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# Fat Points in $\mathbb{P}^1 \times \mathbb{P}^1$ and Their Hilbert Functions

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*Abstract.* We study the Hilbert functions of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. If $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is an arbitrary fat point scheme, then it can be shown that for every $i$ and $j$ the values of the Hilbert function $H_Z(l, j)$ and $H_Z(i, l)$ eventually become constant for $l \gg 0$. We show how to determine these eventual values by using only the multiplicities of the points, and the relative positions of the points in $\mathbb{P}^1 \times \mathbb{P}^1$. This enables us to compute all but a finite number values of $H_Z$ without using the coordinates of points. We also characterize the ACM fat point schemes using our description of the eventual behaviour. In fact, in the case that $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM, then the entire Hilbert function and its minimal free resolution depend solely on knowing the eventual values of the Hilbert function.

## Introduction

The Hilbert function of a fat point scheme in $\mathbb{P}^n$ is the basis for many questions about fat points schemes. Although some facts have been established (see the survey of Harbourne [6] for the case of $n = 2$), we do not have a complete understanding of the Hilbert functions of fat point schemes.

In this paper we investigate the Hilbert functions of fat point schemes in a different space, specifically, in $\mathbb{P}^1 \times \mathbb{P}^1$. Interest in the Hilbert functions of fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$ is motivated, in part, by the work of Catalisano, *et al.* [2] which exhibited a connection between a specific value of the Hilbert function of a special fat point scheme in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and a classical problem of computing the dimension of certain secant varieties to the Segre variety.

The Hilbert functions of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ appear to be first studied by Giuffrida, *et al.* [3]. Some of the results of [3] were extended and generalized to sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by the second author [8, 9]. Unlike the case of sets of simple points in $\mathbb{P}^n$, the problem of characterizing the Hilbert functions of sets of reduced points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, even in the case of $\mathbb{P}^1 \times \mathbb{P}^1$, remains open. Arithmetically Cohen-Macaulay fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ were studied by the first author [5] (which was based upon [4]). Catalisano, *et al.* [2] give some results about fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. However, like the case of fat point schemes in $\mathbb{P}^n$, we do not have a complete understanding of the Hilbert functions of fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

In this paper we are specifically interested in studying the eventual behaviour of the Hilbert function of a fat point scheme $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. If $Z$ is an arbitrary fat point scheme and if $H_Z$ denotes its Hilbert function, then it is not difficult to show that for

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any $i$ or $j$, the values $H_Z(l, j)$ and $H_Z(i, l)$ become constant for $l \gg 0$. Our first main result (Theorem 3.2) is to calculate these eventual values by using numerical information about $Z$. In particular, we show that these values can be calculated directly from the multiplicities of the points, and from the relative positions of the points in the support, that is, if $P, P'$ are in the support, we only need to know if $\pi_i(P) = \pi_i(P')$ for $i = 1, 2$ where $\pi_i$ is the $i$-th projection map. The actual coordinates of the points are therefore not needed to compute all but a finite number of values of $H_Z$.

We then show that the eventual behaviour of $H_Z$ gives us further information about the scheme $Z$. In particular, we show (*cf.* Theorem 4.8) that the eventual values of $H_Z$ can be used to determine if $Z$ is arithmetically Cohen-Macaulay (ACM). In fact, a specific type of eventual behaviour characterizes the ACM fat point schemes of $\mathbb{P}^1 \times \mathbb{P}^1$. We relate our characterization with the results of [3] and [5]. Furthermore, in the case that $Z$ is ACM, the eventual values of $H_Z$ can be used to completely determine the entire Hilbert function, *and* the minimal free resolution, of $Z$.

This paper has five parts. In the first section we recall the relevant facts about bigraded rings and fat point schemes. We also give some elementary properties for the Hilbert function of a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. In the second section we compute the Hilbert function of a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support lies on either a $(0, 1)$-line or a $(1, 0)$-line. In the third section we introduce two tuples $\alpha_Z$ and $\beta_Z$ that contain information about the multiplicities and relative position of the points, and show how to compute all but a finite number of values of the Hilbert function from $\alpha_Z$ and $\beta_Z$. In the fourth section we show how to use $\alpha_Z$ and $\beta_Z$ to determine if $Z$ is ACM. In the final section, we look at some ACM fat point schemes with some extra conditions on their multiplicities.

Many of these results had their genesis in examples. Instrumental in computing these examples was the computer program CoCoA [1]. We would like to thank A. Ragusa for his useful comments and suggestions. We would also like to thank the referee for their helpful comments and suggestions, and especially for suggesting a shorter proof for Theorem 2.2.

## 1 Preliminaries

In this section we recall the necessary definitions and facts about bigraded rings and fat point schemes.

Let $\mathbb{N} := \{0, 1, 2, \dots\}$. It will be useful to consider in $\mathbb{Z} \times \mathbb{Z}$ and in $\mathbb{N} \times \cdots \times \mathbb{N}$ the partial ordering induced by the usual one in $\mathbb{Z}$ and in $\mathbb{N}$ respectively. We will denote it by "$\leq$". Thus, if $(i_1, i_2), (j_1, j_2) \in \mathbb{N}^2$, then we write $(i_1, i_2) \leq (j_1, j_2)$ if $i_k \leq j_k$ for $k = 1, 2$.

We let $\mathbf{k}$ denote an algebraically closed field. Let $R = \mathbf{k}[x_0, x_1, y_0, y_1]$ where $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$. Then the ring $R$ is $\mathbb{N}^2$-*graded*, or simply, *bigraded*, that is,

$$R = \bigoplus_{(i,j) \in \mathbb{N}^2} R_{i,j} \quad \text{and} \quad R_{i_1,i_2} R_{j_1,j_2} \subseteq R_{i_1+j_1, i_2+j_2}$$

were each $R_{i,j}$ consists of all the *bihomogeneous elements* of degree $(i, j)$.

For each $(i, j) \in \mathbb{N}^2$, the set $R_{i,j}$ is a finite dimensional vector space over $\mathbf{k}$. A basis for $R_{i,j}$ is the set of monomials $\{x_0^{a_0}x_1^{a_1}y_0^{b_0}y_1^{b_1} \in R \mid (a_0+a_1, b_0+b_1) = (i,j)\}$. It follows that $\dim_{\mathbf{k}} R_{i,j} = (i+1)(j+1)$ for all $(i,j) \in \mathbb{N}^2$.

Suppose that $I = (F_1, \ldots, F_r) \subseteq R$ is an ideal such that the $F_i$s are bihomogeneous elements. Then $I$ is called a *bihomogeneous ideal*. If $I \subseteq R$ is any ideal, then we define $I_{i,j} := R_{i,j} \cap I$. The set $I_{i,j}$ is a subvector space of $R_{i,j}$. If $I$ is a bihomogeneous ideal, then $I = \bigoplus_{(i,j)} I_{i,j}$.

If $I$ is a bihomogeneous ideal of $S$, then the quotient ring $S = R/I$ is also bigraded, *i.e.*, $S = \bigoplus_{(i,j)} S_{i,j}$ where $S_{i,j} := R_{i,j}/I_{i,j}$ for all $(i,j) \in \mathbb{N}^2$. The numerical function $H_S \colon \mathbb{N}^2 \to \mathbb{N}$ defined by

$$(i,j) \longmapsto \dim_{\mathbf{k}} S_{i,j} = \dim_{\mathbf{k}} R_{i,j} - \dim_{\mathbf{k}} I_{i,j}$$

is the *Hilbert function of* $S = R/I$. We sometimes write the values of the Hilbert function $H_S$ as an infinite matrix $(M_{i,j})$ where $M_{i,j} := H_S(i,j)$. For example, if $I = (0)$, then $H_{R/I}(i,j) = (i+1)(j+1)$, and so we write

$$H_{R/I} = \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots \\ 2 & 4 & 6 & 8 & \cdots \\ 3 & 6 & 9 & 12 & \cdots \\ 4 & 8 & 12 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that we begin the indexing of the rows and columns at 0 rather than 1.

***Remark 1.1*** In [3] the Hilbert function was referred to as the Hilbert matrix. However, we will refer to $(H_S(i,j))$ as the Hilbert function.

We wish to study the Hilbert functions of rings of the form $R/I$ where $I$ is the ideal associated to a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. We now recall the relevant definitions.

Let $\mathbb{P}^1 := \mathbb{P}^1_{\mathbf{k}}$ be the projective line defined over $\mathbf{k}$, and let $\mathbb{P}^1 \times \mathbb{P}^1$ be the product space. The coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ is the bigraded ring $R = \mathbf{k}[x_0, x_1, y_0, y_1]$ where $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$.

Suppose that

$$P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$$

is a point in this space. The ideal $\wp$ associated to $P$ is the bihomogeneous ideal

$$\wp = (a_1 x_0 - a_0 x_1, b_1 y_0 - b_0 y_1).$$

The ideal $\wp$ is a prime ideal of height two that is generated by an element of degree $(1,0)$ and an element of degree $(0,1)$.

If $P = P_1 \times P_2 \in \mathbb{P}^1 \times \mathbb{P}^1$, then we shall sometimes write $L_{P_1}$ and $L_{P_2}$ for the generators of the ideal $\wp = (L_{P_1}, L_{P_2})$ defining $P$ where $L_{P_1}$ is a form of degree $(1,0)$ and $L_{P_2}$ is a form of degree $(0,1)$. Since $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{Q}$, the quadric surface in $\mathbb{P}^3$, it is useful to note that $L_{P_1}$ defines a line in one ruling of $\mathcal{Q}$ and $L_{P_2}$ defines a line in the other ruling, and $P$ is the point of intersection of these two lines.

Let $\mathbb{X}$ be a set of $s$ reduced points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi_1 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projection morphism defined by $P_1 \times P_2 \mapsto P_1$. Let $\pi_2 \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the other projection morphism. The set $\pi_1(\mathbb{X}) = \{R_1, \dots, R_r\}$ is the set of $r \le s$ distinct first coordinates that appear in $\mathbb{X}$. Similarly, the set $\pi_2(\mathbb{X}) = \{Q_1, \dots, Q_t\}$ is the set of $t \le s$ distinct second coordinates. For $i = 1, \dots, r$, let $L_{R_i}$ denote the $(1,0)$ form that vanishes at all the points of $\mathbb{P}^1 \times \mathbb{P}^1$ which have first coordinate $R_i$. Similarly, for $j = 1, \dots, t$, let $L_{Q_j}$ denote the $(0,1)$ form that vanishes at all the points whose second coordinate is $Q_j$.

Let $D := \{(i, j) \mid 1 \le i \le r, 1 \le j \le t\}$. If $P \in \mathbb{X}$, then $I_P = (L_{R_i}, L_{Q_j})$ for some $(i, j) \in D$. (Note that this does not mean that if $(i, j) \in D$, then $P_{ij} \in \mathbb{X}$. There may be a pair $(i, j) \in D$, but $P_{ij} \notin \mathbb{X}$.) For each $(i, j) \in D$, let $m_{ij}$ be a positive integer if $P_{ij} \in \mathbb{X}$, otherwise, let $m_{ij} = 0$. Then we denote by $Z$ the subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the saturated bihomogeneous ideal

$$I_Z = \bigcap_{(i,j)\in D} \wp_{ij}^{m_{ij}}$$

where $\wp_{ij}^0 := (1)$. We say $Z$ is a *fat point scheme* of $\mathbb{P}^1 \times \mathbb{P}^1$. We sometimes say that $Z$ is a *set of fat points*. The integer $m_{ij}$ is called the *multiplicity* of the point $P_{ij}$. We shall sometimes denote the fat point scheme as

$$Z = \{(P_{ij}; m_{ij}) \mid (i, j) \in D\}.$$

In the case all the non-zero $m_{ij}$ are the same, we call $Z$ a *homogeneous fat point scheme*. The *support* of $Z$, written Supp$(Z)$ is the set of points $\mathbb{X}$. If $\mathbb{X} = \mathrm{Supp}(Z)$, then $I_{\mathbb{X}} = \sqrt{I_Z}$.

Let $I_Z$ be the defining ideal of a fat point scheme $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Because the ideal $I_Z \subseteq R$ is a bihomogeneous ideal, we can study its Hilbert function $H_{R/I_Z}$. We sometimes write $H_Z$ to denote $H_{R/I_Z}$, and say $H_Z$ is the *Hilbert function of Z*.

We give some elementary results about the Hilbert function of a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. These results generalize some of the results of [8] about sets of simple points.

It was shown in [8, Lemma 3.3] that if $\mathbb{X}$ is a reduced set of points, then there exists a $(1,0)$ form $L \in R$ (respectively, a $(0,1)$ form $L' \in R$) that is a non-zero divisor of $R/I_{\mathbb{X}}$. The proof of this lemma can extend to the non-reduced case:

***Lemma 1.2*** *Let Z be a fat point scheme of* $\mathbb{P}^1 \times \mathbb{P}^1$. *Then there exists a bihomogeneous element* $L \in R$ *(respectively,* $L' \in R$*) with* $\deg L = (1,0)$ *(respectively,* $\deg L' = (0,1)$*) such that* $\overline{L}$ *(respectively,* $\overline{L}'$*) is a non-zero divisor of* $R/I_Z$.

The existence of these non-zero divisors enables us to prove the following:

***Proposition 1.3*** *Let Z be a fat point scheme in* $\mathbb{P}^1 \times \mathbb{P}^1$ *and suppose that* $H_Z$ *is the Hilbert function of Z. Then*

(i) *for all* $(i, j) \in \mathbb{N}^2$, $H_Z(i, j) \le H_Z(i + 1, j)$, *and* $H_Z(i, j) \le H_Z(i, j + 1)$.

(ii) *if* $H_Z(i,j) = H_Z(i+1,j)$, *then* $H_Z(i+1,j) = H_Z(i+2,j)$.
(iii) *if* $H_Z(i,j) = H_Z(i,j+1)$, *then* $H_Z(i,j+1) = H_Z(i,j+2)$.

**Proof** Let $\overline{L}$ be the non-zero divisor of $R/I_Z$ from Lemma 1.2 with $\deg L = (1,0)$. For any $(i,j) \in \mathbb{N}^2$, the map $(R/I_Z)_{i,j} \xrightarrow{\times\overline{L}} (R/I_Z)_{i+1,j}$ is an injective map of vector spaces because $\overline{L}$ is a non-zero divisor. It then follows that $H_Z(i,j) \leq H_Z(i+1,j)$ for all $(i,j) \in \mathbb{N}^2$. The other statement of (i) is proved similarly.

The proof of (ii) and (iii) are similar, so we will only show (ii). Let $\overline{L}$ be as above. For each $(i,j) \in \mathbb{N}^2$, we have the following short exact sequence of vector spaces:

$$0 \longrightarrow (R/I_Z)_{i,j} \xrightarrow{\times\overline{L}} (R/I_Z)_{i+1,j} \longrightarrow (R/(I_Z,L))_{i+1,j} \longrightarrow 0.$$

If $H_Z(i,j) = H_Z(i+1,j)$, then this implies that the morphism $\times\overline{L}$ is an isomorphism of vector spaces, and thus, $(R/(I_Z,L))_{i+1,j} = 0$, or equivalently, $(I_Z,L)_{i+i,j} = R_{i+1,j}$. But then $(I_Z,L)_{i+2,j} = R_{1,0} \otimes_{\mathbf{k}} R_{i+1,j} = R_{i+2,j}$, and thus, $(R/(I_Z,L))_{i+2,j} = 0$ as well. The exact sequence then implies that $(R/I_Z)_{i+1,j} \cong (R/I_Z)_{i+2,j}$. ■

***Remark 1.4*** Proposition 1.3 implies that the values in the columns and rows of the Hilbert function $H_Z$, written as a matrix, must eventually stabilize, that is, stay constant. However, at least two questions remain. First, where do the rows and columns stabilize? Second, at what values must the columns and rows stabilize? These questions are answered in the following sections (Corollary 3.4).

***Remark 1.5*** Because Lemma 1.2 shows the existence of a non-zero divisor in $R/I_Z$ for any fat point scheme $Z$ of $\mathbb{P}^1 \times \mathbb{P}^1$, it follows that the inequality depth $R/I_Z \geq 1$ always holds. It should be noted that the arguments used in Lemma 1.2 and Proposition 1.3 use nothing special about $\mathbb{P}^1 \times \mathbb{P}^1$ and can be extended to fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Proposition 1.3 could also be deduced from Propositions 2.5 and 2.7 of [3].

## 2 Fat Point Schemes Whose Support Is on a Line

In this section we investigate the Hilbert functions of fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support lies on a line defined either by a form of degree $(1,0)$ or a form of degree $(0,1)$. Because $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathcal{Q}$, the quadric surface of $\mathbb{P}^3$, this is equivalent to studying those fat point schemes whose support is on one of the rulings of the surface. We show that the Hilbert function in this case can be computed directly from the multiplicities of the points. This result is a key component of our proof in the next section describing the eventual behaviour of all fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

So, let $Z$ be the fat point scheme

$$Z = \{(P_{11}; m_{11}), (P_{12}; m_{12}), (P_{13}; m_{13}), \ldots, (P_{1s}; m_{1s})\}$$

of $s$ fat points where $P_{1j} = R_1 \times Q_j$. Then $\mathrm{Supp}(Z) = \{P_{11}, \ldots, P_{1s}\}$. It follows that $\mathrm{Supp}(Z)$ lies on the line defined by the form $L_{R_1} \in R_{1,0}$.

Let $Z'$ denote a fat point scheme whose support lies on a line defined by a form of degree $(0, 1)$, that is, $Z' = \{(Q_1 \times R_1; m_{11}), \dots, (Q_s \times R_1; m_{s1})\}$ with $Q_i$ and $R_1$ as in $Z$. Then, for any $(i, j) \in \mathbb{N}^2$, $(I_Z)_{i,j} \cong (I_{Z'})_{j,i}$, and therefore, $H_Z(i, j) = H_{Z'}(j, i)$. Because of this relation, it is enough to investigate the case that the support of $Z$ is contained on the line defined by a form of degree $(1, 0)$.

***Remark 2.1*** The following result can be recovered from Theorem 4.1 of [3] and Theorem 2.1 in [5] if one first shows that these schemes are arithmetically Cohen-Macaulay. However, we give a new proof of this result that does not depend on knowing that the scheme is Cohen-Macaulay.

***Theorem 2.2*** *Let $Z = \{(P_{11}; m_{11}), (P_{12}; m_{12}), \dots, (P_{1s}, m_{1s})\}$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is on a line defined by a form of degree $(1, 0)$. Set $m = \max\{m_{1j}\}_{j=1}^s$. For $h = 0, \dots, m-1$, set $a_h = \sum_{j=1}^s (m_{1j} - h)_+$ where $(n)_+ := \max\{0, n\}$. Then the Hilbert function of $Z$ is*

$$H_Z = \begin{bmatrix} 1 & 2 & \cdots & a_0 - 1 & a_0 & a_0 & \cdots \\ 1 & 2 & \cdots & a_0 - 1 & a_0 & a_0 & \cdots \\ 1 & 2 & \cdots & a_0 - 1 & a_0 & a_0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & a_1 - 1 & a_1 & a_1 & \cdots \\ 1 & 2 & \cdots & a_1 - 1 & a_1 & a_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} +$$

$$\cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & a_{m-1} - 1 & a_{m-1} & a_{m-1} & \cdots \\ 1 & 2 & \cdots & a_{m-1} - 1 & a_{m-1} & a_{m-1} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Proof** For each $j = 1, \dots, s$, the ideal associated to $P_{1j}$ is $\wp_{1j} = (L_{R_1}, L_{Q_j})$. Set $L = L_{R_1}$ and note that $L$ defines the $(1, 0)$ line in $\mathbb{P}^1 \times \mathbb{P}^1$ on which all the points lie. Now for each $0 \leq h \leq m-1$ we set

$$Z_h = \{(P_{11}; (m_{11} - h)_+), \dots, (P_{1s}; (m_{1s} - h)_+)\}$$

and let $I_{Z_h}$ be the associated ideal. Thus $Z_0 = Z$. Furthermore, we have the identity $L^h \cap I_Z = L^h \cdot I_{Z_h}$ for each $h = 0, \dots, m-1$.

Since $L^m \in I_Z$, we have $0 = \overline{L}^m \cdot S \subseteq \overline{L}^{m-1} \cdot S \subseteq \cdots \subseteq \overline{L} \cdot S \subseteq S$ where $S = R/I_Z$ and $\overline{L}^i$ denotes the image of $L^i$ in $S$. It then follows that

$$H_Z(i, j) = \dim_{\mathbf{k}} S_{i,j} = \sum_{h=0}^{m-1} \dim_{\mathbf{k}} \left( \frac{\overline{L}^h \cdot S}{\overline{L}^{h+1} \cdot S} \right)_{i,j}.$$

Now for each $h = 0, \dots, m-1$,

$$\frac{\overline{L}^h \cdot S}{\overline{L}^{h+1} \cdot S} \cong \frac{L^h R}{L^{h+1} + L^h \cap I_Z} \cong \frac{L^h R}{L^{h+1} + L^h I_{Z_h}} \cong \overline{L}^h \left( \frac{R}{L + I_{Z_h}} \right).$$

Hence $\dim_{\mathbf{k}}\left(\frac{\overline{L}^h\cdot S}{\overline{L}^{h+1}\cdot S}\right)_{i,j} = \dim_{\mathbf{k}}\left(R/(L+I_{Z_h})\right)_{i-h,j}$, and thus

$$H_Z(i,j) = \sum_{h=0}^{m-1} \dim_{\mathbf{k}}\left(R/(L+I_{Z_h})\right)_{i-h,j}.$$

To compute $H_Z$, we thus need to compute the Hilbert function of $R/(L+I_{Z_h})$ for each $h$. We now note that for each $h$,

$$(L+I_{Z_h}) = (L, L_{Q_1}^{(m_{11}-h)_+}\cdots L_{Q_s}^{(m_{1s}-h)_+}),$$

that is, $(L+I_{Z_h})$ is a complete intersection generated by forms of degree $(1,0)$ and $(0,a_h)$. The resolution of $(L+I_{Z_h})$ is given by the *Koszul resolution*, *i.e.*,

$$0 \longrightarrow R(-1,-a_h) \longrightarrow R(-1,0)\oplus R(0,-a_h) \longrightarrow (L+I_{Z_h}) \longrightarrow 0.$$

Hence, the Hilbert function of $R/(L+I_{Z_h})$ is

$$H_{R/(L+I_{Z_h})} = \begin{bmatrix} 1 & 2 & \cdots & a_h-1 & a_h & a_h & \cdots \\ 1 & 2 & \cdots & a_h-1 & a_h & a_h & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This now completes the proof. ■

From now on, if $\alpha = (a_0,\ldots,a_{m-1})$ is a tuple of non-negative integers, then by $a_k \in \alpha$ we shall mean that $a_k$ appears as a coordinate in $\alpha$. The following corollary of Theorem 2.2 will be required in the next section.

***Corollary 2.3*** *With the notation as in Theorem 2.2, let* $\alpha = (a_0,\ldots,a_{m-1})$*. Fix* $j \in \mathbb{N}$*. Then, for all* $i \geq m-1 = \max\{m_{1k}\}_{k=1}^{s} - 1$,

$$H_Z(i,j) = \#\{a_k \in \alpha \mid a_k \geq 1\} + \#\{a_k \in \alpha \mid a_k \geq 2\} + \cdots + \#\{a_k \in \alpha \mid a_k \geq j+1\}.$$

**Proof** Fix a $j \in \mathbb{N}$, and set

$$(*) = \#\{a_k \in \alpha \mid a_k \geq 1\} + \#\{a_k \in \alpha \mid a_k \geq 2\} + \cdots + \#\{a_k \in \alpha \mid a_k \geq j+1\}.$$

From our definition of $a_0,\ldots,a_{m-1}$, it follows that $a_0 \geq a_1 \geq \cdots \geq a_{m-1}$. Let $l$ be the largest index such that $a_0,\ldots,a_{l-1} \geq j+1$ but $a_l,\ldots,a_{m-1} < j+1$. Set $\alpha' = (a_l,\ldots,a_{m-1})$.

For each integer $h = 1,\ldots,j+1$, we have

$$\#\{a_k \in \alpha \mid a_k \geq h\} = l + \#\{a_k \in \alpha' \mid a_k \geq h\}.$$

Thus

$$(*) = (j+1)l + \#\{a_k \in \alpha' \mid a_k \geq 1\} + \cdots + \#\{a_k \in \alpha' \mid a_k \geq a_l\}.$$

If we set $(**) = \#\{a_k \in \alpha' \mid a_k \geq 1\} + \cdots + \#\{a_i \in \alpha' \mid a_k \geq a_l\}$, then

$$\begin{aligned}(**) &= \#\{a_k \in \alpha' \mid a_k = 1\} + 2\#\{a_k \in \alpha' \mid a_k = 2\} + \\ &\qquad \cdots + a_l\#\{a_k \in \alpha' \mid a_k = a_l\} \\ &= a_l + a_{l+1} + \cdots + a_{m-1}.\end{aligned}$$

Hence, $(*) = (j+1)l + a_l + a_{l+1} + \cdots + a_{m-1}$.

On the other hand, by Theorem 2.2, if $i \geq m-1$, then

$$\dim_{\mathbf{k}}(R/I_Z)_{i,j} = \sum_{h=1}^{s} \min\{j+1, a_h\}.$$

Since $a_0, \ldots, a_{l-1} \geq j+1$, it follows that

$$\dim_{\mathbf{k}}(R/I_Z)_{i,j} = (j+1)l + a_l + a_{l+1} + \cdots a_{m-1} = (*)$$

which is what we wished to prove. ∎

## 3 The Eventual Behaviour of the Hilbert Function of a Fat Point Scheme

Let $P_1, \ldots, P_s$ be $s$ distinct points of $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose $m_1, \ldots, m_s$ are arbitrary positive integers. Let $Z = \{(P_1; m_1), \ldots, (P_s; m_s)\}$ be the resulting fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$. In this section we wish to describe the eventual behaviour of the Hilbert function of $Z$. We will show that the eventual values of the Hilbert function depend only upon the numbers $m_1, \ldots, m_s$ and numerical information describing $\mathbb{X} = \mathrm{Supp}(Z)$. This result is a generalization of a result of the second author[8, Corollary 5.13] about sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

We start by defining our notation. If $Z$ is a fat point scheme, let $\mathbb{X}$ denote the support of $Z$. We suppose that $|\mathbb{X}| = s$. Let $\pi_1(\mathbb{X})$ and $\pi_2(\mathbb{X})$ be defined as in the previous section. For each $R_i \in \pi_1(\mathbb{X})$, define

$$Z_{1,R_i} := \{(P_{ij_1}; m_{ij_1}), (P_{ij_2}; m_{ij_2}), \ldots, (P_{ij_{\alpha_i}}; m_{ij_{\alpha_i}})\}$$

where $P_{ij_k} = R_i \times Q_{j_k}$ are those points of $\mathrm{Supp}(Z)$ whose first projection is $R_i$. Thus $\pi_1(\mathrm{Supp}(Z_{1,R_i})) = \{R_i\}$, and furthermore it follows that

$$I_Z = \bigcap_{i=1}^{r} I_{Z_{1,R_i}}.$$

For each $R_i \in \pi_1(\mathbb{X})$ define $l_i := \max\{m_{ij_1}, \dots, m_{ij_{\alpha_i}}\}$. Then, for each integer $0 \le k \le l_i - 1$, we define

$$a_{i,k} := \sum_{t=1}^{\alpha_i} (m_{ij_t} - k)_+ \quad \text{where } (n)_+ := \max\{n, 0\}.$$

Let $\alpha_{R_i} := (a_{i,0}, \dots, a_{i,l_i-1})$ for each $R_i \in \pi_1(\mathbb{X})$. Define

$$\begin{aligned}\alpha_Z &:= (\alpha_{R_1}, \dots, \alpha_{R_r})\\ &= (a_{1,0}, \dots, a_{1,l_1-1}, a_{2,0}, \dots, a_{2,l_2-1}, \dots, a_{r,0}, \dots, a_{r,l_r-1}).\end{aligned}$$

Similarly, for each $Q_j \in \pi_2(\mathbb{X})$, define

$$Z_{2,Q_j} := \{(P_{i_1j}; m_{i_1j}), (P_{i_2j}; m_{i_2j}), \dots, (P_{i_{\beta_j}j}; m_{i_{\beta_j}j})\}$$

where $P_{i_kj} = R_{i_k} \times Q_j$ are those points of $\operatorname{Supp}(Z)$ whose second projection is $Q_j$. Thus $\pi_2(\operatorname{Supp}(Z_{2,Q_j})) = \{Q_j\}$. For $Q_j \in \pi_2(\mathbb{X})$ define $l'_j = \max\{m_{i_1j}, \dots, m_{i_{\beta_j}j}\}$. Then, for each integer $0 \le k \le l'_j - 1$, we define

$$b_{j,k} := \sum_{t=1}^{\beta_j} (m_{i_tj} - k)_+ \quad \text{where } (n)_+ := \max\{n, 0\}.$$

Let $\beta_{Q_j} := (b_{j,0}, \dots, b_{j,l'_j-1})$ for each $Q_j \in \pi_2(\mathbb{X})$. Define

$$\begin{aligned}\beta_Z &:= (\beta_{Q_1}, \dots, \beta_{Q_t})\\ &= (b_{1,0}, \dots, b_{1,l'_1-1}, b_{2,0}, \dots, b_{2,l'_2-1}, \dots, b_{t,0}, \dots, b_{t,l'_t-1}).\end{aligned}$$

***Example 3.1*** With the above notation, let us determine the tuples $\alpha_Z$ and $\beta_Z$ associated to the scheme $Z = \{(P_{11}; 4), (P_{12}; 2), (P_{23}; 3), (P_{32}; 2), (P_{41}; 3)\}$. The subscheme $Z_{1,R_1}$ is

$$Z_{1,R_1} = \{(P_{11}; 4), (P_{12}; 2)\}.$$

We set $l_1 := \max\{4, 2\} = 4$. Then

$$\begin{aligned}a_{1,0} &= 4 + 2 = 6\\ a_{1,1} &= (4-1)_+ + (2-1)_+ = 4\\ a_{1,2} &= (4-2)_+ + (2-2)_+ = 2\\ a_{1,3} &= (4-3)_+ + (2-3)_+ = 1.\end{aligned}$$

Hence, $\alpha_{R_1} = (6, 4, 2, 1)$. For $R_2, R_3$, and $R_4$, we get $\alpha_{R_2} = (3, 2, 1)$, $\alpha_{R_3} = (2, 1)$, $\alpha_{R_4} = (3, 2, 1)$. Hence

$$\alpha_Z = (6, 4, 2, 1, 3, 2, 1, 2, 1, 3, 2, 1).$$

Similarly, for $Q_1, Q_2, Q_3 \in \pi_2(\mathbb{X})$, $l'_1 = 4$, $l'_2 = 2$ and $l'_3 = 3$. So, we have $\beta_{Q_1} = (7, 5, 3, 1)$, $\beta_{Q_2} = (4, 2)$, and $\beta_{Q_3} = (3, 2, 1)$, and therefore,

$$\beta_Z = (7, 5, 3, 1, 4, 2, 3, 2, 1).$$

We now state and prove our main result about the eventual behaviour of the Hilbert function. Recall that if we write $a_k \in \alpha$, where $\alpha$ is a tuple of non-negative integers, then we shall mean that $a_k$ appears as a coordinate in $\alpha$.

***Theorem 3.2*** *Let $Z$ be a fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Then, with the above notation,*

(i) *for a fixed $j \in \mathbb{N}$, if $i \geq (l_1 + \cdots + l_r) - 1$, then*

$$\dim_{\mathbf{k}}(R/I_Z)_{i,j} = \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq 1\} + \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq 2\} + \cdots + \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq j+1\};$$

(ii) *for a fixed $i \in \mathbb{N}$, if $j \geq (l'_1 + \cdots + l'_t) - 1$, then*

$$\dim_{\mathbf{k}}(R/I_Z)_{i,j} = \#\{b_{k,l} \in \beta_Z \mid b_{k,l} \geq 1\} + \#\{b_{k,l} \in \beta_Z \mid b_{k,l} \geq 2\} + \cdots + \#\{b_{k,l} \in \beta_Z \mid b_{k,l} \geq i+1\}.$$

**Proof** We will only prove (i) since the proof of statement of (ii) is similar. Let $Z$ be a set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$, and let $\mathbb{X} = \mathrm{Supp}(Z)$. The proof is by induction on $r = |\pi_1(\mathbb{X})|$. If $r = 1$, *i.e.*, $\pi_1(\mathbb{X}) = \{R_1\}$, the conclusion follows from Corollary 2.3.

So, suppose that $r > 1$, and the theorem holds for all fat point schemes $Z'$ with $|\pi_1(\mathrm{Supp}(Z'))| < r$. For each $R_i \in \pi_1(\mathbb{X})$, we let $I_{Z_{1,R_i}}$ denote the ideal that defines the subscheme $Z_{1,R_i} := \{(P_{ij_1}; m_{ij_1}), (P_{ij_2}; m_{ij_2}), \ldots, (P_{ij_{\alpha_i}}; m_{ij_{\alpha_i}})\}$. We set

$$I_{\mathbb{Y}_1} := \bigcap_{i=1}^{r-1} I_{Z_{1,R_i}} \quad \text{and} \quad I_{\mathbb{Y}_2} := I_{Z_{1,R_r}}.$$

The ideals $I_{\mathbb{Y}_1}$ and $I_{\mathbb{Y}_2}$ are the defining ideals of fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ with $|\pi_1(\mathrm{Supp}(\mathbb{Y}_i))| < r$ for $i = 1, 2$. We shall also require the following result about $I_{\mathbb{Y}_1}$ and $I_{\mathbb{Y}_2}$.

**Claim** For any $j \in \mathbb{N}$, if $i \geq l_1 + \cdots + l_r - 1$, then $(I_{\mathbb{Y}_1} + I_{\mathbb{Y}_2})_{i,j} = R_{i,j}$.

**Proof of the Claim** Set $m = l_1 + \cdots + l_r$. It is enough to show that $(I_{\mathbb{Y}_1} + I_{\mathbb{Y}_2})_{m-1,0} = R_{m-1,0}$. Recall that for each $R_i \in \pi_1(\mathbb{X})$, the integer $l_i$ is defined to be $l_i = \max\{m_{ij_c}\}_{c=1}^{\alpha_i}$ where $Z_{1,R_i}$ is as above. If $(L_{R_i}, L_{Q_{j_c}})$ is the ideal associated to the point $P_{ij_c}$, then $I_{Z_{1,R_i}} = \bigcap_{c=1}^{\alpha_i} (L_{R_i}, L_{Q_{j_c}})^{m_{ij_c}}$. Note that $\deg L_{R_i} = (1, 0)$ and $\deg L_{Q_{j_c}} = (0, 1)$. From this description of $I_{Z_{1,R_i}}$, it follows that $L_{R_i}^{l_i} \in I_{Z_{1,R_i}}$. Thus $L_{R_1}^{l_1} \ldots L_{R_{r-1}}^{l_{r-1}} \in I_{\mathbb{Y}_1}$ and $L_{R_r}^{l_r} \in I_{\mathbb{Y}_2}$.

Set $J := (L_{R_1}^{l_1} \cdots L_{R_{r-1}}^{l_{r-1}}, L_{R_r}^{l_r}) \subseteq I_{\mathbb{Y}_1} + I_{\mathbb{Y}_2}$. Since $J$ is generated by a regular sequence, the bigraded resolution of $J$ is given by the Koszul resolution:

$$0 \longrightarrow R(-m, 0) \longrightarrow R(-m + l_r, 0) \oplus R(-l_r, 0) \longrightarrow J \longrightarrow 0.$$

If we use this exact sequence to calculate the dimension of $J_{m-1,0}$, then we find

$$\begin{aligned}\dim_{\mathbf{k}} J_{m-1,0} &= (m-1-(m-l_r)+1)+(m-1-l_r+1)-(m-1-m+1)\\ &= l_r+m-l_r = m = \dim_{\mathbf{k}} R_{m-1,0}.\end{aligned}$$

Since $\dim_{\mathbf{k}} J_{m-1,0} \leq \dim_{\mathbf{k}}(I_{\mathbb{Y}_1}+I_{\mathbb{Y}_2})_{m-1,0} \leq \dim_{\mathbf{k}} R_{m-1,0}$, the conclusion

$$(I_{\mathbb{Y}_1}+I_{\mathbb{Y}_2})_{m-1,0} = R_{m-1,0}$$

now follows. ■

From the short exact sequence

$$0 \longrightarrow I_{\mathbb{Y}_1} \cap I_{\mathbb{Y}_2} = I_Z \longrightarrow I_{\mathbb{Y}_1} \oplus I_{\mathbb{Y}_2} \longrightarrow I_{\mathbb{Y}_1} + I_{\mathbb{Y}_2} \longrightarrow 0$$

we deduce that

$$\dim_{\mathbf{k}}(I_Z)_{i,j} = \dim_{\mathbf{k}}(I_{\mathbb{Y}_1})_{i,j} + \dim_{\mathbf{k}}(I_{\mathbb{Y}_2})_{i,j} - \dim_{\mathbf{k}}(I_{\mathbb{Y}_1}+I_{\mathbb{Y}_2})_{i,j}$$

for all $(i, j) \in \mathbb{N}^2$. Thus, if $i \geq l_1 + \cdots + l_r - 1$, then by the claim we have

$$\begin{aligned}H_Z(i,j) &= (i+1)(j+1) - \dim_{\mathbf{k}}(I_{\mathbb{Y}_1})_{i,j} - \dim_{\mathbf{k}}(I_{\mathbb{Y}_2})_{i,j} + \dim_{\mathbf{k}}(I_{\mathbb{Y}_1}+I_{\mathbb{Y}_2})_{i,j}\\ &= (i+1)(j+1) - \dim_{\mathbf{k}}(I_{\mathbb{Y}_1})_{i,j} + (i+1)(j+1) - \dim_{\mathbf{k}}(I_{\mathbb{Y}_2})_{i,j}\\ &= H_{\mathbb{Y}_1}(i,j) + H_{\mathbb{Y}_2}(i,j).\end{aligned}$$

For each $h = 1, \ldots, j+1$, it follows that

$$\#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq h\} = \#\{a_{k,l} \in \alpha_{\mathbb{Y}_1} \mid a_{k,l} \geq h\} + \#\{a_{t,l} \in \alpha_{\mathbb{Y}_2} \mid a_{t,l} \geq h\}$$

where $\alpha_{\mathbb{Y}_i}$ is the tuple associated to the fat point scheme $\mathbb{Y}_i$ for $i = 1, 2$. The conclusion now follows by the induction hypothesis and the fact that $H_Z(i, j) = H_{\mathbb{Y}_1}(i, j) + H_{\mathbb{Y}_2}(i, j)$ if $i \geq l_1 + \cdots + l_r - 1$. ■

***Remark 3.3*** Suppose that $Z$ is a set of simple points in $\mathbb{P}^1 \times \mathbb{P}^1$, *i.e.,* the multiplicity of each point in $Z$ is one. So, if $\pi_1(Z) = \{R_1, \ldots, R_r\}$, then

$$Z_{1,R_i} = \{R_i \times Q_{i_1}, \ldots, R_i \times Q_{i_{\alpha_i}}\} \text{ for } i = 1, \ldots, r.$$

So, $l_i = 1$, and thus, $a_{i,0} = \sum_{j=1}^{\alpha_i} 1 = \alpha_i$. So, $\alpha_Z = (\alpha_1, \ldots, \alpha_r)$, which is exactly how $\alpha_Z$ is defined for sets of simple points in [8]. Thus Theorem 3.2 generalizes [8, Proposition 5.11] for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ to fat point schemes in $\mathbb{P}^1 \times \mathbb{P}^1$.

We can rewrite Theorem 3.2 more succinctly.

***Corollary 3.4*** *Let $Z$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. With the notation as in Theorem 3.2, let $m = l_1 + \cdots + l_r$ and $m' = l'_1 + \cdots + l'_t$. Then*

$$H_Z(i,j) = \begin{cases} \sum_{i=1}^{s} \binom{m_i+1}{2} & \text{if } (i,j) \geq (m-1, m'-1), \\ H_Z(m-1, j) & \text{if } i \geq m-1 \text{ and } j < m'-1, \\ H_Z(i, m'-1) & \text{if } j \geq m'-1 \text{ and } i < m-1. \end{cases}$$

**Proof** For any $j \in \mathbb{N}$, if $i \geq m-1$, then Theorem 3.2 implies that $H_Z(i,j) = H_Z(m-1,j)$. Similarly, for any $i \in \mathbb{N}$, if $j \geq m'-1$, then $H_Z(i,j) = H_Z(i,m'-1)$. Thus, for any $(i,j) \geq (m-1, m'-1)$, we have $H_Z(i,j) = H_Z(i,m'-1) = H_Z(m-1,m'-1)$.

All that remains to be shown is that $H_Z(m-1,m'-1) = \sum_{i=1}^{s} \binom{m_i+1}{2}$. From Theorem 3.2 it follows that

$$\begin{aligned} H_Z(m-1,j) &= \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq 1\} + \cdots + \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} \geq j+1\} \\ &= \#\{a_{k,l} \in \alpha_Z \mid a_{k,l} = 1\} + 2\#\{a_{k,l} \in \alpha_Z \mid a_{k,l} = 2\} + \cdots \\ &\quad + (j+1)\#\{a_{k,l} \in \alpha_Z a_{k,l} = j+1\}. \end{aligned}$$

Thus, if $j \gg 0$, then $H_Z(m-1,j) = \sum_{k=1}^{r} \sum_{l=1}^{l_k-1} a_{k,l}$. For any $k \in \{1,\ldots,r\}$

$$\begin{aligned} \sum_{l=1}^{l_k-1} a_{k,l} &= a_{k,0} + a_{k,1} + \cdots + a_{k,l_k-1} \\ &= \left[m_{i_1} + (m_{i_1}-1) + \cdots + 2 + 1\right] + \cdots + \left[m_{i_{\alpha_i}} + (m_{i_{\alpha_i}}-1) + \cdots + 2 + 1\right] \\ &= \binom{m_{i_1}+1}{2} + \cdots + \binom{m_{i_{\alpha_i}}+1}{2}. \end{aligned}$$

It then follows that $H_Z(m-1,j) = \sum_{i=1}^{s} \binom{m_i+1}{2}$ if $j \gg 0$. In particular, $H_Z(m-1,m'-1) = \sum_{i=1}^{s} \binom{m_i+1}{2}$. ■

***Remark 3.5*** From the above corollary, we see that if we know the values of $H_Z(m-1,j)$ for $j = 0,\ldots,m'$ and the values of $H_Z(i,m'-1)$ for $i = 0,\ldots,m$, then we know the entire Hilbert function except at a finite number of values. This observation motivates the next definition.

***Definition 3.6*** Let $Z$ be a fat point scheme and let $\alpha_Z$ and $\beta_Z$ be constructed as described above. If $m = |\alpha_Z|$ and $m' = |\beta_Z|$, then define the following tuples:

$$B_C = \big(H_Z(m-1,0), H_Z(m-1,1), \ldots, H_Z(m-1,m'-1)\big)$$

and

$$B_R = \big(H_Z(0,m'-1), H_Z(1,m'-1), \ldots, H_Z(m-1,m'-1)\big).$$

The tuple $B_C$ is called the *eventual column vector* because it contains the values at which the columns will stabilize. Similarly, $B_R$ is the *eventual row vector*. Set $B_Z := (B_C, B_R)$. The tuple $B_Z$ is called the *border* of the Hilbert function of $Z$.

The notion of a border was first introduced in [8] for sets of simple points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The name is used to describe the fact that once we know the values of border, then we know all the values of the Hilbert function "outside" the border. Thus only values "inside" the border, *i.e.*, those $(i, j) \in \mathbb{N}^2$ with $(i, j) \leq (m-1, m'-1)$, need to be calculated to completely determine the entire Hilbert function.

It follows from Theorem 3.2 that the border can be computed directly from the tuples $\alpha_Z$ and $\beta_Z$. By borrowing some terminology from combinatorics, we can make this connection explicit. Our main reference for this material is Ryser [7]. But first, for the remainder of this paper, we will adopt the following convention about $\alpha_Z$ and $\beta_Z$.

***Convention 3.7*** Let $Z$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, and suppose that $\alpha_Z$ and $\beta_Z$ are constructed from $Z$ as described above. We will assume that the entries of $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$ have been reordered so that $\alpha_i \geq \alpha_{i+1}$ for each $i$. We assume the same for $\beta_Z$.

***Definition 3.8*** A tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of positive integers is a *partition* of an integer $s$ if $\sum \lambda_i = s$ and $\lambda_i \geq \lambda_{i+1}$ for every $i$. We write $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$. The *conjugate* of $\lambda$ is the tuple $\lambda^* = (\lambda_1^*, \ldots, \lambda_{\lambda_1}^*)$ where $\lambda_i^* = \#\{\lambda_j \in \lambda \mid \lambda_j \geq i\}$. Furthermore, $\lambda^* \vdash s$.

***Example 3.9*** If $Z = \{(P_1, m_1), \ldots, (P_s, m_s)\}$ is a fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$, then the tuples $\alpha_Z$ and $\beta_Z$ are partitions of $\deg Z = \sum_{i=1}^{s} \binom{m_i+1}{s}$.

***Definition 3.10*** To any partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash s$ we can associate the following diagram: on an $r \times \lambda_1$ grid, place $\lambda_1$ points on the first line, $\lambda_2$ points on the second, and so on. The resulting diagram is called the *Ferrer's diagram* of $\lambda$.

***Example 3.11*** Suppose $\lambda = (4, 4, 3, 1) \vdash 12$. Then the Ferrer's diagram is

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & & & \end{array}$$

The conjugate of $\lambda$ can be read off the Ferrer's diagram by counting the number of dots in each column as opposed to each row. In this example $\lambda^* = (4, 3, 3, 2)$.

For any tuple $p := (p_1, \ldots, p_k)$, we define $\Delta p := (p_1, p_2 - p_1, \ldots, p_k - p_{k-1})$.

***Corollary 3.12*** *Let $Z$ be a fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Then*

(i) $\Delta B_C = \alpha_Z^*$;
(ii) $\Delta B_R = \beta_Z^*$.

**Proof** We use Theorem 3.2 to calculate $\Delta B_C$:

$$\Delta B_C = (\#\{\alpha_i \in \alpha_Z \mid \alpha_i \geq 1\}, \#\{\alpha_i \in \alpha_Z \mid \alpha_i \geq 2\}, \ldots, \#\{\alpha_i \in \alpha_Z \mid \alpha_i \geq m'\})$$

where $m' = |\beta_Z|$. Since $\#\{\alpha_i \in \alpha_Z \mid \alpha_i \geq h\}$ is by definition the $h^{th}$ coordinate of $\alpha_Z^*$, we have $\Delta B_C = \alpha_Z^*$. The proof of (ii) is the same. ■

***Remark 3.13*** Corollary 3.12 implies that we can compute the Hilbert function of $Z$ at all but a finite number of values from only the multiplicities and the relative positions of the points.

***Example 3.14*** This example illustrates that in $\mathbb{P}^1 \times \mathbb{P}^1$, subschemes with the same border can have different Hilbert functions. Set $R_i = Q_i = [1:i] \in \mathbb{P}^1$, and let $P_{ij}$ denote the point $R_i \times Q_j$. Let

$$Y_1 = \{(P_{11}; 1), (P_{22}; 1), (P_{33}; 1), (P_{45}; 1)\},$$
$$Y_2 = \{(P_{11}; 1), (P_{22}; 1), (P_{33}; 1), (P_{44}, 1)\}.$$

As an exercise one can verify that $\alpha_{Y_1} = \alpha_{Y_2} = (1, 1, 1, 1)$ and $\beta_{Y_1} = \beta_{Y_2} = (1, 1, 1, 1)$. Thus, the two schemes have the same border. The Hilbert function of $H_{Y_1}$ is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \cdots \\ 2 & 4 & 4 & 4 & 4 & \cdots \\ 3 & 4 & 4 & 4 & 4 & \cdots \\ 4 & 4 & 4 & 4 & 4 & \cdots \\ 4 & 4 & 4 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

from which we deduce that $(I_{Y_1})_{1,1} = 0$. On the other hand, the unique $(1, 1)$-form $(x_0 y_1 - y_0 x_1)$ which passes through $P_{11}, P_{22}$, and $P_{33}$ also passes through the point $P_{44}$ but not $P_{45}$. Thus $(I_{Y_2})_{1,1} \neq 0$, and hence, $H_{Y_1} \neq H_{Y_2}$.

As we have seen, the tuples $\alpha_Z$ and $\beta_Z$ give us a lot of information about the Hilbert function of $Z$. It is therefore natural to ask which tuples can arise from a fat point scheme $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Because of Corollary 3.12, this is equivalent to asking what can be the border of the Hilbert function of a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. The following theorem places a necessary condition on the tuples $\alpha_Z$ and $\beta_Z$. We require the following definition.

***Definition 3.15*** Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ and $\delta = (\delta_1, \ldots, \delta_r)$ be two partitions of $s$. If one partition is longer, we add zeroes to the shorter one until they have the same length. We say $\lambda$ *majorizes* $\delta$, written $\lambda \trianglerighteq \delta$, if

$$\lambda_1 + \cdots + \lambda_i \geq \delta_1 + \cdots + \delta_i \text{ for } i = 1, \ldots, \max\{t, r\}.$$

Majorization induces a partial ordering on the set of all partitions of $s$.

***Theorem 3.16*** *Let $Z$ be a scheme of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then*

$$\alpha_Z^* \trianglerighteq \beta_Z.$$

**Proof** We work by induction on $m = |\alpha_Z|$. If $m = 1$, then $Z$ is a scheme of simple points in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus $\alpha_Z^* \trianglerighteq \beta_Z$ by Theorem 5.16 in [8].

So, let us suppose that $m > 1$. We can write $Z$ as

$$Z = \{(P_{ij}; m_{ij}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$$

where $m_{ij} \geq 0$ and $P_{ij} = R_i \times Q_j$ for some $R_i, Q_j \in \mathbb{P}^1$. Recall that if $m_{ij} = 0$, then $P_{ij} \notin \mathrm{Supp}(Z)$.

For each $i = 1, \ldots, r$, set $m_i := \sum_{j=1}^{t} m_{ij}$. After relabeling the $P_{ij}$'s, we can assume that $m_1 = \max\{m_1, \ldots, m_r\}$. Furthermore, we can also suppose that after relabeling, $m_{1j} \neq 0$ for $j = 1, \ldots, k$, and $m_{1j} = 0$ for $j = k+1, \ldots, t$. Thus $m_1 = m_{11} + \cdots + m_{ik}$. Note that $m_1 = \alpha_1$, the first coordinate of $\alpha_Z$.

Let $\mathbb{Y}$ be the following subscheme of $Z$:

$$\mathbb{Y} := \{(P_{ij}; m'_{ij}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$$

where

$$m'_{ij} = \begin{cases} (m_{ij} - 1)_+ & i = 1,\ 1 \leq j \leq t \\ m_{ij} & 2 \leq i \leq r,\ 1 \leq j \leq t \end{cases}$$

with $(n)_+ := \max\{0, n\}$. The subscheme $\mathbb{Y}$ is constructed from $Z$ by subtracting 1 from the multiplicity of each point on the $(1,0)$ line that corresponds to $\alpha_1$ in $\alpha_Z$.

Since $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$, and because $\alpha_1 = m_1$, from our construction of $\mathbb{Y}$ it follows that $\alpha_{\mathbb{Y}} = (\alpha_2, \ldots, \alpha_m)$. Therefore, by induction $\alpha_{\mathbb{Y}}^* \trianglerighteq \beta_{\mathbb{Y}}$.

Let $\beta_{\mathbb{Y}}$ and $\beta_Z$ be the tuples associated to $\mathbb{Y}$ and $Z$, respectively, but for the moment we assume that $\beta_{\mathbb{Y}}$ and $\beta_Z$ have been constructed as first described at the beginning of Section 3, that is, $\beta_{\mathbb{Y}}$ and $\beta_Z$ have not been ordered.

We now describe how $\beta_Z$ and $\beta_{\mathbb{Y}}$ are related. Suppose $\beta_Z = (b_1, b_2, \ldots, b_l)$ and $\beta_{\mathbb{Y}} = (b'_1, b'_2, \ldots, b'_h)$. Clearly $h \leq l$.

If $h = l$, then

$$b_p = b'_p + 1 \quad \text{for all } p = 1, \ldots, l.$$

If $h < l$, we first insert $(l - h)$ zeroes into the tuple $\beta_{\mathbb{Y}}$ at specific locations. For $j = 1, \ldots, t$, set $l'_j := \max\{m_{1j}, m_{2j}, \ldots, m_{rj}\}$, and for $d = 1, \ldots, t$, set $h_d := \sum_{s=1}^{d} l'_s$. Then we insert a zero into the $h_d^{th}$ spot of $\beta_{\mathbb{Y}}$ if $l'_d = m_{1d}$ but $l'_d > m_{id}$ for all $i = 2, \ldots, r$. It then follows from our definition of $\mathbb{Y}$ that we are only adding $(l - h)$ zeroes to $\beta_{\mathbb{Y}}$. Relabel our tuple as $\beta_{\mathbb{Y}} = (c_1, \ldots, c_l)$.

From our construction of $\mathbb{Y}$ from the scheme $Z$, it follows that

$$b_i = \begin{cases} c_i + 1 & \text{for } i = 1, \ldots, m_{11}, l'_1 + 1, \ldots, m_{12}, l'_1 + l'_2 + 1, \ldots \\ & \quad \ldots, m_{13}, \ldots, l'_1 + l'_2 + \cdots + l'_{k-1} + 1, \ldots, m_{1k}, \\ c_i & \text{otherwise.} \end{cases}$$

So $\beta_Z$ can be constructed from $\beta_Y$ by adding 1 to $m_{11} + m_{12} + \cdots + m_{1k} = m_1 = \alpha_1$ distinct coordinates in $\beta_Z$, and then reordering so that $\beta_Z$ is a partition.

Since $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$ and $\alpha_Y = (\alpha_2, \ldots, \alpha_m)$, $\alpha_Z^*$ can be computed from $\alpha_Y^*$ by adding 1 to the first $\alpha_1$ entries of $\alpha_Y^*$. (If $|\alpha_Y^*| < \alpha_1$, we extend $\alpha_Y^*$ by adding zeroes so $|\alpha_Y^*| = \alpha_1$.) By induction, $\alpha_Y^* \trianglerighteq \beta_Y$. So, if $\beta_Y = (c_1, \ldots, c_l)$, then

$$\alpha_Z^* \trianglerighteq (c_1 + 1, \ldots, c_{\alpha_1} + 1, c_{\alpha_1+1}, \ldots, c_l).$$

But since $\beta_Z$ can be recovered from $\beta_Y$ by adding 1 to $m_1 = \alpha_1$ distinct entries of $\beta_Y$ (and not necessarily the first $\alpha_1$ entries) and then reordering, we have

$$\alpha_Z^* \trianglerighteq (c_1 + 1, \ldots, c_{\alpha_1} + 1, c_{\alpha_1+1}, \ldots, c_l) \trianglerighteq \beta_Z.$$

Hence $\alpha_Z^* \trianglerighteq \beta_Z$, as desired. ■

## 4 ACM Fat Point Schemes

For any fat point scheme in $\mathbb{P}^n$, the associated coordinate ring is always Cohen-Macaulay. In contrast, fat point schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with $k \geq 2$ may fail to have this property, even if the support is ACM. See [3, 5, 9] for more details on ACM zero-dimensional schemes in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$.

A fat point scheme is said to be *arithmetically Cohen-Macaulay* (ACM for short) if the associated coordinate ring is Cohen-Macaulay. ACM schemes on a smooth quadric $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$ were studied in [3] and by the first author in [5] (which is based on [4]). In [3] the authors gave a characterization of ACM schemes in terms of their Hilbert functions. In [5], ACM fat points schemes in $\mathbb{P}^1 \times \mathbb{P}^1$ were characterized in terms of the multiplicities of the points. In this section we show that ACM schemes can also be classified using the tuples $\alpha_Z$ and $\beta_Z$ introduced in the previous section. We will also show how these various classifications are related.

We begin by recalling the construction and main result of [5]. Let $Z$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ where $Z = \{(P_{ij}; m_{ij}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$ with $m_{ij} \geq 0$ and $P_{ij} = R_i \times Q_j$ for some $R_i, Q_j \in \mathbb{P}^1$. For each $h \in \mathbb{N}$, and for each tuple $(i, j)$ with $1 \leq i \leq r$ and $1 \leq j \leq t$, define

$$t_{ij}(h) := (m_{ij} - h)_+ = \max\{0, m_{ij} - h\}.$$

The set $\mathcal{S}_Z$ is then defined to be the set of $t$-tuples

$$\mathcal{S}_Z = \left\{ \left( t_{i1}(h), \ldots, t_{it}(h) \right) \,\middle|\, 1 \leq i \leq r,\ h \in \mathbb{N} \right\}.$$

For each integer $1 \leq i \leq r$, set $l_i := \max\{m_{i1}, \ldots, m_{it}\}$. For any fat point scheme, we then have $|\mathcal{S}_Z| = m := \sum_{i=1}^{r} l_i$. For each $i = 1, \ldots, r$ and for all $h \in \mathbb{N}$ we set

$$z_{i,h} := \sum_{j=1}^{t} t_{ij}(h).$$

We then define $u_1 := \max_{i,h}\{z_{i,h}\}$, and we recursively define

$$u_p := \max_{i,h}\left\{ \{z_{i,h}\} \setminus \{u_1, \ldots, u_{p-1}\} \right\} \quad \text{for } p = 2, \ldots, m.$$

***Definition 4.1*** Let $H_Z\colon \mathbb{N}^2 \to \mathbb{N}$ be the Hilbert function of a fat point scheme $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The *first difference function* of $H_Z$, denoted $\Delta H_Z$, is the function defined by

$$\Delta H_Z(i,j) = H_Z(i,j) - H_Z(i-1,j) - H_Z(i,j-1) + H_Z(i-1,j-1)$$

where $H_Z(i,j) = 0$ if $(i,j) \not\geq (0,0)$.

With this notation we can state the main result of [5].

***Theorem 4.2*** ([5, Theorem 2.1]) *Let $Z$ be a fat point scheme on $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then the set $\mathcal{S}_Z$ is totally ordered if and only if $Z$ is* ACM. *In this case, the first difference function of $H_Z$ is:*

$$\Delta H_Z = \begin{bmatrix} \underbrace{1 \quad 1 \quad \cdots \quad 1 \quad 1}_{u_1} & 0 & \cdots \\ \underbrace{1 \quad 1 \quad \cdots \quad 1}_{u_2} \quad 0 & 0 & \cdots \\ \vdots \qquad\qquad \vdots \quad \vdots & \vdots & \\ \underbrace{1 \quad \cdots \quad 1}_{u_m} \quad 0 \quad 0 & 0 & \cdots \\ 0 \quad \cdots \quad 0 \quad 0 \quad 0 & 0 & \cdots \\ \vdots \qquad \vdots \quad \vdots \quad \vdots & \vdots & \ddots \end{bmatrix}$$

*where $u_1,\ldots,u_m$ are defined as above.*

***Remark 4.3*** From the construction of $u_1,\ldots,u_m$, one can verify that the identity $\alpha_Z = (u_1,\ldots,u_m)$ holds.

The following result, required to prove the main result of this section, holds for any ACM scheme of codimension two. Here, we give a proof in the bihomogeneous case.

***Theorem 4.4*** *Suppose that $Z$ is a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$. If $Z$ is* ACM, *then there exists $L_1, L_2 \in R$ such that* $\deg L_1 = (1,0)$ *and* $\deg L_2 = (0,1)$, *and $L_1, L_2$ give rise to a regular sequence in $R/I_Z$.*

**Proof** The Krull dimension of $R/I_Z$ is K-dim $R/I_Z = 2$. Because $Z$ is ACM, it follows that there exists a regular sequence of length 2 in $R/I_Z$. It is therefore sufficient to show that the elements in the regular sequence have the appropriate degrees.

By Lemma 1.2 there exists $L_1 \in R$ such that $\deg L_1 = (1,0)$ and $\overline{L}_1$ is a non-zero divisor of $R/I_Z$. It is therefore enough to show there exists a non-zero divisor $\overline{L}_2 \in R/(I_Z, L_1)$ with $\deg L_2 = (0,1)$.

Let $(I_Z, L_1) = Q_1 \cap \cdots \cap Q_s$ be the primary decomposition of $(I_Z, L_1)$ and set $\wp_i := \sqrt{Q_i}$. We claim that $(x_0, x_1) \subseteq \wp_i$ for each $i$. Indeed, since $L_1$ is a non-zero divisor, we have the following exact graded sequence:

$$0 \longrightarrow (R/I_Z)(-1,0) \xrightarrow{\times L} R/I_Z \longrightarrow R/(I_Z, L) \longrightarrow 0.$$

Thus, $H_{R/(I_Z,L_1)}(i,j) = H_Z(i,j) - H_Z(i-1,j)$ for all $(i,j) \in \mathbb{N}^2$. By Corollary 3.4, if $i \gg 0$, $H_Z(i,0) = H_Z(i-1,0)$, and hence, $H_{R/(I_Z,L_1)}(i,0) = 0$. This implies $(I_Z,L_1)_{i,0} = R_{i,0} = \left[(x_0,x_1)^i\right]_{i,0}$. So, $(x_0,x_1)^i \subseteq Q_j$ for $i \gg 0$ and for each $j = 1,\ldots,s$. Therefore, $(x_0,x_1) \subseteq \wp_j$ for each $j$.

The set of zero divisors of $R/(I_Z,L_1)$, denoted $\mathbf{Z}(R/(I_Z,L_1))$, are precisely the elements of

$$\mathbf{Z}(R/(I_Z,L_1)) = \bigcup_{i=1}^{s} \overline{\wp}_i.$$

Because $\mathbf{k}$ is infinite, it is enough to show that $(\wp_i)_{0,1} \subsetneq R_{0,1}$ for each $i$. If there exists an $i \in \{1,\ldots,s\}$ such that $(\wp_i)_{0,1} = R_{0,1}$, then $(x_0,x_1,y_0,y_1) \subseteq \wp_i$. But then every homogeneous element of $R/(I_Z,L_1)$ is a zero divisor, contradicting the fact that $Z$ is ACM. So $R/(I_Z,L_1)$ has a non-zero divisor of degree $(0,1)$. ■

***Corollary 4.5*** *If $Z$ is an* ACM *fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, then the first difference function $\Delta H_Z$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1,y_1]$.*

**Proof** Let $L_1, L_2$ be the regular sequence of Theorem 4.4. By making a linear change of coordinates in the $x_0,x_1$s, and a linear change of coordinates in the $y_0,y_1$s, we can assume that the $L_1 = x_0, L_2 = y_0$ give rise to a regular sequence in $R/I_Z$.

From the short exact sequences

$$\begin{array}{ccccccccc} 0 & \rightarrow & (R/I_Z)(-1,0) & \xrightarrow{\times \overline{x}_0} & R/I_Z & \rightarrow & R/(I_Z,x_0) & \rightarrow & 0 \\ 0 & \rightarrow & (R/(I_Z,x_0))(0,-1) & \xrightarrow{\times \overline{y}_0} & R/(I_Z,x_0) & \rightarrow & R/(I_Z,x_0,y_0) & \rightarrow & 0 \end{array}$$

it follows that $H_{R/(I_Z,x_0,y_0)}(i,j) = \Delta H_Z(i,j)$ for all $(i,j) \in \mathbb{N}^2$. Moreover,

$$R/(I_Z,x_0,y_0) \cong \frac{R/(x_0,y_0)}{(I_Z,x_0,y_0)/(x_0,y_0)} \cong \mathbf{k}[x_1,y_1]/J$$

where $J$ is a bihomogeneous ideal with $J \cong (I_Z,x_0,y_0)/(x_0,y_0)$. By using Corollary 3.4 it follows that $\Delta H_Z(i,j) = 0$ if $i \gg 0$ or $j \gg 0$. Hence $\mathbf{k}[x_1,y_1]/J$ is an artinian ring. ■

***Lemma 4.6*** *Let $Z$ be a fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$. Set $c_{i,j} := \Delta H_Z(i,j)$. Then*

(i) *for every $0 \le j \le |\beta_Z| - 1$*

$$\alpha^*_{j+1} = \sum_{h \le |\alpha_Z|-1} c_{h,j}$$

*where $\alpha^*_{j+1}$ is the $(j+1)$-th entry of $\alpha^*_Z$, the conjugate of the partition $\alpha_Z$;*

(ii) *for every $0 \le i \le |\alpha_Z| - 1$*

$$\beta^*_{i+1} = \sum_{h \le |\beta_Z|-1} c_{i,h}$$

*where $\beta^*_{i+1}$ is the $(i+1)$-th entry of $\beta^*_Z$, the conjugate of the partition $\beta_Z$.*

**Proof** Fix an integer $j$ such that $0 \leq j \leq |\beta_Z| - 1$ and set $m = |\alpha_Z|$. Using Theorem 3.2 and the identity $H_Z(i,j) = \sum_{(h,k)\leq(i,j)} c_{h,k}$ to compute $\alpha^*_{j+1}$ we have

$$\begin{aligned}\alpha^*_{j+1} &= H_Z(m-1, j) - H_Z(m-1, j-1)\\ &= \sum_{(h,k)\leq(m-1,j)} c_{h,k} - \sum_{(h,k)\leq(m-1,j-1)} c_{h,k} = \sum_{h\leq m-1} c_{h,j}.\end{aligned}$$

The proof for the second statement is the same. ■

***Lemma 4.7*** ([9, Lemma 4.1]) *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_m)$, and suppose that $\alpha, \beta \vdash s$. If $\alpha^* = \beta$, then*

(i) $\alpha_1 = |\beta|$.
(ii) $\beta_1 = |\alpha|$.
(iii) *if $\alpha' = (\alpha_2, \ldots, \alpha_n)$ and $\beta' = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$, then $(\alpha')^* = \beta'$.*

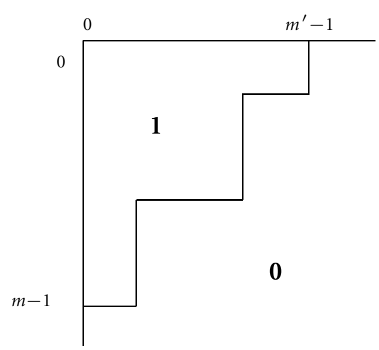
***Theorem 4.8*** *Let $Z$ be a fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert function $H_Z$. Then the following are equivalent:*

(i) *$Z$ is arithmetically Cohen-Macaulay.*
(ii) *$\Delta H_Z$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$.*
(iii) *$\alpha^*_Z = \beta_Z$.*
(iv) *The set $\mathcal{S}_Z$ is totally ordered.*

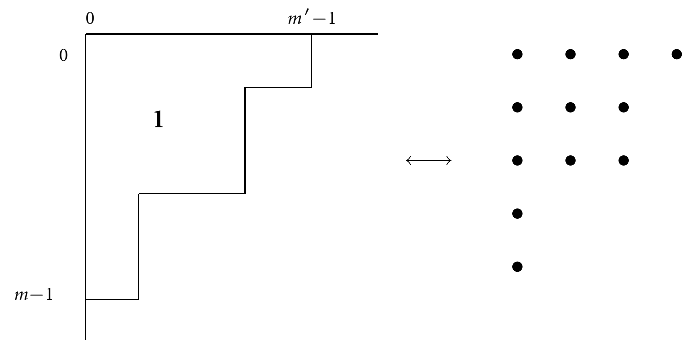
**Proof** In light of Theorem 4.2 and Corollary 4.5, it is enough to prove that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Suppose that $\Delta H_Z$ is the Hilbert function of a bigraded artinian quotient of $\mathbf{k}[x_1, y_1]$. Since $\dim_{\mathbf{k}} \mathbf{k}[x_1, y_1]_{i,j} = 1$ for all $(i,j)$, $\Delta H_Z(i,j) = 1$ or $0$. If we write $\Delta H_Z$ as an infinite matrix whose indexing starts from zero, rather than one, then we have

$\Delta H_Z =$



where $m = |\alpha_Z|$ and $m' = |\beta_Z|$. By Lemma 4.6 the number of 1's in the $(i-1)^{th}$ row of $\Delta H_Z$ for each integer $1 \le i \le m$ is simply the $i^{th}$ coordinate of $\beta_Z^*$. Similarly, the number of 1's in the $(j-1)^{th}$ column of $\Delta H_Z$ for each integer $1 \le j \le m'$ is the $j^{th}$ coordinate of $\alpha_Z^*$. Now $\Delta H_Z$ can be identified with the Ferrer's diagram (see Definition 3.10) by associating each 1 in $\Delta H_Z$ with a dot in the Ferrer's diagram in a natural way:




By using the Ferrer's diagram and Lemma 4.6 we can calculate that $\beta_Z = (\beta_Z^*)^* = \alpha_Z^*$, and so (iii) holds.

Now suppose that $Z$ is a fat point scheme $Z = \{(P_{ij}; m_{ij}) \mid 1 \le i \le r, 1 \le j \le t\}$ where $m_{ij}$ are non-negative numbers and $\alpha_Z^* = \beta_Z$. We will work by induction on $\beta_1 = \max\{\sum_{i=1}^r m_{ij}\}_{j=1}^t$.

If $\beta_1 = 1$, then $Z$ is a set of $s$ distinct simple points with $\alpha_Z = (s)$ and $\beta_Z = (\underbrace{1,\dots,1}_{s})$. So $Z = \{P \times Q_1, \dots, P \times Q_s\}$, in which case it can be easily checked that $\mathcal{S}_Z = \{(1,\dots,1)\}$, and that the set is trivially ordered.

Let us suppose that $\beta_1 > 1$ and the theorem holds for all fat point schemes $\mathbb{Y}$ with $\alpha_{\mathbb{Y}}^* = \beta_{\mathbb{Y}}$, and the first coordinate of $\beta_{\mathbb{Y}}$ is less than $\beta_1$.

Let $k$ be the index in $\{1,\dots,r\}$ such that $\alpha_1 = \sum_{j=1}^t m_{kj}$.

**Claim** $m_{kj} > 0$ for $j = 1,\dots,t$.

**Proof of the Claim** Set $l'_j = \max\{m_{1j},\dots,m_{rj}\}$ for $j = 1,\dots,t$. Then $|\beta_Z| = l'_1 + \cdots + l'_t$. Since $\alpha_Z^* = \beta_Z$, by Lemma 4.7 $\alpha_1 = l'_1 + \cdots + l'_t$. Now suppose that $m_{kc} = 0$ for some $c \in \{1,\dots,t\}$. Since $l'_j \ge m_{kj}$ for each $j = 1,\dots,r$, we would then have

$$\begin{aligned}\alpha_1 = l'_1 + \cdots + l'_t &> l'_1 + \cdots + \widehat{l'_c} + \cdots + l'_t\\ &\ge m_{k1} + \cdots + \widehat{m}_{kc} + \cdots + m_{kt}\\ &= m_{k1} + \cdots + m_{kc} + \dots m_{kt} = \alpha_1\end{aligned}$$

where $\widehat{}$ means the number is omitted. Because of this contradiction, the claim holds. ∎

Let $\mathbb{Y} = \{(P_{ij}; m'_{ij}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$ be the subscheme of $Z$ where

$$m'_{ij} = \begin{cases} m_{ij} & i \neq k, \\ m_{kj} - 1 & i = k. \end{cases}$$

By the claim $m_{kj} - 1 \geq 0$ for all $j = 1, \ldots, t$. Let $\beta$ be the first coordinate of $\beta_{\mathbb{Y}}$. Then $\beta < \beta_1$. In fact, for each $j = 1, \ldots, t$, we have

$$\sum_{i=1}^{r} m'_{ij} = m'_{kj} + \sum_{i \neq k} m_{ij} = \left(\sum_{i=1}^{r} m_{ij}\right) - 1.$$

Furthermore, if $\alpha_Z = (\alpha_1, \ldots \alpha_m)$ and $\beta_Z = (\beta_1, \ldots, \beta_{m'})$, then from our construction $\mathbb{Y}$ we have $\alpha_{\mathbb{Y}} = (\alpha_2, \ldots, \alpha_m)$ and $\beta_{\mathbb{Y}} = (\beta_1 - 1, \ldots, \beta_{\alpha_2} - 1)$. By Lemma 4.7, $\alpha^*_{\mathbb{Y}} = \beta_{\mathbb{Y}}$, and so by induction $\mathcal{S}_{\mathbb{Y}}$ is totally ordered.

The set $\mathcal{S}_Z$ is now obtained from $\mathcal{S}_{\mathbb{Y}}$ by adding the tuple $(m_{k1}, \ldots, m_{kt})$. Moreover, this element is larger than every other element of $\mathcal{S}_{\mathbb{Y}}$ with respect to our ordering, so $\mathcal{S}_Z$ is totally ordered, as desired. ■

***Corollary 4.9*** *If $Z$ is a scheme of fat points whose support is on a line, then $Z$ is* ACM.

**Proof** It easy to check that either the set $\mathcal{S}_Z$ is totally ordered, or $\alpha^*_Z = \beta_Z$. ■

***Corollary 4.10*** *If $Z$ is an* ACM *scheme of fat points with $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$, then the Hilbert function of $Z$ is*

$$H_Z = \begin{bmatrix} 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ 1 & 2 & \cdots & \alpha_1 - 1 & \alpha_1 & \alpha_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ 1 & 2 & \cdots & \alpha_2 - 1 & \alpha_2 & \alpha_2 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix} +$$

$$\cdots + \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 1 & 2 & \cdots & \alpha_m - 1 & \alpha_m & \alpha_m & \cdots \\ 1 & 2 & \cdots & \alpha_m - 1 & \alpha_m & \alpha_m & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Proof** Use Theorem 4.2 and Remark 4.3. ■

From the above corollary, we see that if the fat point scheme $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is ACM, then the entire Hilbert function of $Z$ can be determined from the tuple $\alpha_Z$. This contrasts with the main result of the previous section where we showed that for a general fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, most, but not all, of the values of the Hilbert function can be determined from the tuples $\alpha_Z$ and $\beta_Z$.

In fact, if $Z$ is an ACM fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$, we can even compute the Betti numbers in the minimal free resolution of $I_Z$ directly from the tuple $\alpha_Z$. To state our result, we first develop some suitable notation.

Let $Z$ be an ACM scheme of fat points and let $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$ be the tuple associated to $Z$. Define the following two sets from $\alpha_Z$:

$$C_Z := \{(m, 0), (0, \alpha_1)\} \cup \{(i-1, \alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$$
$$V_Z := \{(m, \alpha_m)\} \cup \{(i-1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$

We take $\alpha_{-1} = 0$. With this notation, we have

***Theorem 4.11*** *Suppose that $Z$ is an ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$. Let $C_Z$ and $V_Z$ be constructed from $\alpha_Z$ as above. Then the bigraded minimal free resolution of $I_Z$ is given by*

$$0 \longrightarrow \bigoplus_{(v_1,v_2)\in V_Z} R(-v_1, -v_2) \longrightarrow \bigoplus_{(c_1,c_2)\in C_Z} R(-c_1, -c_2) \longrightarrow I_Z \longrightarrow 0.$$

**Proof** Using Theorem 4.2, it can be verified that the tuples in the set $C_Z$ are what [3] defined to be the *corners* of $\Delta H_Z$, and the elements in $V_Z$ are precisely the *vertices* of $\Delta H_Z$. The conclusion now follows from Theorem 4.1 in [3] . ■

## 5 Special Configurations of ACM Fat Points

Theorem 4.8 enables us to identify the ACM fat point schemes directly from the tuples $\alpha_Z$ and $\beta_Z$, or from the set $\mathcal{S}_Z$. In this section, we use these characterizations to investigate ACM fat point schemes which have some extra conditions on the multiplicities of the points. We show that some special configurations of ACM fat point schemes can occur only if the support of the scheme has some specific properties.

***Remark 5.1*** By Theorem 2.12 and Theorem 4.1 in [3], we can deduce that $\mathbb{X}$ is not an ACM scheme if and only if there exist two points $P_{11} = [a_1 : a_2] \times [b_1 : b_2]$ and $P_{22} = [c_1 : c_2] \times [d_1 : d_2]$ of $\mathbb{X}$ with $a_i, b_i, c_i, d_i \in \mathbf{k}$ such that $P_{12} = [a_1 : a_2] \times [d_1 : d_2]$ and $P_{21} = [c_1 : c_2] \times [b_1 : b_2] \notin \mathbb{X}$.

***Proposition 5.2*** *If $Z$ is an* ACM *fat point scheme, then* $\operatorname{Supp}(Z)$ *is* ACM.

**Proof** Let us suppose that $\operatorname{Supp}(Z)$ is not ACM. Then by Remark 5.1, in $\mathcal{S}_Z$ we can find tuples of type

$$(*, 1, *, 0, *), (*, 0, *, 1, *)$$

that are incomparable. Therefore, by Theorem 4.8, $Z$ is not ACM. ■

***Remark 5.3*** Theorem 1.2 of [3] showed that for any saturated bihomogeneous ideal $I \subseteq R$ of height two, the minimal generating set for $I$ must contain exactly one form of degree $(m, 0)$ for some $m$, and one form of degree $(0, n)$ for some $n$. If

$F \in I$ is the form of degree $(m, 0)$, then $F \in \mathbf{k}[x_0, x_1] \subseteq R$, and thus $F$ can be written as the product of $(1, 0)$ forms. Similarly, the form of degree $(0, n)$ can be written as a product of forms of degree $(0, 1)$. Thus, following Remark 1.3 of [3], we shall call a set of points $\mathbb{X}$ a *complete intersection* if $I_{\mathbb{X}} = (F, G)$ where $\deg F = (m, 0)$ and $\deg G = (0, n)$.

We now describe the support of the ACM fat point schemes which are *homogeneous*, *i.e.*, all the nonzero multiplicities are equal.

***Theorem 5.4*** *Fix a positive integer $m \geq 2$, and let $Z$ be a homogeneous fat point scheme of $\mathbb{P}^1 \times \mathbb{P}^1$ with all the nonzero multiplicities equal to $m$. Then $Z$ is* ACM *if and only if* $\mathrm{Supp}(Z)$ *is a complete intersection.*

**Proof** If $\mathrm{Supp}(Z)$ is a complete intersection, then $Z$ is ACM by Corollary 2.5 of [5].

Conversely, suppose that $Z$ is ACM, and thus, $\mathcal{S}_Z$ is totally ordered by Theorem 4.8. Because $Z$ is ACM, from Proposition 5.2, $\mathrm{Supp}(Z)$ must also be ACM.

Suppose that $\mathrm{Supp}(Z)$ is not a complete intersection. This implies that $Z$ contains a subscheme of type

$$\begin{aligned}\mathbb{Y} = &\{(P_{i_1 j}; m_{i_1 j}) \mid m_{i_1 j} = m \text{ for } j = 1, \ldots, t\} \\ &\cup \left\{ (P_{i_2 j}; m_{i_2 j}) \;\middle|\; \begin{array}{ll} m_{i_2 j} = m & j = 1, \ldots, h \text{ with } h < t \\ m_{i_2 j} = 0 & \text{otherwise.} \end{array} \right\}.\end{aligned}$$

But then in $\mathcal{S}_Z$ we can find three tuples of the form

$$\left\{ (\underbrace{m, \ldots, m}_{t}), (\underbrace{m, \ldots, m}_{h}, \underbrace{0, \ldots, 0}_{t-h}), (\underbrace{m-1, \ldots, m-1}_{t}) \right\}.$$

But then $\mathcal{S}_Z$ is not totally ordered, which is a contradiction. ∎

***Remark 5.5*** Homogeneous schemes with all $m_{ij} = 2$ have been further investigated by the first author in [5].

***Definition 5.6*** A fat point scheme $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is called an *almost homogeneous fat point scheme* if all the nonzero multiplicities of $Z$ are either $m$ or $m - 1$ for some integer $m > 0$.

We now recall a definition first given in [5].

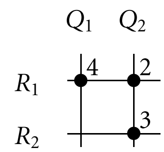
***Definition 5.7*** Let $Z = \{(P_{ij}; m_{ij}) \mid 1 \leq i \leq r, 1 \leq j \leq t\}$ be a fat point scheme. The scheme $Z$ is called a *quasi-homogeneous scheme of fat points* if there exist $r$ integers $t = t_1 \geq t_2 \geq \cdots \geq t_r \geq 1$ such that

$$m_{ij} = \begin{cases} m & j = 1, \ldots, t_i, \\ m - 1 & j = t_{i+1}, \ldots, t_1. \end{cases}.$$

***Remark 5.8*** Note that if $Z$ is a quasi-homogeneous scheme and $m \geq 2$, then $\mathrm{Supp}(Z)$ is the complete intersection $\{P_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq t\}$. If $m = 1$, then a quasi-homogeneous scheme of fat points is an ACM scheme of simple points. However, if $m = 1$, then the support is not a complete intersection. We also observe that any quasi-homogeneous fat point scheme is also an almost homogeneous fat point scheme for any $m$.

***Remark 5.9*** If $Z$ is a quasi-homogeneous fat point scheme, then $Z$ is ACM by Corollary 2.6 in [5] .

Since $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the quadric surface $\mathcal{Q} \subseteq \mathbb{P}^3$, using Remark 5.3, we can draw fat point schemes on $\mathcal{Q}$ as subschemes whose support is contained in the intersection of lines of the two rulings of $\mathcal{Q}$. For example, if $P_{ij} = R_i \times Q_j \in \mathbb{P}^1 \times \mathbb{P}^1$, then the fat point scheme $\mathbb{Z} = \{(P_{11}; 4), (P_{12}; 2), (P_{22}; 3)\}$ can be visualized as

$Z =$ 

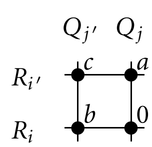

where a dot represents a point in the support, and the number its multiplicity.

***Theorem 5.10*** *Let $Z$ be a fat point scheme. If $Z$ is an* ACM *almost homogeneous fat point scheme with $m \geq 4$, then $Z$ is a quasi-homogeneous scheme of fat points. In particular, the support of $Z$ is a complete intersection.*

**Proof** Suppose that $Z$ is an ACM almost homogeneous fat point scheme.

**Claim** $\mathrm{Supp}(Z)$ is a complete intersection.

**Proof of the Claim** For a contradiction, suppose $\mathrm{Supp}(Z)$ is not a complete intersection. Since $\mathrm{Supp}(Z)$ is contained within a complete intersection, we can find a point $P_{ij} = R_i \times Q_j \notin \mathrm{Supp}(Z)$ but $P_{i'j} = R_{i'} \times Q_j$ and $P_{ij'} = R_i \times Q_{j'}$ in $\mathrm{Supp}(Z)$. So $Z$ contains the following subscheme




where $a$, $b$, and $c$ denote the multiplicities of $R_{i'} \times Q_j, R_i \times Q_{j'}$ and $R_{i'} \times Q_{j'}$ respectively, and 0 denotes the absence of the point $R_i \times Q_j$.

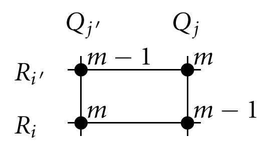
We observe that the tuples $(*, c, *, a, *)$ and $(*, b, *, 0, *)$ are in $\mathcal{S}_Z$ with $c$ and $b$ in the $j'^{th}$ spot and the $a$ and 0 in the $j^{th}$ spot, and where $*$ denotes the other unknown numbers in the tuple. Because $Z$ is ACM, $\mathcal{S}_Z$ is totally ordered, so $m \geq c \geq b \geq m-1$.

We see that $c$ can be either $c > b$ or $c = b$. If $c > b$, then $c = m$ and $b = m - 1$. But then the tuple $(*, m-2, *, a-2, *)$ is also in $\mathcal{S}_Z$ with $a - 2 \geq (m-1) - 2 > 0$

because $m \geq 4$. But then $\mathcal{S}_Z$ is not totally ordered because the tuples $(*, b, *, 0, *)$ and $(*, c-2, *, a-2, *)$ are incomparable.

Similarly, if $c = b$, then the tuple $(*, c-1, *, a-1, *)$ is in $\mathcal{S}_Z$ with $b > c-1$, but $a-1 > 0$, contradicting the fact that $\mathcal{S}_Z$ is totally ordered. So, the support of $Z$ must be a complete intersection. ■

Because of the claim, we can consider subschemes of $Z$ that consist of the following four points: $P_{ij} = R_i \times Q_j$, $P_{i'j} = R_{i'} \times Q_j$, $P_{ij'} = R_i \times Q_{j'}$, and $P_{i'j'} = R_{i'} \times Q_{j'}$. Now no such subscheme will have the form

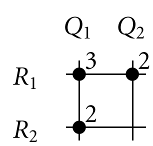



because such a subscheme would contradict the fact that $\mathcal{S}_Z$ is totally ordered. So, if we write only the multiplicities of the points, then the scheme $Z$ must have the form

$$\begin{array}{cccccc}
m & m & \cdots & m & m & m \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
m & m & \cdots & m & m & m \\
m & m & \cdots & m & m & m-1 \\
m & m & \cdots & m & m-1 & m-1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
m & m-1 & \cdots & m-1 & m-1 & m-1
\end{array}$$

that is, $Z$ is a quasi-homogeneous scheme of fat points. ■

***Example 5.11*** One can check that the following scheme




is an almost homogeneous fat point scheme that is also ACM. However, the support is not a complete intersection. So the hypothesis $m \geq 4$ is needed in the above theorem.

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