ELSEVIER

Journal of Computational and Applied Mathematics 113 (2000) 401–410

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

A general variational principle and some of its applications

Biagio Ricceri

Department of Mathematics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy

Received 17 April 1999

Abstract

In this paper, given a reflexive real Banach space X and two sequentially weakly lower semicontinuous functionals Φ, Ψ on X with Ψ strongly continuous and coercive, we are mainly interested in the existence of infinitely many local minima of the functional $\Phi + \lambda\Psi$ for each sufficiently large $\lambda \in \mathbb{R}$.

1. Introduction

The aim of this paper is to establish Theorem 2.1 below and, mainly, to show how one can derive from it, in an absolutely transparent way, a series of consequences about the local minima of a functional of the type $\Phi + \lambda\Psi$, where λ is a positive real number, and, in concrete situations, Φ and Ψ are sequentially weakly lower semicontinuous functionals defined on a subset of a reflexive Banach space.

Ultimately, our end is to apply, via the variational methods, the basic theory developed below to differential equations. In this connection, Theorem 2.5 can be regarded as the main result of this paper. Specific applications of Theorem 2.5 to differential equations will systematically be presented in a series of successive papers. Here, we limit ourselves to give a sample of application to a class of elliptic equations involving the critical Sobolev exponent (Theorem 2.8).

We also derive from Theorem 2.5 a result on fixed points of potential operators in Hilbert spaces (Theorem 2.7) of which the following is a corollary:

Theorem A. *Let X be a real Hilbert space, and let $A : X \to X$ be a potential operator, with sequentially weakly upper semicontinuous potential P satisfying*

$$\liminf_{r\to+\infty} \frac{\sup_{\|x\|\leqslant r} P(x)}{r^2} < \frac{1}{2} < \limsup_{r\to+\infty} \frac{\sup_{\|x\|\leqslant r} P(x)}{r^2}.$$

Then, the set of all fixed points of A is unbounded.

E-mail address: ricceri@dipmat.unict.it (B. Ricceri)

0377-0427/00/$ - see front matter
PII: S0377-0427(99)00269-1

2. Results

Our abstract basic result is as follows:

Theorem 2.1. *Let X be a topological space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two sequentially lower semicontinuous functions. Denote by I the set of all $\rho > \inf_X \Psi$ such that the set $\Psi^{-1}(]-\infty, \rho[)$ is contained in some sequentially compact subset of X. Assume that $I \neq \emptyset$. For each $\rho \in I$, denote by $\mathscr{F}_\rho$ the family of all sequentially compact subsets of X containing $\Psi^{-1}(]-\infty, \rho[)$, and put*

$$\alpha(\rho) = \sup_{K \in \mathscr{F}_\rho} \inf_K \Phi.$$

Then, for each $\rho \in I$ and each λ satisfying

$$\lambda > \inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) - \alpha(\rho)}{\rho - \Psi(x)}$$

the restriction of the function $\Phi + \lambda\Psi$ to $\Psi^{-1}(]-\infty, \rho[)$ has a global minimum.

Proof. Fix $\rho \in I$ and λ as in the conclusion. Observe that

$$\inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) - \alpha(\rho)}{\rho - \Psi(x)} = \inf_{r > -\alpha(\rho)} \inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) + r}{\rho - \Psi(x)}.$$

Consequently, we can fix $r^* > -\alpha(\rho)$ so that

$$\inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) + r^*}{\rho - \Psi(x)} < \lambda. \tag{1}$$

On the other hand, since $-r^* < \alpha(\rho)$, there is a sequentially compact subset K of X containing $\Psi^{-1}(]-\infty, \rho[)$ such that

$$-r^* < \inf_K \Phi.$$

For each $r > -\inf_K \Phi$, put

$$\beta_\rho(r) = \sup_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) + r}{\Psi(x) - \rho}.$$

Observe that the function $(\Phi(\cdot) + r)/(\Psi(\cdot) - \rho)$ is negative in $\Psi^{-1}(]-\infty, \rho[)$. We now show that it there attains the value $\beta_\rho(r)$. To this end, fix a sequence $\{x_n\}$ in $\Psi^{-1}(]-\infty, \rho[)$ such that

$$\lim_{n \to \infty} \frac{\Phi(x_n) + r}{\Psi(x_n) - \rho} = \beta_\rho(r). \tag{2}$$

Since K is sequentially compact, $\{x_n\}$ admits a subsequence $\{x_{n_k}\}$ converging to a point $x^* \in K$. Put

$$l = \liminf_{k \to \infty} \Psi(x_{n_k}).$$

We claim that $l < \rho$. Indeed, if it was $l = \rho$, since

$$l \leqslant \limsup_{k \to \infty} \Psi(x_{n_k}) \leqslant \rho,$$

we would have

$$\lim_{k\to\infty} \Psi(x_{n_k}) = \rho,$$

and so, from (2), we would infer

$$\lim_{k\to\infty} (\Phi(x_{n_k}) + r) = 0.$$

Then, since Φ is sequentially lower semicontinuous, it would follow

$$\Phi(x^*) + r \leqslant 0,$$

from which

$$r \leqslant -\inf_K \Phi,$$

against the choice of r. Since Ψ is sequentially lower semicontinuous, we have

$$\Psi(x^*) \leqslant l < \rho. \tag{3}$$

Now, extract from $\{x_{n_k}\}$ a subsequence $\{x_{n_{k_p}}\}$ so that

$$l = \lim_{p\to\infty} \Psi(x_{n_{k_p}}).$$

From (2), we then get

$$\lim_{p\to\infty} (\Phi(x_{n_{k_p}}) + r) = \beta_\rho(r)(l - \rho).$$

So, by the sequential lower semicontinuity of Φ, we have

$$0 < \Phi(x^*) + r \leqslant \beta_\rho(r)(l - \rho). \tag{4}$$

Now, from (3) and (4), we directly obtain

$$\beta_\rho(r) \leqslant \frac{\Phi(x^*) + r}{\Psi(x^*) - \rho}$$

which shows our claim. Clearly, the function β_ρ, as the supremum of affine functions, is convex in the open interval $]-\inf_K \Phi, +\infty[$. Hence, it is continuous there. Also, observe that it is not bounded below. Indeed, one has

$$\beta_\rho(r) \leqslant \frac{r + \inf_K \Phi}{\inf_K \Psi - \rho}$$

for all $r > -\inf_K \Phi$. Therefore, recalling (1), since $r^* > -\inf_K \Phi$, there exists $r_0 > -\inf_K \Phi$ such that

$$\beta_\rho(r_0) = -\lambda.$$

Finally, let $x_0 \in \Psi^{-1}(]-\infty, \rho[)$ be such that

$$\beta_\rho(r_0) = \frac{\Phi(x_0) + r_0}{\Psi(x_0) - \rho}.$$

Hence, for each $x \in \Psi^{-1}(]-\infty, \rho[)$, one has

$$\frac{\Phi(x) + r_0}{\Psi(x) - \rho} \leqslant -\lambda$$

and so

$$\Phi(x) + r_0 + \lambda(\Psi(x) - \rho) \geqslant 0 = \Phi(x_0) + r_0 + \lambda(\Psi(x_0) - \rho),$$

that gives

$$\Phi(x) + \lambda\Psi(x) \geqslant \Phi(x_0) + \lambda\Psi(x_0).$$

This concludes the proof. □

Remark 2.1. Concerning the statement of Theorem 2.1, observe that when the space X is Hausdorff, the set $H = \bigcap\{K\colon K \in \mathscr{F}_\rho\}$ belongs to $\mathscr{F}_\rho$, and so we have

$$\alpha(\rho) = \inf_H \Phi.$$

One of the most significant features of Theorem 2.1 is the possibility to get from it multiplicity results for local minima. Indeed, we have

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. In addition, suppose*

$$\sup I = +\infty$$

and

$$\gamma < +\infty,$$

where

$$\gamma = \liminf_{\rho \to +\infty} \inf_{x \in \Psi^{-1}(]-\infty, \rho[)} \frac{\Phi(x) - \alpha(\rho)}{\rho - \Psi(x)}.$$

Finally, denote by τ the weakest topology on X for which Ψ is upper semicontinuous.

Then, for each $\lambda > \gamma$, the following alternative holds: either $\Phi + \lambda\Psi$ has a global minimum, or there exists a sequence $\{x_n\}$ of τ-local minima of $\Phi + \lambda\Psi$ such that

$$\lim_{n \to \infty} \Psi(x_n) = +\infty.$$

Proof. Let $\lambda > \gamma$. Then, we can fix a sequence $\{\rho_n\}$ in I, with $\lim_{n\to\infty} \rho_n = +\infty$, such that

$$\inf_{x \in \Psi^{-1}(]-\infty, \rho_n[)} \frac{\Phi(x) - \alpha(\rho_n)}{\rho_n - \Psi(x)} < \lambda$$

for all $n \in \mathbb{N}$. Consequently, owing to Theorem 2.1, for each $n \in \mathbb{N}$, there is $x_n \in \Psi^{-1}(]-\infty, \rho_n[)$ such that

$$\Phi(x_n) + \lambda\Psi(x_n) \leqslant \Phi(x) + \lambda\Psi(x) \tag{5}$$

for all $x \in \Psi^{-1}(]-\infty, \rho_n[)$. Now, if $\lim_{n\to\infty} \Psi(x_n) = +\infty$, we are done, since each set $\Psi^{-1}(]-\infty, \rho_n[)$ is τ-open. Thus, suppose that $\liminf_{n\to\infty} \Psi(x_n) < +\infty$. Hence, there are an increasing sequence $\{n_k\}$ in $\mathbb{N}$ and a constant $c \in I$ such that

$$\Psi(x_{n_k}) < c$$

for all $k \in \mathbb{N}$. But the set $\Psi^{-1}(]-\infty, c[)$ is contained in some sequentially compact subset of X, and so there is a subsequence $\{x_{n_{k_r}}\}$ converging to some $x^* \in X$. Finally, fix $x \in X$. Since we definitively have $\rho_{n_{k_r}} > \Psi(x)$, taking into account the sequential lower semicontinuity of $\Phi + \lambda\Psi$, from (5) we get

$$\Phi(x^*) + \lambda\Psi(x^*) \leqslant \liminf_{r\to+\infty} (\Phi(x_{n_{k_r}}) + \lambda\Psi(x_{n_{k_r}})) \leqslant \Phi(x) + \lambda\Psi(x).$$

Hence, x^* is a global minimum of $\Phi + \lambda\Psi$. □

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *be satisfied. In addition, suppose*

$$\delta < +\infty,$$

where

$$\delta = \liminf_{\rho\to(\inf_X \Psi)^+} \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \alpha(\rho)}{\rho - \Psi(x)}.$$

Finally, denote by τ the weakest topology on X for which Ψ is upper semicontinuous.

Then, for each $\lambda > \delta$, there exists a sequence of τ-local minima of $\Phi + \lambda\Psi$ which converges to a global minimum of Ψ.

Proof. Let $\lambda > \delta$. Fix a sequence $\{\rho_n\}$ in I, with $\lim_{n\to\infty} \rho_n = \inf_X \Psi$, such that

$$\inf_{x\in\Psi^{-1}(]-\infty,\rho_n[)} \frac{\Phi(x) - \alpha(\rho_n)}{\rho_n - \Psi(x)} < \lambda$$

for all $n \in \mathbb{N}$. Thanks to Theorem 2.1, for each $n \in \mathbb{N}$, the restriction of $\Phi + \lambda\Psi$ to $\Psi^{-1}(]-\infty,\rho_n[)$ has a global minimum, say x_n. Hence, x_n is a τ-local minimum of $\Phi+\lambda\Psi$. Finally, the sequence $\{x_n\}$ lies in $\Psi^{-1}(]-\infty, \max_{n\in\mathbb{N}} \rho_n[)$ (which is, in turn, contained in a sequentially compact subset of X), and so it admits a subsequence converging to a point which, by the sequential lower semicontinuity of Ψ, is a global minimum of Ψ. □

We now derive from the previous abstract theorems more concrete results in reflexive Banach spaces.

Theorem 2.4. *Let E be a reflexive real Banach space, X a closed, convex, unbounded subset of E, and $\Phi, \Psi : X \to \mathbb{R}$ two convex functionals, with Φ lower semicontinuous and Ψ continuous and satisfying $\lim_{x\in X, \|x\|\to+\infty} \Psi(x) = +\infty$. Put*

$$\lambda^* = \inf_{\rho>\inf_X \Psi} \inf_{x\in\Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(x) - \inf_{\Psi^{-1}(]-\infty,\rho])} \Phi}{\rho - \Psi(x)}.$$

Then, for each $\lambda > \lambda^*$, the functional $\Phi + \lambda\Psi$ has a global minimum in X. Moreover, if $\lambda^* > 0$, for each $\mu < \lambda^*$, the functional $\Phi + \mu\Psi$ has no global minima in X.

Proof. It is clear that we can apply Theorem 2.1 endowing X with the relativization of the weak topology. In particular, for each $\rho > \inf_X \Psi$, the set $\Psi^{-1}(]-\infty,\rho])$ is sequentially weakly compact owing to the reflexivity of E and to the coercivity of Ψ. Moreover, by the convexity of Ψ, the same set is equal to the closure of $\Psi^{-1}(]-\infty,\rho[)$. From this, it follows that $\Psi^{-1}(]-\infty,\rho])$ is the smallest sequentially weakly compact subset of X containing $\Psi^{-1}(]-\infty,\rho[)$. Hence, by Remark 2.1, we have

$$\alpha(\rho)=\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi.$$

Now, let $\lambda>\lambda^*$ and choose $\rho>\inf_X\Psi$ so that

$$\lambda>\inf_{x\in\Psi^{-1}(]-\infty,\rho[)}\frac{\Phi(x)-\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi}{\rho-\Psi(x)}.$$

Then, Theorem 2.1 ensures that the restriction of the functional $\Phi+\lambda\Psi$ to $\Psi^{-1}(]-\infty,\rho[)$ has a global minimum, say x_0. But, by assumption, Ψ is (strongly) continuous, and so the set $\Psi^{-1}(]-\infty,\rho[)$ is (strongly) open in X. In other words, x_0 is a local minimum for $\Phi+\lambda\Psi$ in the strong topology. Since $\Phi+\lambda\Psi$ is convex, x_0 is actually a global minimum for $\Phi+\lambda\Psi$ in X.

Now, assume that $\lambda^*>0$. Let μ be such that the functional $\Phi+\mu\Psi$ has a global minimum in X, say x_1. We claim that $\mu\geqslant\lambda^*$ from which, of course, the second part of the conclusion follows. Indeed, fix $\rho>\Psi(x_1)$ and choose $x_2\in\Psi^{-1}(]-\infty,\rho])$ so that

$$\Phi(x_2)=\inf_{\Psi^{-1}(]-\infty,\rho])}\Phi.$$

Clearly, since $\lambda^*>0$, we have $\Psi(x_2)=\rho$. Hence, we get

$$\mu\geqslant\frac{\Phi(x_1)-\Phi(x_2)}{\Psi(x_2)-\Psi(x_1)}=\frac{\Phi(x_1)-\alpha(\rho)}{\rho-\Psi(x_1)}\geqslant\inf_{x\in\Psi^{-1}(]-\infty,\rho[)}\frac{\Phi(x)-\alpha(\rho)}{\rho-\Psi(x)}\geqslant\lambda^*,$$

as claimed. □

The next result groups together the versions of Theorems 2.1–2.3 which are directly applicable to differential equations.

Theorem 2.5. *Let X be a reflexive real Banach space, and let $\Phi,\Psi: X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous and Gâteaux differentiable functionals. Assume also that Ψ is* (*strongly*) *continuous and satisfies $\lim_{\|x\|\to+\infty}\Psi(x)=+\infty$. For each $\rho>\inf_X\Psi$, put*

$$\varphi(\rho)=\inf_{x\in\Psi^{-1}(]-\infty,\rho[)}\frac{\Phi(x)-\inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w}\Phi}{\rho-\Psi(x)},$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ is the closure of $\Psi^{-1}(]-\infty,\rho[)$ in the weak topology. Furthermore, set

$$\gamma=\liminf_{\rho\to+\infty}\varphi(\rho)$$

and

$$\delta=\liminf_{\rho\to(\inf_X\Psi)^+}\varphi(\rho).$$

Then, the following conclusions hold:

(a) *For each $\rho>\inf_X\Psi$ and each $\lambda>\varphi(\rho)$, the functional $\Phi+\lambda\Psi$ has a critical point which lies in $\Psi^{-1}(]-\infty,\rho[)$.*

(b) *If $\gamma<+\infty$, then, for each $\lambda>\gamma$, the following alternative holds*: *either $\Phi+\lambda\Psi$ has a global minimum, or there exists a sequence $\{x_n\}$ of critical points of $\Phi+\lambda\Psi$ such that $\lim_{n\to\infty}\Psi(x_n)=+\infty$.*

(c) *If* $\delta < +\infty$, *then*, *for each* $\lambda > \delta$, *the following alternative holds*: *either there exists a global minimum of* Ψ *which is a local minimum of* $\Phi + \lambda\Psi$, *or there exists a sequence of pairwise distinct critical points of* $\Phi + \lambda\Psi$ *which weakly converges to a global minimum of* Ψ.

Proof. Endow X with the weak topology. As in the proof of Theorem 2.4, it is seen that $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ is the smallest sequentially weakly compact subset of X containing $\Psi^{-1}(]-\infty,\rho[)$. So, by Remark 2.1, we have

$$\alpha(\rho) = \inf_{\overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi.$$

Now, (a) follows at once from Theorem 2.1 since any global minimum of the restriction of $\Phi + \lambda\Psi$ to $\Psi^{-1}(]-\infty,\rho[)$ is, by the continuity of Ψ, a local minimum of $\Phi + \lambda\Psi$ in the strong topology.

Assume $\gamma < +\infty$, and let $\lambda > \gamma$. Then, by Theorem 2.2, if $\Phi + \lambda\Psi$ has no global minima, there exists a sequence $\{x_n\}$ of local minima of $\Phi + \lambda\Psi$ in the strong topology such that $\lim_{n\to\infty} \Psi(x_n) = +\infty$, and so (b) follows.

Finally, assume $\delta < +\infty$, and let $\lambda > \delta$. Assume that no global minimum of Ψ is a local minimum of $\Phi + \lambda\Psi$. By Theorem 2.3, there exists a sequence of local minima of $\Phi + \lambda\Psi$ in the strong topology which weakly converges to a global minimum of Ψ. But then, since X equipped with the weak topology is Hausdorff, there is a subsequence whose terms are pairwise distinct, and we are done. □

It is also worth noticing the following corollary of Theorem 2.5.

Theorem 2.6. *Let the assumptions of Theorem* 2.5 *be satisfied. Assume also that* $\delta < +\infty$. *Then*, *either the system*

$$\Phi'(x) = 0,$$

$$\Psi'(x) = 0$$

has at least one solution, *or*, *for each* $\lambda > \delta$, *there exists a sequence of pairwise distinct critical points of* $\Phi + \lambda\Psi$ *which weakly converges to a global minimum of* Ψ. *In particular*, *the first case of the alternative occurs when there exists a Hausdorff vector topology on* X^* *with respect to which the operators* Φ' *and* Ψ' *are sequentially weakly continuous.*

Proof. Assume that the system

$$\Phi'(x) = 0,$$

$$\Psi'(x) = 0$$

has no solutions. Let $\lambda > \delta$. Then, of course, no global minimum of Ψ can also be a local minimum of $\Phi + \lambda\Psi$. Consequently, by Theorem 2.5, there is a sequence $\{x_n\}$ of pairwise distinct critical points of $\Phi + \lambda\Psi$ which weakly converges to a global minimum of Ψ, say x^*. So, we have

$$\Phi'(x_n) + \lambda\Psi'(x_n) = 0 \tag{6}$$

for all $n \in \mathbb{N}$. Finally, observe that there is no Hausdorff vector topology on X^* with respect to which the operators Φ' and Ψ' are sequentially weakly continuous, since, otherwise, passing to the limit in (6), we would have

$$\Phi'(x^*) + \lambda\Psi'(x^*) = 0$$

from which

$$\Phi'(x^*) = 0,$$

$$\Psi'(x^*) = 0,$$

against our assumption. □

We now point out another corollary of Theorem 2.5 concerning fixed points of potential operators in Hilbert spaces. We recall that, given a real Hilbert space X, an operator $A : X \to X$ is said to be a potential operator if there exists a a Gâteaux differentiable functional P on X (which is called a potential of A) such that $P' = A$.

Theorem 2.7. *Let X be a real Hilbert space, and let $A : X \to X$ be a potential operator, with sequentially weakly upper semicontinuous potential P. For each $\rho > 0$, put*

$$\varphi(\rho) = \inf_{||x||^2 < \rho} \frac{\sup_{||y||^2 \leqslant \rho} P(y) - P(x)}{\rho - ||x||^2}.$$

Furthermore, set

$$\gamma = \liminf_{\rho \to +\infty} \varphi(\rho)$$

and

$$\delta = \liminf_{\rho \to 0^+} \varphi(\rho).$$

Then, the following conclusions hold:

(a) *If there is $\rho > 0$ such that $\varphi(\rho) < \frac{1}{2}$, then the operator A has a fixed point whose norm is less than $\sqrt{\rho}$.*

(b) *If $\gamma < \frac{1}{2}$, then the following alternative holds: either the functional $x \to \frac{1}{2}||x||^2 - P(x)$ has a global minimum, or the set of all fixed points of A is unbounded.*

(c) *If $\delta < \frac{1}{2}$, then the following alternative holds: either 0 is a local minimum of the functional $x \to \frac{1}{2}||x||^2 - P(x)$, or there exists a sequence of pairwise distinct fixed points of A which weakly converges to 0.*

Proof. Apply Theorem 2.5 taking $\Phi(x) = -P(x)$ and $\Psi(x) = ||x||^2$ for all $x \in X$, and observe that $\Psi'(x) = 2x$. □

Proof of Theorem A. We have

$$\liminf_{\rho \to +\infty} \inf_{||x||^2 < \rho} \frac{\sup_{||y||^2 \leqslant \rho} P(y) - P(x)}{\rho - ||x||^2} \leqslant \liminf_{\rho \to +\infty} \frac{\sup_{||y||^2 \leqslant \rho} P(y) - P(0)}{\rho} = \liminf_{r \to +\infty} \frac{\sup_{||x|| \leqslant r} P(x)}{r^2} < \frac{1}{2}.$$

On the other hand, since $\limsup_{r\to+\infty}\sup_{||x||\leqslant r} P(x)/r^2 > \frac{1}{2}$, there are a number $c > \frac{1}{2}$ and two sequences $\{r_n\}$, $\{x_n\}$, with $\lim_{n\to\infty} r_n = +\infty$ and $||x_n|| \leqslant r_n$ for all $n \in \mathbb{N}$, such that

$$P(x_n) \geqslant c r_n^2$$

for all $n \in \mathbb{N}$. Consequently, we have

$$P(x_n) - \frac{1}{2}||x_n||^2 \geqslant \left(c - \frac{1}{2}\right) r_n^2$$

for all $n \in \mathbb{N}$. From this, we then infer that the functional $x \to \frac{1}{2}||x||^2 - P(x)$ is unbounded below, and so it has no global minima. Now, the conclusion follows directly from (b) of Theorem 2.7. □

Remark 2.2. Another quite recent result on fixed points of potential operators is provided by Theorem 3.1 of [2]. Among other things, a major difference between Theorem A and Theorem 3.1 of [2] is that this latter deals with weakly continuous potentials.

Our final result is a sample of application of Theorem 2.5 to a class of nonlinear elliptic equations involving the critical Sobolev exponent.

Theorem 2.8. *Let $\Omega \subseteq \mathbb{R}^n$ $(n \geqslant 3)$ be an open bounded set, with smooth boundary, let $\alpha, \beta \in L^{2n/(n+2)}(\Omega)$, and let a, b, c, s, q be five real numbers, with $b, c > 0$, $0 < s < 1$ and $1 < q < (n+2)/(n-2)$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by*

$$f(\xi) = \begin{cases} \xi^{\frac{n+2}{n-2}} & \text{if } \xi \geqslant 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

Then, there exists $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, the problem

$$-\Delta u = a|u|^{s-1}u + \alpha(x) + \lambda(b|u|^{q-1}u - cf(u) + \beta(x)) \text{ in } \Omega,$$
$$u_{|\partial\Omega} = 0$$

has at least one weak solution in $W_0^{1,2}(\Omega)$.

Proof. Consider $W_0^{1,2}(\Omega)$ endowed with the norm $||u|| = (\int_\Omega |\nabla u(x)|^2\,dx)^{1/2}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(\xi) = \begin{cases} \xi^{2n/(n-2)} & \text{if } \xi \geqslant 0, \\ 0 & \text{if } \xi < 0. \end{cases}$$

For each $u \in W_0^{1,2}(\Omega)$, put

$$\Phi(u) = \frac{c(n-2)}{2n}\int_\Omega g(u(x))\,dx - \frac{b}{q+1}\int_\Omega |u(x)|^{q+1}\,dx - \int_\Omega \beta(x)u(x)\,dx$$

and

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,dx - \frac{a}{s+1}\int_\Omega |u(x)|^{s+1}\,dx - \int_\Omega \alpha(x)u(x)\,dx.$$

By a classical result [1, Proposition B.10], the functionals Φ and Ψ are Gâteaux differentiable on $W_0^{1,2}(\Omega)$, the weak solutions of our problem being precisely the critical points of $\Psi + \lambda\Phi$.

Now, observe that, by the Rellich–Kondrachov theorem, the functionals $u \to \int_\Omega |u(x)|^{q+1}\,dx$ and $u \to \int_\Omega |u(x)|^{s+1}\,dx$ are sequentially weakly continuous in $W_0^{1,2}(\Omega)$. Moreover, the functional $x \to \int_\Omega g(u(x))\,dx$ is weakly lower semicontinuous in $W_0^{1,2}(\Omega)$, since it is convex and (strongly) continuous. Hence, the functionals Φ and Ψ are sequentially weakly lower semicontinuous in $W_0^{1,2}(\Omega)$. Finally, observe that, by the Poincaré inequality, since $s < 1$, the functional Ψ is coercive. Therefore, by conclusion (a) of Theorem 2.5, there is a $\mu^* > 0$ such that, for each $\mu > \mu^*$, the functional $\Phi + \mu\Psi$ has a critical point in $W_0^{1,2}(\Omega)$. So, we can take $\lambda^* = 1/\mu^*$ to get the conclusion. □

Remark 2.3. It is worth noticing that, since $b > 0$ and $q > 1$, for each $\lambda > 0$, the functional $\Phi + \lambda\Psi$ in the proof of Theorem 2.8 is unbounded below (and above as well).

References

[1] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. in Math., 65, Amer. Math. Soc., Providence, RI, 1986.
[2] M. Schechter, K. Tintarev, Eigenvalues for semilinear boundary value problems, Arch. Rational Mech. Anal. 113 (1991) 197–208.