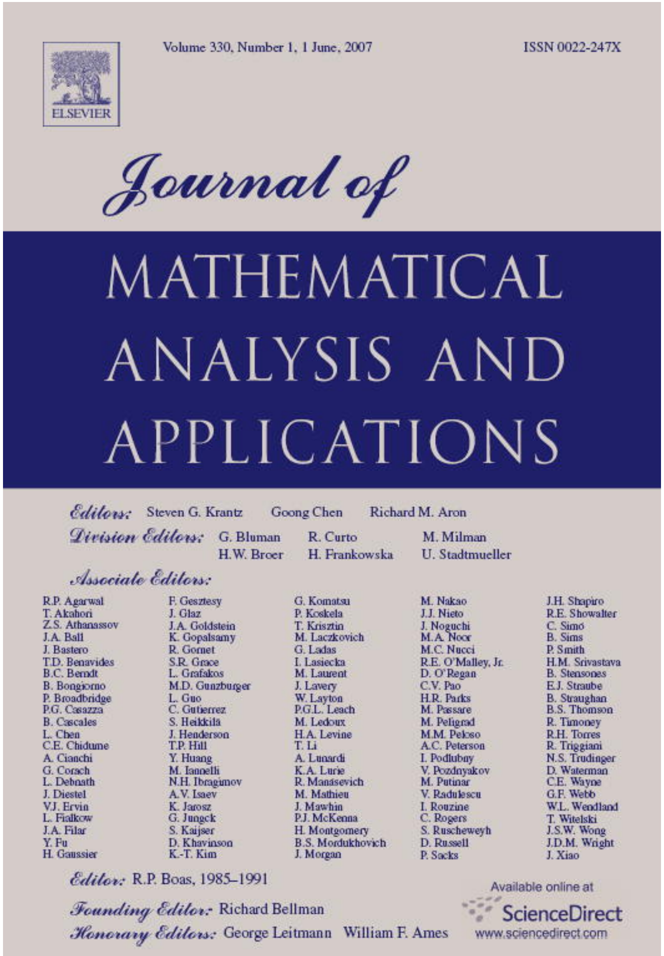
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A multiplicity result for hemivariational inequalities

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Abstract

In this paper we extend a multiplicity result of Ricceri to locally Lipschitz functionals and prove the existence of multiple solutions for a class of hemivariational inequalities.

Keywords: Locally Lipschitz functions; Principle of symmetric criticality; Hemivariational inequalities

1. Introduction

The present paper yields some multiplicity results for a class of hemivariational inequalities on unbounded domains. This problem is studied via variational methods: Clarke's theory of differentiation for locally Lipschitz functionals on Banach spaces is employed, in conjunction with some results in the theory of best approximation in Banach spaces.

A link between best approximation and the (classical) critical point theory was established in the recent works of Tsar'kov [19] and Ricceri [18]. In the latter it is proved that, given a continuously Gâteaux differentiable functional J defined over a real Hilbert space X, for each

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real σ within the range of J and $x_0 \in J^{-1}(]-\infty, \sigma[)$, either there exists $\lambda > 0$ such that the energy functional

$$x \to \frac{\|x - x_0\|^2}{2} - \lambda J(x)$$

admits at least three critical points, or the set $J^{-1}([\sigma, +\infty[)$ has a unique point minimizing the distance from x_0. The alternative is then resolved, under the very general assumption that J admits a non-convex superlevel set: in this case, an application of the results of [19] allows to conclude that the energy functional above has at least three critical points for suitable $x_0 \in X$, $\lambda > 0$. This abstract result has a natural application in the field of differential equations: already in [18], a class of boundary value problems for semilinear equations with smooth nonlinearities on bounded domains was studied, achieving the existence of at least three solutions.

Further developments of Ricceri's result deal with more general classes of problems: in [10], Kristály examines a Schrödinger equation on $\mathbb{R}^N$; in [6], Faraci and Iannizzotto study, through an extension of the abstract result from Hilbert spaces to a wider class of Banach spaces, boundary value problems involving the p-Laplacian on bounded domains. This paper represents a new step in the progress resumed above: the multiplicity result (first in the alternative form) is proved for a class of locally Lipschitz functionals, defined on Banach spaces, and applied to hemivariational inequalities on unbounded domains.

We quickly introduce the class of problems we will be dealing with (see Section 3 below for more details): let Ω be an unbounded domain in $\mathbb{R}^N$, X be a Banach space (compactly embedded in some space $L^r(\Omega)$), A the duality mapping on X induced by the weight function $t \to t^{p-1}$ $(1 < p < N)$, $F : \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function with generalized directional derivative F°, $b : \Omega \to \mathbb{R}$ a non-negative function. In Section 4 we shall prove that, under suitable hypotheses, there exist $u_0 \in X$, $\lambda > 0$ such that the inequality

$$(P_\lambda) \quad \langle A(u - u_0), v \rangle + \lambda \int_\Omega b(x) F^\circ\big(u(x); -v(x)\big)\, dx \geqslant 0 \quad \text{for all } v \in X$$

admits at least three solutions in X.

Hemivariational inequalities were first introduced by Panagiotopoulos in [15] (see also [16]), as a generalization of variational inequalities to the case of non-convex potentials: the theory of generalized directional derivatives for locally Lipschitz functions developed by Clarke was employed, instead of that of subdifferentials for convex functions (in Section 2 we recall some basic definitions and results about locally Lipschitz functions).

The study of hemivariational inequalities on unbounded domains began with the work of Gazzola and Rădulescu [7], followed by the papers of Kristály [11,12], Dályai and Varga [4], and Varga [20]. The main problem arising in this field is the lack of compact embeddings of Sobolev spaces: in general, indeed, if Ω is unbounded, $W^{1,p}(\Omega)$ is *not* compactly embedded in any $L^r(\Omega)$. In Section 5, we show how such a difficulty can be overcome by two different devices: the use of weighted Sobolev spaces, and the application of symmetry groups and of the non-smooth form of Palais' principle of symmetric criticality (see the works of Krawcewicz and Marzantowicz [9], Kobayashi and Otani [8]).

We conclude this section by observing that elliptic equations with discontinuous nonlinearities can be studied with the methods of hemivariational inequalities: for instance, as a consequence of our results, we will prove the existence of a continuous function $g : \mathbb{R}^N \to \mathbb{R}$ and a positive λ such that the equation

$$(E_\lambda) \quad -\Delta u + V(x)u = \lambda b(x) H(u - 1)(\ln u - 1) + g(x) \quad \text{in } \mathbb{R}^N$$

(where V is a positive and coercive potential and H is the Heaviside function) admits at least three solutions in $H^2(\mathbb{R}^N)$ (see Section 5.1 for more details). We point out that the same type of equations, in the case of continuous nonlinearities, was studied by Kristály in [10].

2. Preliminaries

In the sequel, X will denote a (real) Banach space (with norm $\|\cdot\|$) and $X^\star$ its topological dual (with norm $\|\cdot\|_\star$); by $\langle\cdot,\cdot\rangle$ we will denote the duality pairing between $X^\star$ and X.

The next lemma introduces the *duality mapping* on the space X, induced by the weight function $t \to t^{p-1}$:

Lemma 1. [2, Propositions 2.2.2, 2.2.4] *Let X be a Banach space with strictly convex dual, $p > 1$ a real number. Then, there exists a mapping $A : X \to X^\star$ such that for all $x \in X$,*

(DM_1) $\|A(x)\|_\star = \|x\|^{p-1}$;
(DM_2) $\langle A(x), x\rangle = \|A(x)\|_\star \|x\|$.

Moreover, for all $x, y \in X$,

$$\bigl\langle A(x) - A(y), x - y\bigr\rangle \geqslant \bigl(\|x\|^{p-1} - \|y\|^{p-1}\bigr)\bigl(\|x\| - \|y\|\bigr).$$

The functional $x \to \frac{\|x\|^p}{p}$ is Gâteaux differentiable with derivative A.

Now, for the reader's convenience, we give the basic notions from the theory of generalized differentiation for locally Lipschitz functions and non-smooth analysis, as exposed by Motreanu and Panagiotopoulos in [13].

Definition 1. A function $h : X \to \mathbb{R}$ is *locally Lipschitz* if, for every $x \in X$, there exist a neighborhood U of x and a constant $L > 0$ such that

$$\bigl|h(y) - h(z)\bigr| \leqslant L\|y - z\| \quad \text{for all } y, z \in U.$$

Although it is not necessarily differentiable in the classical sense, a locally Lipschitz function admits a derivative, defined as follows:

Definition 2. The *generalized directional derivative* of h at the point $x \in X$ in the direction $y \in X$ is

$$h^\circ(x; y) = \limsup_{z\to x,\ \tau\to 0^+} \frac{h(z + \tau y) - h(z)}{\tau}.$$

The *generalized gradient* of h at $x \in X$ is the set

$$\partial h(x) = \bigl\{x^\star \in X^\star\colon \bigl\langle x^\star, y\bigr\rangle \leqslant h^\circ(x; y) \text{ for all } y \in X\bigr\}.$$

For all $x \in X$, the functional $h^\circ(x, \cdot)$ is subadditive and positively homogeneous: thus, due to the Hahn–Banach theorem, the set $\partial h(x)$ is nonempty. The next lemma resumes the main properties of the generalized derivatives, which will be useful in the sequel.

Lemma 2. *Let $h, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then,*

(h_1) $h^\circ(x; y) = \max\{\langle \xi, y\rangle\colon\ \xi \in \partial h(x)\}$;
(h_2) $(h + g)^\circ(x; y) \leqslant h^\circ(x; y) + g^\circ(x; y)$;
(h_3) $(-h)^\circ(x; y) = h^\circ(x; -y)$.

This notion extends both that of Gâteaux derivative, and that of directional derivative for convex functionals. In particular:

Lemma 3. *Let $h : X \to \mathbb{R}$ be a convex, continuous, Gâteaux differentiable function. Then, h is locally Lipschitz and*

$$\langle h'(x), y\rangle = h^\circ(x; y) \quad \textit{for all } x, y \in X.$$

The next definition generalizes the notion of critical point to the non-smooth context:

Definition 3. A point $x \in X$ is a *critical point* of h, if $0 \in \partial h(x)$, that is,

$$h^\circ(x; y) \geqslant 0 \quad \text{for all } y \in X.$$

Remark 1. Note that every local extremum of h is a critical point of h in the sense above.

A very important tool for our proofs shall be the non-smooth version of the Pucci–Serrin mountain pass theorem. Before stating it, we give a fundamental definition, equivalent to that of [13]:

Definition 4. The function h satisfies the *Palais–Smale condition*, if every sequence $\{x_n\}$ in X such that

(PS_1) $\{h(x_n)\}$ bounded;
(PS_2) there exists a sequence $\{\varepsilon_n\}$ in $]0, +\infty[$ with $\varepsilon_n \to 0$ such that $h^\circ(x_n; y - x_n) + \varepsilon_n \|y - x_n\| \geqslant 0$ for all $y \in X$, $n \in \mathbb{N}$.

admits a convergent subsequence.

The mountain pass theorem, which can be obtained as a consequence of Corollary 3.2 of [14], reads as follows:

Theorem 1. *Let $h : X \to \mathbb{R}$ be a locally Lipschitz function, satisfying the Palais–Smale condition, x and y two local minima of h. Then, h has a critical point in X different from x and y.*

We conclude this section by recalling two results which will be useful in our proofs. The first is a topological minimax theorem:

Theorem 2. [17, Theorem 1 and Remark 1] *Let X be a topological space, Λ a real interval, and $f : X \times \Lambda \to \mathbb{R}$ a function satisfying the following conditions*:

(A_1) *for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous*;

(A_2) *for every* $\lambda \in \Lambda$, *the function* $f(\cdot, \lambda)$ *is lower semicontinuous and each of its local minima is a global minimum*;

(A_3) *there exist* $\rho_0 > \sup_\Lambda \inf_X f$ *and* $\lambda_0 \in \Lambda$ *such that* $\{x \in X\colon f(x, \lambda_0) \leqslant \rho_0\}$ *is compact.*

Then,

$$\sup_\Lambda \inf_X f = \inf_X \sup_\Lambda f.$$

The second is a result from the theory of best approximation which, roughly speaking, assures that, in a certain class of Banach spaces, every sequentially weakly closed Chebyshev set is convex:

Theorem 3. ([19, Theorem 2]; [5, Lemma 1]) *Let X be a uniformly convex Banach space, with strictly convex topological dual, M a sequentially weakly closed, non-convex subset of X. Then, for any convex, dense subset S of X, there exists $x_0 \in S$ such that the set*

$$\bigl\{y \in M\colon \|y - x_0\| = d(x_0, M)\bigr\}$$

contains at least two points.

3. The problem

Now we can give our problem a more precise formulation: let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be an unbounded domain with smooth boundary $\partial\Omega$ (or $\Omega = \mathbb{R}^N$), $p \in]1, N[$ be a real number. Throughout the sequel X denotes a separable, uniformly convex Banach space with strictly convex topological dual; moreover, we assume that

(E) there exists $r \in [p, p^\star[$ such that X is compactly embedded in $L^r(\Omega)$

where $p^\star = \frac{Np}{N-p}$ is the Sobolev critical exponent, and we denote by $\|\cdot\|_r$ the norm in $L^r(\Omega)$ and by c_r the embedding constant. Condition (E) is equivalent to the assumption that X is a linear subspace of $L^r(\Omega)$, endowed with a norm $\|\cdot\|$ such that the identity is a compact operator from $(X, \|\cdot\|)$ into $(L^r(\Omega), \|\cdot\|_r)$.

Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that $F(0) = 0$ and suppose that

(F) there exist $k > 0$, $q \in]0, p-1[$ such that $|\xi| \leqslant k|s|^q$ for all $s \in \mathbb{R}$, $\xi \in \partial F(s)$.

Let $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function such that

(b) $b \in L^1(\Omega) \cap L^\infty(\Omega)$.

Note that in the proofs of our results we will use the inclusion

$$L^1(\Omega) \cap L^\infty(\Omega) \subset L^\nu(\Omega),$$

where $\nu = \frac{r}{r-(q+1)}$, which derives from the inequality $\|u\|_\nu \leqslant \|u\|_\infty^{\frac{\nu-1}{\nu}} \|u\|_1^{\frac{1}{\nu}}$.

Our approach to problem (P_λ) is variational: indeed, we will seek for solutions of (P_λ) as critical points (in the sense of Definition 3) of a suitable energy functional defined on X. Let us define the functional $J : X \to \mathbb{R}$ by putting

$$J(u) = \int_\Omega b(x) F\big(u(x)\big)\, dx$$

for all $u \in X$. The next lemma summarizes the properties of J:

Lemma 4. *The functional J is well defined, locally Lipschitz, sequentially weakly continuous and satisfies*

$$J^\circ(u; v) \leqslant \int_\Omega b(x) F^\circ\big(u(x); v(x)\big)\, dx \quad \text{for all } u, v \in X.$$

Proof. We begin by giving an estimate of the integral which defines J: from the Lebourg mean value theorem [13, Theorem 1.1] it follows that for all $s \in \mathbb{R}$ there exist $t \in \mathbb{R}$, with $0 < |t| < |s|$, and $\xi \in \partial F(t)$ such that $F(s) = \xi s$, so, by (F),

$$\big|F(s)\big| \leqslant k|s|^{q+1}. \tag{1}$$

Thus, for all $u \in X$ we get, by applying Hölder's inequality, that

$$\left|\int_\Omega b(x) F\big(u(x)\big)\, dx\right| \leqslant k \int_\Omega b(x)\big|u(x)\big|^{q+1} dx \leqslant k\|b\|_\nu \|u\|_r^{q+1} \leqslant K\|u\|^{q+1},$$

where $K = c_r^{q+1} k\|b\|_\nu$. Hence, J is well defined.

We prove that J is Lipschitz on bounded sets: let us choose $M > 0$ and $u, v \in X$ with $\|u\|, \|v\| < M$. By using the Lebourg mean value theorem and by condition (F) we get for all $x \in \Omega$,

$$\begin{aligned}\big|F\big(u(x)\big) - F\big(v(x)\big)\big| &\leqslant k \max\big\{\big|u(x)\big|^q, \big|v(x)\big|^q\big\}\big|u(x) - v(x)\big| \\ &\leqslant k\big(\big|u(x)\big| + \big|v(x)\big|\big)^q \big|u(x) - v(x)\big|.\end{aligned}$$

Then, by Hölder's inequality,

$$\begin{aligned}\big|J(u) - J(v)\big| &\leqslant k\|b\|_\nu \left(\int_\Omega \big(\big|u(x)\big| + \big|v(x)\big|\big)^r dx\right)^{\frac{q}{r}} \|u - v\|_r \\ &\leqslant k\|b\|_\nu \big(\|u\|_r + \|v\|_r\big)^q \|u - v\|_r \\ &\leqslant 2^q K M^q \|u - v\|.\end{aligned}$$

We prove now that J is sequentially weakly continuous: let $\{u_n\}$ be a sequence in X, weakly convergent to some $\bar{u} \in X$. It follows by (E) that $\|u_n - \bar{u}\|_r \to 0$. Then, by the above inequalities for J we get $J(u_n) \to J(\bar{u})$.

Finally, the inequality in the thesis follows from Proposition 3.3 of [4], so the proof is concluded. □

Given $u_0 \in X$ and $\lambda > 0$, the energy functional $I : X \to \mathbb{R}$ related to the problem (P_λ) is defined by

$$I(u) = \frac{\|u - u_0\|^p}{p} - \lambda J(u).$$

We observe that, by Lemma 1, the convex functional $u \to \frac{\|u-u_0\|^p}{p}$ is Gâteaux differentiable, so by Lemma 3 it is locally Lipschitz with derivative A: hence, I is locally Lipschitz too. The close relationship between I and (P_λ) is expressed by the following lemma:

Lemma 5. *Let $u \in X$ be a critical point of I. Then, u is a solution of (P_λ).*

Proof. By Lemmas 1, 2 and 4 we get

$$\begin{aligned} I^\circ(u; v) &\leqslant \langle A(u - u_0), v\rangle + \lambda(-J)^\circ(u; v) \\ &= \langle A(u - u_0), v\rangle + \lambda J^\circ(u; -v) \\ &\leqslant \langle A(u - u_0), v\rangle + \lambda \int_\Omega b(x) F^\circ\big(u(x); -v(x)\big)\, dx, \end{aligned}$$

so the conclusion follows. □

4. The main result

Following [18], we first prove our main result in the form of an alternative:

Theorem 4. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$ or $\Omega = \mathbb{R}^N$ $(N \geqslant 2)$, $p \in \,]1, N[$ be a real number, X be a separable, uniformly convex Banach space with strictly convex topological dual, satisfying (E). Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function satisfying $F(0) = 0$ and (F), $b : \Omega \to \mathbb{R}$ be a non-negative, non-zero function satisfying (b).*

Then, for every $\sigma \in \,]\inf_X J, \sup_X J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$ one of the following conditions is true:

(B_1) *there exists $\lambda > 0$ such that the problem (P_λ) has at least three solutions in X;*
(B_2) *there exists $v \in J^{-1}(\sigma)$ such that, for all $u \in J^{-1}([\sigma, +\infty[)$, $u \neq v$,*

$$\|u - u_0\| > \|v - u_0\|.$$

Proof. Fix σ and u_0 as in the thesis, and assume that (B_1) does not hold: we shall prove that (B_2) is true.

Putting $\Lambda = [0, +\infty[$ and endowing X with the weak topology, we define the function $f : X \times \Lambda \to \mathbb{R}$ by

$$f(u, \lambda) = \frac{\|u - u_0\|^p}{p} + \lambda\big(\sigma - J(u)\big),$$

which satisfies all the hypotheses of Theorem 2.

The condition (A_1) is trivial.

The condition (A_3) becomes trivial too, once we know that $\sup_{\lambda\in\Lambda}\inf_{u\in X} f(u,\lambda) < +\infty$: we achieve this inequality by observing that there exists some $u_1 \in X$ such that $J(u_1) > \sigma$, so

$$\sup_{\lambda\in\Lambda}\inf_{u\in X} f(u,\lambda) \leqslant \sup_{\lambda\in\Lambda} f(u_1,\lambda) = \frac{\|u_1-u_0\|^p}{p}.$$

In examining condition (A_2), let $\lambda \geqslant 0$ be fixed: we first observe that, by Lemma 4, the functional $f(\cdot,\lambda)$ is sequentially weakly lower semicontinuous (l.s.c.). Moreover, $f(\cdot,\lambda)$ is coercive: indeed, for all $u \in X$ we have

$$f(u,\lambda) \geqslant \|u\|^p\left(\frac{\|u-u_0\|^p}{p\,\|u\|^p} - \lambda k c_r^{q+1}\|b\|_\nu \|u\|^{(q+1)-p}\right) + \lambda\sigma,$$

and the latter goes to $+\infty$ as $\|u\| \to +\infty$, since $\|u\|^{(q+1)-p} \to 0$ and $\frac{\|u-u_0\|^p}{p\|u\|^p} \to \frac{1}{p}$. As a consequence of the Eberlein–Smulyan theorem, the outcome is that $f(\cdot,\lambda)$ is weakly l.s.c.

We need to check that every local minimum of $f(\cdot,\lambda)$ is a global minimum. Arguing by contradiction, suppose that $f(\cdot,\lambda)$ admits a local, non global minimum; besides, being coercive, it has a global minimum too, that is, it has two strong local minima.

We now prove that $f(\cdot,\lambda)$ fulfills the Palais–Smale condition: let $\{u_n\}$ be a sequence satisfying (PS_1), (PS_2). From (PS_1), together with the coercivity of $f(\cdot,\lambda)$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote $\{u_n\}$, weakly convergent to a point $\bar u \in X$. By condition (E) we can choose $\{u_n\}$ to be convergent to $\bar u$ with respect to the norm of $L^r(\Omega)$. Fix $\varepsilon > 0$. As the sequence $\{\varepsilon_n\}$ from (PS_2) tends to 0, for $n \in \mathbb{N}$ big enough we have

$$\varepsilon_n\|u_n - \bar u\| < \frac{\varepsilon}{2},$$

so, from (PS_2) and Lemma 4 it follows

$$\begin{aligned}0 &\leqslant f^\circ(u_n,\lambda;\bar u - u_n) + \frac{\varepsilon}{2}\\ &\leqslant \big\langle A(u_n-u_0), \bar u - u_n\big\rangle + \lambda\int_\Omega b(x)F^\circ\big(u_n(x); u_n(x)-\bar u(x)\big)\,dx + \frac{\varepsilon}{2},\end{aligned}$$

($f^\circ(\cdot,\lambda;\cdot)$ denotes the generalized directional derivative of the locally Lipschitz functional $f(\cdot,\lambda)$).

We use (h_1) from Lemma 2 and condition (F), together with Hölder's inequality, to get

$$\begin{aligned}\left|\int_\Omega b(x)F^\circ\big(u_n(x); u_n(x)-\bar u(x)\big)\,dx\right| &\leqslant \int_\Omega b(x)\max_{\xi_n(x)\in\partial F(u_n(x))}\big|\xi_n(x)\big(u_n(x)-\bar u(x)\big)\big|\,dx\\ &\leqslant k\int_\Omega b(x)\big|u_n(x)\big|^q\big|u_n(x)-\bar u(x)\big|\,dx\\ &\leqslant k c_r^q\|b\|_\nu\|u_n\|^q\|u_n-\bar u\|_r < \frac{\varepsilon}{2\lambda}.\end{aligned}$$

Hence

$$\big\langle A(u_n-u_0), u_n-\bar u\big\rangle < \varepsilon$$

for $n \in \mathbb{N}$ large enough. On the other hand, $\langle A(\bar u - u_0), u_n - \bar u\rangle$ tends to zero as n goes to infinity. From the above computations, it follows that

$$\limsup_n \big\langle A(u_n-u_0) - A(\bar u - u_0), u_n - \bar u\big\rangle \leqslant 0. \tag{2}$$

Applying Lemma 1, we obtain that

$$\begin{aligned}&\big\langle A(u_n-u_0)-A(\bar u-u_0),u_n-\bar u\big\rangle\\&\quad\geqslant\big(\|u_n-u_0\|^{p-1}-\|\bar u-u_0\|^{p-1}\big)\big(\|u_n-u_0\|-\|\bar u-u_0\|\big)\geqslant 0.\end{aligned}$$

From this inequality and (2), we deduce that $\|u_n-u_0\|\to\|\bar u-u_0\|$, which, together with the weak convergence, implies that $\{u_n\}$ tends to $\bar u$ in X: that is, the Palais–Smale condition is fulfilled.

We can apply Theorem 1, deducing that $f(\cdot,\lambda)$ (or equivalently the energy functional I) admits a third critical point: then, by Lemma 5, the inequality (P_λ) should have at least three solutions in X, against our assumption. Thus, condition (A_2) is fulfilled.

Now Theorem 2 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X}f(u,\lambda)=\inf_{u\in X}\sup_{\lambda\in\Lambda}f(u,\lambda)=:\alpha. \tag{3}$$

Notice that the function $\lambda\to\inf_{u\in X}f(u,\lambda)$ is upper semicontinuous in Λ, and tends to $-\infty$ as $\lambda\to+\infty$ (since $\sigma<\sup_X J$): hence, it attains its supremum in some $\lambda^\star\in\Lambda$, that is,

$$\alpha=\inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p}+\lambda^\star\big(\sigma-J(u)\big)\right). \tag{4}$$

Let us determine the infimum in the right-hand side of (3): since for any $u\in J^{-1}(]-\infty,\sigma[)$ we have $\sup_{\lambda\in\Lambda}f(u,\lambda)=+\infty$, clearly

$$\alpha=\inf_{u\in J^{-1}([\sigma,+\infty[)}f(u,\lambda)=\inf_{u\in J^{-1}([\sigma,+\infty[)}\frac{\|u-u_0\|^p}{p};$$

moreover, since the functional $u\to\frac{\|u-u_0\|^p}{p}$ is coercive and sequentially weakly l.s.c. while the set $J^{-1}([\sigma,+\infty[)$ is sequentially weakly closed, there exists $v\in J^{-1}([\sigma,+\infty[)$ such that

$$\alpha=\frac{\|v-u_0\|^p}{p}.$$

It is easily seen that $v\in J^{-1}(\sigma)$. Hence

$$\alpha=\inf_{u\in J^{-1}(\sigma)}\frac{\|u-u_0\|^p}{p}\quad\text{(in particular }\alpha>0). \tag{5}$$

By (4) and (5) it follows that

$$\inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p}-\lambda^\star J(u)\right)=\inf_{u\in J^{-1}(\sigma)}\left(\frac{\|u-u_0\|^p}{p}-\lambda^\star J(u)\right). \tag{6}$$

We deduce that $\lambda^\star>0$: if $\lambda^\star=0$, indeed, (6) would become $\alpha=0$, against (5).

Now we can prove (B_2): namely, we prove that v defined above is the only point of $J^{-1}([\sigma,+\infty[)$ minimizing the distance from u_0. Arguing by contradiction, let $w\in J^{-1}([\sigma,+\infty[)\setminus\{v\}$ be such that $\|w-u_0\|=\|v-u_0\|$. As above, we have that $w\in J^{-1}(\sigma)$, and so both w and v are global minima of the functional I (for $\lambda=\lambda^\star$) over $J^{-1}(\sigma)$, hence, by (6), w and v are global minima of I over X. Thus, applying Theorem 1, we obtain that I has at least three critical points, against the assumption that (B_1) does not hold (recall that $\lambda^\star$ is positive). This concludes the proof. □

In the next corollary, the alternative of Theorem 4 is resolved, under a very general assumption on the functional J, so we are led to a multiplicity result for the hemivariational inequality (P_λ) (for suitable data u_0, λ):

Corollary 1. *Let Ω, p, X, F, b be as in Theorem* 4 *and let S be a convex, dense subset of X. Moreover, let $J^{-1}([\sigma, +\infty[)$ be not convex for some $\sigma \in]\inf_X J, \sup_X J[$.*

Then, there exist $u_0 \in J^{-1}(]-\infty, \sigma[) \cap S$ and $\lambda > 0$ such that problem (P_λ) admits at least three solutions in X.

Proof. Since J is sequentially weakly continuous (Lemma 4), the set $M = J^{-1}([\sigma, +\infty[)$ is sequentially weakly closed. Since M is not convex, by Theorem 3 we get that, for some $u_0 \in S$, there exist two points $v_1, v_2 \in M$, with $v_1 \neq v_2$, satisfying

$$\|v_1 - u_0\| = \|v_2 - u_0\| = \operatorname{dist}(u_0, M).$$

Clearly $u_0 \notin M$, that is, $J(u_0) < \sigma$. In the framework of Theorem 4, condition (B_2) is false, so (B_1) must be true: there exists $\lambda > 0$ such that (P_λ) has at least three solutions in X. □

5. Applications

In order to give some applications of the results of the previous section, we need to express their hypotheses in a slightly different form, directly related to the data of (P_λ).

We can obtain the non-convexity of a convenient superlevel set of the functional J (the main hypothesis of Corollary 1) by assuming the very general condition that F is not a quasi-concave function, that is:

(C) there exists $\rho \in]\inf_{\mathbb{R}} F, \sup_{\mathbb{R}} F[$ such that $F^{-1}([\rho, +\infty[)$ is not convex.

Concerning condition (E), the lack of compact embedding of $W_0^{1,p}(\Omega)$ into any $L^r(\Omega)$ for unbounded Ω suggests us to restrict the study to special cases: namely, we will examine a modified Sobolev space with a weighted norm and a subspace of $W_0^{1,p}(\Omega)$ characterized by a symmetry property with respect to a group of isometries (in the latter case we will make use of the principle of symmetric criticality).

5.1. Weighted Sobolev spaces

Let Ω be $\mathbb{R}^N$ or an unbounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ with C^1 bounded boundary, $p \in]1, N[$, $V : \Omega \to \mathbb{R}$ be a continuous potential satisfying the following conditions:

(V_1) $\inf_\Omega V > 0$;
(V_2) for every $M > 0$ the set $\{x \in \Omega\colon\ V(x) \leqslant M\}$ has finite Lebesgue measure

(note that in particular, condition (V_2) is fulfilled whenever V is coercive). We introduce the space

$$X = \left\{ u \in W^{1,p}(\Omega)\colon \int_\Omega \left(\left|\nabla u(x)\right|^p + V(x)\left|u(x)\right|^p \right) dx < +\infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_\Omega \left(\left|\nabla u(x)\right|^p + V(x)\left|u(x)\right|^p \right) dx \right)^{\frac{1}{p}}.$$

With the definitions above, for all $u_0 \in X$, $\lambda > 0$, our problem (P_λ) reads as follows:

$$\begin{aligned}
&\int_\Omega \Big(\left|\nabla\big(u(x) - u_0(x)\big)\right|^{p-2} \nabla\big(u(x) - u_0(x)\big) \cdot \nabla v(x) \\
&\quad + V(x)\left|u(x) - u_0(x)\right|^{p-2}\big(u(x) - u_0(x)\big)v(x)\Big)\, dx \\
&\quad + \lambda \int_\Omega b(x) F^\circ\big(u(x); -v(x)\big)\, dx \geqslant 0 \quad \text{for all } v \in X.
\end{aligned}$$

We can deduce the following multiplicity result:

Corollary 2. *Let Ω, p, V, X be as above*; *F, b be as in Theorem* 4; *S be a convex, dense subset of X. Moreover, assume that condition (C) is satisfied. Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions in X.*

Proof. We observe that X is a separable, uniformly convex Banach space with strictly convex topological dual, and that $C_c^\infty(\Omega) \subset X$; moreover, the conditions (V_1), (V_2) guarantee that the space X is compactly embedded in $L^p(\Omega)$ (see [1] for the case $p = 2$), so condition (E) is satisfied with $r = p$. Since b is not zero, there exist a point $x_0 \in \Omega$ and $R > 0$ such that

$$b_1 = \int_B b(x)\, dx > 0,$$

where B is the open ball centered in x_0 with radius R, contained in Ω.

By condition (C), we can assume, without loss of generality, that there exist real numbers $s_1 < s_2 < s_3$, with $s_1 \neq 0$, such that $F(s_1), F(s_3) > \rho$, $F(s_2) < \rho$. Now we prove that the functional J admits a non-convex superlevel set. Choose $\varepsilon > 0$, $R_1 > R$ with

$$\|b\|_\infty M \operatorname{meas}(A) < \varepsilon < b_1\left|F(s_i) - \rho\right| \quad (i = 1, 2, 3),$$

where $A = \{x \in \Omega\colon\ R < |x - x_0| < R_1\}$ and $M = \max\{|F(t)|\colon\ |t| \leqslant |s_i|,\ i = 1, 2, 3\}$. There exists $u_1 \in C_c^\infty(\Omega)$ such that

$$u_1(x) = \begin{cases} s_1 & \text{if } x \in B, \\ 0 & \text{if } x \in \Omega \setminus (A \cup B) \end{cases}$$

and $\|u_1\|_\infty = |s_1|$; define, also, $u_2, u_3 \in C_c^\infty(\Omega)$ by putting $u_2 = \frac{s_2}{s_1}u_1$, $u_3 = \frac{s_3}{s_1}u_1$. Thus,

$$\begin{aligned}
J(u_1) &= \int_B b(x)F(s_1)\, dx + \int_A b(x)F\big(u_1(x)\big)\, dx \\
&\geqslant b_1 F(s_1) - M\|b\|_\infty \operatorname{meas}(A) \\
&\geqslant b_1 F(s_1) - \varepsilon \\
&> b_1 \rho.
\end{aligned}$$

By analogous computations, we get

$$J(u_2) < b_1\rho < J(u_3).$$

Then, since u_2 lies on the segment joining u_1 and u_3, it is proved that $J^{-1}([b_1\rho, +\infty[)$ is not convex. An application of Corollary 1 yields the existence of a function $u_0 \in J^{-1}(]-\infty, b_1\rho[) \cap S$ and $\lambda > 0$ such that (P_λ) has at least three solutions in X. □

We go back to the equation (E_λ) considered in the Introduction, and clarify what stated there:

Example 1. Let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous, positive and coercive function, X be as above with $p = 2 < N$, b be as in Theorem 4. Recall that the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(s) = \begin{cases} 0 & \text{if } s \leqslant 0, \\ 1 & \text{if } s > 0, \end{cases}$$

and put

$$f(s) = H(s-1)(\ln s - 1) \quad \text{for all } s \in \mathbb{R}$$

(with obvious meaning for $s \leqslant 1$). We denote, for all $s \in \mathbb{R}$,

$$f_-(s) = \lim_{\delta \to 0^+} \inf_{|t-s|<\delta} f(t), \qquad f_+(s) = \lim_{\delta \to 0^+} \sup_{|t-s|<\delta} f(t).$$

Following Chang [3], for all continuous $g : \mathbb{R}^N \to \mathbb{R}$ and $\lambda > 0$, by a weak solution of (E_λ) we mean a function $u \in H^2(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$,

$$-\Delta u(x) + V(x)u(x) \in g(x) + \lambda b(x)\big[f_-\big(u(x)\big), f_+\big(u(x)\big)\big]. \tag{7}$$

It is easily seen that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(t)\,dt$$

is locally Lipschitz and satisfies the condition (F) with arbitrary $q \in]0, 1[$ for k big enough; moreover, for all $\rho \in]2-e, 0]$ the set $F^{-1}([\rho, +\infty[)$ is not convex, so condition (C) is fulfilled. Taking $S = C_c^\infty(\mathbb{R}^N)$, we can apply Corollary 2: thus, we find $u_0 \in S$ and $\lambda > 0$ such that the hemivariational inequality

$$\int_{\mathbb{R}^N} \big(\nabla\big(u(x) - u_0(x)\big) \cdot \nabla v(x) + V(x)\big(u(x) - u_0(x)\big)v(x)\big)\,dx$$
$$+ \lambda \int_{\mathbb{R}^N} b(x) F^\circ\big(u(x); -v(x)\big)\,dx \geqslant 0 \quad \text{for all } v \in X$$

admits at least three solutions in X. Let u be one of these: by standard regularity results, we get $u \in H_0^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$; arguing as in [3], we find that u satisfies (7) with

$$g(x) = -\Delta u_0(x) + V(x)u_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

Thus, (E_λ) has at least three weak solutions.

5.2. Symmetry groups

Let Ω be an unbounded domain in $\mathbb{R}^N$ $(N > 2)$ with smooth boundary, such that $0 \in \Omega$, and let G be a closed subgroup of $O(N)$ which leaves Ω invariant, i.e. $g(\Omega) = \Omega$ for all $g \in G$. We assume that Ω is *compatible* with G, that is, there exists $r > 0$ such that

$$m(x, r, G) \to \infty \quad \text{as } \operatorname{dist}(x, \Omega) \leqslant r,\ |x| \to \infty,$$

where

$$m(x, r, G) = \sup\bigl\{n \in \mathbb{N}\colon \exists g_1, g_2, \ldots, g_n \in G \text{ s.t. } B(g_i x, r) \cap B(g_j x, r) = \emptyset \text{ if } i \neq j\bigr\}.$$

We consider the space $X = W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \biggl(\int_\Omega \bigl(\bigl|\nabla u(x)\bigr|^p + \bigl|u(x)\bigr|^p\bigr)\,dx\biggr)^{\frac{1}{p}}.$$

In the present setting, our problem (P_λ), for all $u_0 \in X$, $\lambda > 0$, reads as follows:

$$\begin{aligned}
&\int_\Omega \bigl(\bigl|\nabla\bigl(u(x) - u_0(x)\bigr)\bigr|^{p-2}\nabla\bigl(u(x) - u_0(x)\bigr) \cdot \nabla v(x) \\
&\quad + \bigl|u(x) - u_0(x)\bigr|^{p-2}\bigl(u(x) - u_0(x)\bigr)v(x)\bigr)\,dx \\
&\quad + \lambda \int_\Omega b(x) F^\circ\bigl(u(x); -v(x)\bigr)\,dx \geqslant 0 \quad \text{for all } v \in X.
\end{aligned}$$

We define the action of the group G over the space X by putting

$$gu(x) = u\bigl(g^{-1}x\bigr) \quad \text{for all } g \in G,\ u \in X,\ x \in \Omega.$$

We observe that G acts *linearly* and *isometrically* on X, i.e., the action $G \times X \to X$ which maps (g, u) into gu is continuous and, for every $g \in G$, the map $u \to gu$ is linear and $\|gu\| = \|u\|$ for every $u \in X$. The group G induces an action of the same type on the dual space $X^\star$ defined by $\langle gu^\star, u\rangle = \langle u^\star, g^{-1}u\rangle$ for every $g \in G$, $u \in X$ and $u^\star \in X^\star$.

We introduce the set

$$X^G = \bigl\{u \in X\colon\ gu = u \text{ for all } g \in G\bigr\}$$

of the fixed points of X under the action of G, and observe that X^G is a Banach space (which inherits all the properties of X), whose dual coincides with the fixed point set of $X^\star$ under the action of G, denoted $(X^G)^\star$. By reducing ourselves to considering symmetric functions, as said before, we easily overcome the problem of verifying condition (E):

Lemma 6. [8, Proposition 4.2] *X^G is compactly embedded in $L^r(\Omega)$ for all $r \in\,]p, p^\star[$.*

Now we define the class of functionals we will be dealing with:

Definition 5. A functional $h : X \to \mathbb{R}$ is *G-invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$.

An important property of invariant functionals is expressed by the principle of symmetric criticality. This principle, proved by Palais for Gâteaux differentiable functionals and then extended to locally Lipschitz functionals (see [9]), for our purposes, can be stated as follows:

Theorem 5. *Let X be a Banach space, let G be a compact topological group acting linearly and isometrically on X, and $h : X \to \mathbb{R}$ be a locally Lipschitz, G-invariant functional. Then, every critical point of $h|_{X^G}$ is also a critical point of h.*

From our general result we deduce an analogous of Corollary 2, which assures the existence of at least three symmetric solutions:

Corollary 3. *Let Ω, p, X, G be as above, S be a convex, dense subset of X^G. Let F be as in Theorem 4 and satisfying condition (C). Also, let $b : \Omega \to \mathbb{R}$ be a non-negative, G-invariant function (that is, $b(gx) = b(x)$ for all $g \in G$, $x \in \Omega$) satisfying condition (b) and such that*

$$\int_B b(x)\,dx > 0 \quad \bigl(B = B(0, R) \text{ for some } R > 0 \text{ small enough}\bigr).$$

Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions lying in X^G.

Proof. We are going to apply Corollary 1 to the space X^G and to the functional $J|_{X^G}$: first, we note that X^G is separable and uniformly convex, and that $(X^G)^\star$ is strictly convex (as a subspace of $X^\star$); moreover, by Lemma 6, the space X^G satisfies condition (E) for any $r \in \,]p, p^\star[$.

To see that $J|_{X^G}$ admits a non-convex superlevel set, we argue as in the proof of Corollary 2, putting $x_0 = 0$ and choosing the functions $u_1, u_2, u_3 \in C_c^\infty(\Omega)$ radially symmetric (so, in particular, lying in X^G).

Thus, by Corollary 1, there exist $u_0 \in S$ and $\lambda > 0$ such that the energy functional $I|_{X^G}$ has at least three critical points in X^G.

Now we prove that I is G-invariant on X. Let $g \in G$ and $u \in X$; recalling that $u_0 \in X^G$, G acts isometrically over X and b is G-invariant, we obtain the following equalities:

$$\begin{aligned}
I(gu) &= \frac{1}{p}\|gu - u_0\|^p - \int_\Omega b(x) F\bigl(gu(x)\bigr)\,dx \\
&= \frac{1}{p}\bigl\|g(u - u_0)\bigr\|^p - \int_\Omega b(x) F\bigl(u\bigl(g^{-1}x\bigr)\bigr)\,dx \\
&= \frac{1}{p}\|u - u_0\|^p - \int_\Omega b(y) F\bigl(u(y)\bigr)\,dy \\
&= I(u).
\end{aligned}$$

Then, applying Theorem 5, we deduce that the critical points of $I|_{X^G}$ are actually critical points of I. We can conclude that problem (P_λ) has at least three symmetric solutions. □

Next we give an example, in order to highlight the generality of our hypotheses:

Example 2. Put $N = 3$ and define the unbounded domain

$$\Omega = \bigl\{(x_1, x_2, x_3) \in \mathbb{R}^3:\ |x_3| < x_1^2 + x_2^2 + 1\bigr\}.$$

Then, consider the closed subgroup of $O(3)$ defined by $G = O(2) \times \{\mathrm{id}\}$, whose action on $X = W_0^{1,p}(\Omega)$ $(1 < p < N)$ is expressed as follows: for all $g = (\tilde{g}, \mathrm{id}) \in G$, and for all $u \in X$, $(x_1, x_2, x_3) \in \Omega$ we set

$$gu(x_1, x_2, x_3) = u\big(\tilde{g}^{-1}(x_1, x_2), x_3\big).$$

It is easily seen that Ω is G-invariant and compatible with G, and that the subspace X^G of the fixed points of X under the action of G is the set of all $u \in X$ with a *cylindric symmetry*, that is, satisfying

$$u(x_1, x_2, x_3) = u(y_1, y_2, x_3) \quad \text{whenever } x_1^2 + x_2^2 = y_1^2 + y_1^2.$$

Let $q \in]0, p-1[$ be a real number, $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = 1 - \big||s|^{q+1} - 1\big| \quad \text{for all } s \in \mathbb{R}.$$

It is easily seen that F is a locally Lipschitz function, satisfying $F(0) = 0$ and conditions (F) (with $k = q + 1$) and (C) (for all $\rho \in]0, 1]$).

Moreover, we consider a non-negative function $b : \Omega \to \mathbb{R}$, having a cylindric symmetry and satisfying condition (b), and we assume that b is positive in a neighborhood of 0.

In such a setting, Corollary 3 applies: thus, there exist $u_0 \in X^G$, $\lambda > 0$ such that the hemivariational inequality (P_λ) admits at least three solutions, and each of them has a cylindric symmetry.

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References

[1] T. Bartsch, Z.Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[2] J. Chabrowski, Variational Methods for Potential Operator Equations, de Gruyter, Berlin, 1997.
[3] K.C. Chang, Variational methods for non-differentiable functionals and their applications to partial differential equations, J. Math. Anal. Appl. 80 (1981) 102–129.
[4] Z. Dályai, Cs. Varga, An existence result for hemivariational inequalities, Electron. J. Differential Equations 37 (2004) 1–17.
[5] N.V. Efimov, S.B. Stechkin, Approximate compactness and Chebyshev sets, Soviet Math. Dokl. 2 (1961) 1226–1228.
[6] F. Faraci, A. Iannizzotto, An extension of a multiplicity theorem by Ricceri with an application to a class of quasilinear equations, Studia Math. 172 (2006) 275–287.
[7] F. Gazzola, V. Rădulescu, A non-smooth critical point theory approach to some nonlinear elliptic equations in $\mathbb{R}^N$, Differential Integral Equations 13 (2000) 47–60.
[8] J. Kobayashi, M. Otani, The principle of symmetric criticality for non-differentiable mappings, J. Funct. Anal. 214 (2004) 428–449.
[9] W. Krawcewicz, W. Marzantowicz, Some remarks on the Lusternik–Schnirelman method for non-differentiable functionals invariant with respect to a finite group action, Rocky Mountain J. Math. 20 (1990) 1041–1049.
[10] A. Kristály, Multiple solutions for a sublinear Schrödinger equation, preprint.
[11] A. Kristály, Infinitely many radial and non-radial solutions for a class of hemivariational inequalities, Rocky Mountain J. Math. 35 (2005) 1173–1190.
[12] A. Kristály, Multiplicity results for an eigenvalue problem for hemivariational inequalities in strip-like domains, Set-Valued Anal. 13 (2005) 85–103.
[13] D. Motreanu, P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Nonconvex Optim. Appl., vol. 29, Kluwer Academic Publishers, Dordrecht, 1999.
[14] D. Motreanu, Cs. Varga, A nonsmooth equivariant minimax principle, Comm. Appl. Anal. 3 (1999) 115–130.

[15] P.D. Panagiotopulos, Inequality Problems in Mechanics and Applications – Convex and Nonconvex Energy Functionals, Birkhäuser-Verlag, Basel, 1985.
[16] P.D. Panagiotopulos, Hemivariational Inequalities. Applications in Mechanics and Engineering, Springer-Verlag, Berlin, 1993.
[17] B. Ricceri, A further improvement of a minimax theorem of Borenshtein and Shul'man, J. Nonlinear Convex Anal. 2 (2001) 279–283.
[18] B. Ricceri, A general multiplicity theorem for certain nonlinear equations in Hilbert spaces, Proc. Amer. Math. Soc. 133 (2005) 3255–3261.
[19] I.G. Tsar'kov, Nonunique solvability of certain differential equations and their connection with geometric approximation theory, Math. Notes 75 (2004) 259–271.
[20] Cs. Varga, Existence and infinitely many solutions for an abstract class of hemivariational inequalities, J. Inequal. Appl. 2 (2005) 89–105.