty(B)$, we have*

$$\left(\frac{1}{w(B)}\int_B \phi^{2\tau} w\,dx\right)^{\frac{1}{2\tau}} \leqslant C_1(\operatorname{diam} B)\left(\frac{1}{w(B)}\int_B |\nabla\phi|^2 w\,dx\right)^{\frac{1}{2}}$$

for some $\tau > 1$, independent of ϕ and B;

(b) *for all $\phi \in C^1(\bar{B})$, we have*

$$\left(\frac{1}{w(B)}\int_B |\phi - \phi_B|^2 w\,dx\right)^{\frac{1}{2}} \leqslant C_2(\operatorname{diam} B)\left(\frac{1}{w(B)}\int_B |\nabla\phi|^2 w\,dx\right)^{\frac{1}{2}},$$

where $\phi_B = \frac{1}{w(B)}\int_B \phi w\,dx$.

Let Ω be an open bounded set in $\mathbb{R}^n$. Because of the local character of our results it is sufficient to assume $\Omega = B(0, R)$.

Let $w \in A_2$. We give the definitions of the weighted spaces $L^p(\Omega, w)$, $H^{1,p}(\Omega, w)$, $H^{1,p}_{loc}(\Omega, w)$, $H^{1,p}_0(\Omega, w)$, $H^{-1,p}(\Omega, w)$, $p \in [1, +\infty[$ (see also [6]).

$L^p(\Omega, w)$ is the space of measurable u in Ω, such that

$$\|u\|_{L^p(\Omega,w)} = \left(\int_\Omega |u(x)|^p w(x)\,dx\right)^{\frac{1}{p}} < +\infty.$$

Lip $(\bar{\Omega})$ denotes the class of Lipschitz functions in $\bar{\Omega}$. Lip_0 (Ω) denotes the class of functions $\phi \in \text{Lip}\,(\bar{\Omega})$ with compact support contained in Ω. If ϕ belongs to Lip $(\bar{\Omega})$ we can define the norm

$$\|\phi\|_{H^{1,p}(\Omega,w)} := \|\phi\|_{L^p(\Omega,w)} + \sum_{i=1}^n \|\phi_{x_i}\|_{L^p(\Omega,w)}. \tag{2.2}$$

$H^{1,p}(\Omega, w)$ denotes the closure of Lip $(\bar{\Omega})$ under norm (2.2). We say that $u \in H^{1,p}_{loc}(\Omega, w)$ if $u \in H^{1,p}(\Omega', w)$ for every $\Omega' \subset\subset \Omega$. $H^{1,p}_0(\Omega, w)$ denotes the closure of Lip_0 (Ω) under norm (2.2). $H^{-1,p'}(\Omega, w)$ is the dual space of $H^{1,p}_0(\Omega, w)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We have $T \in H^{-1,p'}(\Omega, w)$ if there exist f_i such that $\frac{f_i}{w} \in L^{p'}(\Omega, w)$, $i = 1, 2, \ldots, n$, with $T = \sum_{i=1}^n (f_i)_{x_i}$.

We now give some more definitions.

DEFINITION 2.3 [See Gutiérrez [9]]. Let Ω be a bounded domain of R^n and $w \in L^1_{loc}(\Omega)$. We set

$$S(\Omega, w) = \left\{ V \in L^1(\Omega, w): \sup_{\substack{x\in\Omega \\ 0<r<2R}} \int_{\{y\in\Omega:\ |x-y|<r\}} |V(y)| \right.$$
$$\left. \times \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))} ds\, w(y)\, dy = \eta(r) \to 0 \text{ for } r \to 0 \right\}.$$

DEFINITION 2.4 [See Vitanza and Zamboni [16]]. Let $\sigma \in \mathbb{R}$. We set

$$M_\sigma(\Omega, w) = \left\{ V \in L^1(\Omega, w): \|V\|_{\sigma,\Omega} = \sup_{\substack{x\in\Omega \\ 0<r<2R}} \frac{1}{r^\sigma} \int_{\{y\in\Omega:\ |x-y|<r\}} |V(y)| \right.$$
$$\left. \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))} ds\, w(y)\, dy < +\infty \right\}.$$

Remark 2.5. We note that in the nondegenerate case, i.e. $w = 1$, $S(\Omega, w)$ and $M_\sigma(\Omega, w)$ coincide with the classical Kato–Stummel class and Morrey space $L^{1,\lambda}$, for some appropriate λ, respectively. In particular, for $\sigma > 0$, we obtain $L^{1,\lambda}$ with $\lambda = n - 2 + \sigma$.

Remark 2.6. The inclusion $M_\sigma(\Omega, w) \subset S(\Omega, w)$, $\sigma > 0$, is trivial. We now compare $M_\sigma(\Omega, w)$ with the space $L^p(\Omega, w)$ for $p > \frac{\kappa}{2}$. We shall show that there exists $\sigma > 0$ such that

$$L^p(\Omega, w) \subset M_\sigma(\Omega, w).$$

Indeed, for every $x \in \Omega$ and $0 < r < 2R$, we have

$$\int_{\{y\in\Omega:\ |x-y|<r\}} |V(y)| \int_{|x-y|}^{4R} \frac{s}{w(B(x,s))} ds\, w(y)\, dy$$
$$= \int_{\{y\in\Omega:\ |x-y|<r\}} |V(y)| \int_{|x-y|}^{r} \frac{s}{w(B(x,s))} ds\, w(y)\, dy$$
$$+ \int_{\{y\in\Omega:\ |x-y|<r\}} |V(y)| \int_{r}^{4R} \frac{s}{w(B(x,s))} ds\, w(y)\, dy = \mathrm{I} + \mathrm{II}. \quad (2.3)$$

By Fubini's theorem, Hölder's inequality and Lemma 2.1(a) we have

$$\mathrm{I}=\int_0^r \frac{s}{w(B(x,s))}\int_{\{y\in\Omega:\ |x-y|<s\}}|V(y)|w(y)\,dy\,ds$$

$$\leqslant \|V\|_{L^p(\Omega,w)}\int_0^r \frac{s}{w(B(x,s))^{\frac{1}{p}}}\,ds \leqslant C(C_d,R)\|V\|_{L^p(\Omega,w)}\int_0^r s^{1-\frac{\kappa}{p}}\,ds$$

$$=C(C_d,R)\|V\|_{L^p(\Omega,w)}r^{2-\frac{\kappa}{p}}. \tag{2.4}$$

We now estimate II. We write

$$\mathrm{II}=\int_r^{4R}\frac{s}{w(B(x,s))}\,ds\int_{\{y\in\Omega:\ |x-y|<r\}}|V(y)|w(y)\,dy=AB. \tag{2.5}$$

By Hölder's inequality

$$B\leqslant \|V\|_{L^p(\Omega,w)}\,w(B(x,r))^{\frac{1}{p'}}.$$

To estimate B, pick q such that $p>q>\kappa/2$ and write

$$A=\int_r^{4R}\frac{s}{w(B(x,s))^{\frac{1}{q}}\,w(B(x,s))^{\frac{1}{q'}}}\,ds.$$

Since $s<4R$, by Lemma 2.1(a) it follows that $w(B(x,s))\geqslant C(C_d,R)s^{\kappa}$, and consequently,

$$A\leqslant \frac{1}{C(C_d,R)w(B(x,r))^{\frac{1}{q'}}}\int_r^{4R}s^{1-\frac{\kappa}{q}}\,ds.$$

Therefore,

$$\mathrm{II}\leqslant C(C_d,R,\|V\|_{L^p(\Omega,w)})w(B(x,r))^{\varepsilon}$$

with $\varepsilon=\frac{1}{q}-\frac{1}{p}$. On the other hand, by reverse doubling, Lemma 2.1(b), we have $w(B(x,r))\leqslant C(C_a,R)r^{\mu}$, and consequently, we obtain

$$\mathrm{II}\leqslant Cr^{\varepsilon\mu}.$$

The desired inclusion then follows with $\sigma=\min\{2-\frac{\kappa}{p},\varepsilon\mu\}$.

In general, we have $M_\sigma(\Omega,w)\not\equiv L^p(\Omega,w)$ for which it is sufficient to note that if $w\equiv 1$ then $\kappa\equiv n$, $M_\sigma(\Omega,1)\equiv L^{1,n-2+\sigma}(\Omega)$ and it is known (see [12]) that $L^p(\Omega)L^{1,n-2+\varepsilon}(\Omega)$ with $p>\frac{n}{2}$.

The following theorem (see [9, Lemma 3.3]) is the main tool in the proofs of the next sections.

THEOREM 2.7. *Let* $V: \Omega \to \mathbb{R}^n$ *be a function such that* $\frac{V}{w} \in M_\sigma(\Omega, w)$. *Then for any* $0 < \varepsilon < 1$ *there exists* $\delta > 0$ *such that*

$$\int_\Omega |V(x)| u^2(x)\, dx \leqslant \varepsilon \int_\Omega |\nabla u(x)|^2 w(x)\, dx + C\varepsilon^{-\delta} \int_\Omega u^2(x) w(x)\, dx$$

for all $u \in C_0^\infty(\Omega)$, *where* C *is a constant depending on* v, σ, n *and* $\|\frac{V}{w}\|_{\sigma,\Omega}$.

Proof. It is sufficient to note that $\frac{V}{w} \in S(\Omega, w)$ with $\eta(r) \sim \|\frac{V}{w}\|_{\sigma,\Omega} r^\sigma$. Then by Lemma 3.3, p. 408 of [9], it follows that for any $0 < \varepsilon < 1$

$$\int_\Omega |V(x)| u^2(x)\, dx \leqslant \varepsilon \int_\Omega |\nabla u(x)|^2 w(x)\, dx + \rho(\varepsilon)^{-2-n} \int_\Omega u^2(x) w(x)\, dx.$$

Fixing $\rho(\varepsilon)$ in such a way that

$$C(v)\left(\left\|\frac{V}{w}\right\|_{\sigma,\Omega} \rho^\sigma\right)^2 = \frac{\varepsilon}{2},$$

the conclusion follows with $\delta = \frac{n+2}{2\sigma}$. ∎

3. LOCAL BOUNDEDNESS OF SOLUTIONS

Let $a_{ij}(x)$ be measurable functions such that (1.2) and (1.3) hold, and b_i, c, d_i, f and f_i $(i = 1, 2, \ldots, n)$ real functions defined in Ω such that

$$\left(\frac{b_i}{w}\right)^2, \frac{c}{w}, \left(\frac{d_i}{w}\right)^2, \frac{f}{w}, \left(\frac{f_i}{w}\right)^2 \in M_\sigma(\Omega, w), \qquad \sigma > 0. \tag{3.1}$$

We will say that $u \in H^{1,2}_{loc}(\Omega, w)$ is a *local weak solution* of the equation

$$-\sum_{i,j=1}^n (a_{ij} u_{x_i} + d_j u)_{x_j} + \sum_{i=1}^n b_i u_{x_i} + cu = f - \sum_{i=1}^n (f_i)_{x_i} \tag{3.2}$$

if

$$\int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} u_{x_i} \psi_{x_j} + \sum_{j=1}^n d_j u \psi_{x_j} + \sum_{i=1}^n d_i u_{x_i} \psi + cu\psi \right\} dx$$
$$= \int_\Omega \left\{ f\psi + \sum_{i=1}^n f_i \psi_{x_i} \right\} dx \qquad \forall \psi \in C_0^\infty(\Omega). \tag{3.3}$$

We note that (3.3) is meaningful by Theorem 2.7.

The purpose of this section is to show that weak solutions of Eq. (3.2) are locally bounded. To do this we will follow the technique by Serrin [13].

THEOREM 3.1. *Let u be a weak solution of Eq.* (3.2) *defined in some ball $B_{2r} \subset\subset \Omega$. We assume that conditions* (1.2), (1.3) *and* (3.1) *hold. Then there exists a positive constant C, depending on v, σ, n, $r^\sigma \sum_{i=1}^n \|(\frac{b_i}{w})^2\|_{\sigma,B_{2r}}$, $r^\sigma\|\frac{c}{w}\|_{\sigma,B_{2r}}$, $r^\sigma \sum_{i=1}^n \|(\frac{d_i}{w})^2\|_{\sigma,B_{2r}}$, C_1 and τ, such that*

$$\|u\|_{L^\infty(B_r,w)} \leqslant C\left\{\left(\frac{1}{w(B_{2r})}\int_{B_{2r}} |u|^2 w\,dx\right)^{\frac{1}{2}} + r^\sigma\left\|\frac{f}{w}\right\|_{\sigma,B_{2r}} + \left(r^\sigma \sum_{i=1}^n \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_{2r}}\right)^{\frac{1}{2}}\right\},$$

where B_r is a ball with the same center of B_{2r}.

Proof. We prove the theorem when $r = 1$ with the solution correspondingly defined in B_2. The general case $r \neq 1$ follows by dilations.

Let $q \geqslant 1$, h a positive number that will be determined later and $l > h$. We consider the function

$$G(u) = \operatorname{sign} u\{F(v)F'(v) - qh^{2q-1}\}, \quad u \in]-\infty, +\infty[,$$

where

$$v = |u| + h$$

and

$$F(v) = \begin{cases} v^q & \text{if } h \leqslant v \leqslant l, \\ ql^{q-1}v - (q-1)l^q & \text{if } l \leqslant v. \end{cases}$$

Let $\psi(x) = \eta^2(x)G(u)$, where $\eta(x) \in C_0^\infty(\Omega)$ is such that $0 \leqslant \eta(x) \leqslant 1$ and $\operatorname{supp}\eta(x) \subseteq B_2$. Substituting ψ in (3.3) yields

$$\sum_{i,j=1}^n \int_\Omega a_{ij}u_{x_i}[2\eta\eta_{x_j}G(u) + \eta^2 G'(u)u_{x_j}]\,dx + \sum_{j=1}^n \int_\Omega d_j u[2\eta\eta_{x_j}G(u) + \eta^2 G'(u)u_{x_j}]\,dx$$

$$+\ \sum_{i=1}^{n}\ \int_\Omega b_i u_{x_i}\eta^2 G(u)\,dx + \int_\Omega cu\eta^2 G(u)\,dx$$

$$=\ \int_\Omega f\eta^2 G(u)\,dx + \sum_{i=1}^{n}\ \int_\Omega f_i[2\eta\eta_{x_i}G(u) + \eta^2 G'(u)u_{x_i}]\,dx.$$

Using (1.2) and (1.3) we obtain

$$\begin{aligned}
&v^{-1}\ \int_\Omega |\nabla u|^2\eta^2 G'(u)w\,dx\\
&\leqslant 2v\int_\Omega |\nabla u|\,|\nabla\eta|\eta\,|G(u)|w\,dx\\
&\quad+\ \sum_{j=1}^{n}\ \int_\Omega |d_j|\,|u|\eta^2|G'(u)|\,|u_{x_j}|\,dx + 2\ \sum_{j=1}^{n}\ \int_\Omega |d_j|\,|u|\eta|\eta_{x_j}|\,|G(u)|\,dx\\
&\quad+\ \sum_{i=1}^{n}\ \int_\Omega |b_i|\,|u_{x_i}|\eta^2|G(u)|\,dx + \int_\Omega |c|\,|u|\eta^2|G(u)|\,dx + \int_\Omega |f|\eta^2|G(u)|\,dx\\
&\quad+\ 2\sum_{i=1}^{n}\ \int_\Omega |f_i|\eta|\eta_{x_i}|\,|G(u)|\,dx + \sum_{i=1}^{n}\ \int_\Omega |f_i|\eta^2|u_{x_i}|\,|G'(u)|\,dx.
\end{aligned}$$

Recalling that $v = |u| + h$ and using that (see [13, Theorem 1, p. 257])

$$G' = \begin{cases} (2-1/q)(F')^2 & \text{if } |u| < l-h,\\ (F')^2 & \text{if } |u| > l-h, \end{cases}$$

$$|G| \leqslant F(F'),$$
$$vF' \leqslant qF,$$

we obtain

$$\begin{aligned}
&\int_\Omega |\nabla v|^2\eta^2(F')^2 w\,dx\\
&\leqslant 2v^2\int_\Omega |\nabla v|\,|\nabla\eta|\eta F(F')w\,dx\\
&\quad+\ \left(2-\frac{1}{q}\right)qv\ \sum_{j=1}^{n}\ \int_\Omega |d_j|\eta^2 F(F')|v_{x_j}|\,dx + 2qv\ \sum_{j=1}^{n}\ \int_\Omega |d_j|\eta|\eta_{x_j}|F^2\,dx
\end{aligned}$$

$$+\ v\sum_{i=1}^{n}\int_{\Omega}|b_i|\,|v_{x_i}|\eta^2F(F')\,dx+vq\int_{\Omega}|c|\eta^2F^2\,dx+h^{-1}vq\int_{\Omega}|f|\eta^2F^2\,dx$$
$$+\ 2qvh^{-1}\sum_{i=1}^{n}\int_{\Omega}|f_i|\eta|\eta_{x_i}|F^2\,dx$$
$$+\ \left(2-\frac{1}{q}\right)qvh^{-1}\sum_{i=1}^{n}\int_{\Omega}|f_i|\eta^2|v_{x_i}|F(F')\,dx.$$

Using the inequality

$$ab\leqslant\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2,\qquad \varepsilon>0$$

and setting

$$V=\sum_{i=1}^{n}\frac{|b_i|^2}{w}+|c|+\sum_{j=1}^{n}\frac{|d_j|^2}{w}+h^{-1}|f|+h^{-2}\sum_{i=1}^{n}\frac{|f_i|^2}{w},$$

it follows that

$$\int_{\Omega}|\nabla v|^2\eta^2(F')^2w\,dx\leqslant C(v)q^2\left\{\int_{\Omega}|\nabla\eta|^2F^2w\,dx+\int_{\Omega}V\eta^2F^2\,dx\right\}. \quad (3.4)$$

We observe now that $\frac{V}{w}\in M_\sigma(\Omega,w)$ and

$$\left\|\frac{V}{w}\right\|_{\sigma,B_2}\leqslant\sum_{i=1}^{n}\left\|\left(\frac{b_i}{w}\right)^2\right\|_{\sigma,B_2}+\left\|\frac{c}{w}\right\|_{\sigma,B_2}+\sum_{j=1}^{n}\left\|\left(\frac{d_j}{w}\right)^2\right\|_{\sigma,B_2}$$
$$+\ h^{-1}\left\|\frac{f}{w}\right\|_{\sigma,B_2}+h^{-2}\sum_{i=1}^{n}\left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_2}.$$

Thus, setting

$$h=\left\|\frac{f}{w}\right\|_{\sigma,B_2}+\left(\sum_{i=1}^{n}\left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_2}\right)^{\frac{1}{2}}$$

we obtain

$$\left\|\frac{V}{w}\right\|_{\sigma,B_2}\leqslant\sum_{i=1}^{n}\left\|\left(\frac{b_i}{w}\right)^2\right\|_{\sigma,B_2}+\left\|\frac{c}{w}\right\|_{\sigma,B_2}+\sum_{j=1}^{n}\left\|\left(\frac{d_j}{w}\right)^2\right\|_{\sigma,B_2}+2.$$

If we set

$$U(x) = F(v),$$

then from (3.4) it follows that

$$\int_\Omega \eta^2 |\nabla U|^2 w\,dx \leqslant C(v) q^2 \left\{ \int_\Omega |\nabla \eta|^2 U^2 w\,dx + \int_\Omega V \eta^2 U^2\,dx \right\}$$

and using Theorem 2.7 we get

$$\int_\Omega \eta^2 |\nabla U|^2 w\,dx \leqslant C(v) q^2 \Bigg\{ (1+\varepsilon) \int_\Omega |\nabla \eta|^2 U^2 w\,dx + \varepsilon \int_\Omega \eta^2 |\nabla U|^2 w\,dx. + C\left(v, \sigma, n, \left\| \frac{V}{w} \right\|_{\sigma, B_2} \right) \varepsilon^{-\delta} \int_\Omega \eta^2 U^2 w\,dx \Bigg\} \quad \forall 0 < \varepsilon < 1.$$

By choosing $\varepsilon = \frac{1}{2C(v)q^2}$, we obtain

$$\int_\Omega \eta^2 |\nabla U|^2 w\,dx \leqslant C \left\{ q^2 \int_\Omega |\nabla \eta|^2 U^2 w\,dx + + q^{2+2\delta} \int_\Omega \eta^2 U^2 w\,dx \right\}, \tag{3.5}$$

where C is a positive constant depending on v, σ, n and $\|\frac{V}{w}\|_{\sigma,B_2}$.
From Lemma 2.2(a) and (3.5) we have

$$\left(\int_{B_2} |\eta U|^{2\tau} w\,dx \right)^{\frac{1}{\tau}} \leqslant C w(B_2)^{\frac{1}{\tau}-1} \left\{ q^2 \int_{B_2} |\nabla \eta|^2 U^2 w\,dx + q^{2+2\delta} \int_{B_2} |\eta U|^2 w\,dx \right\} \tag{3.6}$$

with $\tau > 1$ and C depending on v, σ, n, $\|\frac{V}{w}\|_{\sigma,B_2}$, and the constant C_1 in Lemma 2.2(a).

Let r_1 and r_2 be such that $1 \leqslant r_1 \leqslant r_2 \leqslant 2$, choosing $\eta(x)$ in such a way that $\eta(x) = 1$ in B_{r_1}, $0 \leqslant \eta(x) \leqslant 1$ in B_{r_2} and $|\nabla \eta| \leqslant \frac{2}{r_2 - r_1}$ we obtain

$$\left(\int_{B_{r_1}} U^{2\tau} w\,dx \right)^{\frac{1}{\tau}} \leqslant C w(B_2)^{\frac{1}{\tau}-1} \frac{1}{(r_2 - r_1)^2} q^{2+2\delta} \int_{B_{r_2}} U^2 w\,dx.$$

Taking the $\frac{1}{2q}$th root on each side and letting $l \to +\infty$, we have

$$\left(\int_{B_{r_1}} v^{2q\tau} w \, dx \right)^{\frac{1}{2q\tau}}$$

$$\leqslant C^{\frac{1}{2q}} w(B_2)^{\frac{1}{2}(\frac{1}{\tau}-1)\frac{1}{q}} \left(\frac{1}{r_2 - r_1} \right)^{\frac{1}{q}} q^{\frac{1}{q}(1+\delta)} \left(\int_{B_{r_2}} v^{2q} w \, dx \right)^{\frac{1}{2q}}.$$

Set $\gamma = 2q$, we have

$$\|v\|_{L^{\tau\gamma}(B_{r_1},w)} \leqslant C^{\frac{1}{\gamma}} w(B_2)^{(\frac{1}{\tau}-1)\frac{1}{\gamma}} \left(\frac{1}{r_2 - r_1} \right)^{\frac{2}{\gamma}} \left(\frac{\gamma}{2} \right)^{\frac{2}{\gamma}(1+\delta)} \|v\|_{L^{\gamma}(B_{r_2},w)}.$$

We set, for $i = 1, 2, \ldots,$

$$\gamma_i = 2\tau^i$$

and

$$r_i = 1 + \frac{1}{2^i}.$$

Hence, the previous inequality becomes

$$\|v\|_{L^{\gamma_{i+1}}(B_{r_{i+1}},w)} \leqslant C^{\frac{1}{\gamma_i}} w(B_2)^{(\frac{1}{\tau}-1)\frac{1}{\gamma_i}} (2^{i+1})^{\frac{2}{\gamma_i}} \left(\frac{\gamma_i}{2} \right)^{\frac{2}{\gamma_i}(1+\delta)} \|v\|_{L^{\gamma_i}(B_{r_i},w)}.$$

Iteration yields

$$\|v\|_{L^\infty(B_1,w)} \leqslant C \left(\frac{1}{w(B_2)} \int_{B_2} v^2 w \, dx \right)^{\frac{1}{2}}$$

and recalling that $v = |u| + h$ we get

$$\|u\|_{L^\infty(B_1,w)} \leqslant C \left\{ \left(\frac{1}{w(B_2,w)} \int_{B_2} |u|^2 w \, dx \right)^{\frac{1}{2}} + \left\| \frac{f}{w} \right\|_{\sigma,B_2} + \left(\sum_{i=1}^n \left\| \frac{f_i}{w} \right\|_{\sigma,B_2} \right)^{\frac{1}{2}} \right\}. \quad \blacksquare \tag{3.7}$$

4. HARNACK'S INEQUALITY OF SOLUTIONS

In this section we prove the following theorem.

THEOREM 4.1. *Let u be a nonnegative weak solution of Eq.* (3.2) *defined in some ball* $B_{3r} \subset\subset \Omega$. *We assume that conditions* (1.2), (1.3) *and* (3.1) *hold. Then there exists a positive constant C depending on* ν, σ, n, $r^\sigma \sum_{i=1}^n \|(\frac{b_i}{w})^2\|_{\sigma,B_{3r}}$, $r^\sigma\|\frac{c}{w}\|_{\sigma,B_{3r}}$, $r^\sigma \sum_{i=1}^n \|(\frac{d_i}{w})^2\|_{\sigma,B_{3r}}$, C_1, C_2 *and* τ *such that*

$$\max_{B_r} u \leqslant C\left\{\min_{B_r} u + r^\sigma\left\|\frac{f}{w}\right\|_{\sigma,B_{3r}} + \left(r^\sigma \sum_{i=1}^n \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_{3r}}\right)^{\frac{1}{2}}\right\}.$$

Proof. Also in this case we prove the theorem assuming $r = 1$. Proceeding as in the proof of Theorem 3.1, setting $v = u + h$, where

$$h = \left\|\frac{f}{w}\right\|_{\sigma,B_3} + \left(\sum_{i=1}^n \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_3}\right)^{\frac{1}{2}}$$

and taking as a test function in (3.3) $\psi(x) = \eta^2(x)v^\beta(x)$, where $\eta(x)$ is a nonnegative smooth function such that $\operatorname{supp}\eta(x) \subseteq B_3$ and $\beta \in \mathbb{R}$, we obtain

$$\begin{aligned} &\nu^{-1}|\beta| \int_{B_3} |\nabla v|^2\eta^2 v^{\beta-1} w\,dx \\ &\leqslant 2\nu \int_{B_3} |\nabla v|\,|\nabla\eta|\eta v^\beta w\,dx \\ &\quad + |\beta| \sum_{j=1}^n \int_{B_3} |d_j|\,|v_{x_j}|\eta^2 v^\beta\,dx + 2\sum_{j=1}^n \int_{B_3} |d_j|\eta|\eta_{x_j}|v^{\beta+1}\,dx \\ &\quad + \sum_{i=1}^n \int_{B_3} |b_i|\eta^2|v_{x_i}|v_\beta\,dx + \int_{B_3} |c|\eta^2 v^{\beta+1}\,dx + \int_{B_3} h^{-1}|f|\eta^2 v^{\beta+1}\,dx \\ &\quad + |\beta| \sum_{i=1}^n \int_{B_3} h^{-1}|f_i|\eta^2\,|v_{x_i}|v^\beta\,dx + 2\sum_{i=1}^n \int_{B_3} h^{-1}|f_i|\eta|\eta_{x_i}|v^\beta\,dx. \end{aligned} \tag{4.1}$$

Using, as usual, the inequality

$$0 \leqslant ab \leqslant \frac{a^2}{2}\varepsilon + \frac{b^2}{2\varepsilon},$$

it follows that

$$\int_{B_3} |\nabla v|^2 \eta^2 v^{\beta+1} w\, dx$$
$$\leqslant C(v)\left\{\frac{|\beta|+1}{\beta^2}\int_{B_3} |\nabla \eta|^2 v^{\beta+1} w\, dx + \left(\frac{|\beta|+1}{\beta}\right)^2 \int_{B_3} V\eta^2 v^{\beta+1}\, dx\right\}, \qquad (4.2)$$

where

$$V = \sum_{i=1}^n \frac{|b_i|^2}{w} + |c| + \sum_{i=1}^n \frac{|d_i|^2}{w} + h^{-1}|f| + h^{-2}\sum_{i=1}^n \frac{|f_i|^2}{w}.$$

Set

$$U(x) = \begin{cases} v(x)^{(\beta+1)/2} & \text{if } \beta \neq -1, \\ \log v(x) & \text{if } \beta = -1. \end{cases}$$

By (4.2) we have that

$$\int_{B_3} |\nabla U|^2 \eta^2 w\, dx \leqslant C(v)\left\{\frac{(|\beta|+1)^3}{\beta^2}\int_{B_3} |\nabla \eta|^2 U^2 w\, dx + \left(\frac{|\beta|+1}{\beta}\right)^2 \int_{B_3} V\eta^2 U^2\, dx\right\} \quad \text{if } \beta \neq -1 \qquad (4.3)$$

and

$$\int_{B_3} |\nabla U|^2 \eta^2 w\, dx \leqslant C(v)\left\{\int_{B_3} |\nabla \eta|^2 w\, dx + \int_{B_3} V\eta^2\, dx\right\} \quad \text{if } \beta = 1. \qquad (4.4)$$

We begin to examine (4.4). By Theorem 2.7

$$\int_{B_3} |\nabla U|^2 \eta^2 w\, dx \leqslant C\left\{\int_{B_3} |\nabla \eta|^2 w\, dx + \int_{B_3} \eta^2 w\, dx\right\}, \qquad (4.5)$$

where C is a positive constant depending on v, σ, n and $\|\frac{V}{w}\|_{\sigma,B_3}$. Choosing $\eta(x)$ in such a way that $\eta(x) = 1$ in B_ρ, $\operatorname{supp}\eta \subset B_{2\rho} \subset B_3$, and $|\nabla\eta(x)| \leqslant \frac{3}{\rho}$, where B_ρ is an arbitrary open ball contained in B_2, by (4.5) and doubling we get

$$\left(\frac{1}{w(B_\rho)}\int_{B_\rho} |\nabla U|^2 w\, dx\right)^{\frac{1}{2}} \leqslant C\frac{1}{\rho}.$$

Thus, by Lemma 2.2(b) we obtain

$$\left(\frac{1}{w(B_\rho)}\int_{B_\rho}|U-U_{B_\rho}|^2 w\,dx\right)^{\frac{1}{2}} \leqslant C$$

for every $B_\rho \subseteq B_2$, with C depending on v, σ, n, $\|\frac{V}{w}\|_{\sigma,B_3}$ and C_2, where C_2 is the constant in Lemma 2.2(b). The John and Nirenberg Lemma for BMO (w) (see [11]) yields that there exist two positive constants, p_0 and $\bar{C}$, depending on the same arguments of C, such that

$$\left(\frac{1}{w(B_2)}\int_{B_2} e^{p_0 U} w\,dx\right)^{\frac{1}{p_0}}\left(\frac{1}{w(B_2)}\int_{B_2} e^{-p_0 U} w\,dx\right)^{\frac{1}{p_0}} \leqslant \bar{C}. \tag{4.6}$$

Set

$$\Phi(p,h) = \left(\int_{B_h} v^p w\,dx\right)^{\frac{1}{p}}$$

for any real number $p \neq 0$ and $h > 0$, by (4.6), recalling that $U = \log v$, we have

$$w^{\frac{-1}{p_0}}(B_2)\Phi(p_0,2) \leqslant \bar{C} w^{\frac{1}{p_0}}(B_2)\Phi(-p_0,2). \tag{4.7}$$

We consider now (4.3). By Theorem 2.7 we obtain

$$\int_{B_3}|\nabla U|^2\eta^2 w\,dx \leqslant C\left\{(|\beta|+1)^3\left(1+\frac{1}{|\beta|}\right)^2\int_{B_3}|\nabla\eta|^2 U^2 w\,dx \right.$$
$$\left. +(|\beta|+1)^{4+4\delta}\left(1+\frac{1}{|\beta|}\right)^{2+2\delta}\int_{B_3}\eta^2 U^2 w\,dx\right\},$$

where C depends on v, σ, n and $\|\frac{V}{w}\|_{1,\sigma,B_3}$; using Lemma 2.2(a) we have

$$\left(\int_{B_3}|\eta U|^{2\tau} w\,dx\right)^{\frac{1}{\tau}} \leqslant C w^{\frac{1}{\tau}-1}(B_3)(|\beta|+1)^{4+4\delta}\left(1+\frac{1}{|\beta|}\right)^{2+2\delta}$$
$$\times\left\{\int_{B_3}|\nabla\eta|^2 U^2 w\,dx + \int_{B_3}\eta^2 U^2 w\,dx\right\}, \tag{4.8}$$

where C depends on v, σ, n, $\|\frac{V}{w}\|_{\sigma,B_3}$ and C_1, C_1 being the constant in Lemma 2.2(a).

Let r_1 and r_2 be real numbers such that $0 < r_1 < r_2 \leqslant 2$. Let the function η be chosen so that $\eta(x) = 1$ in B_{r_1}, $0 \leqslant \eta(x) \leqslant 1$ in B_{r_2}, $\eta(x) = 0$ outside B_{r_2},

$|\nabla\eta(x)| \leqslant \frac{2}{r_2 - r_1}$. By (4.8) we have

$$\left(\int_{B_{r_1}} |U|^{2\tau} w \, dx \right)^{\frac{1}{\tau}}$$
$$\leqslant C w^{\frac{1}{\tau - 1}}(B_3)(|\beta| + 1)^{4+4\delta} \left(1 + \frac{1}{|\beta|}\right)^{2+2\delta} \frac{1}{(r_2 - r_1)^2} \int_{B_{r_2}} U^2 w \, dx. \tag{4.9}$$

Putting $p = \beta + 1$ and taking the pth root of each side of (4.9), we obtain, recalling that $U^2(x) = v^{\beta+1}(x) = v^p(x)$,

$$\Phi(\tau p, r_1) \leqslant C^{\frac{1}{p}} w^{\frac{1}{p}\left(\frac{1}{\tau} - 1\right)}(B_3)(p + 2)^{\frac{4+4\delta}{p}} \left(1 + \frac{1}{|\beta|}\right)^{\frac{2+2\delta}{p}}$$
$$\frac{1}{(r_2 - r_1)^{\frac{2}{p}}} \Phi(p, r_2) \tag{4.10}$$

for positive $p \neq 1$, and

$$\Phi(\tau p, r_1) \geqslant C^1 p w^{\frac{1}{p}\left(\frac{1}{\tau} - 1\right)}(B_3)(|p| + 2)^{\frac{4+4\delta}{p}} \frac{1}{(r_2 - r_1)^{\frac{2}{p}}} \Phi(p, r_2) \tag{4.11}$$

for negative p. These are the inequalities which we wish to iterate. In order that (4.10) be applicable at each stage, we choose an initial value $p_0' \leqslant p_0$ in such a way that the point $p = 1$ lies midway between two consecutive iterates of p_0' and for $i = 0, 1, \ldots,$ we let

$$p_i = \tau^i p_0'$$

and

$$r_i = 1 + \frac{1}{2^i}.$$

Thus we also obtain

$$|\beta| \geqslant \frac{\tau - 1}{1 + \tau}.$$

Hence, by iteration of (4.10) we get

$$\max_{B_1} v \leqslant C w^{\frac{-1}{p_0'}}(B_3)\Phi(p_0', 2), \tag{4.12}$$

where C is a positive constant depending on ν, σ, n, $\|\frac{V}{w}\|_{\sigma,B_3}$, C_1 and τ. Now if $p_i = \tau^i p_0$ and $r_i = 1 + \frac{1}{2^i}$, then iteration of (4.11) yields

$$\min_{B_1} v \geqslant C w^{\frac{1}{p_0}}(B_3)\Phi(-p_0, 2), \tag{4.13}$$

where C is a positive constant depending on ν, σ, n, $\|\frac{V}{w}\|_{\sigma,B_3}$, C_1 and τ.

Therefore, from (4.7), (4.12), (4.13), and noting that from Hölder's inequality

$$\Phi(p_0', 2) \leqslant \Phi(p_0, 2) w(B_2)^{\frac{1}{p_0'} - \frac{1}{p_0}},$$

we obtain

$$\max_{B_1} v \leqslant C \min_{B_1} v$$

with C depending on ν, σ, n, $\|\frac{V}{w}\|_{\sigma,B_3}$, C_1 and τ. Since $v = u + h$ we have

$$\max_{B_1} u \leqslant C \left\{ \min_{B_1} u + \left\| \frac{f}{w} \right\|_{\sigma,B_3} + \left(\sum_{i=1}^{n} \left\| \frac{f_i}{w} \right\|_{\sigma,B_3} \right)^{\frac{1}{2}} \right\},$$

which concludes the proof of the theorem. ∎

5. HÖLDER CONTINUITY OF SOLUTIONS

Before proving the Hölder continuity result we recall the following lemma (see [15]).

LEMMA 5.1. *Let* $0 < \theta < 1$, $H > 0$ *and* $\omega :]0, +\infty[\to]0, +\infty[$ *be a non-decreasing function such that*

$$\omega(\rho) \leqslant \theta \omega(4\rho) + H\rho^{\alpha} \qquad \forall \rho < \rho_0 < 1.$$

Then there exist positive constants λ *and* K *such that*

$$\omega(\rho) \leqslant K\rho^{\lambda}.$$

THEOREM 5.2. *Let* u *be a weak solution of* (3.2) *in* Ω. *If we assume that conditions* (3.1) *hold, then* u *is locally continuous in* Ω.

Proof. By Theorem 3.1 we have that

$$|u(x)| \leqslant L$$

in every arbitrary subset Ω' of Ω, where L is a positive constant depending on ν, σ, n, $\sum_{i=1}^n \|(\frac{b_i}{w})^2\|_{\sigma,\Omega}$, $\|\frac{c}{w}\|_{\sigma,\Omega}$, $\sum_{i=1}^n \|(\frac{d_i}{w})^2\|_{\sigma,\Omega}$, $\|\frac{f}{w}\|_{\sigma,\Omega}$, $\sum_{i=1}^n \|(\frac{f_i}{w})^2\|_{\sigma,\Omega}$, C_1, C_2, τ and Ω'.

Let B_r be an arbitrary ball contained in Ω' and the functions

$$M = M(r) = \max_{B_r} u$$

and

$$m = m(r) = \min_{B_r} u.$$

Set

$$\bar{u} = M(r) - u,$$

it is not difficult to show that $\bar{u}$ is a nonnegative solution in B_r of the equation

$$-(a_{ij}u_{x_i})_{x_j} - \sum_{j=1}^n (d_j u)_{x_j} + \sum_{i=1}^n b_i u_{x_i} + cu = (Mc - f) - \sum_{i=1}^n (Md_i - f_i)_{x_i}.$$

We note that

$$\frac{Mc-f}{w}, \left(\frac{Md_i - f_i}{w}\right)^2 \in M_\sigma(\Omega, w), \qquad \sigma > 0$$

and

$$\left\|\frac{Mc-f}{w}\right\|_{\sigma,B_\rho} \leqslant L\left\|\frac{c}{w}\right\|_{\sigma,B_\rho} + \left\|\frac{f}{w}\right\|_{\sigma,B_\rho},$$

$$\sum_{i=1}^n \left\|\left(\frac{Md_i - f_i}{w}\right)^2\right\|_{\sigma,B_\rho} \leqslant 2L^2 \sum_{i=1}^n \left\|\left(\frac{d_i}{w}\right)^2\right\|_{\sigma,B_\rho} + 2\sum_{i=1}^n \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,B_\rho}$$

for every $B_\rho \subseteq \Omega'$.

By Harnack's inequality we obtain

$$\max_{B_{\frac{r}{3}}} \bar{u} \leqslant C\left\{\min_{B_{\frac{r}{3}}} \bar{u} + \left(\frac{r}{3}\right)^\sigma \left(L\left\|\frac{c}{w}\right\|_{\sigma,\Omega} + \left\|\frac{f}{w}\right\|_{\sigma,\Omega}\right)\right.$$

$$+ \left[\left(\frac{r}{3}\right)^{\sigma}\left(2L^2 \sum_{i=1}^{n} \left\|\left(\frac{d_i}{w}\right)^2\right\|_{\sigma,\Omega} + 2 \sum_{i=1}^{n} \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,\Omega}\right)\right]^{\frac{1}{2}}\Bigg\},$$

where C is a positive constant depending on v, σ, n, $\sum_{i=1}^{n} \|(\frac{b_i}{w})^2\|_{\sigma,\Omega}$, $\|\frac{c}{w}\|_{\sigma,\Omega}$, $\sum_{i=1}^{n} \|(\frac{d_i}{w})^2\|_{\sigma,\Omega}$, C_1, C_2, τ.

Then

$$M(r) - m\left(\frac{r}{3}\right) \leqslant C\left\{M(r) - M\left(\frac{r}{3}\right) + Hr^{\frac{\sigma}{2}}\right\}, \tag{5.1}$$

where $(r \leqslant R)$

$$H = \left(\frac{1}{3}\right)^{\sigma} R^{\frac{\sigma}{2}}\left(L\left\|\frac{c}{w}\right\|_{1,\sigma,\Omega} + \left\|\frac{f}{w}\right\|_{1,\sigma,\Omega}\right)$$

$$+ \left(\frac{1}{3}\right)^{\sigma}\left(2L^2 \sum_{i=1}^{n} \left\|\left(\frac{d_i}{w}\right)^2\right\|_{\sigma,\Omega} + 2 \sum_{i=1}^{n} \left\|\left(\frac{f_i}{w}\right)^2\right\|_{\sigma,\Omega}\right)^{\frac{1}{2}}.$$

In the same way, setting

$$\bar{\bar{u}} = u - m(r),$$

we obtain

$$M\left(\frac{r}{3}\right) - m(r) \leqslant C\left\{m\left(\frac{r}{3}\right) - m(r) + Hr^{\frac{\sigma}{2}}\right\}, \tag{5.2}$$

where C and H are the same as those in (5.1).

Adding (5.1) and (5.2) we get

$$M\left(\frac{r}{3}\right) - m\left(\frac{r}{3}\right) \leqslant \frac{C-1}{C+1}[M(r) - m(r)] + \frac{2C}{C+1}Hr^{\frac{\sigma}{2}}.$$

Set, for $\rho > 0$

$$\omega(\rho) = M(\rho) - m(\rho),$$

$$\theta = \frac{C-1}{C+1}$$

and

$$K = \frac{2C}{C+1}H,$$

we have

$$\omega\left(\frac{r}{4}\right) \leqslant \omega\left(\frac{r}{3}\right) \leqslant \theta\omega(r) + Kr^{\frac{\sigma}{2}}$$

and the conclusion follows by Lemma 5.1. ∎

ACKNOWLEDGMENT

We wish to express our gratitude to Professor C. E. Gutiérrez for some useful talks and suggestions.

REFERENCES

1. F. Chiarenza, E. Fabes, and N. Garofalo Harnack's inequality for Schrödinger operators and continuity of solutions, *Proc. Amer. Math. Soc.* **98** (1986), 415–425.
2. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241–250.
3. R. Coifman and G. Weiss, "Analyse Harmonique Noncommutative sur Certaines Espaces Homogenes," Lecture Notes in Mathematics, Vol. 242, Springer Verlag, Berlin, 1971.
4. V. De Cicco and M. A. Vivaldi, Harnack inequalities for Fuchsian type weighted elliptic equations, *Comm. Partial Differential Equations* **21** (1996), 1321–1347.
5. G. Di Fazio, Hölder continuity of solutions for some Schrödinger equations, *Rend. Sem. Mat. Univ. Padova* **79** (1988), 173–183.
6. E. Fabes, D. Jerison, and C. Kenig, The Wiener test for degenerate elliptic equations, *Ann. Inst. Fourier* (*Grenoble*) **32** (1982), 151–182.
7. E. Fabes, C. Kenig, and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** (1982), 77–116.
8. J. Garcia Cuerva and J. L. Rubio de Francia, "Weighted Norm Inequalities and Related Topics," North-Holland, Amsterdam, 1985.
9. C. E. Gutiérrez, Harnack's inequality for degenerate Schrödinger operators, *Trans. Amer. Math. Soc.* **312** (1989), 403–419.
10. J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591.
11. B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for fractional integrals, *Trans. Amer. Math. Soc.* **192** (1974), 261–274.
12. L. C. Piccinini, Inclusioni tra spazi di Morrey, *Boll. Un. Mat. Ital.* **2** (1969), 95–99.
13. J. Serrin, Local behavior of solutions of quasilinear equations, *Acta Math.* **111** (1964), 247–302.
14. C. Simader, An elementary proof of Harnack's inequality for Schrödinger operators and related topics, *Math. Z.* **203** (1990), 129–152.
15. G. Stampacchia, Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus, *Ann. Inst. Fourier Grenoble* **15** (1965), 198–258.
16. C. Vitanza and P. Zamboni, Necessary and sufficient conditions for Hölder continuity of solutions of degenerate Schrödinger operators, *Le Matematiche* **LII** (1997), 393–409.
17. P. Zamboni, Local behavior of solutions of quasilinear elliptic equations with coefficients in Morrey spaces, *Rend. Mat. Appl. Ser.* VII **15** (1995), 251–262.