Identification of Concentrated Damages in Euler-Bernoulli Beams under Static Loads

G. Buda[1] and S. Caddemi[2]

Abstract: An identification procedure of concentrated damages in Euler-Bernoulli beams under static loads is presented in this work. The direct analysis problem is solved first by modeling concentrated damages as Dirac's delta distributions in the flexural stiffness. Closed-form solutions for both statically determinate and indeterminate beams are presented in terms of damage intensities and positions. On this basis, for the inverse damage identification problem, a nonquadratic optimization procedure is proposed. The presented procedure relies on the minimization of an error function measuring the error between the analytical model response and experimental data. The procedure allows to recognize "a posteriori" some sufficient conditions for the uniqueness of the solution of the damage identification problem. The influence of the instrumental noise on the identified parameters is also explored.

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Introduction

In the literature of the last decades the problem of damage identification has been the object of several studies in view of its applicability to those cases in which a simple visual inspection of the damaged structural element is not allowed. The appearance of damage implies a loss of the structural stiffness inducing variation of both static and dynamic response. Response measurements, hence, represent crucial data for damage identification. Experimental data can be obtained by nondestructive tests that represent the starting point of damage identification procedures. Single nondestructive tests in dynamic regime provide, in general, a large amount of information, and furthermore, since they are easily repeatable, encourage a wide research work devoted to the study of dynamic identification procedures (Vestroni and Capecchi 1996, 2000; Sinha et al. 2002; Patil and Maiti 2003). However, in cases of simple structural systems, such as straight beams subject to damage, static tests are easily executable and provide additional information to dynamic identification without any introduction of uncertainties due to masses and damping ratios. In the literature there are in fact studies, although less numerous, proposing identification procedures based on measurements by static tests aiming at identification of both physical and geometrical parameters of structural systems, and also discretized by means of finite elements (Banan et al. 1994a,b; Hjelmstad and Shin 1997; Sanayei and Scampoli 1991).

An optimization procedure for damage identification in straight beams by means of bending moment measurements by static tests has been recently proposed (Di Paola and Bilello 2004). In the latter procedure the damage has been modeled as a distortion superimposed to the undamaged beam. Since the distortion is function of the stress distribution in the damage beam, a different treatment is required by statically determinate and indeterminate beams.

In this study the identification problem of concentrated damages, such as cracks in Euler-Bernoulli beams by means of static response measurements, is dealt with. Since no crack closure phenomenon is considered, a linear behavior of the damaged beam is assumed.

First, the direct analysis problem under static loads is treated by considering the concentrated damage as a singularity of the flexural stiffness, without introducing any restriction concerning the damage intensity. The above-mentioned singularity is modeled by means of the well known Dirac's delta distribution. The treated case requires ad hoc integration rules of distributions recently discussed (Biondi and Caddemi 2005). The proposed approach for the direct analysis problem leads to an explicit response in terms of intensity and position of the damage.

The inverse identification problem is here studied by means of an optimization procedure of a function measuring the error of the model response with respect to experimental measurements. The error function, on the basis of the approach adopted for the direct analysis problem, is formulated as an explicit nonquadratic function of the parameters to be identified. Application of the proposed optimization procedure permits important indications concerning the position of measurements and the types of load conditions for the execution of static tests. Finally, sensitivity of the identification procedure to instrumental noise affecting the experimental data is also studied by modeling the noise as a random variable and adopting suitable probabilistic indices of the identified parameters.

[1]Dept. of Environmental and Civil Engineering, Univ. of Catania, Catania, Italy.

[2]Dept. of Environmental and Civil Engineering, Univ. of Catania, Catania, Italy (corresponding author). E-mail: scaddemi@dica.unict.it

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Concentrated Damage Model for Euler-Bernoulli Beams

The presence of damage in a continuous body determines a loss of some physical parameters of the material to be held in an appropriate constitutive model. In the case of damage caused by the presence of a crack it is well known that, besides the stress concentration occurring at the crack tip, there is a zone, adjacent to the crack, denoted as "ineffective" in view of its low stress level.

If a straight beam is subjected to a concentrated damage, such as a crack or a saw cut at a certain cross section, the presence of the ineffective zone in the crack vicinity can be accounted for by a loss of the flexural stiffness in the Euler-Bernoulli beam theory. In this study, since the final aim concerns the detection of the crack on the basis of suitably designed static tests, phenomena such as closure or propagation of the crack will not be considered; hence, the beam will show a linear behavior. In the literature, several models of flexural stiffness variation over a finite leg of the beam in the vicinity of the crack have been proposed (Anifantis and Dimarogonas 1993; Chondros et al. 1998; Christides and Barr 1984; Ostachowicz and Krawczuk 1991; Paipetis and Dimarogonas 1986; Sinha et al. 2002). A comparison of these models has been reported by Cerri and Vestroni (2003). Whatever variation law is adopted for the flexural stiffness over the finite interval of the beam, an equivalence criterion allows modeling of the considered concentrated damage as an internal hinge located at the same cross section endowed with rotational spring whose "equivalent" stiffness is dependent on the damage intensity. In this study, for the case of a uniform rectangular cross-section beam, the following expression for the stiffness of the equivalent rotational spring is applied (Bilello 2001)

$$k_{eq} = \frac{EI_0}{h}\frac{0.9(\beta-1)^2}{\beta(2-\beta)} \tag{1}$$

where E=Young's modulus of the material; I_0=inertia moment of the undamaged section; h=cross section height; and $\beta = d/h$, subjected to inequalities $0 \leqslant \beta \leqslant 1$, is the ratio between the crack depth d and the cross section height h. Henceforth, the β parameter will be addressed as "damage intensity parameter."

The adoption of Eq. (1) for the stiffness of the equivalent rotational spring, proposed in the case of the rectangular cross section with a crack normal to the beam axis (Bilello 2001), does not represent a limitation for the present study. In fact, for different cross sections, different equivalent stiffness models can also be adopted in the damage identification procedure proposed in this paper. Furthermore, different types of damage (such as cracks not normal to the beam axis or even diffused) can also be treated provided they can be modeled by means of an equivalent rotational spring whose stiffness should replace that adopted in Eq. (1).

The Euler-Bernoulli beam model, adopted in this study in order to perform the damage identification procedure, under static loads and for the general case of variable inertia moment $I(x)$, is governed by the following equations

$$T'(x) = q(x), \tag{2a}$$

$$M'(x) = T(x) \tag{2b}$$

$$\varphi(x) = -u'(x), \tag{2c}$$

$$\chi(x) = \varphi'(x) \tag{2d}$$

$$\chi(x) = M(x)/EI(x) \tag{2e}$$

where $q(x)$=external vertical load; $T(x)$ and $M(x)$=shear force and the bending moment, respectively; $u(x)$, $\varphi(x)$, and $\chi(x)$ are the deflection, slope, and curvature functions, respectively; and the prime denotes differentiation with respect to the spatial coordinate x, spanning from 0 to the length L of the beam.

The differential Eqs. (2*a*)–(2*d*) represent the equilibrium and the compatibility equations, while the algebraic Eq. (2*e*) is the constitutive equation relating curvature and bending moment through the spatial variable flexural stiffness $EI(x)$.

Combining the equilibrium, the compatibility and the constitutive equations yields to the following fourth-order differential governing equation in terms of deflection only

$$[EI(x)u''(x)]'' = q(x) \tag{3}$$

where the spatial variability of the flexural stiffness has to be accounted for.

In this study, Eq. (3), holding over the entire domain $0 \leqslant x \leqslant L$, will be adopted in order to describe an Euler-Bernoulli beam showing a slope discontinuity at x_0 equivalent to the presence of a concentrated damage. However, with this aim, a constant inertia moment I_0 of the cross section along the beam span, showing a singularity at x_0, is considered as follows

$$I(x) = I_0[1 - \alpha\delta(x - x_0)] \tag{4}$$

where the singularity is represented by a Dirac's delta distribution $\delta(x-x_0)$, centred at x_0, multiplied by a dimensional parameter α.

The model introduced by means of Eq. (4) indicates that the inertia moment is a distribution; hence, according to the distribution theory (Guelfand and Chilov 1972; Hoskins 1979; Lighthill 1958), Eq. (4) is a synthetic notation implying that $I(x)$ is not defined at x_0 but its properties are defined by integration after multiplication by test functions. In this sense, according to Eq. (4) the inertia moment does not take negative values at x_0, but integration operation used in this work can be performed by means of integration rules of Dirac's delta. However, even though Eq. (4) is considered as representative of a function, rather than a distribution, in this section it is proved that the parameter α does not assume values greater or equal to 1, hence the inertia moment does not take negative values.

A deeper interpretation of the inertia moment model adopted in Eq. (4) is not straightforward and requires a careful insight into the distribution theory presented in a recent work (Biondi and Caddemi 2005).

Substitution of Eq. (4) into the fourth-order governing differential Eq. (3) leads to

$$\{EI_0[1 - \alpha\delta(x - x_0)]u''(x)\}'' = q(x) \tag{5}$$

Double integration of Eq. (5), in view of the compatibility Eqs. (2*c*) and (2*d*), leads to the following equation

$$\chi(x) = -u''(x) = -\frac{q^{[2]}(x) + c_1 + c_2 x}{EI_0} - \alpha u''(x)\delta(x - x_0) \tag{6}$$

where $q^{[k]}$ indicates a primitive of order k of the external load function $q(x)$; and c_1 and c_2 are integration constants. It has to be noted that the primitive of order two, $q^{[2]}$, of the load function $q(x)$ is usually a continuous function, even in those cases showing singularities of the external load such as abrupt variations of the external load (modeled as unit step functions) or concentrated loads (modeled as Dirac's deltas) (Yavari et al. 2000; Falsone 2002).

The loss of continuity of $q^{[2]}(x)$ is due only to the presence of concentrated external moments (modeled as doublet distributions). However, in the latter case the discontinuities are never assumed to be coincident with the singularity of the inertia moment at x_0.

The curvature function $\chi(x)$ expressed under the form of Eq. (6) does not explicitly justify the inertia moment model adopted in Eq. (4), in view of the presence of the term $u''(x)\delta(x-x_0)$ on the right-hand side. However, multiplying both sides of Eq. (6) by $\delta(x-x_0)$ the following expression is obtained

$$u''(x)\delta(x-x_0) = \frac{q^{[2]}(x)+c_1+c_2x}{EI_0}\delta(x-x_0) + \alpha u''(x)\delta(x-x_0)\delta(x-x_0) \tag{7}$$

The first term on the right-hand side of Eq. (7) can be considered as a standard Dirac's delta distribution in view of the continuity of $q^{[2]}(x)$ at x_0, previously discussed. The second term contains the product of two Dirac's deltas both centred at x_0. In order to give Eq. (7) some mathematical meaning, the product definition of two Dirac's deltas proposed by Bagarello (1995, 2002) is adopted here. Bagarello indicates that the product of two Dirac's deltas both centred at x_0 can be reduced to a single Dirac's delta multiplied by a constant A

$$\delta(x-x_0)\delta(x-x_0) = A\delta(x-x_0) \tag{8}$$

A set of values for the quantity A, defined by Bagarello (1995), for which Eq. (8) holds, is reported in the Appendix.

Since this study, besides the adopted distribution theory, aims at capturing physical aspect of an engineering problem, the constant A is required to be dimensional, and as shown later, consistent with the dimension of the Dirac's delta.

Replacing Eq. (8) in Eq. (7) leads to

$$u''(x)\delta(x-x_0) = \frac{1}{1-\alpha A}\frac{q^{[2]}(x)+c_1+c_2x}{EI_0}\delta(x-x_0) \tag{9}$$

Substitution of Eq. (9) in Eq. (6) provides the following explicit expression for the curvature $\chi(x)$

$$\chi(x) = -u''(x) = -\frac{q^{[2]}(x)+c_1+c_2x}{EI_0}\left[1+\frac{\alpha}{1-\alpha A}\delta(x-x_0)\right] \tag{10}$$

The importance of Eq. (10) must be highlighted in view of its capability of justifying the inertia moment model adopted in Eq. (4). In fact, according to Eq. (10) the curvature $\chi(x)$ is given as superimposition of a Dirac's delta distribution, centred at x_0, to the function $[q^{[2]}(x)+c_1+c_2x]/EI_0$ definitely continuous at x_0. As a consequence, the slope function $\varphi(x)$, in view of Eq. (2*d*), shows a discontinuity at x_0. The latter circumstance indicates that the choice of Eq. (4) for the inertia moment represents a beam with an internal hinge with rotational spring at the abscissa x_0.

Double integration of Eq. (10) provides the following expressions for the slope function $\varphi(x)$ and the deflection function $u(x)$ (Biondi and Caddemi 2005)

$$\begin{aligned}\varphi(x) &= -u'(x)\\ &= -c_3 - \frac{c_1}{EI_0}\left[x+\frac{\alpha}{1-\alpha A}U(x-x_0)\right] + \\ &\quad -\frac{c_2}{2EI_0}\left[x^2+2\frac{\alpha}{1-\alpha A}x_0U(x-x_0)\right] - \frac{q^{[2]}(x)}{EI_0}\\ &\quad -\frac{\alpha}{1-\alpha A}\frac{q^{[2]}(x_0)}{EI_0}U(x-x_0)\end{aligned} \tag{11}$$

$$\begin{aligned}u(x) &= c_4 + c_3x + \frac{c_1}{2EI_0}\left[x^2+2\frac{\alpha}{1-\alpha A}(x-x_0)U(x-x_0)\right]\\ &\quad +\frac{c_2}{6EI_0}\left[x^3+6\frac{\alpha}{1-\alpha A}x_0(x-x_0)U(x-x_0)\right] + \frac{q^{[4]}(x)}{EI_0}\\ &\quad +\frac{\alpha}{1-\alpha A}\frac{q^{[2]}(x_0)}{EI_0}(x-x_0)U(x-x_0)\end{aligned} \tag{12}$$

where $U(x-x_0)$ indicates the well-known unit step distribution, also known in the literature as Heaviside's function, representing the formal primitive of the Dirac's delta, showing discontinuity at x_0, defined as $U(x-x_0)=0$ for $x < x_0$, $U(x-x_0)=1$ for $x \geqslant x_0$. Furthermore, constants c_1, c_2, c_3, and c_4 appearing in Eqs. (11) and (12) can be obtained by means of enforcement of boundary conditions. In particular, mechanical conditions require the knowledge of the bending moment $M(x)$ and the shear force $T(x)$ functions.

The bending moment $M(x)$ is obtained by means of Eq. (2*e*) and Eq. (10), after simple algebra and accounting for the product of two Dirac's deltas, as follows

$$M(x) = EI(x)\chi(x) = -\lfloor c_1 + c_2x + q^{[2]}(x)\rfloor \tag{13}$$

Differentiation of Eq. (13), in view of Eq. (2*b*), leads to the following expression for the shear force $T(x)$

$$T(x) = -c_2 + q^{[1]}(x) \tag{14}$$

In the case of statically determinate beams, the influence of the discontinuity on the response by means of the parameter α is expected through the constants c_3 and c_4 only, once the boundary conditions are enforced, since it is well known that in this case $M(x)$ and $T(x)$, where c_1 and c_2 appear, should not depend on the physical characteristics of the beam.

The slope function $\varphi(x)$ provided by Eq. (11) shows at x_0 the following discontinuity $\Delta\varphi(x_0)$

$$\Delta\varphi(x_0) = \varphi(x_0^+) - \varphi(x_0^-) = -\frac{1}{EI_0}\frac{\alpha}{1-\alpha A}[c_1 + c_2x_0 + q^{[2]}(x_0)] \tag{15}$$

where x_0^+ and x_0^-=abscissae on the right and on the left of x_0, respectively. The discontinuity $\Delta\varphi(x_0)$ provided by Eq. (15) represents the relative rotation between the cross sections at x_0^+ and x_0^-, a consequence of the inertia moment model adopted in Eq. (4).

Comparison of Eq. (13) and Eq. (15) leads to

$$\Delta\varphi(x_0) = \frac{\alpha}{1-\alpha A}\frac{M(x_0)}{EI_0} \tag{16}$$

Eq. (16) provides the relationship between the relative rotation $\Delta\varphi(x_0)$ and the bending moment $M(x_0)$ at x_0, suggesting the interpretation of inertia moment model of Eq. (4) as an internal hinge at x_0 endowed with rotational spring stiffness K_φ, given as

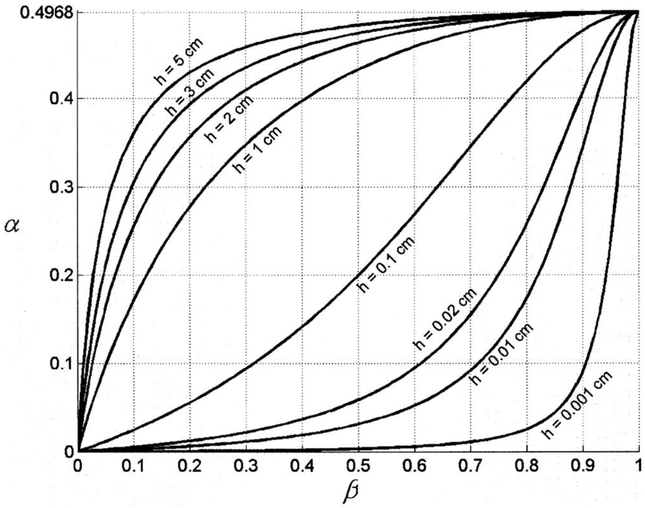

Fig. 1. Variation of the parameter α, adopted in the inertia moment model, with the damage intensity parameter β for different values of the cross section height h

$$K_\varphi = \frac{1-\alpha A}{\alpha} EI_0 \tag{17}$$

Dimensional analysis of Eq. (17) requires the A constant to be measured as the inverse of a length, the parameter α as a length, and as a consequence the Dirac's delta, appearing in Eq. (4) is a distribution to be measured as the inverse of a length.

The stiffness K_φ of the rotational spring at x_0 given by Eq. (17) is dependent on the parameter α introduced in the inertia moment model of Eq. (4). The stiffness K_φ is here adopted to reproduce the stiffness K_{eq} of the rotational spring at x_0, given by Eq. (1), equivalent to the damage intensity β. Hence, for a given rotational spring stiffness K_{eq}, the related value of the parameter α has to be obtained by Eq. (17) as $\alpha = EI_0/(K_{eq}+AEI_0)$ for a value of the quantity A among those proposed by Bagarello (1995).

However, a direct relationship between the damage intensity parameter β and the parameter α can be obtained by equating Eqs. (1) and (17), leading to the following expression for the parameter α

$$\alpha = \frac{2\beta - \beta^2}{(0.9/h - A)\beta^2 - (0.9/h - A)2\beta + (0.9/h)} \tag{18}$$

Eq. (18), for a value of the quantity A among those proposed by Bagarello (1995), provides the parameter α, to be adopted in the model with singularity given by Eq. (4) as function of the damage intensity parameter β.

According to Eq. (18), for damage intensity parameter $\beta=0$ (i.e., no damage), it is also $\alpha=0$, and, in view of Eq. (17), the rotational spring stiffness $K_\varphi=\infty$. Under the latter circumstance, any relative rotation is forbidden at x_0 and the case of constant inertia moment I_0 of the entire beam is recovered.

On the other hand, for damage intensity parameter $\beta=1$ (i.e., the damage affects the entire height of the cross section), the parameter α obtained by Eq. (18) attains the value $1/A$ leading to a rotational spring stiffness equal to zero.

The expression in Eq. (18), dependent on the height h of the cross section, is a monotonic function in the range of $0 \leq \beta \leq 1$, plotted in Fig. 1, where for the quantity A the first value among those proposed by Bagarello (1995) has been chosen (evaluated as $A=2.013\ \text{cm}^{-1}$, as shown in the appendix, for indices $j=2$ and $m=2$ in Bagarello's paper). Hence, for significant values of damage intensity factor $0 \leq \beta \leq 1$, the parameter α is such that $0 \leq \alpha \leq 1/A$, where $1/A < 1$.

It has to be pointed out that curves plotted in Fig. 1 have been obtained by expressing the height h in centimeters. Since Eq. (18) is dependent on the adopted unit of measure, the value of α can be different even for a fixed value of β. However, the value of α, provided by Eq. (18) and replaced in Eq. (17) will provide the correct value of stiffness K_φ equal to the rotational spring stiffness K_{eq} in Eq. (1), equivalent to the concentrated damage.

The model presented in this section for a sole singularity of the inertia moment leads to the explicit solution, reported in Eq. (12), as function of the intensity and position damage parameters. Eq. (12), solution of the direct analysis problem, will be adopted in the next section for the inverse problem aiming at damage identification.

However, since the presence of multiple damage in a beam can occur, generalization of the explicit solution to the case of multiple singularities is also presented in this section.

The model of inertia moment $I(x)$ with a singularity proposed in Eq. (4) can be extended to the presence of n singularities as follows

$$I(x) = I_0\left[1 - \sum_{i=1}^{n} \alpha_i \delta(x - x_{0i})\right] \tag{19}$$

where $\alpha_i (i=1,\ldots,n)$ parameters are the intensities of the singularities present at abscissae $x_{0i} (i=1,\ldots,n)$. The model adopted in Eq. (19), in light of Eq. (17), corresponds to the presence of n internal hinges at x_{0i} endowed with rotational spring stiffnesses $K_{\varphi i} = EI_0(1-\alpha_i A)/\alpha_i$.

Integration of the governing equation of Euler-Bernoulli beam with inertia moment given by Eq. (19) can be conducted according to a procedure analogous to that presented for a sole singularity, leading to the following expression for the deflection function

$$u(x) = c_4 + c_3 x + c_1\left[x^2 + 2\sum_{i=1}^{n} \frac{\alpha_i}{1-\alpha_i A}(x - x_{0i})U(x - x_{0i})\right] + c_2\left[x^3 + 6\sum_{i=1}^{n} \frac{\alpha_i}{1-\alpha_i A} x_{0i}(x - x_{0i})U(x - x_{0i})\right] + \frac{q^{[4]}(x)}{EI_0} + \sum_{i=1}^{n} \frac{\alpha_i}{1-\alpha_i A}\frac{q^{[2]}(x_{0i})}{EI_0}(x - x_{0i})U(x - x_{0i}) \tag{20}$$

Eq. (20) represents the solution of the direct analysis problem for the case of multiple damages if the parameters $\alpha_i (i=1,\ldots,n)$ are strictly related to n damages with intensities β_i according to the relationship given by Eq. (18). Eq. (20) will be adopted in the following section to perform a solution procedure of the damage identification inverse problem.

Optimization Procedure for Damage Identification

The aim of this section is identification of the intensity and position of concentrated damages in a beam, if measurements of the deflection at selected cross sections are given by static tests. In particular, it is supposed that deflection measurements at nm cross sections are given by the execution of static tests for nlc different load conditions.

The identification procedure adopted is based on the explicit solution in terms of deflection function of the direct analysis problem proposed earlier.

Damage identification is achieved by means of the following minimization problem

$$\min_{\alpha_i, x_{0i}} \prod (\alpha_i, x_{0i}) = \sum_{l=1}^{nlc} \sum_{m=1}^{nm} [u(\alpha_i, x_{0i}, x_{m,l}) - u^E(x_{m,l})]^2$$

subject to

$$0 \leqslant \alpha_i \leqslant 1/A, \quad 0 \leqslant x_{0i} \leqslant L(i=1, \dots, n) \tag{21}$$

Therefore, given a beam with n damaged cross sections, with unknown intensity and position of damages, the optimization problem represented by Eq. (21) provides the sought values of α_i and $x_{0i}(i=1, \dots, n)$ as those that minimize the function Π. The error function $\Pi(\alpha_i, x_{0i})$ is defined in Eq. (21) as the square of the difference between deflection of the Euler-Bernoulli damaged beam model $u(\alpha_i, x_{0i}, x)$, presented in Eq. (20), and the experimental deflection measurements $u^E(x_{m,l})$ at nm different cross sections for nlc different load conditions indicated as $x_{m,l}$ $(m=1, \dots, nm; l=1, \dots, nlc)$. The problem reported in Eq. (21) is not a quadratic optimization problem in view of the nonlinear dependence of the deflection function given in Eq. (20) on intensities α_i and positions x_{0i} of damages. Additional nonlinearities are hidden in the c_1, c_2, c_3, and c_4 constants to be evaluated in Eq. (20) by means of enforcement of boundary conditions.

In this study, rather than exploring aspects regarding solution algorithms of Eq. (21), the solution procedure proposed in the literature for identification procedures in dynamic field (Vestroni and Capecchi 1996, 2000; Cerri and Vestroni 2000), and recently recast for static identification problems (Di Paola and Bilello 2004), is adopted.

The mentioned solution procedure is efficient for the search of the solution of Eq. (21) with particular regard to the position variables x_{0i} appearing as arguments of the unit step functions in Eq. (20). The solution procedure is performed according to the following two phases: (1) minimization of the error function Π, with respect to the α_i parameters only, for fixed values of damage positions x_{0i}, leading to the reduced error function $\tilde{\Pi}(x_{0i}) = \min_{\alpha_i} \Pi(\alpha_i, x_{0i})$, where each value of $\tilde{\Pi}(x_{0i})$ will be coupled to the optimal values of α_i; and (2) minimization of the reduced error function $\tilde{\Pi}(x_{0i})$ with respect to the positions x_{0i}.

The solution of the inverse identification problem is provided by $\min_{x_{0i}} \tilde{\Pi}(x_{0i})$ together with its coupled values α_i.

It is known (Banan et al. 1994a; Hjelmstad and Shin 1997; Vestroni and Capecchi 2000) that the number of measurements required for the solution of the inverse identification problem must be greater or equal to the number of parameters to be identified; hence the following inequality must be satisfied

$$nlc \times nm \geqslant 2n \tag{22}$$

Eq. (22) is understood as a necessary condition for the solution of the damage identification problem. To the authors' knowledge no sufficient conditions have been formulated neither regarding the position of the measurements nor the position and distribution of the external loads.

In what follows, by making use of the above-mentioned optimization procedure, attention will be devoted to the analysis of the results of single and double damage identification problem. Aim of the study is testing the proposed identification procedure, rather than the adopted damaged beam model, and providing, in addition to Eq. (22), indications concerning measurement and concentrated load positions for a correct formulation of the identification problem. To this aim, the given deflection measurements $u^E(x_{m,l})$ appearing in Eq. (21) are generated by means of the explicit solution in Eq. (20) of the adopted model; subsequently, the influence of experimental errors on the proposed procedure is investigated by means of superimposition of a random variable to the generated deflection measurements. Performance of the proposed identification procedure on the basis of acquisition of deflection measurements by real laboratory static tests will be subject of future work.

Identification of a Single Concentrated Damage

In this section a steel beam with Young's modulus $E=210$ GPa, length $L=100$ cm, and square cross section with height $h=5$ cm under different boundary conditions is considered. A single concentrated damage due to a crack with depth $d=2.5$ cm, at the cross section $x_0=40$ cm, is object of identification by means of the proposed procedure.

The ratio between the crack depth d and the cross-section height h, named damage intensity parameter β, takes the value $\beta=0.5$. From now on, for the constant A the first value among those proposed by Bagarello (1995) $A=2.013\ \text{cm}^{-1}$ will be adopted. For $\beta=0.5$ and $A=2.013\ \text{cm}^{-1}$, Eq. (18) provides $\alpha=0.4824$ cm.

The first damage identification case is solved on the basis of the knowledge of two deflection measurements $(nm=2)$ at x_1 and x_2 here obtained by means of the explicit solution provided by Eq. (20) for a single load condition $(nlc=1)$ given by a concentrated load $P=2$ kN at abscissa x_P. It has to be noted that the choice of two measurements $(nm=2)$ and a single load condition $(nlc=1)$ is in accordance with the necessary condition in Eq. (22) required for the solution of the inverse identification problem of single damage $(n=1)$. However, since the existence of sufficient conditions regarding the positions x_1 and x_2 of deflection measurements has not been proved in the literature, in this section correct choices of measurement positions leading to a unique solution of the identification problem will be studied.

In the case of a pinned-pinned beam two deflection measurements at $x_1=30$ cm and $x_2=70$ cm bridging the crack to be identified are considered first. Equation (12), for the load $P=2$ kN concentrated at $x_P=60$ cm, provides the following deflection measurements $u^E(x_1)=u_1=0.0362$ cm at $x_1=30$ cm and $u^E(x_2)=u_2=0.0362$ cm at $x_2=70$ cm. The identification procedure presented previously has been applied to identify the position x_0 and the parameter α of the concentrated damage. In Figs. 2(a and b) the reduced error function $\tilde{\Pi}(x_0)$ and the error function $\Pi(\alpha, x_0)$ with its contour lines are plotted. The reduced error function in Fig. 2(a) takes the zero value, as minimum value, at $x_0=40$ cm, and it is coupled to the optimal value of $\alpha=0.4824$ cm. The results are confirmed by the error function $\Pi(\alpha, x_0)$ in Fig. 2(b), where the contour lines, in the range $0 \leqslant \alpha \leqslant 1/A=0.4968$ cm and $0 \leqslant x_0 \leqslant L=100$ cm show the unique minimum for $x_0=40$ cm and $\alpha=0.4824$ cm, which is the exact solution.

The same identification problem has been also solved by considering two deflection measurements of two cross sections both lying on the right hand side of the crack to be identified. In particular, the identification procedure has been performed given the following deflection measurements $u^E(x_1)=u_1=0.0360$ cm at x_1

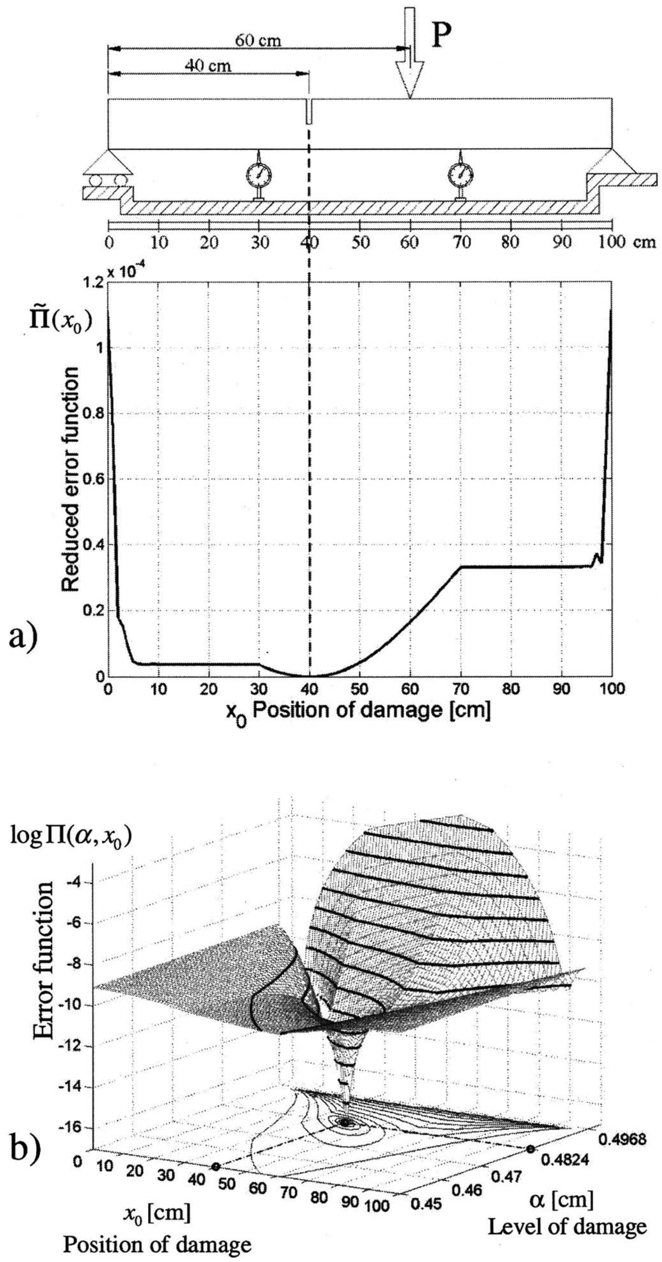

Fig. 2. Pinned-pinned beam with a single damage and two deflection measurements bridging the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

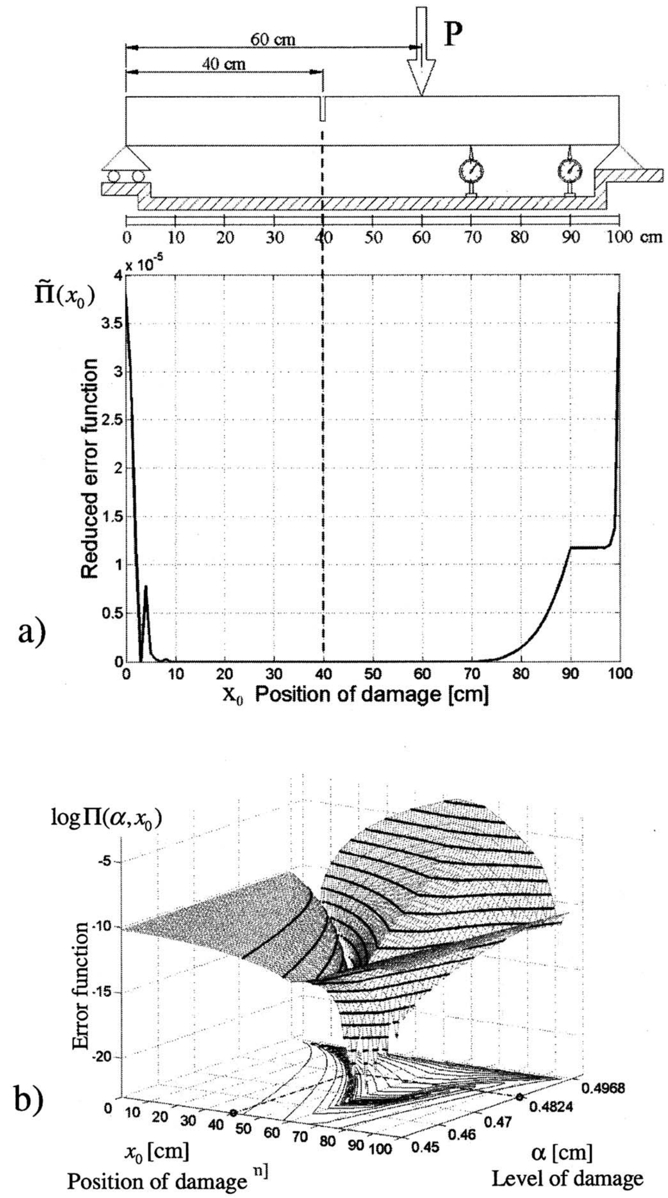

Fig. 3. Pinned-pinned beam with a single damage and two deflection measurements on one side of the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

$=70$ cm and $u^E(x_2)=u_2=0.0135$ cm at $x_2=90$ cm obtained by means of Eq. (20).

The reduced error function $\tilde{\Pi}(x_0)$, plotted in Fig. 3(a), attains the zero value in the region approximately between 10 and 70 cm; hence a unique solution of the damage identification problem cannot be recognized. In fact, the contour lines of the error function $\Pi(\alpha, x_0)$ plotted in Fig. 3(b) show the presence of an entire valley where the absolute minimum in reached.

Further identification problems on the pinned-pinned beam, based on different couples of deflection measurements, although not reported here, have been performed and confirmed the uniqueness of the solution for couples of measurements bridging the crack to be identified.

In order to study whether the positions of two measurements bridging the crack can be considered sufficient for the uniqueness of the solution of the identification problem, a clamped-pinned beam, a clamped-clamped beam, and a clamped-free beam have been also considered.

For a clamped-pinned beam under the concentrated load P at $x_p=80$ cm, if two measurements at $x_1=30$ cm and $x_2=70$ cm are available, the reduced error function $\tilde{\Pi}(x_0)$ and error function $\Pi(\alpha, x_0)$, plotted in Fig. 4, shows the existence of a unique minimum correspondent to the exact solution.

For a clamped-clamped beam under the concentrated load P at $x_p=50$ cm, it is shown in Fig. 5 that once again, a unique minimum, correspondent to the exact solution, is reached by the error function for two measurements bridging the crack.

The case of a clamped-free beam shows different peculiarities since, for two measurements bridging the crack $x_1=30$ cm and $x_2=70$ cm and a concentrated load at the free end, the error function in Fig. 6 shows several peaks defining an entire region where the exact solution cannot be uniquely identified. In this case dif-

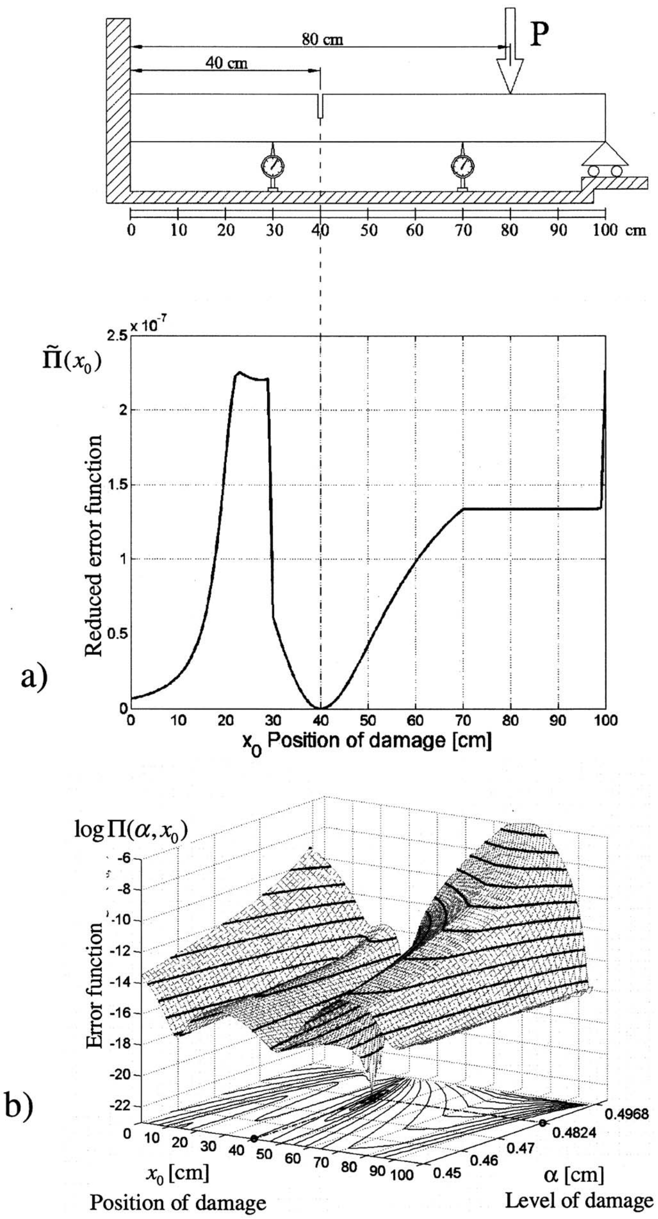

Fig. 4. Clamped-pinned beam with a single damage and two deflection measurements bridging the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

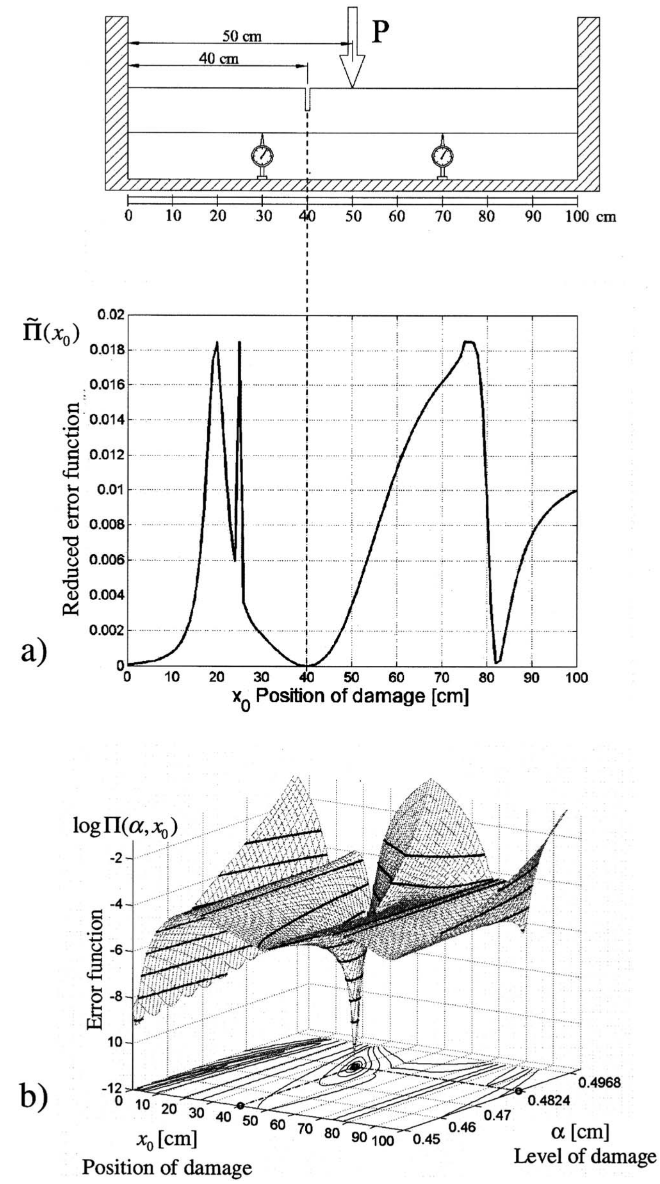

Fig. 5. Clamped-clamped beam with a single damage and two deflection measurements bridging the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

ferent measurement positions have been also considered in order to find a unique and exact solution. In Fig. 7 the error function for two measurements both on the left of the crack, $x_1=10$ cm and $x_2=30$ cm, still shows the existence of infinite solutions of the identification problem. Finally, the case of two measurements taken on the right of the crack at $x_1=60$ cm and $x_2=90$ cm, shown in Fig. 8, leads to a unique minimum of the error function coincident with the exact solution.

It can be noted that measurements have to be taken one at the left and one at the right of the crack for the cases of clamped-clamped, clamped-pinned, and pinned-pinned beams, while for clamped free beams, couples of measurements have to be taken on the side of the crack toward the free end, provided that the load position allows the crack to be open.

If real experimental tests are executed to identify a single crack whose position is not a priori known, it is suggested to choose two deflection measurements at x_1 and x_2 close to the left end and to the right end, respectively, for cases of pinned-pinned, clamped-pinned, and clamped-clamped beams in such a way that the damage will most likely lie between x_1 and x_2. In the case of a cantilever beam, it is instead suggested to choose measurements next to the free edge of the beam. The second damage identification case is solved with one deflection measurement ($nm=1$) only at $x_1=70$ cm. In order to enforce the necessary condition given by Eq. (22) at least two load conditions ($nlc=2$) have to be accounted for.

In this case suitable choices of the load conditions to be adopted in experimental tests are studied in order to obtain uniqueness of the solution of the damage identification problem. The case of a pinned-pinned beam under a concentrated load type $P=2$ kN is presented. In particular, two concentrated load conditions at $x_{P1}=30$ cm and $x_{P2}=60$ cm, bridging the crack, are considered first. The reduced error function $\tilde{\Pi}(x_0)$ plotted in Fig. 9(a)

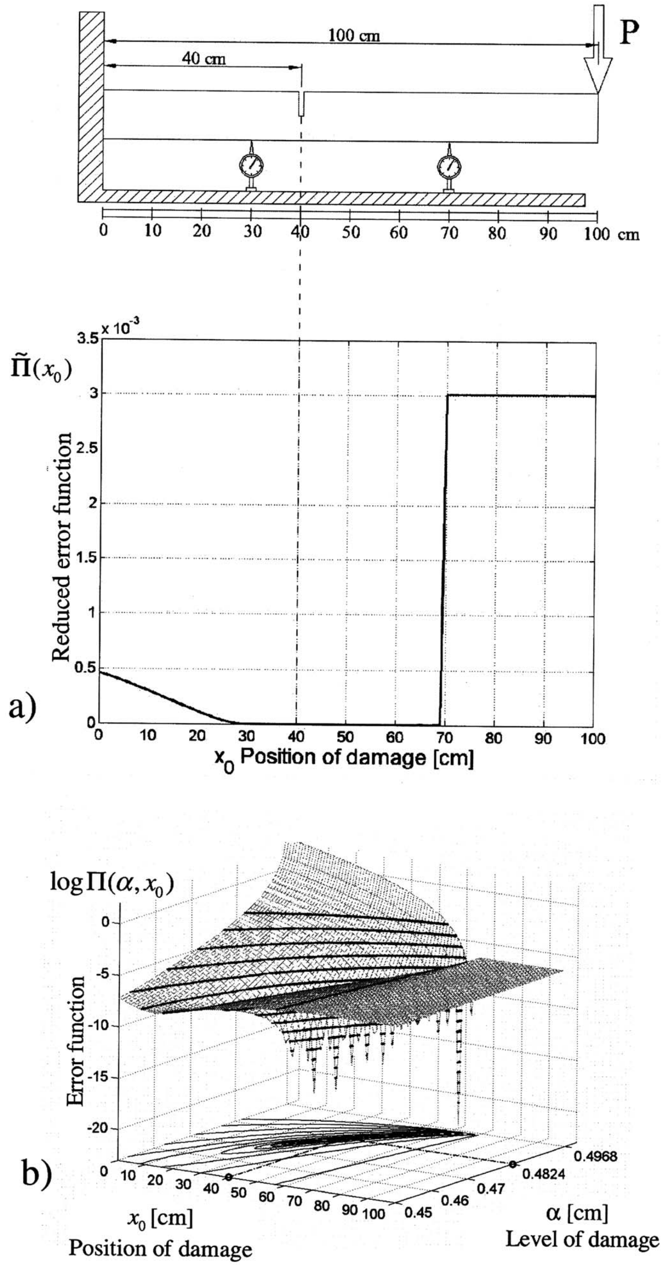

Fig. 6. Clamped-free beam with a single damage and two deflection measurements bridging the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

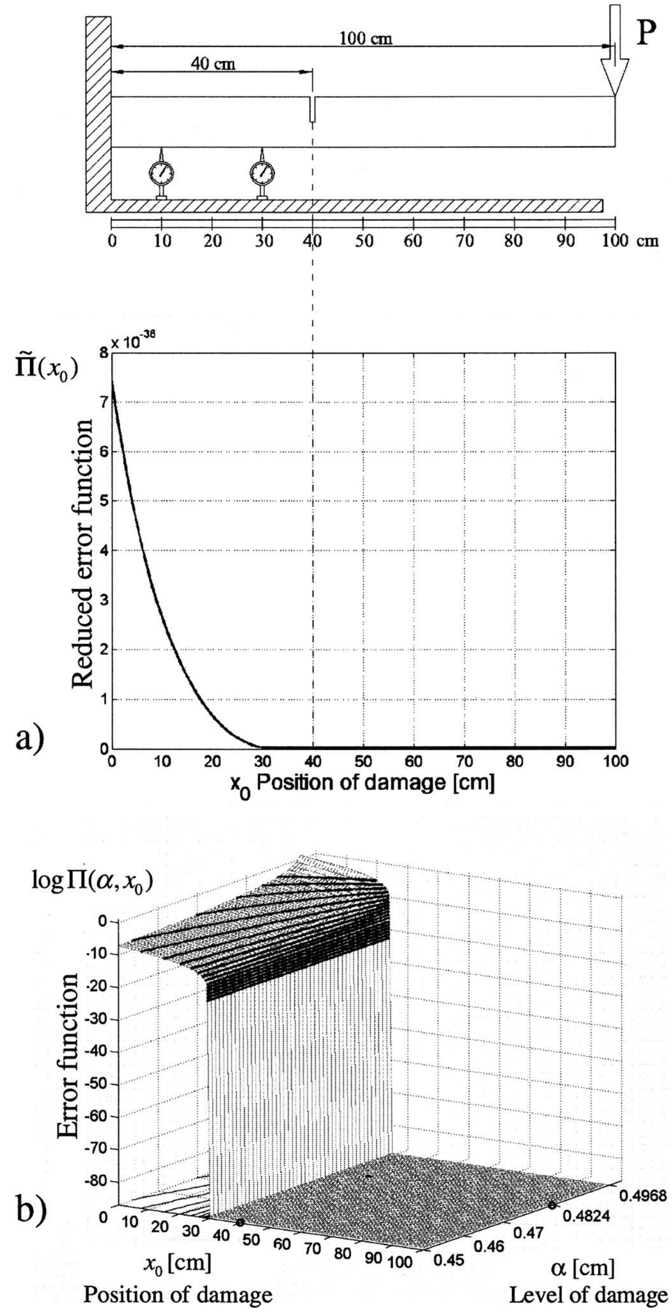

Fig. 7. Clamped-free beam with a single damage and two deflection measurements on the left of the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

shows a zero value as absolute minimum at $x_0=40$ cm. The error function $\Pi(\alpha x_0)$ plotted in Fig. 9(b) shows a unique minimum for $x_0=40$ cm and $\alpha=0.4824$ cm, correspondent to the exact solution of the damage identification problem.

Then, two concentrated load conditions, both on the right hand side of the crack, at $x_{P1}=50$ cm and $x_{P2}=60$ cm, are considered. The damage identification procedure in this case provides the reduced error function $\tilde{\Pi}(x_0)$ plotted in Fig. 10(a), not showing a unique minimum, but on the contrary, attaining the zero value in the entire region approximately between 10 and 50 cm. The error function $\Pi(\alpha, x_0)$ plotted in Fig. 10(b), in fact, shows a valley of absolute minima including the exact solution.

The latter case indicates that concentrated load conditions require loads to be applied on opposite sides of the crack in order to provide uniqueness of the solution. The latter circumstance explains the reason why in Fig. 7 in the original work by Di Paola and Bilello (2004), after the introduction of noise in deflection measurements, the highest mean error has been found for a load sequence of concentrated loads equally spaced from the left hand $x=0$ of the beam, the damage being located at $x=0.75L$.

Identification of Two Concentrated Damages

The extension of the solution of the direct analysis problem to the case of multiple damaged beam, presented in Eq. (20) in terms of deflection function, allows the treatment of the identification

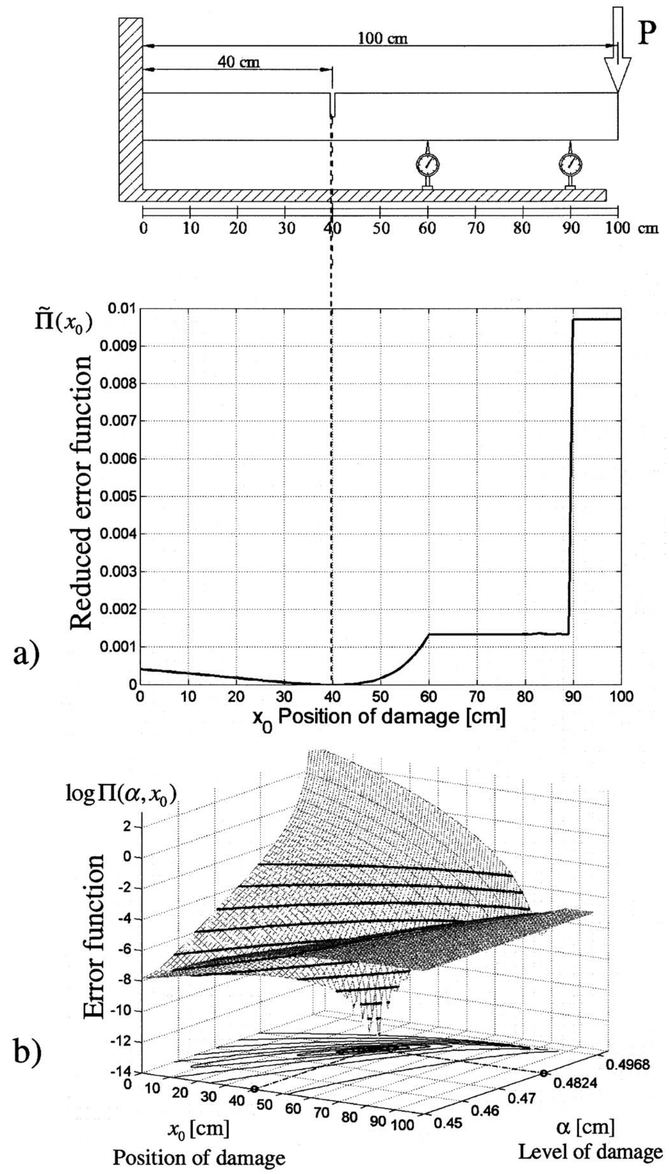

Fig. 8. Clamped-free beam with a single damage and two deflection measurements on the right of the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

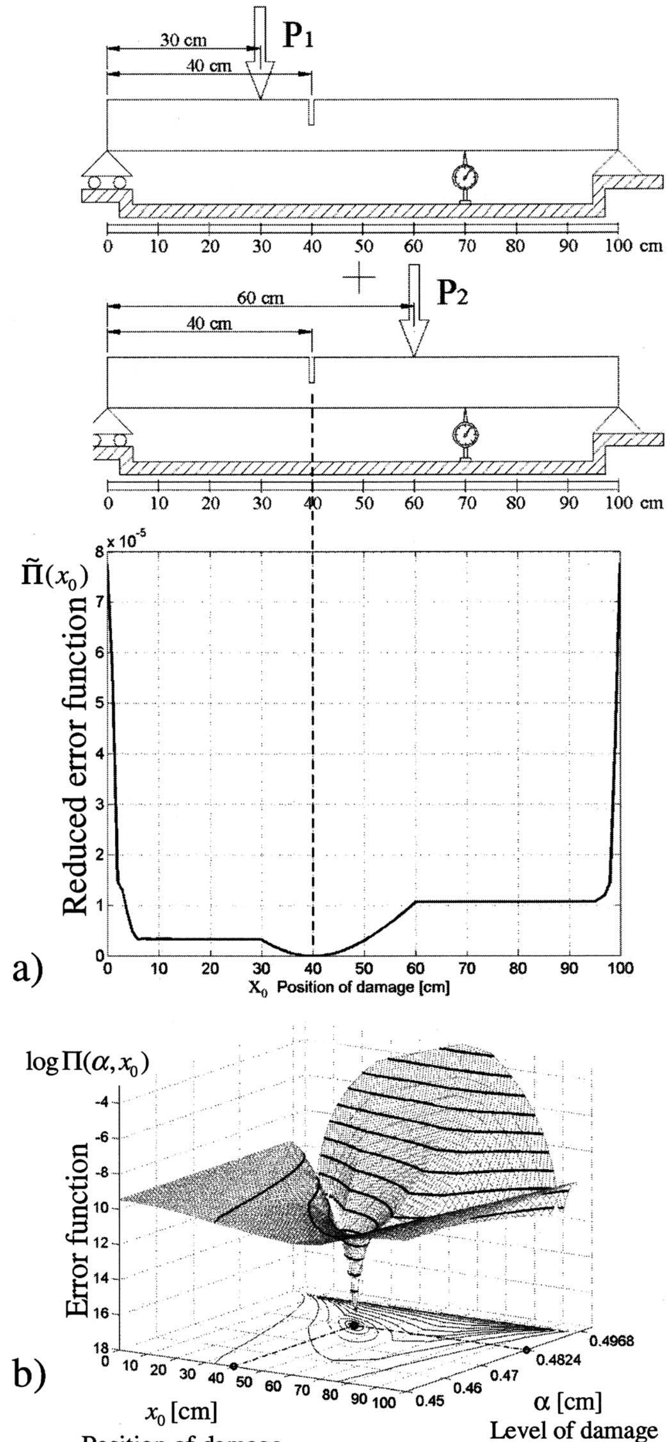

Fig. 9. Pinned-pinned beam with a single damage, one deflection measurement, and two concentrated load conditions bridging the crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

problem of two cracks as presented in this section. The same steel beam, treated in the previous section, is here considered under the pinned-pinned boundary conditions; however, two cracks $n=2$ concentrated at sections $x_{01}=25$ cm and $x_{02}=70$ cm are supposed. Both cracks reach the same depth $d_1=d_2=2.2$ cm leading to the damage intensity parameters $\beta_1=\beta_2=0.5$. According to the presented damage model, the parameters to be identified, besides the positions x_{01} and x_{02}, are $\alpha_1=\alpha_2=0.4824$ cm.

In this double damage identification problem a single concentrated load condition ($nlc=1$) is considered as follows: $P=2$ kN at $x_P=50$ cm. The necessary condition provided by Eq. (22) indicates that at least $nm=4$ deflection measurements are needed. In particular, two couples of measurements, each bridging a crack, are generated by means of Eq. (20) at the following cross sections: $x_1=10$ cm, $x_2=30$ cm, $x_3=60$ cm, and $x_4=85$ cm (such that $x_1 < x_{01} < x_2 < x_3 < x_{02} < x_4$).

The first step of the identification procedure presented previously has led to a reduced error function $\tilde{\Pi}(x_{01}, x_{02})$, plotted in Fig. 11. Two minima can be recognized in Fig. 11 (since crack 1 can be exchanged with crack 2), both correspondent to the exact solution of the identification problem $x_{01}=25$ cm and $\alpha_1=0.4824$ cm, and $x_{02}=70$ cm and $\alpha_2=0.4824$ cm. It is important to note that any other type of choice of four deflection measurements, besides the one above described, would lead to indeterminate solution of the problem. In fact, if the two cracks are con-

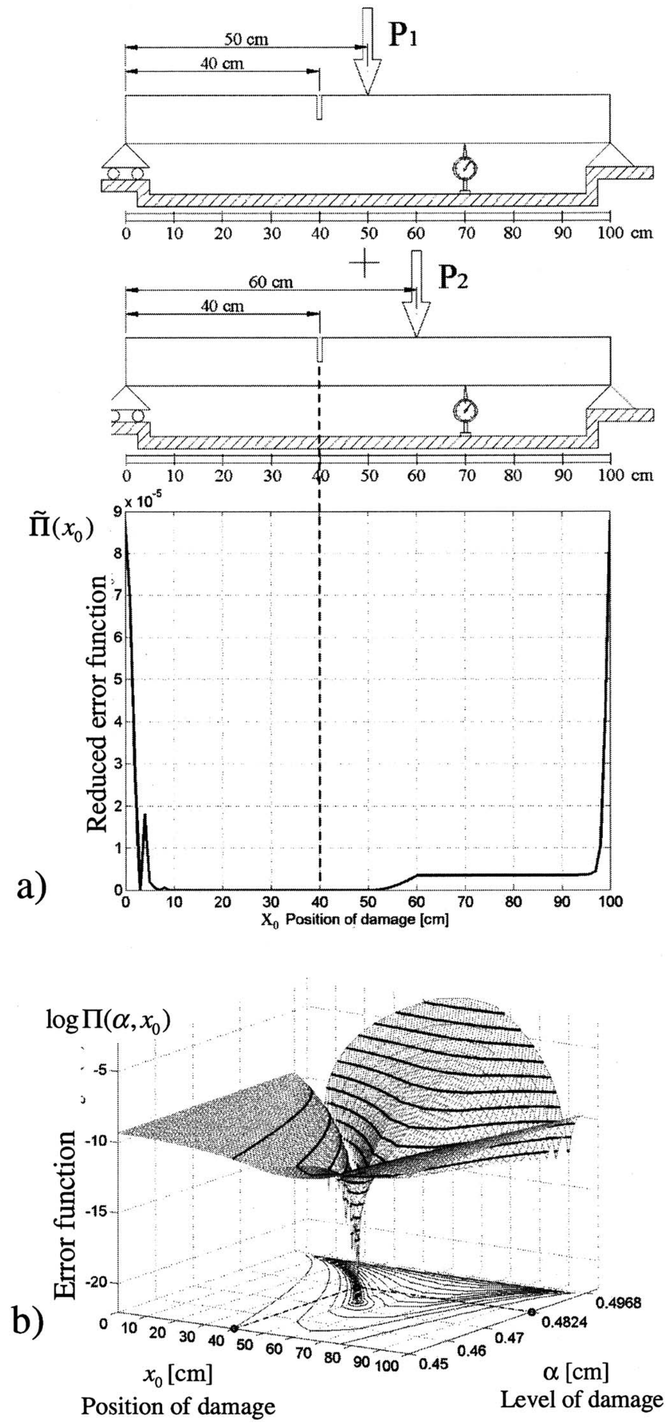

Fig. 10. Pinned-pinned beam with a single damage, one deflection measurement, and two concentrated load conditions on one side of crack: (a) reduced error function $\tilde{\Pi}(x_0)$; (b) error function $\log \Pi(\alpha, x_0)$ with its contour lines

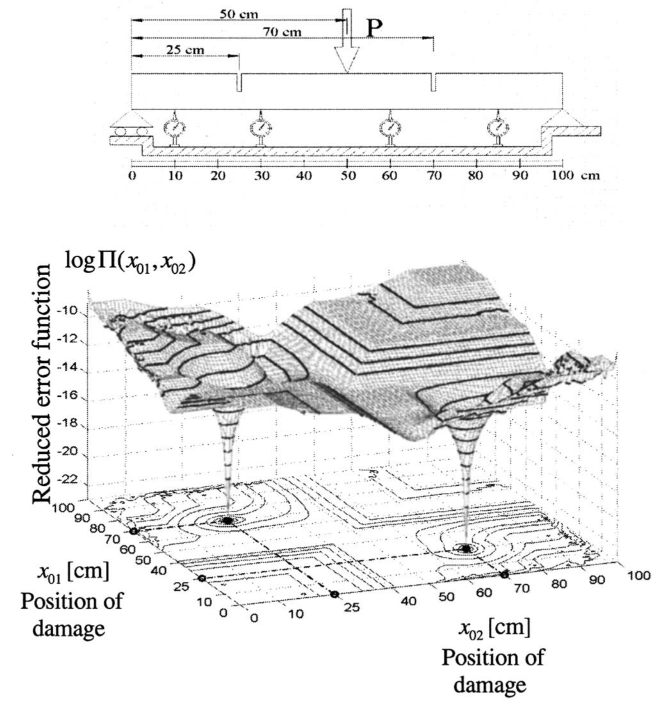

Fig. 11. Pinned-pinned beam with double damage and two couples of deflection measurements each bridging a crack: reduced error function $\log \tilde{\Pi}(x_{01}, x_{02})$ with its contour lines

centrated at $x_{01}=40$ cm and $x_{02}=70$ cm and the measurements are chosen at $x_1=10$, $x_2=30$, $x_3=60$, $x_4=85$ cm (such that $x_1 < x_2 < x_{01} < x_3 < x_{02} < x_4$), the results shown in Fig. 12, for this case, do not allow a unique identification of the damage parameters.

The nature of the damage identification problem implies that the number of cracks is not known a priori; hence, in what follows, the performance of the identification procedure with the double damage model when only one crack appears is explored.

For the beam presented in the previous section a single crack at $x_0=70$ cm and an intensity parameter $\alpha=0.4824$ cm; to be identified, is considered under pinned-pinned boundary conditions. The error function $\Pi(\alpha_i, x_{0i})(i=1,2)$ is obtained by means of a double damage model on the basis of a single concentrated load condition $P=2$ kN at $x_P=50$ cm and four deflection measurements at $x_1=10$, $x_2=30$, $x_3=60$, and $x_4=85$ cm. The reduced error function $\tilde{\Pi}(x_{01}, x_{02})$ obtained by means of minimization with respect to α_1, α_2 is plotted in Fig. 13.

Inspection of Fig. 13 reveals that the reduced error function does not show a unique minimum; on the contrary, the zero value is reached over an entire valley. However, it has to be noted, first, that all the minima belong to the straight line $x_{01}=70$ cm (and the symmetric line $x_{02}=70$ cm), indicating that one damage is surely located at 70 cm (which is the exact solution). Furthermore, for each couple x_{01}, x_{02}, where the minimum is attained, the associated intensity parameters α_1 and α_2, although both different from the exact solution $\alpha=0.4824$ cm, verify the following relationship

$$\frac{\alpha_1}{1-\alpha_1 A} + \frac{\alpha_2}{1-\alpha_2 A} = \frac{\alpha}{1-\alpha A} \tag{23}$$

which indicates that by adding the identified stiffnesses $\alpha_1/(1-\alpha_1 A)$ and $\alpha_2/(1-\alpha_2 A)$ the stiffness $\alpha/(1-\alpha A)$ equivalent to the exact damage is recovered.

It can be concluded that the damage is unique, is located at $x_0=70$ cm, and the exact identified intensity parameter $\alpha=0.4824$ cm can be obtained by solving Eq. (23) with respect to α, as follows

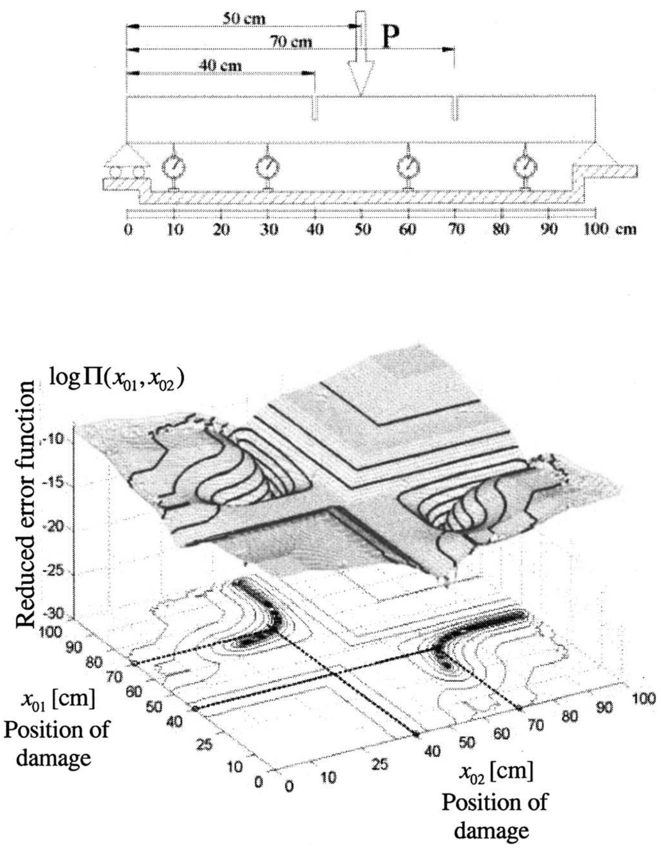

Fig. 12. Pinned-pinned beam with double damage and only one deflection measurements between the two cracks: reduced error function $\log \tilde{\Pi}(x_{01}, x_{02})$ with its contour lines

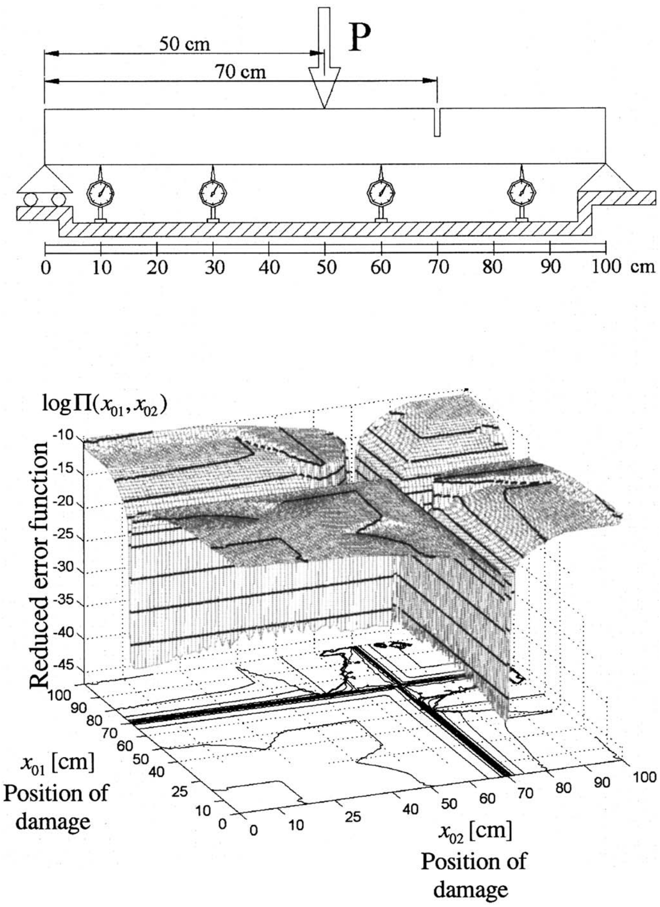

Fig. 13. Pinned-pinned beam with a single damage and four deflection measurements: reduced error function $\log \tilde{\Pi}(x_{01}, x_{02})$ evaluated by means of the model with two cracks

$$\alpha = \left(\frac{\alpha_1}{1 - \alpha_1 A} + \frac{\alpha_2}{1 - \alpha_2 A} \right) \Big/ \left(1 + \frac{\alpha_1}{1 - \alpha_1 A} + \frac{\alpha_2}{1 - \alpha_2 A} \right) \quad (24)$$

where α_1 and α_2 take the values associated to those couples x_{01} and x_{02}, where the reduced error function attains the minimum.

Finally, in order to further test the proposed identification procedure when the number of cracks is not a priori known, the case of beams in presence of two cracks while the identification model is limited to a single crack is also discussed. It is obviously expected that the identification procedure does not identify the exact solution since the single crack model cannot reproduce the actual damage configuration.

Two actual damages with intensities $\alpha_1 = \alpha_2 = 0.4824$ cm, concentrated at $x_{01} = 40$ and $x_{02} = 60$ cm, are supposed in the previously considered pinned-pinned beam. A damage model with a single crack is adopted; hence, two deflection measurements and a single load condition fulfil the necessary condition [Eq. (22)] to identify a single crack.

Since it has been shown that a single crack must lie between the two measurements to have a unique solution, displacements are measured at the left and at the right of the actual cracks ($x_1 = 30$ cm $< x_{01} < x_{02} < x_2 = 70$ cm). The result of the identification procedure is reported in Fig. 14 where the reduced error function shows a unique minimum at $x_0 = 50$ cm, which is an intermediate position between the two damages; furthermore, the error function, also showing a unique minimum, indicates an intensity parameter $\alpha = 0.4877$ cm greater than the value of the actual intensities $\alpha_1 = \alpha_2 = 0.4824$ cm.

Sensitivity to Experimental Noise

In the previous section the capability of the proposed identification procedure of providing exact solutions on the basis of exact measurements has been tested. However, if experimental measurements are given by real tests they are expected to be affected by noise that is the source of error influencing the results of the identification procedure. Hence in this section the sensitivity of the proposed identification procedure to the variability of the experimental data is explored. The noise will be considered as a random variable, and the identified damage parameters, provided by the proposed identification procedure, have to be considered as random variables too. For this study experimental deflection measurements $u^E(x)$ are simulated as follows:

$$u^E(x_{m,l}) = u(\alpha_i, x_{0i}, x_{m,l})[1 + R_{m,l}]$$

$$(i = 1, \ldots, n; m = 1, \ldots, nm; l = 1, \ldots, nlc) \quad (25)$$

where $u(\alpha_i, x_{0i}, x_{m,l})$=exact deflection values at $x_{m,l}$ abscissae, provided by the solution in Eq. (20) of the adopted damaged beam model, with α_i and x_{0i} being the actual damage parameters; $R_{m,l}$=uniformly distributed random variables independent of each other with zero mean and given amplitude range. For each sample

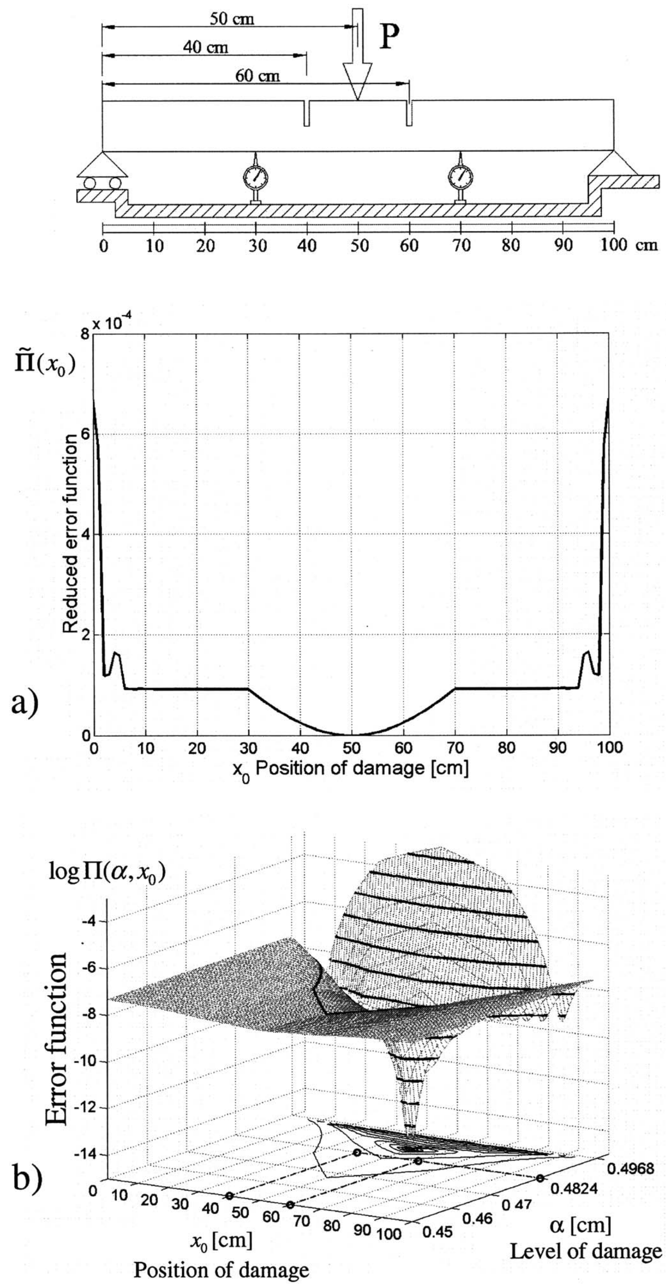

Fig. 14. Pinned-pinned beam with double damage and two deflection measurements: reduced error function $\log \tilde{\Pi}(x_{01}, x_{02})$ evaluated by means of the model with a single crack

of generated random variables $R_{m,l}^{(k)}$, a sample $u^{E(k)}(x_{m,l})$ of experimental data will be obtained by means of Eq. (25) and the identification procedure will provide a sample of perturbed identified damage parameters $\alpha_i^{(k)}$, $x_{0i}^{(k)}$. Generating ns samples and performing ns identification procedures represent the well known Monte Carlo simulation. Sensitivity of the solution of the identification procedure to experimental measurement noise is studied by means of the normalized average mean error (AME) and the normalized average standard deviation (ASD) defined as follows (Banan et al. 1994b)

$$AME = \frac{1}{2n}\sum_{i=1}^{n}\left\{\frac{|E[\alpha_i^{(k)}] - \alpha_i|}{\alpha_i} + \frac{|E[x_{0i}^{(k)}] - x_{0i}|}{x_{0i}}\right\} \qquad (26a)$$

$$ASD = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\sigma_{\alpha_i^{(k)}}}{\alpha_i} + \frac{\sigma_{x_{0i}^{(k)}}}{x_{0i}}\right) \qquad (26b)$$

where $E[\alpha_i^{(k)}]$ and $E[x_{0i}^{(k)}]$ indicate the mean; and $\sigma_{\alpha_i^{(k)}}$ and $\sigma_{x_{0i}^{(k)}}$ = standard deviation of the identified intensity and position damage parameters, respectively. It has to be noted that the error parameters adopted in Eqs. (26) are normalized with respect to the actual damage parameters, intensity and position (α_i and x_{0i}); hence, absolute errors do not count equally for different damage parameters. However, error parameters adopted to evaluate the influence of experimental noise have been chosen as in Eq. (26) in order to provide results consistent with those presented by Banan et al. (1994b) and Di Paola and Bilello (2004). Furthermore, normalized errors expressed in percentage provide an error measure to be compared with the experimental noise amplitude introduced in the experimental deflection measurements. However, in case absolute errors on damage parameters are needed, denominators appearing in Eq. (26) should be left out.

The beam with single damage described previously under the pinned-pinned boundary condition and reported in Fig. 2(a) is here considered by assuming that experimental deflection measurements are taken at $x_1 = 30$ cm and $x_2 = 70$ cm and simulated according to Eq. (25). Two levels of proportional noise are analyzed by considering the amplitude ranges ±5 and ±10% for the random variables $R_{m,l}$. The proposed identification procedure has been performed by reproducing experimental tests for eight different load conditions ($nlc = 8$). The load conditions have been obtained by means of a concentrated load $P = 2$ kN at eight different positions x_{Pi}, and denoted as load conditions number $k = 1$ through to $k = 8$ as reported in Table 1.

For each single load condition ($k = 1, \ldots, 8$), with noise amplitude 5%, Monte Carlo simulation of the damage identification procedure for an increasing number of samples up to $ns = 2{,}000$ has been performed, and results in terms of AME and ASD are reported in Figs. 15(a and b). Inspection of Figs. 15(a and b) reveals that, for each single load condition, the normalized average mean error AME and standard deviation ASD of the identification procedure solution tend to establish to fixed values as the number of samples increases, as also indicated by Banan et al. (1994a) and called "bias error." However, the above-mentioned fixed values of AME and ASD reached for an increasing number of samples, and denoted $\overline{AME}$ and $\overline{ASD}$ in what follows, depend on the position of the concentrated load. Inspection of Figs. 15(a and b) reveals that, in order to recognize the correct noise influence on the identified parameters and compare the sensitivity to the noise for different load conditions, a number of samples greater than 1,500 should be considered. As a consequence, for the limit quantities $\overline{AME}$ and $\overline{ASD}$, the values obtained by Monte Carlo simulation for 2,000 samples have been assumed and plotted in Figs. 16(a and b).

Table 1. Concentrated Load Condition Numeration with respect to Position x_p.

Concentrated load condition number k	3	5	7	Crack	8	4	6	1	2
Position of the concentrated load x_P [cm]	10	35	38	40	42	45	50	60	80

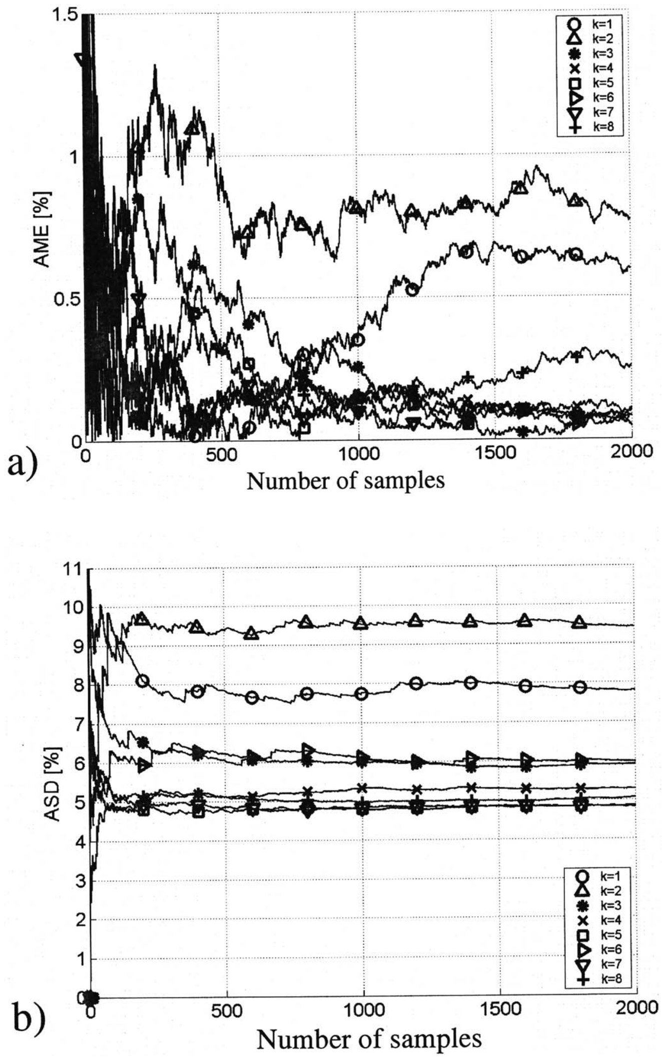

Fig. 15. (a) Average mean error; (b) average standard deviation for eight different loading conditions $k=1,\ldots,8$, listed in Table 1, against the number of samples in the Monte Carlo simulation for noise amplitude 5%

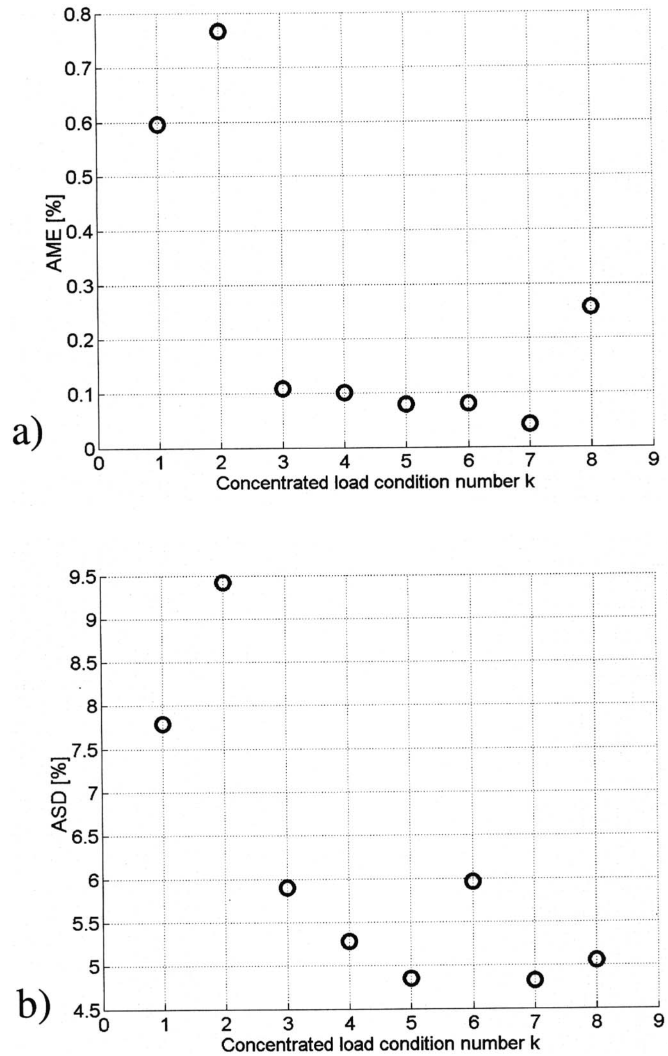

Fig. 16. (a) Average; (b) average standard deviation reached, at 2000 samples, by the eight loading conditions $k=1,\ldots,8$, listed in Table 1, with noise amplitude 5%

Analysis of Figs. 16(a and b) shows that load conditions denoted as $k=1$ and $k=2$ (corresponding to concentrated loads at $x_{p_1}=60$ cm and $x_{p_2}=80$ cm, hence distant from the crack) lead to mean error $\overline{AME}$ and dispersion of data $\overline{ASD}$ higher than those produced by other concentrated loads. In any case, for any load condition, it can be noted that the average mean error $\overline{AME}$ of the identified damage parameters is always ($<0.8\%$) less than the assigned level (5%) of the experimental noise. Hence, the proposed identification procedure tends to reduce the error affecting the input data. Furthermore, the proposed identification procedure based on the minimization problem in Eq. (21) allows to take into account all the displacement data obtained by static tests for different load conditions; in fact, summation in Eq. (21) is extended up to *nlc* load conditions. It is expected that the more load conditions that are considered in the optimization problem, the better is the reduction of the noise effect on the identified parameters. In particular, the proposed identification procedure has been performed by introducing in Eq. (21) displacement data for an increasing number of load conditions according to a prescribed sequence given by k, in Table 1, increasing from 1 to 8. The results of the Monte Carlo simulation in terms of $\overline{AME}$ and $\overline{ASD}$ for an increasing number of load conditions in the summation in Eq. (21) are reported in Figs. 17(a and b) for noise levels 5% and 10%. The results reported in Figs. 17(a and b) show that adding load conditions, according to the prescribed load sequence, provides a substantial decrement of the noise effect both in terms of $\overline{AME}$ and $\overline{ASD}$.

Conclusions

In this work the problem of the concentrated damage identification in Euler-Bernoulli beams has been addressed. A linear behavior of the damaged beam has been considered and no restrictions concerning the damage intensity has been introduced. First the direct analysis problem under the influence of static loads has been solved in closed form by modeling concentrated damages as Dirac's delta distributions in the flexural stiffness. On the basis of the proposed direct analysis solution the inverse identification problem has been tackled by means of an optimization procedure. The proposed model led to an explicit nonquadratic function to be minimized and formulated in explicit form in terms of position and intensity parameters to be identified, measuring the error be-

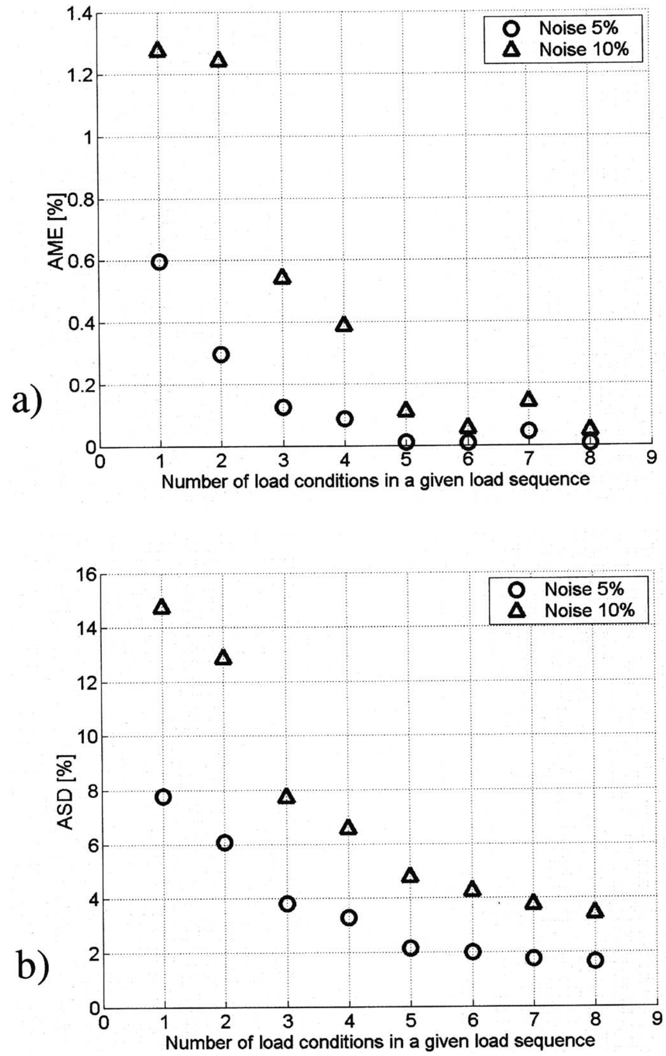

Fig. 17. (a) Average mean error; (b) average standard deviation for load sequence 1-2-3-4-5-6-7-8

tween the analytical model response and experimental data.

The presented approach allows an easy solution of the damage identification problem. It has to be remarked that the explicit error function to be minimized hold for both statically determinate and indeterminate beams without any additional computational effort. The difference between the two cases lies in the expression of the constants dependent on the boundary conditions only.

A few cases regarding beams with different boundary conditions with single and double concentrated damages have been solved. In particular, different positions of deflection measurements and concentrated loads have been analyzed and some sufficient conditions, dependent on the boundary conditions, for the uniqueness of the solution have been "a posteriori" recognized.

Analysis of the results provided useful suggestions concerning the position of the deflection measurers to be adopted in real experimental tests. The presented procedure provides also encouraging results in those cases in which the number of concentrated damages is not known "a priori."

Finally, the performance of the proposed procedure, where the experimental data are affected by instrumental noise, has been analyzed. The noise has been modeled by means of a random variable and the average mean error and average standard deviation of the identified parameters have been studied. It has been shown that, under different load conditions, the instrumental noise affects, to a different extent, the mean error and the dispersion of the identified parameters. Increasing the number of load conditions in the optimization problem provides the average mean error and standard deviation of the identified parameters approaching the zero value.

Acknowledgments

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Appendix. Definition of the Product of Two Dirac's Deltas

In the classical theory of distributions, although the product of distributions is well defined, the proposed definitions cannot be extended to the product of two Dirac's deltas centered at the same point.

In the literature some theories address the problem of definition of new classes of multiplication of distributions to be applied to two or more Dirac's deltas centered at the same point. Usually, theories attempting a definition of the product of distributions rely on: (1) regularization of the distributions in order to obtain continuous functions able to return to the original distributions by means of a limiting procedure; (2) multiplication, in the sense of distributions, of the regularized distributions; (3) definition of the product of two or more distributions by means of a limiting procedure applied to the multiplication of the regularized distributions as defined in step 2. The theory proposed by Bremermann and Durand (1961) is based on a regularization of distributions by means of the so called analytic continuation of a distribution. The Colombeau's theory (Colombeau 1984) follows a different approach to define a regularized version of a distribution called *the sequential completion*. The latter makes use of the so-called δ-sequences, and the regularized distribution is defined as the convolution of the original distribution with the δ-sequences. It has to be remarked that the previously mentioned theories for regularized distributions do not allow the definition of the product of Dirac's deltas.

In this appendix a different approach, proposed by Bagarello (1995, 2002), which makes use of both previously mentioned definitions of regularized distributions in order to introduce a new multiplication for distributions is reported. The multiplication introduced by Bagarello applies only to distributions for which both analytic continuation, dependent on an ε parameter, and convolution with δ-sequences, dependent on a γ parameter, exist, and it has been proved to apply to Dirac's delta and its derivatives. In particular, according to Bagarello (1995), a regularized distribution δ_{red} of a Dirac's delta is considered first by means of an analytic continuation as follows

$$\delta_{red}\left(x,\frac{1}{n^{\varepsilon}}\right)=\frac{1}{\pi n^{\varepsilon}}\frac{1}{x^2+\dfrac{1}{n^{2\varepsilon}}} \qquad (27)$$

and then another regularized distribution $\delta_n^{(\gamma)}$ is considered by means of the following δ-sequence

$$\delta_n^{(\gamma)}(x) = n^{\gamma}\Phi(n^{\gamma}x) \tag{28}$$

for any fixed n and where $\Phi(x)$ is a suitable chosen function with support $[-1,1]$ and such that $\int_{-1}^{1}\Phi(x)dx=1$.

According to the multiplication for distributions proposed by Bagarello (1995) the product of two Dirac's deltas, making use of the regularized distributions reported in Eqs. (27) and (28), depending on the choice of the parameters ε and γ, is defined as follows

$$[\delta(x)\delta(x)]_{\varepsilon,\gamma}[\Psi(x)] = \lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n^{(\gamma)}(x)\delta_{red}\left(x,\frac{1}{n^{\varepsilon}}\right)\Psi(x)dx \tag{29}$$

for any test function $\Psi(x)$.

The limit of the sequence defined in Eq. (29) exists if we require the function $\Phi(x)$ appearing in Eq. (28) to be of the form

$$\Phi(x) = \begin{cases} \dfrac{x^m}{F}\exp\left\{\dfrac{1}{x^2-1}\right\} & |x| < 1 \\ 0 & |x| \geqslant 1 \end{cases} \tag{30}$$

where m=natural number; and F=normalization constant and the fulfillment of the inequality

$$\varepsilon - 2\gamma \geqslant 0 \tag{31}$$

The limit of the sequence in Eq. (29) under the conditions provided by Eqs. (30) and (31) defines the product of two Dirac's deltas as follows (Bagarello 1995, 2002):

$$[\delta(x)\delta(x)]_{\varepsilon,\gamma}(\Psi(x)) = \begin{cases} A_j\delta(x)[\Psi(x)] & \varepsilon = 2\gamma \\ 0 & \varepsilon > 2\gamma \end{cases} \tag{32}$$

where

$$A_j = \frac{1}{\pi}\int_{-1}^{1}\frac{\Phi(x)}{x^j}dx \tag{33}$$

In this paper we adopt the first option provided by Eq. (32) as the product of two Dirac's deltas, which returns the properties of a single Dirac's delta if $\varepsilon=2\gamma$ is assumed. Furthermore, in order to guarantee the existence of the integral in Eq. (33), it is assumed $j=2$ and $m=2$ appearing in Eq. (30), such that

$$A_2 = \frac{1}{\pi}\int_{-1}^{1}\frac{\Phi(x)}{x^2}dx = 2.013 \tag{34}$$

According to Eq. (32) for $\varepsilon=2\gamma$, the product of two Dirac's deltas both centered at x_0 is a single Dirac's delta and will be adopted throughout the paper by means of the following formal expression

$$\delta(x-x_0)\delta(x-x_0) = A\delta(x-x_0) \tag{35}$$

where the application of the Dirac's delta to any test function is implicitly assumed and where the constant $A=A_2=2.013$ defined by Eq. (34) is adopted.

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