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Embeddings into pseudocompact spaces of countable tightness

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Abstract

A method for embedding certain countable spaces into a pseudocompact Tychonoff space of countable tightness is given. In particular, this permits construction under [CH] of two pseudocompact Tychonoff spaces of countable tightness, one does not have countable fan tightness and the other contains a non-discrete extremally disconnected subspace.

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It is well known that if a regular space X is countably compact then many properties of X actually imply stronger properties both in X and its subspaces.

The starting point for the results presented here was in the attempt to generalize the following two theorems:

Theorem A [1, Corollary 2]. *Any countably compact regular space of countable tightness has also countable fan tightness.*

Theorem B [3, Corollary 2]. *Every extremally disconnected subspace of a countably compact regular space of countable tightness is discrete.*

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A topological space X has countable fan tightness provided that whenever $x \in \bigcap\{\overline{A_n}\colon n < \omega\}$ then there are finite sets $K_n \subseteq A_n$ such that $x \in \overline{\bigcup\{K_n\colon n < \omega\}}$.

It is rather easy to see that theorems A and B are no longer true for Hausdorff spaces. For this, let M be a countable regular maximal space. The existence of such space was first shown in [4]. It is well known that M is extremally disconnected. Moreover, Proposition 3.1 in [2] shows that M does not have countable fan tightness.

Now, consider the Čech–Stone compactification βM of M. Theorem 1.1 of [7] shows that there exists a strengthening of the topology of βM in such a way that the resulting space X has the following properties:

(1) M is a subspace of X;
(2) X is locally countable;
(3) each closed infinite subset of X has cardinality $2^{\mathfrak{c}}$.

So, we get a countably compact Hausdorff space of countable tightness which does not have countable fan tightness and contains the non-discrete extremally disconnected subspace M.

It is a much harder job to show that the above two theorems may fail for pseudocompact regular spaces.

All the basic notions used here can be found in [6,5].

As usual, the formula $Q \subseteq^* P$ means that the set $Q \setminus P$ is finite.

A collection $\{f_\beta\colon \beta < \omega_1\}$ of functions from ω into ω is a scale of cardinality ω_1 if $f_\alpha(m) < f_\beta(m)$ for cofinitely many $m \in \omega$ whenever $\alpha < \beta$ and for every map $f : \omega \to \omega$, there is α such that $f(m) < f_\alpha(m)$ for cofinitely many $m \in \omega$.

The existence of scales of cardinality ω_1 is equivalent to the assumption that the dominating number $\mathfrak{d}$ is equal to ω_1 (see [5]). Furthermore, CH implies $\mathfrak{d} = \omega_1$.

If $\mathcal{F}$ is a free filter on ω, then $\omega \cup \{\mathcal{F}\}$ indicates the space with the only non-isolated point $\mathcal{F}$, where a local base at $\mathcal{F}$ is the family $\{\{\mathcal{F}\} \cup F\colon F \in \mathcal{F}\}$.

Theorem 1. $[\mathfrak{d} = \omega_1]$ *If $\mathcal{F}$ is a free filter on ω having a base which is well-ordered by $\subseteq^*$ in type ω_1 then there exists a pseudocompact zero-dimensional T_1-space of countable tightness which has a closed subspace homeomorphic to $\omega \cup \{\mathcal{F}\}$.*

Proof. Let $\{F_\alpha\colon \alpha < \omega_1\}$ be a base for $\mathcal{F}$, where $F_\alpha \subseteq^* F_\beta$ whenever $\beta \leqslant \alpha$. Fix a scale $\{f_\beta\colon \beta < \omega_1\}$ in ${}^\omega\omega$.

Let α, β be two countable ordinals and let $G \subseteq \omega \times \omega$. Let us say that the set G *is parametrized by* $\langle\alpha, \beta\rangle$ if the following holds:

(1) G is a partial function $\omega \to \omega$ and $\mathrm{dom}(G)$ is infinite;
(2) $\mathrm{dom}(G) \cap F_\alpha = \emptyset$;
(3) for each $\gamma < \alpha$, $\mathrm{dom}(G) \subseteq^* F_\gamma$;
(4) $G < f_\beta$, i.e., $G(n) < f_\beta(n)$ for each $n \in \mathrm{dom}(G)$;
(5) for all $\gamma < \beta$, the set $\{n\colon f_\gamma(n) > G(n)\}$ is finite.

If a set $G \subseteq \omega \times \omega$ is parametrized by $\langle\alpha, \beta\rangle$ then we will write $\phi_1(G) = \alpha$ and $\phi_2(G) = \beta$.

Let $\mathcal{G}$ be a maximal almost disjoint collection of parametrized subsets of $\omega \times \omega$.

For any $\alpha < \omega_1$, let $R_\alpha = \{(n, x) \in \omega \times (\omega+1)\colon n \notin F_\alpha,\ x \geqslant f_\alpha(n)\} \cup \{G \in \mathcal{G}\colon \phi_1(G) \leqslant \alpha < \phi_2(G)\}$.

Let $X = \omega \times (\omega + 1) \cup \mathcal{G} \cup \{a\}$, where a is a point not belonging to $\omega \times (\omega + 1) \cup \mathcal{G}$ and define a topology on the set X as follows:

- $\omega \times (\omega + 1)$ is the usual product of a discrete space with a convergent sequence and it is open in X;
- A neighbourhood base at a point $G \in \mathcal{G}$ consists of all sets $\{G\} \cup G \setminus D$, where D is finite;
- A neighbourhood subbase at a consists of all sets $X \setminus R_\alpha$ for $\alpha < \omega_1$, of all sets $X \setminus (n \times (\omega + 1))$, for $n \in \omega$, and of all sets $X \setminus (G \cup \{G\})$, where $G \in \mathcal{G}$.

It is clear that the space $\omega \times (\omega + 1) \cup \mathcal{G}$, defined according to the first two conditions, is T_1 and zero-dimensional. Taking this into account, it is evident that the above neighbourhood assignment gives to the set X a zero-dimensional T_1-topology upon the verification that any set R_α is clopen in the space $\omega \times (\omega + 1) \cup \mathcal{G}$.

If $n \in \omega$ then the set $\{n\} \times (\omega + 1)$ satisfies either $\{n\} \times (\omega + 1) \cap R_\alpha = \emptyset$ (case $n \in F_\alpha$) or $\{n\} \times (\omega + 1) \setminus R_\alpha = \{n\} \times f_\alpha(n)$. So, we see that a point of the form (n, ω) is in the interior either of $X \setminus R_\alpha$ or of R_α.

Let us try now to do the same for a point $G \in \mathcal{G}$. We have to distinguish three cases.

Case 1: $\alpha < \phi_1(G)$. Because of (3), we have $\operatorname{dom}(G) \subseteq^* F_\alpha$ and consequently $G \cap R_\alpha \subseteq \{(n, G(n))\colon n \in \operatorname{dom}(G) \setminus F_\alpha\}$. Since the latter set is finite, we see that the point G is in the interior of $X \setminus R_\alpha$.

Case 2: $\phi_2(G) \leqslant \alpha$. Because of (4), there is an integer n_0 such that for any $n \geqslant n_0$ we have $G(n) < f_\alpha(n)$. Consequently $G \cap R_\alpha \subseteq \{(n, G(n))\colon n < n_0\}$ and so G is in the interior of $X \setminus R_\alpha$.

Case 3: $\phi_1(G) \leqslant \alpha < \phi_2(G)$, i.e., $G \in R_\alpha$. Because of (2) and (5), we have that both sets $\operatorname{dom}(G) \cap F_\alpha$ and $\{n \in \operatorname{dom}(G)\colon (G(n) < f_\alpha(n)\}$ are finite. Now, if $k \in \operatorname{dom}(G) \setminus (\operatorname{dom}(G) \cap F_\alpha \cup \{n\colon G(n) < f_\alpha(n)\})$, we immediately see that $(k, G(k)) \in R_\alpha$ and hence $G \setminus R_\alpha$ is finite. So, the point G is in the interior of R_α.

Now that we have proved that X is a zero-dimensional T_1-space, we may proceed to the verification of the other required properties.

- X is pseudocompact. For this, it suffices to show that any infinite subset $A \subseteq \omega \times \omega$ has an accumulation point in X. If $\operatorname{dom}(A)$ is finite, then there is some $n \in \omega$ such that the set $B = A \cap \{n\} \times \omega$ is infinite. So, we have $(n, \omega) \in \overline{B} \subseteq \overline{A}$.

If $\operatorname{dom}(A)$ is infinite, then let α be the smallest ordinal such that $\operatorname{dom}(A) \setminus F_\alpha$ is infinite and put $B = \operatorname{dom}(A) \setminus F_\alpha$. Now, denote by C a partial function $\omega \to \omega$ satisfying $\operatorname{dom}(C) = B$ and $C \subseteq A$. Since the family $\{f_\beta\colon \beta < \omega_1\}$ is a scale, we can find an ordinal $\beta < \omega_1$ which is the smallest ordinal for which the set $D = \{n \in B\colon C(n) < f_\beta(n)\}$ is infinite. Then let $E = C \cap D \times \omega$. It is immediate to check that the set E is parametrized by the pair $\langle \alpha, \beta \rangle$. But then, the maximality of the family $\mathcal{G}$ leads to the existence of some $G \in \mathcal{G}$ for which $G \cap E$ is infinite. This clearly implies $G \in \overline{E} \subseteq \overline{A}$. Thus, X is pseudocompact.

- X contains a closed subspace homeomorphic to $\omega \cup \{\mathcal{F}\}$. Let us consider the subspace $Y = \omega \times \{\omega\} \cup \{a\} \subseteq X$. Each point of the form (n, ω) is obviously isolated in Y and the trace of the neighbourhood system given at a on Y is the family $\{(F_\alpha \setminus n) \times \{\omega\}\colon \alpha < \omega_1,\ n < \omega\}$. Thus, the subspace Y is homeomorphic to $\omega \cup \{\mathcal{F}\}$.
- X has countable tightness. The space X is first countable at each point different than a. Since $\omega \times (\omega + 1)$ is countable, we need only to check that if $a \in \overline{T}$, for some $T \subseteq \mathcal{G}$, then there is a countable $S \subseteq T$ such that $a \in \overline{S}$.

If we put $V_\alpha = \{G \in \mathcal{G}\colon\ \phi_1(G) > \alpha \text{ or } \phi_2(G) \leqslant \alpha\}$, then the family $\{V_\alpha \setminus A\colon \alpha < \omega_1,\ A \in [\mathcal{G}]^{<\omega}\}$ is just the trace of a local base at a on the subspace $\mathcal{G} \cup \{a\}$. Case 1: the set T' of all elements $G \in T$ for which $\phi_1(G) \geqslant \phi_2(G)$ is infinite. In this case, it follows that $T' \subseteq V_\alpha$ for every $\alpha < \omega_1$ and so a is in the closure of any countable infinite set $S \subseteq T'$.

If case 1 fails, then we may assume that $\phi_1(G) < \phi_2(G)$ holds for each $G \in T$. Let γ be the smallest ordinal such that $a \in \overline{\{G \in T\colon\ \phi_2(G) < \gamma\}}$.

Case 2. γ is a successor ordinal. Then the minimality of γ implies that $a \in \overline{T}'$, where $T' = \{G \in T\colon\ \phi_2(G) = \gamma - 1\}$. Since $V_\alpha \cap T' \neq \emptyset$, for each $\alpha < \gamma - 1$, it follows that $\sup\{\phi_1(G)\colon\ G \in T'\} = \gamma - 1$. Therefore, for any $n < \omega$ we may pick $G_n \in T'$ in such a way that $\{\phi_1(G_n)\colon\ n < \omega\}$ is an increasing sequence converging to $\gamma - 1$. It is clear that $a \in \overline{S}$ where $S = \{G_n\colon n < \omega\}$.

Case 3. γ is limit. Fix $\alpha < \gamma$ and denote the set $\{G \in T\colon\ \alpha < \phi_2(G) < \gamma\}$ by T'. Then $a \in \overline{T}'$ by the minimality of γ. Hence $T' \cap V_\alpha \neq \emptyset$. It follows from the definitions of T' and V_α that $\phi_1(G) > \alpha$ for some $G \in T'$. Also we are assuming that $\phi_1(G) < \phi_2(G)$ for every such G. So for every $\alpha < \gamma$, there is $G \in T$ such that $\alpha < \phi_1(G) < \phi_2(G) < \gamma$. Then there is a sequence $S = \{G_n \in P\colon\ n \in \omega\}$ such that $\phi_1(G_n) > \phi_2(G_k)$ whenever $n > k$. Then $|S \cap U_\beta| \leqslant 1$ for every $\beta < \omega_1$. Therefore $a \in \overline{S}$. □

Remark. If $\mathcal{F}$ is a filter on ω which has a base which is well-ordered by almost inclusion in type ω_1, then no space of the form $\omega \cup \{\mathcal{F}\}$ can be densely embedded into a pseudocompact Tychonoff space of countable tightness.

Proof. By contradiction, assume there exists a pseudocompact Tychonoff space X which contains $\omega \cup \{\mathcal{F}\}$ as a dense subspace and let $\{F_\alpha\colon\ \alpha < \omega_1\}$ be a base of $\mathcal{F}$ which is well-ordered by $\supseteq^*$. Let T be the set of all points of X which are in the closure of some infinite subset of $\omega \setminus F_\alpha$ for some $\alpha < \omega_1$. The regularity of X guarantees that $\mathcal{F} \in \overline{T}$. Now, let S be a countable subset of T and for each $s \in S$ let α_s be an ordinal for which $s \in \overline{\omega \setminus F_{\alpha_s}}$. If $\gamma = \sup\{\alpha_s\colon\ s \in S\}$ and U is an open neighbourhood of $\mathcal{F}$ such that $U \cap \omega = F_\gamma$, then we have $U \cap S = \emptyset$. Therefore X cannot have countable tightness. □

Recall that a space X is sequential if any non-closed set $S \subseteq X$ contains a sequence converging to a point outside S. The class of sequential spaces is properly contained in the class of spaces of countable tightness.

For any $f \in {}^{\omega}\omega$ let $U(f) = \{(m,n) \in \omega \times \omega\colon\ n \geqslant f(m)\}$. The sequential fan S_ω is the set $\omega \times \omega \cup \{p\}$ topologized in such a way that every point of $\omega \times \omega$ is isolated and a fundamental system of neighbourhoods at p is the family $\{\{p\} \cup U(f)\colon\ f \in {}^{\omega}\omega\}$. The space S_ω is a standard example of a countable space which does not have countable fan tightness.

Let $\mathcal{F}$ be the neighbourhood filter of p in S_ω. Assuming the existence of a scale $\{f_\alpha\colon\ \alpha < \omega_1\}$, it is easy to realize that the family $\{\{p\} \cup U(f_\alpha) \setminus n \times n\colon\ \alpha < \omega_1,\ n < \omega\}$ is a local base at p in S_ω. Notice that, if $\alpha < \beta$ then $U(f_\beta) \subseteq^* U(f_\alpha)$.

By identifying $\omega \times \omega$ with ω, the space S_ω takes the form $\omega \cup \{\mathcal{F}\}$.

Corollary 1. $[\mathfrak{d} = \omega_1]$ *There exists a pseudocompact Tychonoff sequential space X which contains a copy of S_ω. Then X does not have countable fan tightness.*

Proof. Consider S_ω in the form $\omega \cup \{\mathcal{F}\}$ and apply Theorem 1. We get a pseudocompact Tychonoff space X which contains a closed subspace homeomorphic to S_ω. Following the same notation in the proof of Theorem 1, we will check that this X is actually sequential. To this end, let A be a non-closed subset of $X = \omega \times (\omega + 1) \cup \mathcal{G} \cup \{a\}$. If there is some $x \neq a$ in $\overline{A} \setminus A$, then the first countability at x does the job. Thus, we proceed by assuming that A is closed in $X \setminus \{a\}$.

Case 1: $a \in \overline{A \cap \mathcal{G}}$. By applying the reasoning used to check the countable tightness in Theorem 1, we get a set $S \subseteq A \cap \mathcal{G}$ such that $a \in \overline{S}$. However, it is easy to realize that S is actually a sequence converging to a.

Case 2: $a \notin \overline{A \cap \mathcal{G}}$. We may fix a closed neighbourhood U of a such that $U \cap A \cap \mathcal{G} = \emptyset$. By replacing A with $U \cap A$, we may then assume that $A \subseteq \omega \times (\omega + 1)$. The set $A' = A \cap \omega \times \omega$ must satisfy $A' \subseteq n \times \omega$ for some $n < \omega$. The reason is that otherwise $\operatorname{dom}(A')$ should be infinite and then, arguing as in the proof of pseudocompactness of X in Theorem 1, we could find an element $G \in \mathcal{G}$ in the closure of A'. The condition $A' \subseteq n \times \omega$ clearly implies $a \notin \overline{A'}$ and so $a \in \overline{A \cap \omega \times \{\omega\}} = \overline{A \cap Y}$. But, Y is just the sequential fan and we are done. $\square$

Recall that a point $p \in \beta\omega \setminus \omega$ is a P-point provided that for any family $\{P_n\colon\ n < \omega\} \subseteq p$ there exists $Q \in p$ such that $Q \subseteq^* P_n$ for each n. It is known that P-points exist under CH. Moreover, assuming CH, any P-point has a base which is well-ordered by $\supseteq^*$ in type ω_1. Thus, we immediately get:

Corollary 2. [CH] *If $p \in \beta\omega \setminus \omega$ is a P-point then there exists a pseudocompact Tychonoff space of countable tightness which contains a closed copy of the subspace $\omega \cup \{p\} \subseteq \beta\omega$.*

Finally, taking into account the well-known fact that any space of the form $\omega \cup \{p\} \subseteq \beta\omega$ is extremally disconnected, we have:

Corollary 3. [CH] *There exists a pseudocompact Tychonoff space of countable tightness which contains a non-discrete extremally disconnected subspace.*

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