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THE POINCARÉ SERIES OF THE MODULE OF DERIVATIONS OF SOME MONOMIAL RINGS

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Abstract

Let R be a quasi-homogeneous k-algebra and M be a finitely generated graded R-module. The formal power series $\sum_i \dim_k \operatorname{Tor}_i^R(k, M)z^i$ is called the Poincaré series of M and it is denoted by $P_M^R(z)$. We remark that the Poincaré series of the module of derivations of a monomial ring is rational and determine it in some cases.

1. Introduction

For any commutative k-algebra R, the *module of derivations* is the set given by $\operatorname{Der}_k(R) = \{\rho \in \operatorname{Hom}_k(R, R) \mid \rho(ab) = a\rho(b) + \rho(a)b \text{ for all } a, b \in R\}$. This set has a natural R-module structure.

Let R be a quasi-homogeneous k-algebra. For any finitely generated graded R-module M, the *Poincaré series* of M is the formal power series $P_M^R(z) = \sum_i \dim_k \operatorname{Tor}_i^R(k, M)z^i$.

In this paper, our object of study is the Poincaré series of the module of derivations $\operatorname{Der}_k(R)$ of a monomial k-algebra R.

In Section 2, we state a theorem due to Brumatti and Simis that represents the starting point of our paper. We also remark that it follows from a theorem due to Lescot that the Poincaré series of the module of derivations $\operatorname{Der}_k(R)$ is rational for any monomial k-algebra R.

In Section 3, we calculate the Poincaré series of the module of derivations for a large class of Stanley-Reisner rings of dimension one or two.

In Section 4, we determine the Poincaré series of the module of derivations for some further cases of monomial rings.

In Section 5, we give formulas for the Poincaré series of the module of derivations when $R = k[\Delta]$ is the Stanley-Reisner ring of a join $\Delta = \Delta_1 * \Delta_2$ or a disjoint union $\Delta = \Delta_1 \cup \Delta_2$ of simplicial complexes.

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2. Preliminaries

A monomial algebra over a field k is an algebra of the form $R = k[x_1, \dots, x_n]/I$, where I is an ideal generated by monomials. For any monomial k-algebra, $\mathrm{Der}_k(R)$ has a natural Z^n-grading, induced by the Z^n-grading of R. Hence it follows from [11, Theorem 1] that the Poincaré series of $\mathrm{Der}_k(R)$ is rational.

The starting point of our paper is the following theorem due to Brumatti and Simis in [2], Theorem 2.2.1:

THEOREM 2.1. *Let $R = k[x_1, \dots, x_n]/I$ be a monomial k-algebra. If the ideal I is generated by monomials whose exponents are prime to the characteristic of k, then*

$$\mathrm{Der}_k(R) = \bigoplus_{i=1}^{n} (0 : (0 : x_i))\partial_i$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

REMARK 2.2. Since $P^R_{M \oplus N}(z) = P^R_M(z) \oplus P^R_N(z)$ for any finitely generated graded R-module M, N, it is enough to consider Poincaré series of the type $P^R_{0:(0:x_i)}(z)$.

Our aim is to derive explicit formulas for the Poincaré series of the module of derivations over some algebras using this result. We shall repeatedly use the following lemma:

LEMMA 2.3. *Let R be a ring and let J be an ideal in R. Then $P^R_J(z) = (P^R_{R/J}(z) - 1)/z$.*

3. Stanley-Reisner rings of dimension one or two

In this section we consider Stanley-Reisner rings of dimension one or two. In Section 4 we will consider some Stanley-Reisner ring of higher dimension.

A *(finite) simplicial complex* consists of a finite set V of vertices and a collection Δ of subsets of V called *faces* or *simplices* such that:

(i) If $v \in V$, then $\{v\} \in \Delta$.

(ii) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Let Δ be a simplicial complex and $F \in \Delta$, then the *dimensions* of F and Δ are defined by $\dim(F) = |F| - 1$ and $\dim(\Delta) = \sup\{\dim(F) \mid F \in \Delta\}$ respectively. A face of dimension q is sometimes refered to as a q-face.

A face F of Δ is said to be a *facet* if F is not properly contained in any other face of Δ. The *q-skeleton* of a simplicial complex Δ is the simplicial complex Δ^q consisting of all p-faces of Δ with $p \leq q$.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let Δ be a simplicial complex with vertex set $V = \{1, \ldots, n\}$. The *Stanley-Reisner ring* $k[\Delta]$ is defined as the quotient ring S/I_Δ, where

$$I_\Delta = (\{x_{i_1} \cdots x_{i_r} \mid i_1 < \cdots < i_r,\ \{i_1 \cdots i_r\} \notin \Delta\})$$

and I_Δ is called *Stanley-Reisner ideal* of Δ. By [14, Corollary 5.3.11], $\dim k[\Delta] = \dim(\Delta) + 1$.

For a general reference to properties of simplicial complexes and of Stanley-Reisner rings, see [14, Chapter 5].

To calculate Hilbert series of a Stanley-Reisner ring $R = k[\Delta]$, where Δ is a simplicial complex of dimension $n-1$, we often use the formula given in [13, Theorem II.1.4],

$$H_{k[\Delta]}(z) = \sum_{i=-1}^{n-1} f_i z^{i+1}/(1-z)^{i+1}$$

where we write f_i for the number of i-dimensional faces of Δ for $0 \le i \le n-1$, and put $f_{-1} = 1$.

Example 3.1. Let Δ be a graph, that is a simplicial complex of dimension 1, with n vertices and d edges. Then $H_{k[\Delta]}(z) = 1 + \frac{nz}{1-z} + \frac{dz^2}{(1-z)^2}$.

Let $R = k[x_1, \ldots, x_n]/I$ be a monomial k-algebra and let $\mathbf{b} \subseteq R$ be an ideal generated by a subset of $\{x_1, \ldots, x_n\}$. If I is generated by monomials of degree two, then it follows from [4, Proposition 1.2] that $\mathbf{b}$ has a linear free R-resolution. Moreover, a costruction of a linear resolution of $\mathbf{b}$ is given in ([6, Section 3]) in case $\mathbf{b} = (x_1, \ldots, x_n)$.

Our next aim is to relate $P^R_{\mathbf{b}}(z)$ to the Hilbert series $H_{R/\mathbf{b}}(z)$ and $H_R(z)$.

Theorem 3.2. *Let $R = k[x_1, \ldots, x_n]/I$ be a monomial k-algebra and let $\mathbf{b} \subseteq R$ be an ideal generated by a subset of $\{x_1, \ldots, x_n\}$. Then $H_R(z)P^R_{\frac{R}{\mathbf{b}}}(-z) = H_{\frac{R}{\mathbf{b}}}(z)$.*

Proof. By what it is written above, $R/\mathbf{b}$ has a free linear R-resolution

$$\cdots \longrightarrow R^{b_3}[-3] \longrightarrow R^{b_2}[-2] \longrightarrow R^{b_1}[-1] \longrightarrow R^{b_0} = R \longrightarrow R/\mathfrak{b} \longrightarrow 0.$$

Let $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ and $R/\mathbf{b} = k \oplus [R/\mathbf{b}]_1 \oplus [R/\mathbf{b}]_2 \oplus \cdots$, then we have the following graded version of the resolution above

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
0 & \longleftarrow & [R/\mathbf{b}]_3 & \longleftarrow & R_3 & \longleftarrow & R_2^{b_1} & \longleftarrow & R_1^{b_2} & \longleftarrow & k^{b_3} & \longleftarrow & 0 \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & \oplus & & \\
0 & \longleftarrow & [R/\mathbf{b}]_2 & \longleftarrow & R_2 & \longleftarrow & R_1^{b_1} & \longleftarrow & k^{b_2} & \longleftarrow & 0 & & \\
 & & \oplus & & \oplus & & \oplus & & \oplus & & & & \\
0 & \longleftarrow & [R/\mathbf{b}]_1 & \longleftarrow & R_1 & \longleftarrow & k^{b_1} & \longleftarrow & 0 & & & & \\
 & & \oplus & & \oplus & & \oplus & & & & & & \\
0 & \longleftarrow & k & \longleftarrow & k & \longleftarrow & 0 & & & & & &
\end{array}
$$

Hence we get the following exact sequence of vector spaces (with $m > 0$)

$$
\begin{aligned}
0 \longrightarrow k^{b_m} \longrightarrow R_1^{b_{m-1}} \longrightarrow R_2^{b_{m-2}} \longrightarrow& \\
\cdots \longrightarrow R_{m-1}^{b_1} \longrightarrow R_m^{b_0} \longrightarrow [R/\mathbf{b}]_m \longrightarrow 0.&
\end{aligned}
$$

Let $\dim_k R_i = h_i$ and let $\dim_k [R/\mathbf{b}]_i = r_i$ and in particular $h_0 = r_0 = 1$. Then, for every $i \geq 0$, we have $r_i = h_i b_0 - h_{i-1} b_1 + \cdots + (-1)^i h_0 b_i$, hence $(h_0 + h_1 z + h_2 z^2 + \cdots)(b_0 - b_1 z + b_2 z^2 - \cdots) = (r_0 + r_1 z + r_2 z^2 + \cdots)$.

Corollary 3.3. *Let R and $\mathbf{b}$ be as in Theorem 3.2. Then*

$$P_{\mathbf{b}}^{R}(z) = \big(H_{\frac{R}{\mathbf{b}}}(-z)/H_R(-z) - 1\big)/z.$$

Now we give a characterization of Stanley-Reisner ideals generated by monomials of degrees 2. Of course this is always the case if $\dim \Delta = 0$. So we can consider the case $\dim \Delta \geq 1$.

Proposition 3.4. *Let Δ be a simplicial complex with $\dim \Delta \geq 1$. The Stanley-Reisner ideal I_Δ is generated by monomials of degrees two if and only if Δ is the maximal complex supported by its 1-skeleton.*

Proof. The ideal I_Δ is not generated by monomials of degrees two if and only if there is a monomial $x_{i(1)} x_{i(2)} \cdots x_{i(d)} \in I_\Delta$ of degree $d \geq 3$ such that $x_{i(a)} x_{i(b)} \notin I_\Delta$ for $1 \leq a < b \leq d$, or equivalently that $\{i(1), i(2), \ldots, i(d)\} \notin \Delta$ with $d \geq 3$ but $\{i(a), i(b)\} \in \Delta$ for $1 \leq a < b \leq d$. Hence I_Δ is not generated by monomials of degree two if and only if Δ is not the maximal complex supported by its 1-skeleton.

We note that if $\dim \Delta = 1$, then it follows from Proposition 3.4 that I_Δ is generated by monomials of degree two if and only if Δ is triangle free.

THEOREM 3.5. *Let Δ be a simplicial complex with* $\dim(\Delta) \leq 1$ *and let R be the Stanley-Reisner ring of Δ. Then either* $(0 : (0 : x_i)) = R$ *or* $(0 : (0 : x_i))$ *is generated by a subset of* $\{x_1, \dots, x_n\}$ *that may depends on i.*

PROOF. Of course, if $\dim \Delta = 0$, then $(0 : (0 : x_i)) = (x_i)$ for every i. Suppose that $\dim \Delta = 1$. If the theorem is not true for $(0 : (0 : x_i))$, then we can suppose that $x_j, x_k \notin (0 : (0 : x_i))$ and $x_j x_k \in (0 : (0 : x_i))$ with $x_j x_k \neq 0$. As $x_i \in (0 : (0 : x_i))$, x_i, x_j, x_k are distinct. Since $\dim \Delta = 1$, then necessary $x_i x_j x_k = 0$ and therefore $x_j x_k \in (0 : x_i)$. This is impossible since $(x_j x_k)^2 \neq 0$.

REMARK 3.6. We note that the theorem above in general is not true when $\dim \Delta > 1$, even if I_Δ is generated by monomials of degree 2. Indeed let Δ be a 2-dimensional simplicial complex with vertex set $V = \{1, \dots, 5\}$ and facets $\{\{1, 2, 5\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Then $I_\Delta = (x_1x_3, x_1x_4, x_2x_4, x_3x_5)$ and $(0 : (0 : x_1)) = (x_1, x_2x_5)$.

Now we are ready to calculate the Poincaré series of the module of derivations of Stanley-Reisner rings $k[\Delta]$ of dimension one or two (i.e. $\dim \Delta \leq 1$).

Let us start with the case of simplicial complexes Δ of dimension zero (hence $\dim k[\Delta] = 1$) with vertex set $\{1, \dots, n\}$. Then we have that $R = k[\Delta] = k[x_1, \dots, x_n]/(x_i x_j, i \neq j)$ and $(0 : (0 : x_i)) = (x_i)$ for all i. By Corollary 3.3 (that we can use because of Theorem 3.5), we get that

$$P^R_{(x_1)}(z) = \frac{1}{1 - (n-1)z}$$

and by Theorem 2.1 and Remark 2.2, we have that $P^R_{\mathrm{Der}_k(R)}(z) = n P^R_{(x_1)}(z)$.

Let us now consider complexes of dimension one, that is graphs (hence $\dim k[\Delta] = 2$). As above, because of Theorem 3.5, we can use Corollary 3.3 together with Theorem 2.1 in order to give a method to determine the Poincaré series of $\mathrm{Der}_k\, k[\Delta]$ for the Stanley-Reisner ring of some of these 1-dimensional complexes.

Let us start with the case of a star graph.

EXAMPLE 3.7. With the same argument as for the 0-dimensional case we have that the Poincaré series of the module of derivations for a star graph with vertex set $V = \{1, \dots, n\}$ and with center vertex n is

$$P^R_{\mathrm{Der}_k(R)}(z) = \frac{n-(n-2)z}{1-(n-2)z}$$

if $n \geq 3$ and it is equal to 2 if $n = 2$ (as in this case $(0 : (0 : x_i)) = R$ and $\mathrm{Der}_k(R) \simeq R^2$).

Let v be a vertex of a graph Δ. The degree of v, $\deg(v)$, is the number of edges at v. We denote by $N_\Delta(v)$ the set of neighbors of a vertex v.

PROPOSITION 3.8. *Let Δ be a triangle free graph that is not a star graph, with n vertices and d edges. Then*

$$P^R_{\mathrm{Der}_k(R)}(z) = \frac{(r+n)+(r+n-2d)z}{1-(n-2)z+(d+1-n)z^2}$$

where r is the number of vertices of degree 1.

PROOF. For any vertices i and j in Δ, let $l(i, j)$ be the minimal length of a path connecting i and j, and let $l(i, j) = \infty$ if no such path exists. We claim that $(0 : (0 : x_i)) = (x_k \mid k \in I_i)$, where $I_i = \{i\} \cup \{k \mid l(i,k) = 1 \text{ and } \deg(k) = 1\}$. In fact, this claim follows directly from the fact that $(0 : (0 : x_i)) = (x_k \mid l(i,k) \geq 2)$. Now it follows that:

$$P^R_{(0:(0:x_i))}(z) = \frac{1}{z}\left(\frac{H_{R/(0:(0:x_i))}(-z)}{H_R(-z)} - 1\right) = \frac{(r_i+1)+(r_i+1-d_i)z}{1-(n-2)z+(d+1-n)z^2}$$

with $d_i = \deg(i)$. As $\sum_{i=1}^n r_i = r$ and $\sum_{i=1}^n d_i = 2d$, the formula for $P^R_{\mathrm{Der}_k(R)}(z)$ follows from Theorem 2.1.

Let us now consider the cases of a cycle and of a complete bipartite graph.

EXAMPLE 3.9. Let Δ be a cycle with vertex set $V = \{1, \dots, n\}$, $n \geq 3$. If $n = 3$, then $R = k[\Delta] = k[x_1, x_2, x_3]/(x_1x_2x_3)$ is a complete intersection and $P^R_{\mathrm{Der}_k(R)}(z) = 3/(1-z)$ (cf. Subsection 4.1).

If $n \geq 4$, then, by Proposition 3.8

$$P^R_{\mathrm{Der}_k(R)}(z) = \frac{n(1-z)}{1-(n-2)z+z^2}.$$

EXAMPLE 3.10. Let now Δ be a complete bipartite graph $K_{m,n}$ with vertex set $V = \{1, \dots, m+n\}$ and edges $\{i, j\}$ with $i = 1, \dots, m$ and $j = m +$

$1, \ldots, m+n$. Since the cases $k_{n,1}$ and $k_{1,m}$ were treated in the Example 3.7, we can suppose $n, m \geq 2$. Then by Proposition 3.8 we have that

$$P^R_{\mathrm{Der}_k(R)}(z) = \frac{m+n+(m+n-2mn)z}{1-(m+n-2)z+(mn-m-n+1)z^2)}.$$

4. Further cases

In this section we determine the Poincaré series of the module of derivations for some further cases of monomial ring. Before we do it we need one more result.

In [12], Levin introduces the idea of a large homomorphism of graded (or local) rings as a dual notion to small homomorphisms of graded rings introduced in [1]. Namely, if A and B are quasi-homogeneous rings and $f : A \longrightarrow B$ is a graded homomorphism which is surjective, then f is *large* if $f_* : \mathrm{Tor}^A(k,k) \longrightarrow \mathrm{Tor}^B(k,k)$ is surjective.

It follows from [12, Theorem 1.1] that $f : A \longrightarrow B$ is large if and only if $P^A_M(z) = P^A_B(z) P^B_M(z)$ for all finitely generated graded B-module M.

For the rest of the paper we only need that the map $f : R \longrightarrow R/(x_i)$ is large for any monomial ring $R = k[x_1, \ldots, x_n]/I$ and for any $1 \leq i \leq n$. However, we prove a little more.

PROPOSITION 4.1. *Let $R = k[x_1, \ldots, x_n]/I$ be a monomial ring. Then, for all j, $1 \leq j \leq n$, the map $f : R \longrightarrow R/(x_1, \ldots, x_j)$ is large.*

PROOF. Since the composition of large homomorphisms is large, it is enough to prove that the map $f : R \longrightarrow R/(x_1)$ is large. Let us consider the minimal free R-resolution of k

$$0 \longleftarrow k \longleftarrow R \longleftarrow R^{b_1} \longleftarrow \cdots.$$

We may choose this resolution to be multigraded. If we kill everything of degree grater than zero in x_1, we get the minimal free $R/(x_1)$-resolution of k

$$0 \longleftarrow k \longleftarrow R/(x_1) \longleftarrow [R/(x_1)]^{b'_1} \longleftarrow \cdots.$$

Since all vertical maps

$$\begin{array}{ccccccccc}
0 & \longleftarrow & k & \longleftarrow & R & \longleftarrow & R^{b_1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & k & \longleftarrow & R/(x_1) & \longleftarrow & [R/(x_1)]^{b'_1} & \longleftarrow & \cdots
\end{array}$$

are surjective, the homomorphism $f_* : \mathrm{Tor}^R(k,k) \longrightarrow \mathrm{Tor}^{R/(x_1)}(k,k)$ is surjective.

In Subsections 4.1 and 4.2 we consider graded rings R for which $P_k^R(z)$ it is known. So we may calculate $P^R_{(x_i)}(z)$ using Lemma 2.3 and the following formula.

COROLLARY 4.2. *Let R and x_i ($i = 1, \dots, n$) be as in Proposition 4.1, then*

$$P^R_{R/(x_i)}(z) = \frac{P_k^R(z)}{P_k^{R/(x_i)}(z)}.$$

4.1. *The complete intersection case*

In order to determine the Poincaré series of the module of derivations of a complete intersection, we use Corollary 4.2 together with the fact, due to Tate (cf. [14, Theorem 6]; [9, Corollary 3.4.3]), that

$$P_k^R(z) = \frac{(1+z)^n}{(1-z^2)^m}$$

for any graded complete intersection $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$.

Let $R = k[x_1, \dots, x_n]/I$ be a monomial ring that is also a complete intersection. Then R has the form

$$\frac{k[x_1, \dots, x_n]}{\left(x_1^{n_1} \cdots x_{m_1}^{n_{m_1}}, x_{m_1+1}^{n_{m_1+1}} \cdots x_{m_2}^{n_{m_2}}, \dots, x_{m_{r-1}+1}^{n_{m_{r-1}+1}} \cdots x_{m_r}^{n_{m_r}}\right)}$$

with $m_r \le n$.

By Corollary 4.2 and Lemma 2.3, we have that $P^R_{(x_i)}(z) = 1/(1-z)$ for every $i = 1, \dots, n$. Moreover, we see that $(0 : (0 : x_i)) = (x_i)$ for $1 \le i \le m_r$ and $(0 : (0 : x_i)) = R$ for $m_r < i \le n$.

Assume that the exponents $n_1, \dots, n_{m_r}$ are prime to the characteristic of k. Then $\mathrm{Der}_k(R) = (x_1)\partial_1 \oplus \cdots \oplus (x_{m_r})\partial_{m_r} \oplus R\partial_{m_r+1} \oplus \cdots \oplus R\partial_n$ by Theorem 2.1, and we get

$$P^R_{\mathrm{Der}_k(R)}(z) = \frac{n + (m_r - n)z}{1-z}.$$

4.2. *The case of $k[X_1, \dots, X_n]/(X_1, \dots, X_n)^l$*

In [8, p. 748], Golod showed, in particular, that for algebras R of the form $k[X_1, \dots, X_n]/(X_1, \dots, X_n)^l$, the Poincaré series is

$$P_k^R(z) = \frac{(1+z)^n}{1 - \sum_{i=1}^n \binom{i+l-2}{l-1}\binom{n+l-1}{i+l-1} z^{i+1}}.$$

We see that $(0 : x_i) = \mathbf{m}^{l-1}$, (with $\mathbf{m} = (x_1, \ldots, x_n)$), hence $(0 : (0 : x_i)) = \mathbf{m}$.

Assume that the characteristic of k is either 0 or a prime $p > l$. Then $\mathrm{Der}_k(R) = \mathbf{m}\partial_1 \oplus \cdots \oplus \mathbf{m}\partial_n$ by Theorem 2.1. Using Lemma 2.3, we get

$$P^R_{\mathrm{Der}_k(R)}(z) = n\frac{\frac{(1+z)^n}{1-\sum_{i=1}^n \binom{i+l-2}{l-1}\binom{n+l-1}{i+l-1}z^{i+1}} - 1}{z}.$$

4.3. The case of skeletons of a simplex

A simplicial complex Δ with vertex set V and with $|V| = m$ is called *simplex* if $\dim \Delta = m - 1$. In this subsection we determine the Poincaré series of the module of derivations for a Stanley-Reisner ring R of the skeleton of a simplex. We can also think of R as the factor ring of the polynomial ring modulo all squarefree monomials of a certain degree.

Let Δ^q_{n-1} be the q-dimensional skeleton of a $(n-1)$-dimensional simplex Δ_{n-1}. If $q = n-1$, then $R = k[x_1, \ldots, x_n]$, $\mathrm{Der}_k(R) \simeq R^n$ and $P^R_{\mathrm{Der}_k(R)}(z) = n$. Hence let us suppose that $q < n-1$. Then we have that $R = k[\Delta] = k[x_1, \ldots, x_n]/(x_{m_1}x_{m_2}\cdots x_{m_{q+2}} \mid m_1 < m_2 < \cdots < m_{q+2})$.

For $1 \le i \le n$, we easily see that $(0 : (0 : x_i)) = (x_i)$ and, by Theorem 2.1, we get $\mathrm{Der}_k(R) = (x_1)\partial_1 \oplus \ldots \oplus (x_n)\partial_n$. Moreover, by [14, Proposition 5.3.14], we have that R is Cohen-Macaulay. Finally $H_R(z) = \sum_{i=0}^{q+1}\binom{n}{i}z^i/(1-z)^i$.

Assume that k is an infinite field. Since $\dim R = q+1$, we can find a regular sequence $\{a_1, \ldots, a_{q+1}\}$ of linear elements of length $q+1$. Let $R' = R/(a_1, \ldots, a_{q+1})$. Then

$$\begin{aligned} H_{R'}(z) &= (1-z)^{q+1}H_R(z) \\ &= 1 + \binom{n-(q+1)}{1}z + \binom{n-(q+1)+1}{2}z^2 + \cdots + \binom{n-1}{q+1}z^{q+1} \end{aligned}$$

and all graded rings with such a Hilbert series are isomorphic to the ring $\overline{R} = k[y_1, \ldots, y_{n-(q+1)}]/(y_1, \ldots, y_{n-(q+1)})^{q+2}$. Hence $R' \simeq \overline{R}$. Since $\{a_1, \ldots, a_{q+1}\}$ is a regular sequence and using the results in Subsection 4.2, we get

$$P^R_k(z) = (1+z)^{q+1}P^{R'}_k(z) = \frac{(1+z)^n}{1-\sum_{i=1}^{n-(q+1)}\binom{i+q}{q+1}\binom{n}{i+q+1}z^{i+1}}.$$

Finally, using Theorem 2.1, Lemma 2.3 and Corollary 4.2 we can derive a formula for $P^R_{\mathrm{Der}_k(R)}(z)$.

5. Disjoint unions and joins of simplicial complexes

Let $\{\Delta_i\}_{i=1,\dots,r}$ be a family of simplicial complexes with disjoint vertex sets V_i. Then the *disjoint union* is a simplicial complex $\cup\Delta_i$ on the vertex set $\cup V_i$. The *join*, $*\Delta_i$, is the simplicial complex on the vertex set $\cup V_i$ with faces $F_1 \cup \dots \cup F_r$ where $F_i \in \Delta_i$ for $i \le i \le r$.

In this section, we derive formulas for the Poincaré series of the module of derivations $\mathrm{Der}_k(R)$ when R is the Stanley-Reisner ring of a disjoint union or join of r simplicial complexes. We only consider the case $r = 2$, as the general case can be obtained by induction.

PROPOSITION 5.1. *Let Δ_1 and Δ_2 be simplicial complexes and let $\Delta = \Delta_1 * \Delta_2$. Then $P^R_{\mathrm{Der}_k(R)}(z) = P^{R_1}_{\mathrm{Der}_k(R_1)}(z) + P^{R_2}_{\mathrm{Der}_k(R_2)}(z)$ where $R = k[\Delta]$ and $R_i = k[\Delta_i]$.*

PROOF. Let Δ_1 and Δ_2 be simplicial complexes on $V_1 = \{1, \dots, n\}$ and $V_2 = \{\overline{1}, \dots, \overline{m}\}$ respectively. First we note that $R \simeq R_1 \otimes R_2$. Indeed $R_1 = k[x_1, \dots, x_n]/I_{\Delta_1}$ and $R_2 = k[y_{\overline{1}}, \dots, y_{\overline{m}}]/I_{\Delta_2}$. Then we have that $R_1 \otimes R_2 = k[x_1, \dots, x_n, y_{\overline{1}}, \dots, y_{\overline{m}}]/I_\Delta$, where I_Δ is generated by those $x_{i_1} \cdots x_{i_k} \cdot y_{\overline{j_1}} \cdots y_{\overline{j_l}} (i_1 < \dots < i_k, \overline{j_1} < \dots < \overline{j_l})$ for which $\{i_1, \dots, i_k, \overline{j_1}, \dots, \overline{j_l}\} \notin \Delta$, that is for which $\{i_1, \dots, i_k\} \notin \Delta_1$ and $\{\overline{j_1}, \dots, \overline{j_l}\} \notin \Delta_2$. This gives that I_Δ is the sum of the extension of I_{Δ_1} and the extension of I_{Δ_2} to $k[x_1, \dots, x_n, y_{\overline{1}}, \dots, y_{\overline{m}}]$ so that $R \simeq R_1 \otimes R_2$.

Let $i \in V_1$. It easy to check that $(0 :_R (0 :_R x_i \otimes 1)) \simeq (0 :_{R_1} (0 :_{R_1} x_i)) \otimes_k R_2$. As the functor $\bullet \otimes_k R_2$ is exact, then $P^R_{(0:_R(0:_R x_i \otimes 1))}(z) = P^{R_1}_{(0:_{R_1}(0:_{R_1} x_i))}(z)$. As a similar equation holds for $\overline{j} \in V_2$, the asserted formula follows from Theorem 2.1.

PROPOSITION 5.2. *Let Δ_1 and Δ_2 be simplicial complexes, and let $\Delta = \Delta_1 \cup \Delta_2$. Then*

$$\frac{P^R_{\mathrm{Der}_k(R)}(z)}{P^R_k(z)} = \frac{P^{R_1}_{\mathrm{Der}_k(R_1)}(z)}{P^{R_1}_k(z)} + \frac{P^{R_2}_{\mathrm{Der}_k(R_2)}(z)}{P^{R_2}_k(z)} + r_1 \frac{P^R_k(z) - (1+z)P^{R_2}_k(z)}{zP^R_k(z)P^{R_2}_k(z)} + r_2 \frac{P^R_k(z) - (1+z)P^{R_1}_k(z)}{zP^R_k(z)P^{R_1}_k(z)}$$

where $R = k[\Delta]$, $R_i = k[\Delta_i]$ and r_i is the number of vertices in Δ_i connected with every other vertex in Δ_i for $i = 1, 2$.

PROOF. The ring R is nothing but the fiber product of R_1 and R_2 over k (cf. [5] and [10] for local rings. The extension to the case of graded rings and graded module is immediate). The natural projections $p_i : R \longrightarrow R_i$ $(i = 1, 2)$ are large homomorphisms and we consider any R_i-module as a R-module via p_i. Denoting by $\mathbf{m}$, $\mathbf{m}_1$, $\mathbf{m}_2$ the maximal graded ideals of R, R_1, R_2 respectively, we have $\mathbf{m} = \mathbf{m}_1 \oplus \mathbf{m}_2$. It follows that the Poincaré series of R, R_1, R_2 are related ([5, Satz 1]):

$$\frac{1}{P_k^R(z)} = \frac{1}{P_k^{R_1}(z)} + \frac{1}{P_k^{R_2}(z)} - 1.$$

Let $i \in V_1$. Since $(0 :_R x_i) = (0 :_{R_1} x_i) \oplus \mathbf{m}_2$, it follows that $(0 :_R (0 :_R x_i)) = (0 :_{R_1} (0 :_{R_1} x_i))$ except if $(0 :_R x_i) = 0$; in this case we have $(0 :_R (0 :_R x_i)) = \mathbf{m}_1 \oplus 0 \simeq \mathbf{m}_1$. As the map p_1 is large, we get

$$\frac{P^R_{(0:_{R_1}(0:_{R_1}x_i))}(z)}{P_k^R(z)} = \frac{P^{R_1}_{(0:_{R_1}(0:_{R_1}x_i))}(z)}{P_k^{R_1}(z)}$$

and

$$\frac{P^R_{\mathbf{m}_1}(z)}{P_k^R(z)} = \frac{P^{R_1}_{\mathbf{m}_1}(z)}{P_k^{R_1}(z)}.$$

A similar formula is obtained for $\overline{j} \in V_2$. Then using Theorem 2.1, we obtain the required formula for $P^R_{\mathrm{Der}_k(R)}(z)$.

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