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Existence of Three Solutions for a Class of Elliptic Eigenvalue Problems

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Abstract—In this paper, we consider a problem of the type $-\Delta u = \lambda(f(u)+\mu g(u))$ in Ω, $u|_{\partial\Omega} = 0$, where $\Omega \subseteq \mathbf{R}^n$ is an open-bounded set, f, g are continuous real functions on $\mathbf{R}$, and $\lambda, \mu \in \mathbf{R}$. As an application of a new approach to nonlinear eigenvalues problems, we prove that, under suitable hypotheses, if $|\mu|$ is small enough, then there is some $\lambda > 0$ such that the above problem has at least three distinct weak solutions in $W_0^{1,2}(\Omega)$.

Keywords—Nonlinear eigenvalue problems, Local minima, Multifunctions, Connectedness, Minimax inequalities.

1. INTRODUCTION

The aim of this paper is to establish the following result.

THEOREM 1.1. *Let $\Omega \subseteq \mathbf{R}^n$ be an open-bounded set, with smooth boundary, and $f, g\colon \mathbf{R} \to \mathbf{R}$ two continuous functions, with $\sup_{\xi\in\mathbf{R}} \int_0^\xi f(t)\,dt > 0$. Assume that there are four positive constants a, q, s, γ, with $q < (n+2)/(n-2)$ (if $n > 2$), $s < 2$, and $\gamma > 2$, such that*

$$\max\{|f(\xi)|, |g(\xi)|\} \le a(1+|\xi|^q), \qquad \forall \xi \in \mathbf{R}, \tag{1.1}$$

$$\max\left\{\int_0^\xi f(t)\,dt, \left|\int_0^\xi g(t)\,dt\right|\right\} \le a(1+|\xi|^s), \qquad \forall \xi \in \mathbf{R}, \tag{1.2}$$

and

$$\limsup_{\xi\to 0} \frac{\int_0^\xi f(t)\,dt}{|\xi|^\gamma} < +\infty. \tag{1.3}$$

Then, there exists $\delta > 0$ such that, for each $\mu \in [-\delta, \delta]$, there exists $\lambda_0 > 0$ such that the problem

$$-\Delta u = \lambda_0(f(u) + \mu g(u)), \qquad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0$$

has at least three distinct weak solutions in $W_0^{1,2}(\Omega)$.

Theorem 1.1, in a slightly simpler (but less general) form, has been announced in [1], as an example of application of a new approach to nonlinear eigenvalue problems.

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Here is the plan for the present paper. Given the novelty of our approach, we report, in Section 2, the basic theory developed in [1], giving complete proofs. In Section 3, combining our theory with Corollary 1 in [2], we get a three critical points theorem which is the key to proving Theorem 1.1. We also establish a technical proposition which is most useful for the applications of the basic theory. The proof of Theorem 1.1 is given in Section 4. Finally, Section 5 is devoted to various remarks.

2. THE BASIC THEORY

Our method is based on the following general principle.

THEOREM 2.1. *Let X be a topological space, $I \subseteq \mathbf{R}$ an interval, and $f : X \times I \to \mathbf{R}$ a given function. Assume that*

(a) *for each $x \in X$, the set*
$$\{\lambda \in I : f(x,\lambda) \geq 0\}$$
is closed in I and the set
$$\{\lambda \in I : f(x,\lambda) > 0\}$$
is nonempty, connected, and open in I;

(b) *for each $\lambda \in I$, the set*
$$\{x \in X : f(x,\lambda) \leq 0\}$$
is nonempty, closed, and sequentially compact;

(c) *there is $\lambda_0 \in I$ such that the set*
$$\{x \in X : f(x,\lambda_0) \leq 0\}$$
is connected.

Under such assumptions, there exists $x^* \in X$, $\lambda^* \in I$, a sequence $\{\lambda_n\}$ in I converging to λ^*, and a neighbourhood U of x^* such that $f(x^*,\lambda^*) = 0$ and $f(x,\lambda_n) > 0$ for all $n \in \mathbf{N}$, $x \in U$. So, in particular, x^* is a local minimum for $f(\cdot,\lambda^*)$.

Our proof of Theorem 2.1 is fully based on set-valued analysis. So, for the reader's convenience, we first recall some basic notions on multifunctions. Let X, Y be two topological spaces and $F : X \to 2^Y$ a multifunction. For any $\Omega \subseteq Y$, we put
$$F^-(\Omega) = \{x \in X : F(x) \cap \Omega \neq \emptyset\}.$$
We say that F is lower (respectively, upper) semicontinuous if for every open set $\Omega \subseteq Y$, the set $F^-(\Omega)$ is open (respectively, closed). The graph of F (denoted by $\mathrm{gr}(F)$) is the set $\{(x,y) \in X \times Y : y \in F(x)\}$. We say that F is sequentially upper semicontinuous if for every sequence $\{(x_n,y_n)\}$ in $\mathrm{gr}(F)$, with $\{x_n\}$ converging to some $x_0 \in X$, there is a subsequence of $\{y_n\}$ converging to some $y_0 \in F(x_0)$. It is easy to check that when X is first-countable, the sequential upper semicontinuity of F implies the upper semicontinuity of F.

We now point out four propositions whose use will be the key for proving Theorem 2.1.

PROPOSITION 2.1. *(See [3, Theorem 2.2].) Let X be a topological space, $I \subseteq \mathbf{R}$ a compact interval, S a connected subset of $X \times I$ whose projection on I is equal to I, and $F : X \to 2^I$ a lower semicontinuous multifunction, with nonempty connected values.*

Then, the graph of F intersects S.

PROPOSITION 2.2. *Let X, Y be two topological spaces, with X connected and first-countable, and let $F : X \to 2^Y$ be a lower semicontinuous and sequentially upper semicontinuous multifunction. Assume that there is some $x_0 \in X$ such that $F(x_0)$ is nonempty and connected.*

Then, the graph of F is connected.

PROOF. Consider the multifunction $G : X \to 2^{X \times Y}$ defined by

$$G(x) = \{x\} \times F(x),$$

for all $x \in X$. It is clear that G is lower semicontinuous and sequentially upper semicontinuous. So, since X is first-countable, it is upper semicontinuous. Then, taking into account that $G(x_0)$ is nonempty and connected, Theorem 1 of [4] ensures that the set $\bigcup_{x \in X} G(x)$, that is the graph of F, is connected, as claimed. ∎

PROPOSITION 2.3. *Let X, Y be two topological spaces, and let $F : X \to 2^Y$ be a multifunction, with nonempty values, such that $F^-(y)$ is open for all $y \in Y$ and $X \setminus F^-(y_0)$ is sequentially compact for some $y_0 \in Y$.*

Then, for every nondecreasing sequence $\{Y_n\}$ of subsets of Y, with $\bigcup_{n \in \mathbf{N}} Y_n = Y$, there exists $n \in \mathbf{N}$ such that $F^-(Y_n) = X$.

PROOF. Arguing by contradiction, assume that for every $n \in \mathbf{N}$, there is $x_n \in X$ such that

$$F(x_n) \cap Y_n = \emptyset. \tag{2.1}$$

Fix $\nu \in \mathbf{N}$ such that $y_0 \in Y_\nu$. Thus, for each $n > \nu$, one has $y_0 \notin F(x_n)$, that is $x_n \in X \setminus F^-(y_0)$. Consequently, since this latter set is sequentially compact, the sequence $\{x_n\}$ has a cluster point $x^* \in X \setminus F^-(y_0)$. Fix $y^* \in F(x^*)$ and choose $n_1 \in \mathbf{N}$ such that $y^* \in Y_{n_1}$. Since, by assumption, $F^-(y^*)$ is a neighbourhood of x^*, there exists $n_2 > n_1$ such that $x_{n_2} \in F^-(y^*)$. Hence, $y^* \in F(x_{n_2}) \cap Y_{n_2}$, against (2.1). ∎

PROPOSITION 2.4. *Let $I \subseteq \mathbf{R}$ be an interval, X a topological space, and $F : I \to 2^X$ a multifunction, with sequentially compact and sequentially closed values, such that for each $x \in X$, $I \setminus F^-(x)$ is connected and open in I.*

Then, F is sequentially upper semicontinuous.

PROOF. Let $\{(\lambda_n, x_n)\}$ be a sequence in $\mathrm{gr}(F)$, with $\{\lambda_n\}$ converging to some $\lambda_0 \in I$. So, one has

$$x_n \in F(\lambda_n), \qquad \forall n \in \mathbf{N}. \tag{2.2}$$

Consider the sets

$$\mathbf{N}_1 = \{n \in \mathbf{N} : \lambda_n \le \lambda_0\}$$

and

$$\mathbf{N}_2 = \{n \in \mathbf{N} : \lambda_n \ge \lambda_0\}.$$

One of them is infinite. Suppose, for instance, that $\mathbf{N}_1$ is so (the reasoning is similar if $\mathbf{N}_2$ is infinite). Put

$$A = \{\lambda \in I : \{n \in \mathbf{N}_1 : x_n \in F(\lambda)\} \text{ is infinite}\}.$$

First, assume that $\lambda_0 \in A$. Then, since $F(\lambda_0)$ is sequentially compact, there is a subsequence of $\{x_n\}$ converging to some $x_0 \in F(\lambda_0)$, and we are done. Now, assume that $\lambda_0 \notin A$. So, there is $\nu \in \mathbf{N}$ such that

$$x_n \notin F(\lambda_0), \qquad \forall n > \nu, \quad \text{with } n \in \mathbf{N}_1. \tag{2.3}$$

In view of (2.3), $\lambda_0 > \inf I$. Let $r = \inf_{n \in \mathbf{N}} \lambda_n$. We claim that $r \in A$. Assume the contrary. Thus, there is $\nu_1 > \nu$ such that

$$x_n \notin F(r), \qquad \forall n > \nu_1, \quad \text{with } n \in \mathbf{N}_1. \tag{2.4}$$

Then, if $n \in \mathbf{N}_1$ and $n > \nu_1$, from (2.3) and (2.4) we get that both r and λ_0 lie in $I \setminus F^-(x_n)$ which is, by assumption, connected. But $r \leq \lambda_n \leq \lambda_0$, and hence, $\lambda_n \in I \setminus F^-(x_n)$, against (2.2). Since $F(r)$ is sequentially compact, there is an increasing sequence $\{n_k\}$ in $\mathbf{N}_1$ such that the sequence $\{x_{n_k}\}$ converges to some $x^* \in F(r)$. Now, reasoning as above, we see that for each $\rho \in]r, \lambda_0[$, the set $\{k \in \mathbf{N} : x_{n_k} \in F(\rho)\}$ is infinite. Consequently, since $F(\rho)$ is sequentially closed, we have $x^* \in F(\rho)$. But $F^-(x^*)$ is closed, and so $x^* \in F(\lambda_0)$, which concludes the proof. ∎

We now can prove Theorem 2.1.

PROOF OF THEOREM 2.1. Consider the multifunctions $F : I \to 2^X$ and $G : X \to 2^I$ defined by

$$F(\lambda) = \{x \in X : f(x, \lambda) \leq 0\},$$

and

$$G(x) = \{\lambda \in I : f(x, \lambda) > 0\}.$$

Thanks to (a) and (b), we can apply Proposition 2.3 to G. Thus, if $\{I_k\}$ is a nondecreasing sequence of compact intervals with $\lambda_0 \in I_1$ and $\bigcup_{k \in \mathbf{N}} I_k = I$, there is $k \in \mathbf{N}$ such that $G^-(I_k) = X$. Put

$$S = \{(x, \lambda) \in X \times I_k : f(x, \lambda) \leq 0\},$$

as well as

$$\Phi(x) = G(x) \cap I_k,$$

for all $x \in X$. Observe that Φ is a lower semicontinuous multifunction with nonempty connected values whose graph does not intersect S. Moreover, by (b), the projection of S on I_k is equal to I_k. Consequently, by Proposition 2.1, S is disconnected. But S is homeomorphic to $\mathrm{gr}(F_{|I_k})$, and so this latter set is disconnected. From (a) and (b), we directly get that the multifunction F satisfies the assumptions of Proposition 2.4. Consequently, it is sequentially upper semicontinuous. Since (by (c)) $F(\lambda_0)$ is connected, we then conclude, in view of Proposition 2.2, that $F_{|I_k}$ is not lower semicontinuous. Hence, there exist $\lambda^* \in I_k$, a sequence $\{\lambda_n\}$ in I_k converging to λ^*, and an open set $U \subseteq X$ intersecting $F(\lambda^*)$ such that $F(\lambda_n) \cap U = \emptyset$ for all $n \in \mathbf{N}$. Of course, that means $f(x, \lambda_n) > 0$ for all $x \in U$, $n \in \mathbf{N}$. By (a), we infer that $f(x, \lambda^*) \geq 0$ for all $x \in U$. Finally, let x^* be any point in $F(\lambda^*) \cap U$. So, $f(x^*, \lambda^*) = 0$, and hence x^* is a local minimum for $f(\cdot, \lambda^*)$. ∎

The next result is a consequence of Theorem 2.1 which is particularly useful in view of applications to differential equations.

THEOREM 2.2. *Let X be an unbounded, closed, convex set in a separable and reflexive real Banach space E; $\Phi : X \to \mathbf{R}$ a lower semicontinuous convex functional; $\Psi : X \to \mathbf{R}$ a sequentially weakly continuous functional; $I \subseteq \mathbf{R}$ an interval; $\mu_0 \in I$; and $h : I \to \mathbf{R}$ a continuous concave function. Put*

$$a = \sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + (\lambda - \mu_0)\Psi(x) + h(\lambda)), \qquad b = \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + (\lambda - \mu_0)\Psi(x) + h(\lambda)),$$

and assume that $a < b$. Moreover, suppose that

$$\lim_{x \in X,\ \|x\| \to +\infty} (\Phi(x) + (\lambda - \mu_0)\Psi(x)) = +\infty,$$

for all $\lambda \in I$. Then, for each $r \in]a, b[$, there exist $\lambda^* \in I \setminus \{\mu_0\}$ and $x^* \in X$ such that

$$\Phi(x^*) + (\lambda^* - \mu_0)\Psi(x^*) + h(\lambda^*) = r,$$

and x^* is a local, nonabsolute minimum for $\Phi + (\lambda^* - \mu_0)\Psi$ in the relative weak topology of X.

PROOF. Fix $r \in]a, b[$, and put

$$f(x, \lambda) = \Phi(x) + (\lambda - \mu_0)\Psi(x) + h(\lambda) - r,$$

for all $(x, \lambda) \in X \times I$. We wish to apply Theorem 2.1 to such an f, endowing X with the relative weak topology. Since $r < b$ and h is continuous and concave, condition (a) is clearly satisfied. Moreover, since Φ is convex, we can satisfy condition (c) taking $\lambda_0 = \mu_0$. Now, fix $\lambda \in I$, and consider the set

$$L = \{x \in X : \Phi(x) + (\lambda - \mu_0)\Psi(x) \leq r - h(\lambda)\}.$$

It is clear that L is nonempty (since $a < r$) and bounded, in view of the coercivity of the functional $\Phi + (\lambda - \mu_0)\Psi$. Let us show that L is weakly closed in X (and so in E, since X is weakly closed). So, let $x_0 \in X$ be such that

$$\Phi(x_0) + (\lambda - \mu_0)\Psi(x_0) > r - h(\lambda).$$

Choose $\rho > 0$ so that

$$\Phi(x) + (\lambda - \mu_0)\Psi(x) > r - h(\lambda),$$

for all $x \in X$, with $\|x - x_0\| > \rho$. Since E is separable and reflexive, the set $\{x \in X : \|x - x_0\| \leq \rho\}$, equipped with the relative weak topology, is metrizable. Consequently, the restriction to this set of the functional $\Phi + (\lambda - \mu_0)\Psi$ is weakly continuous. Hence, there is a neighbourhood U of x_0 in the weak topology of E such that

$$\Phi(x) + (\lambda - \mu_0)\Psi(x) > r - h(\lambda),$$

for all $x \in U \cap X$, with $\|x - x_0\| \leq \rho$. This inequality is so satisfied for all $x \in U \cap X$, showing that $X \setminus L$ is weakly open in X. Then, by reflexivity and by the Eberlein-Smulyan theorem, L is also sequentially weakly compact, and so condition (b) is satisfied. At this point, the existence of λ^* and x^* as in the conclusion follows directly from Theorem 2.1. In particular, the fact that x^* is not an absolute minimum for $\Phi + (\lambda^* - \mu_0)\Psi$ follows from the inequality $a < r$. Moreover, the fact that $\lambda^* \neq \mu_0$ follows from the convexity of Φ. ∎

REMARK 2.1. From the above proof, it is clear that when $\mu_0 = \inf I$ (respectively, $\mu_0 = \sup I$), it suffices to assume that Ψ is sequentially weakly lower (respectively, upper) semicontinuous.

3. A THREE CRITICAL POINTS THEOREM

The key to proving Theorem 1.1 is provided by the following result which comes from a joint application of Theorem 2.2 with Corollary 1 in [2].

THEOREM 3.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbf{R}$ a continuously Gâteaux differentiable and convex functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbf{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbf{R}$ an interval; and $\mu_0 \in I$. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) + (\lambda - \mu_0)\Psi(x)) = +\infty, \tag{3.1}$$

for all $\lambda \in I$, and that there exists a continuous concave function $h : I \to \mathbf{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\Phi(x) + (\lambda - \mu_0)\Psi(x) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (\Phi(x) + (\lambda - \mu_0)\Psi(x) + h(\lambda)). \tag{3.2}$$

Then, there exists $\lambda^* \in I \setminus \{\mu_0\}$ such that the equation

$$\Phi'(x) + (\lambda^* - \mu_0)\Psi'(x) = 0$$

has at least three solutions in X.

PROOF. Clearly, Φ and Ψ satisfy the hypotheses of Theorem 2.2. In particular, the fact that Ψ' is compact implies that Ψ is sequentially weakly continuous [5, Corollary 41.9]. Hence, there exists $\lambda^* \in I \setminus \{\mu_0\}$ such that the functional $\Phi + (\lambda^* - \mu_0)\Psi$ has a local, nonabsolute minimum in the weak (and so in the strong) topology of X. By (3.1), this functional has an absolute minimum, too. Moreover, our assumptions ensure that it satisfies the Palais-Smale condition (see, for instance, [5, Example 38.25]). Then, by Corollary 1 in [2], the same functional admits a third critical point, as claimed. ∎

The application of Theorem 3.1 in proving Theorem 1.1 is made possible by the following proposition.

PROPOSITION 3.1. *Let X be a nonempty set, and Φ, J two real functions on X. Assume that there are $r > 0$, $x_0, x_1 \in X$, such that*

$$\Phi(x_0) = J(x_0) = 0, \qquad \Phi(x_1) > r,$$
$$\sup_{x \in \Phi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}. \tag{3.3}$$

Then, for each ρ satisfying

$$\sup_{x \in \Phi^{-1}(]-\infty,r])} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)}, \tag{3.4}$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\rho - J(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\rho - J(x))).$$

PROOF. First of all, observe that

$$\inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\rho - J(x))) = \inf_{x\in J^{-1}([\rho,+\infty[)} \Phi(x).$$

Next, note that by (3.4), one has

$$r \leq \inf_{x\in J^{-1}([\rho,+\infty[)} \Phi(x).$$

Moreover, since $\Phi(x_1) > r$, from (3.4), we infer that $J(x_1) > \rho$. This implies that the function $\lambda \to \inf_{x\in X}(\Phi(x) + \lambda(\rho - J(x)))$ tends to $-\infty$ as $\lambda \to +\infty$. But, this function is upper semicontinuous in $[0, +\infty[$, and hence, it attains its supremum at a point $\bar{\lambda}$. We now distinguish two cases. If $0 \leq \bar{\lambda} < r/\rho$ (note that $\rho > 0$ since $\Phi(x_0) = J(x_0) = 0$), then $\Phi(x_0) + \bar{\lambda}(\rho - J(x_0)) = \bar{\lambda}\rho < r$. If $r/\rho \leq \bar{\lambda}$, then since (by (3.4) again) $(r - \Phi(x_1))/(\rho - J(x_1)) < r/\rho$, we have $\Phi(x_1) + \bar{\lambda}(\rho - J(x_1)) < r$, and the proof is complete. ∎

REMARK 3.1. Let X be a nonempty set, and Φ, J two real functions on X having a common zero. Consider the function $\varphi :]0, +\infty[\to [0, +\infty]$ defined by putting

$$\varphi(t) = \frac{\sup_{x\in\Phi^{-1}(]-\infty,t])} J(x)}{t},$$

for all $t > 0$. Then, it is easy to check that the following conditions are equivalent.

(i) The function φ is not nonincreasing.

(ii) There exist $r > 0$ and $x_1 \in X$, with $\Phi(x_1) > r$, such that

$$\sup_{x \in \Phi^{-1}(]-\infty,r])} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

4. PROOF OF THEOREM 1.1

We are going to apply Theorem 3.1, taking $I = [0, +\infty[$, $\mu_0 = 0$, and $X = W_0^{1,2}(\Omega)$, with the norm $\|u\| = (\int_\Omega |\nabla u(x)|^2\, dx)^{1/2}$. For each $u \in X$, put

$$J_1(u) = \int_\Omega \left(\int_0^{u(x)} f(t)\, dt \right) dx$$

and

$$J_2(u) = \int_\Omega \left(\int_0^{u(x)} g(t)\, dt \right) dx.$$

By (1.3), there are $\eta \in]0, 1]$ and $c > 0$ such that

$$\int_0^\xi f(t)\, dt \le c|\xi|^\gamma,$$

for all $\xi \in [-\eta, \eta]$. Of course, it is not restrictive to assume that $\gamma < 2n/(n-2)$ (if $n > 2$). In view of (1.2), if we put

$$c_1 = \max\left\{ c, \sup_{|\xi|>\eta} \frac{a(1+|\xi|^s)}{|\xi|^\gamma} \right\},$$

we have

$$\int_0^\xi f(t)\, dt \le c_1|\xi|^\gamma,$$

for all $\xi \in \mathbf{R}$. Consequently, if $r > 0$ and $\|u\|^2 \le 2r$, by the Sobolev embedding theorem, we have (for suitable constants c_2, c_3)

$$J_1(u) \le c_1 \int_\Omega |u(x)|^\gamma\, dx \le c_2 \left(\int_\Omega |\nabla u(x)|^2\, dx \right)^{\gamma/2} \le c_3 r^{\gamma/2}.$$

Hence, we have

$$\lim_{r \to 0^+} \frac{\sup_{\|u\|^2 \le 2r} J_1(u)}{r} = 0.$$

Since, by assumption, $\sup_{\xi \in \mathbf{R}} \int_0^\xi f(t)\, dt > 0$, we can choose $w \in X \setminus \{0\}$ in such a way that $J_1(w) > 0$. Now, fix $r, \epsilon > 0$, with $r < (1/2)\|w\|^2$, so that

$$\sup_{\|u\|^2 \le 2r} J_1(u) \le 2r \frac{J_1(w)}{\|w\|^2} - \epsilon.$$

Next, fix $\delta > 0$ satisfying

$$\delta \left(\sup_{\|u\| \le 2r} |J_2(u)| + 2r \frac{|J_2(w)|}{\|w\|^2} \right) < \epsilon.$$

Then, if we put

$$\sigma = \epsilon - \delta \left(\sup_{\|u\|^2 \le 2r} |J_2(u)| + 2r \frac{|J_2(w)|}{\|w\|^2} \right),$$

we readily have

$$\sup_{\|u\|^2 \le 2r} (J_1(u) + \mu J_2(u)) \le 2r \frac{J_1(w) + \mu J_2(w)}{\|w\|^2} - \sigma,$$

for all $\mu \in [-\delta, \delta]$. Finally, fix $\mu \in [-\delta, \delta]$, and define $\Phi, \Psi : X \to \mathbf{R}$, and $h : [0, +\infty[\to [0, +\infty[$, by

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|^2, \\ \Psi(u) &= -(J_1(u) + \mu J_2(u)), \\ h(\lambda) &= \rho\lambda, \end{aligned}$$

where ρ is a fixed number satisfying

$$\sup_{\|u\|^2 \leq 2r} (J_1(u) + \mu J_2(u)) < \rho < 2r \frac{J_1(w) + \mu J_2(w)}{\|w\|^2}.$$

By (1.1), it follows that the functional Ψ is continuously Gâteaux differentiable, with compact Gâteaux derivative, the weak solutions of the problem

$$\begin{aligned} -\Delta u &= \lambda(f(u) + \mu g(u)), \qquad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0 \end{aligned}$$

being precisely the critical points in X of the functional $\Phi + \lambda\Psi$ [6, Proposition B.10]. Furthermore, observe that thanks to (1.2) and to the Poincaré inequality, one readily has

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty,$$

for all $\lambda \geq 0$. Consequently, thanks to Proposition 3.1, Φ, Ψ, and h satisfy all the assumptions of Theorem 3.1, and our conclusion then follows directly from the latter. ∎

5. REMARKS

We begin this section with the following remark.

REMARK 5.1. In Theorem 1.1, the condition $\sup_{\xi \in \mathbf{R}} \int_0^\xi f(t)\,dt > 0$ is essential. Indeed, if we drop this condition, then we can assume $f = g = 0$, and, of course, the conclusion does not hold. Likewise, we cannot drop condition (1.3). To see this, take $f = 1$ and $g = 0$. Note that when $n = 1$, condition (1.1) can be removed. Concerning condition (1.2), we believe that some slightly more sophisticated examples should show that it cannot be omitted. Moreover, the conclusion of Theorem 1.1 cannot hold, in general, for any $\mu \in \mathbf{R}$. To see this, it suffices to take first an f satisfying (together with $-f$) the assumptions, and then $g = f$, $\mu = -1$.

REMARK 5.2. Proposition 3.1 provides a first, natural way of finding a suitable continuous concave function h for which inequality (3.2) does hold. Precisely, it provides an h which is linear. It would be very interesting to find a pair of functionals Φ, Ψ satisfying the assumptions of Theorem 3.1 and such that for each linear function h on $\mathbf{R}$, inequality (3.2) does not hold.

REMARK 5.3. Clearly, if Φ, J are two real functions on a set X having a common zero x_0, the simplest way of satisfying condition (3.3) in Proposition 3.1 is to assume that

$$\lim_{r \to 0^+} \frac{\sup_{x \in \Phi^{-1}(]-\infty, r])} J(x)}{r} = 0,$$

and that there is some $x_1 \in X$ such that $\min\{\Phi(x_1), J(x_1)\} > 0$. Now, besides these assumptions, suppose that X is a topological space, Ψ is continuous at x_0, and x_0 is a strict local minimum for Φ. Then, for every $\lambda > \Phi(x_1)/J(x_1)$, x_0 is a strict local, nonabsolute minimum for $\Phi + \lambda J$.

To see this, fix an $\eta > 0$ such that

$$\frac{\sup_{x \in \Phi^{-1}(]-\infty, r])} J(x)}{r} < \frac{1}{\lambda}, \tag{5.1}$$

for all $r \in]0, \eta[$. Next, choose a neighbourhood U of x_0 such that

$$0 < \Phi(x) < \eta,$$

for all $x \in U \setminus \{x_0\}$. Consequently, in view of (5.1), if $x \in U \setminus \{x_0\}$, we have

$$\frac{J(x)}{\Phi(x)} < \frac{1}{\lambda},$$

that is

$$\Phi(x) - \lambda J(x) > 0.$$

From this, recalling that $\Phi(x_1) - \lambda J(x_1) < 0$, our claim follows.

At this point, the meaning of the present remark is clear. That is to say, when we are in a setting of the type described above, the conclusion of Theorem 2.2 (even in a more precise form) follows at once, without resorting to Theorem 2.1. Consequently, always in the same setting, the nontriviality of the conclusion of Theorem 3.1 would be due to Corollary 1 in [2] only. As it comes out from the proof, this remark applies to Theorem 1.1 when $g = 0$. In other words, the full power of Theorem 1.1 resides in the cases where $g \neq 0$.

REMARK 5.4. The final remark concerns the position of Theorem 1.1 with respect to known results. So, given a continuous function $\varphi : \mathbf{R} \to \mathbf{R}$, let us consider the problem

$$\begin{aligned} -\Delta u &= \lambda \varphi(u), \qquad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0. \end{aligned}$$

In checking the (by now boundless) literature on multiplicity results for this problem, we can realize two distinct categories of them, according to whether the conditions imposed on φ imply that $\varphi(0) = 0$ or not. The results where $\varphi(0) = 0$ are certainly the majority. We refer, for instance, to [7–13], and, more generally, to the monographs [6,14,15].

In our Theorem 1.1, condition (1.3) implies that $f(0) = 0$, but we can have $g(0) \neq 0$, and so for our φ (that is $f + \mu g$) we can have $\varphi(0) \neq 0$. In another numerous category of papers, the ancestor of which is [16], it is required that the limits $\lim_{\xi \to +\infty}(\varphi(\xi)/\xi)$ and $\lim_{\xi \to -\infty}(\varphi(\xi)/\xi)$ exist, in order to do suitable comparisons with the eigenvalues of the problem $-\Delta u = \lambda u$, $u_{|\partial\Omega} = 0$. We refer, for instance, to the papers quoted in [17] and, furthermore, to [18–21] and the references therein.

The hypotheses of Theorem 1.1 do not imply that the above limits exist. In conclusion, assuming $n \geq 3$, consider the specific problem

$$\begin{aligned} -\Delta u &= \lambda \left(|u|^{\gamma-2} u \cos |u|^{\gamma} + \mu \left(a|u|^{q} + b \right) \right), \qquad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0, \end{aligned}$$

where $a, b, \mu \in \mathbf{R}$, $0 < q < 1$, and $2 < \gamma < 2n/(n-2)$.

Clearly, Theorem 1.1 applies. Accordingly, if $|\mu|$ is small enough, there is $\lambda > 0$ for which the problem has at least three weak solutions. Actually, these solutions are classical since the right-hand side of the equation is locally Hölder continuous.

Such a conclusion, for the problem under consideration, can be deduced from none of the results we have quoted above.

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